MA1522 Linear Algebra for Computing
AY23/24, Y1S1
Notes

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Contents

1 Linear Equations

1.1 Linear Equation

A **linear equation** is an equation of the form $ax_1 + bx_2 + cx_3 + \ldots + zx_z = n(x_0)$.

- $a, b, c, \ldots$ are the **coefficients**.
- $x_1, x_2, x_3, \ldots$ are the **variables**.

2 Matrix Algebra

2.1 Linear Systems / Matrices

A system of linear equations can be represented by a matrix.

2.1.1 Consistency of Linear Systems

- **Inconsistent:** No solution.
- **Consistent:** Has at least one solution.
  - **One solution**.
  - **Infinite solutions** (e.g., the equations describe the same line).

2.1.2 Augmented Matrix

An **Augmented Matrix** is a matrix that contains the constants from the right-hand side of the linear equations, separated from the coefficient matrix by a vertical line.

2.1.3 Matrix Naming and Size

- A matrix is referred to as $m \times n$ (read as "m by n"), where m is the number of **rows** and n is the number of **columns**.
- A **Square matrix** of order n has n rows and n columns.

2.2 Row Operations and Echelon Forms

2.2.1 Fundamental Row Operations

- **Replacement:** Adding or subtracting a multiple of one row to another row (e.g., $R_i + cR_j \to R_i$).
- **Interchange:** Swapping two rows ($R_i \leftrightarrow R_j$).
- **Scaling:** Multiplying a row by a non-zero scalar (e.g., $cR_x \to R_x$).

2.2.2 Echelon Forms

- **Echelon form:**
  - Any row of all zeros is at the bottom of the matrix.
  - The leading non-zero entry (pivot) of each non-zero row is in a column to the right of the leading entry of the row above it (staircase pattern).
- **Reduced Row Echelon Form (RREF):** An echelon form matrix where:
  - Each leading entry is 1 (a **leading 1**).
  - Each leading 1 is the only non-zero entry in its column.
- **Pivot Columns:** Columns that contain a leading 1.
  - Variables corresponding to pivot columns are **Basic Variables**.
  - Other variables are **Free Variables**.

2.3 Vectors and Span

2.3.1 Vectors

- A **vector** is an $n \times 1$ matrix. The set of all $n \times 1$ matrices is denoted R^n.
- A vector represents a line segment from the origin to a point in n-dimensional space.
- The **zero vector** ($\mathbf{0}$) or **null vector** is a vector where all entries are zero.

2.3.2 Span

- Span$\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$ (points on the line that passes through $\mathbf{0}$ and $\mathbf{v}$).
- Span$\{\mathbf{u}, \mathbf{v}\}$ will describe a plane in R^n space, provided $\mathbf{u}$ is not a scalar multiple of $\mathbf{v}$.
- Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ is the set of all possible linear combinations $a\mathbf{v}_1 + b\mathbf{v}_2 + \ldots$.

2.4 Matrix and Vector Equations

2.4.1 Matrix Equation

A **matrix equation** is written as $A\mathbf{x} = \mathbf{b}$, where:

- A is an $m \times n$ matrix.
- $\mathbf{x}$ is an $n \times 1$ vector (A must have as many columns as $\mathbf{x}$ has rows).
- $\mathbf{b}$ is an $m \times 1$ vector.

2.4.2 Homogeneous Systems

- A system is **homogeneous** if $A\mathbf{x} = \mathbf{0}$.
- A homogeneous system always has the **trivial solution** ($\mathbf{x} = \mathbf{0}$).
- If a homogeneous system has a free variable, it has infinite solutions.
- The solution set of a homogeneous system is a **subspace** called the **nullspace** ($\text{Null}(A)$).

2.4.3 Parametric Forms

The solution set of a linear system can be expressed in **parametric forms**:

- **Line:** $\mathbf{x} = s\mathbf{u}$.
- **Plane:** $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$.

2.5 Matrix Operations

2.5.1 Matrix Equality

Two matrices are equal if they have the same size (rows and columns) and their corresponding entries are equal.

2.5.2 Basic Operations

- **Addition:** Only possible if the matrices have the same size.
- **Scalar Multiplication:** Multiplying every entry in a matrix by a scalar.

2.5.3 Matrix Multiplication

- AB is only possible if the number of columns of A equals the number of rows of B.
- The entry in row i and column j of the product AB is given by $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots$.
- **Right multiplying** B to A means AB.
- **Left multiplying** B to A means BA.

