Introduction to Applied Mathematics
- Advance Calculus II -

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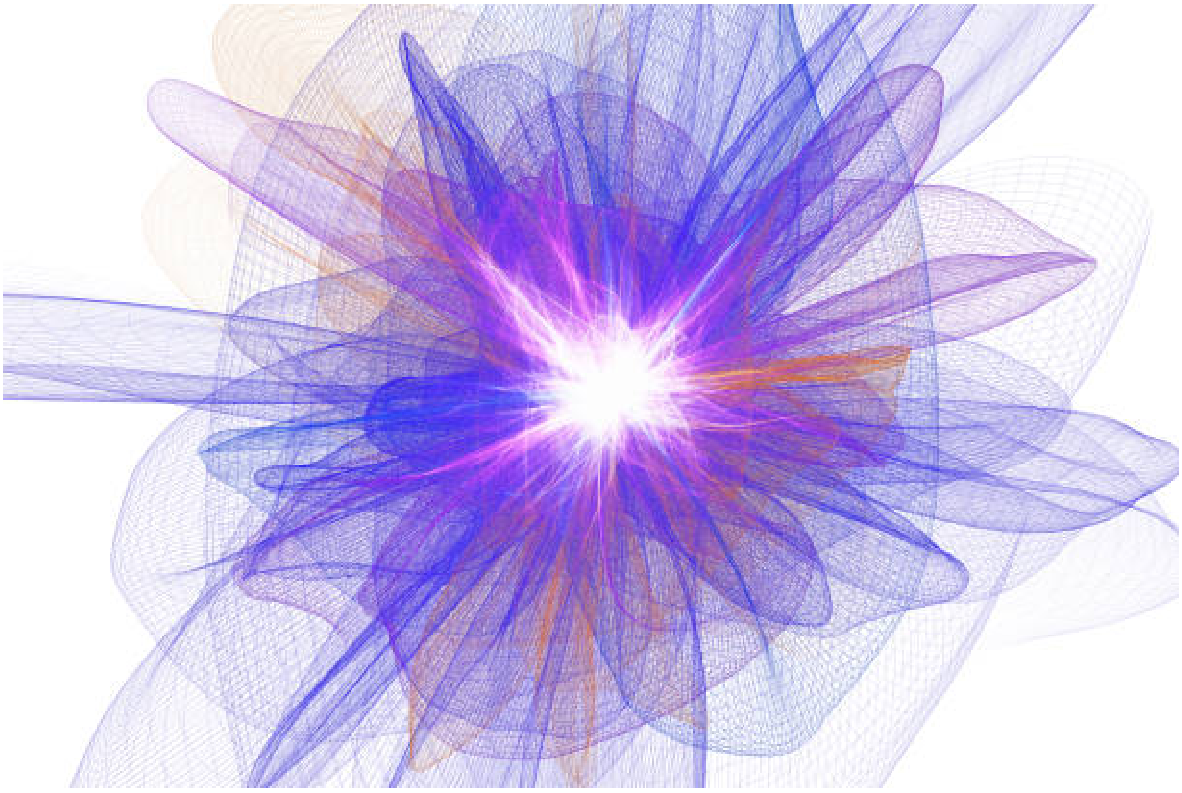

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Chapter 1

Differentiation

1.1 Derivative and Carathéodory's Theorem

Derivative

Definition 1.1. Let $f : I \longrightarrow \mathbb{R}$ and $a \in I$. We say that $L \in \mathbb{R}$ is the **derivative of** f at a if

$$\forall \epsilon > 0 : \exists \delta > 0 : x \in \mathcal{N}_\delta^*(a) \cap I \implies \left| \frac{f(x) - f(a)}{x - a} - L \right| < \epsilon.$$

Remark 1.1. We say that f is **differentiable** at a, and we write $L = f'(a)$. In other words, $f'(a) = \lim\limits_{x \longrightarrow a} \dfrac{f(x) - f(a)}{x - a}$.

Proposition 1.1. *If $f : I \longrightarrow \mathbb{R}$ has a derivative at $a \in I$ then f is continuous at a. That is,*

$$\exists f'(a) \implies f(a) = \lim_{x \longrightarrow a} f(x).$$

Proof. Let $\exists f'(a)$. Then

$$\begin{aligned} \lim_{x \longrightarrow a} [f(x) - f(a)] &= \lim_{x \longrightarrow a} \left[\frac{f(x) - f(a)}{x - a} \cdot (x - a) \right] \qquad \because x \in \mathcal{N}_\delta^*(a) \Rightarrow x \neq a \\ &= \lim_{x \longrightarrow a} \frac{f(x) - f(a)}{x - a} \lim_{x \longrightarrow a} (x - a) \\ &= f'(a) \cdot 0 = 0. \end{aligned}$$

$\square$

Remark 1.2. The continuity of $f : I \longrightarrow \mathbb{R}$ at point does not assure the existence of the derivative at that point, e.g., $f(x) := |x|$ for $x \in \mathbb{R}$.

⋆ Carathéodory's Theorem ⋆

Theorem 1.2. *Let f be defined on an interval I containing the point a. Then*

$$\exists f'(a) \iff \exists \varphi \in \mathbb{R}^I \quad \text{such that} \quad \begin{cases} \varphi \text{ is continuous on } I & \cdots(1) \\ f(x) - f(a) = \varphi(x)(x-a) & \cdots(2) \end{cases}$$

In this case, we have $\varphi(a) = f'(a)$.

Proof. ($\Rightarrow$) Assume that $\exists f'(a)$. Define a function $\varphi : I \to \mathbb{R}$ as following

$$\varphi(x) = \begin{cases} \dfrac{f(x)-f(a)}{x-a} & : x \neq a \\ f'(a) & : x = a. \end{cases}$$

Then

(i) ϕ is continuous on I, i.e., for all $a \in I$,

$$\lim_{x\to a} \varphi(x) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a) = \varphi(a).$$

(ii)

$$\begin{cases} f(x) - f(a) = \varphi(x)(x-a) & : x \neq a \\ 0 = \varphi(x) \cdot 0 & : x = a. \end{cases}$$

($\Leftarrow$) Let $x \neq a$ and $x \to a$. The continuity of φ gives that

$$\exists \phi(a) = \lim_{x\to a} \varphi(x) = \lim_{x\to a} \frac{\varphi(x)(x-a)}{(x-a)} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a).$$

That is, f is differentiable at a and $f'(a) = \varphi(a)$.

□

Example 1.1. Let us consider the function f defined by $f(x) := x^3$ for $x \in \mathbb{R}$. For any $a \in \mathbb{R}$, we see from the factorization

$$f(x) - f(a) = x^3 - a^3 = (x^2 + ax + a^2)(x-a)$$

that $\varphi(x) := x^2 + ax + a^2$ satisfies the condition of Carathéodory's Theorem. Therefore, we conclude that f is differentiable at $a \in \mathbb{R}$ and that $f'(a) = \varphi(a) = 3a^2$.

