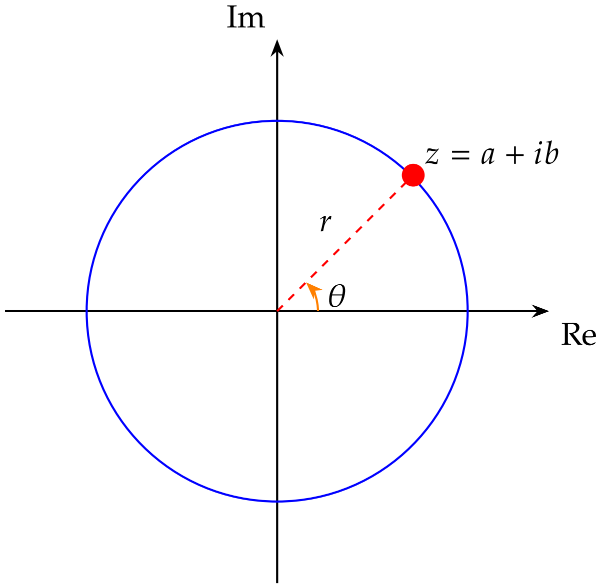
Complex Analysis

Ji Yong-Hyeon



Ji Yong-Hyeon
June 3, 2023

Contents

Chapter 1

The Complex Number System

The complex number system is an extension of the real number system that includes a new type of number called the complex number. A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers and i is the imaginary unit, which is defined as the square root of -1.

The real part of a complex number a + bi is a, and the imaginary part is b. We can represent complex numbers geometrically using the complex plane, which is a two-dimensional plane where the horizontal axis represents the real part of a complex number and the vertical axis represents the imaginary part.

Addition and subtraction of complex numbers are performed by adding or subtracting their real and imaginary parts separately. Multiplication of complex numbers is performed using the distributive property and the fact that $i^2 = -1$. Division of complex numbers is also possible by multiplying both the numerator and denominator by the complex conjugate of the denominator.

The absolute value or modulus of a complex number is the distance between the origin and the point representing the complex number on the complex plane. It is defined as:

$$|a + bi| = \sqrt{a^2 + b^2}$$

The argument or phase of a complex number is the angle that the line connecting the origin to the point representing the complex number makes with the positive real axis. It is defined as:

$$\theta = \arg(a + bi) = \arctan\left(\frac{b}{a}\right)$$

The complex number system is important in mathematics, physics, engineering, and many other fields. It is used to represent quantities that have both a magnitude and a direction, such as electrical currents and electromagnetic waves. Complex numbers also have applications in signal processing, control theory, and cryptography, among others.

1.1 The Field of Complex Numbers

The set of complex numbers, denoted by $\mathbb{C}$, is defined as the collection of all ordered pairs (x, y) where $x, y \in \mathbb{R}$. The operations of addition and multiplication are defined by:

$$(x_1, y_1) + (x_2 + y_2) = (x_1 + x_2, y_1 + y_2),$$
$$(x_1, y_1) \cdot (x_2 + y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).$$

We verify that the axioms for a field are met by the definitions given for $\mathbb{C}$:

(F1) $(\mathbb{C}, +)$ is an "Abelian group",

(F2) $(\mathbb{C} \setminus \{0\}, \cdot)$ is an Abelian group, and

(F3) the distributive law holds: $x, y, z \in \mathbb{C} \implies (x + y) \cdot z = x \cdot z + y \cdot z$.

In (F1), an Abelian group refers to the fact that the operation + on $\mathbb{C}$ satisfies the properties of associativity and commutativity, and

$$\exists e := (0, 0) \in \mathbb{C} : [(x, y) \in \mathbb{C} \implies (x, y) + e = (x, y) = e + (x, y)].$$

Additionally,

$$(x, y) \in \mathbb{C} \implies \exists(-x, -y) \in \mathbb{C} : [(x, y) + (-x, -y) = (0, 0) = (-x, -y) + (x, y)].$$

In condition (F2), the multiplicative identity is (1,0), and the multiplicative inverse of any complex number (x,y) in $\mathbb{C} \setminus \{(0, 0)\}$ is determined by

$$\left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right). \tag{1.1}$$

Exercise 1.1.1. Verify that (1.1) is indeed the inverse of $(x, y) \in \mathbb{C} \setminus \{(0, 0)\}$.

Sol. Let $(x, y) \in \mathbb{C}$. Then

$$\begin{aligned} (x, y) \cdot \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) &= \left(\frac{x^2}{x^2 + y^2} - \frac{-y^2}{x^2 + y^2}, \frac{-xy}{x^2 + y^2} + \frac{xy}{x^2 + y^2} \right) \\ &= \left(\frac{x^2 + y^2}{x^2 + y^2}, \frac{-xy + xy}{x^2 + y^2} \right) \\ &= (1, 0). \end{aligned}$$

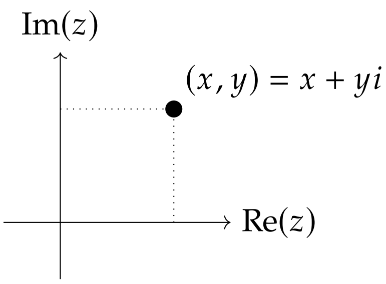
□

Complex Numbers are Field

Proposition 1.1. $(\mathbb{C}, +, \cdot)$ *is a field.*

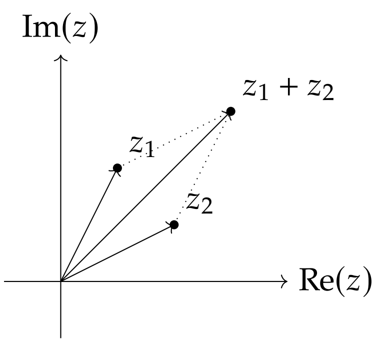
1.2 Geometric Representation of Complex Numbers

Note that $\mathbb{C} \approx \mathbb{R}^2$ (vector space):



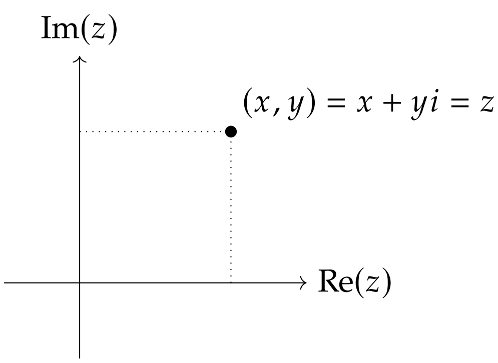
1.2.1 Addition

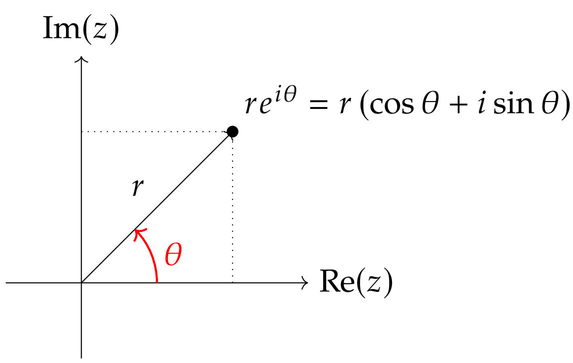
Addition $\leftrightarrow$ Vector Addition:



1.2.2 Polar Coordinate

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad \begin{cases} r = \sqrt{x^2 + y^2} \geq 0 \\ \theta \in [0, 2\pi). \end{cases} \qquad 0$$





$$x + yi = z \iff r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

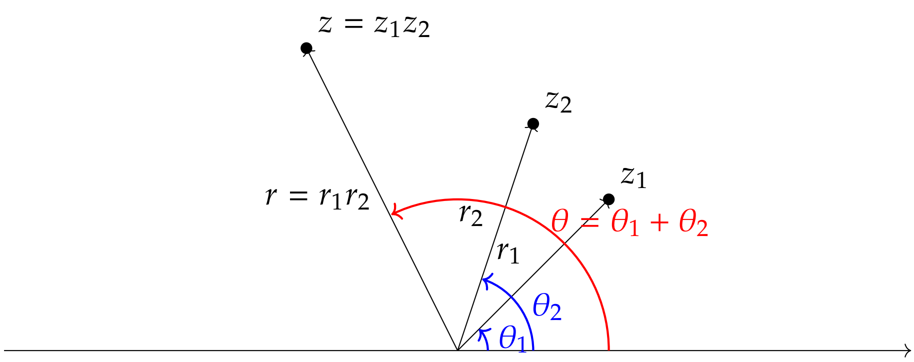
1.2.3 Multiplication

Let

$$\begin{aligned} z_1 &= (x_1, y_1) \Leftrightarrow r_1 (\cos\theta_1 + i\sin\theta_1)\,, \\ z_2 &= (x_2, y_2) \Leftrightarrow r_2 (\cos\theta_2 + i\sin\theta_2)\,. \end{aligned}$$

Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 \left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right] \\ &= r_1 r_2 \left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right] \\ &= r(\cos\theta + \sin\theta) \text{ with } \begin{cases} r = r_1 r_2 \\ \theta = \theta_1 + \theta_2. \end{cases} \end{aligned}$$



1.2.4 De Movire's Formula

De Moivre's Formula

Proposition 1.2. *Let $n \in \mathbb{N}$. Then*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + \sin(n\theta).$$

That is,

$$n \in \mathbb{N} \implies \left(e^{i\theta}\right)^n = e^{in\theta}.$$

Remark 1.2.1 (Approximation of π)**.** Let $y = \tan x$ then $\frac{d}{dx}y = \sec^2 x = 1 + \tan^2 x = 1 + y^2$. Since $x = \arctan y$, we have $\frac{d}{dy}x = \frac{1}{1+y^2}$, that is, $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$. Note that

$$\begin{aligned}\arctan x = \int \frac{d}{dx}(\arctan x)\,dx = \int \frac{1}{1+x^2}dx &= \int \sum_{n=0}^{\infty}\left(-x^2\right)^n dx \quad \because \frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \\ &= \sum_{n=0}^{\infty}\int (-1)^n x^{2n}\,dx \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}x^{2n+1} \\ &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots .\end{aligned}$$

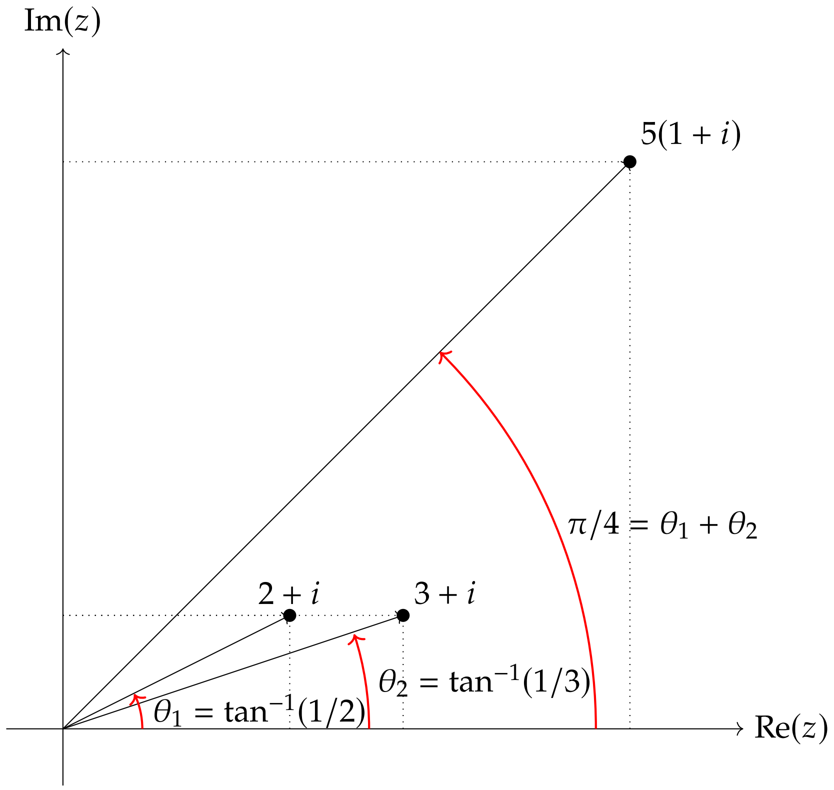
Since $\tan\frac{\pi}{4} = 1 \Leftrightarrow \arctan(1) = \frac{\pi}{4}$, we have

$$\pi = 4\cdot\arctan(1) = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\right) + \cdots .$$

Exercise 1.2.1. Show that

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}.$$

Sol. Note that $(2+i)(3+i) = 6 + 5i - 1 = 5(1+i)$.



□

1.2.5 n-th roots

Note that ω is a n-th root of z if $\omega^n = z$. Let

$$z = r(\cos\theta + i\sin\theta) \text{ with } r \geq 0 \text{ and } \theta \in [0, 2\pi),$$
$$w = \rho(\cos\alpha + i\sin\alpha) \text{ with } \rho \geq 0 \text{ and } \alpha \in [0, 2\pi).$$

Then

$$\omega^n = z \Rightarrow \rho^n(\cos n\alpha + i\sin n\alpha) = r(\cos\theta + i\sin\theta) \Rightarrow \begin{cases} \rho^n = r \\ n\alpha = \theta + 2k\pi, \ k \in \mathbb{Z}. \end{cases}$$

Thus,

$$w = \rho(\cos\alpha + i\sin\alpha) = \boxed{\sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]}.$$

Example 1.2.1. Find all value of ω such that $\omega^4 = -1$.

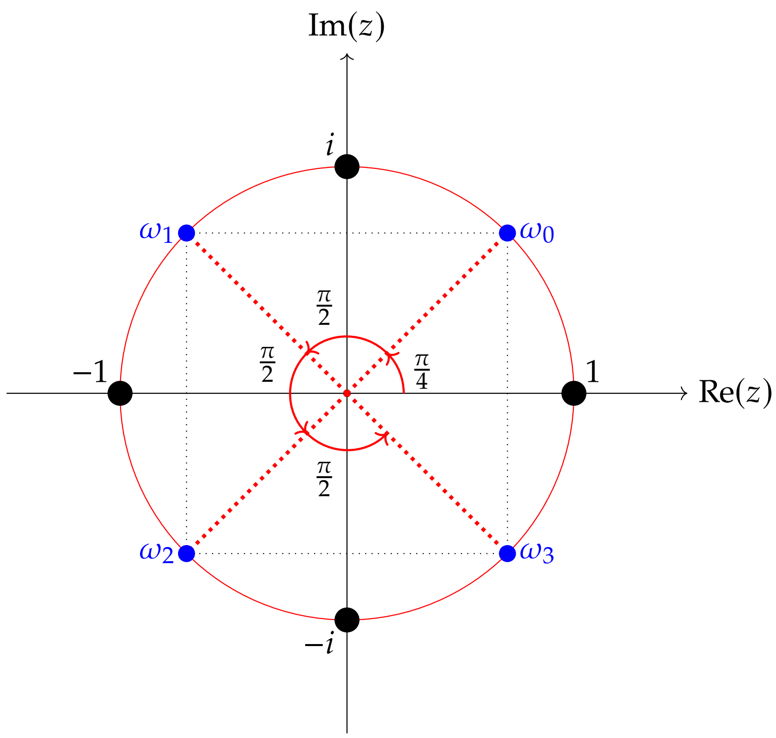
Sol. Let $\omega = re^{i\theta}$ then $w^4 = r^4 e^{i4\theta} = -1$, and so $\begin{cases} r = 1 \\ 4\theta = \pi + 2\pi \cdot k, \ k \in \mathbb{Z}. \end{cases}$

Thus,

$$\omega_k = \exp\left(i\left(\frac{\pi}{4} + \frac{\pi}{2} \cdot k\right)\right), \quad k = 0, 1, 2, 3.$$

That is,

$$w_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad w_1 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad w_2 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \quad w_3 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i.$$



□

1.2.6 Absolute(Modulus) and Conjugate

Let $z = x + iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$. Then

- (Absolute or Modulus) $|z| = \sqrt{x^2 + y^2}$.
- (Conjugate) $\overline{z} = x - yi$.

Proposition 1.3. *Let $z, z_1, z_2 \in \mathbb{C}$. Then*

1. $|z_1 z_2| = |z_1|\,|z_2|$
2. $\overline{\overline{z}} = z$
3. $\boxed{z\overline{z} = |z|^2}$
4. $\begin{cases} \mathrm{Re}(z) = \frac{z+\overline{z}}{2} \\ \mathrm{Im}(z) = \frac{z-\overline{z}}{2i}. \end{cases}$

Remark 1.2.2 (A polynomial with real coefficient)**.** Let

$$P(z) = \sum_{i=0}^{d} c_i z^i$$

with $z \in \mathbb{C}$ and $c_i \in \mathbb{R}$. Then

$$P(\omega) = 0 \Longleftarrow P(\overline{\omega}) = 0$$

for all $\omega \in \mathbb{Z}$.

Proof.

$$\overline{P(w)} = \overline{0} \Longleftrightarrow \overline{\sum_{i=0}^{d} c_i w^i} = 0 \Longleftrightarrow \sum_{i=0}^{d} c_i \overline{w}^i = 0.$$

□

1.3 Topology of $\mathbb{C}$

- $d(z_1, z_2) = |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 - (y_1 - y_2)^2}$
- $|z_1 + z_2| \leq |z_1| + |z_2|$
- $|z_1 - z_2| \geq ||z_1| - |z_2||$

Let $S \subseteq C$.

- Interior Point z_1: $\exists \varepsilon > 0 : D(z_1, \varepsilon) \subseteq S$
- Exterior Point z_2: $\exists \varepsilon > 0 : D(z_2, \varepsilon) \cap S = \emptyset$
- Boundary Point z_3: $\forall \varepsilon > 0 : D(z_3, \varepsilon) \cap S \neq \emptyset \wedge D(z_3, \varepsilon) \cap S^C \neq \emptyset$
- $U (\subseteq \mathbb{C})$ is open if, for all $z \in U$, z is an interior point, that is,

$$z \in U \implies \exists \varepsilon > 0 : D(z, \varepsilon) \subseteq U.$$

- V is closed if V^C is open.
- A is bounded if $\exists M > 0 : D(0, M) \supset A$.
- K is compact if it is bounded closed.

1.4 Elementary Functions : exp, Log, etc.

Summary

Let $z = x + iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$.

(1) The Complex Exponential Function;

$$e^{z} = e^{x+iy} := e^{x}\left(\cos x + i \sin y\right).$$

(2) Complex Trigonometric Functions;

- $\cos z := \dfrac{e^{iz} + e^{iz}}{2}$.
- $\sin z := \dfrac{e^{iz} - e^{iz}}{2i}$.
- $\tan z := \dfrac{\sin z}{\cos z}$.

(3) Complex Hyperbolic Functions;

- $\cosh z := \dfrac{e^{z} + e^{z}}{2}$.
- $\sinh z := \dfrac{e^{z} - e^{z}}{2}$.
- $\tanh z := \dfrac{\sinh z}{\cosh z}$.

(4) Logarithm Functions;

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$$

(5) Principal Value;

$$\mathrm{P.V.}\, z^{c} = \exp(\mathrm{Log}\, z^{c}) = \exp\left(c\,\mathrm{Log}\, z\right).$$

1.4.1 The exponential $\exp z$

Complex Exponential

Definition 1.1. The **complex exponential function** is defined as

$$\exp z = \exp(x + iy) \triangleq e^x(\cos y + i \sin y),$$

where $x, y \in \mathbb{R}$ and i is the imaginary unit, $i^2 = -1$.

