

Complex Analysis - #HW4

1. Answer the following questions for the function $f(z) = \frac{1}{z^3 + 1}$.

(a) Find the residues of the function $f(z) = \frac{1}{z^3 + 1}$ at the isolated singularities w_0, w_1 , and w_2 .

Sol. Let $z = e^{i\theta}$. Then $w_0 = e^{\frac{\pi}{3}i}$, $w_1 = e^{\pi i} = -1$ and $w_2 = e^{\frac{5\pi}{3}i}$.

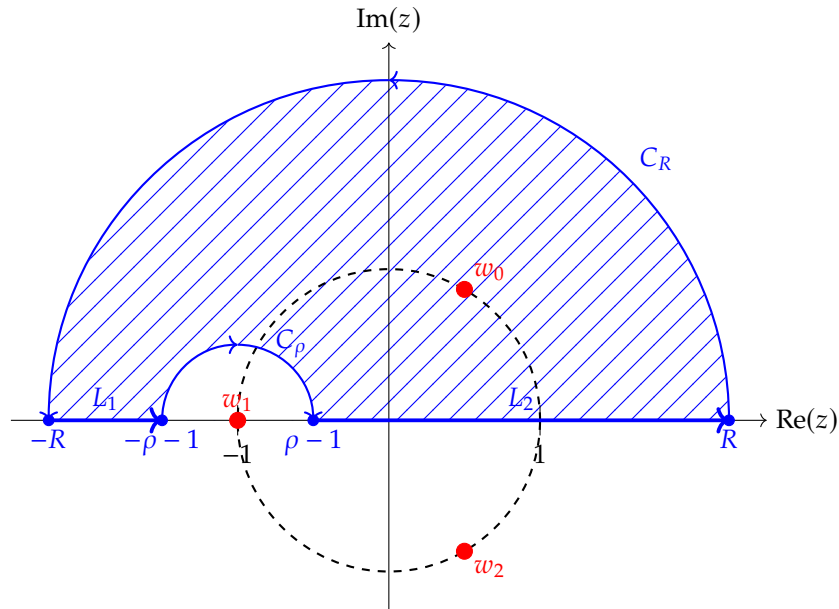
$$\text{res}(f, w_0) = \lim_{z \rightarrow w_0} (z - w_0) f(z) = \lim_{z \rightarrow w_0} \frac{z - w_0}{z^3 + 1} = \lim_{z \rightarrow w_0} \frac{1}{3z^2} = \frac{1}{3} e^{-\frac{2\pi}{3}i}$$

$$\text{res}(f, w_1) = \lim_{z \rightarrow -1} \frac{1}{3z^2} = \frac{1}{3}$$

$$\text{res}(f, w_2) = \lim_{z \rightarrow w_2} \frac{1}{3z^2} = \frac{1}{3} e^{-\frac{10\pi}{3}i} = \frac{1}{3} e^{-\frac{4\pi}{3}i}$$

□

(b) Referring to the figure below



demonstrate the following improper integral:

$$\int_{-\infty}^{\infty} \frac{1}{x^3 + 1} dx = \frac{\pi}{\sqrt{3}}.$$

Sol. Let $f(z) := \frac{1}{z^3 + 1} = \frac{1}{(z + 1)(z - e^{\pi i/3})(z - e^{5\pi i/3})}$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{x^3 + 1} dx &= 2\pi i \text{res}(f, w_0) + \pi i \text{res}(f, w_1) \\ &= 2\pi i \cdot \frac{1}{3} \cdot \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) + \frac{\pi i}{3} \\ &= -\frac{\pi i}{3} + \frac{\sqrt{3}\pi}{3} + \frac{\pi i}{3} \\ &= \frac{\pi}{\sqrt{3}}. \end{aligned}$$

Consider a path $C := C_R + L_1 + C_\rho + L_2$ where

$$C_R : z(\theta) = Re^{i\theta} \quad (\theta \in [0, \pi]),$$

$$L_1 : [-R, -\rho - 1],$$

$$L_2 : [\rho - 1, R],$$

$$C_\rho : z(t) = \rho e^{i(\pi-t)} \quad (t \in [0, \pi]).$$

Then $\oint_C f(z) dz = 0$ by the Cauchy-Goursat Theorem. Consider

$$0 = \oint_C f(z) dz = \underbrace{\int_{C_R} f(z) dz}_{=(1)} + \underbrace{\int_{L_1 \cup L_2} f(z) dz}_{=(2)} + \underbrace{\int_{C_\rho} f(z) dz}_{=(3)}$$

(1) Note that $|f(z)| = \left| \frac{1}{z^3+1} \right| \leq \frac{1}{|z|^3+1} \frac{1}{R^3+1} =: M_R \rightarrow 0$ as $R \rightarrow \infty$. So

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$$

by Jordan's Lemma.