Chain Rule

Theorem 1.3. *Let I, J be intervals in $\mathbb{R}$, let $g : J \to \mathbb{R}$ and $f : I \to \mathbb{R}$ be functions such that $f[I] \subseteq J$, and let $a \in I$. Then*

$$\exists f'(a) \exists g'(f(a)) \implies \exists (g \circ f)'(a)$$

and $(g \circ f)'(a) = g'(f(a)) f'(a)$.

Proof. We must show that there exists a continuous function $\varphi(x)$ s.t.

$$g(f(x)) - g(f(a)) = \varphi(x)(x - a).$$

(1) Since $\exists f'(a)$, by Carathéodory's Theorem, $\exists \sigma : I \to \mathbb{R}$ s.t.

- (i) σ is continuous at $a \in I$;
- (ii) $f(x) - f(a) = \sigma(x)(x - a)$;
- (iii) $f'(a) = \sigma(a)$.

(2) Since $\exists g'(f(a))$, by Carathéodory's Theorem, $\exists \tau : J \to \mathbb{R}$ s.t.

- (i) τ is continuous at $f(a) \in J$;
- (ii) $g(f(x)) - g(f(a)) = \tau(f(x))(f(x) - f(a))$;
- (iii) $g'(f(a)) = \tau(f(a))$.

Then

$$\begin{aligned} g(f(x)) - g(f(a)) &= \tau(f(x))(f(x) - f(a)) \quad \text{by (2)-(ii)} \\ &= \tau(f(x))\sigma(x)(x - a) \quad \text{by (1)-(ii)}. \end{aligned}$$

Let $\varphi(x) := \tau(f(x))\sigma(x)$. Then

- (i) $\phi : I \to \mathbb{R}$ is continuous at a and
- (ii) $g(f(x)) - g(f(a)) = \varphi(x)(x - a)$,

and so, by Carathéodory's Theorem,

$$\exists (g \circ f)'(a) = \varphi(a) = \tau(f(a)) \cdot \sigma(a) = g'(f(a)) \cdot f'(a).$$

□

Remark 1.3. If f is a differentiable function, then the chain rule implies that the function $g \circ f = |f|$ is also differentiable at all points x where $f(x) \neq 0$, and its derivative is given by

$$|f(x)|'(x) = \mathsf{sgn}(f(x)) \cdot f'(x) = \begin{cases} f'(x) & : f(x) > 0, \\ -f'(x) & : f(x) < 0. \end{cases}$$

Remark 1.4. *A function f that is differentiable at every point of $\mathbb{R}$* ***need not*** *have a continuous derivative f'.*

Differentiablility of The Inverse Function

Theorem 1.4. *Let $f : I \longrightarrow \mathbb{R}$ be strictly monotone and continuous on I. Let $J := f[I]$ and $g : J \longrightarrow \mathbb{R}$ be the strictly monotone and continuous function inverse to f. Then*

$$\exists f'(a) \neq 0 \implies \exists g'(f(a)) = \frac{1}{f'(a)}.$$

Proof. Since $\exists f'(a)$, by Carathéodory's Theorem, $\exists \sigma : I \longrightarrow \mathbb{R}$ s.t.

(i) σ is continuous at $a \in I$;

(ii) $f(x) - f(a) = \sigma(x)(x - a)$;

(iii) $f'(a) = \sigma(a) \neq 0$.

Since $\sigma(a) \neq 0$, $\exists \delta > 0$ s.t. $\sigma(x) \neq 0$, $x \in \mathcal{N}_\delta(a) \cap I$. Let $\Omega := f[\mathcal{N}_\delta(a) \cap I]$. Since $g = f^{-1}$, we have

$$\begin{aligned} f(x) - f(a) &= f((g \circ f)(x)) - f((g \circ f)(a)) && \because f \circ g = \mathrm{id} \\ &= \sigma((g \circ f)(x))((g \circ f)(x) - (g \circ f)(a)) && \text{by (ii)}. \end{aligned}$$

Since $f(x) \in \Omega \Rightarrow \sigma(x) \neq 0 \Rightarrow \sigma((g \circ f)(x)) \neq 0$,

$$g(f(x)) - g(f(a)) = \frac{1}{\sigma((g \circ f)(x))}(f(x) - f(a)).$$

Let $\varphi(x) := 1/\sigma((g \circ f)(x))$. Then φ is continuous at $f(a)$. By Carathéodory's Theorem,

$$g'(f(a)) = \varphi(a) = \frac{1}{\sigma((g \circ f)(a))} = \frac{1}{\sigma(a)} = \frac{1}{f'(a)}.$$

$\square$

1.2 The Rolle's Theorem and the Mean Value Theorem

Absolute and Local Maxi/Mini-mum

Definition 1.2. Let $f : I \to \mathbb{R}$ be a function.

- f has an **absolute maximum** at $a \in I$ if $x \in I \implies f(x) \le f(a)$.
- f has an **absolute minimum** at $a \in I$ if $x \in I \implies f(a) \le f(x)$.
- f is said to have a **local (or relative) maximum** at $a \in I$ if
$$\exists \mathcal{N}_\delta(a) : f(x) \le f(a),\ x \in \mathcal{N}_\delta(a) \cap I.$$
- f is said to have a **local (or relative) minimum** at $a \in I$ if
$$\exists \mathcal{N}_\delta(a) : f(a) \le f(x),\ x \in \mathcal{N}_\delta(a) \cap I.$$
- f has a **local (or relative extremum)** at $a \in I$ either a relative maximum or a relative minimum at a.

Interior Extremum Theorem

Theorem 1.5. *Let $f : (a, b) \to \mathbb{R}$ has a relative extremum and $c \in (a, b)$. Then*
$$\exists f'(c) \implies f'(c) = 0.$$

Proof. Let f has a relative maximum at c, i.e.,
$$\exists \mathcal{N}_\delta(a) : x \in \mathcal{N}_\delta(a) \cap (a, b) \implies f(x) \le f(a).$$
Assume that $f'(c) > 0$ then
$$\exists \mathcal{N}_\delta(c) \subseteq (a, b) : x \in \mathcal{N}^*_\delta(c) \Rightarrow \frac{f(x) - f(c)}{x - c} > 0.$$
If $c \in \mathcal{N}_\delta(c)$ and $x > c$, then we have
$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f(c)}{x - c} > 0.$$
But this contradicts the hypothesis that f has a relative maximum at c. Similarly if $f'(c) < 0$ then we have a contradiction. Hence $f'(c) = 0$. □

Corollary 1.5.1. *Let $f : (a, b) \to \mathbb{R}$ be continuous on (a, b) and suppose that f has a relative extremum at $c \in (a, b)$. Then either*
$$\nexists f'(c) \quad \text{or} \quad f'(c) = 0.$$

⋆ Rolle's Theorem

Theorem 1.6. *Let f is continuous on $I = [a, b]$, and let f is differentiable on (a, b). Then*

$$f(a) = 0 = f(b) \implies \exists c \in (a, b) : f'(c) = 0.$$

⋆ Mean Value Theorem of Differential Calculus ⋆

Theorem 1.7. *Let f is continuous on $I = [a, b]$, and let f is differentiable on (a, b). Then*

$$\exists c \in (a, b) : f(b) - f(a) = f'(c)(b - a).$$

Proof. Consider the function whose graph is the line segment joining the points $(a, f(a))$ and $(b, f(b))$:

$$f(x) - f(a) = \frac{f(b) - f(a)}{b - a}(x - a).$$

Define a function $g : [a, b] \to \mathbb{R}$ s.t.

$$g(x) := f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then

(i) g is continuous on $[a, b]$;

(ii) g is differentiable on (a, b);

(iii) $g(a) = 0 = g(b)$.

