

MA1521 Calculus for Computing
AY23/24, Y1S1
Notes

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1 Real Numbers and Functions

1.1 Numbers

- Complex Numbers (C)
- Real Numbers (R)
 - Transcendental
 - Irrational
 - Rational (Q)
 - Integers (Z)
 - Natural Numbers (N)
- Absolute Value

1.2 Functions

- A function $f : A \rightarrow B$ assigns each element $a \in A$ one specific member $f(a) \in B$.
 - **Domain:** The set A . **Maximal Domain:** The largest possible set for which $f(x)$ is defined. **Subdomain.**
 - **Range:** The set of all values $f(a)$ for $a \in A$.
 - **Composition:** $(f \circ g)(x) = f(g(x))$, $(g \circ f)(x) = g(f(x))$.
 - **Polynomials:** A polynomial of degree n has at most n real roots, and exactly n roots when counting complex roots (with multiplicity).
 - **Inverse Function** (f^{-1}): Exists if f is a bijection.
 - **Injective** (One-to-One)
 - **Surjective** (Onto)
 - **Bijective** (Both Injective and Surjective)
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2 Limits and Continuity

2.1 Limits

- **Left-hand limit:** $x \rightarrow c^-$, when x approaches c from the left (negative side).
- **Right-hand limit:** $x \rightarrow c^+$, when x approaches c from the right (positive side).

- $\lim_{x \rightarrow c} f(x)$ exists and equals L if $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$.
- **Asymptotes:** Occur as x approaches $\pm\infty$ or as $x \rightarrow n$ where $f(x) = \pm\infty$.
- **Squeeze Theorem**

2.2 Continuity

- A function is **Continuous** at c if $\lim_{x \rightarrow c} f(x) = f(c)$.
 - **Discontinuous**
 - **Continuity on an Interval** ($[a, b]$): Continuous on (a, b) , right continuous at a , and left continuous at b .
 - **Intermediate Value Theorem (IVT):** If f is continuous on $[a, b]$, it takes on every value between $f(a)$ and $f(b)$.
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3 Derivatives and Applications

3.1 Derivatives

- The derivative $f'(x)$ is used to find the **gradient** of the tangent line to $f(x)$ at any point using limits.
- **Rules:**
 - **Chain Rule**
 - **Product Rule**
 - **Quotient Rule**
- **Implicit Differentiation:** Differentiate both sides with respect to x , treating y as a function of x (so $\frac{d}{dx}y = y'\frac{dy}{dx}$), then make $\frac{dy}{dx}$ the subject.
- **Inverse Differentiation:** $\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}$.
- **Parametric Equations:** $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. Examples include **Ellipses**, **Circles**, and **Hyperbolas**.

3.2 Applications of Differentiation

- **Tangents and Normals:**
 - Tangent line gradient m is $f'(x)$.

- **Normal** line gradient is $-1/m$ (negative reciprocal).
 - **Increasing/Decreasing:** $f(x)$ is increasing on (a, b) if $f'(x) > 0$; decreasing if $f'(x) < 0$.
 - **Concavity:**
 - **Concave Upward** if $f''(x) > 0$.
 - **Concave Downward** if $f''(x) < 0$.
 - **Related Rates**
 - **Critical Points:** Occur when $f'(x) = 0$ or $f'(x)$ is undefined.
 - **Local Maximum:** $f'(x)$ changes from positive to negative ($f''(x) < 0$, concave down).
 - **Local Minimum:** $f'(x)$ changes from negative to positive ($f''(x) > 0$, concave up).
 - **Points of Inflexion:** Concavity changes (e.g., $f''(x) = 0$, sometimes).
 - **Rolle's Theorem / Mean Value Theorem (MVT):** If $f(x)$ is differentiable and continuous on $[a, b]$, and $f(a) = f(b)$, then there is at least one c , $a < c < b$, where $f'(c) = 0$.
 - **L'Hôpital's Rule:** Used to evaluate limits of indeterminate forms ($\frac{0}{0}$ or $\frac{\infty}{\infty}$).
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4 Integrals and Applications

4.1 Integrals

- **Antiderivative:** $\int f(x)dx$ is the reverse of differentiation. Remember the constant of integration, $+C$.
- **Fundamental Theorem of Calculus (FTC) / Riemann Sums:** $\int_a^b x^2 dx = \left[\frac{x^3}{3} \right]_a^b = \frac{b^3}{3} - \frac{a^3}{3}$.
- **Improper Integrals:** Used when the bounds are $\pm\infty$ or an asymptote. Calculated using limits, e.g., $\lim_{t \rightarrow \infty} \int_a^t f(x)dx$.

