Analysis

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Part I
Introduction to Real Number System

1 The Real Number System

1.1 Principle of Mathematical Induction

Axiom. (*Well-ordering Principle*) Every nonempty subset of $\mathbb{N}$(or $\mathbb{Z}_{\geq 0}$) has a least element.

1.2 The Algebraic Properties of Real Number $\mathbb{R}$

Definition. A **binary operation** B on a set F is a function from $F \times F$ into F.

1.3 The Order Properties of Real Number $\mathbb{R}$

Axiom. (*Axiom of order*) A relation $<$ defined on $\mathbb{R} \times \mathbb{R}$ satisfies the following axiom of order:

1. For $a, b \in \mathbb{R}$, exactly one of the following holds (property of trichotomy):
$$a = b, \quad a < b \quad \text{or} \quad b < a.$$
2. For $a, b \in \mathbb{R}$, if $0 < a$ and $0 < b$ then $0 < a + b$ and $0 < ab$.
3. For $a, b, c \in \mathbb{R}$, if $a < b$ then $a + c < b + c$.

Theorem. For $a, b \in \mathbb{R}$, if $a < b$ then
$$a < \frac{1}{2}(a + b) < b.$$

Proof. $a = \frac{1}{2}(a + a) < \frac{1}{2}(a + b) < \frac{1}{2}(b + b) = b$. □

Corollary. For $a \in \mathbb{R}^+$
$$0 < \frac{1}{2} < a.$$

Corollary. If $a \in \mathbb{R}$ satisfies $0 \leq a < \varepsilon$ for every $\varepsilon > 0$ then
$$a = 0.$$

If $a \neq 0$, then one of the number a and $-a$ is strictly positive by the trichotomy property. The **absolute value** of $a \neq 0$ is defined to be the strictly positive one of the pair $\{a, -a\}$, and the absolute value of 0 is defined to be 0.

Definition. If $a \in \mathbb{R}$, the **absolute value** of a is denoted by $|a|$ and is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Theorem. (**Triangle inequality**) If $a, b \in \mathbb{R}$ then

$$|a + b| \leq |a| + |b|.$$

Corollary. If $a, b \in \mathbb{R}$ then

1. $||a| - |b|| \leq |a| - |b|$.
2. $|a - b| \leq |a| + |b|$.

Corollary. If $a_1, a_2, \cdots, a_n \in \mathbb{R}$ then

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|.$$

Theorem. Let $a, b \in \mathbb{R}$. For arbitrary $\varepsilon > 0$, if $|a - b| < \varepsilon$ then

$$a = b.$$

Note. In analysis, $a = b \iff \forall \varepsilon > 0, |a - b| < \varepsilon$.

1.4 The Completeness Property of Real Number $\mathbb{R}$

Definition. Let X be a nonempty subset of $\mathbb{R}$.

1. The set X is said to be **bounded above** if $\exists a \in \mathbb{R}$ such that $x \leq a$ for all $x \in X$. Each number a is called an **upper bound** of X.

2. The set X is said to be **bounded below** if $\exists b \in \mathbb{R}$ such that $b \leq x$ for all $x \in X$. Each number a is called a **lower bound** of X.

3. The set X is said to be bounded if it is both bound above and bounded below.

Definition. Let X be a nonempty subset of $\mathbb{R}$.

1. If X is bounded above then a number a is said to be a **supremum**(or a **least upper bound**) of X if it satisfies the following conditions:

   (a) a is an upper bound of X

   (b) if b is any upper bound of X then $a \leq b$.

2. If X is bounded below then a number b is said to be a **infimum**(or a **greatest lower bound**) of X if it satisfies the following conditions:

   (a) b is a lower bound of X

   (b) if a is any lower bound of X then $a \geq b$.

Theorem. Let A be a bounded above, nonempty subset of $\mathbb{R}$ and $a \in \mathbb{R}$ is an upper bound of A. Then the following statements are equivalent:

1. $a = \sup A$.

2. $\forall b \in \mathbb{R}$ satisfying $b < a$, $\exists x \in A$ such that $b < x \leq a$.

Axiom. (*Completeness property of real number*) Every non-empty set of real numbers which has an upper bound also has a supremum in $\mathbb{R}$.

This axiom is also called *supremum property* of real number.

e.g. Assume that only rational numbers $\mathbb{Q}$ exists. Let us consider a set

$$X = \{1.4, 1.41, 1.414, 1.4142, 1.414213, 1.4142135, \cdots\}.$$

Then X has an upper bound 2. However, $\sup X = \sqrt{2} \notin \mathbb{Q}$. It means that $\mathbb{Q}$ is not complete.

Theorem. Every nonempty set of real numbers which has a lower bound has an infimum in $\mathbb{R}$.

Part II
Sequences and Series in Real Number

1 Sequence in Real Number

1.1 Convergent Sequences

Definition. **A sequence of real number**(or **a sequence** in $\mathbb{R}$) is a function defined on the set $\mathbb{N}$ whose range is contained in the set $\mathbb{R}$.

Consider

$$\lim_{n\to\infty} a_n = L.$$

To define the limit of a sequence, we need to make the concepts **close to** and **for all large positive integers** n precise.

1. $\forall \varepsilon > 0, |a_n - L| < \varepsilon$.

e.g. Consider $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. Let $\varepsilon = 0.1$. Then $|a_n - L| < 0.1$ and so

$$\frac{1}{n} < 0.1 \Rightarrow n > 10.$$

Hence, after 11-th term, $1/n = 0$ $(a_n = L)$.

2. "there exists $N(\varepsilon) \in \mathbb{N}$ such that if $n \geq N(\varepsilon)$" then $a_n = L$.

Definition A sequence $\{a_n\}$ in $\mathbb{R}$ is said to **converge** to $L \in \mathbb{R}$ or L is said to be a **limit** of $\{a_n\}$, if for every $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$, the term a_n satisfy

$$|a_n - L| < \varepsilon.$$

If a sequence has a limit, we say that the sequence is **convergent**; if it has no limit, we say that the sequence is **divergent**.

Note.

$$\lim_{n\to\infty} a_n = L \iff \forall \varepsilon > 0,\ \exists N(\varepsilon) \in \mathbb{N} \text{ such that if } n \geq N(\varepsilon) \text{ then } |a_n - L| < \varepsilon.$$

e.g. Prove that the sequence

$$\{a_n\} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

converges to 0.

Sol. Let $\varepsilon > 0$. Then $\exists N > \dfrac{1}{\varepsilon}$ such that if $n \geq N$,

$$|a_n - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

□

Theorem. (**Uniqueness of limits**) The limit of a sequence in $\mathbb{R}$ is unique. That is, if a sequence $\{a_n\}$ has limit L_1 and L_2 then $L_1 = L_2$.

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = L_1$, $\exists N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then $|a_n - L_1| < \frac{\varepsilon}{2}$. And since $\lim_{n\to\infty} a_n = L_2$, $\exists N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then $|a_n - L_2| < \frac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$,

$$\begin{aligned}
|L_1 - L_2| &= |L_1 - L_2 + a_n - a_n| \\
&= \big|(a_n - L_2) - (a_n - L_1)\big| \\
&= |a_n - L_1| + |a_n - L_2| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

Definition. Let $\{a_n\}$ be a sequence of real numbers.

1. $\{a_n\}$ is said to be **bounded above** if there exists a real number M such that for all $n \in \mathbb{N}$,
$$a_n \leq M.$$
2. $\{a_n\}$ is said to be **bounded below** if there exists a real number M such that for all $n \in \mathbb{N}$,
$$a_n \geq M.$$
3. $\{a_n\}$ is said to be **bounded** when it is both bounded above and bounded below, i.e., if there exists a real number $M > 0$ such that for all $n \in \mathbb{N}$,
$$|a_n| \leq M.$$

Theorem. A convergent sequence of real numbers is bounded. That is, if $\{a_n\}$ converges to L, there exists $M > 0$ such that
$$|a_n| \leq M$$
for all $n \in \mathbb{N}$.

Proof. Since $\lim_{n\to\infty} a_n = L$, for $\varepsilon = 1 > 0$, $\exists N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - L| < 1$. Note that

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| < 1 + |L|.$$

Let $M = \max\left\{|a_1|, |a_2|, \cdots, |a_{N-1}|, 1 + |L|\right\}$. Then

$$|a_n| \leq M$$

for all $n \in \mathbb{N}$. □

1.2 Limit Theorem

Theorem. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers that converges to x and y, respectively. Then

1. For $k \in \mathbb{R}$, $\{ka_n\}$ converges to kx.
2. $\{a_n + b_n\}$ converges to $x + y$.
3. $\{a_n b_n\}$ converges to xy.
4. If $\{b_n\}$ is a sequence of non-zero numbers that converges to non-zero number y then $\{a_n/b_n\}$ converges to x/y.

Proof.

3. $\lim_{n\to\infty} a_n b_n = xy$.

   Note that, since $\{a_n\}$ converges, $\exists M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.
   Let $\varepsilon > 0$. Since $\lim_{n\to\infty} b_n = y$, $\exists N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then $|b_n - y| < \frac{\varepsilon}{2M}$. And since $\lim_{n\to\infty} a_n = x$, $\exists N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then $|a_n - x| < \frac{\varepsilon}{2|y|+1}$. Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$,

$$\begin{aligned}|a_n b_n - xy| &= |a_n b_n - a_n y + a_n y - xy| \\ &\leq |a_n||b_n - y| + |a_n - x||y| \\ &< M\frac{\varepsilon}{2M} + \frac{\varepsilon}{2|y|+1}|y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

4. $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{y}$ with $b_n \neq 0$ and $y \neq 0$.

   Since $\lim_{n\to\infty} b_n = y$, for $\frac{|y|}{2}$, $\exists N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then $|b_n - y| < \frac{|y|}{2}$.
   By corollary $-|a - b| \leq |a| - |b|$,

$$|b_n| \geq |y| - |b_n - y| > |y| - \frac{|y|}{2},$$

   and so $\frac{1}{|b_n|} < \frac{2}{|y|}$.

   Let $\varepsilon > 0$. Since $\{b_n\}$ converges, $\exists N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then $|b_n - y| < \frac{|y|^2}{2}\varepsilon$. Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$,

$$\left|\frac{1}{b_n} - \frac{1}{y}\right| = \frac{|b_n - y|}{|b_n||y|} < \frac{2}{|y|} \cdot \frac{1}{|y|} \cdot \frac{|y|^2}{2}\varepsilon = \varepsilon.$$

□

Theorem. (**Squeeze Theorem**) Let $\{a_n\}$ and $\{b_n\}$ are convergent sequences of real numbers such that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L.$$

If $\{c_n\}$ be a sequence of real numbers such that $a_n \le c_n \le b_n$ for all $n \in \mathbb{N}$ then

$$\lim_{n\to\infty} c_n = L.$$

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = L$, $\exists N_1 \in \mathbb{N}$ such that if $n \ge N_1$ then

$$|a_n - L| < \varepsilon \text{ implies } L - \varepsilon < a_n.$$

Since $\lim_{n\to\infty} b_n = L$, $\exists N_2 \in \mathbb{N}$ such that if $n \ge N_2$ then

$$|b_n - L| < \varepsilon \text{ implies } b_n < L + \varepsilon.$$

Let $N = \max\{N_1, N_2\}$. Then if $n \ge N$,

$$L - \varepsilon < a_n \le c_n \le b_n < L + \varepsilon,$$

and so $|c_n - L| < \varepsilon$. □

e.g. Prove that

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1.$$

Sol. Let $a_n = n^{\frac{1}{n}} - 1$. Then

$$n = (1 + a_n)^n = 1 + \binom{n}{1} a_n + \binom{n}{2} (a_n)^2 + \cdots + (a_n)^n.$$

Note that $a_n \ge 0$. Since $n > \binom{n}{2}(a_n)^2 = \dfrac{n(n-1)}{2}(a_n)^2$,

$$0 \le a_n < \sqrt{\frac{2}{n-1}}.$$

Since $\lim_{n\to\infty} \sqrt{\dfrac{2}{n-1}} = 0$, by squeeze theorem, $\lim_{n\to\infty} a_n = 0$, and so $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. □

1.3 Monotone Sequences

Definition. Let $\{a_n\}$ be a sequence of real numbers. $\{a_n\}$ is (strictly) **monotone** if it is either (strictly) increasing or (strictly) decreasing.