By Rolle's Theorem, $\exists c \in (a, b) : g'(c) = 0$. Then

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \implies g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \implies f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

Example 1.2. Prove that $e^x \geq 1 + x$ for $x \in \mathbb{R}$.

Solution. (1) $x = 0 \implies e^x = 1 + x$.

(2) Let $x > 0$ and $f(x) = e^x$. Then, by MVT,

$$\exists c \in (0, x) : f(x) - f(0) = f'(c)(x - 0),$$

and so

$$e^x - 1 = e^c x > x \implies e^x > 1 + x.$$

(3) Let $x < 0$ and $f(x) = e^x$. Then, by MVT,

$$\exists c \in (x, 0) : f(0) - f(x) = f'(c)(0 - x),$$

and so

$$1 - e^x = e^c(-x) < -x \implies 1 + x < e^x.$$

□

1.3 L'Hôspital's Rules

Theorem 1.8. *Let f and g be defined on $[a, b]$, let $f(a) = 0 = g(a)$, and let $g(x) \neq 0$ for $x \in (a, b)$. If f and g are differentiable at a if $g'(a) \neq 0$, then the limit f/g at a exits and is equal to $f'(a)/g'(a)$. Thus*

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof. Since $f(a) = 0 = g(a)$,

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} \frac{f(x) - f(a)}{g(x) - f(a)} = \lim_{x \to a+} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-f(a)}{x-a}} = \frac{f'(a)}{g'(a)}.$$

□

Remark 1.5. In L'Hôspital Rules, the hypothesis $f(a) = 0 = g(a)$ is essential. For example, it $f(x) := x + 17$ and $g(x) := 2x + 3$ for $x \in \mathbb{R}$ then,

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{17}{3} \quad \text{while} \quad \frac{f'(0)}{g'(0)} = \frac{1}{2}.$$

Cauchy's Mean Value Thoerem of Differential Calculus

Theorem 1.9. *Let f and g be continuous on $[a, b]$ and differentiable on (a, b), and assume that $g'(x) \neq 0$ for all $x \in (a, b)$. Then*

$$\exists c \in (a, b) : \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Since $g'(x) \neq 0$ for $x \in (a, b)$, $g(a) \neq g(b)$ by Rolle's Theorem. Define $h : [a, b] \to \mathbb{R}$ such that

$$h(x) := f(x) - g(a) - \frac{f(b) - f(a)}{g(b) - g(a)} \left(g(x) - g(a)\right).$$

Then

(i) h is continuous on $[a, b]$;

(ii) h is differentiable on (a, b);

(iii) $h(a) = 0 = h(b)$.

By Rolle's Theorem,

$$\exists c \in (a, b) : h'(c) = 0.$$

Since $h'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(x)$, we have

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) \implies \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

□

Remark 1.6. Note that if $g(x) := x$ then the Cauchy's mean value theorem reduces to the mean value theorem.

Remark 1.7. By the Mean Value Theorem,

$$\exists \alpha, \beta \in (a,b) : \begin{cases} f(b) - f(a) = f'(\alpha)(b-a) \\ \\ f(b) - f(a) = g'(\beta)(b-a) \end{cases} .$$

If $g'(x) \neq 0$ for $x \in (a,b)$, we have $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(\alpha)}{g'(\beta)}$.

L'Hôpital's Rule - 1st

Theorem 1.10. *Let $-\infty \le a < b \le \infty$ and let f, g be differentiable on (a,b) such that $g'(x) \neq 0$ for all $x \in (a,b)$. Suppose that*

$$\lim_{x \to a+} f(x) = 0 = \lim_{x \to a+} g(x).$$

Then

(1) $\displaystyle\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to a+} \frac{f(x)}{g(x)} = L.$

(2) $\displaystyle\lim_{x \to a+} \frac{f'(x)}{g'(x)} = \pm\infty \implies \lim_{x \to a+} \frac{f(x)}{g(x)} = \pm\infty.$

Proof. We must show that $\lim\limits_{x \to a+} \frac{f(x)}{g(x)} = L$, i.e.,

$$\forall \varepsilon > 0 : \exists \delta > 0 : x \in (a, a+\delta) \implies \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon$$
$$\iff \forall \varepsilon > 0 : \exists c \in (a,b) : x \in (a,c) \implies \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon.$$

Since $g'(x) \neq 0$ for $x \in (a,b)$,

$$a < \alpha < x < b \implies g(x) - g(\alpha) \neq 0.$$

By Cauchy's Mean-Value Theorem,

$$\exists \gamma \in (\alpha, x) : \frac{f(x)-f(\alpha)}{g(x)-g(\alpha)} = \frac{f'(\gamma)}{g'(\gamma)}.$$

Let $\varepsilon > 0$. Then

$$\lim_{\gamma \to a+} \frac{f'(\gamma)}{g'(\gamma)} = L \implies \exists c \in (a,b) : \left[a < \gamma < x < c \implies \left| \frac{f'(\gamma)}{g'(\gamma)} - L \right| < \frac{\varepsilon}{2} \right]$$

Then

$$\begin{aligned}
&L - \frac{\varepsilon}{2} < \frac{f'(\gamma)}{g'(\gamma)} < L + \frac{\varepsilon}{2} \\
\implies &L - \frac{\varepsilon}{2} < \frac{f(x) - f(\alpha)}{g(x) - g(\alpha)} < L + \frac{\varepsilon}{2} \\
\implies &\lim_{\alpha \to a+} \left(L - \frac{\varepsilon}{2}\right) \leq \lim_{\alpha \to a+} \frac{f(x) - f(\alpha)}{g(x) - g(\alpha)} \leq \lim_{\alpha \to a+} \left(L + \frac{\varepsilon}{2}\right) \quad \because \lim_{\alpha \to a+} f(x) = 0 = \lim_{\alpha \to a+} g(x) \\
\implies &L - \frac{\varepsilon}{2} < L - \varepsilon \leq \frac{f(x)}{g(x)} \leq L + \frac{\varepsilon}{2} < L + \varepsilon \\
\implies &\left|\frac{f(x)}{g(x)} - L\right| < \varepsilon.
\end{aligned}$$

Thus, $\displaystyle\lim_{x \to a+} \frac{f(x)}{g(x)} = L$. □

Example 1.3. Let $I := (0, \pi/2)$. Then evaluate

$$\lim_{x \to 0+} \left(\frac{1}{x} - \frac{1}{\sin x}\right),$$

which has the indeterminate form $\infty - \infty$.