4.2 Integration Techniques

- **Partial Fractions:**

1. Ensure it is a proper fraction.

2. Factorize the denominator.
 3. Decompose the fraction into simpler terms (e.g., $\frac{A}{g(x)} + \frac{B}{h(x)}$).
 4. Integrate each term, often resulting in $A \ln |g(x)| + \dots + C$.
- **By Substitution** (u -substitution): Let u be the inner function. Find $du = f'(x)dx$, and substitute u and du into the integral.
 - **By Parts:** $\int u dv = uv - \int v du$. Differentiate the "easier" function (u) until it is 0, and integrate the "harder" function (dv).

4.3 Applications of Integrals

- **Volume of Solid of Revolution:**
 - **Disk Method:** $V = \pi \int_a^b [f(x)]^2 dx$. Stacks disks with radius $f(x)$ along the axis of rotation (e.g., x -axis).
 - **Shell Method:** $V = 2\pi \int_a^b x f(x) dx$. Integrates the surface area of nested cylindrical shells with height $f(x)$ and radius x .
- **Arc Length:** Calculated using the distance formula (Pythagoras theorem) on infinitesimal changes in x and y :

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

5 Sequences and Series

5.1 Tests for Convergence

Let $\sum a_n$ be an infinite series.

- **n -th Term Test for Divergence:** If $\lim_{n \rightarrow \infty} a_n \neq 0$, the series diverges. If the limit is 0, the test is inconclusive.
- **Integral Test:** If $f(x)$ is continuous, positive, and decreasing, and $a_n = f(n)$, the series $\sum a_n$ converges if and only if $\int_1^\infty f(x) dx$ converges.
- **Comparison Test:** If $0 < a_n < b_n$:
 - If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.
 - If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

- **Ratio Test:** $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$.
 - If $0 \leq L < 1$, the series is **absolutely convergent**.
 - If $L > 1$, the series is **divergent**.
 - If $L = 1$, the test is inconclusive.
- **Root Test:** $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$. Conclusion is the same as the Ratio Test.
- **Absolute Convergence:** If $\sum |a_n|$ converges, then $\sum a_n$ converges.
- **Alternating Series Test:** For an alternating series $\sum (-1)^{n-1} b_n$ (where $b_n > 0$), the series converges if $b_n \geq b_{n+1}$ (decreasing) and $\lim_{n \rightarrow \infty} b_n = 0$.

5.2 Power Series

- **Power Series:** $\sum c_n(x - a)^n$.
- **Interval of Convergence:** The series converges at:
 1. Only $x = a$.
 2. Everywhere (R).
 3. An interval with a radius of convergence R , i.e., $-R + a$ to $R + a$ (checking endpoints).
- **Taylor Series:** Represents a function $f(x)$ as a power series centered at a :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

- **Maclaurin Series:** The special case of the Taylor series when the center is $a = 0$.
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6 Vectors and Geometry of Space

6.1 Vectors and Dot Product

- **Distance between points** in an n -dimensional space.
- **Equation of a 3D Sphere:** $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$.
- Vectors are **closed under scalar multiplication and addition**.
- **Zero Vector (0)**.

- **Unit Vector (\mathbf{u}):** A vector with the same direction but a length of 1. Calculated as $\mathbf{v}/\|\mathbf{v}\|$.
- **Dot Product ($\mathbf{a} \cdot \mathbf{b}$):**
 - $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$, where θ is the angle between \mathbf{a} and \mathbf{b} .
 - $\theta = \cos^{-1} \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \right)$.
- **Projection**

6.2 Cross Product and Lines

- **Cross Product ($\mathbf{a} \times \mathbf{b}$):** A vector **perpendicular (orthogonal)** to both \mathbf{a} and \mathbf{b} . The direction is determined by the **right-hand rule**.
- Magnitude: $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta$.
- **Parametric Equation of a Line** in 3D: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$.