2.5.4 Matrix Powers

- **Power:** Multiplying a square matrix by itself ($A^m = A \cdot A \cdot \ldots \cdot A$).
- **Power 0:** $A^0 = I$, where I is the **Identity Matrix** (a square matrix with 1s on the main diagonal and 0s elsewhere).

2.5.5 Transpose

- **Transpose (A^T):** Rows become columns, and columns become rows.
- **Property:** $(AB)^T = B^T A^T$.

3 Determinants and Inverses

3.1 Matrix Inverse

3.1.1 Definition and Calculation

- **Inverse (A^{-1}):** For a square matrix A, C is the inverse if $AC = I$ and $CA = I$ (written $C = A^{-1}$).
- A is **invertible** if and only if $\det(A) \neq 0$.
- **For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:**

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

.

- **Solution to $A\mathbf{x} = \mathbf{b}$:** If A is invertible, the unique solution is $\mathbf{x} = A^{-1}\mathbf{b}$.
- **Inverse of a Product:** $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

3.1.2 Elementary Matrices

- An **Elementary Matrix** (E) is a matrix obtained by performing a single row operation on the identity matrix I.
- If $A \sim E_1A \sim E_2E_1A = I$, then $A^{-1} = E_k \dots E_2E_1$.
- **Finding A^{-1}:** The inverse can be found by row reducing the augmented matrix $(A|I)$ to $(I|A^{-1})$.

3.1.3 Adjoint Matrix

- **Adjoint (adj(A)):** The transpose of the matrix of cofactors.
- **Inverse Formula:** $A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$.

3.2 Determinant

3.2.1 Calculation

- **Cofactor:** $C_{ij} = (-1)^{i+j} \det(A_{ij})$, where A_{ij} is the submatrix obtained by removing row i and column j.
- **Cofactor Expansion (Theorem 3.1):** The determinant of A can be calculated by cofactor expansion along any row i or column j:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots$$

.

- **Strategy:** Choose the row or column with the most zeros.
- **Triangular Matrix:** The determinant is the product of the diagonal entries.

3.2.2 Determinant Algebra

- **Row Replacement ($R_i + cR_j \to R_i$):** The determinant remains unchanged.
- **Row Interchange ($R_i \leftrightarrow R_j$):** The determinant changes sign (becomes negative).
- **Row Scaling ($cR_i \to R_i$):** The new determinant is $c \cdot \det(A)$.
- **Transpose (A^T):** $\det(A^T) = \det(A)$.
- **Product Rule:** $\det(AB) = \det(A)\det(B)$.
- **Inverse Rule:** $\det(A^{-1}) = (\det(A))^{-1}$.

3.2.3 Cramer's Rule

Cramer's Rule provides a formula for the unique solution $\mathbf{x} = (x_1, \dots, x_n)$ of $A\mathbf{x} = \mathbf{b}$ when $\det(A) \neq 0$:

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)}$$

where $A_i(\mathbf{b})$ is the matrix formed by replacing the i-th column of A with the vector $\mathbf{b}$.

3.3 LU Factorization

- **Factorization:** $A = LU$, where L is a **lower triangular matrix** (1s on the diagonal and non-zero entries only below) and U is an **upper triangular matrix** (the Echelon form of A using only row replacements).
- **Solving $A\mathbf{x} = \mathbf{b}$:** This is solved in two steps:
  1. Solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ (using forward substitution or RREF of $(L|\mathbf{b})$).
  2. Solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ (using backward substitution or RREF of $(U|\mathbf{y})$).

4 Vector Spaces and Subspaces

4.1 Subspaces

A subset V of R^n is a **subspace** if it is a linear span of a set of vectors and satisfies:

1. V **contains the origin** ($\mathbf{0} \in V$).
2. V is **closed under addition** (if $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$).
3. V is **closed under scalar multiplication** (if $\mathbf{u} \in V$ and $a \in R$, then $a\mathbf{u} \in V$).

- **Spans as Subspaces:** $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ is always a subspace.

4.2 Linear Independence and Basis

4.2.1 Linear Independence

- A set of vectors is **linearly independent** if the only solution to the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots = \mathbf{0}$ is the trivial solution $c_1 = c_2 = \ldots = 0$.
- A set is linearly independent if the RREF of the matrix formed by the vectors as columns has **no non-pivot columns**.
- A set of more than n vectors in R^n space is always **linearly dependent**.
- The zero vector $\mathbf{0}$ always makes a set linearly dependent.

4.2.2 Basis

A set S is a **basis** for a vector space V if:

1. $\text{Span}(S) = V$.
2. S is **linearly independent**.