Remark 1.4.1. Recall that Taylor series $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. Note that

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{1}{2!}(iy)^2 + \frac{1}{3!}(iy)^3 + \frac{1}{4!}(iy)^4 + \frac{1}{5!}(iy)^5 + \cdots \\ &= \left(1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 - \cdots\right) + i\left(y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \cdots\right) \\ &= \cos y + i \sin y. \end{aligned}$$

Then $e^{iy} = \cos y + i \sin y$. Thus, we have $\exp z = e^x(\cos y + i \sin y) = e^{x+iy}$.

Properties of Complex Expoenential

Proposition 1.4. *Let $z, z_1, z_2 \in \mathbb{C}$.*

(1) $\exp 0 = 1$.

(2) $\exp(z_1 + z_2) = (\exp z_1)(\exp z_2)$.

(3) $\exp z \neq 0 \implies (\exp z)^{-1} = \exp(-z)$.

(4) $\exp(z + 2\pi i) = \exp z$.

(5) $|\exp z| = e^{\mathrm{Re}(z)}$

Proof. (2) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\begin{aligned} \exp(z_1 + z_2) = \exp(x_1 + x_2 + i(y_1 + y_2)) &= e^{x_1+x_2+i(y_1+y_2)} \\ &= e^{x_1+iy_1} e^{x_2+iy_2} \\ &= (\exp z_1)(\exp z_2). \end{aligned}$$

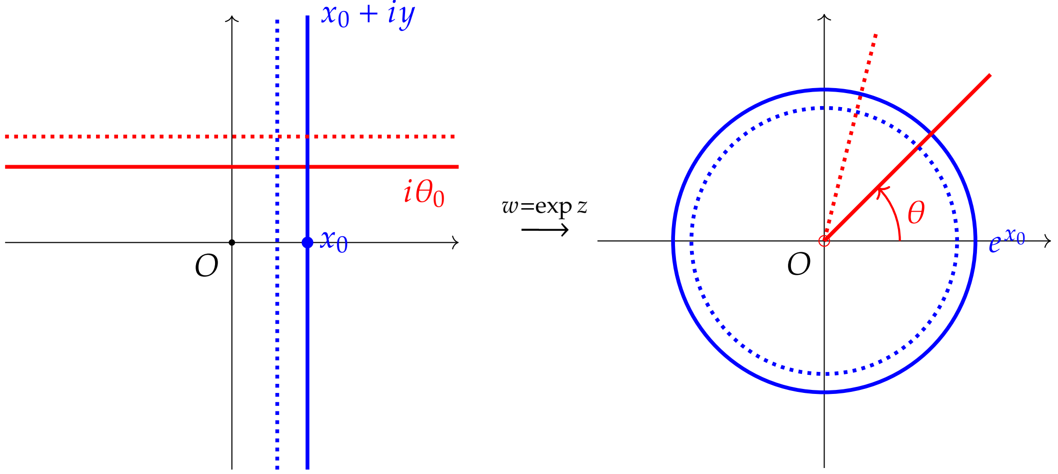
(3) $1 = \exp 0 = (\exp z)(\exp(-z))$.

(4) $\exp(z + 2\pi) = e^z(\cos 2\pi + i \sin 2\pi) = \exp z(1 + i \cdot 0) = \exp z$.

(5) $|\exp(z)| = |e^x \cos y + e^x \sin y| = \sqrt{e^{2x}\cos^2 y + e^{2x}\sin^2 y} = e^x = e^{\mathrm{Re}(z)}$.

$\square$

Remark 1.4.2 (Conformality).



1.4.2 Trigonometric Functions

The complex exponential function is intimately connected to trigonometry. The trigonometric functions are defined using the complex exponential function. Let $x \in \mathbb{R}$ then

$$\exp(ix) = \cos x + i \sin x \quad \text{and} \quad \exp(-ix) = \cos x - i \sin x.$$

This gives

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}.$$

Complex Trigonometric

Definition 1.2. Let $z \in \mathbb{C}$. Then

$$\cos(z) := \frac{1}{2}\left[\exp(iz) + \exp(-iz)\right], \quad \sin(z) := \frac{1}{2i}\left[\exp(iz) - \exp(-iz)\right].$$

Properties of Complex Trigonometric

Proposition 1.5. *Let $z, z_1, z_2 \in \mathbb{C}$.*

(1) $\sin\left(\frac{\pi}{2} - z\right) = \cos z$

(2) $\cos z$ *and* $\sin z$ *are not bounded (by Liouville's Theorem).*

(3) $\cos^z + \sin^2 z = 1$

Remark 1.4.3. Let $z, z_1, z_2 \in \mathbb{C}$.

- $\begin{cases} \sin z = 0 \Leftrightarrow z = n\pi & (n \in \mathbb{Z}), \\ \cos z = 0 \Leftrightarrow z = \left(n + \frac{1}{2}\right)\pi & (n \in \mathbb{Z}). \end{cases}$

- $\begin{cases} \sin(3z) = 3\sin z - 4\sin^3 z, \\ \cos(3z) = 4\cos^3 z - 3\cos z. \end{cases}$

- $\begin{cases} \sin^2 \frac{z}{2} & = \dfrac{1-\cos z}{2} \\ \cos^2 \frac{z}{2} & = \dfrac{1+\cos z}{2} \\ \tan^2 \frac{z}{2} & = \dfrac{1-\cos z}{1+\cos z} \end{cases}$

- $\begin{cases} \cos(z_1+z_2) & = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \\ \sin(z_1+z_2) & = \sin z_1 \cos z_2 - \cos z_1 \sin z_2 \\ \tan(z_1+z_2) & = \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2} \end{cases}$

- $\begin{cases} \sin(-z) & = -\sin z \\ \cos(-z) & = \cos z \\ \tan(-z) & = -\tan z \end{cases}$

1.4.3 Hyperbolic Functions

Complex Hyperbolic

Definition 1.3. Let $z \in \mathbb{C}$. Then

$$\cosh(z) := \cos(iz) = \frac{1}{2}(e^z + e^{-z}),$$
$$\sinh(z) := \frac{1}{i}\sin(iz) = \frac{1}{2}(e^z - e^{-z}).$$

Properties of Complex Hyperbolic

Proposition 1.6. *Let $z, z_1, z_2 \in \mathbb{C}$.*

(1) $\cosh^z - \sinh^2 z = 1$

(2) $\begin{cases} \sinh(z_1+z_2) & = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2, \\ \cosh(z_1+z_2) & = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2. \end{cases}$

(3) $\begin{cases} \cosh(-z) & = \cosh(z) \\ \sinh(-z) & = -\sinh(z) \end{cases}$

1.4.4 Logarithm Function

Principal Argument

Definition 1.4. The **principal argument** of a complex number z, denoted by $\mathrm{Arg}(z)$, is defined to be the unique value $\theta \in (-\pi, \pi]$ such that

$$z = |z|\, e^{i\theta} = |z|\,(\cos\theta + i\sin\theta)\,.$$

Example 1.4.1.

$$\mathrm{Arg}(1) = 0, \quad \mathrm{Arg}(-1) = \pi, \quad \mathrm{Arg}(i) = \frac{\pi}{2}, \quad \mathrm{Arg}(-i) = \frac{3}{2}\pi.$$

Principal Logarithm

Definition 1.5. The **principal logarithm** $\mathrm{Log}\, z$ $(z \neq 0)$ is defined by

$$\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}(z).$$

Remark 1.4.4. The principal logarithm satisfies the following properties:

- $\mathrm{Log}(z_1 z_2) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$ for all $z_1, z_2 \in \mathbb{C} \setminus (-\infty, 0]$.
- $\mathrm{Log}(e^z) = z$ for all $z \in \mathbb{C}$.
- $\exp(\mathrm{Log}(z)) = z$ for all $z \in \mathbb{C} \setminus (-\infty, 0]$.

Remark 1.4.5. Let z be a non-zero complex number. Then the logarithm of z, denoted by $\log z$, is defined to be any complex number w such that $e^w = z$.

However, since $e^{w+2\pi i} = e^w$, there are infinitely many possible values of $\log z$. In fact, the set of all possible values of $\log z$ is given by

$$\log z = \ln|z| + i\arg z + 2\pi i k$$

where $\ln|z|$ is the natural logarithm of the modulus of z, $\arg z$ is the argument of z, and k is any integer.

Note that the complex logarithm is not continuous on the entire complex plane, since there is a branch cut along the negative real axis. However, it is analytic on any simply connected domain that does not contain the origin.

The principal logarithm of a complex number z, denoted by $\mathrm{Log}(z)$, is defined to be the complex number $w = \ln|z| + i\,\mathrm{Arg}(z)$.

Note that the principal logarithm is a single-valued function defined on the domain $\mathbb{C} \setminus (-\infty, 0]$.

Chapter 2

Complex Differentiability

2.1 Complex Differentiability

Complex Differentiability

Definition 2.1. A complex function $f : U(\subseteq \mathbb{C}) \to \mathbb{C}$, U is an open subset, is said to be **complex differentiable** at a point $z_0 \in U$ if $\exists f'(z_0)$ defined by

$$f'(z_0) = \left.\frac{df}{dz}\right|_{z=z_0} := \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Remark 2.1.1. We say that a complex function $f(z)$ is differentiable at a point $z_0 \in U$ if

$$\exists f'(z_0) \in \mathbb{C} : \left[\forall \varepsilon > 0 : \exists \delta > 0 : \forall z \in U : 0 < |z - z_0| < \delta \Rightarrow \left|\frac{f(z) - f(0)}{z - z_0} - f'(z_0)\right| < \varepsilon\right].$$

Example 2.1.1 (★★★).

(1) Let $f(z) = z^2$ then

$$\lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z\to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z\to 0}(2z + \Delta z) = 2z.$$

(2) Let $f(z) = \overline{z}$ then

$$\lim_{\Delta z\to 0} \frac{\overline{z+\Delta z} - \overline{z}}{\Delta z} = \lim \lim_{\Delta z\to 0} \frac{\overline{z} + \overline{\Delta z} - \overline{z}}{\Delta z} = \lim_{\Delta z\to 0}\left(\frac{\overline{\Delta z}}{\Delta z}\right) = \begin{cases} 1 & \Delta z = \Delta x + i\cdot 0, \\ -1 & \Delta z = 0 + i\cdot \Delta y. \end{cases}$$

Thus $\nexists f'(z)$ for all $z \in \mathbb{C}$.

(3) Let $f(z) = |z|^2 = z\overline{z}$ then

$$\lim_{\Delta z\to 0} \frac{(z+\Delta z)(\overline{z} + \overline{\Delta z}) - z\overline{z}}{\Delta z} = \lim_{\Delta z\to 0} \frac{z\overline{\Delta z} + \overline{z}\Delta z + \Delta z\overline{\Delta z}}{\Delta z} = \lim_{\Delta z\to 0}\left(z\cdot\left(\frac{\overline{\Delta z}}{\Delta z}\right) + \overline{z} + \overline{\Delta z}\right).$$

Thus $z = 0 \implies \exists f'(z)$.

Equivalence of Complex Differentiability

Lemma 2.1. *Let U be an open set in $\mathbb{C}$, $z_0 \in U$, and $f : U \to C$. Then the following are equivalent:*

(1) f is complex differentiable at z_0

(2) There exists an $r > 0$, and function $h : D(z_0, r) \to \mathbb{C}$, where $D(z_0, r) := \{z \in C : |z - z_0| < r\}$, such that

(a) $f(z) = f(z_0) + [f'(z_0) + h(z)](z - z_0)$ for $|z - z_0| < r$ and

(b) $\lim_{z \to z_0} h(z) = 0$.

Proof. ($\Longrightarrow$) Suppose that the complex derivative $f'(z_0)$ exists:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

We want to show that there an $r > 0$ and a function $h : D(z_0, r)$ satisfying condition (a) and (b). Let $\varepsilon > 0$. Define the function $h : D(z_0, \delta) \to \mathbb{C}$ by

$$h(z) = \begin{cases} \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) & : z \neq z_0, \\ 0 & : z = z_0. \end{cases}$$

Then

(Case I) ($z \neq z_0$, i.e., $0 < |z - z_0| < \delta$)

$$h(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \overset{\text{Rearraning}}{\Longrightarrow} f(z) = f(z_0) + [f'(z_0) + h(z)](z - z_0).$$

(Case II) ($z = z_0$, i.e., $0 = |z - z_0|$)

$$f(z) = f(z_0) + [f'(z_0) + h(z)](z - z_0) \leftrightsquigarrow f(z_0) = f(z_0) + \cancelto{0}{[f'(z_0) + 0]} \cdot 0.$$

Thus, $f(z) = f(z_0) + [f'(z_0) + h(z)](z - z_0)$ holds whenever $|z - z_0| < \delta$. By the definition of $f'(z_0)$, we have

$$\left| \frac{f(z) - f(0)}{z - z_0} - f'(z_0) \right| = |h| = |h(z) - 0| < \varepsilon.$$

($\Longleftarrow$) For $z \in D(z_0, r) \setminus \{z_0\}$, we have, upon rearranging, that

$$\frac{f(z) - f(0)}{z - z_0} - f'(z_0) = h(z) \xrightarrow{z \to z_0} 0$$

and so $\exists f'(z_0)$.

$\square$

Proposition 2.2. *Let U be an open subset of $\mathbb{C}$. Let $f, g : U \to \mathbb{C}$ are complex differentiable at $z_0 \in U$. Let $\alpha, \beta \in \mathbb{C}$.*

- *(Linearity)*
$$(\alpha f \pm \beta g)'(z_0) = \alpha f'(z_0) \pm \beta g'(z_0).$$

- *(Product Rule)*
$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0).$$

- *(Quotient Rule)*
$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.$$

Chain Rule

Proposition 2.3. *Let $f : I_f \to \mathbb{R}$ and $g : I_g \to \mathbb{R}$ satisfies*

(i) $f(I_f) \subseteq I_g$;

(ii) f *is differentiable at* $x = c$;

(iii) g *is differentaible at* $y = f(c)$.

Define $h : I_f \to \mathbb{R}$ as follows:
$$h(t) = g(f(t)) := g \circ f(t)$$
with $t \in I_f$. Then
$$h'(c) = g'(f(c))f'(c),$$
i.e.,
$$(g \circ f)'(c) = g'(f(c))f'(c).$$

2.2 Cauchy-Riemann Equations

Cauchy-Riemann Equations

Theorem 2.4. *Let U ($\subseteq \mathbb{C}$) be an open set, and let*

$$f : U \longrightarrow \mathbb{C} : f(z) = f(x + yi) = u(x, y) + iv(x, y)$$

be a complex-valued function, where $u(x, y)$ and $v(x, y)$ are real-valued functions, that is,

$$\begin{aligned} u &: U \longrightarrow \mathbb{R} : (x, y) \mapsto \operatorname{Re}(f(x + iy)) \quad \textit{and} \\ v &: U \longrightarrow \mathbb{R} : (x, y) \mapsto \operatorname{Im}(f(x + iy)). \end{aligned}$$

If $f(z)$ is differentiable at a point $z_0 = x_0 + iy_0$, then the partial derivatives of $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations:

$$u_x = v_y \quad \textit{and} \quad u_y = -v_x \quad \textit{at } (x_0, y_0).$$

Proof. Let $f'(z_0)$ be the complex derivative of $f(z)$ at z_0. By definition, we have

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \begin{cases} \displaystyle\lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} & \cdots\cdots (1) \\ \displaystyle\lim_{\substack{\Delta y \to 0 \\ \Delta x = 0}} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} & \cdots\cdots (2). \end{cases}$$

(1) $(\Delta z = \Delta x + i \cdot 0)$;

$$\begin{aligned} (1) &= \lim_{\Delta x \to 0} \frac{[u(x_0 + \Delta x, y_0) - u(x_0, y_0)] - i[v(x_0 + \Delta x, y_0) - v(x_0, y_0)]}{\Delta x} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + \frac{\partial v}{\partial x}(x_0, y_0) \\ &= u_x + iv_x \Big|_{z = z_0}. \end{aligned}$$

(2) $(\Delta z = 0 + i\Delta y)$;

$$\begin{aligned} (2) &= \lim_{\Delta y \to 0} \frac{[u(x_0, y_0 + \Delta y) - u(x_0, y_0)] - i[v(x_0, y_0 + \Delta y) - v(x_0, y_0)]}{i\Delta y} \\ &= \frac{1}{i}\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) \\ &= v_y - iu_y \Big|_{z = z_0} \quad \text{by multiplying } 1 = i/i. \end{aligned}$$

Hence we have

$$(1) = (2) \implies u_x = v_y, \quad u_y = -v_x.$$

□

Remark 2.2.1.

$$\exists f'(z_0) : f'(0) = u_x + iv_x \Big|_{z=z_0} \iff \begin{cases} \text{(i) } u, v \in C^1 \\ \text{(ii) CR-Eq hold at } (x_0, y_0). \end{cases}$$

Remark 2.2.2. Let $z = x + yi$.

$f(z)$	$u(x,y)$	$v(x,y)$	u_x	u_y	v_x	v_y	$u_x = v_y$?	$u_y = -v_y$?
z^2	$x^2 - y^2$	$2xy$	$2x$	$-2y$	$2y$	$2x$	O	O
$\overline{z}$	x	$-y$	1	0	0	-1	X	X
$\lvert z\rvert^2$	$x^2 + y^2$	0	$2x$	$2y$	0	0	if $z = 0$	if $z = 0$

Remark 2.2.3.

	Real Function	**Complex Function**
1. Existence of limit of sequence	Sub-seqns Criterion	Sub-seqns Criterion
2. Existence of limit of function	Comparison of Left-Right Limit	Comparison of Approaches
3. Differentiability	Comp. of. LR derivatives	CR-Eqs

Example 2.2.1. The function $f : \mathbb{C} \to \mathbb{C}$ is defined by

$$f(z) = \exp z = e^{x+iy} = e^x (\cos y + i \sin y)$$

where $z \in \mathbb{C}$. Then we have

$$u(x,y) = \text{Re}\left(e^{x+iy}\right) = e^x \cos y,$$
$$v(x,y) = \text{Im}\left(e^{x+iy}\right) = e^x \sin y.$$

Thus,

$$\frac{\partial u}{\partial x}(x,y) = e^x \cos y = \frac{\partial v}{\partial y}(x,y),$$
$$\frac{\partial u}{\partial y}(x,y) = -e^x \sin y = -\frac{\partial v}{\partial x}(x,y),$$

which shows that the Cauchy-Riemann equations hold in $\mathbb{C}$. Since

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = e^x \cos y + ie^x \sin y = \exp z,$$

we also obtain that

$$\frac{d}{dz} \exp z = \exp z$$

for $z \in \mathbb{C}$.

Example 2.2.2 (★ ★ ★)**.** Consider a complex function $f : D \to \mathbb{C}$. Assume that

(i) f is holomorphic in D and

(ii) $|f(z)| = c \in \mathbb{C}$, that is, $|f(z)|$ is constant.

Show that $f(z)$ is also constant.