Theorem. (**Monotone convergence theorem**) A monotone sequence of real numbers is convergent if and only if it is bounded. Further:

1. If $\{a_n\}$ is bounded increasing sequence then
$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$
2. If $\{a_n\}$ is bounded decreasing sequence then
$$\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}.$$
3. Bounded monotone sequence is convergent.

Proof. Let $S = \{a_n : n \in \mathbb{N}\}$. Then $S \neq \varnothing$ and, since $\{a_n\}$ is bounded, S has an upper bound. Thus, $\exists \sup S = \sup\{a_n : n \in \mathbb{N}\} = L$.

Let $\varepsilon > 0$. Since $L - \varepsilon$ is not an upper bound of S, $\exists N \in \mathbb{N}$ such that
$$L - \varepsilon < a_N.$$
Since $\{a_n\}$ is increasing sequence, $a_N \leq a_n$ whenever $n \geq N$, so that for all $n \geq N$,
$$L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon.$$
Thus, $\exists N \in \mathbb{N}$ such that if $n \geq N$,
$$|a_n - L| < \varepsilon.$$
Hence,
$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$
□

e.g. **(Recurrence formula)** Let $\{b_n\}$ be defined inductively by
$$b_1 = 3, \quad b_{n+1} = \frac{b_n}{2} + \frac{3}{b_n}$$
for all $n \geq 1$. Show that $\{b_n\}$ is convergent and $\lim_{n\to\infty} b_n = \sqrt{6}$.

Sol. Since $\{b_n\}$ is decreasing and $0 < b_n \leq 3$, i.e., $\{b_n\}$ is bounded, $\exists \lim_{n\to\infty} b_n = L$. Then
$$L = \frac{L}{2} + \frac{3}{L},$$
so that $L^2 = 6$, i.e., $L = \sqrt{6}$. Hence $\lim_{n\to\infty} b_n = \sqrt{6}$. □

1.4 Subsequences and the Cauchy Criterion

Definition. Let $\{a_n\}$ be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a strictly increasing sequence of natural numbers. Then $\left\{a_{n_k}\right\} := \left\{a_{n_k}\right\}_{k=1}^{\infty}$ is called **subsequence** of $\{a_n\}$.

Theorem. If a sequence $\{a_n\}$ of real numbers converges to a real number L if and only if any subsequence $\left\{a_{n_k}\right\}$ of $\{a_n\}$ converges to L.

Proof. We show that

$$\lim_{n\to\infty} a_n = L \iff \exists \lim_{k\to\infty} a_{n_k} = L.$$

($\Rightarrow$) Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = L$, $\exists N \in \mathbb{N}$ such that if $k \geq N$ then $|a_k - L| < \varepsilon$. Since $n_k \geq k$, if $n_k \geq k \geq N$ then $\left|a_{n_k} - L\right| < \varepsilon$. Thus, $\lim_{k\to\infty} a_{n_k} = L$.

($\Leftarrow$) Since $\{a_n\}$ is subsequence of $\{a_n\}$, $\exists \lim_{n\to\infty} a_n = L$. $\square$

Corollary. Let $\{a_n\}$ be a sequence of real numbers.

1. If $\{a_n\}$ converges and there exists a subsequence converging to L then $\{a_n\}$ converges to L.
2. If $\{a_n\}$ has two convergent subsequence whose limits are not equal then $\{a_n\}$ diverges.
3. If a subsequence of $\{a_n\}$ diverges then $\{a_n\}$ diverges.

Definition. We say that a sequence of intervals $\{I_n : n \in \mathbb{N}\}$ is **nested** if the following chain of inclusions holds $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$.

Theorem. (**Nested intervals property**) If $I_n = [a_n, b_n]$ is a nested sequence of closed bounded intervals then there exists a $x \in \mathbb{R}$ such that $x \in I_n$ for all $n \in \mathbb{N}$.

Theorem. If $I_n = [a_n, b_n]$ is a nested sequence of closed bounded intervals such that the lengths $b_n - a_n$ of I_n satisfy

$$\lim_{n\to\infty}(b_n - a_n) = 0$$

then the number $x \in I_n$ for all $n \in \mathbb{N}$ is unique.

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$. Since $A \neq \varnothing$ and A has an upper bound, $\exists \sup A = a$. Similarly, $\exists \inf B = b$. Since $a_n \leq a \leq b \leq b_n$,

$$I = [a, b] \subseteq I_n$$

for all $n \in \mathbb{N}$. Since $\{a_n\}$ is bounded and increasing, $\lim\limits_{n\to\infty} a_n = a$. Similarly, $\lim\limits_{n\to\infty} b_n = b$. Since $a_n \leq a$ and $b \leq b_n$, for any $n \in \mathbb{N}$,

$$0 \leq b - a \leq b_n - a_n.$$

By squeeze theorem,

$$\lim_{n\to\infty}(b - a) = 0,$$

that is, $a = b$. $\square$

Theorem. (**Bolzano-Weierstrass theorem**) A bounded sequence of real numbers has a convergent subsequence.

Definition. A sequence $\{a_n\}$ of real number is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists a natural number N such that for all natural numbers $n, m \geq N$, the terms a_n and a_m satisfy

$$|a_n - a_m| < \varepsilon.$$

Note. $\{a_n\}$ is a Cauchy sequence : $\forall\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n, m \geq N$ then $|a_n - a_m| < \varepsilon$.

Lemma. If $\{a_n\}$ is a convergent sequence of real numbers, then $\{a_n\}$ is a Cauchy sequence.

Proof. Let $\lim_{n\to\infty} a_n = L$ and let $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - L| < \frac{\varepsilon}{2}$. If $m, n \geq N$ then

$$|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |a_m - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\{a_n\}$ is a Cauchy sequence. □

Lemma. A Cauchy sequence of real number is bounded.

Proof. Let $\varepsilon = 1$. Since $\{a_n\}$ is a Cauchy sequence, $\exists N \in \mathbb{N}$ such that if $n, N \geq N$ then

$$|a_n - a_N| < 1 \text{ implies } |a_n| < 1 + |a_N|.$$

Let $M = \max\left\{|a_1|, |a_2|, \cdots, |a_{N-1}|, 1 + |a_N|\right\}$. Then

$$|a_n| \leq M$$

for all $n \in \mathbb{N}$. Hence $\{a_n\}$ is bounded. □

Theorem. (**Cauchy convergence criterion**) A sequence of real number is convergent if and only if it is a Cauchy sequence.

Proof. ($\Rightarrow$) It is proved in **Lemma**.
($\Leftarrow$) Let $\varepsilon > 0$. Since $\{a_n\}$ is bounded, by Bolzano-Weierstrass Theorem, $\exists \left\{a_{n_k}\right\}$, a subsequence of $\{a_n\}$ such that $\lim_{k\to\infty} a_{n_k} = L$. This implies $\exists N_1 \in \mathbb{N}$ such that if $n_k \geq k \geq N$ then $\left|a_{n_k} - L\right| < \frac{\varepsilon}{2}$. Since $\{a_n\}$ is a Cauchy sequence, $\exists N_2 \in \mathbb{N}$ such that if $n, m \geq N_2$ then $|a_n - a_m| < \frac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. Then if $n_k \geq k \geq N$,

$$|a_k - L| = \left|a_k - a_{n_k} + a_{n_k} - L\right| \leq \left|a_k - a_{n_k}\right| + \left|a_{n_k} - L\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Definition. We say that sequence $\{a_n\}$ of real numbers is **contractive** if there exists a constant α, $0 < \alpha < 1$ such that

$$|a_{n+2} - a_{n+1}| \le \alpha |a_{n+1} - a_n|$$

for all $n \in \mathbb{N}$. The number α is called the **constant of the contractive sequence**.

Theorem. Every contractive sequence is a Cauchy sequence, and therefore is convergent.

Proof. Let $\varepsilon > 0$. Since $\{a_n\}$ is contractive, $\exists \alpha$, $0 < \alpha < 1$, such that

$$\begin{aligned}|a_{n+2} - a_{n+1}| &\le \alpha |a_{n+1} - a_n| \\ &\le \alpha^2 |a_n - a_{n-1}| \\ &\le \alpha^3 |a_{n-1} - a_{n-2}| \\ &\le \cdots \\ &\le \alpha^n |a_2 - a_1| .\end{aligned}$$

Then, for $m > n \ge N$,

$$\begin{aligned}|a_m - a_n| &= |a_m - a_{m-1} + a_{m-1} + \cdots - a_{n+1} + a_{n+1} - a_n| \\ &\le |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+2} - a_{n+1}| + |a_{n+1} - a_n| \\ &\le \alpha^{m-2}|a_2 - a_1| + \alpha^{m-3}|a_2 - a_1| + \cdots + \alpha^n |a_2 - a_1| + \alpha^{n-1}|a_2 - a_1| \\ &\le (\alpha^{n-1} + \alpha^n + \cdots + \alpha^{m-3} + \alpha^{m-2})|a_2 - a_1| \\ &\le \frac{\alpha^{n-1}(1 - \alpha^{m-n})}{1 - \alpha}|a_2 - a_1| \\ &\le \frac{\alpha^{n-1}}{1 - \alpha}|a_2 - a_1| \\ &\le \frac{\alpha^{N}}{1 - \alpha}|a_2 - a_1| < \varepsilon.\end{aligned}$$

This implies

$$N < \frac{\ln\left(\dfrac{\varepsilon(1-\alpha)}{|a_2 - a_1|}\right)}{\ln \alpha}.$$

By *the completeness axiom of real number*, there exists N. Hence $\{a_n\}$ is a Cauchy sequence. $\square$

Definition. Let $\{a_n\}$ be a sequence of real numbers.

1. We say that $\{a_n\}$ **diverges to infinity**(or **tends to infinity**) if for every $M \in \mathbb{R}$, there exists a natural number N such that if $n \geq N$ then
$$a_n > M$$
and write
$$\lim_{n\to\infty} a_n = +\infty.$$
2. We say that $\{a_n\}$ **diverges to minus infinity**(or **tends to minus infinity**) if for every $M \in \mathbb{R}$, there exists a natural number N such that if $n \geq N$ then
$$a_n < M$$
and write
$$\lim_{n\to\infty} a_n = -\infty.$$
3. We say that $\{a_n\}$ is **properly divergent** in case we have either
$$\lim_{n\to\infty} a_n = +\infty \quad \text{or} \quad \lim_{n\to\infty} a_n = -\infty.$$

Note. $\lim\limits_{n\to\infty} a_n = \pm\infty \iff \forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ such that if $n \geq N$ then $a_n > M$(or $a_n < M$).