6.3 Planes

- A plane is represented by its **normal vector** and a point on the plane, or by the equation $ax + by + cz + d = 0$.
 - The **angle between two planes** is the angle between their normal vectors.
 - **Cases for 2 Planes:** Identical ($a = b$), parallel ($a // b$), or intersect on a line (a, c intersects on a line).
 - **Cases for 3 Planes:** Range from $a = b = c$, $a = b // c$, $a // b // c$, $a // b$ and c intersects both on a line, $a = b$ and c intersects both on a line, each intersects each at a line, all intersect at a line, or intersect at a point.
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7 Multivariable Functions

7.1 Basic Definitions

- **Domain of R^2 :** The set of all ordered pairs (x, y) for which $f(x, y)$ is defined.
- **Level Curve:** $f(x, y) = k$ for some constant k .
- **Contour Plot:** Multiple level curves for a bunch of k 's.
- **Cylinder:** A surface where all planes parallel to a plane P intersect the surface in the same curve.

- **Quadric Surface:** A surface defined by a second-degree equation in x, y, z . Examples include **Elliptic Paraboloid** and **Ellipsoid** (cuboid but spherical).

7.2 Partial Derivatives

- **Partial Derivative f_x :** Treat y as a constant and differentiate with respect to x .
- **Second Partial Derivatives:** $f_{xx}, f_{yy}, f_{xy}, f_{yx}$.
- **Clairaut's Theorem:** $f_{xy} = f_{yx}$ if continuous (always true for MA1521).
- **Tangent Planes:** The equation for the tangent plane to $z = f(x, y)$ at (a, b) is:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- **Chain Rule (Multivariable):**

$$\frac{dw}{du} = \frac{\partial w}{\partial x} \frac{dx}{du} + \frac{\partial w}{\partial y} \frac{dy}{du}$$

- **Implicit Differentiation (Level Surface):** For $F(x, y, z) = 0$, if $F_z(x, y, z) \neq 0$, then $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$.

7.3 Directional Derivatives and Critical Points

- **Gradient Vector:** $\nabla f(x, y) = \langle f_x, f_y \rangle$. Is **normal to the level curve**.
- **Directional Derivative ($D_{\mathbf{u}}f(x, y)$):** $D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$, where \mathbf{u} is a unit vector.
- Rate of Increase is maximised when direction is $\mathbf{P} = \nabla f(x, y)$.
- Rate of Decrease is maximised when direction is $\mathbf{P} = -\nabla f(x, y)$.
- **Critical Points:** Occur where $f_x(a, b) = f_y(a, b) = 0$, or if they do not exist.
- **Second Derivative Test:** Use the discriminant $D = f_{xx}f_{yy} - f_{xy}^2$.
 - If $D > 0$ and $f_{xx} > 0$: **Local Minimum**.
 - If $D > 0$ and $f_{xx} < 0$: **Local Maximum**.
 - If $D < 0$: **Saddle Point**.
 - If $D = 0$: No conclusion.

8 Double Integrals

8.1 Double Integrals over General Regions

- **Type 1 Region:** x is well-behaved, y follows a function of x (Up-Down integration).
- **Type 2 Region:** y is well-behaved, x follows a function of y (Left-Right integration).
- If it is both, pick the easier one or **interchange order of integration**.
- If it is neither, **decompose** into smaller sub-areas and double integrate separately.
- **Polar Coordinates:**

$$- r^2 = x^2 + y^2$$

$$- x = r \cos \theta$$

$$- y = r \sin \theta$$

- **Surface Area:** Can convert to polar too, if the domain D is a polar rectangle.
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9 Ordinary Differential Equations (ODEs)

9.1 First-Order ODEs

- Separable
- Linear (Integrating Factor)
- Bernoulli

9.2 More on ODEs

- Additional advanced topics in ODEs.