(2)

$$\begin{aligned} \int_{L_1 \cup L_2} f(z) dz &= \int_{-R}^{-\rho-1} \frac{1}{x^3+1} dx + \int_{\rho-1}^R \frac{1}{x^3+1} dx \\ &\rightarrow \int_{-\infty}^{\infty} \frac{1}{x^3+1} dx \quad \text{as} \quad \begin{cases} R \rightarrow \infty, \\ \rho \rightarrow 0. \end{cases} \end{aligned}$$

(3) Note that

$$\begin{aligned} f(z) &= \frac{1}{z^3+1} = \frac{1}{z+1} (\operatorname{arg} 1) \\ &= \frac{1}{z} + \left(i + \frac{1}{2!} i^2 z + \frac{1}{3!} i^3 z^2 + \dots \right) \\ &= \frac{1}{z} + g(z), \end{aligned}$$

where $g(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} i^{n+1} z^n$ is analytic at $z = 0$. Consider

$$\int_{C_\rho} f(z) e^{iz} dz = \underbrace{\int_{C_\rho} \frac{1}{z} dz}_{=(a)} + \underbrace{\int_{C_\rho} g(z) dz}_{=(b)}.$$

A.

$$\int_{C_\rho} \frac{1}{z} dz = \int_0^\pi \frac{1}{\rho e^{i(\pi-t)}} \cdot i \rho e^{i(\pi-t)} (-1) dt = -i \int_0^\pi dt = -\pi i.$$

B. Since $\exists M : |g(z)| \leq M$ as $|z| \leq \rho_0$, we have

$$\int_{C_\rho} g(z) dz \leq M \cdot \pi \rho \rightarrow 0 \quad \text{as } \rho \rightarrow 0.$$

Therefore, by (1), (2) and (3), we obtain

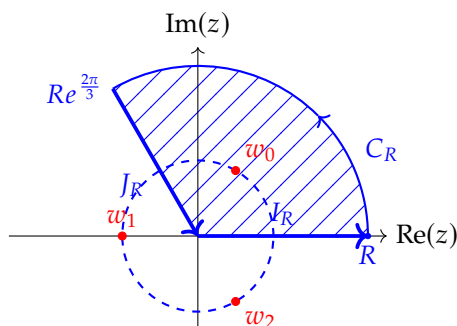
$$0 = \lim_{\substack{R \rightarrow \infty \\ \rho \rightarrow 0}} \oint_C f(z) e^{iz} dz = 0 + \int_{-\infty}^{\infty} \frac{\cos x}{x} dx + i \int_{-\infty}^{\infty} \frac{\sin x}{x} dx - \pi i,$$

and so

$$\int_{-\infty}^{\infty} \frac{\cos x}{x} dx + i \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 0 + \pi i \implies \boxed{\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi}.$$

□

(c) Referring to the figure below



demonstrate the following improper integral:

$$\int_0^{\infty} \frac{1}{x^3 + 1} dx = \frac{2\pi}{3\sqrt{3}}.$$

Sol. Firstly, let's find the isolated singularities (poles) of $f(z) = \frac{1}{z^3 + 1}$.

Setting the denominator $z^3 + 1$ equal to zero, we find that $z^3 = -1$ which gives us the roots $z = w_k = \exp\left(\frac{i(2k+1)\pi}{3}\right)$ for $k = 0, 1, 2$.

Let's calculate the residue at each w_k .

The residue $\text{Res}(f(z); w_k)$ of $f(z)$ at the pole w_k is defined as the coefficient of $(z - w_k)^{-1}$ in the Laurent series of $f(z)$ centered at w_k .