Theorem. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that
$$\lim_{n\to\infty} a_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} b_n > 0$$
then
$$\lim_{n\to\infty} a_n b_n = +\infty.$$

Proof. Let $M > 0$. Since $\lim\limits_{n\to\infty} b_n > 0$, $\exists L = \frac{1}{2}\lim\limits_{n\to\infty} b_n \in \mathbb{R}$ such that $0 < L < \lim\limits_{n\to\infty} b_n$ and $\exists N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then $b_n > L$. Since $\lim\limits_{n\to\infty} a_n = +\infty$, $\exists N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then $a_n > \frac{M}{L}$. Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$,
$$a_n b_n > \frac{M}{L} L = M.$$
Hence $\lim\limits_{n\to\infty} a_n b_n = +\infty$. □

Theorem. A monotone sequence of real numbers is properly divergent if and only if it is unbounded.

1. If $\{a_n\}$ is an bounded increasing sequence then
$$\lim_{n\to\infty} a_n = +\infty.$$
2. If $\{a_n\}$ is an bounded decreasing sequence then
$$\lim_{n\to\infty} a_n = -\infty.$$

Proof. Let $M \in \mathbb{R}$. Since $\{a_n\}$ is increasing and unbounded $\exists N \in \mathbb{N}$ such that $a_N > M$. If $n \geq N$,
$$a_n \geq a_N > M.$$
Hence, $\lim\limits_{n\to\infty} a_n = +\infty$. □

Theorem. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers and suppose that for all $n \in \mathbb{N}$,
$$a_n \leq b_n.$$
Then the followings are holds:
$$\text{If } \lim_{n\to\infty} a_n = +\infty \text{ then } \lim_{n\to\infty} b_n = +\infty$$
$$\text{If } \lim_{n\to\infty} b_n = -\infty \text{ then } \lim_{n\to\infty} a_n = -\infty$$

Proof. Let $M \in \mathbb{R}$. Since $\lim\limits_{n\to\infty} a_n = +\infty$, $\exists N \in \mathbb{N}$ such that if $n \geq N$ then $a_n > M$. Since $a_n \leq b_n$, $b_n > M$. Hence $\lim\limits_{n\to\infty} b_n = +\infty$. □

Theorem. (**Limit comparison theorem**) Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive real numbers and suppose that for some positive real number $L > 0$, we have
$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$
Then
$$\lim_{n\to\infty} a_n = +\infty \text{ if and only if } \lim_{n\to\infty} b_n = +\infty.$$

Theorem. Let $\{a_n\}$ be a sequence of real numbers such that $a_n > 0$ for all $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} a_n = +\infty \text{ if and only if } \lim_{n\to\infty} \frac{1}{a_n} = 0.$$

Proof. ($\Rightarrow$) Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = +\infty$, for $M(=\frac{1}{\varepsilon}) \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that if $n \geq N$ then $a_n > M(=\frac{1}{\varepsilon})$ implies

$$\left|\frac{1}{a_n}\right| = \frac{1}{a_n} < \varepsilon.$$

($\Leftarrow$) Let $M \in \mathbb{R}^+$. Since $\lim_{n\to\infty} \frac{1}{a_n} = 0$, for $\varepsilon(=\frac{1}{M}) > 0$, $\exists N \in \mathbb{N}$ such that if $n \geq N$ then

$$\left|\frac{1}{a_n} - 0\right| = \frac{1}{a_n} < \varepsilon(=\frac{1}{M})$$

implies

$$a_n > M.$$

$\square$

Definition. Let $\{a_n\}$ be a sequence of real numbers.

1. Let $A_k = \sup\{a_k, a_{k+1}, \cdots\} = \sup\{a_n : n \geq k\}$. Then L is the **limit superior** of $\{a_n\}$ if

$$L := \lim_{k\to\infty} A_k = \lim_{k\to\infty} \sup a_k.$$

2. Let $B_k = \inf\{a_k, a_{k+1}, \cdots\} = \inf\{a_n : n \geq k\}$. Then L is the **limit inferior** of $\{a_n\}$ if

$$L := \lim_{k\to\infty} B_k = \lim_{k\to\infty} \inf a_k.$$

Theorem. Let $\{a_n\}$ be a bounded sequence of real numbers. Then

$$\lim_{n\to\infty} a_n = L \quad \text{if and only if} \quad L = \limsup a_n = \liminf a_n.$$

Proof. ($\Rightarrow$) Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} a_n = L$, $\exists N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - L| < \dfrac{\varepsilon}{2}$, i.e.,

$$L - \frac{\varepsilon}{2} < a_n < L + \frac{\varepsilon}{2}.$$

Thus, if $n \geq N$,

$$L - \varepsilon < L - \frac{\varepsilon}{2} < \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\} \leq L + \frac{\varepsilon}{2} < L + \varepsilon,$$

and so $\big|\sup\{a_n, a_{n+1}, a_{n+2}, \cdots\} - L\big| < \varepsilon$. Hence $\limsup a_n = L$. Similarly, $\liminf a_n = L$.

($\Leftarrow$) Let $\varepsilon > 0$. Since $\limsup a_n = L$, $\exists N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then $\big|\sup\{a_n, a_{n+1}, a_{n+2}, \cdots\} - L\big| < \varepsilon$. This implies

$$a_n \leq \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\} < L + \varepsilon.$$

Since $\liminf a_n = L$, $\exists N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then $\big|\inf\{a_n, a_{n+1}, a_{n+2}, \cdots\} - L\big| < \varepsilon$. This implies

$$L - \varepsilon < \inf\{a_n, a_{n+1}, a_{n+2}, \cdots\} \leq a_n.$$

Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$,

$$L - \varepsilon < a_n < \text{Ł} + \varepsilon, \quad \text{i.e.,} \quad |a_n - L| < \varepsilon.$$

Hence $\lim\limits_{n\to\infty} a_n = L$. $\square$

2 Infinite Series

2.1 Introduction to Infinite Series

Definition. If $\{a_n\}$ is a sequence is $\mathbb{R}$ then the **infinite series**(or simply the series) generated by $\{a_n\}$ is the sequence $\{S_k\}$ defined by

$$\begin{aligned} S_1 &:= a_1 \\ S_2 &:= a_1 + a_2 \\ &\vdots \\ S_k &:= a_1 + a_2 + \cdots + a_{k-1} + a_k \\ &\vdots \end{aligned}$$

The number of a_k are called the **terms** of the series and the numbers S_k are called the **partial sums** of this series. If

$$\lim_{k\to\infty} S_k$$

exists, we say that his series is **convergent** and call this limit the **sum** or the **value** of this series. If this limit does not exits, we say that the series $\{S_k\}$ is **divergent**.

*** The n-th Term Test**

Theorem. (**The n-th term test**) If the series $\sum a_n$ converges then

$$\lim_{n\to\infty} a_n = 0.$$

Proof. Let $S_n = \sum_{k=1}^{n} a_k$ and let $\lim\limits_{n\to\infty} S_n = L$. Then, since $a_n = S_n - S_{n-1}$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = L - L = 0.$$

$\square$

Corollary. If $\lim\limits_{n\to\infty} a_n \neq 0$ then the series $\sum a_n$ diverges.

*** Cauchy criterion for series**

Recall.

$$\{a_n\} \text{ is a Cauchy sequence } \iff \forall\varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. if } n, m \geq N, \text{ then } |a_n - a_m| < \varepsilon.$$

Theorem. (**Cauchy criterion for series**) The series $\sum a_n$ converges if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m > n \geq N$ then

$$|S_m - S_n| = |a_{n+1} + a_{n+2} + \cdots + a_m| < \varepsilon.$$

Corollary. The series $\sum a_n$ converges if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ then

$$\sum_{k=n}^{\infty} |a_k| < \varepsilon.$$

2.2 Convergence Test Part I: Comparison, Limit Comparison & Integral Tests

Theorem. (**Comparison test**) Let $\{a_n\}$ and $\{b_n\}$ be real sequences and suppose that $0 \leq a_n \leq b_n$ for $n \in \mathbb{N}$. Then

1. The convergence of $\sum b_n$ implies the convergence of $\sum a_n$.
2. The divergence of $\sum a_n$ implies the divergence of $\sum b_n$.

Proof. Let $\varepsilon > 0$. Since $\exists \sum_{n=1}^{\infty} b_n$, $\exists N \in \mathbb{N}$ such that if $m > n \geq N$ then

$$|b_{n+1} + b_{n+2} + \cdots + b_m| < \varepsilon.$$

Since $0 \leq a_n \leq b_n$,

$$\begin{aligned} |a_{n+1} + a_{n+2} + \cdots + a_m| &= a_{n+1} + a_{n+2} + \cdots + a_m \\ &\leq b_{n+1} + b_{n+2} + \cdots + b_m \\ &= |b_{n+1} + b_{n+2} + \cdots + b_m| < \varepsilon. \end{aligned}$$

Hence, $\exists \sum_{n=1}^{\infty} a_n$. $\square$

e.g. (**The p-series**) The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges when $p > 1$ and diverges when $(0 <)p \leq 1$.

e.g. The series

$$\sum_{n=1}^{\infty} \frac{1}{3n^3}$$

converges. Then the series

$$\sum_{n=1}^{\infty} \frac{1}{3n^3 - 1}$$

converges?

Sol. Since $\dfrac{1}{3n^3 - 1} < \dfrac{1}{n^3}$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^3 - 1}$ converges. $\square$

Theorem. (**Limit comparison test**) Let $\{a_n\}$ and $\{b_n\}$ are strictly positive sequences and suppose that the following limit exists in $\mathbb{R}$

$$r = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

1. If $r \neq 0$ then $\sum a_n$ is convergent (divergent) if and only if $\sum b_n$ is convergent (divergent).
2. If $r = 0$ and if $\sum b_n$ is convergent then $\sum a_n$ is convergent.

Proof. Let $a_n = \dfrac{1}{n^p}$ and $b_n = \dfrac{1}{n^q}$. Then $r = \lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{n^q}{n^p}$.

1. Let $r \neq 0$, i.e., $p = q$. Thus,
$$\exists \sum_{n=1}^{\infty} a_n \iff \exists \sum_{n=1}^{\infty} b_n.$$
2. Let $r = 0$, i.e., $p > q$. If $\exists \sum_{n=1}^{\infty} b_n$ with $q > 1$, then
$$\exists \sum_{n=1}^{\infty} a_n \text{ with } p > q > 1.$$

(**Another proof**) For $\varepsilon = 1 > 0$, $\exists N \in \mathbb{N}$ such that if $n \geq N$ then

$$\left| \frac{a_n}{b_n} - r \right| < 1, \text{ i.e., } (r-1)b_n < a_n < (r+1)b_n$$

That is, for $c \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that if $n \geq N$ then

$$\frac{r}{c} b_n \leq a_n \leq c r b_n.$$

So

$$\frac{r}{c} \sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n \leq cr \sum_{n=1}^{\infty} b_n.$$

$\square$

Theorem. Let $f : [1, \infty) \to \mathbb{R}$ be a positive, decreasing function on $[1, \infty)$. Then the series $\sum_{k=1}^{\infty} f(k)$ converges if and only if the improper integral

$$\int_1^{\infty} f(x)\ dx = \lim_{b \to \infty} \int_1^b f(x)\ dx$$

exists. In the case of convergence, the partial sum $S_n = \sum_{k=1}^{n} f(k)$ and the sum $S = \sum_{k=1}^{\infty} f(k)$ satisfy the estimate

$$\int_{n+1}^{\infty} f(x)\ dx \leq S - S_n \leq \int_n^{\infty} f(x)\ dx.$$

Proof. Since f is positive and decreasing on the interval $[k-1, k]$, we have

$$f(k) \leq \int_{k-1}^{k} f(x)\ dx \leq f(k-1).$$

And then

$$\sum_{k=2}^{n} f(k) \leq \sum_{k=2}^{n} \int_{k-1}^{k} f(x)\ dx \leq \sum_{k=2}^{n} f(k-1),$$

and so

$$S_n - f(1) \leq \int_1^n f(x)\ dx \leq S_{n-1}.$$

Consequently,

$$\lim_{n \to \infty} S_n - f(1) \leq \int_1^{\infty} f(x)\ dx \leq \lim_{n \to \infty} S_{n-1}.$$

By Comparison test,

$$\exists \sum_{n=1}^{\infty} f(n) \iff \exists \int_1^{\infty} f(x)\ dx.$$

□

2.3 Absolute Convergence

Definition. Let $\{a_n\}$ be a sequence in $\mathbb{R}$. We say that the series $\sum a_n$ is **absolutely convergent** if the series $\sum|a_n|$ is convergent in $\mathbb{R}$. A series is said to be **conditionally**(or **non-absolutely**) **convergent** if it is convergent, but it is not absolutely convergent.

