MA3238 / STATS217 Stochastic Processes I
AY24/25, ST
Notes

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Contents

1 Probability Review

1.1 Sample Space and Probability Distribution

The sample space is a nonempty set representing the collection of all possible outcomes of an experiment. The probability distribution / measure / law assigns to each subset (event) E a value between 0 and 1.

- $P(0) = 0$
- $P(S) = 1$
- If E1, E2, E3... are pairwise disjoint, then the probability of their union is the sum of the probabilities.
- If E is a subset of F, then $P(E) \leq P(F)$
- $P(E) = 1 - P(E')$
- The probability of the union of many events is always less than or equal to the sum of the probabilities of each event.

1.1.1 Monotone Sequences

A sequence $(E_n)_{n\geq 1}$ is monotone if it is either increasing or decreasing:

- $E1 \subset E2 \subset E3...$ where E is the union of all E up to n Vice versa for decreasing

1.1.2 Limit Superior and Inferior

The limit superior represents the set of points that belong to infinitely many events in En, used to capture the most persistent or long lasting elements of the sequence. It is the set of outcomes that keep reappearing as n goes to infinity.

For example, flipping a coin infinitely many times and having a success occur when it lands on heads.

The limit inferior represents a set of points that belong to all but finitely many events of En. If the set is in Ek, then it is in all sets for Ek+1, and so on,

For example, a coin that progressively gets less likely to show heads at a rate of $(\frac{1}{n^2}$. Since this is a geometric series, it can be said that the expected number of heads is finite.

1.2 Independence and the Borel-Cantelli Lemma

1.2.1 Independence

Events are independent if $P(E_1 \cap E_2...) = P(E_1)P(E_2)...$ Note that if all elements in the family are pairwise independent, it does not imply that the entire family together is independent.

1.2.2 Borel-Cantelli Lemma

(a) If $\sum_{n=1}^{\infty} P(E_n) < \infty,$ then $P\left(\limsup_{n\to\infty} E_n\right) = 0.$

If the total probability of the events is finite, then the probability that infinitely many of these events occur is zero. (If the events become less likely as n increases, then the chances of infinitely many of them happening approaches 0, for example $\frac{1}{n^2}$ from earlier.

(b) If $\sum_{n=1}^{\infty} P(E_n) = \infty$ and the events E_n are independent, then $P\left(\limsup_{n\to\infty} E_n\right) = 1.$

If the total probability of the events are infinite, and the events are independent, then the probability that infinitely many of these events occur is one (guaranteed).

1.3 Random Variables, their Distributions, and Related Quantities

1.4 Conditional Probabilities and Expectations

A function X mapping from the sample space to R is called a random variable. This function is bijective, and the function X has an 'inverse' function that will get the pre image of the output. When converting to a distribution, it is unique and can result in the CDF, and the distribution of a random variable is completely characterized by its cdf F.

- F is nondecreasing
- F is right continuous
- The limit at neg inf of F is 0 and the limit at inf of F is 1.

X can be discrete or continuous.

1.5 Lack of Memory Property and Exponential Distributions

2 Discrete-time Markov Chains

2.1 Markov Chains

A sequence taking values in a at most countable set S is called a time homogenous Markov Chains if it satisfies these properties:

- $P(M_{n+1} = j|M_n = i, M_{n-1} = i_2 \dots) = P(M_{n+1} = j|M_n = i$, or that the next state only depends on the current state. This is the markov property
- This property is independent of n, or temporal, or time homogenous.

A markov chain is fully characterized by the transition probabilities P_{ij} and the initial distribution M_0

2.1.1 Chapman-Kolmogorov

$\forall m, n \in N_0, i, j \in S, P_{ij}^{m+n} = \sum_{k \in S} P_{ik}^m P_{kj}^n$

With matrices, it is simply $P^{m+n} = P^m \cdot P^n$

2.1.2 Stopping times

TODO. Something non anticipation property.The first entrance time, and the last entrance time? These are important as it allows us to analyze the behaviors of processes better

2.1.3 Strong Markov Property

TODO.

3 Transition Matrix and Classification of States

3.1

4 Stationary Distribution and Ergodic Theorems

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5 Absorption Problems

5.1 Gambler's Ruin

5.2 Branching Processes

6 Reversibility and Detailed Balance Equations

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7 Homogeneous and non-homogeneous Poisson Processes

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8 Applications and Properties of Poisson Processes

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9 Continuous-time Markov Chains

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10 Interpretation (Clock Mechanism) of the Q-Matrix

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11 Continuous-time Markov Chains in queuing

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