Complex Analysis

1. We define the differentiation and integration of the function $f : \mathbb{R} \to \mathbb{C}$ as follows:

$$f'(t) = \frac{d}{dt}[u(t) + iv(t)] = u'(t) + iv'(t), \quad \int f(t)\,dt = \int u(t)\,dt + i\int v(t)\,dt.$$

   (a) From this definition, show that the derivative of the function $f : \mathbb{R} \to \mathbb{C}$ defined by $f(t) = e^{z_0 t}$ is $f'(t) = z_0 e^{z_0 t}$. Here, $z_0 = a + ib$ is a complex number with $a, b \in \mathbb{R}$.
   (b) For real numbers a and b, what is the integral of the function $f(t) = e^{(a+bi)t}$?
   (c) Using the above result, find the integral of the following real-valued function:

$$\int e^{at}\cos(bt)\,dt, \quad \int e^{at}\sin(bt)\,dt.$$

Sol. (a) Note that

$$f(t) = e^{z_0 t} = e^{(a+ib)t} = e^{at}e^{ibt} = e^{at}(\cos(bt) + i\sin(bt)).$$

Then

$$\begin{aligned} u(t) &= \mathrm{Re}\left(e^{z_0 t}\right) = e^{at}\cos(bt), \\ v(t) &= \mathrm{Im}\left(e^{z_0 t}\right) = e^{at}\sin(bt), \end{aligned}$$

and so

$$\begin{aligned} u'(t) &= \frac{d}{dt}\left[e^{at}\cos(bt)\right] = ae^{at}\cos(bt) - be^{at}\sin(bt), \\ v'(t) &= \frac{d}{dt}\left[e^{at}\sin(bt)\right] = ae^{at}\sin(bt) + be^{at}\cos(bt). \end{aligned}$$

Thus

$$\begin{aligned} f'(t) = u'(t) + iv'(t) &= ae^{at}\cos(bt) - be^{at}\sin(bt) + iae^{at}\sin(bt) + ibe^{at}\cos(bt) \\ &= ae^{at}\cos(bt) - i^2be^{at}\sin(bt) + iae^{at}\sin(bt) + ibe^{at}\cos(bt) \\ &= \left(ae^{at} + ibe^{at}\right)(\cos(bt) + i\sin(bt)) \\ &= (a + bi)\cdot e^{at}e^{ibt} \\ &= (a + bi)\,e^{(a+bi)t} \\ &= z_0 e^{z_0 t}. \end{aligned}$$

(b)

$$\int e^{z_0 t}dt = \frac{1}{z_0}e^{z_0 t} = \frac{1}{a + bi}e^{(a+bi)t} + C.$$

(c) (i)

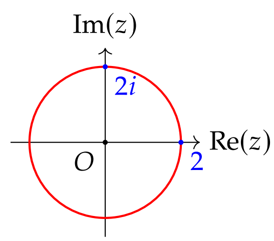
$$\begin{aligned} \int e^{at}\cos(bt)\,dt &= \frac{1}{2}\int e^{at}\left(e^{ibt} + e^{-ibt}\right)dt \\ &= \frac{1}{2}\left(\int e^{(a+bi)t}dt + \int e^{(a-bi)t}dt\right) \\ &= \frac{1}{2}\left(\frac{1}{a+bi}e^{(a+bi)t} + \frac{1}{a-bi}e^{(a-bi)t}\right) + C_1 \\ &= \frac{e^{at}}{2}\left(\frac{a-bi}{a^2+b^2}e^{ibt} + \frac{a+bi}{a^2+b^2}e^{-ibt}\right) + C_1 \\ &= \frac{e^{at}}{2\left(a^2+b^2\right)}\left[(a-bi)\,e^{ibt} + (a+bi)\,e^{-ibt}\right] + C_1. \end{aligned}$$

(ii) Similarly,

$$\int e^{at}\sin(bt)\,dt = \frac{1}{2i}\int e^{at}\left(e^{ibt}-e^{-ibt}\right)dt = \frac{e^{at}}{2\left(a^2+b^2\right)i}\left[(a-bi)\,e^{ibt}-(a+bi)\,e^{-ibt}\right]+C_2.$$

□

2. Let C be the path in the complex plane defined as the counter-clockwise rotation of a circle with center at the origin and radius 2, represented by the function $z(t) = 2e^{it}$ for $t \in [0, 2\pi]$.



Calculate the value of the following integral:

(a) $\displaystyle\int_C \frac{e^z}{z}dz$ (b) $\displaystyle\int_C \frac{z^2}{z-1}dz$ (c) $\displaystyle\int_C \frac{z}{z-3}dz$ (d) $\displaystyle\int_C \frac{\cos z}{z\left(z^2+9\right)}dz$

Sol. (a) Let $f(z) = e^z$ then

$$\oint_C \frac{e^z}{z}dz = \oint_C \frac{f(z)}{z-0}dz = 2\pi i \cdot e^0 = 2\pi i.$$

by the Cauchy integral formula.

(b) Let $f(z) = z^2$ then

$$\oint_C \frac{z^2}{z-1}dz = \oint_C \frac{f(z)}{z-1}dz = 2\pi i \cdot (1)^2 = 2\pi i.$$

by the Cauchy integral formula.

(c) Since $z_0 = 3$ is not inside the curve C, $\displaystyle\oint_C \frac{z}{z-3}dz = 0$ by Cauchy-Goursat theorem.