Theorem. (**Absolute convergence test**) If a series $\sum a_n$ in $\mathbb{R}$ is absolutely convergent then it is convergent.

Proof. Let $\varepsilon > 0$. Since $\exists \sum_{n=1}^{\infty}|a_n|$, $\exists N \in \mathbb{N}$ such that if $m > n \geq N$,

$$\big||a_{n+1}| + |a_{n+2}| + \cdots + |a_m|\big| < \varepsilon.$$

By triangle inequality,

$$\big||a_{n+1}| + |a_{n+2}| + \cdots + |a_m|\big| \leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_m| < \varepsilon.$$

Hence, $\exists \sum_{n=1}^{\infty} a_n$ by Cauchy criterion. $\square$

2.4 Convergence Test Part II: Root and Ratio Tests

Theorem. (**Root test**) Let $\sum a_n$ be a series such that

$$r = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

1. If $r < 1$ then the series $\sum a_n$ is absolutely convergent.
2. If $r > 1$ then the series $\sum a_n$ is divergent.
3. If $r = 1$ then this test gives no information.

Proof.

1. Let $r < 1$. Since $\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = r$, for $\varepsilon > 0$ such that $r + \varepsilon < 1$, $\exists N \in \mathbb{N}$ such that if $n \geq N$, then
$$\left| |a_n|^{\frac{1}{n}} - r \right| < \varepsilon.$$
This implies that
$$\begin{aligned} 0 \leq &|a_n|^{\frac{1}{n}} < r + \varepsilon, \\ 0 \leq &|a_n| < (r+\varepsilon)^n. \end{aligned}$$
Since $r + \varepsilon < 1$, $\exists \sum_{n=N}^{\infty} (r+\varepsilon)^n$. By the comparison test, $\exists \sum_{n=N}^{\infty} |a_n|$. Since $\exists \sum_{n=1}^{N-1} |a_n|$,
$$\exists \sum_{n=1}^{\infty} |a_n|.$$

2. Let $r > 1$. Since $\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = r$, for $\varepsilon > 0$ such that $r - \varepsilon > 1$, $\exists N \in \mathbb{N}$ such that if $n \geq N$, then
$$\left| |a_n|^{\frac{1}{n}} - r \right| < \varepsilon.$$
This implies that
$$\begin{aligned} r - \varepsilon &< |a_n|^{\frac{1}{n}}, \\ (r-\varepsilon)^n &< |a_n|. \end{aligned}$$
Since $r - \varepsilon > 1$ and by comparison test,
$$\nexists \sum_{n=N}^{\infty} (r-\varepsilon)^n \implies \nexists \sum_{n=N}^{\infty} |a_n| \implies \nexists \sum_{n=1}^{\infty} |a_n|.$$

$\square$

Theorem. (**Ratio test**) Let $\sum a_n$ be a series such that

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

1. If $r < 1$ then the series $\sum a_n$ is absolutely convergent.
2. If $r > 1$ then the series $\sum a_n$ is divergent.
3. If $r = 1$ then this test gives no information.

Proof. It is similar to Root test. □

e.g. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

converges.

Proof. Let $a_n = \dfrac{1}{n!}$. Since

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{1}{(n+1)!} \cdot n! \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1,$$

$\sum_{n=1}^{\infty} a_n$ converges. □

2.5 Alternating Series

Definition (**Alternating series**) A sequence $\{a_n\}$ of nonzero real numbers is said to be **alternating** if the terms $(-1)^{n+1}a_n, n \in \mathbb{N}$, are all positive(or all negative) real numbers. If the sequence $\{a_n\}$ is alternating, we say that the series $\sum a_n$ it generates is an **alternating series**.

Theorem. (**Alternating series test**) Let $\{a_n\}$ be a decreasing sequence of strictly positive numbers with $\lim a_n = 0$. Then the alternating series $\sum (-1)^{n+1}a_n$ is convergent.

Proof. Let

$$S_n := \sum_{k=1}^{n} (-1)^{k+1}a_k.$$

Then since

$$\begin{aligned} S_{2n} &= (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \cdots + (a_{2n} - a_{2n}) \\ &\leq (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \cdots + (a_{2n} - a_{2n}) + (a_{2n+1} + a_{2n+2}) = S_{2(n+1)}, \end{aligned}$$

$\{S_{2n}\}$ is increasing. Since

$$\begin{aligned} S_{2n} &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \\ &\leq a_1, \end{aligned}$$

$\{S_{2n}\}$ is bounded. By monotone convergence theorem, $\exists \lim_{n\to\infty} S_{2n} = S$.

We must show that $\lim_{n\to\infty} S_{2n+1} = S$. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} S_{2n} = S$, $\exists N_1 \in \mathbb{N}$ such that if $(2n >)n \geq N_1$ then

$$|S_{2n} - S| < \frac{\varepsilon}{2}.$$

Since $\exists \lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} \sum_{k=1}^{2n} (-1)^{k+1}a_k$, by the n-th term test,

$$\lim_{n\to\infty} (-1)^{n+1}a_n = 0.$$

Then $\exists N_2 \in \mathbb{N}$ such that if $(2n + 1) > n \geq N_2$, then

$$\left|(-1)^{2n+2}a_{2n+1}\right| < \frac{\varepsilon}{2} \implies |a_{2n+1}| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$,

$$\begin{aligned} |S_{2n+1} - S| =& |S_{2n} + a_{2n+1} - S| \\ \leq& |S_{2n} - S| + |a_{2n+1}| < \varepsilon. \end{aligned}$$

Hence, $\lim_{n\to\infty} S_{2n+1} = S$. □

Lemma. (**Abel's lemma**) Let $\{a_n\}$ and $\{b_n\}$ sequences in $\mathbb{R}$ and let the parital sums of $\sum b_n$ be denoted by $\{S_n\}$ with $S_0 = 0$. If $m > n$ then

$$\sum_{k=n+1}^{m} a_k b_k = (a_m S_m - a_{n+1} S_n) + \sum_{k=n+1}^{m-1} (a_k - a_{k+1}) S_k.$$

Proof. Let $S_n = \sum_{k=1}^{n} b_k$ with $S_0 = 0$. Then

$$\begin{aligned}
\sum_{k=n+1}^{m} a_k b_k &= \sum_{k=n+1}^{m} a_k (S_k - S_{k-1}) \\
&= \sum_{k=n+1}^{m} a_k S_k - \sum_{k=n+1}^{m} a_k S_{k-1} \\
&= a_{n+1} S_{n+1} + a_{n+2} S_{n+2} + \cdots + a_{m-1} S_{m-1} + a_m S_m \\
-(a_{n+1} S_n + a_{n+2} S_{n+1} &+ a_{n+3} S_{n+2} + \cdots + a_m S_{m-1}) \\
&= a_m S_m - a_{n+1} S_n + \sum_{k=n+1}^{m-1} (a_k - a_{k+1}) S_k
\end{aligned}$$

□

Theorem. (**Dirichlet's test**) If $\{a_n\}$ is a decreasing sequence with $\lim a_n = 0$ and if the partial sums $\{S_n\}$ of $\sum b_n$ are bounded then $\sum a_n b_n$ is convergent.

Proof. Let $\varepsilon > 0$. Since S_n is bounded, $\exists B > 0$ such that

$$|S_n| \le M$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} a_n = 0$, $\exists N \in \mathbb{N}$ such that if $n \ge N$ then

$$|a_n| < \frac{\varepsilon}{2B}.$$

Since $\sum_{k=n+1}^{m} a_k b_k = (a_m S_m - a_{n+1} S_n) + \sum_{k=n+1}^{m-1} (a_k - a_{k+1}) S_k$, if $m > n \ge N$, then

$$\begin{aligned}
\left|\sum_{k=n+1}^{m} a_k b_k\right| &= \left|(a_m S_m - a_{n+1} S_n) + \sum_{k=n+1}^{m-1} (a_k - a_{k+1}) S_k\right| \\
&= |a_m S_m| + |a_{n+1} S_n| + \left|\sum_{k=n+1}^{m-1} (a_k - a_{k+1}) S_k\right| \\
&\le a_m |S_m| + a_{n+1} |S_n| + \sum_{k=n+1}^{m-1} (a_k - a_{k+1}) |S_k| \\
&\le a_m B + a_{n+1} B + \sum_{k=n+1}^{m-1} (a_k - a_{k+1}) B \\
&= (a_m + a_{n+1}) B + (a_{n+1} + a_m) B \\
&= 2a_{n+1} B < \varepsilon.
\end{aligned}$$

Hence, $\exists \sum_{n=1}^{\infty} a_n b_n$. $\square$

Theorem. (**Abel's test**) If $\{a_n\}$ is a convergent monotone sequence and the series $\sum b_n$ is convergent, then the series $\sum a_n b_n$ is also convergent.

Proof. Let $S_n = \displaystyle\sum_{k=1}^{n} b_k$. Since $\exists \lim\limits_{n\to\infty} S_n$, $\sum_{k=1}^{n} b_k$ is bounded.

1. Let $\{a_n\}$ be a decreasing sequence and $\lim\limits_{n\to\infty} a_n = L$.

   Let $c_n = a_n - L$. Then $\{c_n\}$ is decreasing and
   $$\lim_{n\to\infty} c_n = 0.$$
   Since $\displaystyle\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} (c_n - L) b_n = \sum_{n=1}^{\infty} c_n b_n + \sum_{n=1}^{\infty} L b_n$, $\exists \sum_{n=1}^{\infty} a_n b_n$.

2. Let $\{a_n\}$ be a increasing sequence and $\lim\limits_{n\to\infty} a_n = L$.

   Let $d_n = L - a_n$. Then $\{d_n\}$ is decreasing and
   $$\lim_{n\to\infty} d_n = 0.$$
   Since $\displaystyle\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} (L - d_n) b_n = \sum_{n=1}^{\infty} L b_n - \sum_{n=1}^{\infty} d_n b_n$, $\exists \sum_{n=1}^{\infty} a_n b_n$.

$\square$

Theorem. (**Cauchy Condensation Test**) Let $\sum a_n$ be a series of monotone decreasing positive numbers. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges if and only if } \sum_{n=1}^{\infty} 2^n a_{2^n} \text{ converges.}$$

Proof. Let $S_n = \sum_{k=1}^{n} a_k$. Then

$$S_{2^n} = (a_1 + a_2 + \cdots + a_{2^n}) \quad \text{and} \quad \sum_{n=1}^{\infty} S_{2^n} = \sum_{n=1}^{\infty} S_n.$$

Then

$$\begin{aligned} S_{2^n} &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots + a_{2^n} \\ &\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + (\underbrace{a_{2^{n-1}} + a_{2^{n-1}} + \cdots + a_{2^{n-1}}}_{2^{n-1} \text{ times}}) + a_{2^n} \\ &= a_1 + 2a_2 + 4a_4 + \cdots + 2^{n-1} a_{2^{n-1}} + a_{2^n} \end{aligned}$$

. Since

$$\begin{aligned} S_{2^n} &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots + a_{2^n} \\ &\geq \frac{1}{2} a_1 + a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) + \cdots + (\underbrace{a_{2^n} + a_{2^n} + \cdots + a_{2^n}}_{2^n \text{ times}}) \\ &= \frac{1}{2}(a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots + 2^n a_{2^n}), \end{aligned}$$

by comparison test, we can conclude that

$$\exists \sum_{n=1}^{\infty} a_n \iff \exists \sum_{n=1}^{\infty} 2^n a_{2^n}.$$

$\square$

Part III

Functions in Real number

1 Limits of Functions

1.1 Limits of Functions

Definition. Let $a \in \mathbb{R}$ and let $\varepsilon > 0$.