4.3 Fundamental Subspaces

Let A be an $m \times n$ matrix.

4.3.1 Column Space

- **Column Space (Col(A)):** The span of the columns of A. $\text{Col}(A) \subseteq R^m$.
- $\text{Col}(A) = \{\mathbf{v} \in R^m \mid A\mathbf{x} = \mathbf{v} \text{ is consistent}\}$.
- **Basis of Col(A):** The pivot columns of the original matrix A (not RREF(A)).
- **Note:** $\text{Col}(A) \neq \text{Col}(\text{RREF}(A))$, as row operations do not preserve the column space.

4.3.2 Row Space

- **Row Space (Row(A)):** The span of the rows of A. $\text{Row}(A) \subseteq R^n$.
- $\text{Row}(A) = \text{Row}(\text{RREF}(A))$.
- **Basis of Row(A):** The non-zero rows of $\text{RREF}(A)$. /ul

4.3.3 Nullspace

- **Nullspace (Null(A)) or Kernel (Ker(T)):** The solution space to the homogeneous system $A\mathbf{x} = \mathbf{0}$.
- $\text{Null}(A) = \{\mathbf{v} \in R^n \mid A\mathbf{v} = \mathbf{0}\}$.

4.4 Rank and Nullity

- **Rank (rank(A)):** $\dim(\text{Col}(A))$. It equals the number of pivot columns in $\text{RREF}(A)$ and the number of non-zero rows in $\text{RREF}(A)$ ($\dim(\text{Row}(A))$).
- **Nullity (nullity(A)):** $\dim(\text{Null}(A))$. It equals the number of non-pivot columns (free variables) in $\text{RREF}(A)$.
- **Rank-Nullity Theorem:** $\text{rank}(A) + \text{nullity}(A) = \text{number of columns of } A$.

4.4.1 Consistency and Rank

The system $A\mathbf{x} = \mathbf{b}$ is **consistent** if and only if $\text{rank}(A) = \text{rank}([A\mathbf{b}])$. If inconsistent, $\text{rank}(A) < \text{rank}([A\mathbf{b}])$.

4.4.2 Full Rank (Invertibility Conditions)

- **If A is $n \times n$ (square):** A is invertible $\iff$ $\text{rank}(A) = n$ ($\iff$ $\text{RREF}(A) = I_n$).
- **If A is $m \times n$ and rank$(A) = n$ (full column rank, $m \geq n$):**
  - $\text{Col}(A)$ is R^m.
  - Columns of A are linearly independent.
  - $\text{Null}(A) = \{\mathbf{0}\}$ ($\text{nullity}(A) = 0$).
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  - $A^T A$ is an invertible matrix of order n.
  - A has a **left inverse** B such that $BA = I_n$.
- **If A is $m \times n$ and rank$(A) = m$ (full row rank, $m \leq n$):**

* $\mathrm{Col}(A) = R^m$.
* $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in R^m.
* Rows of A are linearly independent.
* $\mathrm{RREF}(A)$ has no zero rows.
* AA^T is an invertible matrix of order m.
* A has a **right inverse** B such that $AB = I_m$.

5 Orthogonal Projection and Least Squares

5.1 Dot Product, Norm, and Angle

- **Inner Product (Dot Product):** For vectors $\mathbf{u}, \mathbf{v}$, the inner product is $\mathbf{u}^T\mathbf{v}$, which results in a 1×1 matrix (a scalar value).
- **Outer Product:** $\mathbf{u}\mathbf{v}^T$, which results in an $n \times n$ matrix.
- **Length / Norm ($||\mathbf{u}||$):** The distance of $\mathbf{u}$ from the origin. Calculated as $||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$.
- **Unit Vector:** A vector $\mathbf{u}$ such that $||\mathbf{u}|| = 1$. Found by $\mathbf{u} = \mathbf{v}/||\mathbf{v}||$.
- **Angle between u and v:**

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \cdot ||\mathbf{v}||}\right)$$

.

5.2 Orthogonality

- **Orthogonal Vectors:** $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if their dot product is zero ($\mathbf{u} \cdot \mathbf{v} = 0$). The zero vector is orthogonal to every vector.
- **Orthogonal Set (Basis S):** A set of non-zero vectors where every pair of distinct vectors is orthogonal.
- **Orthonormal Set (Basis S):** An orthogonal set where every vector is a unit vector.
- **Orthogonal Matrix (A):** A square matrix whose columns are an orthonormal basis for R^n. This implies $A^T A = I = AA^T$ and $A^T = A^{-1}$.