Solution.

$$\lim_{x \to 0+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x \to 0+} \frac{\sin x - 1}{x \sin x} = \lim_{x \to 0+} \frac{\cos x - 1}{\sin x + x \cos x} = \lim_{x \to 0+} \frac{-\sin x}{2 \cos x - x \sin x} = 0.$$

□

Example 1.4. Let $I := (0, \infty)$. Then evaluate

$$\lim_{x \to 0+} x \ln x,$$

which has the indeterminate form $0 \times \infty$.

Solution.

$$\lim_{x \to 0+} x \ln x = \lim_{x \to 0+} \frac{\ln x}{1/x} = \lim_{x \to 0+} \frac{1/x}{-1/x^2} = \lim_{x \to 0+} (-x) = 0.$$

□

Example 1.5. Let $I := (0, \infty)$ and consider

$$\lim_{x \to 0+} x^x$$

which has the indeterminate form 0^0.

Solution. Let $f(x) := x^x$ then $\ln f(x) = x \ln x$. Then

$$\lim_{x \to 0+} (x \ln x) = \lim_{x \to 0+} \frac{\ln x}{1/x} = \lim_{x \to 0+} \frac{1/x}{-1/x^2} = \lim_{x \to 0+} (-x) = 0.$$

Thus, $\lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0+} e^{\ln f(x)} = e^0 = 1.$ □

Example 1.6. Let $I := (0, \infty)$. Then evaluate

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x,$$

which has the indeterminate form 1^∞.

Solution. Let $f(x) := \left(1 + \frac{1}{x}\right)^x$ then $\ln f(x) = x \ln\left(1 + \frac{1}{x}\right)$. Then

$$\lim_{x \to \infty} x \ln\left(1 + \frac{1}{x}\right) \overset{t=1/x}{=} \lim_{t \to 0+} \frac{\ln(1+t)}{t} = \lim_{t \to 0+} \frac{\frac{1}{1+t}}{1} = 1.$$

Thus, $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} e^{\ln f(x)} = e^1 = e.$ □

Example 1.7. Let $I := (0, \infty)$. Then evaluate

$$\lim_{x \to \infty} (1 + x)^{\frac{1}{x}},$$

which has the indeterminate form ∞^0.

Solution. Let $f(x) := (1 + x)^{1/x}$ then $\ln f(x) = \frac{\ln(1+x)}{x}$. Then

$$\lim_{x \to \infty} \frac{\ln(1 + x)}{x} = \lim_{x \to \infty} \frac{\frac{1}{1+x}}{1} = 0.$$

Thus, $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} e^{\ln f(x)} = e^0 = 1.$ □

1.4 Taylor's Theorem

⋆ **Talyor's Theorem** ⋆

Theorem 1.11. *Let $n \in \mathbb{N}$ and $f : [a,b] \to \mathbb{R}$ be such that f and its derivatives $f', f'', \dots, f^{(n)}$ are continuous on $[a,b]$ and that $f^{(n+1)}$ exists on (a,b). Then*

$$t \in [a,b] \implies \forall x \in [a,b] : \exists c \in (t,x) : f(x) = \sum_{i=0}^{n} \frac{f^{(n)}(t)}{i!}(x-t)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-t)^{n+1}.$$

Proof. Define a function $F : [a,b] \to \mathbb{R}$ such that

$$\begin{aligned} F(t) &= f(x) - \sum_{i=0}^{n} \frac{f^{(n)}(t)}{i!}(x-t)^n \\ &= f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n. \end{aligned}$$

We claim that

$$\exists c \in (a,x) : F(a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Define $G : [a,b] \to \mathbb{R}$ such that

$$G(t) = F(t) - \left(\frac{x-t}{x-a}\right)^{n+1} F(a).$$

Then

(i) G is continuous on $[a,b]$;

(ii) G is differentiable on $[a,b]$;

(iii) $G(a) = 0 = G(b)$.

By Rolle's Theorem, $\exists c \in (a,x) : G'(c) = 0$. Then

$$G'(t) = F'(t) + \frac{(n+1)(x-t)^n}{(x-a)^{n+1}}F(a) \implies F(a) = -\frac{(x-a)^{n+1}}{(n+1)(x-c)^n}F'(c).$$

Since

$$\begin{aligned} F'(t) = &- f'(t) \\ &- f''(t)(x-t) + f'(t) \\ &- \frac{f'''(t)}{2!}(x-t)^2 + f''(t)(x-t) \\ &- \cdots \\ &- \frac{f^{(n+1)}(t)}{n!}(x-t)^n + \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}, \end{aligned}$$

we have

$$F'(c) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n.$$

Hence $F(a) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$. □

Example 1.8 (Numerical Estimation)**.** Approximate the number e with error less than 10^{-5}.

Solution. Let $f(x) = e^x$. Then

$$P_n(x) = \sum_{i=0}^{n} \frac{x^n}{i!} = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n.$$

By Taylor's theorem,

$$\exists c \in (0, x) : f(x) = P_n(x) + R_n(x), \text{ where } R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}.$$

For $c \in (0, 1)$

$$R_n(1) = \frac{e^c}{n+1}! < \frac{3}{(n+1)!} < 10^{-5} \implies n = 8.$$

□

Example 1.9. For any $k \in \mathbb{N}$ and for all $x > 0$, prove that

$$x - \frac{1}{2}x^2 + \cdots - \frac{1}{2k}x^{2k} < \ln(1 + x) < x - \frac{1}{2}x^2 + \cdots + \frac{1}{2k+1}x^{2k+1}.$$

Solution. Let $g(x) := \ln(1 + x)$ for $x > 0$. Then

$$g'(x) = \frac{1}{1+x} \implies \begin{cases} P_n(x) = x - \frac{1}{2}x^2 + \cdots + \frac{(-1)^{n-1}}{n}x^n & \text{with } a = 0 \\ R_n(x) = \frac{(-1)^n c^{n+1}}{n+1}x^{n+1} & \text{for some } c \in (0, x) \end{cases}$$

Thus for any $x > 0$,

(1) $n = 2k \implies R_{2k}(x) > 0$,

(2) $n = 2k + 1 \implies R_{2k+1}(x) < 0$.

□

1.5 Exercises

Exercise 1.1. Prove that
$$(\cos^{-1})'(x) = -\frac{1}{\sqrt{1-x^2}}$$
for $x \in (-1, 1)$.

Solution. Let $y := \cos^{-1}(x)$, i.e., $x = \cos y$. Then

$$\begin{aligned}\frac{d}{dx}x = \frac{d}{dx}\left[\cos y\right] &\implies 1 = -\sin y \cdot \frac{dy}{dx} \\ &\implies -\frac{1}{\sin y} = \frac{dy}{dx} \quad \because x \in (-1,1) \Rightarrow y = \cos^{-1}(x) \in (0,\pi) \Rightarrow \sin y \neq 0.\end{aligned}$$

By Pythagorean identity,

$$\sin^2(y) + \cos^2(y) = 1 \implies \sin^2(y) = 1 - \cos^2(y) \implies \sin(y) = \sqrt{1-x^2}$$

and so
$$(\cos^{-1})'(x) = \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}.$$

□

Exercise 1.2. Let $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$f(x) := \begin{cases} x^2 \sin(x^{-2}) & : x \neq 0 \\ 0 & : x = 0. \end{cases}$$

Then, prove that f is differentiable on $\mathbb{R}$ and f' is discontinuous on $[-1, 1]$.