1. The **ε-neighborhood** of a is the set

$$\mathcal{N}_\varepsilon(a) := \left\{x \in \mathbb{R} : |x - a| < \varepsilon\right\} = \{x \in \mathbb{N} : a - \varepsilon < x < a + \varepsilon\}.$$

2. D is called the **neighborhood** of a if there exists an ε-neighborhood $\mathcal{N}_\varepsilon(a)$ such that

$$\mathcal{N}_\varepsilon(a) \subset D.$$

3. The **ε-deleted neighborhood** of a is the set

$$\mathcal{N}^*_\varepsilon(a) := \left\{x \in \mathbb{R} : 0 < |x - a| < \varepsilon\right\} = \{x \in \mathbb{N} : a - \varepsilon < x < a + \varepsilon\} \setminus \{a\}.$$

Definition. Let $D \in \mathbb{R}$. A point a is an **accumulation point** or **cluster point** (or **limit point**) of D if for every δ-neighborhood $\mathcal{N}_\delta(a)$ of a contains at least one point of D distinct from a, i.e.,

$$(a - \delta, a + \delta) \cap (D \setminus \{a\}) \neq \varnothing,$$

Theorem. A number $a \in \mathbb{R}$ is an accumulation point of a subset $D \subseteq \mathbb{R}$ if any only if there exists a sequence $\{a_n\}$ in D such that for all $n \in \mathbb{N}$

$$\lim_{n\to\infty} a_n = a \quad \text{and} \quad a_n \neq a.$$

Proof. ($\Rightarrow$) Let a be an accumulation point of $D \in \mathbb{R}$. Then, for $n \in \mathbb{N}$, $\exists a_n \in D$ such that

$$a_n \in \left(a - \frac{1}{n},\ a + \frac{1}{n}\right) - \{a\}.$$

And this implies

$$0 < |a_n - a| < \frac{1}{n}.$$

By squeeze theorem, $\lim\limits_{n\to\infty} a_n = a$.

($\Leftarrow$) Let $\exists \{a_n\}$ such that $\lim\limits_{n\to\infty} a_n = a$ and $a_n \neq a$. Let $\varepsilon > 0/$ Since $\lim\limits_{n\to\infty} a_n = a$, $\exists N \in \mathbb{N}$ such that if $n \geq N$ then $0 < |a_n - a| < \varepsilon$ and $0 < |a_N - a| < \varepsilon$ also. Hence, a is an accumulation point of D. $\square$

Definition. Let $D \in \mathbb{R}$ and a be an accumulation point of D. A function $f : D \to \mathbb{R}$, a real number L is said to be a **limit** of f at a if for given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that if $x \in D$ and $0 < |x - a| < \delta(\varepsilon)$ then

$$\left|f(x) - L\right| < \varepsilon.$$

If the limit of f at a doest not exists, we say that f **diverges** at a.

Note.

- (**Limit of sequence**) $\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that

$$\text{if} \quad n \geq N(\varepsilon) \quad \text{then} \quad |a_n - L| < \varepsilon.$$

- (**Limit of function**) $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad \left|f(x) - L\right| < \varepsilon.$$

Theorem. (**Uniqueness of limits**) Let $f : D \to \mathbb{R}$ be a function and if a is an accumulation point of D then f can have only one limit at a.

Proof. Let $\lim\limits_{x\to a} f(x) = L_1$ and $\lim\limits_{x\to a} f(x) = L_2$. Let $\varepsilon > 0$. Since $\lim\limits_{x\to a} f(x) = L_1$, $\exists \delta_1 > 0$ such that if $0 < |x - a| < \delta_1$ then $\left|f(x) - L_1\right| < \dfrac{\varepsilon}{2}$. Since $\lim\limits_{x\to a} f(x) = L_2$, $\exists \delta_2 > 0$ such that if $0 < |x - a| < \delta_1$ then $\left|f(x) - L_2\right| < \dfrac{\varepsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $0 < |x - a| < \delta$ then

$$|L_1 - L_2| = \left|L_1 - f(x) + f(x) - L_2\right| \leq \left|f(x) - L_1\right| + \left|f(x) - L_2\right| < \varepsilon.$$

Hence $L_1 = L_2$. □

Definition. Let $f : D \to \mathbb{R}$ be a function.

1. If a is an accumulation point of $D \cap (a, \infty)$, then we say that $L \in \mathbb{R}$ is a **right-hand limit** of f at a if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that all $x \in D$ with $a < x < a + \delta$,
$$\left|f(x) - L\right| < \varepsilon.$$
In this case, we write
$$\lim_{x \to a+} f(x) = L \quad \text{or} \quad f(a+) = L.$$

2. If a is an accumulation point of $D \cap (-\infty, a)$, then we say that $L \in \mathbb{R}$ is a **left-hand limit** of f at a if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that all $x \in D$ with $a - \delta < x < a$,
$$\left|f(x) - L\right| < \varepsilon.$$
In this case, we write
$$\lim_{x \to a-} f(x) = L \quad \text{or} \quad f(a-) = L.$$

Note.

$$\lim_{x \to a+} f(x) = L \iff \forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that if } a < x < a + \delta \text{ then } \left|f(x) - L\right| < \varepsilon.$$
$$\lim_{x \to a-} f(x) = L \iff \forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that if } a - \delta < x < a \text{ then } \left|f(x) - L\right| < \varepsilon.$$

Theorem. Let $f : D \to \mathbb{R}$ be a function and a be an accumulation point of $D \cap (a, \infty)$ and $D \cap (-\infty, a)$. Then
$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a+} = L = \lim_{x \to a-} f(x).$$

Proof. $(\Rightarrow)$ Let $\varepsilon > 0$. Since $\lim_{x \to a} f(x) = L$, $\exists \delta > 0$ such that
$$\text{if } 0 < |x - a| < \delta \text{ then } \left|f(x) - L\right| < \varepsilon.$$
Thus
$$\text{if } a < x < a + \delta \text{ then } \left|f(x) - L\right| < \varepsilon, \text{ i.e., } \lim_{x \to a+} f(x) = L,$$
$$\text{if } a - \delta < x < a \text{ then } \left|f(x) - L\right| < \varepsilon, \text{ i.e., } \lim_{x \to a-} f(x) = L.$$

$(\Leftarrow)$ Let $\varepsilon > 0$. Since $\lim_{x \to a+} f(x) = L$, $\exists \delta_1 > 0$ such that
$$\text{if } a < x < a + \delta_1 \text{ then } \left|f(x) - L\right| < \varepsilon.$$
Since $\lim_{x \to a-} f(x) = L$, $\exists \delta_2 > 0$ such that
$$\text{if } a - \delta_2 < x < a \text{ then } \left|f(x) - L\right| < \varepsilon.$$
Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $0 < |x - a| < \delta$, $\left|f(x) - L\right| < \varepsilon$. $\therefore \lim_{x \to a} f(x) = L$. $\square$

1.2 Some Properties

Theorem. (**Sequential criterion**) Let $f : D \to \mathbb{R}$ be a function and a be an accumulation point of D. Then the following are equivalent.

1. $\lim_{x\to a} f(x) = L$.
2. For every sequence $\{x_n\}$ in D that converges to a such that $x_n \neq a$ for all $n \in \mathbb{N}$,
$$\lim_{n\to\infty} f(x_n) = L.$$

Proof. ($\Rightarrow$) Let $\varepsilon > 0$. Since $\lim_{x\to a} = L$, $\exists\delta > 0$ such that if $0 < |x - a| < \delta$ then
$$\big|f(x) - L\big| < \varepsilon.$$
Since $\lim_{n\to\infty} x_n = a$ and $x_n \neq a$, for given $\delta > 0$, $\exists N \in \mathbb{N}$ such that if $n \geq N$ then
$$0 < |x_n - a| < \delta.$$
Since $x_n \in D, \big|f(x) - L\big| < \varepsilon$. $\therefore \lim_{n\to\infty} f(x_n) = L$.

($\Leftarrow$) Assume that $\lim_{x\to a} f(x) \neq L$, i.e., $\exists\varepsilon > 0$, $\forall\delta > 0$, $\exists x \in D$ such that
$$0 < |x - a| < \delta \quad \text{but} \quad \big|f(x) - L\big| \geq \varepsilon.$$
Since $\lim_{n\to\infty} x_n = a$ and $x_n \neq a$, for $n \in \mathbb{N}$, $\exists x_n \in D$ such that
$$0 < |x_n - a| < \frac{1}{n} \quad \Longrightarrow \quad \big|f(x_n) - L\big| \geq \varepsilon.$$
Since $\lim_{n\to\infty} f(x_n) = L$, it is contradiction. Hence, $\lim_{x\to a} f(x) = L$. $\square$

Note.

$$\lim_{x\to a} f(x) = L \iff \forall\varepsilon > 0, \exists\delta > 0 \text{ such that if } 0 < |x - a| < \delta, \text{ for } x \in D, \text{ then } \big|f(x) - L\big| < \varepsilon.$$
$$\lim_{x\to a} f(x) \neq L \iff \exists\varepsilon > 0, \forall\delta > 0, \ \exists x \in D \text{ such that if } 0 < |x - a| < \delta \text{ but } \big|f(x) - L\big| \geq \varepsilon.$$

Theorem. (**Divergence criterion**) Let $f : D \to \mathbb{R}$ be a function and a be an accumulation point of D. Then the following are equivalent.

1. $\lim_{x\to a} f(x) \neq L$.
2. There exists a sequence $\{x_n\}$ in D with $x_n \neq a$ for all $n \in \mathbb{N}$ such that
$$\lim_{n\to\infty} x_n = a \quad \text{but} \quad \lim_{n\to\infty} f(x_n) \neq L.$$

Definition. Let $f : D \to \mathbb{R}$ be a function and let a be an accumulation point of D. We say that f is **bounded** on a neighborhood of a if there exists a $\mathcal{N}_\delta(a)$ and a constant $M > 0$ such that for all $x \in D \cap \mathcal{N}_\delta(a)$,

$$\left|f(x)\right| \leq M.$$

Theorem. Let $f : D \to \mathbb{R}$ be a function and let a be an accumulation point of D. If

$$\lim_{x \to a} f(x) = L,$$

then f is bounded on some neighborhood of a.

Proof. Let $\varepsilon = 1$. Since $\lim_{x \to a} f(x) = L$, $\exists \delta > 0$ such that if $0 < |x - a| < \delta$ then $\left|f(x) - L\right| < \varepsilon$. Let $x \in (a - \delta, a + \delta) \cap D$ and let

$$M := \begin{cases} 1 + |L|, & a \notin D \\ \sup\left\{1 + |L|, \left|f(a)\right|\right\}, & a \in D \end{cases}.$$

Then $\left|f(x)\right| \leq M$, for $x \in (a - \delta, a + \delta) \cap D$. □

Theorem. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be functions and a be an accumulation point of D. Further, let $k \in \mathbb{R}$. If

$$\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = M$$

then:

(i) $\lim_{x \to a} (fg)(x) = LM$.