Proof. Let $f = u + iv$ then $|f(z)| = u^2 + v^2 = c^2$. Note that

$$\begin{cases} 2u \cdot u_x + 2v \cdot v_x = 0 \\ 2u \cdot u_y + 2v \cdot v_y = 0 \end{cases} \overset{\text{by CR-Eq}}{\Longrightarrow} \begin{cases} 2uu_x + 2v(-u_y) = 0 \cdots\cdots (1) \\ 2uu_y + 2vu_x = 0 \cdots\cdots (2) \end{cases}.$$

By computing $(1) \cdot u + (2) \cdot v$, we have

$$2u^2u_x + 2v^2u_x = 0 \implies 2(u^2 + v^2)u_x = 0 \implies 2 \cdot c^2 \cdot u_x = 0.$$

(Case 1) $(c = 0)$ It is trivial.

(Case 2) $(c \neq 0)$ u_x must be 0.

Similarly, we obtain $u_y = 0$, and so

$$u_x = u_y = 0, \quad v_x = v_y = 0.$$

That is, f is constant. □

2.3 Geometric Meaning of the Complex Derivative

In $\mathbb{R}$,

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0)\Rightarrow\frac{f(x)-f(x_0)}{x-x_0}\approx f'(x_0)\,\text{for}\,x\in\mathcal{N}_\delta(x_0)\Rightarrow f(x)-f(x_0)\approx f'(x_0)(x-x_0).$$

In $\mathbb{C}$,

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=f'(z_0)\implies f(z)\approx f(z_0)+f'(z_0)(z-z_0).$$

Recall that, for $z_1=e^{i\theta_1}$ and $z_2=r_2e^{i\theta_2}$

$$\begin{cases} z_1z_2=r_1r_2e^{i(\theta_1+\theta_2)} \\ \arg(z_1z_2)=\arg(z_1)+\arg(z_2). \end{cases}$$

Then

1. $\arg(f(z)-f(0)\approx\arg\left(f'(z)(z-z_0)\right)=\arg(f'(z_0))+\arg(z-z_0)$.
2. $|f(z)-f(z_0)|=|f'(z_0)|\,|z-z_0|$.

2.4 The d-bar operator

$$\boxed{\frac{\partial}{\partial z}:=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)},\quad \boxed{\frac{\partial}{\partial\overline{z}}:=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)}.$$

Let f is complex differentiable. Then $\frac{d}{d\overline{z}}f=0$.

Proof.

$$\frac{\partial}{\partial\overline{z}}f=\frac{\partial}{\partial\overline{z}}u+i\frac{\partial}{\partial\overline{z}}v=\frac{1}{2}(u_x+iu_y)+i\frac{1}{2}(u_x+iv_y)=\frac{1}{2}(u_x-v_y)=\frac{1}{2}\left(u_y+v_x\right)=0.$$

□

Example 2.4.1.

1. For $f(z)=z^2$, $\frac{\partial}{\partial\overline{z}}f=0$
2. For $f(z)=\overline{z}=\frac{\partial}{\partial\overline{z}}f=1\neq0$.
3. For $f(z)=|z|^2$, $\frac{\partial}{\partial\overline{z}}f=z$, i.e., f is differentiable at $z=0$ only.

Chapter 3

Cauchy Integral Theorem

3.1 Definition of the Contour Integral

Path Integral $\mathbb{R} \to \mathbb{C}$

Definition 3.1. Define a function $f : [a, b] \to \mathbb{C} : t \mapsto f(t) = u(t) + iv(t)$. Then

$$\int_a^b f(t)\, dt = \int_a^b (u(t) + iv(t))\, dt := \int_a^b u(t) + i \int_a^b v(t)\, dt.$$

Example 3.1.1. Compute $\int_0^1 (t+i)^3\, dt$.

Sol. (S1)

$$\begin{aligned}\int_0^1 (t+i)^3\, dt = \int_0^1 \left(t^3 + 3t^2 i - 3t - i\right) dt &= \int_0^1 \left(t^3 - 3t\right) dt + i \int_0^1 \left(3t^2 - 1\right) dt \\ &= \frac{1}{4}t^4 - \frac{3}{2}t^3 \Big|_0^1 + i\left(t^3 - t\Big|_0^1\right) \\ &= \frac{1}{4} - \frac{3}{2} \\ &= -\frac{5}{4}.\end{aligned}$$

(S2)

$$\begin{aligned}\int_0^1 (t+i)^3\, dt = \frac{1}{4}(t+i)^4\Big|_0^1 = \frac{1}{4}\left((1+i)^4 - i^4\right) &= \frac{1}{4}\left(\left(1 + 4i + 6i^2 + 4i^3 + 1\right) - 1\right) \\ &= \frac{1}{4}(1-6) \\ &= -\frac{5}{4}.\end{aligned}$$

□

Length of Curve

Definition 3.2. Let γ be a smooth curve such that

$$[a,b] \to \mathbb{C} : z(t) = x(t) + iy(t).$$

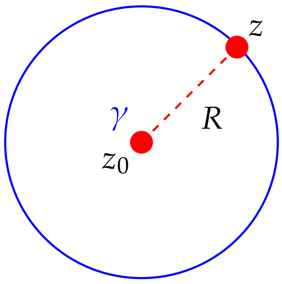
We define a length L of curve γ as follows:

$$L := \int_{\gamma} |dz| = \int_a^b |z'(t)|\, dt.$$

Remark 3.1.1.

$$\begin{aligned}
\int_{\gamma} |dz| &= \int_a^b |z'(t)|\, dt \\
&= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\left(\frac{\Delta x_k}{\Delta t}\right)^2 + \left(\frac{\Delta y_k}{\Delta t}\right)^2}\, \Delta t.
\end{aligned}$$

Example 3.1.2. Consider a circle γ with center z_0 and radius R:



For $t \in [0, 2\pi]$,

$$z(t) = z_0 + Re^{it} = z_0 + R\cos t + iR\sin t.$$

Then

$$\begin{aligned}
\int_{\gamma} |dz| = \int_0^{2\pi} |z'(t)|\, dt = \int_0^{2\pi} \left|\frac{d}{dt}\left(z_0 + Re^{it}\right)\right| dt &= \int_0^{2\pi} \left|Rie^{it}\right| dt \\
&= \int_0^{2\pi} |Ri| \left|e^{it}\right| dt \\
&= \int_0^{2\pi} \sqrt{R^2}\sqrt{\cos^2 t + \sin^2 t}\, dt \\
&= \int_0^{2\pi} R\, dt \\
&= Rt\Big|_0^{2\pi} \\
&= 2\pi R.
\end{aligned}$$

Contour Integral

Definition 3.3. Let D ($\subseteq \mathbb{C}$) be a domain. Given

1. a continuous function
$$f : D \to \mathbb{C} : f(z) = u(x,y) + iv(x,y)$$
with $u, v : D \to \mathbb{R}$, and

2. a smooth path
$$\gamma : [a,b] \to D : \gamma(t) = x(t) + iy(t)$$
with $x, y : [a,b] \to \mathbb{R}$,

we define

$$\begin{aligned}
\int_\gamma f(z)\,dz :&= \int_a^b f(\gamma(t))\,\gamma'(t)\,dt \\
&= \int_a^b (u(\gamma(t)) + iv(\gamma(t))) \cdot (x'(t) + iy'(t))\,dt \\
&= \int_a^b (u(\gamma(t)) \cdot x'(t) - v(\gamma(t)) \cdot y'(t))\,dt \\
&\quad + i\int_a^b (u(\gamma(t)) \cdot x'(t) + v(\gamma(t)) \cdot y'(t))\,dt.
\end{aligned}$$

Remark 3.1.2. For $c_k \in [z_{k-1}, z_k]$,

$$\begin{aligned}
\int_\gamma f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^n f(c_k)\,\Delta z_k = \lim_{n\to\infty} \sum_{k=1}^n f(c_k) \left(\frac{\Delta x}{\Delta t} + i\frac{\Delta y}{\Delta t}\right) \Delta t &= \int_a^b f(\gamma(t))\,(x'(t) + iy'(t)) \\
&= \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.
\end{aligned}$$

Example 3.1.3. Consider a function

$$f : \mathbb{C} \setminus \{0\} \to \mathbb{C} : f(z) = \frac{1}{z}$$

and two smooth paths

$$\begin{aligned}
\gamma_1 &= \exp(it), \quad t \in [0, 2\pi] \\
\gamma_2 &= \exp(2it), \quad t \in [0, \pi].
\end{aligned}$$

Then

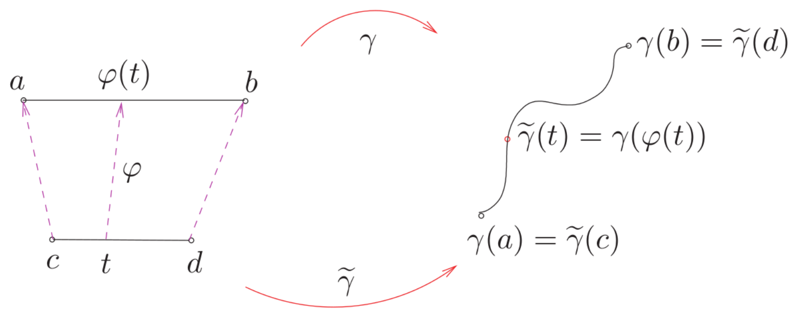
$$\begin{aligned}
\int_{\gamma_1} f(z)\,dz &= \int_0^{2\pi} f\big(\exp(it)\big)\, i\exp(it)\,dt = \int_0^{2\pi} i dt = it\Big|_0^{2\pi} = 2\pi i, \\
\int_{\gamma_2} f(z)\,dz &= \int_0^{\pi} f\big(\exp(2it)\big)\, 2i\exp(2it)\,dt = \int_0^{\pi} 2i dt = 2it\Big|_0^{\pi} = 2\pi i.
\end{aligned}$$

Equivalent paths give the same integral

Proposition 3.1. *Consider two smooth paths:*

$$\gamma : [a, b] \to \mathbb{C} \quad \text{and} \quad \tilde{\gamma} : [c, d] \to \mathbb{C}$$

such that there is a continuously differentiable function $\varphi : [c, d] \longrightarrow [a, b]$ such that $a = \varphi(c)$, $b = \varphi(d)$, and $\tilde{\gamma}(t) = (\gamma \circ \varphi)(t)$ for $t \in [c, d]$.



$$\begin{aligned}
\int_{\tilde{\gamma}} f(z)\, dz &= \int_a^b f(\tilde{\gamma}(t))\, \tilde{\gamma}'(t) dt \\
&= \int_a^b f(\gamma(\varphi(t)))\, \gamma'(\varphi(t))\, \varphi'(t)\, dt \\
&= \int_c^d f(\gamma(\tau))\, \gamma'(\tau)\, d\tau \quad \text{by } \tau = \varphi(t) \\
&= \int_\gamma f(z)\, dz.
\end{aligned}$$

An Important Integral

Theorem 3.2. *Let C be a circular path with center z_0 and radius $r > 0$ traversed in the anti-clockwise direction. Then*

$$\int_C (z - z_0)^n \, dz = \begin{cases} 2\pi i & : n = -1, \\ 0 & : n \neq -1. \end{cases}$$

Proof. (1) Let $n \neq -1$ then

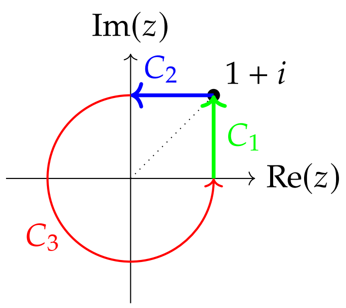
$$\begin{aligned} \int_C (z - z_0)^n \, dz &= \int_0^{2\pi} \left(z_0 + re^{it} - z_0\right)^n \cdot ire^{it} dt \\ &= \int_0^{2\pi} r^n e^{int} \cdot ire^{it} dt \\ &= ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt \\ &= ir^{n+1} \left[\frac{1}{i(n+1)} e^{i(n+1)t}\right]_0^{2\pi} \\ &= 0. \end{aligned}$$

(2) Let $n = -1$ then

$$\int_C (z - z_0)^{-1} \, dz = \int_0^{2\pi} \left(re^{it}\right)^{-1} \cdot ire^{it} dt = \int_0^{2\pi} i dt = 2\pi i.$$

□

Example 3.1.4. Consider the following path:



with

$$\begin{aligned} &C_1 : z_1(t) = 1 + ti (0 \leq t \leq 1) \\ &C_2 : z_2(t) = (1 - t) + i (0 \leq t \leq 1) \\ &C_3 : z_3(t) = e^{it} (\pi/2 \leq t \leq 2\pi). \end{aligned}$$

Let $C = C_1 + C_2 + C_3$. Find $\int_C \bar{z} dz$.

Sol.

$$\int_C \bar{z} dz = 2i \cdot (\text{Area of } C = \partial R) = 2i \cdot \left(\frac{3\pi}{4} + 1\right) = \left(2 + \frac{3\pi}{2}\right) i.$$

□

3.2 Properties of Contour Integration

Linearity of Integration

Proposition 3.3. *Let D be a domain in $\mathbb{C}$ and $\gamma : [a,b] \to D$ be a piecewise smooth path. Then the following hold: for all continuous $f, g : D \to \mathbb{C}$ and all $\alpha \in \mathbb{C}$,*

$$\int_\gamma (\alpha f + \beta g)(z)\, dz = \alpha \int_\gamma f(z)\, dz + \beta \int_\gamma g(z)\, dz.$$

Proposition 3.4. *Let $\gamma : [a,b] \to D$ be a smooth path in a domain D and $f : D \to \mathbb{C}$ be a continuous function. Then*

$$\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz.$$

Proof. Note that $-\gamma : [a,b] \to \mathbb{C}$ is defined by

$$(-\gamma)(t) = \gamma(a+b-t).$$

$$\begin{aligned}
\int_{-\gamma} f(z)\, dz &= \int_a^b f((-\gamma)(t))(-\gamma)'(t)\, dt \\
&= \int_a^b f(\gamma(a+b-t)) \frac{d}{dt}[(-\gamma)(t)]\, dt \\
&= \int_a^b f(\gamma(a+b-t))\gamma'(a+b-t)(-1)\, dt \\
&= \int_b^a f(\gamma(\tau))\gamma'(\tau)\, d\tau \quad \text{by } \tau := a+b-t \text{, i.e., } d\tau = -dt \\
&= -\int_a^b f(z)\, dz.
\end{aligned}$$

□

Concatenation of Paths

Proposition 3.5. *Let $\gamma_1 : [a_1, b_1] \to D$ and $\gamma_2 : [a_2, b_2] \to D$ be two paths such that $\gamma_1(b_1) = \gamma_2(a_2)$. Define the concatenation of paths $\gamma_1 + \gamma_2 : [a_1, b_1 + b_2 - a_2]$ as follows:*

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & : t \in [a_1, b_1], \\ \gamma_2(t - b_1 + a_2) & : t \in [b_1, b_1 + b_2 - a_2]. \end{cases}$$

Then

$$\int_{\gamma_1+\gamma_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz.$$

Proposition 3.6. *Let*

1. *D be a domain in $\mathbb{C}$;*
2. *$\gamma : [a,b] \to D$ be a piecewise smooth path and*
3. *$f : D \to \mathbb{C}$ be a continuous function.*

Then

$$\left| \int_\gamma f(z)\, dz \right| \leq \left(\max_{t\in[a,b]} |f(\gamma(t))| \right) \cdot \int_\gamma |dz|$$

Proof. Consider first a curve $\varphi : [a,b] \to \mathbb{C}$. We claim that

$$\left| \int_a^b \varphi(t)\, dt \right| \leq \int_a^b |\varphi(t)|\, dt.$$

Let $\int_a^b \varphi(t)\, dt = re^{i\theta}$, where $r \geq 0$ and $\theta \in (-\pi, \pi]$. Then

$$\begin{aligned}
\left| \int_a^b \varphi(t)\, dt \right| = r = e^{-i\theta} \int_a^b \varphi(t)\, dt & \quad \because \int_a^b \varphi(t)\, dt = re^{i\theta} \\
= \int_a^b e^{-i\theta} \varphi(t)\, dt & \\
= \int_a^b \operatorname{Re}\left(e^{-i\theta} \varphi(t) \right) dt & \quad \because \left| \int_a^b \varphi(t)\, dt \right| \in \mathbb{R} \\
\leq \int_a^b \left| \operatorname{Re}\left(e^{-i\theta} \varphi(t) \right) \right| dt & \\
\leq \int_a^b \left| \left(e^{-i\theta} \varphi(t) \right) \right| dt & \quad \because |\operatorname{Re}(z)| \leq |z| \\
= \int_a^b |\varphi(t)|\, dt & \quad \because \left| e^{-i\theta} \right| = 1.
\end{aligned}$$

Let $\varphi(t) := f(\gamma(t)) \cdot \gamma'(t)$ with $t \in [a,b]$. Then

$$\begin{aligned}
\left| \int_\gamma f(z)\, dz \right| &= \left| \int_a^b f(\gamma(t))\, \gamma'(t)\, dt \right| \\
&\leq \int_a^b |f(\gamma(t))\, \gamma'(t)|\, dt \\
&= \int_a^b |f(\gamma(t))|\, |\gamma'(t)|\, dt \\
&\leq \max_{t\in[a,b]} |f(\gamma(t))| \cdot \int_a^b |\gamma'(t)|\, dt.
\end{aligned}$$

□

3.3 Fundamental Theorem of Contour Integration

Fundamental Theorem of Contour Integration

Theorem 3.7. *Let*

(i) D be a domain in $\mathbb{C}$;

(ii) $\gamma : [a,b] \longrightarrow D$ be a piecewise smooth path;

(iii) $f : D \longrightarrow \mathbb{C}$ be a continuous in D;

(iv) $F : D \longrightarrow \mathbb{C}$ be a holomorphic function such that $F' = f$ in D.