5.3 Orthogonal Projection

- **Coordinates relative to an Orthogonal Basis** $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$**:** If $\mathbf{v} = c_1\mathbf{u}_1 + \ldots + c_n\mathbf{u}_n$, the coordinate c_i is found by:

$$c_i = \frac{\mathbf{u}_i \cdot \mathbf{v}}{||\mathbf{u}_i||^2} = \frac{\mathbf{u}_i \cdot \mathbf{v}}{\mathbf{u}_i \cdot \mathbf{u}_i}$$

.

- **Coordinates relative to an Orthonormal Basis:** If the basis is orthonormal, $c_i = \mathbf{u}_i \cdot \mathbf{v}$.
- **Projection of w onto Subspace** $V = \textbf{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$**:** If B is an orthogonal basis for V, the projection $\mathbf{w}_p$ is:

$$\mathbf{w}_p = \sum_{i=1}^{k} \frac{\mathbf{u}_i \cdot \mathbf{w}}{||\mathbf{u}_i||^2}\mathbf{u}_i$$

.

- **Decomposition:** Any vector $\mathbf{w}$ can be decomposed as $\mathbf{w} = \mathbf{w}_p + \mathbf{w}_n$, where $\mathbf{w}_p$ is the projection of $\mathbf{w}$ onto the subspace (in the subspace) and $\mathbf{w}_n$ is the component normal (orthogonal) to the subspace.

5.4 Gram-Schmidt Process

The **Gram-Schmidt Process** converts a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ for a subspace into an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

- $\mathbf{v}_1 = \mathbf{u}_1$.
- $\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1$.
- $\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{\mathbf{v}_1}\mathbf{u}_3 - \text{proj}_{\mathbf{v}_2}\mathbf{u}_3$.
- If any resulting $\mathbf{v}_i = \mathbf{0}$, the original set was linearly dependent.

5.5 QR Factorization

- **Factorization:** $A = QR$, where A must have linearly independent columns.
- Q is an orthogonal matrix obtained by performing the Gram-Schmidt process on the columns of A. $Q^TQ = I$.
- R is an upper triangular matrix with positive diagonal entries, calculated as $R = Q^TA$.

5.6 Least Squares Approximation

- **Least-Squares Solution ($\hat{\mathbf{x}}$):** A vector $\hat{\mathbf{x}}$ is a least-squares solution to $A\mathbf{x} = \mathbf{b}$ if $||A\hat{\mathbf{x}} - \mathbf{b}|| \leq ||A\mathbf{v} - \mathbf{b}||$ for all $\mathbf{v}$.
- $A\hat{\mathbf{x}}$ is the vector in $\text{Col}(A)$ that is **closest to b**, i.e., $A\hat{\mathbf{x}} = \text{proj}_{\text{Col}(A)}\mathbf{b}$.
- **Normal Equations:** The least-squares solutions are the solutions $\hat{\mathbf{x}}$ to the system:
$$A^T A\mathbf{x} = A^T\mathbf{b}$$
.
- **Unique Solution:** The least-squares solution $\hat{\mathbf{x}}$ is unique if $A^T A$ is invertible. In this case, $\hat{\mathbf{x}} = (A^T A)^{-1} A^T\mathbf{b}$.

6 Diagonalization and Eigenspaces

6.1 Eigenvalues and Eigenvectors

- **Eigenvalue (λ):** A scalar λ is an eigenvalue of an $n \times n$ matrix A if there is a non-zero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.
- **Eigenvector ($\mathbf{v}$):** The non-zero vector $\mathbf{v}$ is the eigenvector corresponding to the eigenvalue λ.

6.2 Characteristic Polynomial and Eigenspace

- **Characteristic Polynomial (char(A)):** $\det(A - \lambda I)$ or $\det(\lambda I - A)$.
- **Finding Eigenvalues:** The eigenvalues are the roots λ of the characteristic equation $\det(A - \lambda I) = 0$.
- **Eigenspace (E_λ):** The set of all vectors $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. It is the $\text{Null}(A - \lambda I)$ (the solution space to the homogeneous system $(A - \lambda I)\mathbf{x} = \mathbf{0}$).

6.3 Multiplicities

- **Algebraic Multiplicity (AM):** The number of times λ appears as a root of the characteristic polynomial.
- **Geometric Multiplicity (GM):** $\dim(E_\lambda) = \text{nullity}(A - \lambda I)$ (the number of free variables when solving for the eigenspace).
- **Relationship:** $1 \leq$ Geometric Multiplicity $\leq$ Algebraic Multiplicity. If AM $=$ 1, then GM $=$ 1.