(ii) $\lim_{x \to a} (f/g)(x) = L/M$ where $M \neq 0$.

Proof. Let $\varepsilon > 0$.

(i) Since $\lim_{x \to a} f(x) = L$, $\exists \delta_1 > 0$ and $\exists F > 0$ such that if $0 < |x - a| < \delta_1$ then $\big|f(x)\big| \leq F$.
Since $\lim_{x \to a} g(x) = M$, $\exists \delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ then $\big|g(x) - M\big| < \dfrac{\varepsilon}{2F}$.
Since $\lim_{x \to a} f(x) = L$, $\exists \delta_3 > 0$ such that if $0 < |x - a| < \delta_3$ then $\big|f(x) - L\big| < \dfrac{\varepsilon}{2|M| + 1}$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then if $0 < |x - a| < \delta$, then

$$\begin{aligned} \big|f(x)g(x) - LM\big| &\leq \big|f(x)\big|\big|g(x) - M\big| + \big|f(x) - L\big||M| \\ &< F \cdot \frac{\varepsilon}{2F} + \frac{\varepsilon}{2|M| + 1} \cdot |M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

(ii) Since $\lim_{x \to a} g(x) = M$, for given $\varepsilon = \dfrac{|M|}{2} > 0$, $\exists \delta_1 > 0$ such that if $0 < |x - a| < \delta_1$ then

$$\Big|\big|g(x)\big| - |M|\Big| \leq \big|g(x) - M\big| < \frac{|M|}{2} \quad \Longrightarrow \quad \frac{|M|}{2} < \big|g(x)\big| \quad \Longrightarrow \quad \frac{1}{\big|g(x)\big|} < \frac{2}{|M|}.$$

Since $\lim_{x \to a} g(x) = M$, $\exists \delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ then

$$\big|g(x) - M\big| < \frac{|M|^2}{2}\varepsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $0 < |x - a| < \delta$, then

$$\begin{aligned} \left|\frac{1}{g(x)} - \frac{1}{M}\right| &= \frac{1}{|M|} \frac{1}{\big|g(x)\big|} \big|g(x) - M\big| \\ &< \frac{1}{|M|} \frac{2}{|M|} \frac{|M|^2}{2} \varepsilon = \varepsilon. \end{aligned}$$

□

Definition. Let $f : D \to \mathbb{R}$ be a function and a be an accumulation point of D.

1. We say that f **approaches to infinity**(or **tend to infinity**) as $x \longrightarrow a$ if for every $M \in \mathbb{R}$ there exists $\delta = \delta(M) > 0$ such that for all $x \in D$ with $0 < |x - a| < \delta$ then
$$f(x) > M \quad \text{and write} \quad \lim_{x \to a} f(x) = \infty (\text{or } +\infty).$$

2. We say that f **approaches to minus infinity**(or **tend to minus infinity**) as $x \longrightarrow a$ if for every $M \in \mathbb{R}$ there exists $\delta = \delta(M) > 0$ such that for all $x \in D$ with $0 < |x - a| < \delta$ then
$$f(x) < M \quad \text{and write} \quad \lim_{x \to a} f(x) = -\infty.$$

Note.

$$\lim_{x \to a} f(x) = \infty \iff \forall M \in \mathbb{R},\ \exists \delta > 0 \text{ such that if } 0 < |x - a| < \delta \text{ then } f(x) > M.$$

Definition. Let $f : D \to \mathbb{R}$ be a function

1. We say that L is a **limit** of f as $x \longrightarrow \infty$ if given $\varepsilon > 0$ there exists M such that for any $x > M$, then
$$\left|f(x) - L\right| < \varepsilon \quad \text{and write} \quad \lim_{x \to \infty} f(x) = L.$$

2. We say that f **approaches to infinity**(or **tend to infinity**) as $x \longrightarrow \infty$ if given any $M \in \mathbb{R}$ there exists $K \in \mathbb{R}$ such that for any $x > K$, then
$$f(x) > M \quad \text{and write} \quad \lim_{x \to \infty} f(x) = \infty.$$

Note.

$$\lim_{x \to \infty} f(x) = L \iff \forall \varepsilon > 0,\ \exists N \in \mathbb{R} \text{ such that if } x > N \text{ then } \left|f(x) - L\right| < \varepsilon.$$
$$\lim_{x \to \infty} f(x) = \infty \iff \forall M \in \mathbb{R},\ \exists N \in \mathbb{R} \text{ such that if } x > N \text{ then } f(x) > M.$$

2 Continuous Functions

2.1 Continuous Functions

Definition. Let $f : D \to \mathbb{R}$ be a function and let $a \in D$. W say that f is **continuous at** a if, given any number $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in D$ satisfying $|x - a| < \delta$ then

$$\big|f(x) - f(x)\big| < \varepsilon.$$

If f is continuous on every point of D, then we say that f is **continuous on** D. If f fails to be continuous at a, then we say that f is **discontinuous at** a.

Note.

$$\lim_{x \to a} f(x) = f(a) \iff \forall \varepsilon > 0, \exists \delta > 0 \text{ such that if } |x - a| < \delta \text{ then } \big|f(x) - f(a)\big| < \varepsilon.$$

Theorem. (**Sequential criterion for continuity**) A function $f : D \to \mathbb{R}$ is continuous at the point $a \in D$ if and only if for every sequence $\{x_n\}$ in D that converges to a, the sequence $\big\{f(x_n)\big\}$ converges to $f(a)$.

Theorem. (**Discontinuity criterion**) A function $f : D \to \mathbb{R}$ is discontinuous at the point $a \in D$ if and only if for every sequence $\{x_n\}$ in D that converges to a, but the sequence $\big\{f(x_n)\big\}$ does not converges to $f(a)$.

Theorem. Let $A, B \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$. If f and g are continuous at $a \in A$ and $b = f(a) \in B$, respectively, then the composition

$$g \circ f : A \to \mathbb{R}$$

is continuous at a.

Proof. Let $\varepsilon > 0$. Since g is continuous on B, $\exists \delta_1 > 0$ such that if $|y - b| < \delta_1$, $y, b \in B$, then

$$\big|g(y) - g(b)\big| < \varepsilon.$$

Since f is continuous at a, for given δ_1, $\exists \delta > 0$ such that if $|x - a| < \delta$, $x, a \in A$, then

$$\big|f(x) - f(a)\big| < \delta_1.$$

Let $y = f(x)$ and $g = f(a)$. Since $f[A] \subseteq B$, $f(x), f(a) \in B$,

$$\big|f(x) - f(a)\big| < \delta_1 \implies \big|g(f(x)) - g(f(a))\big| < \varepsilon.$$

Hence, $g \circ f$ is continuous at $x = a$. $\square$

2.2 Properties of Continuous Functions

Definition. A function $f : D \to \mathbb{R}$ is said to be **bounded** on D if there exists a constant $M > 0$ such that

$$\left| f(x) \right| \le M$$

for all $x \in D$. On the other hand, f is said to be **unbounded** on D if given $M > 0$, there exists a point $x \in D$ such that

$$\left| f(x) \right| > M.$$

Theorem. (**Boundedness theorem**) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then f is bounded on $[a, b]$.

Proof. Assume that f is unbounded on $[a, b]$. For $n \in \mathbb{N}$, $\exists x_n \in [a, b]$ such that

$$\left| f(x_n) \right| > n.$$

This implies that

$$\lim_{n\to\infty} \left| f(x_n) \right| \ge \lim_{n\to\infty} n = \infty.$$

Since $a \le x_n \le b$ for all $n \in \mathbb{N}$, $\{x_n\}$ is a bounded sequence in $[a, b]$. By Bolzano-Weierstrass theorem, $\exists \left\{x_{n_k}\right\}$, a subsequence of $\{x_n\}$, such that

$$\lim_{k\to\infty} x_{n_k} = c \in [a, b].$$

Since f is continuous at $c \in [a, b]$,

$$\lim_{k\to\infty} f(x_{n_k}) = f(c) \implies \lim_{k\to\infty} \left| f(x_{n_k}) \right| = \left| f(c) \right|.$$

But $\lim_{k\to\infty} \left| f(x_{n_k}) \right| \ge \lim_{k\to\infty} n_k = \infty$. It is contradiction. Hence, f is bounded on $[a, b]$. □

Definition. Let $f : D \to \mathbb{R}$ be a function.

1. We say that f has an **absolute maximum** on D if there is a point x^* such that

   $$f(x^*) \ge f(x) \text{ for all } x \in D.$$

   In this case, x^* is called an **absolute maximum point** for f on D if it exists.

2. We say that f has an **absolute minimum** on D if there is a point x^* such that

   $$f(x^*) \le f(x) \text{ for all } x \in D.$$

   In this case, x^* is called an **absolute minimum point** for f on D if it exists.

Theorem. (**Maximum-Minimum theorem**) Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Then f has an absolute maximum and an absolute minimum on $[a, b]$, i.e., there exists $p, q \in [a, b]$ such that

$$f(p) \leq f(x) \leq f(q).$$

Proof. Let $I = [a, b]$. Since f is continuous on I, $\exists M > 0$ such that

$$\big|f(x)\big| \leq M$$

for all $x \in I$. Let $f(I) = \big\{f(x) : x \in I\big\}$. Then $f(I)$ satisfies

(i) $f(I) \neq \varnothing$;

(ii) $f(I)$ has an upperbound M.

By completeness of $\mathbb{R}$, $\exists \sup f(I) = s^*$. Let $n \in \mathbb{N}$. Since $s^* - \dfrac{1}{n}$ is not an upperbound of $f(I)$, $\exists x_n \in I$ such that

$$s^* - \frac{1}{n} < f(x_n) \leq s^*$$

By squeeze theorem, $\lim\limits_{n\to\infty} f(x_n) = s^*$. Since $\{x_n\}$ is bounded on I, by Bolzano-Weierstrass theorem, $\exists \big\{x_{n_k}\big\}$, a subsequence of $\{x_n\}$, such that

$$\lim_{k\to\infty} x_{n_k} = q \in I.$$

Since $s^* - \dfrac{1}{n_k} < f(x_{n_k}) \leq s^*$,

$$\lim_{k\to\infty} f(x_{n_k}) = s^*.$$

Since f is continuous at $q \in [a, b]$,

$$\lim_{k\to\infty} f(x_{n_k}) = f(q) \implies f(x) \leq f(q) = s^*.$$

$\square$

Theorem. (**Bolzano's intermediate value theorem**) Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ and $f(a) < f(b)$. If $r \in \mathbb{R}$ satisfies $f(a) < r < f(b)$, then there exists a point $c \in (a, b)$ such that

$$f(c) = r.$$

Proof. Let $A = \{x \in [a, b] : f(x) < r\}$. Then

(i) Since $a \in A$, $A \neq \varnothing$;

(ii) A has an upperbound b.

Thus, $\exists c \in \sup A$. Let $n \in \mathbb{N}$. Since $c - \dfrac{1}{n}$ is not an upperbound of A, $\exists x_n \in A$ such that

$$c - \frac{1}{n} < x_n \leq c.$$

Then

$$\lim_{n\to\infty} x_n = c \in [a, b] \implies \lim_{n\to\infty} f(x_n) = f(c).$$

Since $f(x_n) < r$, $\lim_{n\to\infty} f(x_n) \leq r$, and so $f(c) \leq r$.