Then

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof. For $z = x + iy \in D$, where $x, y \in \mathbb{R}$, we define the real-valued functions U, V, u, v by

$$F(x+iy) = U(x,y) + iV(x,y),$$
$$f(x+iy) = u(x,y) + iv(x,y).$$

Also, set $\gamma(t) = x(t) + iy(t)$ ($t \in [a,b]$). Then

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_a^b f(\gamma(t))\,\gamma'(t)\,dt \\
&= \int_a^b (u+iv)(x'+iy')\,dt \\
&= \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy')\,dt \\
&= \int_a^b (U_x x' - V_x y')\,dt + i\int_a^b (V_x x' + U_x y')\,dt \qquad \because F' = U_x + iV_x = f = u + iv \\
&= \int_a^b (U_x x' + U_y y')\,dt + i\int_a^b (V_x x' + V_y y')\,dt \quad \text{by CR-Eqs: } U_x = V_y, U_y = -V_x \\
&= \int_a^b \frac{d}{dt}[U(x,y)]\,dt + i\int_a^b \frac{d}{dt}[V(x,y)]\,dt \\
&= U(x(b),y(b)) - U(x(a),y(a)) + i\left(V(x(b),y(b)) - V(x(a),y(a))\right) \\
&= \left(U(x(b),y(b)) + U(x(a),y(a))\right) - \left(V(x(b),y(b)) + V(x(a),y(a))\right) \\
&= F(\gamma(b)) - F(\gamma(a)).
\end{aligned}$$

$\square$

3.4 The Cauchy Integral Theorem

Path Homotopy

Definition 3.4. Consider two closed paths $\gamma_0, \gamma_1 : [0,1] \longrightarrow D$. γ_0 is *D-homotopic to* γ_1 if there exists a continuous function $H : [0,1]^2 \longrightarrow D$ such that

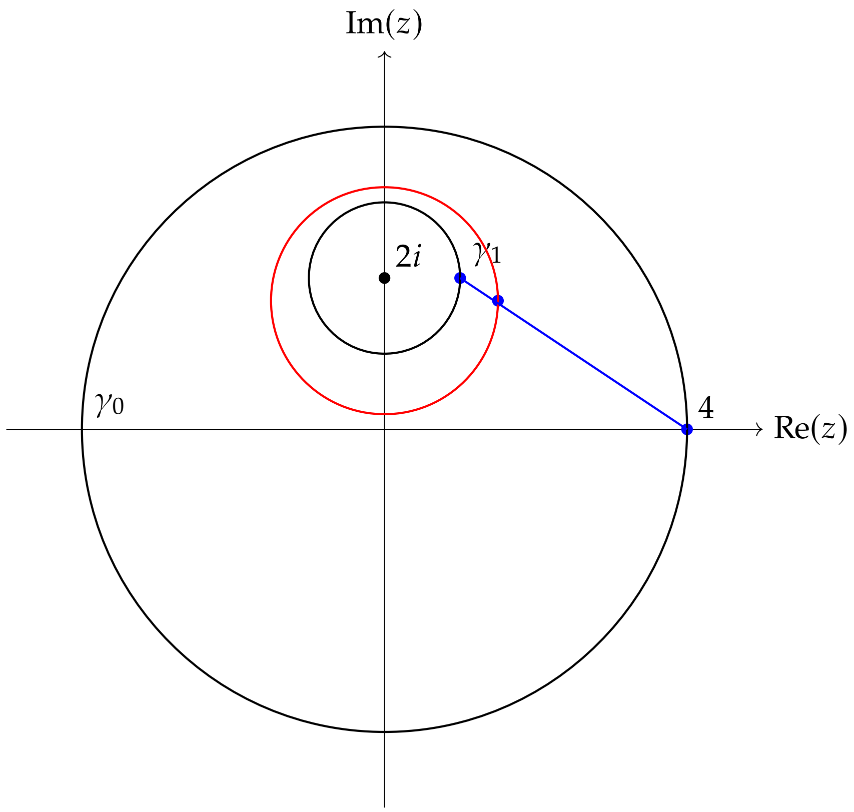
(H1) $\forall t \in [0,1] : H(t,0) = \gamma_0(t)$;

(H2) $\forall t \in [0,1] : H(t,1) = \gamma_1(t)$;

(H3) $\forall s \in [0,1] : H(0,s) = H(1,s)$.

Example 3.4.1. Consider

$$\gamma_0 : [0,1] \to \mathbb{C} : \gamma_0(t) = 4e^{2\pi i t},$$
$$\gamma_1 : [0,1] \to \mathbb{C} : \gamma_1(t) = 2i + e^{2\pi i t}.$$



Then γ_0 is $\mathbb{C}$-homotopic to γ_1 by

$$H(t,s) = (1-s)\gamma_0(t) + s\gamma_1(t).$$

γ_0 is not $\mathbb{C}\setminus\{0\}$-homotopic to γ_1.

The Cauchy Integral Theorem

Theorem 3.8. *Let*

(i) D be a domain in $\mathbb{C}$; (ii) $f : D \longrightarrow \mathbb{C}$ be holomorphic in D, and

(iii) $\gamma_0, \gamma_1 : [0,1] \longrightarrow \mathbb{C}$ be two closed, piecewise smooth, D-homotopic paths.

Then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

Proof. Consider a path homotopy $H : [0,1]^2 \to D$ s.t. $H \in C^2$. Let $\gamma_s := H(\cdot, s)$ be a closed path with fixed point s. Define

$$I(s) := \int_{\gamma_s} f(z)\,dz, \quad s \in [0,1].$$

We must show that $I(0) = I(1)$. Note that

$$\left[\forall s \in [0,1] : \frac{d}{ds}[I(s)] = 0\right] \implies I(0) = I(1).$$

We claim that $\frac{d}{ds}[I(s)] = 0$ for $s \in [0,1]$:

$$\begin{aligned}
\frac{d}{ds}[I(s)] &= \frac{d}{ds}\left[\int_{\gamma_s} f(z)\,dz\right] = \frac{d}{ds}\left[\int_0^1 f(\gamma_s(t))\,\gamma_s'(t)\,dt\right] \\
&= \frac{d}{ds}\left[\int_0^1 f(H(t,s))\,H_t(t,s)\,dt\right] \\
&= \int_0^1 \frac{\partial}{\partial s}\left[f(H(t,s))\,\frac{\partial}{\partial t}H(t,s)\right] dt \\
&= \int_0^1 \left(f'(H(t,s)) \cdot \frac{\partial}{\partial s}H(t,s) \cdot \frac{\partial}{\partial t}H(t,s) + f(H(t,s)) \cdot \frac{\partial^2}{\partial s \partial t}H(t,s)\right) dt \\
&= \int_0^1 \frac{\partial}{\partial t}\left[f(H(t,s))\,\frac{\partial}{\partial s}H(t,s)\right] dt \qquad \because H \in C^2 \\
&= \left[f(H(t,s))\,\frac{\partial}{\partial s}H(t,s)\right]_0^1 \\
&= f(H(1,s))\,\frac{\partial}{\partial s}H(1,s) - f(H(0,s))\,\frac{\partial}{\partial s}H(0,s) \\
&= 0,
\end{aligned}$$

since

(i) By (H3), $H(1,s) = H(0,s)$ holds.

(ii) $\displaystyle \frac{\partial}{\partial s}H(1,s) = \lim_{h \to 0} \frac{H(1,s+h) - H(1,s)}{h} = \lim_{h \to 0} \frac{H(0,s+h) - H(0,s)}{h} = \frac{\partial}{\partial s}H(0,s).$

Hence

$$\frac{d}{ds}[I(s)] = 0 \implies I(0) = I(1) \implies \int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

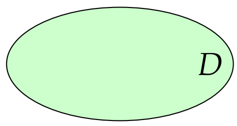
□

Cauchy-Goursat Theorem

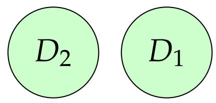
Theorem 3.9. *Let $f : D \to \mathbb{C}$ be a holomorphic function, where $D \subseteq \mathbb{C}$ is a simply connected domain. Let C be a closed contour in D. Then*

$$\oint_C f(z)\,dz = 0.$$

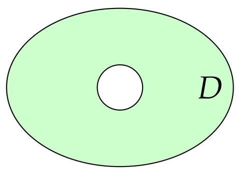
Remark 3.4.1 (Simply Connected Domain)**.**



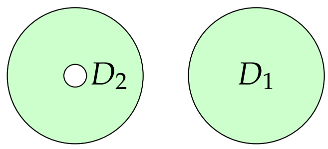
Connected (O)
Simply connected (O)



Connected (X)
Simply connected (O)



Connected (O)
Simply connected (X)



Connected (X)
Simply connected (X)

3.5 Existence of Primitive

Anti-derivative Theorem

Theorem 3.10. *Let*

(i) D is a simply connected domain and

(ii) $f : D \longrightarrow \mathbb{C}$ is holomorphic.

Then there is a holomorphic function $F : D \longrightarrow \mathbb{C}$ such that

$$z \in D \implies F'(z) = f(z).$$

Proof. Fixed a point $p \in D$. Define a function $F : D \longrightarrow \mathbb{C}$ as follows:

$$F(z) = \int_{\gamma_z} f(\zeta)\, d\zeta$$

where γ_z is a path joining p to z.

(i) (F is well-defined) Clearly, $\gamma := \gamma_z + (-\tilde{\gamma}_z)$ is closed. Then

$$\int_{\gamma} f(\zeta)\, d\zeta = 0 \implies \int_{\gamma_z + (-\tilde{\gamma}_z)} f(\zeta)\, d\zeta = 0$$
$$\implies \int_{\gamma} f(\zeta)\, d\zeta = \int_{\tilde{\gamma}_z} f(\zeta)\, d\zeta$$

That is, Cauchy Integral Theorem gives F is well-defined.

(ii) (Holomorphicity of F and $F' = f$ in D) Since f is holomorphic in D, it is also continuous there. Let $\varepsilon > 0$. Then

$$\exists \delta > 0 : \forall z \in D : |w - z| < \delta \implies |f(w) - f(z)| < \varepsilon.$$

We take a w such that $0 < |w - z| < \delta$. Then

$$\frac{F(w) - F(z)}{w - z} = \frac{1}{w - z}\left(\int_{\gamma_w} f(\zeta)\, d\zeta - \int_{\gamma_z} f(\zeta)\, d\zeta\right).$$

Let γ_{zw} is a straight line path joining z to w. By the Cauchy Integral Theorem, we obtain

$$0 = \int_{\gamma_z + \gamma_{zw} - \gamma_w} f(\zeta)\, d\zeta \implies \int_{\gamma_{zw}} f(\zeta)\, d\zeta = \int_{\gamma_w} f(\zeta)\, d\zeta - \int_{\gamma_z} f(\zeta)\, d\zeta.$$

Note that

$$w - z = \zeta\Big|_z^w = \int_{\gamma_{zw}} \frac{d}{d\zeta}[\zeta]\, d\zeta = \int_{\gamma_{zw}} 1 d\zeta.$$

Then

$$
\begin{aligned}
\frac{F(w)-F(z)}{w-z}-f(z) &= \frac{1}{w-z}\int_{\gamma_{zw}} f(\zeta)\,d\zeta - f(z)\\
&= \frac{1}{w-z}\int_{\gamma_{zw}} f(\zeta)\,d\zeta - f(z)\cdot\frac{1}{w-z}\int_{\gamma_{zw}} 1 d\zeta\\
&= \frac{1}{w-z}\int_{\gamma_{zw}} \left(f(\zeta)-f(z)\right)d\zeta,
\end{aligned}
$$

and so

$$
\begin{aligned}
\left|\frac{F(w)-F(z)}{w-z}-f(z)\right| &= \left|\frac{1}{w-z}\int_{\gamma_{zw}} \left(f(\zeta)-f(z)\right)d\zeta\right|\\
&= \frac{1}{|w-z|}\left|\int_{\gamma_{zw}} \left(f(\zeta)-f(z)\right)d\zeta\right|\\
&\le \frac{1}{|w-z|}\cdot\max_{\zeta\in\gamma_{zw}}|f(\zeta)-f(z)|\cdot\int_{\gamma_{zw}}|dz|\\
&< \frac{1}{|w-z|}\cdot\varepsilon\cdot|w-z|\\
&= \varepsilon.
\end{aligned}
$$

Thus $F'(z)=f(z)$, and F is holomorphic.

$\square$

3.6 The Cauchy Integral Formula

Proposition 3.11. *Let*

(1) D be a domain;

(2) $f : D \to \mathbb{C}$ be holomorphic in $D \setminus \{0\}$, and continuous on D;

(3) the disc $\Delta := \{z \in \mathbb{C} : |z - z_0| \leq r\} \subset D$ with $r > 0$ and $z_0 \in D$.

Then

$$f(z_0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - z_0} dz, \quad |z - z_0| < r,$$

where C_r is the circular path $C_r(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$, with enter z_0 and radius $r > 0$ traversed in the anti-clockwise direction.

Proof. Let $\varepsilon > 0$. We must show that

$$\left| \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - z_0} dz - f(z_0) \right| < \varepsilon.$$

The continuity of f on D gives

$$\exists \delta : |z - z_0| < \varepsilon \implies |f(z) - f(z_0)| < \varepsilon.$$

Since C_r is $D \setminus \{z_0\}$-homotopic to C_δ, we have

$$\int_{C_r} \frac{f(z)}{z - z_0} dz = \int_{C_\delta} \frac{f(z)}{z - z_0} dz$$

Note that

$$\int \frac{1}{z - z_0} dz = 2\pi i.$$

Thus,

$$\begin{aligned}
\left| \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - z_0} dz - f(z_0) \right| &= \left| \frac{1}{2\pi i} \int_{C_\delta} \frac{f(z)}{z - z_0} dz - f(z_0) \right| \\
&= \left| \frac{1}{2\pi i} \int_{C_\delta} \frac{f(z)}{z - z_0} dz - \frac{1}{2\pi i} \cdot f(z_0) \cdot \int_{C_\delta} \frac{1}{z - z_0} dz \right| \\
&= \left| \frac{1}{2\pi i} \int_{C_\delta} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \\
&\leq \frac{1}{|2\pi i|} \cdot \max_{\substack{z \in C_\delta \\ (|z - z_0| = \delta)}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \cdot \int_{C_\delta} |dz| \\
&< \frac{1}{2\pi} \cdot \frac{\varepsilon}{\delta} \cdot 2\pi\delta \\
&= \varepsilon.
\end{aligned}$$

□

The Cauchy Integral Formula for Circular Paths

Theorem 3.12. *Let*

(1) D be a domain;

(2) $f : D \to \mathbb{C}$ be holomorphic in D and $z_0 \in D$;

(3) the disc $\Delta := \{z \in \mathbb{C} : |z - z_0| \leq r\} \subset D$ with $r > 0$ and $z_0 \in D$.

Then

$$f(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - w} dz, \quad |w - z_0| < r,$$

where C_r is the circular path $C_r(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$, with enter z_0 and radius $r > 0$ traversed in the anti-clockwise direction.

Proof. Since $\dfrac{f(z)}{z - w}$ is holomorphic in $D \setminus \{w\}$ and C_δ is $D \setminus \{w\}$-homotopic to C_r,

$$f(w) = \frac{1}{2\pi i} \int_{C_\delta} \frac{f(z)}{z - w} dz = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - w} dz.$$

$\square$

The Cauchy Integral Formula for General Paths

Corollary 3.12.1. *Let*

(1) D be a domain;

(2) $f : D \to \mathbb{C}$ be holomorphic in D;

(3) γ be a closed path in D which is $D \setminus \{z_0\}$-homotopic to a circular path C centered at z_0, such that C and its interior is contained in D.

Then $f(z_0) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{f(z)}{z - z_0} dz.$

3.7 Holomorphic Functions are Infinitely Differentiable

Corollary 3.12.2. *Let*

(1) D be a domain;

(2) $f : D \longrightarrow \mathbb{C}$ be holomorphic in D;

Then f' is holomorphic in D.

Remark 3.7.1. The above gives the following chain of implications:

$$\boxed{f \in \mathrm{Hol}\,(D)} \Rightarrow \boxed{f' \in \mathrm{Hol}\,(D)} \Rightarrow \boxed{f'' \in \mathrm{Hol}\,(D)} \Rightarrow \cdots \boxed{f^{(n)} \in \mathrm{Hol}\,(D)} \Rightarrow \cdots .$$

Proof. (Naive Proof) Let $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$. Then

$$\begin{aligned} f'(z) &= \frac{1}{2\pi i} \frac{d}{dz} \left[\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta \right] \\ &= \frac{1}{2\pi i} \int_\gamma \frac{d}{dz} \left[\frac{f(\zeta)}{\zeta - z} \right] d\zeta \quad \text{(an assumption)} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} d\zeta, \end{aligned}$$

and $f''(z) = \dfrac{1}{2\pi i} \cdot 2 \displaystyle\int_\gamma \frac{f(\zeta)}{(\zeta - z)^3} d\zeta$. Thus

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

□

3.8 Liouville's Theorem; F.T.A.

Liouville's Theorem

Theorem 3.13. *Every bounded entire function is constant.*

Proof. Let $M \geq 0$ be such that $\forall z \in \mathbb{C} : |f(z)| \leq M$. Choose $w \in \mathbb{C}$, and let

$$\gamma(t) = w + Re^{it}, \quad t \in [0, 2\pi].$$

By generalized Cauchy integral theorem,

$$f'(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^2} dz,$$

and so

$$\begin{aligned} |f'(w)| = \left| \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^2} dz \right| &\leq \left| \frac{1}{2\pi i} \right| \cdot \max_{z \in \gamma} \left| \frac{f(z)}{(z-w)^2} \right| \cdot \int_\gamma |dz| \\ &\leq \frac{1}{2\pi} \cdot \frac{M}{R^2} \cdot 2\pi R \\ &= \frac{M}{R}. \end{aligned}$$

Since $R > 0$ was arbitrary, it follows that $f'(w) = 0$, and hence f is constant. □

Fundamental Theorem of Algebra

Corollary 3.13.1. *Every polynomial of degree* ≥ 1 *has a root in* $\mathbb{C}$.

Proof. Let $P : \mathbb{C} \to \mathbb{C} : P(z) = \sum_{i=1}^d c_i z^i = c_0 + c_1 z + \cdots + c_d z^d$ is a polynomial with $d \geq 1$. Suppose that $P(z)$ has no root in $\mathbb{C}$, that is, for all $z \in \mathbb{C}$, $P(z) \neq 0$. Define the function f by $f(z) = 1/P(z)$ $(z \in \mathbb{C})$, is entire. Note that

$$|z| > R \implies \exists M, R > 0 : |P(z)| \geq M |z|^d.$$

And so

$$|P(z)| \leq \max\left\{ \frac{1}{MR^d}, \frac{1}{m} \right\}, \quad z \in \mathbb{C}.$$

By Liouville's Theorem, f must be constant, and so P must be a constant, a contradiction to the fact that $d \geq 1$. □

3.9 Morera's Theorem

Morera's Theorem

Theorem 3.14. *Let*

(i) *D is a domain;*

(ii) $f : D \longrightarrow \mathbb{C}$ *is a continuous function such that*

(iii) *for every closed path* γ *in every disc contained in* D, $\oint_\gamma f(z)\,dz = 0$.

Then f *is holomorphic in* D.