Here, we have a simple pole, so we can use the following formula for the residue at w_k :

$$\text{Res}\left(\frac{1}{z^3 + 1}; w_k\right) = \lim_{z \rightarrow w_k} (z - w_k) f(z) = \lim_{z \rightarrow w_k} \frac{z - w_k}{z^3 + 1} \text{Res}(f(z); w_k) = \lim_{z \rightarrow w_k} \frac{z - w_k}{z^3 + 1} f(z)$$

So we have:

$$\text{Res}(f(z); w_0) = \lim_{z \rightarrow w_0} (z - w_0) f(z) \quad \text{Res}(f(z); w_1) = \lim_{z \rightarrow w_1} (z - w_1) f(z) \quad \text{Res}(f(z); w_2) = \lim_{z \rightarrow w_2} (z - w_2) f(z)$$

Solving each, we find:

$$\text{Res}(f(z); w_0) = \lim_{z \rightarrow w_0} (z - w_0) \frac{1}{z^3 + 1} = \frac{1}{3(w_0^2 - w_0 w_1 - w_0 w_2)} \quad \text{Res}(f(z); w_1) = \lim_{z \rightarrow w_1} (z - w_1) \frac{1}{z^3 + 1} = \frac{1}{3(w_1^2 - w_1 w_0 - w_1 w_2)}$$

$\text{Res}(f(z); w_2) = \lim_{z \rightarrow w_2} (z - w_2) \frac{1}{z^3 + 1} = \frac{1}{3(w_2^2 - w_2 w_0 - w_2 w_1)}$ But since w_0, w_1 and w_2 are roots of the equation $z^3 = -1$, we know that $w_0 + w_1 + w_2 = 0$ and $w_0 w_1 + w_0 w_2 + w_1 w_2 = 0$ (from Vieta's formulas). Hence the residues at each singularity are:

$$\text{Res}(f(z); w_0) = -\frac{1}{3w_0} \quad \text{Res}(f(z); w_1) = -\frac{1}{3w_1} \quad \text{Res}(f(z); w_2) = -\frac{1}{3w_2}$$

User demonstrate the following improper integral:

$$\int_0^{\infty} \frac{1}{x^3 + 1} dx = \frac{2\pi}{3\sqrt{3}}.$$

ChatGPT Sure, let's solve the integral using the method of contour integration.

Let's consider the contour integral of $f(z) = \frac{1}{z^3+1}$ around a keyhole contour Γ which starts from the positive real axis, encircles the origin in the positive direction, returns along the negative real axis, and then circles back to the positive real axis around a semicircular arc in the upper half-plane. We know that $f(z)$ has poles at w_0, w_1 , and w_2 .

As per the Residue theorem, the integral of $f(z)$ around this contour is $2\pi i$ times the sum of the residues at its poles within the contour. Here, only w_0 and w_1 are within the contour, so:

$$\oint_{\Gamma} f(z), dz = 2\pi i (\text{Res}(f(z); w_0) + \text{Res}(f(z); w_1)) = 2\pi i \left(-\frac{1}{3w_0} - \frac{1}{3w_1}\right)$$

The keyhole contour integral can be split into four parts: $\int_R^{1/R}$, \int_{C_R} , $\int_{1/R}^R$, and $\int_{C_{1/R}}$, where C_R and $C_{1/R}$ are the semicircular contours with radii R and $1/R$, respectively. As $R \rightarrow \infty$ and $1/R \rightarrow 0$, the integrals over C_R and $C_{1/R}$ vanish (since $|f(z)|$ approaches 0 faster than $1/R$ on C_R and faster than R on $C_{1/R}$), so we have:

$$\int_0^{\infty} f(x), dx - \int_0^{\infty} f(-x), dx = 2\pi i \left(-\frac{1}{3w_0} - \frac{1}{3w_1}\right)$$

Since $f(x)$ is an even function, we have $\int_0^{\infty} f(x), dx = \int_0^{\infty} f(-x), dx$, and therefore:

$$2 \int_0^{\infty} f(x), dx = 2\pi i \left(-\frac{1}{3w_0} - \frac{1}{3w_1}\right)$$

This simplifies to:

$$\int_0^{\infty} \frac{1}{x^3+1}, dx = \frac{\pi i}{3} \left(-\frac{1}{w_0} - \frac{1}{w_1}\right)$$

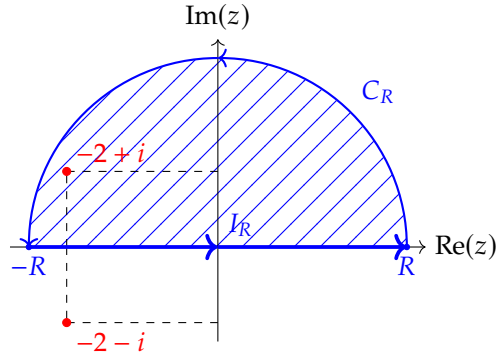
Substituting $w_0 = \exp\left(\frac{i\pi}{3}\right)$ and $w_1 = \exp\left(-\frac{i\pi}{3}\right)$, we get:

□

2. Demonstrate the following improper integral:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5} dx = -\frac{\pi}{e} \sin 2.$$

Sol. Let $f(z) := \frac{1}{z^2 + 4z + 5} = \frac{1}{(z + 2 - i)} \cdot \frac{1}{(z + 2 + i)}$, and let $f(z)e^{iz} = \frac{\phi(z)}{z + 2 - i}$ with $\phi(z) = \frac{e^{iz}}{z + 2 + i}$. Here, ϕ is analytic at $z = -2 + i$. Let $C := C_R + I_R$:



then

$$\text{Res}_{z=-2+i} [f(z)e^{iz}] = \text{Res}_{z=-2+i} \left[\frac{\phi(z)}{z + 2 - i} \right] = \phi(-2+i) = \frac{e^{i(-2+i)}}{2i} = \frac{e^{-2i}e^{-1}}{2i} = \frac{\cos(-2) + i \sin(-2)}{2ei} = \frac{\cos 2 - i \sin 2}{2ei}.$$

Consider

$$\oint_C f(z)e^{iz} dz = \underbrace{\int_{C_R} f(z)e^{iz} dz}_{=(1)} + \underbrace{\int_{I_R} f(z)e^{iz} dz}_{=(2)}.$$

(1) Note that $|f(z)| = \left| \frac{1}{z^2 + 4z + 5} \right| \leq \frac{1}{|z|^2 + |4z| + 5} = \frac{1}{R^2 + 4R + 5} \rightarrow 0$ as $R \rightarrow \infty$, and so

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z)e^{iz} dz = 0$$

by Jordan's Lemma.

(2)

$$\int_{I_R} f(z)e^{iz} dz = \int_{-R}^R \frac{1}{x^2 + 4x + 5} \cos x dx + i \int_{-R}^R \frac{1}{x^2 + 4x + 5} \sin x dx.$$

Therefore, by (1) and (2),

$$\begin{aligned} \lim_{R \rightarrow \infty} \oint_C f(z)e^{iz} dz &= \int_{-\infty}^{\infty} \frac{1}{x^2 + 4x + 5} \cos x dx + i \int_{-\infty}^{\infty} \frac{1}{x^2 + 4x + 5} \sin x dx \\ &= 2\pi i \cdot \text{res} \left(f(z)e^{iz}, -2 + i \right) \quad \text{by Residue Theorem} \\ &= 2\pi i \cdot \frac{\cos 2 - i \sin 2}{2ei} \\ &= \frac{\pi \cos 2}{e} + i \left(-\frac{\pi \sin 2}{e} \right). \end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5} dx = -\frac{\pi}{e} \sin 2.$$

□

3. Demonstrate the following improper integral:

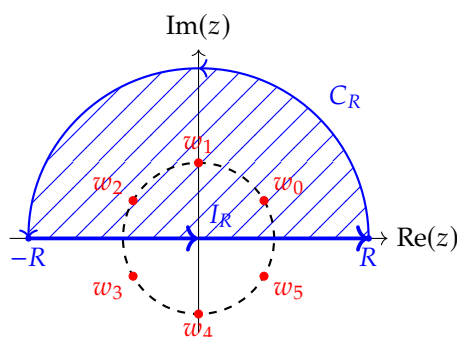
$$\int_0^{\infty} \frac{x^2}{x^6 + 1} dx = \frac{\pi}{6}.$$

Sol. Let $f(z) := \frac{z^2}{z^6 + 1} \cdot e^{\frac{\pi i}{6}}, e^{\frac{2\pi i}{3}}, e^{\pi i}, e^{\frac{4\pi i}{3}}, e^{\frac{5\pi i}{6}}, e^{2\pi i}$

$$\text{res}(f, w_0) = \lim_{z \rightarrow w_0} (z - w_0)f(z) = \lim_{z \rightarrow w_0} \frac{z^3 - z^2 w_0}{z^6 + 1} = \lim_{z \rightarrow w_0} \frac{3z^2 - 2zw_0}{6z^5} = \frac{w_0^2}{6w_0^5} = \frac{1}{6}w_0^{-3} = \frac{1}{6} \left(e^{\frac{\pi i}{6}} \right)^{-3} = \frac{1}{6} e^{-\frac{\pi}{2} i}$$

$$\text{res}(f, w_1) = \frac{1}{6}w_1^{-3} = \frac{1}{6} \left(e^{\frac{3\pi i}{6}} \right)^{-3} = \frac{1}{6} e^{-\frac{3\pi}{2} i}$$