Solution.

(1) **Differentiability of f on $\mathbb{R}$:** Let $x \neq 0$. Since $f(x) = x^2 \sin \frac{1}{x^2}$,

$$f'(x) = 2x \sin\frac{1}{x^2} + x^2 \cos\frac{1}{x^2} \cdot (-2)\frac{1}{x^3} = 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}.$$

And
$$f'(0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{h^2 \sin\frac{1}{h^2}}{h} = \lim_{h\to 0} h \sin\frac{1}{h^2} = 0$$
because $\left|\sin(h^{-2})\right| \le 1 \Rightarrow 0 \le \left|h \sin(h^{-2})\right| \le |h|$. $\therefore \forall x \in \mathbb{R} : \exists f'(x)$.

(2) **Discontinuity of f' on** $[-1,1]$**:** Let $n \in \mathbb{N}$. Then $\frac{1}{\sqrt{2n\pi}} \in [-1,1] \setminus \{0\}$. Note that

$$f'\left(\frac{1}{\sqrt{2n\pi}}\right) = \frac{2}{\sqrt{2n\pi}} \sin(2n\pi) - 2\sqrt{2n\pi}\cos(2n\pi) = -2\sqrt{2n\pi} \neq 0.$$

Then

$$\lim_{n\to\infty} \lim_{n\to\infty} \frac{1}{\sqrt{2n\pi}} = 0 \quad \text{and} \quad f'\left(\frac{1}{\sqrt{2n\pi}}\right) \neq 0 \quad \text{but} \quad f'(0) = 0.$$

□

Exercise 1.3. Let $f : I \to \mathbb{R}$ be differentiable at $c \in I$. Establish the **Straddle Lemma:** Given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $u, v \in I$ satisfy $c - \delta < u \leq c \leq v < c + \delta$, then

$$f(v) - f(u) - (v-u)f'(c) \leq \varepsilon(v-u).$$

[Hint: use the term $f(c) - cf'(c)$ and apply the Triangle Inequality.]

Solution. Let $\varepsilon > 0$. Since f is differentiable at c,

$$\exists \delta > 0 : 0 < |x-c| < \delta \implies \left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \varepsilon.$$

Then

$$\left|f(x) - f(c) - (x-c)f'(c)\right| < \varepsilon |x-c|. \tag{*}$$

Let $u, v \in I$ satisfies $c - \delta < u \leq c \leq v < c + \delta$. Then

$$\begin{aligned}
\left|f(v) - f(u) - (v-u)f'(c)\right| &= \left|f(v) - f(c) + f(c) - f(u) - (v - c + c - u)f'(c)\right| \\
&= \left|f(v) - f(c) - (v-c)f'(c) - \big(f(u) - f(c) - (u-c)f'(c)\big)\right| \\
&\leq \left|f(v) - f(c) - (v-c)f'(c)\right| + \left|f(u) - f(c) - (u-c)f'(c)\right| \\
&< \varepsilon|v-c| + \varepsilon|u-c| \quad \text{by (*)} \\
&= \varepsilon(v-c) - \varepsilon(u-c) \quad \because u \leq c \leq v \\
&= \varepsilon(v-u).
\end{aligned}$$

□

Exercise 1.4. Let $a > b > 0$ and $n \in \mathbb{N}$. Prove that

$$\sqrt[n]{a} - \sqrt[n]{b} < \sqrt[n]{a-b}$$

for $n \geq 2$.

Solution. Define $f : \mathbb{R}_{\geq 1} \to \mathbb{R}$ by

$$f(x) := \sqrt[n]{x} - \sqrt[n]{x-1}$$

for $n \geq 2$. Then

$$\begin{aligned} f'(x) &= \frac{1}{n} x^{\frac{1-n}{n}} - \frac{1}{n}(x-1)^{\frac{1-n}{n}} \\ &= \frac{1}{n}\left[\left(x^{\frac{n-1}{n}}\right)^{-1} - \left((x-1)^{\frac{n-1}{n}}\right)^{-1}\right] \\ &= \frac{1}{n}\left[\frac{1}{x^{\frac{n-1}{n}}} - \frac{1}{(x-1)^{\frac{n-1}{n}}}\right] \\ &= \frac{1}{n}\left[\frac{(x-1)^{\frac{n-1}{n}} - x^{\frac{n-1}{n}}}{x^{\frac{n-1}{n}} \cdot (x-1)^{\frac{n-1}{n}}}\right]. \end{aligned}$$

Note that

$$x > 1 \implies 0 < x - 1 < x \implies (x-1)^{\frac{n-1}{n}} < x^{\frac{n-1}{n}}.$$

Thus, $f'(x) < 0$ for $x > 1$. That is, f is decreasing for $x \geq 1$. Then

$$a > b > 0 \implies 1 < \frac{a}{b} \implies f\left(a/b\right) < f(1) \implies \sqrt[n]{a/b} - \sqrt[n]{a/b - 1} < 1.$$

Multiplying by $\sqrt[n]{b}$, we have

$$\sqrt[n]{a} - \sqrt[n]{a-b} < \sqrt[n]{b} \implies \sqrt[n]{a} - \sqrt[n]{b} < \sqrt[n]{a-b}.$$

□

Exercise 1.5. Use the Mean Value Theorem to show that

$$\frac{x-1}{x} < \ln x < x-1$$

for $x > 1$.

Solution.

(1) Let

$$f(x) := \ln x - \frac{x-1}{x} = \ln x - 1 + \frac{1}{x}.$$

Then $f'(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2}$. Since $x > 1$ and $f'(x) > 0$, by the Mean Value Theorem,

$$\exists c \in (1, x) : f(x) - f(1) = f'(c)(x-1),$$

i.e., $f(x) - f(1) > 0$. Thus

$$f(x) = \ln x - \frac{x-1}{x} > 0 = f(1) \implies \ln x > \frac{x-1}{x}.$$

(2) Let

$$g(x) := (x-1) - \ln x.$$

Then $g'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$. Since $x > 1$ and $g'(x) > 0$,

$$g(x) > g(1) = 0 \implies x - 1 > \ln x.$$

□

Exercise 1.6. Prove of disprove: If f is differentiable and uniformly continuous on I then f is a Lipschitz function on I.

Solution. **Counterexample:** Let $f(x) := \sqrt{x}$ for $x \in (0,1)$. Then f is uniformly continuous on $(0,1)$ by continuous extension theorem. Then

$$\exists f^*(x) = \begin{cases} f(x) = \sqrt{x} & : x \in (0,1) \\ 0 & : x = 0 \\ 1 & : x = 1. \end{cases}$$

But f is not a Lipschitz function on $(0,1)$. □

Exercise 1.7. Let f, g be differentiable on R and suppose that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x > 0$. Show that $f(x) \leq g(x)$ for all $x > 0$.