Proof. Let $z_0 \in D$. Consider $z \in D$ with $z \neq z_0$. For two distinct path γ_1, γ_2 joining z_0 to z, define $\gamma := \gamma_1 + (-\gamma_2)$. Then

$$\begin{aligned} &\oint_\gamma f(\zeta)\,d\zeta = \int_{\gamma_1} f(\zeta)\,d\zeta - \int_{\gamma_2} f(\zeta)\,d\zeta = 0 \\ \Longrightarrow &\int_{\gamma_1} f(\zeta)\,d\zeta = \int_{\gamma_2} f(\zeta)\,d\zeta. \end{aligned}$$

We define

$$F(z) := \int_{z_0}^{z} f(\zeta)\,d\zeta.$$

Then $F'(z) = f(z)$ and $F \in \mathrm{Hol}$, and so $F^{(n)} \in \mathrm{Hol}$. Thus, $\exists F''(z)$ and then $F''(z) = f'(z)$. Hence f is holomorphic in D. $\square$

3.10 Special Content

3.10.1 Line Integral of Real function

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} := & \int_a^b \mathbf{F}\left(x(t), y(t)\right)\cdot\left(x'(t), y'(t)\right) dt \\ = & \int_a^b P\left(x(t), y(t)\right)\frac{dx(t)}{dt}dt + \int_a^b Q\left(x(t), y(t)\right)\frac{dy(t)}{dt}dt \\ = & \int_C Pdx + Qdy.\end{aligned}$$

Example 3.10.1. Let $\mathbf{F}(x, y) = (-y, x)$, and let $C(t) = (a\cos t, b\sin t)$ for $t \in [0, 2\pi]$. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}(-b\sin t, a\cos t)\cdot(-a\sin t, b\cos t)\, dt = \int_0^{2\pi} abdt = 2\pi ab.$$

3.10.2 Green's Theorem

Let $C = \partial R$ be a simple close contour (counter-clockwise). Consider two functions $P, Q : D \to \mathbb{R}$ with $P, Q \in C^1$. Then

$$\int_{C=\partial R} Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

3.10.3 Fundamental Theorem of Calculus (Generalized ver.)

$$\boxed{\int_{\partial R} f = \iint_R df}$$

Remark 3.10.1. Let $f := Pdx + Qdy$ then

$$\begin{aligned} df = d\left(Pdx + Qdy\right) &= \left(dPdx + Pd(dx)\right) + \left(dQdy + Qd(dy)\right) \\ &= dPdx + dQdy \quad \because d(dx) = 0 = d(dy) \\ &= \left(P_x dx + P_y dy\right)dx + \left(Q_x dx + Q_y dy\right)dy \\ &= P_y dydx + Q_x dxdy \quad \because dxdx = 0 = dydy \\ &= \left(Q_x - P_y\right)dxdy \quad \because dxdy = -dydx.\end{aligned}$$

Example 3.10.2.

$$\int_{C=\partial R}\left(x^2 + y^2\right)dx + (2xy)\,dy = \iint_R \left[\frac{\partial}{\partial x}2xy - \frac{\partial}{\partial y}\left(x^2 + y^2\right)\right]dxdy = \iint_R (2y - 2y)\,dxdy = 0.$$

Remark 3.10.2 (Area)**.** Area(C):= $\frac{1}{2}\oint_{C=\partial R} xdy - ydx.$

Proof.

$$\oint_{C=\partial R} xdy - ydx = \iint_R \left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y)\right) dxdy = 2\iint_R dxdy = 2 \cdot \text{Area}(C).$$

□

Example 3.10.3. Let $C(t) = (a\cos t, b\sin t)$ for $t \in [0, 2\pi]$ then

$$\oint_C xdy - ydx = \int_0^{2\pi} a\cos t \cdot b\cos t dt - \int_0^{2\pi} b\sin t \cdot (-a)\sin t dt = \int_0^{2\pi} abdt = 2\pi ab.$$

Thus the area is $S = \pi ab$.

Example 3.10.4. Let $C(t) = a\cos t + ib\sin t$ for $t \in [0, 2\pi]$. Then

$$\begin{aligned}
\int_C \bar{z}dz &= \int_0^{2\pi} (x(t) - iy(t))(x'(t) + iy'(t))\,dt \\
&= \int_0^{2\pi} (xx' + yy')\,dt + i\int_0^{2\pi} (-yx' + xy')\,dt \\
&= \oint_C xdx + ydy + i\oint_C (-y)\,dx + xdy \\
&= \iint_R \left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right) dxdy + i\iint_R \left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y)\right) dxdy \\
&= 0 + i2\iint_R dxdy \\
&= 2i \cdot \text{Area}(C).
\end{aligned}$$

Thus,

$$\text{Area}(C) = \frac{1}{2i}\int_C \bar{z}dz.$$

3.10.4 Cauchy-Goursat Theorem for Multiply-connected Domain

Let f is holomorphic in simply counter-clockwise connected contours C,C_1 and C_2. Then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

Proof. Let $\tilde{C} := C - C_1 - C_2$. By Cauchy-Goursat Theorem,

$$\int_{\tilde{C}} f(z)\,dz = 0,$$

and so

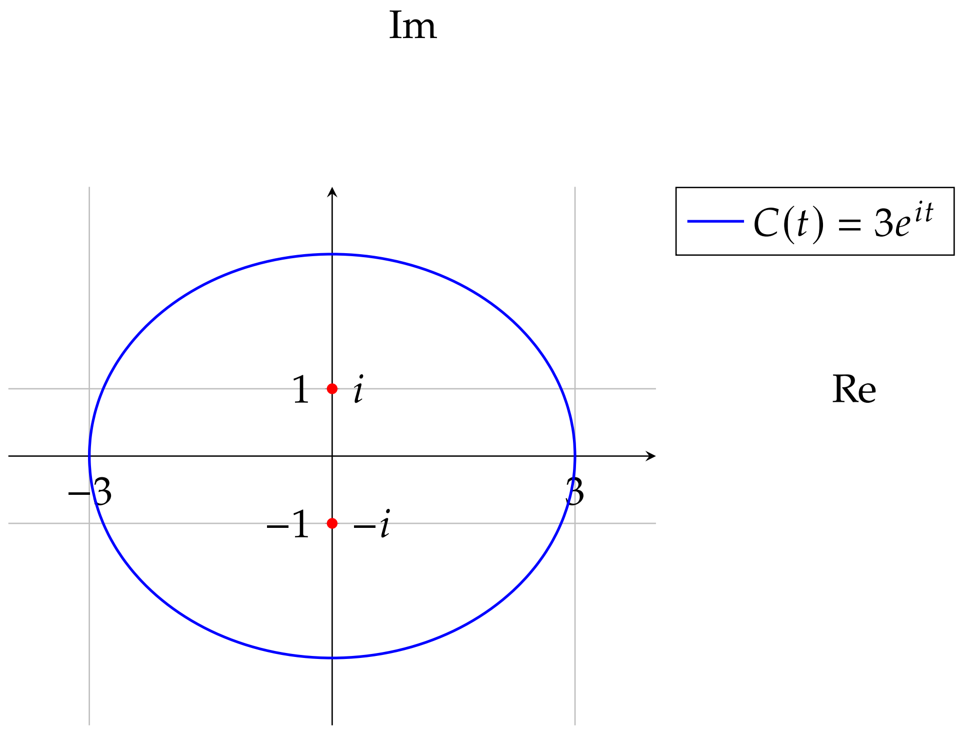
$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

□

Exercise 3.10.1. Find

$$\frac{1}{2\pi i}\oint_C \frac{e^{\alpha z}}{z^2+1}dz$$

with $C(t) = 3e^{it}$ for $t \in [0, 2\pi]$.



Sol. Note that

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right).$$

Let $f(z) = e^{\alpha z}$ then

$$\begin{aligned}
\frac{1}{2\pi i}\oint_C \frac{e^{\alpha z}}{z^2+1}dz &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)dz \\
&= \frac{1}{2\pi i}\frac{1}{2i}\oint_C \left(\frac{f(z)}{z-i} - \frac{f(z)}{z+i}\right)dz \\
&= \frac{1}{2i}\left[\frac{1}{2\pi i}\int_C \frac{f(z)}{z-i}dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{z+i}dz\right] \\
&= \frac{1}{2i}\left(f(i) - f(-i)\right) \quad \text{by Cauchy Integral formula} \\
&= \frac{e^{\alpha i} - e^{-\alpha i}}{2i} \\
&= \sin\alpha.
\end{aligned}$$

□

Chapter 4

Taylor and Laurent series

Note (Convergence of Sequence)**.**

$$\lim_{n\to\infty} a_n = A \overset{\text{def.}}{\iff} \forall \varepsilon > 0 : \exists N \in \mathbb{N} : [n \geq N \implies |a_n - A| < \varepsilon] .$$

4.1 Series

Let $\{a_n\}$ is a sequence in $\mathbb{C}$. The sequence $\{s_k\}$ defined by

$$\begin{aligned} s_1 &:= a_1 \\ s_2 &:= a_1 + a_2 \\ &\vdots \\ s_k &:= a_1 + a_2 + \cdots + a_{k-1} + a_k \\ &\vdots \end{aligned}$$

The numbers s_k are called the **partial sums**.

Convergence of Series

Definition 4.1.

(1) The series $\sum_{n=1}^{\infty} a_n$ **converges** if $\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\sum_{k=1}^{n} a_k\right)$.

(2) The series $\sum_{n=1}^{\infty} a_n$ **diverges** if $\{s_n\}_{n\in\mathbb{N}}$ is diverges.

(3) The series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if the real series $\sum_{n=1}^{\infty} |a_n|$ converges.

Note. Let $\{a_n\}_{n\in\mathbb{N}}$ is positive bounded. Then

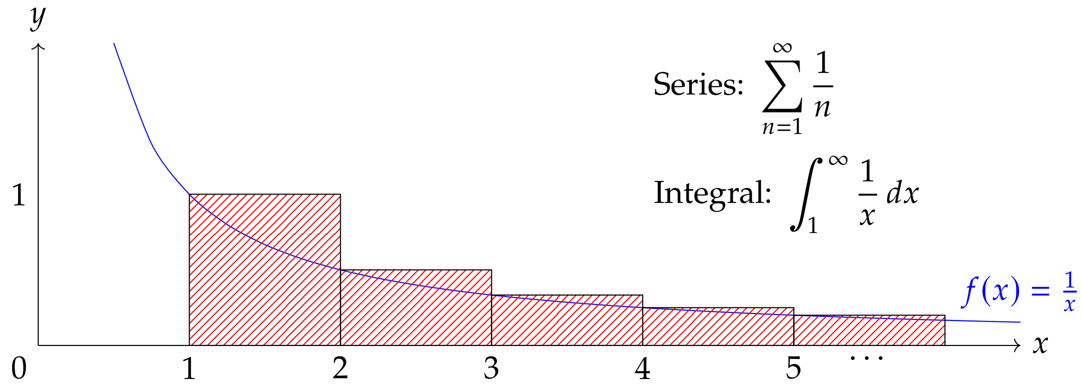
$$\sum a_n \text{ converges } \iff \exists M \in \mathbb{C} : \left[n \in \mathbb{N} \implies \sum_{k=1}^{\infty} \leq M \right].$$

Note (Integral Test). Let $f : [1,\infty) \to \mathbb{R}^+$ be a decreasing function on $[1,\infty)$. Then the series $\sum_{k=1}^{\infty} f(k)$ converges if and only if the improper integral

$$\int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^b f(x)\,dx$$

exists. In the case of convergence, the partial sum $S_n = \sum_{k=1}^{n} f(k)$ and the sum $S = \sum_{k=1}^{\infty} f(k)$ satisfy the estimate

$$\int_{n+1}^{\infty} f(x)\,dx \le S - S_n \le \int_n^{\infty} f(x)\,dx.$$



Example 4.1.1 (p-series)**.** The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

converges when $p > 1$ and diverges when $p \le 1$.

Note (Ratio Test)**.** Let $\sum a_n$ be a series such that

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

1. If $r < 1$ then the series $\sum a_n$ is absolutely convergent.
2. If $r > 1$ then the series $\sum a_n$ is divergent.
3. If $r = 1$ then this test gives no information.

Example 4.1.2. Determine whether $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ converges.

Sol. Let $a_n = \frac{n^n}{n!}$ then

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n.$$

Thus, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e > 1$, and so $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges. $\square$

Note (Root Test)**.** Let $\sum a_n$ be a series such that

$$r = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

1. If $r < 1$ then the series $\sum a_n$ is absolutely convergent.

2. If $r > 1$ then the series $\sum a_n$ is divergent.

3. If $r = 1$ then this test gives no information.

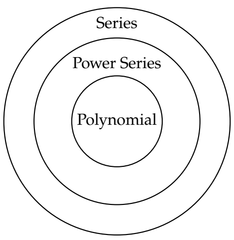
Example 4.1.3. Determine whether $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ converges.

Sol. Since

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{(\ln n)^n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 < 1,$$

$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ converges. □

4.2 Power Series



Polynomial ⊆ Power Series ⊆ Series

Power Seires

Let $\{c_n\}_{n\in\mathbb{N}}$ be a complex sequence (thought of as a sequence of coefficients). An expression of the type

$$\sum_{n=0}^{\infty} c_n z^n$$

is called a **power series** in the complex variable z.

Existence of Radius of Convergence

Theorem 4.1. *For $\sum_{n=0}^{\infty} c_n z^n$, exactly one of the following hold:*

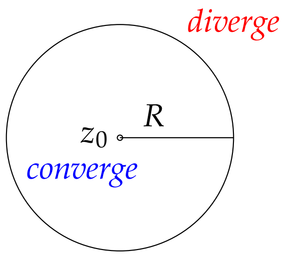
(1) *Either it is absolutely convergence for all $z \in \mathbb{C}$.*

(2) *Or there is a unique non-negative real number R (**radius of convergence**) such that*

   (a) *$\sum_{n=0}^{\infty} c_n z^n$ is absolutely convergent for all $z \in \mathbb{C}$ with $|z| < R$, and*

   (b) *$\sum_{n=0}^{\infty} c_n z^n$ is divergent for all $z \in \mathbb{C}$ with $|z| > R$.*

If the power series converges for all $z \in \mathbb{C}$, we say that the power series has an infinite radius of convergence, and write $R = \infty$.



Theorem 4.2. *Consider the power series $\sum_{n=0}^{\infty} c_n z^n$. Let R is the radius of convergence, and let $L := \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$ exists. Then*

(1) $L \neq 0 \implies R = 1/L$.

(2) $L = 0 \implies R = \infty$.

Theorem 4.3. *Let $f(z) := \sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < R(> 0)$. Then*

$$f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1} \quad \text{for } |z| < R.$$

Corollary 4.3.1. *Let $f(z) := \sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < R(> 0)$. Then for $k \geq 1$,*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \left[\left(\prod_{i=0}^{k-1} (n-i) \right) c_n z^{n-k} \right] \quad \text{for } |z| < R.$$

In particular, for $n \geq 0$, $c_n = \dfrac{1}{n!} f^{(n)}(0)$.

Corollary 4.3.2. *Let* $z_0 \in \mathbb{C}$*, and let* $f(z) := \sum_{n=0}^{\infty} c_n (z - z_0)^n$ *converges for* $|z - z_0| < R(> 0)$*. Then for* $k > 1$*,*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \left[\left(\prod_{i=0}^{k-1} (n - i) \right) c_n (z - z_0)^{n-k} \right] \quad \text{for } |z - z_0| < R.$$

In particular, for $n \geq 0$*,* $c_n = \dfrac{1}{n!} f^{(n)}(z_0)$*.*

4.3 Taylor Series

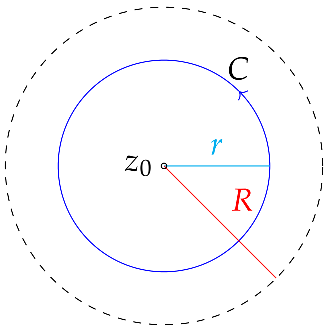
Theorem 4.4. *Let f is holomorphic in $D(z_0, R) := \{z \in \mathbb{C} : |z - z_0| < R\}$. Then*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n = c_0 + c_1 (z - z_0) + c_2 (z - z_0)^2 + \cdots$$

for $z \in D(z_0, R)$, where for $n \geq 0$,

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta,$$

and C is the circular path with center z_0 and radius r, where $0 < r < R$ traversed in the anti-clockwise direction.



Proof. Let $z \in D(z_0, R)$. Initially, let r be such that $|z - z_0| < r < R$. Then by Cauchy's Integral Formula,

$$\begin{aligned} f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta &= \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z_0 + z_0 - z} d\zeta \\ &= \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)\left(1 - \dfrac{z - z_0}{\zeta - z_0}\right)} d\zeta \\ &= \frac{1}{2\pi i} \oint_C \left[\frac{f(\zeta)}{(\zeta - z_0)} \cdot \frac{1}{1 - \dfrac{z - z_0}{\zeta - z_0}} \right] d\zeta. \end{aligned}$$

Set $w := \dfrac{z-z_0}{\zeta - z_0}$ then $|w| = \dfrac{|z-z_0|}{r} < 1$. Thus

$$\begin{aligned}
\frac{1}{1-\dfrac{z-z_0}{\zeta-z_0}} = \frac{1}{1-w} = \sum_{k=0}^{n-1} w^k + \frac{w^n}{1-w} &= 1 + \sum_{k=1}^{n-1} \frac{(z-z_0)^k}{(\zeta-z_0)^k} + \frac{\left(\dfrac{z-z_0}{\zeta-z_0}\right)^n}{1-\dfrac{z-z_0}{\zeta-z_0}} \\
&= 1 + \sum_{k=1}^{n-1} \frac{(z-z_0)^k}{(\zeta-z_0)^k} + \frac{\left(\dfrac{z-z_0}{\zeta-z_0}\right)^n}{\dfrac{\zeta-z_0-z+z_0}{\zeta-z_0}} \\
&= 1 + \sum_{k=1}^{n-1} \frac{(z-z_0)^k}{(\zeta-z_0)^k} + \frac{(z-z_0)^n}{(\zeta-z_0)^n} \cdot \frac{\zeta-z_0}{\zeta-z} \\
&= 1 + \sum_{k=1}^{n-1} \frac{(z-z_0)^k}{(\zeta-z_0)^k} + \frac{(z-z_0)^n}{(\zeta-z_0)^{n-1}(\zeta-z)}.
\end{aligned}$$

Plugging this in the above, we obtain

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \oint_C \left[\frac{f(\zeta)}{(\zeta-z_0)} \cdot \left[1 + \sum_{k=1}^{n-1} \frac{(z-z_0)^k}{(\zeta-z_0)^k} + \frac{(z-z_0)^n}{(\zeta-z_0)^{n-1}(\zeta-z)} \right] \right] d\zeta \\
&= \frac{1}{2\pi i} \oint_C \left[f(\zeta) \cdot \left[\sum_{k=0}^{n-1} \frac{(z-z_0)^k}{(\zeta-z_0)^{k+1}} + \frac{(z-z_0)^n}{(\zeta-z_0)^n(\zeta-z)} \right] \right] d\zeta. \\
&= \sum_{k=0}^{n-1} \left[\left(\frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta-z_0)^{k+1}}\, d\zeta \right) (z-z_0)^k \right] + \frac{1}{2\pi i} \oint_C \frac{f(\zeta)(z-z_0)^n}{(\zeta-z_0)^n(\zeta-z)}\, d\zeta \\
&= \sum_{k=0}^{n-1} \left[\left(\frac{f^{(k)}(z_0)}{k!} \right) (z-z_0)^k \right] + \frac{1}{2\pi i} \oint_C \frac{f(\zeta)(z-z_0)^n}{(\zeta-z_0)^n(\zeta-z)}\, d\zeta \\
&= \left(\sum_{k=0}^{n-1} c_k (z-z_0)^k \right) + R_n(z).
\end{aligned}$$

We must show that $R_n(z) \to 0$ as $n \to \infty$: Note that

$$|R_n(z)| = \left| \frac{1}{2\pi i} \oint_C \frac{f(\zeta)(z-z_0)^n}{(\zeta-z_0)^n(\zeta-z)}\, d\zeta \right| \leq \frac{1}{2\pi} \cdot \max_{\zeta\in C} \left| \frac{(z-z_0)^n}{(\zeta-z_0)^n} \cdot \frac{f(\zeta)}{\zeta-z} \right| \cdot \int_C |d\zeta|\,.$$

□

Taylor Series

Corollary 4.4.1. *Let*

(1) D be a domain;

(2) $f : D \longrightarrow \mathbb{C}$ is holomorphic, and

(3) $z_0 \in D$.

Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots ,$$

$|z - z_0| < R$, where R is the radius of the largest open disk with center z_0 contained in D. Also,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^n + 1}\, dz,$$

where C is the circular path with center z_0 and radius r, where $0 < r < R$ traversed in the anti-clockwise direction.