6.4 Diagonalization

- **Diagonalizable:** A is **diagonalizable** if it is similar to a diagonal matrix D, meaning $A = PDP^{-1}$ or $P^{-1}AP = D$.
- D is the diagonal matrix made up of the eigenvalues of A.
- P is the matrix whose columns are the corresponding eigenvectors of A.
- **Condition for Diagonalization:** A is diagonalizable if and only if the sum of the geometric multiplicities of all eigenvalues equals n, or if A has n distinct eigenvalues.
- **Application:** Calculating powers of A: $A^m = PD^mP^{-1}$, where D^m is easy to calculate.

6.5 Orthogonal Diagonalization

- **Orthogonally Diagonalizable:** A is **orthogonally diagonalizable** if $A = PDP^T$ (where $P^T = P^{-1}$).
- **Condition:** A must be a **symmetric matrix** ($A^T = A$).
- **Property:** Eigenvectors from different eigenspaces are orthogonal to each other if A is symmetric.
- **Process:** Use Gram-Schmidt to find an orthonormal basis for each eigenspace.

7 Linear Transformations

7.1 Definition and Standard Matrix

- **Linear Transformation (T):** A function $T : R^n \to R^m$ that satisfies:
  1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
  2. $T(c\mathbf{u}) = cT(\mathbf{u})$

  This is equivalent to $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$.
- **Standard Matrix (A):** Every linear transformation T can be represented by an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.
- **Domain and Codomain:** R^n is the **domain**, and R^m is the **codomain**.
- **Image:** $T(\mathbf{x})$ is the image of $\mathbf{x}$ in R^m.

7.2 Range and Kernel

- **Range ($R(T)$):** The set of all images $T(\mathbf{x})$. $R(T) = \text{Col}(A)$. The range is a subspace of the codomain R^m.
- **Kernel (Ker(T)):** The set of vectors $\mathbf{u}$ in the domain R^n that are sent to $\mathbf{0}$ in the codomain. $\text{Ker}(T) = \text{Null}(A)$. The kernel is a subspace of the domain R^n.
- **Rank-Nullity Theorem for Transformations:** $\dim(\text{Range}(T)) + \dim(\text{Ker}(T)) = n$ (dimension of the domain).

7.3 Injective, Surjective, and Bijective

- **Surjective (Onto):** T is surjective if each $\mathbf{b}$ in R^m is the image of at least one $\mathbf{a}$ in R^n.
  - $\iff \text{Range}(T) = R^m$
  - $\iff \text{rank}(T) = m$.
- **Injective (One-to-One):** T is injective if each $\mathbf{b}$ in R^m is the image of at most one $\mathbf{a}$ in R^n.
  - $\iff T(\mathbf{x}) = \mathbf{0}$ only has the trivial solution $\mathbf{x} = \mathbf{0}$.
  - $\iff \text{Ker}(T) = \{\mathbf{0}\}$.
  - $\iff \text{nullity}(T) = 0$.
- **Bijective:** T is bijective if it is both injective and surjective.

7.4 Invertible Transformations

- A linear transformation $T : R^n \to R^n$ (square) is **invertible** if and only if its standard matrix A is invertible.
- **Area/Volume Scaling:** If S is a set in R^n, the volume of $T(S)$ is $|\det(A)| \times$ volume of S.

8 Invertible Matrix Theorem (Summary)

Let A be a square matrix of order n. The following statements are equivalent to A being invertible:

- **Definition / Inverse:**
  - A is invertible.
  - There exists an $n \times n$ matrix C such that $CA = I_n$ (Left Inverse).

    - There exists an $n \times n$ matrix D such that $AD = I_n$ (Right Inverse).
    - A^T is invertible.
  - **Linear Systems:**
    - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.
    - $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ for every $\mathbf{b}$.
  - **Row Operations / Echelon Form:**
    - $\text{RREF}(A) = I_n$.
    - A is a product of elementary matrices.
    - A has n pivot positions.
  - **Determinants / Eigenspaces:**
    - $\det(A) \neq 0$.
    - 0 is not an eigenvalue of A.
  - **Vector Spaces / Rank:**
    - The columns of A form a linearly independent set.
    - The rows of A form a linearly independent set.
    - The columns (or rows) of A form a basis for R^n.
    - $\text{rank}(A) = n$.
    - $\text{nullity}(A) = 0$.
    - A has no non-pivot columns.
  - **Linear Transformations:**
    - The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is **bijective** (one-to-one and onto).
    - $T(\mathbf{x}) = A\mathbf{x}$ maps R^n **onto** R^n (surjective).