Solution. Let $h(x) := g(x) - f(x)$. Since $h'(x) = g'(x) - f'(x) \geq 0$, h is an increasing function on $x > 0$. Thus, $g(x) \geq f(x)$ for all $x > 0$. □

Exercise 1.8. Show that

$$\lim_{x \to c} \frac{x^c - c^x}{x^x - c^c} = \frac{1 - \ln c}{1 + \ln c}$$

for $c > 0$.

Solution. Note that

$$\begin{aligned} y := x^x &\implies \ln y = x \ln x \\ &\implies \frac{y'}{y} = \ln x + 1 \\ &\implies y' = x^x(\ln x + 1). \end{aligned} \qquad \begin{aligned} y := c^x &\implies \ln y = x \ln c \\ &\implies \frac{y'}{y} = \ln c \\ &\implies y' = c^x(\ln c). \end{aligned}$$

By L'Hôpital's rule, we have

$$\lim_{x \to c} \frac{cx^{c-1} - c^x \ln c}{x^x(\ln x + 1)} = \frac{c^c - c^c \ln c}{c^c(\ln c + 1)} = \frac{c^c(1 - \ln c)}{c^c(1 + \ln c)} = \frac{1 - \ln c}{1 + \ln c}$$

□

Exercise 1.9. Let $f : (0, 1) \to \mathbb{R}$ be differentiable on $(0, \infty)$ and suppose that

$$\lim_{x \to \infty} \big(f(x) + f'(x)\big) = L.$$

Then prove that

$$\lim_{x \to \infty} f(x) = L \quad \text{and} \quad \lim_{x \to \infty} f'(x) = 0.$$

[Hint: $f(x) = \frac{e^x f(x)}{e^x}$.]

Solution. Since $f(x) = \frac{e^x f(x)}{e^x}$,

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{e^x f(x)}{e^x} \overset{\text{L'Hôpital's rule}}{=} \lim_{x \to \infty} \frac{e^x f(x) + e^x f'(x)}{e^x} = \lim_{x \to \infty} \big(f(x) + f'(x)\big) = L$$

and so $\lim_{x \to \infty} f'(x) = 0$. □

Exercise 1.10. Let $I \subseteq \mathbb{R}$ be an open interval, let $f : I \to \mathbb{R}$ be differentiable on I, and suppose $f''(a)$ exists at $a \in I$. Show that

$$f''(a) = \lim_{h \to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2}.$$

Solution.

$$\begin{aligned}
\lim_{h \to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} &\overset{\text{L'Hôpital's rule}}{=} \lim_{h \to 0} \frac{f'(a+h) - f'(a-h)}{2h} \\
&= \lim_{h \to 0} \left(\frac{1}{2} \cdot \frac{f'(a+h) - f(a) + f(a) - f'(a-h)}{h} \right) \\
&= \frac{1}{2} \left(\lim_{h \to 0} \frac{f'(a+h) - f(a)}{h} + \lim_{h \to 0} \frac{f'(a-h) - f'(a)}{-h} \right) \\
&= \frac{1}{2} \left(f''(a) + f''(a) \right) \\
&= f''(a).
\end{aligned}$$

$\square$

Chapter 2

The Riemann Integral

2.1 Introduction to Riemann Integral

Parition

Definition 2.1. Consider a closed bounded interval $[a, b] \subseteq \mathbb{R}$. A **partition** of $[a, b]$ is a finite ordered set

$$P := \{a = x_0, x_1, \cdots, x_{n-1}, x_n = b\} \text{ s.t. } x_0 < x_1 < \cdots < x_{n-1} < x_n.$$

Upper and Lower Sum

Definition 2.2. Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and $P = \{x_0, x_1, \cdots, x_{n-1}, x_n\}$ be a partition of $[a, b]$.

(1) The **upper sum** of f for the partition P is the sum

$$U(f, P) := \sum_{i=1}^{n} M_i[f](x_i - x_{i-1}), \quad M_i[f] := \sup\left\{f(x) : x \in [x_{i-1}, x_i]\right\}$$

for $i = 1, 2, \cdots, n$.

(2) The **lower sum** of f for the partition P is the sum

$$L(f, P) := \sum_{i=1}^{n} m_i[f](x_i - x_{i-1}), \quad m_i[f] := \inf\left\{f(x) : x \in [x_{i-1}, x_i]\right\}$$

for $i = 1, 2, \cdots, n$.

Proposition 2.1. *Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and P be a partition of $[a, b]$. Then*

$$L(f, P) \le U(f, P).$$

Proof. $M_i[f] \ge m_i[f] \implies L(f, P) \le U(f, P)$. □

Refinement

Definition 2.3. Let Q and P are partitions of $[a,b]$ and $P \subseteq Q$. We say that Q is a **refinement** of P.

Theorem 2.2. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be bounded on $[a,b]$ and $P = \{a = x_0, x_1, \cdots, x_n = b\}$ be a partition of $[a,b]$. Let Q is a refinement of P. Then*

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$

Proof. Assume that $Q := P \cup \{x*\}$ s.t.

$$\begin{aligned} Q &= \{a = x_0, x_1, x_2, \cdots, x_n = b\} \cup \{x^*\} \\ &= \left\{a, x_1, x_2, \cdots, x_{j-1}, x^*, x_j, x_{j+1}, \cdots, x_{n-1}, b\right\}. \end{aligned}$$

Let $M_j^L = \sup\left\{f(x) : [x_{j-1}, x^*]\right\}$ and $M_j^R = \sup\left\{f(x) : [x^*, x_j]\right\}$ then

$$M_j^L[f] \leq M_j[f] \quad \text{and} \quad M_j^R[f] \leq M_j[f],$$

and we have

$$\begin{aligned} U(f,Q) &= \left(\sum_{i=1}^{j-1} M_f[f]\Delta x_i\right) + \left(M_j^L[f](x^* - x_{j-1})\right) + \left(M_j^R(x_j - x^*)\right) + \left(\sum_{i=j+1}^{n} M_i[f]\Delta x_i\right) \\ &\leq \left(\sum_{i=1}^{j-1} M_f[f]\Delta x_i\right) + M_j[f](x^* - x_{j-1}) + M_j(x_j - x^*) + \left(\sum_{i=j+1}^{n} M_i[f]\Delta x_i\right) \\ &= \sum_{i=1}^{n} M_i[f]\Delta x_i = U(f,P). \end{aligned}$$

Similarly, we have $L(f,P) \leq L(f,Q)$. □

Corollary 2.2.1. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be bounded on $[a,b]$ and P and Q are partitions of $[a,b]$ then*

$$L(f,Q) \leq U(f,P).$$

Proof. Let $R = P \cup Q$. By **Theorem 2.2**, we have

$$L(f,Q) \leq L(f,R) \leq U(f,R) \leq U(f,P)$$

since R is a refinement of both P and Q. □

Remark 2.1. By the completeness property of real number, there exist the followings:

$$L(f) := \sup \left\{ L(f, P) : P \text{ is a partition of } [a, b] \right\},$$
$$U(f) := \inf \left\{ U(f, P) : P \text{ is a partition of } [a, b] \right\}.$$

Moreover, $L(f) \leq U(f)$.