Cauchy's Inequality

Corollary 4.4.2. *Let*

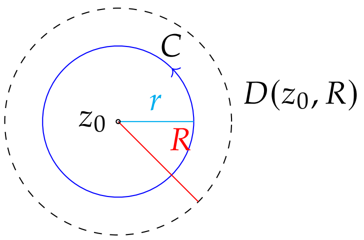
(1) f is holomorphic in $D(z_0, R) := \{z \in \mathbb{C} : |z - z_0| < R\}$ and

(2) $\forall z \in D(z_0, R) : |f(z)| \leq M$.

Then

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!M}{R^n} \quad \textit{for } n \geq 0.$$

Proof. Let C be the circle with center z_0 and radius $r < R$:



Then

$$\begin{aligned}\left| f^{(n)}(z_0) \right| = \left| \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \right| &\leq \frac{n!}{2\pi} \max_{z \in C} \left| \frac{f(z)}{(z - z_0)^{n+1}} \right| \cdot 2\pi r \\ &= \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r = \frac{n!M}{r^n}.\end{aligned}$$

The claim now follows by passing the limit $r \nearrow R$. $\square$

4.4 Classification of Zeros

Example 4.4.1.

(1) $\exp z$ has no zeros in $\mathbb{C}$. Indeed, $\left|\exp z\right| = e^{\mathrm{Re}(z)} > 0$ for all $z \in \mathbb{C}$.

(2) The polynomial p, $p(z) = (z+1)^3 z^9 (z-1)^9$, has zeros at $-1, 1, 0$.

(3) $(\cos z) - 3$ has infinitely may zeros in $\mathbb{C}$ at $2\pi n \pm i \ln(3 + 2\sqrt{2})$ for $n \in \mathbb{N}$:

Let $f(z) = (\cos z) - 3$. Note that $f(z) = 0 \Rightarrow \cos z = \frac{e^{iz}+e^{-iz}}{2} = 3$. Let $X = e^{iz}$ then

$$\frac{X + \frac{1}{X}}{2} \implies X^2 - 6X + 1 \implies X = 3 \pm 2\sqrt{2}.$$

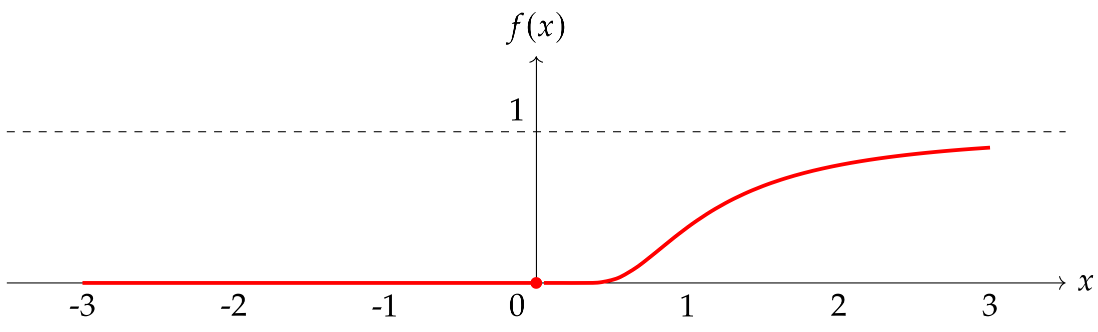
Let $z = x + iy$ then

$$e^{iz} = e^{-y+ix} = e^{-y}(\cos x + i \sin x) = (3 \pm 2\sqrt{2}) \cdot \underbrace{e^{i2n\pi}}_{=1}$$

$$\Rightarrow \begin{cases} x = 2n\pi \\ y = -\ln(3 \pm 2\sqrt{2}) = \pm\ln(3 + 2\sqrt{2}). \end{cases}$$

Thus, $z = x + yi = 2n\pi \pm i \ln(3 + 2\sqrt{2})$.

(4) Consider the real function $f(x) = \begin{cases} e^{-1/x^2} & : x > 0 \\ 0 & : x \leq 0. \end{cases}$



Note that

$$f(0) = 0 = f'(0) = f''(0) = \cdots = f^{(n)} = 0 = \cdots,$$

i.e., $f^{(n)} = 0$ for all $n \in \mathbb{Z}_{\geq 0}$. Thus, $f \in C^\infty$ and $f^{(n)}(x) = 0$ $(x \leq 0)$. Hence $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$ with

$$c_n = \frac{f^{(n)}(n)}{n!} = 0$$

for $n \geq 0$. That is $f \equiv 0$.

Zero

Definition 4.2. Let D be a domain and $f : D \longrightarrow \mathbb{C}$ be holomorphic in D. A point $z_0 \in D$ is called a **zero** of f if $f(z_0) = 0$. If there is a smallest $m \in \mathbb{N}$ such that

(1) $f^{(m)}(z_0) \neq 0$;

(2) $f^{(0)}(z_0) = \cdots = f^{m-1}(z_0) = 0$,

then z_0 is said to be a **zero of f of order** m

Example 4.4.2. Consider $f(z) = \sin z$. Since $f(0) = 0$ but

$$f'(0) = \cos 0 = 1 \neq 0,$$

0 is a zero of f of order 1.

Classification of Zeros

Proposition 4.5. *Let*

(1) $f : D \longrightarrow \mathbb{C}$ be holomorphic in domain D and

(2) $z_0 \in D$ be a zero of f, that is, $f(z_0) = 0$.

Then there are exactly two possibilities:

1. $\exists R > 0 : \forall z \in B(z_0, R) : f(z) = 0.$
2. $\exists m \in \mathbb{N}$ *such that z_0 is a zero of f of order m, and there exists a holomorphic function $g : D \longrightarrow \mathbb{C}$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^m g(z)$ for all $z \in D$.*

Proof. We have a power series expansion for f in $D(z_0, R > 0)$:

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots$$

for $|z - z_0| < R$. Since $f(z_0) = 0$, we know that $c_0 = 0$.

(Case I) $\forall n \in \mathbb{N} : c_n = 0$.

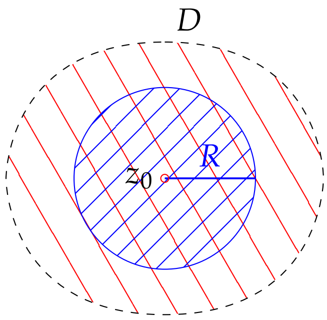
$$f(z) = 0 \quad \text{whenever} \quad |z - z_0| < R.$$

(Case II) $\exists m \in \mathbb{N}_{\geq 1} : [c_m \neq 0 \text{ and } c_0 = c_1 = \cdots = c_{m-1} = 0]$.

$$f(z) = c_m(z - z_0)^m + c_{m+1}(z - z_0)^{m+1} + \cdots = (z - z_0)^m \sum_{k=0}^{\infty} c_{m+k}(z - z_0)^k$$

for $|z - z_0| < R$. Define $g : D \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-z_0)^m} & : z \neq z_0, \\ \displaystyle\sum_{k=0}^{\infty} c_{m+k}(z-z_0)^k & : |z - z_0| < R. \end{cases}$$



We claim that g is well-defined on $0 < |z - z_0| < R$:

$$\begin{aligned} f(z) &= c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots \\ &= (z-z_0)^m \sum_{k=0}^{\infty} c_{m+k}(z-z_0)^k \quad \because (c_0 = c_1 = \cdots = c_{n-1} = 0) \\ &= (z-z_0)^m g(z). \end{aligned}$$

□

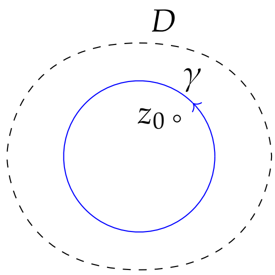
Example 4.4.3. $\exp(z^2) - 1$ has a zero at 0 since $\exp(0^2) - 1 = 1 - 1 = 0$. What is its order?

Sol. We have

$$\exp(z^2) = \sum_{n=0}^{\infty} \frac{(z^2)^n}{n!} = 1 + \frac{z^2}{1!} + \frac{z^4}{2!} + \cdots, \quad z \in \mathbb{C},$$

and so $\exp(z^2) - 1 = z^2 g(z)$ $(z \in \mathbb{C})$, where $g(z) := \dfrac{1}{1!} + \dfrac{z^2}{2!} + \cdots$. g is given by a power series that converges in $\mathbb{C}$, and so g is entire. Also, $g(0) = 1 \neq 0$. Thus the order of 0 as a zero of $\exp(z^2) - 1$ is 2. □

Example 4.4.4. Let f be holomorphic in a disc that contains a circle γ in its interior.



Suppose there is exactly one zero z_0 of order 1 of f, which lies in the interior of γ. Prove that

$$z_0 = \frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)}\,dz.$$

Proof. Note that $f(z) = (z - z_0)g(z)$ with $g(z) \neq 0$ since z_0 is a zero of order 1. Then $f'(z) = (z - z_0)g'(z) + g(z)$. Thus

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(z)}\,dz &= \frac{1}{2\pi i}\int_\gamma \frac{z(z-z_0)g'(z) + zg(z)}{(z-z_0)g(z)}\,dz\\ &= \frac{1}{2\pi i}\int_\gamma \frac{zg'(z)}{g(z)}\,dz + \frac{1}{2\pi i}\int_\gamma \frac{z}{z-z_0}\,dz\\ &= 0 + z_0\\ &= z_0.\end{aligned}$$

□

4.5 The Identity Theorem

Theorem 4.6. *Let*

(1) $f : D \to \mathbb{C}$ be a holomorphic function in a domain D;

(2) $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of distinct zeros of f which converges to $z_* \in D$.

Then f is identically zero in D, that is, $f \equiv 0$.

Proof. sdf □

Identity Theorem

Corollary 4.6.1. *Let*

(1) $f : D \to \mathbb{C}$ be a holomorphic function in a domain D;

(2) $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of distinct zeros of f which converges to $z_* \in D$, and such that $n \in \mathbb{N} \implies f(z_n) = g(z_n)$.

Then $f(z) = g(z)$ for all $z \in D$.

Example 4.5.1. We know that $\exp : \mathbb{C} \to \mathbb{C}$ defined by

$$\exp(z) = \exp(x + iy) := e^x (\cos y + i \sin y), \quad z = x + iy \in \mathbb{C},$$

is an entire function such that $\exp x = e^x$ for $x \in \mathbb{R}$. In other words, exp is an entire extension of the usual real exponential function. Is there any other entire extension possible? No! Suppose that $g : \mathbb{C} \to \mathbb{C}$ is entire and $g(x) = e^x$ for real x. But then $\exp x = g(x)$ for all $x \in \mathbb{R}$. In particular,

$$\exp\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right), \quad n \in \mathbb{N},$$

and $1/n \to 0 \in \mathbb{C}$. By Identity Theorem, $\exp z = g(z)$ for all $z \in \mathbb{C}$. So there is *only one* entire function whose restriction to $\mathbb{R}$ to e^x.

4.6 The Maximum Modulus Theorem

Maximum Modulus Theorem

Theorem 4.7. *Let*

(1) $f : D \to \mathbb{C}$ be holomorphic in a domain D,

(2) $\exists z_0 \in D : \forall z \in D : |f(z_0)| \geq |f(z)|$.

Then f is constant on D.

Proof. Let $r > 0$ be such that the disc with center z_0 and radius $2r$ is contained in D. Let C_r be the circular path $C_r(t) = z_0 + r\exp(it)$, $t \in [0, 2\pi]$. Then by the Cauchy Integral Formula,

$$\begin{aligned}
f(z_0) &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}\,dz \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + r\exp(it))}{r\exp(it)} ir\exp(it)\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi i} f(z_0 + r\exp(it))\,dt.
\end{aligned}$$

Since $\left|f(z_0 + r\exp(it))\right| \leq |f(z_0)|$ for all t, we have

$$\begin{aligned}
|f(z_0)| &= \left|\frac{1}{2\pi}\int_0^{2\pi i} f(z_0 + r\exp(it))\,dt\right| \\
&\leq \frac{1}{2\pi} \cdot \max_{t\in[0,2\pi]} \left|f(z_0 + r\exp(it))\right| \cdot 2\pi \\
&\leq |f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\ dt
\end{aligned}$$

Thus,

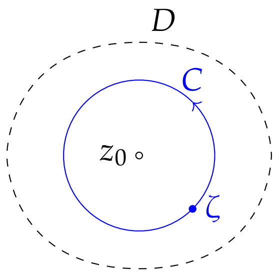
$$\frac{1}{2\pi}\int_0^{2\pi} \left(|f(z_0)| - \left|f(z_0 + r\exp(it))\right|\right)\ dt = 0.$$

□

Summary

(1) (**Cauchy Integral Formula**) If the value of $f(z)$ is known at the boundary, the function values can be determined at all points within the interior.

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$



(2) (**Taylor Series**) Knowing the information at a point allows us to determine the function value.

$$f(z_0), f'(z_0), f''(z_0), \cdots, f^{(n)}(z_0), \cdots \rightsquigarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

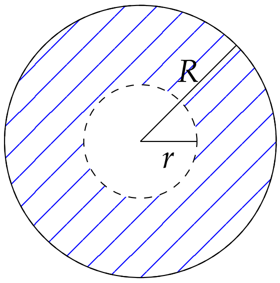
(3) (**Identity Theorem**) $f(z_n) \rightsquigarrow \forall z \in D : f(z)$.

4.7 Laurent Series

Laurent series generalize Taylor series. A **Laurent series** is an expression of the type

$$\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = \sum_{n\in\mathbb{Z}} c_n(z-z_0)^n = \cdots + c_{-1}(z-z_0)^{-1} + c_0 + c_1(z-z_0)^1 + \cdots,$$

which has negative powers of $z - z_0$ too.



We will see that

(1) Laurent series "converge" in an annulus $\{z \in \mathbb{C} : r < |z - z_0| < R\}$ with center z_0 and gives a holomorphic function there, and

(2) conversely, if we have a holomorphic function in an annulus with center z0 and it has singularities that lie in the "hole" inside the annulus, then the function has a Laurent series expansion in the annulus. example,

Example 4.7.1. we know that for all $z \in \mathbb{C}$,

$$\exp z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots,$$

and so for $z \neq 0$, we have the "Laurent series expansion"

$$\exp\frac{1}{z} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots$$

Note that $\exp(1/z)$ is holomorphic in $\mathbb{C} \setminus \{0\}$, which is a degenerate annulus centered at 0 with inner radius $r = 0$ and out radius $R = +\infty$.

Example 4.7.2. content...

Convergence of Laurent series

Definition 4.3. The Laurent series $\sum_{n\in\mathbb{Z}} c_n(z-z_0)^n$ **converges** for z if

$$\textstyle\sum_{n=1}^{\infty} c_{-n}(z-z_0)^{-n} \text{ converges and } \sum_{n=0}^{\infty} c_n(z-z_0)^n \text{ converges.}$$

If $\sum_{n\in\mathbb{Z}} c_n(z-z_0)^n$ converges, then we write

$$\sum_{n\in\mathbb{Z}} c_n(z-z_0)^n = \sum_{n=1}^{\infty} c_{-n}(z-z_0)^{-n} + \sum_{n=0}^{\infty} c_n(z-z_0)^n,$$

and call this the sum of the Laurent series.

Theorem 4.8. *Let f is holomorphic in $\mathbb{A} := \{z \in \mathbb{C} : r < |z - z_0| < R\}$. Then*

$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z - z_0)^n \quad \text{for } z \in \mathbb{A},$$

where

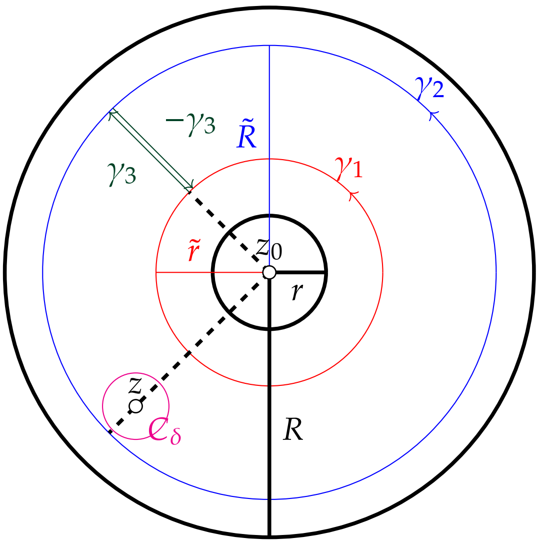
(1) $c_n = \dfrac{1}{2\pi i} \displaystyle\oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta,$

(2) *C is the circular path given by $C(t) = z_0 + \rho e^{it}$, $t \in [0, 2\pi]$,*

(3) *ρ is any number such that $r < \rho < R$.*

Moreover, the coefficients are unique.

Proof.



(Existence) Fix $z \in \mathbb{A}$. Choose $\tilde{r}$ and $\tilde{R}$ such that $r < \tilde{r} < |z - z_0| < \tilde{R} < R$. Let γ_1 and γ_2 be the circular paths

$$\gamma_1(t) = z_0 + \tilde{r}e^{it},$$
$$\gamma_2(t) = z_0 + \tilde{R}e^{it},$$

for $t \in [\theta, 2\pi + \theta]$, and $\theta = \operatorname{Arg}(z) + \pi/2$. Let $\gamma_3 : [\tilde{r}, \tilde{R}] \to \mathbb{A}$ be the path

$$\gamma_3(t) = ti\frac{z - z_0}{|z - z_0|}.$$

Clearly the path $\gamma := \gamma_2 - \gamma_3 - \gamma_1 + \gamma_3$ is $\mathbb{A} \setminus \{z\}$-homotopic to a small circle C_δ centered at z. Also, $\frac{f(*)}{*-z}$ is holomorphic in $\mathbb{A} \setminus \{z\}$, and so

$$\oint_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \oint_{C_\delta} \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z) \cdot 2\pi i,$$

by the Cauchy Integral Theorem. Thus,

$$f(z) = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \underbrace{\frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta}_{\text{(I)}} \underbrace{-\frac{1}{2\pi i}\oint_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta}_{\text{(II)}}.$$

(I) We will show that $\frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \sum_{n=0}^{\infty} c_n (z - z_0)^n$.

We have for $\zeta \in \gamma_2$ that

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - z_0 + z_0 - z} = \frac{f(\zeta)}{(\zeta - z_0)\left(1 - \frac{z - z_0}{\zeta - z_0}\right)} = \frac{f(\zeta)}{(\zeta - z_0)(1 - w)}$$

where $w = \frac{z - z_0}{\zeta - z_0}$. We have $|w| = \frac{|z - z_0|}{|\zeta - z_0|} = \frac{|z - z_0|}{\tilde{R}} < 1$, and so $\frac{1}{1 - w} = \sum_{k=0}^{n-1} w^k + \frac{w^n}{1 - w}$. Using this, we obtain

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - z_0}\left(\sum_{k=0}^{n-1} w^k + \frac{w^n}{1 - w}\right) = \sum_{k=0}^{n-1}\left[\frac{f(\zeta)}{(\zeta - z_0)^{k+1}}(z - z_0)^k\right] + \frac{f(\zeta)(z - z_0)^n}{(\zeta - z_0)^n(\zeta - z)}.$$

Thus

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta &= \sum_{k=0}^{n-1}\left[\frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\,d\zeta \cdot (z - z_0)^k\right] \\
&\quad + \frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(\zeta)}{(\zeta - z_0)^n(\zeta - z)}\,d\zeta \cdot (z - z_0)^n \\
&= \sum_{k=0}^{n-1} c_k (z - z_0)^k + R_n(z),
\end{aligned}$$

where

$$R_n(z) := \frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(\zeta)}{(\zeta - z_0)^n(\zeta - z)}\,d\zeta.$$

(II)

□

Example 4.7.3. fdg

4.8 Classification of Singularities

Isolated Singularity

Definition 4.4. Let f be a complex valued function which is not defined at a point z_0. Suppose that f is holomorphic in $\mathcal{B}^*(z_0, R)$. Then we call z_0 an **isolated singularity** of f.