$$\text{res}(f, w_2) = \frac{1}{6}w_2^{-3} = \frac{1}{6} \left(e^{\frac{5\pi i}{6}} \right)^{-3} = \frac{1}{6} e^{-\frac{5\pi}{2} i}.$$



$$\begin{aligned} \int_{-\infty}^{\infty} \frac{x^2}{x^6 + 1} dx &= 2\pi i \sum_{i=0}^3 \text{res}(f, w_i) \\ &= 2\pi i \cdot \frac{1}{6} \cdot \left(e^{-\frac{\pi}{2} i} + e^{-\frac{3\pi}{2} i} + e^{-\frac{5\pi}{2} i} \right) \\ &= 2\pi i \cdot \frac{1}{6} \cdot ((-i) + (i) + (-i)) \\ &= \frac{\pi}{3}. \end{aligned}$$

Hence

$$\int_0^{\infty} \frac{x^2}{x^6 + 1} dx = \frac{1}{2} \cdot \frac{\pi}{3} = \frac{\pi}{6}.$$

□

4. Using the Rouché's theorem, demonstrate the number of roots the equation $z^3 - z - 4 = 0$ has within the domain $\{z \in \mathbb{C} : |z| < 2\}$.

Rouché's Theorem

Theorem. Let

- (1) Let D be the region enclosed by simply closed curve γ ;
- (2) f and g be complex-valued function which are holomorphic in D
- (3) $|g(z)| < |f(z)|$ on γ .

Then f and $f + g$ have the same number of zeros in D counted up to multiply.

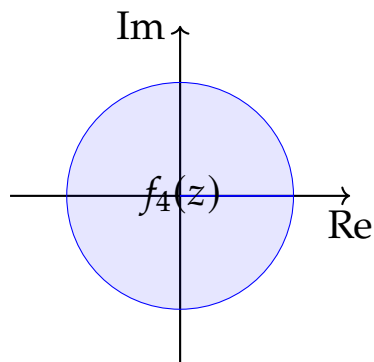
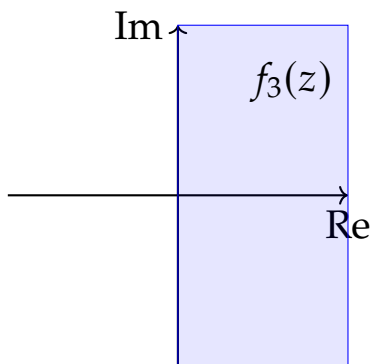
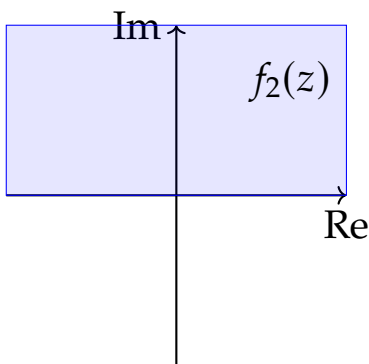
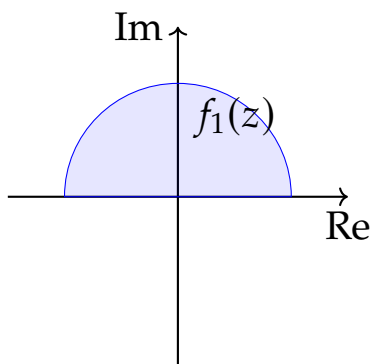
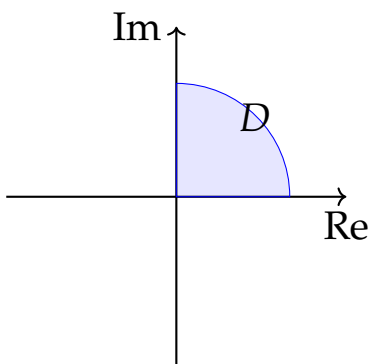
Sol. Let $f(z) + g(z) = 0$ where $f(z) = z^3$ and $g(z) = -z - 4$. For $|z| = 2$, we have