(d) Let $f(z) = \dfrac{\cos z}{(z+3i)(z-3i)}$ then

$$\oint_C \frac{(\cos z)}{z\left(z^2+9\right)}dz = 2\pi i \cdot \frac{\cos 0}{3i\cdot(-3i)} = \frac{2\pi i}{9}.$$

□

3. Let C be the path in the complex plane defined as the counter-clockwise rotation of a circle with center at the origin and radius 3, represented by the function $z(t) = 3e^{it}$ for $t \in [0, 2\pi]$. Define the function $g : \mathbb{C} \setminus \{C\} \to \mathbb{C}$ by

$$g(z) = \int_C \frac{2\zeta^2-\zeta-2}{\zeta - z}d\zeta, \quad (|z| = 3).$$

Show that $g(2) = 8\pi i$.

Sol. Let $f(\zeta) = 2\zeta^2 - \zeta - 2$. Since $z = 2$ is inside the curve C,

$$g(2) = \oint_C \frac{2\zeta^2-\zeta-2}{\zeta-2} = 2\pi i \cdot f(2) = 2\pi i \cdot \left(2\cdot 2^2 - 2 - 2\right) = 8\pi i$$

by the Cauchy integral formula. □

4. Find the following integral.

   (a) $\displaystyle\int_0^{2\pi} e^{e^{i\theta}}\, d\theta.$

   (b) $\displaystyle\int_0^{2\pi} e^{-i\theta} e^{e^{i\theta}}\, d\theta.$

Sol. Recall that Cauchy integral formula for a function $f(z)$ that is analytic inside and on a simple closed contour C:

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz.$$

(a) Let $z = e^{i\theta}$ and $dz = ie^{i\theta}d\theta$ (i.e., $d\theta = \frac{1}{ie^{i\theta}}dz$). Then, the integral becomes:

$$\begin{aligned}
\int_0^{2\pi} e^{e^{i\theta}}\, d\theta &= \oint_{|z|=1} e^z \cdot \frac{1}{iz}dz \\
&= \frac{1}{i}\oint_{|z|=1} \frac{e^z}{z-0}dz \\
&= \frac{1}{i}\cdot 2\pi i \cdot e^0 \quad \text{by the Cauchy integral formula} \\
&= 2\pi.
\end{aligned}$$

(b) Let $z = e^{i\theta}$ and $dz = ie^{i\theta}d\theta$ (i.e., $d\theta = \frac{1}{ie^{i\theta}}dz$). Then, the integral becomes:

$$\begin{aligned}
\int_0^{2\pi} e^{-i\theta} e^{e^{i\theta}}\, d\theta &= \oint_{|z|=1} z^{-1} \cdot e^z \cdot \frac{1}{iz}dz \\
&= \frac{1}{i}\oint_{|z|=1} \frac{e^z}{z^2}dz \\
&= \frac{1}{i}\oint_{|z|=1} \frac{e^z}{(z-0)^2}dz \\
&= \frac{1}{i}\cdot 2\pi i \cdot e^0 \qquad \because f^{(1)}(a) = \frac{1!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{1+1}}dz \\
&= 2\pi.
\end{aligned}$$

□

5. For a simple closed curve C in the complex plane, let the function $g : \mathbb{C} \setminus \{C\} \to \mathbb{C}$ be defined by

$$g(z) = \int_C \frac{\zeta^3 + 2\zeta}{(\zeta - z)^3}d\zeta, \quad (|z| \neq 3).$$

Show that

$$g(z) = \begin{cases} 6\pi i z & \text{if } z \text{ is in interior of } C, \\ 0 & \text{if } z \text{ is in exterior of } C. \end{cases}$$

holds.

Sol. Let $f(\zeta) = \zeta^3 + 2\zeta$ then

$$f'(\zeta) = 3\zeta^2 + 2, \quad f''(\zeta) = 6\zeta.$$

Note that $f^{(n)}(z) = \dfrac{n!}{2\pi i} \oint_C \dfrac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$. Then we have

$$g(z) = \frac{2\pi i}{2!} \cdot f''(\zeta) = \pi i \cdot (6\zeta).$$

Thus

(i) (z is in interior of C) By the generalized Cauchy integral formula,

$$g(z) = \pi i \cdot 6z = 6\pi i z.$$

(ii) (z is in exterior of C) By the Cauchy-Goursat theorem,

$$g(z) = 0.$$

□

6. Let C be the curve in the complex plane defined by the circular arc $z(t) = 2e^{it}$ for $t \in [0, \frac{\pi}{2}]$ from $z = 2$ to $z = 2i$. Show that the following inequality holds without directly computing the integral.

Sol. Note that

$$\left| \int_C f(z) dz \right| \le ML,$$

where $M = \max_{t \in [a,b]} |f(\gamma(t))|$ and L = length of C. For $z(t) = 2e^{it}$ with $t \in [0, \frac{\pi}{2}]$, we obtain the length of the contour C:

$$L = \int_0^{\frac{\pi}{2}} |z'(t)| dt = \int_0^{\frac{\pi}{2}} 2dt = \pi.$$

Let $f(z) = \dfrac{z+4}{z^3 - 1}$ then $f(2e^{it}) = \dfrac{2e^{it} + 4}{(2e^{it})^3 - 1}$. Since

$$\left| f(2e^{it}) \right| = \left| \frac{2e^{it} + 4}{(2e^{it})^3 - 1} \right| = \frac{|2e^{it} + 4|}{|8e^{3it} - 1|} \le \frac{2+4}{|8e^{3it} - 1|},$$

We have:

$$\begin{aligned} M &= \max_{t \in [0, \frac{\pi}{2}]} \frac{6}{|8e^{3it} - 1|} \\ &= \frac{6}{|8 - 1|} \quad \text{when } t = 0 \\ &= \frac{6}{7}. \end{aligned}$$

Hence

$$\left| \int_C \frac{z+4}{z^3 - 1} dz \right| = \left| \int_C f(z)\, dz \right| \le ML = \frac{6}{7} \cdot \pi = \frac{6\pi}{7}.$$

□