Assume that $f(c) < r$ then $c \in A$. Since f is continuous at $c \in [a, b]$, for $\varepsilon = \dfrac{1}{2}(r - f(c)) > 0$, $\exists \delta > 0$ such that if $|x - c| < \delta$ then

$$\left|f(x) - f(c)\right| < \varepsilon.$$

For $x \in (c, c + \delta) \cap (c, b]$,

$$\begin{aligned} f(x) < f(c) + \varepsilon &= f(c) + \frac{1}{2}(r - f(c)) \\ &= \frac{1}{2}(r + f(c)) \\ &< \frac{1}{2}(r + r) = r. \end{aligned}$$

Thus $x \in A$. It is contradiction. Hence $f(c) = r$. □

Definition. Let $f : D \to \mathbb{R}$ be a continuous function. A point x is said to be a **fixed point** of f in case

$$f(x) = x.$$

Theorem. (**Fixed point theorem**) Let $f : [0,1] \to [0,1]$ be a continuous function then there exists a point $c \in [0,1]$ such that

$$f(c) = c.$$

Proof. If $f(0) = 0$ or $f(1) = 1$, then $x = 0$ or $x = 1$, respectively. Let $f(0) \neq 0$ and $f(1) \neq 1$. Define a function $g : [0,1] \to \mathbb{R}$ such that

$$g(x) = f(x) - x.$$

Then

1. g is continuous on $[0,1]$;
2. $g(0) = f(0) - 0 > 0$;
3. $g(1) = f(1) - 1 < 0$.

By Bolzano's intermediate value theorem, $\exists c \in (0,1)$ such that $g(c) = 0$. This implies that

$$f(c) = c.$$

$\square$

2.3 Uniformly Continuous Functions

Definition. (**Continuous function - revisited**) Let $f : D \to \mathbb{R}$ be a continuous function. Then, the following statements are equivalent:

1. f is continuous at every point $a \in D$.
2. Given $\varepsilon > 0$ and $a \in D$, there exists $\delta(\varepsilon, a) > 0$ such that for all $x \in D$ and $|x - a| < \delta(\varepsilon, a)$ then
$$\big|f(x) - f(a)\big| < \varepsilon.$$

Definition. (**Uniformly continuous function**) We say that $f : D \to \mathbb{R}$ is **uniformly continuous** on D if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $|x - y| < \delta$, $x, y \in D$, then

$$\big|f(x) - f(y)\big| < \varepsilon.$$

Note.

$$f \text{ is uniformly continuous on } D$$
$$\Longleftrightarrow$$
$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 \text{ such that if for } x, y \in D, |x - a| < \delta \text{ then } \big|f(x) - f(y)\big| < \varepsilon.$$

Theorem. (**Non-uniform continuity criteria**) Let $f : D \to \mathbb{R}$ be a function then the following statements are equivalent:

1. f is not uniformly continuous on D.
2. There exists an ε_0 such that for every delta $\delta > 0$ there are points $x, y \in D$ such that
$$|x - y| < \delta \quad \text{and} \quad \big|f(x) - f(y)\big| \geq \varepsilon_0.$$
3. There exists an $\varepsilon_0 > 0$ and two sequences $\{x_n\}$ and $\{y_n\}$ in D such that
$$\lim_{n \to \infty} (x_n - y_n) = 0 \quad \text{and} \quad \big|f(x_n) - f(y_n)\big| \geq \varepsilon_0$$
for all $n \in \mathbb{N}$.

Theorem. (**Uniform continuity theorem**) Let $I = [a, b]$ be a closed interval and $f : I \to \mathbb{R}$ be a continuous function on I. Then f is uniformly continuous on I.

Proof. Assume that f is not uniformly continuous on I. Then $\exists \varepsilon > 0$ and $\exists \{x_n\}, \{y_n\}$ in I such that

$$|x_n - y_n| < \frac{1}{n} \quad \text{but} \quad \big|f(x_n) - f(y_n)\big| \geq \varepsilon.$$

for all $n \in \mathbb{N}$. Since $a \leq x_n \leq b$, $\{x_n\}$ is bounded on I and so $\exists \{x_{n_k}\}$, a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty} x_{n_k} = c \in I.$$

Then since

$$0 \leq \big|y_{n_k} - c\big| \leq \big|\underbrace{y_{n_k} - x_{n_k}}_{\to 0}\big| + \big|\underbrace{x_{n_k} - c}_{\to 0}\big|,$$

we have $\lim\limits_{k\to\infty} y_{n_k} = c$. Since f is continuous at $c \in I$,

$$\lim_{k\to\infty} f(x_{n_k}) = f(c) = \lim_{k\to\infty} f(y_{n_k}).$$

But since

$$\big|f(x_{n_k}) - f(y_{n_k})\big| \geq \varepsilon$$

for all $n \in \mathbb{N}$, it is contradiction. Hence f is uniformly continuous on I. □

Definition. (**Lipschitz function**) Let $f : D \to \mathbb{R}$ be a function. If there exists a constant $K > 0$ such that

$$\big|f(x) - f(y)\big| \leq K|x - y|$$

for all $x, y \in D$, then f is said to be a **Lipschitz function** or to satisfy a **Lipschitz condition** on D.

Theorem. If $f : D \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on D.

Proof. Let $\varepsilon > 0$. Since f is a Lipschitz function, for $x, y \in D$, $\exists K > 0$ such that

$$\big|f(x) - f(y)\big| \leq K|x - y|.$$

Let $\delta = \dfrac{\varepsilon}{K} > 0$ Then if $|x - a| < \delta$, then

$$\big|f(x) - f(y)\big| \leq K|x - y| < K\delta = K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

Hence, f is uniformly continuous on D. □

Theorem. If $f : D \to \mathbb{R}$ is uniformly continuous on D and if $\{x_n\}$ is a Cauchy sequence in D, then $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

Proof. Let $\varepsilon > 0$. Since f is uniformly continuous on D, $\exists \delta(= \delta(\varepsilon)) > 0$ such that if $|x - y| < \delta$, $x, y \in D$, then

$$\left|f(x) - f(y)\right| < \varepsilon.$$

Since $\{x_n\}$ is a Cauchy sequence in D, for given $\delta > 0$, $\exists N \in \mathbb{N}$ such that if $m, n \geq N$ then

$$|x_m - x_n| < \delta.$$

Since $x_m, x_n \in D$,

$$\left|f(x_m) - f(x_n)\right| < \varepsilon.$$

Hence, $\{f(x_n)\}$ is a Cauchy sequence. $\square$

Remark. Note that $f(x) = x^{-1}$ is not uniformly continuous on $(0, 1)$. Let

$$x_n = \frac{1}{n}.$$

Then $\{x_n\}$ is a Cauchy sequence in $(0, 1)$ but

$$\{f(x_n)\} = \{n\}$$

is not a Cauchy sequence.

Theorem. (**Continuous extension theorem**) A function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on (a, b) if and only if it can be defined at the endpoints a and b such that the extended function f^* is continuous on $[a, b]$.

Proof. ($\Leftarrow$) Since f^* is uniformly continuous on $[a, b]$ by the uniform continuity theorem, $\forall \varepsilon > 0$, $\exists \delta (= \delta(\varepsilon)) > 0$ such that if $|x - y| < \delta$ and $x, y \in [a, b]$, then

$$\big|f^*(x) - f^*(y)\big| < \varepsilon.$$

If $x, y \in (a, b) \subset [a, b]$ and $|x - y| < \delta$, then

$$\big|f(x) - f(y)\big| < \varepsilon,$$

since $f^*(x) = f(x)$ for all $x \in (a, b)$.

($\Rightarrow$) Assume that f is uniformly continuous on (a, b). Let $\{x_n\}$ be a sequence in (a, b) such that

$$\lim_{n \to \infty} x_n = a.$$

Then since $\{x_n\}$ is Cauchy sequence in (a, b), $\big\{f(x_n)\big\}$ is a Cauchy sequence in $\mathbb{R}$, i.e.,

$$\exists \lim_{n \to \infty} f(x_n) = p.$$

Let $\{y_n\}$ be a any other sequence in (a, b) such that

$$\lim_{n \to \infty} y_n = a.$$

Then $\lim\limits_{n \to \infty} (x_n - y_n) = 0$ and so

$$\begin{aligned}
\lim_{n \to \infty} f(y_n) &= \lim_{n \to \infty} \{f(y_n) - f(x_n) + f(x_n)\} \\
&= \lim_{n \to \infty} \{f(y_n) - f(x_n)\} + \lim_{n \to \infty} f(x_n) = p.
\end{aligned}$$

That is, $\lim\limits_{n \to \infty} f(x_n) = \lim\limits_{x \to a} f(x) = p$. Similarly, $\exists \lim\limits_{x \to b} f(x) = q$. Define $f^* : [a, b] \to \mathbb{R}$ such that

$$f^*(x) := \begin{cases} f(x) & , x \in (a, b) \\ p & , x = a \\ q & , x = b. \end{cases}$$

$\square$

Definition. (**One-sided continuous function**) Let $f : D \to \mathbb{R}$ be a function and $a \in D$.

1. We say that f is **right-continuous function** at a if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in D$ with $a < x < a + \delta$ then
$$\big|f(x) - f(a)\big| < \varepsilon.$$

2. We say that f is **left-continuous function** at a if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in D$ with $a - \delta < x < a$ then
$$\big|f(x) - f(a)\big| < \varepsilon.$$

Theorem. Let $f : (a, b) \to \mathbb{R}$ be a function and $c \in (a, b)$. Then f is right-continuous function at c if and only if there exists $f(c+)$ and $f(c+) = f(c)$.

Theorem. Let $f : (a, b) \to \mathbb{R}$ be a function and $c \in (a, b)$. Then f is left-continuous function at c if and only if there exists $f(c-)$ and $f(c-) = f(c)$.

Definition. Let $f : D \to \mathbb{R}$ be a function and $c \in D$.

1. We say that f is **discontinuous** at a if f does not defined at a or there exists $\lim\limits_{x \to a} f(x)$ but does not equal to $f(a)$. In this cases, the point a is called **removable discontinuous point**.

2. We say that f is **jump discontinuous** at a if there exists $f(a+)$ and $f(a-)$ but $f(a+) \neq f(a-)$.

Theorem. Let $I \in \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be an increasing function on I. Then for all $c \in I$ there exists $f(c+), f(c-)$ and

$$\sup \left\{ f(x) : x < c, x \in I \right\} = f(-c) \leq f(c) \leq f(c+) = \inf \left\{ f(x) : c < x, x \in I \right\}.$$

Moreover, if $c, d \in I$ satisfies $c < d$ then $f(c+) \leq f(d-)$.

Proof. Let $S = \left\{ f(x) : x < c \right\}$. Then since

1. $S \neq \varnothing$;
2. S has an upperbound $f(c)$,

$\exists \sup S = L$. We claim that $L = \lim_{x \to c-} f(x)$.

Let $\varepsilon > 0$. Since $L - \varepsilon$ is not upperbound of S, $\exists x_c \in I$ such that

$$x_c < c \quad \text{and} \quad L - \varepsilon < f(x_c) \leq L.$$

let $\delta = c - x_c > 0$. Then if $c - \delta < x < c$,

$$L - \varepsilon < f(x_c) \leq f(x) \leq L < L + \varepsilon,$$

i.e., $\left| f(x) - L \right| < \varepsilon$. Hence,

$$\lim_{x \to c-} f(x) = L \iff f(c-) = \sup \left\{ f(x) : x < c \right\}$$

and $\sup \left\{ f(x), x < c \right\} \leq f(c)$. $\square$

Theorem. (**Continuous inverse theorem**) Let $I \in \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be strictly increasing (or decreasing) and continuous on I. Then the function f^{-1} inverse to to f strictly increasing (or decreasing) and continuous on $J := f(I)$.

