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18.02 Multivariable Calculus Fall 2007

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18.02 Practice Exam 3B – Solutions

1. a)
$$y = 2x$$
 (1,2) $x = 1$ (1,1) $y = x$

b)
$$\int_0^1 \int_{y/2}^y dx dy + \int_1^2 \int_{y/2}^1 dx dy$$
.

1. a) $y = 2x / \begin{cases} x = 1 \\ (1,1) \end{cases}$ b) $\int_{0}^{1} \int_{y/2}^{y} dx dy + \int_{1}^{2} \int_{y/2}^{1} dx dy.$ (the first integral corresponds to the bottom half $0 \le y \le 1$, the second in the second half 1 < y < 2.)

2. a)
$$\delta dA = \frac{r\sin\theta}{r^2} r dr d\theta = \sin\theta dr d\theta$$
.

$$M = \iint_{R} \delta dA = \int_{0}^{\pi} \int_{1}^{3} \sin \theta \ dr d\theta = \int_{0}^{\pi} 2 \sin \theta d\theta = \left[-2 \cos \theta \right]_{0}^{\pi} = 4.$$

b)
$$\bar{x} = \frac{1}{M} \iint_R x \delta dA = \frac{1}{4} \int_0^{\pi} \int_1^3 r \cos \theta \sin \theta dr d\theta$$

The reason why one knows that $\bar{x} = 0$ without computation is that the region and the density are symmetric with respect to the y-axis $(\delta(x,y) = \delta(-x,y))$.

- **3.** a) $N_x = -12y = M_y$, hence **F** is conservative.
- b) $f_x = 3x^2 6y^2 \Rightarrow f = x^3 6y^2x + c(y) \Rightarrow f_y = -12xy + c'(y) = -12xy + 4y$. So c'(y) = 4y, thus $c(y) = 2y^2$ (+ constant). In conclusion

$$f = x^3 - 6xy^2 + 2y^2$$
 (+ constant).

c) The curve C starts at (1,0) and ends at (1,1), therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1) - f(1,0) = (1-6+2) - 1 = -4.$$

4. a) The parametrization of the circle C is $x = \cos t$, $y = \sin t$, for $0 \le t < 2\pi$; then $dx = \cos t$ $-\sin t dt$, $dy = \cos t dt$ and

$$W = \int_0^{2\pi} (5\cos t + 3\sin t)(-\sin t)dt + (1 + \cos(\sin t))\cos tdt.$$

b) Let R be the unit disc inside C;

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (N_x - M_y) dA = \iint_R (0 - 3) dA = -3 \operatorname{area}(R) = -3\pi.$$

5. a)
$$(0,4)$$

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b) On C_4 , x = 0, so $\mathbf{F} = -\sin y \,\hat{\mathbf{j}}$, whereas $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$. Hence $\mathbf{F} \cdot \hat{\mathbf{n}} = 0$. Therefore the flux of \mathbf{F} through C_4 equals 0. Thus

$$\int_{C_1+C_2+C_3} \mathbf{F} \cdot \hat{\mathbf{n}} \, ds = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}} \, ds - \int_{C_4} \mathbf{F} \cdot \hat{\mathbf{n}} \, ds = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}} \, ds ;$$

and the total flux through $C_1 + C_2 + C_3$ is equal to the flux through C.

6. Let
$$u = 2x - y$$
 and $v = x + y - 1$. The Jacobian $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3$.

Hence dudv = 3dxdy and $dxdy = \frac{1}{3}dudv$, so that

$$\begin{split} V &= \iint\limits_{(2x-y)^2 + (x+y-1)^2 < 4} (4 - (2x-y)^2 - (x+y-1)^2) \, dx dy \\ &= \iint\limits_{u^2 + v^2 < 4} (4 - u^2 - v^2) \frac{1}{3} du dv \\ &= \int_0^{2\pi} \int_0^2 (4 - r^2) \frac{1}{3} r dr d\theta = \int_0^{2\pi} \left[\frac{2}{3} r^2 - \frac{1}{12} r^4 \right]_0^2 d\theta \\ &= \int_0^{2\pi} \frac{4}{3} d\theta = \frac{8\pi}{3}. \end{split}$$