Example 4.8.1. Each of the functions

$$\frac{\sin z}{z}, \quad \frac{1}{z^3}, \quad \exp\frac{1}{z},$$

has an isolated singularity at 0.

Example 4.8.2. f given by

$$f(z) = \frac{1}{\sin\left(\frac{1}{z}\right)}$$

has a singularity at 0, but it is not an isolated singularity.

$$\sin\left(\frac{1}{z}\right) = 0 \rightsquigarrow z = \frac{1}{n\pi}\ (n \in \mathbb{Z}) \rightsquigarrow z = 0,\ \pm\frac{1}{\pi},\ \pm\frac{1}{2\pi},\ \cdots.$$

Definition 4.5. An isolated singularity z_0 of f is called

(1) a **removable singularity** of f if there is a function F, holomorphic in $B(z_0, R)$ such that $F = f$ in $B^*(z_0, R)$.

(2) a **pole** of f if $\lim_{z\to z_0} |f(z)| = +\infty$, that is,

$$\forall M > 0 : \exists \delta > 0 : z \in B^*(z_0, \delta) \implies |f(z)| > M.$$

(3) an **essential singularity** of f if z_0 is neither removable nor a pole.

Example 4.8.3.

(1) The function $f(z) = \dfrac{\sin z}{z}$ has a removable singularity at 0, since for $z \neq 0$, we have

$$\frac{\sin z}{z} = \frac{1}{z}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}$$

(2) The function $\dfrac{1}{z^3}$ has a pole at 0, since $\lim_{z\to 0} \dfrac{1}{|z|^3} = +\infty$.

(3) The function $\exp \frac{1}{z}$ has an essential singularity at 0. Indeed,

(a) 0 is not a removable singularity, because $\lim_{x \searrow 0} e^{1/x} = +\infty$.

(b) 0 is also not a pole, since $\lim \lim_{x \nearrow 0} e^{1/x} = 0$, and so it can not be that $\lim_{z \to 0} |f(z)| = +\infty$.

Classification of Singularities via Limits

Theorem 4.9. *Let z_0 is an isolated singularity of f. Then*

(1) z_0 is removable $\iff \lim_{z \to z_0} f(z) = 0$.

(2) z_0 is a pole $\iff$

(a)

(b)

(3) z_0 is essential $\iff$

Classification of Singularities via Laurent Coefficients

Theorem 4.10. *Let z_0 be an isolated singularity of f, and*

$$f(z) = \sum_{n\in\mathbb{Z}} c_n(z-z_0)^n \quad \textit{for} \quad z \in B^*(z_0, R),$$

for some $R > 0$. Then

(1) z_0 is removable $\iff \forall n < 0 : c_n = 0$, i.e., $c_{-1} = c_{-2} = \cdots = 0$.

(2) z_0 is pole (of order m) $\iff c_m \neq 0$ *and* $\forall n < -m : c_n = 0$.

(3) z_0 is essential $\iff \forall n < 0 : c_n \neq 0$.

Example 4.8.4.

1. (removable) $f(z) = \dfrac{\sin z}{z} = \dfrac{1}{z}\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots\right) = 1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} - \cdots$.

2. (pole of order 2) $g(z) = \dfrac{1}{z^2} + \dfrac{1}{z} + z$.

3. (essential) $h(z) = \exp\left(\dfrac{1}{z}\right) = 1 + \left(\dfrac{1}{z}\right) + \dfrac{1}{2!}\left(\dfrac{1}{z^2}\right) + \dfrac{1}{3!}\left(\dfrac{1}{z^3}\right) + \cdots$.

Exercise 4.8.1. Let D be a domain an $z_0 \in D$. Suppose that f has a pole of order m at z_0 and that f has the Laurent series expansion

$$f(z) = \sum_{n\in\mathbb{Z}} c_n(z-z_0)^n \quad \text{for} \quad z \in B^*(z_0, R),$$

where $R > 0$. Show that

$$c_{-1} = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right].$$

Sol. Since f has a pole of order m at z_0, we have

$$f(z) = c_{-m}(z-z_0)^{-m} + \cdots + c_{-1}(z-z_0)^{-1} + c_0 + c_1(z-z_0) + \cdots,$$
$$(z-z_0)^m f(z) = c_{-m} + \cdots + c_{-1}(z-z_0)^{m-1} + c_0(z-z_0)^m + c_1(z-z_0)^{m+1} + \cdots.$$

Note that

$$\frac{d^{m-1}}{dz^{m-1}}\left[c_{-1}(z-z_0)^{m-1}\right] = c_{-1}(m-1)(m-2)\cdots 2\cdot 1 = c_{-1}(m-1)!.$$

Thus

$$\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right] = c_{-1}(m-1)! + c_0 m!(z-z_0) + \cdots$$

and then

$$c_{-1} = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right].$$

$\square$

Example 4.8.5.

1. Let f has pole of order 1 at z_0. That is, $f(z) = c_{-1}(z-z_0)^{-1}+\sum_{n=0}^{\infty} c_n(z-z_0)^n$. Then
$$c_{-1} = \lim_{z\to z_0}(z-z_0)f(z).$$

2. Let f has pole of order 2 at z_0. That is, $f(z) = c_{-2} + (z-z_0)^{-2} + c_{-1}(z-z_0)^{-1} + \sum_{n=0}^{\infty} c_n(z-z_0)^n$. Then
$$c_{-1} = \lim_{z\to z_0}\frac{d}{dz}\left[(z-z_0)^2 f(z)\right].$$

3. Consider $f(z) = \dfrac{\sin z}{z^2}$. Since
$$\frac{\sin z}{z^2} = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots,$$
f has a pole of order 1 at $z = 0$. Then
$$c_{-1} = \lim_{z\to 0}(z-0)f(z) = \lim_{z\to 0}\left(z\cdot\frac{\sin z}{z^2}\right) = \lim_{z\to 0}\frac{\sin z}{z} = 1.$$

4. Consider $f(z) = \dfrac{1}{z^2-1}$. Since
$$\frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right) = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{2+z-1}\right),$$
f has a pole of order 1 at $z = 1$. Then
$$c_{-1} = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\left((z-1)\cdot\frac{1}{z^2-1}\right) = \lim_{z\to 1}\frac{1}{z+1} = \frac{1}{2}.$$

4.8.1 Wild Behaviour near Essential Singularities

Casorati-Weierstrass

Theorem 4.11. *Suppose that z_0 is an essential singularity of f. Then*

$$\forall w \in \mathbb{C} : \forall \delta > 0 : \forall \varepsilon > 0 : \exists z \in \mathbb{C} : z \in B(z_0, \delta) \wedge f(z) \in B(w, \varepsilon).$$

4.9 Residue Theorem

Residue

Definition 4.6. Suppose that D is a domain and that a holomorphic $f : D \setminus \{z_0\} \to \mathbb{C}$ has an isolated singularity at z_0. Let

$$f(z) = \sum_{n\in\mathbb{Z}} c_n(z-z_0)^n \quad \text{for} \quad z \in B^*(z_0, R).$$

We call the coefficient c_{-1} the **residue of** f **at** z_0, and denote it by $\mathrm{res}(f, z_0)$.

Remark 4.9.1. We know that

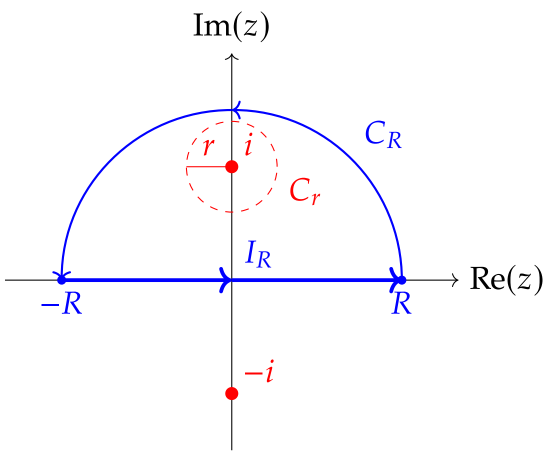
$$\oint_{C_r} \left(\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n \right) dz = \oint_{C_r} f(z)\, dz = \oint_{C_r} \frac{f(z)}{(z-z_0)^{-1+1}}\, dz = 2\pi i c_{-1},$$

where C_r is given by $C_r(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$, and $r < R$. Note that we have

$$\oint_{C_r} \left(c_n(z-z_0)^n\right)\, dz = \begin{cases} 2\pi i c_{-1} & n = -1, \\ 0 & n \neq -1. \end{cases}$$

Example 4.9.1. Find $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx$.

Sol. Let $f(z) = \dfrac{1}{z^2+1} = \dfrac{1}{(z-i)(z+i)}$.



$$\begin{aligned} f(z) = \frac{1}{z^2+1} &= \frac{1}{z-i} \cdot \frac{1}{z+i} \\ &= \frac{1}{z-i}\left(a_0 + a_1(z-i) + a_2(z-i)^2 + \cdots\right) \qquad \because g(z) := \frac{1}{z+i} \text{ is analytic at } z = i \end{aligned}$$

$$\oint_{C=C_R+I_R} f(z)\, dz = \oint_{C_r} f(z)\, dz = 2\pi i \cdot c_{-1} = 2\pi i \cdot \frac{1}{2i} = \pi.$$

□

Residue Theorem

Theorem 4.12. *Let*

(1) *D be a domain;*

(2) *f be holomorphic in $D \setminus \{p_1, \ldots, p_k\}$;*

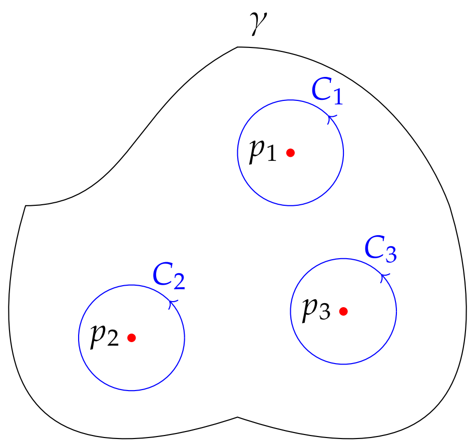
(3) *f have poles at $p_1, \ldots, p_k$ of order $m_1, \ldots, m_k$, respectively;*

(4) *γ be a closed path in $D \setminus \{p_1, \ldots, p_k\}$ and*

(5) *γ be such that for each $j = 1, 2, \ldots, k$, γ is $D \setminus \{p_j\}$-homotopic to a circle C_j centered at p_j such that the interior of C_j is contained in D and contains only pole p_j.*

Then

$$\oint_\gamma f(z)\, dz = 2\pi i \sum_{j=1}^{k} \operatorname{res}(f, p_k)\,.$$



Proof. $\square$

4.10 Improper Integral using Residue

4.10.1 Type1 : Basic Form

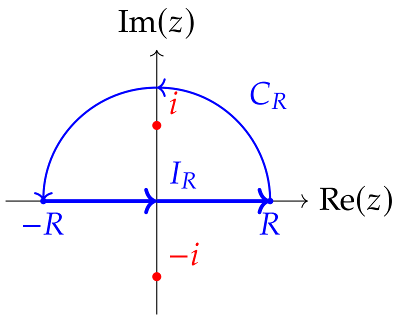
Basic Form

Let $P(x)$ and $Q(x)$ are polynomial with $\deg Q \geq \deg P + 2$.

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx.$$

Example 4.10.1. Find $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx$.

Sol. Let $f(z) = \dfrac{1}{(z^2+1)^2} = \dfrac{1}{(z-i)^2(z+i)^2}$ and $f(z) = \frac{\phi(z)}{(z-i)^2}$ with $\phi(z) = \frac{1}{(z+i)^2}$. Here ϕ is analytic at $z = i$. Note that $\phi'(z) = \frac{-2}{(z+i)^3}$. Let $C := C_R + I_R$:



then

$$\oint_C f(z)\, dz = 2\pi i \cdot \operatorname{res}(f, i) = 2\pi i \cdot \phi'(i) = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2}.$$

Consider

$$\oint_C f(z)\, dz = \underbrace{\int_{C_R} f(z)\, dz}_{=(1)} + \underbrace{\int_{I_R} f(z)\, dz}_{=(2)}.$$

(1) Note that $|f(z)| = \left|\frac{1}{(z^2+1)^2}\right| \leq \frac{1}{(|z|^2-1)^2} = \frac{1}{(R^2-1)^2}$. Thus

$$\left|\int_{C_R} f(z)\, dz\right| \leq \frac{1}{(R^2-1)^2} \cdot \pi R \longrightarrow 0 \quad \text{as } R \longrightarrow \infty.$$

(2)

$$\int_{I_R} f(z)\, dz = \int_{-R}^{R} f(x)\, dx = \int_{-R}^{R} \frac{1}{(x^2+1)^2}\, dx \longrightarrow \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx \quad \text{as } R \longrightarrow \infty.$$

Hence

$$\lim_{R\to\infty} \oint_C f(z)\, dz = \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx = \frac{\pi}{2}.$$

□

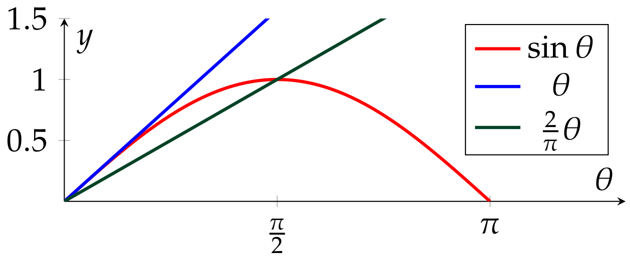
4.10.2 Type2 : Fourier Form

Jordan's Inequality

Proposition 4.13.

$$\int_0^\pi e^{-R\sin\theta}\,d\theta \le \frac{\pi}{R} \quad (R>0).$$

Proof.



(1) Consider $\theta \in \left[0, \frac{\pi}{2}\right]$. Then

$$\frac{2}{\pi}\theta \le \sin\theta \le \theta \implies -\frac{2R}{\pi}\theta \ge -R\sin\theta.$$

Thus

$$\begin{aligned}\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \le \int_0^{\pi/2} e^{-\frac{2R}{\pi}\theta}\,d\theta &= -\frac{\pi}{2R}e^{-\frac{2R}{\pi}\theta}\Big|_0^{\pi/2} \\ &= -\frac{\pi}{2R}\left(e^{-R}-1\right) \\ &= \frac{\pi}{2R}-\frac{\pi}{2R}e^{-R} \\ &< \frac{\pi}{2R} \qquad \because \frac{\pi}{2R}e^{-R}>0.\end{aligned}$$

(2) Consider $\theta \in \left[\frac{\pi}{2}, \pi\right]$. Then

$$\begin{aligned}\int_{\pi/2}^{\pi} e^{-R\sin\theta}\,d\theta &= \int_{\pi/2}^{0} e^{-R\sin(\pi-t)}(-1)dt \quad \text{by substituting } \theta=\pi-t \\ &= \int_0^{\pi/2} e^{-R\sin t}\,dt \\ &< \frac{\pi}{2R} \quad \text{by (1).}\end{aligned}$$

Hence, by (1) and (2),

$$\int_0^\pi e^{-R\sin\theta}\,d\theta < \frac{\pi}{2R}+\frac{\pi}{2R}=\frac{\pi}{R}.$$

$\square$

Jordan's Lemma

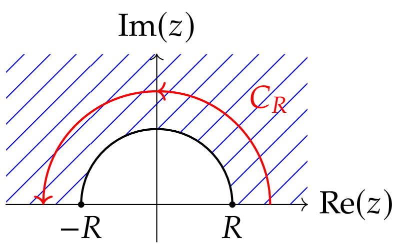
Lemma 4.14. *Let*

(1) $f(z)$ *is holomorphic in* $|z| > R_0$ *and* $\mathrm{Im}(z) \geq 0$;

(2) $C_{R>R_0}$ *satisfies* $z(\theta) = Re^{i\theta}$ *with* $\theta \in [0, \pi]$ *and*

(3) *for* $z \in C_R$,

$$\exists M_R > 0 : |f(z)| \leq M_R \wedge \lim_{R\to\infty} M_R = 0.$$



Then

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}\, dz = 0$$

for any $a > 0$.

Proof. By (2), we have

$$\int_{C_R} f(z)e^{iaz}\, dz = \int_0^\pi \underbrace{f(Re^{i\theta})}_{=\text{(i)}} \cdot \underbrace{e^{iaRe^{i\theta}}}_{=\text{(ii)}} \cdot \underbrace{iRe^{i\theta}}_{=\text{(iii)}}\, d\theta.$$

Then

(i) $|\text{(i)}| = \left|f(Re^{i\theta})\right| \leq M_R \to 0$ as $R \to \infty$ by (3)

(ii) $|\text{(ii)}| = \left|e^{iaR(\cos\theta + i\sin\theta)}\right| = \left|e^{iaR\cos\theta}\right|\left|e^{-aR\sin\theta}\right| = \left|e^{-aR\sin\theta}\right|$.

(iii) $|\text{(iii)}| = R$.

Thus,

$$\begin{aligned}
\left|\int_{C_R} f(z)e^{iaz}\, dz\right| &\leq M_R \left|\int_0^\pi e^{-aR\sin\theta}\, d\theta\right| \cdot R \\
&< M_R \cdot \frac{\pi}{aR} \cdot R \quad \text{by Jordan's Inequality} \\
&= M_R \cdot \frac{\pi}{a} \to 0 \quad \text{as } R \to \infty
\end{aligned}$$

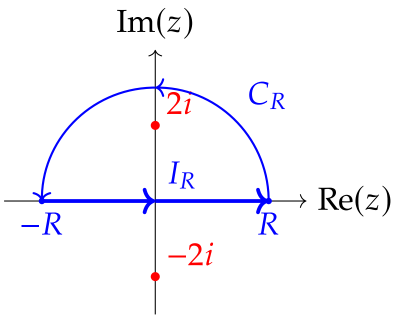
□

Fourier Form

$$\int_{-\infty}^{\infty} f(x)\sin(ax)\,dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\cos(ax)\,dx.$$

Example 4.10.2. Find $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+4}\,dx$.