Upper and Lower Integral

Definition 2.4. Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$.

(1) The **upper integral** of f on $[a, b]$ is defined by

$$\overline{\int_a^b} f(x)dx := U(f) = \inf \left\{ U(f, P) : P \text{ is a partition of } [a, b] \right\}.$$

(2) The **lower integral** of f on $[a, b]$ is defined by

$$\underline{\int_a^b} f(x)dx := L(f) = \sup \left\{ L(f, P) : P \text{ is a partition of } [a, b] \right\}.$$

Riemann Integral

Definition 2.5. Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$. We say that f is **Riemann integrable** (or **integrable**) on $[a, b]$ if $L(f) = U(f)$. We define the **Riemann integral** of f on $[a, b]$ as follow:

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

Example 2.1. Let $f : [0, 1] \to \mathbb{R}$ be a Dirichlet's discontinuous function defined by

$$f(x) := \begin{cases} 1 & : x \in \mathbb{Q}, \\ 0 & : x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

We claim that f is not Riemann integrable.

Solution. Let $P = \{x_0, x_1, \cdots, x_{n-1}, x_n\}$ be a partition of $[0, 1]$. Note that

$$M_i[f] \equiv 1 \quad \text{and} \quad m_i[f] \equiv 0$$

for $i = 1, 2, \ldots, n$. Then

$$L(f, P) = \sum_{i=1}^{n} m_i[f]\Delta x_i = \sum_{i=1}^{n} (0 \cdot \Delta x_i) = 0,$$
$$U(f, P) = \sum_{i=1}^{n} M_i[f]\Delta x_i = \sum_{i=1}^{n} (1 \cdot \Delta x_i) = 1.$$

Therefore $L(f) = 0 \neq 1 = U(f)$, and so f is not Riemann integrable on $[0, 1]$. □

⋆ Riemann's Condition ⋆

Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$. Then*

$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx \iff \forall \varepsilon > 0 : \exists P : U(f, P) - L(f, P) < \varepsilon.$$

Proof. ($\Rightarrow$) Let $\varepsilon > 0$. Then $\exists P_1, P_2$ such that

$$L(f) - \frac{\varepsilon}{2} < L(f, P_1) \quad \text{and} \quad U(f, P_2) < U(f) + \frac{\varepsilon}{2}.$$

Let $P := P_1 \cup P_2$. Since $L(f) = U(f)$, we have

$$\begin{aligned} U(f, P) - L(f, P) &\le U(f, P_2) - L(f, P_1) \\ &< U(f) + \frac{\varepsilon}{2} - \left(L(f) - \frac{\varepsilon}{2}\right) \\ &= \varepsilon. \end{aligned}$$

($\Leftarrow$) Let P be a partition of $[a, b]$. Since $U(f) \le U(f, P)$ and $L(f, P) \le L(f)$, for $\varepsilon > 0$,

$$0 \le U(f) - L(f) \le U(f, P) - (f, P) < \varepsilon.$$

That is, $L(f) = U(f)$.

□

2.2 Properties of Riemann Integral

Theorem 2.4. *If $f : [a, b] \to \mathbb{R}$ is is monotone on $[a, b]$ then f is Riemann integrable on $[a, b]$.*

Proof. Suppose that f is increasing on $[a, b]$. Let $\varepsilon > 0$. By the completeness property of $\mathbb{R}$,

$$\exists N \in \mathbb{N} : \left[f(b) - f(a)\right] \frac{b - a}{N} < \varepsilon.$$

Correspondingly, there exists a partition $P_N = \{x_0, x_1, \ldots, x_{N-1}, x_N\}$ such that

$$\Delta x_i = x_i - x_{i-1} = \frac{b - a}{N}$$

for $i = 1, 2, \cdots, N$. Since $\begin{cases} M_i[f] = \sup\left\{f(x) : x \in [x_{i-1}, x_i]\right\} = f(x_i) \\ m_i[f] = \inf\left\{f(x) : x \in [x_{i-1}, x_i]\right\} = f(x_{i-1}) \end{cases}$,

$$\begin{aligned} U(f, P_N) - L(f, P_N) &= \sum_{i=1}^{N} M_i[f]\Delta x_i - \sum_{i=1}^{N} m_i[f]\Delta x_i \\ &= \sum_{i=1}^{N} \left[f(x_i) - f(x_{i-1})\right] \Delta x_i \\ &= \left[f(b) - f(a)\right] \frac{b - a}{N} < \varepsilon. \end{aligned}$$

By Riemann's Condition, f is Riemann integrable. Similarly a decreasing function on $[a, b]$ is also Riemann integrable on $[a, b]$. □

Uniform Continuity Theorem

If $f : [a, b] \to \mathbb{R}$ is is continuous on $[a, b]$ then f is uniformly continuous on $[a, b]$.

Maximum-Minimum Theorem

Let $f : [a.b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Then

$$\exists p, q \in [a, b] : f(p) \leq f(x) \leq f(q).$$

Theorem 2.5. *If $f : [a, b] \to \mathbb{R}$ is is continuous on $[a, b]$ then f is Riemann integrable on $[a, b]$.*

Proof. Let $\varepsilon > 0$. Since f is continuous on $[a, b]$, f is uniformly continuous on $[a, b]$. Then

$$\exists \delta : \forall x, y \in [a, b] : \left|x - y\right| < \delta \implies \left|f(x) - f(y)\right| < \frac{\varepsilon}{b - a}.$$

Let $P = \{a = x_0, x_1, \cdots, x_n = b\}$ be a partition of $[a, b]$ such that

$$\Delta_i = x_i - x_{i-1} < \delta \quad \text{for} \quad i = 1, 2, \ldots, n.$$

By Maximum-Minimum Theorem,

$$\exists s_i, t_i \in [x_{i-1}, x_i] : m_i[f] = f(s_i) \wedge M_i[f] = f(t_i) \quad \text{for} \quad i = 1, 2, \cdots, n.$$

Since $|s_i - t_i| < \delta$, we have

$$0 \leq M_i[f] - m_i[f] = f(t_i) - f(s_i) < \frac{\varepsilon}{b - a} \quad \text{for} \quad i = 1, 2, \cdots, n.$$

Therefore,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} \left(M_i[f] - m_i[f]\right) \Delta x_i \\ &< \sum_{i=1}^{n} \left(\frac{\varepsilon}{b - a}\right) \Delta x_i = \frac{\varepsilon}{b - a}(b - a) = \varepsilon. \end{aligned}$$

□

Example 2.2. Let $f : [0, 1] \to \mathbb{R}$ be a function defined as

$$f(x) = \begin{cases} x \sin \frac{1}{x} & : x \neq 0 \\ 0 & : x = 0. \end{cases}$$

Since f is continuous on $[0, 1]$, f is Riemann integrable on $[a, b]$.