$$|f(z)| = |z^3| = 8,$$

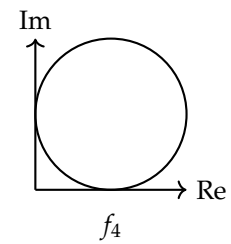
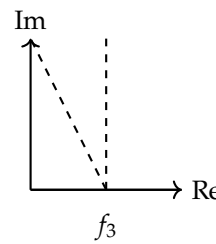
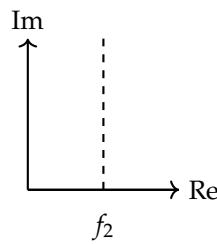
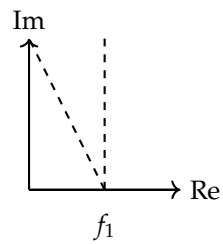
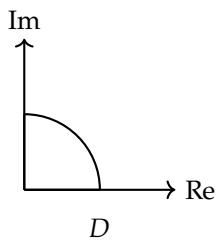
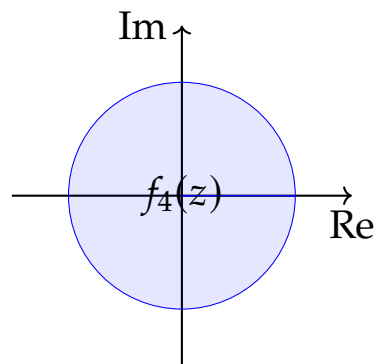
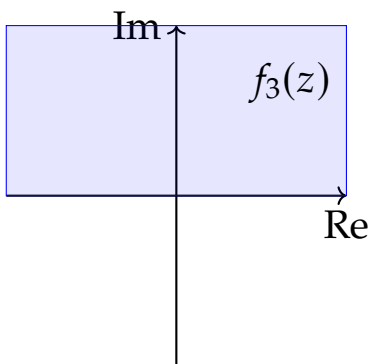
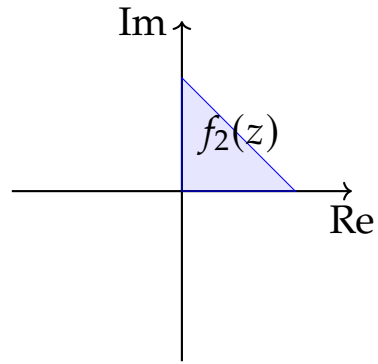
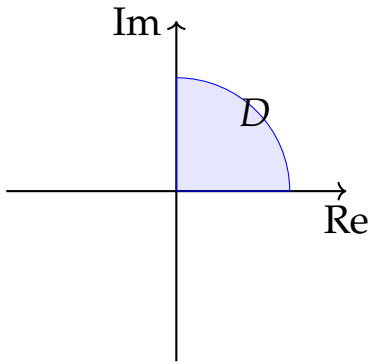
$$|g(z)| = |-z - 4| = 6.$$

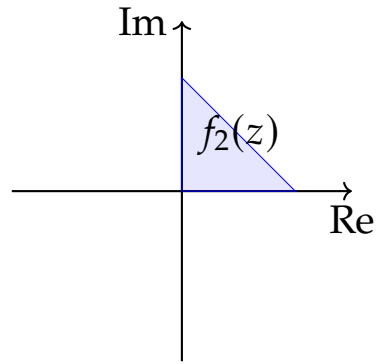
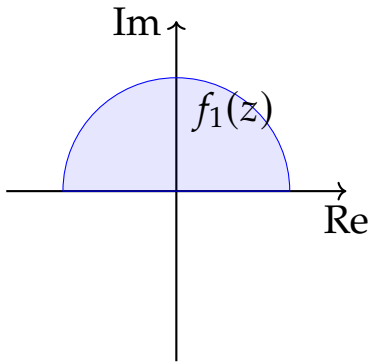
Then $|f(z)| > |g(z)|$ for all z with $|z| = 2$. Since $f(z) = z^3$ has 3 roots in the complex plane, so $f(z) + g(z) = z^3 - z - 4$ must also have 3 roots in the disc $|z| < 2$. Hence, by Rouché's Theorem, the equation $z^3 - z - 4 = 0$ has three roots within the domain $|z| < 2$. \square

5. **Sol.** Find the conformal mapping that sends the region $D = \{re^{i\theta} \in \mathbb{C} : 0 \leq r < 1, 0 < \theta < \pi/2\}$ to the open disc $U = \{z \in \mathbb{C} : |z| < 1\}$ [Hint: It can be created by composing functions $f_1(z)$ to $f_4(z)$. The function $f_2(z)$ is the inverse of a conformal mapping that sends the right half-plane to the interior of the unit disc. Also, two among $f_1(z)$ to $f_4(z)$ are the function $z \mapsto z^2$.]



\square





6. Sol. content...

□



Department of Information Security, Cryptography and Mathematics
College of Science and Technology
Kookmin University