Sol. Let $f(z) := \frac{z}{z^2+4} = \frac{z}{(z-2i)(z+2i)}$, and let $f(z)e^{iz} = \frac{\phi(z)}{z-2i}$ with $\phi(z) = \frac{ze^{iz}}{z+2i}$. Here ϕ is analytic at $z = 2i$. Let $C := C_R + I_R$:



then

$$\operatorname{res}\left(f(z)e^{iz}, 2i\right) = \operatorname{res}\left(\frac{\phi(z)}{z-2i}, 2i\right) = \phi(2i) = \frac{2ie^{-2}}{4i} = \frac{1}{2}e^{-2}.$$

Consider $\displaystyle\oint_C f(z)e^{iz}\,dz = \underbrace{\int_{C_R} f(z)e^{iz}\,dz}_{=(1)} + \underbrace{\int_{I_R} f(z)e^{iz}\,dz}_{=(2)}$.

(1) Note that $|f(z)| = \left|\frac{z}{z^2+4}\right| \le \frac{|z|}{|z|^2-4} = \frac{R}{R^2-4} =: M_R$. Then

$$|f(z)| \le M_R \quad \text{and} \quad M_R = \frac{R}{R^2-4} \to 0 \quad \text{as } R \to \infty,$$

and so $\lim\limits_{R\to\infty} \int_{C_R} f(z)e^{iz}\,dz = 0$ by Jordan's Lemma.

(2)

$$\int_{I_R} f(z)e^{iz}\,dz = \int_{-R}^{R} \frac{x}{x^2+4}\cos x\,dx + i\int_{-R}^{R} \frac{x}{x^2+4}\sin x\,dx.$$

Therefore

$$\begin{aligned}\lim_{R\to\infty}\oint_C f(z)e^{iz}\,dz &= \int_{-\infty}^{\infty} \frac{x}{x^2+4}\cos x\,dx + i\int_{-\infty}^{\infty} \frac{x}{x^2+4}\sin x\,dx \\ &= 2\pi i\cdot\operatorname{res}\left(f(z)e^{iz}, 2i\right) \\ &= \pi e^{-2} i.\end{aligned}$$

Hence

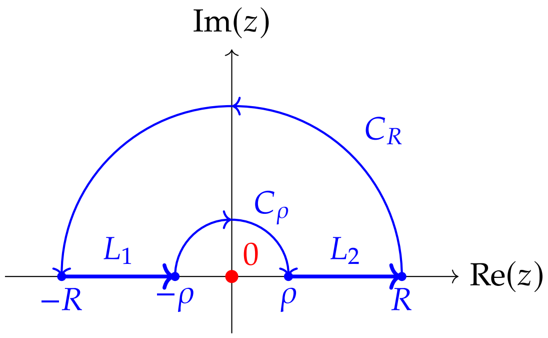
$$\int_{-\infty}^{\infty} \frac{x}{x^2+4}\sin x\,dx = \frac{\pi}{e^2}.$$

□

4.10.3 Type3 : Indented Path, Half Residue

Example 4.10.3. Find $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$.

Sol. Let $f(z) := \frac{1}{z}$. Consider a path $C := C_R + L_1 + C_\rho + L_2$:



where

$$\begin{aligned} &C_R : z(\theta) = Re^{i\theta} \quad (\theta \in [0, \pi]),\\ &L_1 : [-R, \rho],\\ &L_2 : [\rho, R],\\ &C_\rho : z(t) = \rho e^{i(\pi - t)} \quad (t \in [0, \pi]). \end{aligned}$$

Then

$$\oint_C f(z)e^{iz}\, dz = 0$$

by the Cauchy-Goursat Theorem. Consider

$$0 = \oint_C f(z)e^{iz}\, dz = \underbrace{\int_{C_R} f(z)e^{iz}\, dz}_{=(1)} + \underbrace{\int_{L_1 \cup L_2} f(z)e^{iz}\, dz}_{=(2)} + \underbrace{\int_{C_\rho} f(z)e^{iz}\, dz}_{=(3)}$$

(1) Note that $|f(z)| = \left|\frac{1}{z}\right| = \frac{1}{R} =: M_R$, i.e., $M_R \to 0$ as $R \to \infty$. So

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iz}\, dz = 0$$

by Jordan's Lemma.

(2)

$$\begin{aligned} \int_{L_1 \cup L_2} f(z)e^{iz}\, dz &= \int_{L_1 \cup L_2} f(z) \cos z\, dz + i \int_{L_1 \cup L_2} f(z) \sin z\, dz\\ &= \left[\int_{-R}^{-\rho} \frac{\cos x}{x}\, dx + \int_{\rho}^{R} \frac{\cos x}{x}\, dx\right] + i \left[\int_{-R}^{-\rho} \frac{\sin x}{x}\, dx + \int_{\rho}^{R} \frac{\sin x}{x}\, dx\right]\\ &\to \int_{-\infty}^{\infty} \frac{\cos x}{x} dx + i \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx \quad \text{as} \quad \begin{cases} R \to \infty,\\ \rho \to 0. \end{cases} \end{aligned}$$

(3) Note that

$$\begin{aligned} f(z)e^{iz} = \frac{e^{iz}}{z} &= \frac{1}{z}\left(1 + (iz) + \frac{1}{2!}(iz)^2 + \frac{1}{3!}(iz)^3 + \cdots\right) \\ &= \frac{1}{z} + \left(i + \frac{1}{2!}i^2 z + \frac{1}{3!}i^3 z^2 + \cdots\right) \\ &= \frac{1}{z} + g(z), \end{aligned}$$

where $g(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} i^{n+1} z^n$ is analytic at $z = 0$. Consider

$$\int_{C_\rho} f(z)e^{iz}\, dz = \underbrace{\int_{C_\rho} \frac{1}{z}\, dz}_{=\text{(a)}} + \underbrace{\int_{C_\rho} g(z)\, dz}_{=\text{(b)}}.$$

(a)
$$\int_{C_\rho} \frac{1}{z}\, dz = \int_0^{\pi} \frac{1}{\rho e^{i(\pi - t)}} \cdot i\rho e^{i(\pi - t)}(-1)\, dt = -i\int_0^{\pi} dt = -\pi i.$$

(b) Since $\exists M : |g(z)| \le M$ as $|z| \le \rho_0$, we have

$$\int_{C_\rho} g(z)\, dz \le M \cdot \pi\rho \to 0 \quad \text{as } \rho \to 0.$$

Therefore, by (1), (2) and (3), we obtain

$$0 = \lim_{\substack{R\to\infty \\ \rho\to 0}} \oint_C f(z)e^{iz}\, dz = 0 + \int_{-\infty}^{\infty} \frac{\cos x}{x} dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx - \pi i,$$

and so

$$\int_{-\infty}^{\infty} \frac{\cos x}{x} dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = 0 + \pi i \implies \boxed{\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi}.$$

$\square$

4.10.4 Type4 : Sine/Cosine on $[0, 2\pi]$

Sine/Cosine Form

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta = \oint_C F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{1}{iz}\, dz$$

where C is the unit circle, that is, $C : z(\theta) = e^{i\theta}$ with $\theta \in [0, 2\pi]$.

Example 4.10.4. Find $\displaystyle\int_0^{2\pi} \frac{1}{1 + a\sin\theta}\, d\theta$ with $a \in (-1, 1)$.

Sol. Let $a = 0$ then $\int_0^{2\pi} = \frac{1}{1+0}\, d\theta = 2\pi$. Suppose that $a \neq 0$. Let C be the unit circle, that is, $C : z(\theta) = e^{i\theta}$ with $\theta \in [0, 2\pi]$. Then

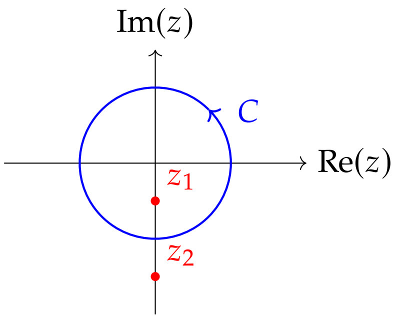
$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} &= \int_0^{2\pi} \frac{1}{1 + a\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)}\, d\theta \\
&= \oint_C \frac{1}{1 + a\left(\frac{(z - z^{-1})}{2i}\right)} \frac{1}{iz}\, dz \quad \text{by substituting } z = e^{i\theta} \\
&= \oint_C \frac{1}{\frac{2i + az - az^{-1}}{2i} \cdot iz}\, dz \\
&= \oint_C \frac{2}{az^2 + 2iz - a}\, dz \\
&= \oint_C \frac{a/2}{z^2 + (2i/a)z - 1}\, dz \\
&= \oint_C \frac{a/2}{(z - z_1)(z - z_2)}\, dz,
\end{aligned}$$

where z_1, z_2 are solutions of quadratic equation $z^2 + \left(\frac{2i}{a}\right) z - 1$. We solve this equation:

$$\begin{aligned}
z^2 + \left(\frac{2i}{a}\right) z - 1 &= 0 \\
z^2 + 2\frac{i}{a} z + \left(\frac{i}{a}\right)^2 &= 1 + \left(\frac{i}{a}\right)^2 \\
\left(z + \frac{i}{a}\right)^2 &= \frac{a^2 - 1}{a^2} = -\frac{1 - a^2}{a^2} \qquad (|a| < 1)
\end{aligned}$$

Thus, $z = -\dfrac{i}{a} \pm \sqrt{-\dfrac{1 - a^2}{a^2}} = -\dfrac{i}{a} \pm \dfrac{\sqrt{1 - a^2}}{|a|} i$. Let

$$z_1 = \frac{-1 + \sqrt{1 - a^2}}{a} i, \quad z_2 = \frac{-1 - \sqrt{1 - a^2}}{a} i.$$



Thus,

$$\begin{aligned}\oint_C f(z)\,dz &= \oint_C \frac{2/a}{(z-z_1)(z-z_2)}\,dz \\ &= \oint \frac{g(z)}{(z-z_1)}\,dz \quad \text{with} \quad g(z) = \frac{2/a}{z-z_2} \\ &= 2\pi i \cdot \operatorname{res}(f, z_1).\end{aligned}$$

Note that

$$\operatorname{res}(f, z_1) = \phi(z_1) = \frac{2/a}{z_1 - z_2} = \frac{2/a}{\frac{2\sqrt{1-a^2}}{a} i} = \frac{1}{\sqrt{1-a^2} i}.$$

Hence

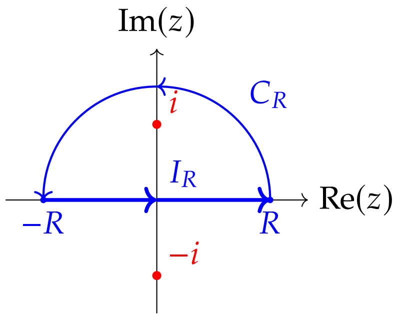
$$\int_0^{2\pi} \frac{1}{1 + a\sin\theta}\,d\theta = \oint_C f(z)\,dz = 2\pi \cdot \operatorname{res}(f, z_1) = 2\pi i \cdot \frac{1}{\sqrt{1-a^2} i} = \frac{2\pi}{\sqrt{1-a^2}}.$$

$\square$

Exercise 4.10.1.

1. Find $\displaystyle\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\, dx$.

   Sol. Let $f(z) := \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$, and let $f(z)e^{iaz} = \frac{\phi(z)}{z-i}$ with $\phi(z) = \frac{e^{iaz}}{z+i}$. Here ϕ is analytic at $z = i$. Let $C := C_R + I_R$:



   then

$$\mathrm{res}\left(f(z)e^{iaz}, i\right) = \mathrm{res}\left(\frac{\phi(z)}{z-i}, i\right) = \phi(i) = \frac{e^{-a}}{2i} = -\frac{e^{-a}}{2} i = \begin{cases} -\frac{e^{-a}}{2} i & : a \geq 0 \\ -\frac{e^{a}}{2} i & : a < 0 \end{cases}.$$

   Consider $\displaystyle\oint_C f(z)e^{iaz}\, dz = \underbrace{\int_{C_R} f(z)e^{iaz}\, dz}_{=(1)} + \underbrace{\int_{I_R} f(z)e^{iaz}\, dz}_{=(2)}$.

   (1) Note that $|f(z)| = \left|\frac{1}{z^2+1}\right| \leq \frac{1}{|z|^2-1} = \frac{1}{R^2-1} =: M_R$. Then

$$|f(z)| \leq M_R \quad \text{and} \quad M_R = \frac{1}{R^2-1} \longrightarrow 0 \quad \text{as } R \longrightarrow \infty,$$

   and so $\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}\, dz = 0$ by Jordan's Lemma.

   (2)

$$\int_{I_R} f(z)e^{iaz}\, dz = \int_{-R}^{R} \frac{1}{x^2+1} \cos(ax)\, dx + i \int_{-R}^{R} \frac{1}{x^2+1} \sin(ax)\, dx.$$

   Therefore

$$\begin{aligned} \lim_{R\to\infty} \oint_C f(z)e^{iaz}\, dz &= \int_{-\infty}^{\infty} \frac{1}{x^2+1} \cos(ax)\, dx + i \int_{-\infty}^{\infty} \frac{1}{x^2+1} \sin(ax)\, dx \\ &= 2\pi i \cdot \mathrm{res}\left(f(z)e^{iaz}, i\right) \\ &= \begin{cases} \pi e^{-a} & : a \geq 0 \\ \pi e^{a} & : a < 0 \end{cases} = \begin{cases} \frac{\pi}{e^a} & : a \geq 0 \\ \frac{\pi}{e^{-a}} & : a < 0 \end{cases} = \frac{\pi}{e^{|a|}}. \end{aligned}$$

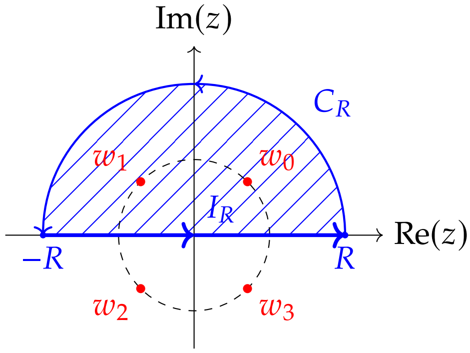
   Hence

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} \cos(ax)\, dx = \frac{\pi}{e^{|a|}}.$$

   □

2. Find $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx$.

Sol. Let $f(z) = \dfrac{z^2}{1+z^4}$There are two simple poles $w_0 = e^{\frac{\pi}{4}i}$ and $w_1 = e^{\frac{3\pi}{4}i}$ on the upper semicircle of the region $C := C_R + I_R$.



Note that

$$\operatorname{res}(f, w_0) = \lim_{z\to w_0} \frac{z^3 - w_0 z^2}{z^4+1} = \lim_{z\to w_0} \frac{3z^2 - 2w_0 z}{4z^3} = \frac{w_0^2}{4w_0^3} = \frac{1}{4}e^{-\frac{\pi}{4}i} = \frac{1}{4}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right),$$

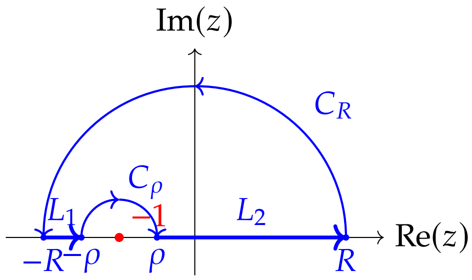
$$\operatorname{res}(f, w_1) = \frac{1}{4}e^{-\frac{3\pi}{4}i} = \frac{1}{4}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right).$$

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx = 2\pi i \cdot (\operatorname{res}(f, w_0) + \operatorname{res}(f, w_1)) = 2\pi i \cdot \frac{1}{4}\left(-\frac{2}{\sqrt{2}}i\right) = \frac{\pi}{\sqrt{2}}.$$

□

3. Find $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^3+1}\,dx$.

   Sol. Let $f(z) := \frac{1}{z^3+1}$. Consider a path $C := C_R + L_1 + C_\rho + L_2$:



   where

$$\begin{aligned}
C_R &: z(\theta) = Re^{i\theta} \quad (\theta \in [0,\pi]),\\
L_1 &: [-R, \rho],\\
L_2 &: [\rho, R],\\
C_\rho &: z(t) = \rho e^{i(\pi - t)} \quad (t \in [0,\pi]).
\end{aligned}$$

   Then
$$\oint_C f(z)e^{iz}\,dz = 0$$
   by the Cauchy-Goursat Theorem. Consider

$$0 = \oint_C f(z)e^{iz}\,dz = \underbrace{\int_{C_R} f(z)e^{iz}\,dz}_{=(1)} + \underbrace{\int_{L_1\cup L_2} f(z)e^{iz}\,dz}_{=(2)} + \underbrace{\int_{C_\rho} f(z)e^{iz}\,dz}_{(3)}$$

□

4. Find $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+3\cos^2\theta}$.

Sol. content... $\square$

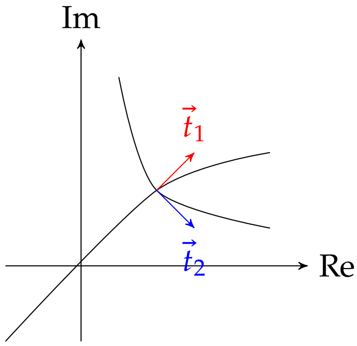
Chapter 5

Conformal Mapping

Conformal

Definition 5.1. A conformal mapping is a complex function $f : \mathbb{C} \to \mathbb{C}$ that is both analytic and bijective at a point z_0, and additionally, the derivative of the function at that point, $f'(z_0)$, is not zero.

Remark 5.0.1.



Angle-Preserving Property

Theorem 5.1. *Let*

(1) f is analytic at z_0;

(2) $f'(z_0) \neq 0$.

Then f is conformal at z_0.

Proof. content... □

5.1 Linear Transformation

Linear Transformation

Let $w = f(z)$. Consider

$$w = Az + B$$

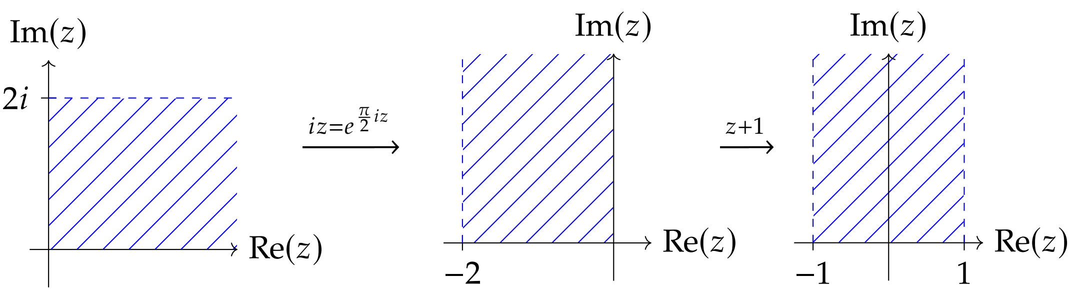
where $A \in \mathbb{C} \setminus \{0\}$ and $B \in \mathbb{C}$.

(1) $(w = Az)$;

$$\begin{cases} |w| = |A|\,|z| & \cdots\cdots \text{Zoom in/out by } |A| \\ \arg(w) = \arg(Az) = \arg(z) + \arg(A) & \cdots\cdots \text{Rotation} \end{cases}$$

(2) $(w = z + B)$; Parallel movement by B.

Example 5.1.1. Let $D := \{x + iy : x > 0, 0 < y < 2\}$ be a semi-infinite strip. Consider $w = iz + 1$:



5.2 Reciprocal Transformation

5.3 Linear Fractional Transformation

5.4 Non-linear Transformation

Bibliography