Part IV
Differentiations and Integrations

1 Differentiation

1.1 Derivative & Carathéodory's Theorem

Definition. Let $I \in \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, and let $a \in I$. We say that a real number L is the **derivative of** f at a if given any $\varepsilon > 0$ there exists $\delta := \delta(\varepsilon) > 0$ such that if $x \in I$ satisfies $0 < |x - a| < \delta$, then

$$\left| \frac{f(x) - f(a)}{x - a} - L \right| < \varepsilon.$$

In this case we say that f is **differentiable** at a, and we write $f'(a)$ for L. In other words, the derivative of f at a is given by the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided this limit exists.

Theorem. If $f : I \to \mathbb{R}$ has a derivative at $a \in I$, then f is continuous at a.

Proof. Since $\exists f'(a)$,

$$\begin{aligned} \lim_{x \to a}(f(x) - f(a)) &= \lim_{x \to a} \frac{f(x) - f(a)}{(x - a)}(x - a) \\ &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a}(x - a) \\ &= f'(a) \cdot 0 = 0. \end{aligned}$$

Hence, $\lim_{x \to a} f(x) = f(a)$. $\square$

Theorem. (**Carathéodory's theorem**) Let f be defined on an interval I containing the point a. Then f is differentiable at a if and only if there exists a function φ on I that is continuous at a and satisfies

$$f(x) - f(a) = \varphi(x)(x-a) \quad \text{for} \quad x \in I.$$

In this case, we have $\varphi(a) = f'(a)$.

Proof. $(\Rightarrow)$ Assume that $\exists f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$. Define $\varphi : I \to \mathbb{R}$ such that

$$\varphi(x) = \begin{cases} \dfrac{f(x)-f(a)}{x-a} & , x \neq a \\ \\ f'(a) & , x = a \end{cases}$$

Then

(*i*) If $x \neq a$, then $f(x) - f(a) = \varphi(x)(x-a)$.
If $x = a$, then $0 = \varphi(a) \cdot 0$.

(*ii*) φ is continuous at a.

Moreover, $\varphi(a) = f'(a)$.

$(\Leftarrow)$ Let $x \neq a$ and $x \to a$. The continuity of φ implies that

$$\varphi(a) = \lim_{x\to a} \varphi(x) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

exists. $\therefore$ f is differentiable at a and $\phi(a) = f'(a)$. $\square$

e.g. Let us consider the function f defined by $f(x) := x^3$ for $x \in \mathbb{R}$. For $a \in \mathbb{R}$, we see from the factorization

$$f(x) - f(a) = x^3 - a^3 = (x^2 + ax + a^2)(x-a)$$

that $\varpi(x) := x^2 + ax + a^2$ satisfies the condition of Caratheódory's theorem. Therefore, we conclude that f differentiable at $a \in \mathbb{R}$ and that

$$f'(a) = \varphi(a) = 3a^2.$$

Theorem. (**Chain rule**) Let I, J be intervals in $\mathbb{R}$, let $g : J \to \mathbb{R}$ and $f : I \to \mathbb{R}$ be functions such that $f(I) \subseteq J$, and let $a \in I$. If f is differentiable at a and if g is differentiable at $f(a)$, then the composite function $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

Proof. Since $\exists f'(a)$, by Carathéodory's theorem, $\exists h : I \to \mathbb{R}$ such that

(*i*) h is continuous at $a \in I$;

(*ii*) $f(x) - f(a) = h(x)(x - a)$;

(*iii*) $f'(a) = h(a)$.

Since $\exists g'(b)$ for $b = f(a) \in J$, by Carathéodory's theorem, $\exists H : I \to \mathbb{R}$ such that

(*i*) H is continuous at $b \in J$;

(*ii*) $g(y) - g(b) = H(y)(y - b)$;

(*iii*) $g'(y) = H(y)$.

Since $f(I) \subseteq J$, $f(x)$ and $f(a)$ are in J. Then

$$\begin{aligned} g(f(x)) - g(f(a)) &= H(f(x))(f(x) - f(a)) \\ &= H(f(x))h(x)(x - a). \end{aligned}$$

Let $\varphi(x) = H(f(x))h(x)$. Then $\varphi : I \to \mathbb{R}$ is continuous at a and

$$g(f(x)) - g(f(a)) = \varphi(x)(x - a).$$

Thus $g(f(x))$ is differentiable at $a \in I$ and

$$(g \circ f)'(a) = \varphi(a) = H(f(a))h(a) = g'(f(a))f'(a).$$

$\square$

Theorem. (**Differentiability of the inverse functions**) Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Let $J := f(I)$ and let $g : J \to \mathbb{R}$ be the strictly monotone and continuous function inverse to f. If f is differentiable at $a \in I$ and $f'(a) \neq 0$, then g is differentiable at $b := f(a)$ and

$$g'(f(a)) = \frac{1}{f'(a)}.$$

Proof. Since $\exists f'(a) \neq 0$, $\exists h : I \to \mathbb{R}$ such that

(i) h is continuous at $a \in I$;

(ii) $f(x) - f(a) = h(x)(x-a)$l

(iii) $f'(a) = h(a) \neq 0$.

Since $h(a) \neq 0$, $\exists \delta > 0$ such that

$$h(x) \neq 0, \quad x \in (a-\delta, a+\delta) \cap I.$$

Let $\Omega = f[(a-\delta, a+\delta) \cap I]$. Then, for $y, b \in \Omega$ such that $y = f(x), b = f(a)$,

$$f(g(y)) = y \quad \text{and} \quad f(g(b)) = b$$

holds. Then

$$y - b = f(g(y)) - f(g(b)) = h(g(y))(g(y) - g(b)).$$

Since $h(g(y)) \neq 0$,

$$g(y) - g(b) = \frac{1}{h(g(y))}(y-b), \text{ where } \varphi(y) = \frac{1}{h(g(y))}.$$

Thus $g = f^{-1}$ is differentiable at $b = f(a)$ and

$$g'(b) = g'(f(a)) = \frac{1}{h(a)} = \frac{1}{f'(a)}.$$

□

1.2 Rolle's and Mean Value Theorem

Theorem. (**Interior extremum theorem**) let c be an interior point of the interval $I = (a, b)$ at which $f : I \to \mathbb{R}$ has a relative extremum. If the derivative of f at c exists, then

$$f'(c) = 0.$$

Proof. If $f'(c) > 0$, then $\exists N_\delta(c) \in I$ of c such that

$$\frac{f(x) - f(c)}{x - c} > 0 \quad \text{for} \quad x \in N_\delta(c), x \neq c.$$

If $c \in N_\delta(c)$ and $x > c$, then

$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f(c)}{x - c} > 0.$$

But this contradicts the hypothesis that f has a relative maximum at c. Similarly, we can not have $f'(c) < 0$. Hence, $f'(c) = 0$. □

Remark.

1. Let us notice that the converse of Interior extremum theorem does not hold. For example, if $f(x) := x^3$ for $x \in \mathbb{R}$, then there exists $f'(0) = 0$ but f does not have relative extrema.
2. If $f := |x|$ on $[-1, 1]$, then f has an relative minimum at $x = 0$; however, the derivative of f fail to exists at $x = 0$.

Corollary. Let $f : (a, b) \to \mathbb{R}$ be continuous on an interval (a, b) and suppose that f has a relative extremum at an interior point c of (a, b). Then either the derivative of f at c does not exist, or it is equal to zero.

Theorem. (**Rolle's theorem**) Suppose that f is continuous on a closed interval $I = [a, b]$, that the derivative f' exists at every point of the open interval (a, b), and that $f(a) = f(b) = 0$. Then there exists at least one point c in (a, b) such that

$$f'(c) = 0.$$

Theorem. (**Mean value theorem of differential calculus**) Suppose that f is continuous on a closed interval $I = [a, b]$, and that f has a derivative in the open interval (a, b). Then there exists at least one point c such that

$$f(a) - f(b) = f'(c)(b - a).$$

Proof. Define $g : [a, b] \to \mathbb{R}$ such that

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then

(*i*) g is continuous on $[a, b]$;

(*ii*) g is differentiable;

(*iii*) $g(a) = 0 = g(b)$.

By Rolle's theorem, $\exists c \in (a, b)$ such that $g'(c) = 0$. Since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Hence, $f(b) - f(a) = f'(c)(b - a)$. □

Theorem. Let $f : I \to \mathbb{R}$ be differentiable on the interval I and if the derivative f' is bounded on I then f satisfies a Lipschitz condition on I so that f is uniformly continuous on I.

Proof. Since $\exists f'(x), x \in I$ and f' is bounded, $\exists K > 0$ such that

$$\left|f'(x)\right| \leq K, \quad x \in I.$$

Let $a, b \in I$, $a < b$. Then by Mean-value theorem, $\exists c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

This implies that

$$\begin{aligned} \left|f(b) - f(a)\right| &= \left|f'(c)\right||b - a| \\ &\leq K|b - a|\,. \end{aligned}$$

Hence, f is a Lipschitz function on I. □

1.3 L'Hospital's Rules

Theorem. (**Cauchy's mean value theorem of differential calculus**) Let f and g be continuous on $[a, b]$ and differentiable on (a, b), and assume that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Theorem. (**L'Hopital's Rule - First**) Let $-\infty \leq a < b \leq \infty$ and let f, g be differentiable on (a, b) such that $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose that

$$\lim_{x \to a+} f(x) = \lim_{x \to a+} g(x) = 0.$$

Then

1. If $\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L$ then $\lim_{x \to a+} \frac{f(x)}{g(x)} = L$.
2. If $\lim_{x \to a+} \frac{f'(x)}{g'(x)} = \pm\infty$ then $\lim_{x \to a+} \frac{f(x)}{g(x)} = \pm\infty$.

1.4 Taylor's Theorem

Theorem. (**Taylor's theorem**) Let $n \in \mathbb{N}$, $I := [a,b]$ and $f : I \to \mathbb{R}$ be such that f and its derivatives $f', f'', \cdots, f^{(n)}$ are continuous on I and that $f^{(n+1)}$ exists on (a,b). If $a \in I$ then for any $x \in I$ there exists a point c between a and x such that

$$\begin{aligned} f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \end{aligned}$$

Proof. Define $F : [a,x] \to \mathbb{R}$ such that

$$\begin{aligned} F(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 \\ - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n. \end{aligned}$$

We claim that $\exists c \in (a,x)$ such that

$$F(a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Define $G : [a,x] \to \mathbb{R}$ such that

$$G(t) = F(t) - \left(\frac{x-t}{x-a}\right)^{n+1} F(a).$$

Then

(*i*) G is continuous on $[a,x]$;

(*ii*) G is differentiable on (a,x);

(*iii*) $G(a) = 0 = G(x)$.

By Rolle's theorem, $\exists c \in (a,x)$ such that

$$G'(c) = 0.$$

Since $G'(t) = F'(t) + \dfrac{(n+1)(x-t)^n}{(x-a)^{n+1}}F(a)$, we have $F(a) = -\dfrac{(x-a)^{n+1}}{(n+1)(x-c)^n}F'(c)$. Since

$$\begin{aligned} F'(t) = &- f'(t) \\ &- f''(t)(x-t) + f'(t) \\ &- \frac{f'''(t)}{2!}(x-t)^2 + f''(t)(x-t) \\ &- \cdots \\ &- \frac{f^{(n+1)}(t)}{n!}(x-t)^n + \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}, \end{aligned}$$

we have

$$F'(c) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n.$$

Hence,

$$F(a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

□

We shall use the notation $P_n(x)$ for the $n-th$ Taylor polynomial of f

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

We shall use the notation $R_n(x)$ for the remainder of f. Thus we may write the conclusion of Taylor's theorem as

$$f(x) = P_n(x) + R_n(x)$$

where R_n is given by

$$R_n(x) := \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some point c between x and x_0.

This formula for R_n is referrer to as the **Lagrange form**(or the derivative form) of the remainder.