Linearity of Riemann Integral

Theorem 2.6. *Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable functions.*

(1) For $\alpha \in \mathbb{R}$, αf is Riemann integrable and

$$\int_a^b \alpha f(x)\, dx = \alpha \int_a^b f(x)\, dx.$$

(2) $f + g$ is Riemann integrable and

$$\int_a^b (f+g)(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

Proof. (1) We must show that $U(\alpha f) = L(\alpha f)$. Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$.

(i) $(\alpha = 0)$ $U(\alpha f) = 0 = L(\alpha f)$.

(ii) $(\alpha > 0)$ Since

$$\begin{aligned} M_i[\alpha f] &= \sup\left\{\alpha f(x) : x \in [x_{i-1}, x_i]\right\} \\ &= \alpha \sup\left\{f(x) : x \in [x_{i-1}, x_i]\right\} \\ &= \alpha M_i[f], \end{aligned}$$

we have

$$\begin{aligned} U(\alpha f) &= \inf\left\{U(\alpha f, P) : P \text{ be a partition of } [a,b]\right\} \\ &= \inf\left\{\alpha U(f, P) : P \text{ be a partition of } [a,b]\right\} \quad \because \sum_{i=1}^n M_i[\alpha f]\Delta x_i = \alpha \sum_{i=1}^n M_i[f]\Delta x_i \\ &= \alpha \inf\left\{U(f, P) : P \text{ be a partition of } [a,b]\right\} \\ &= \alpha U(f). \end{aligned}$$

Similarly, $L(\alpha f) = \alpha L(f)$. Since f is Riemann integrable, i.e., $L(f) = U(f)$, thus,

$$U(\alpha f) = \alpha U(f) = \alpha L(f) = L(\alpha f).$$

(iii) $(\alpha < 0)$ Similarly, it holds.

Moreover,

$$\int_a^b \alpha f(x)\, dx = U(\alpha f) = \alpha U(f) = \alpha \int_a^b f(x)\, dx.$$

(2) We must show that

$$\forall \varepsilon > 0 : \exists P : U(f+g, P) - L(f+g, P) < \varepsilon.$$

Let $\varepsilon > 0$. Since f, g are Riemann integrable on $[a, b]$, $\exists P_1, P_2$ such that

$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2} \quad \text{and} \quad U(g, P_2) - L(g, P_2) < \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2 = \{x_0, x_1, \ldots, x_n\}$. Then P is a partition of $[a, b]$, and is a refinement of P_1 and P_2. Since

$$m_i[f] + m_i[g] \le m_i[f+g] \le M_i[f+g] \le M_i[f] + M_i[g],$$

we have

$$L(f,P) + L(g,P) \le L(f+g,P) \le U(f+g,P) \le U(f,P) + U(g,P).$$

Hence

$$\begin{aligned}
U(f+g,P) - L(f+g,P) &\le U(f,P) + U(g,P) - \big[L(f,P) + L(g,P)\big] \\
&\le U(f,P_1) + U(g,P_2) - \big[L(f,P_1) + L(g,P_2)\big] \\
&= \big[U(f,P_1) - L(f,P_2)\big] + \big[U(g,P_2) - L(g,P_2)\big] \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

We want to show that

$$\forall \varepsilon > 0 : \left| \int_a^b (f+g)(x)\,dx - \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right| < \varepsilon.$$

□

Corollary 2.6.1. *Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable functions. Then for $\alpha, \beta \in \mathbb{R}$,*

$$\int_a^b (\alpha f + \beta g)(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$$

Theorem 2.7. *Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable function.*

(1)

$$\big(\forall x \in [a,b] : f(x) \geq 0\big) \implies \int_a^b f(x)\,dx \geq 0.$$

(2)

$$\big(\forall x \in [a,b] : f(x) \leq g(x)\big) \implies \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Proof. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$.

(1) Since $f(x) \geq 0$ for all $x \in [a,b]$ and $m_i[f] \geq 0$ for $i = 1, \dots, n$, we have

$$\int_a^b f(x)\,dx = L(f) \geq L(f,P) = \sum_{i=1}^n m_i[f]\Delta x_i \geq 0.$$

(2) Since $g(x) - f(x) \geq 0$, by (1),

$$0 \leq \int_a^b (g-f)(x)\,dx = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx \implies \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

□

Example 2.3.

(1) Let $f(x) = 0$ and $g(x) = x$ for $x \in [-1,3]$. Then

$$\int_{-1}^3 f(x)\,dx = 0 < 4 = \int_{-1}^3 g(x)\,dx \quad \text{but } f(x) > g(x) \text{ for } x \in [-1,0).$$

(2) Let $f(x) = 0$ and $g(x) = \sin x$ for $x \in [0,2\pi]$. Then

$$\int_0^{2\pi} f(x)\,dx = 0 = \int_0^{2\pi} g(x)\,dx \quad \text{but } f(x) \neq g(x) \text{ for } x \in (0,2\pi) \setminus \{\pi\}.$$

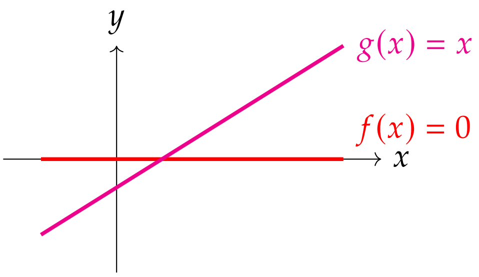


Figure 2.1: **Example 2.3.** - (1)

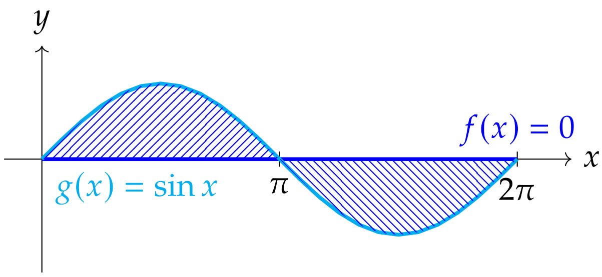


Figure 2.2: **Example 2.3.** - (2)

Theorem 2.8. *Let $f : [a,b] \to \mathbb{R}$ be a function and $c \in (a,b)$. If f is Riemann integrable for closed sub-intervals $[a,c]$ and $[c,b]$ of $[a,b]$ then f is Riemann integrable on $[a,b]$. Moreover,*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof. Let $\varepsilon > 0$. Since f is Riemann integrable on $[a,c]$,

$$\exists P_1, \text{ partition of } [a,c], \text{ such that } U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}.$$

Since f is Riemann integrable on $[c,b]$,

$$\exists P_2, \text{ partition of } [c,b], \text{ such that } U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{2}.$$

Let $P := P_1 \cup P_2$ be a partition of $[a,b]$. Then

$$\begin{aligned} U(f,P) - L(f,P) &= U(f,P_1) + U(f,P_2) - \left[L(f,P_1) + L(f,P_2)\right] \\ &= U(f,P_1) - L(f,P_1) + U(f,P_2) - L(f,P_2) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, f is Riemann integrable on $[a,b]$. By Riemann's condition,

$$\begin{aligned} \int_a^b f(x)\,dx \le U(f,P) &= U(f,P_1) + U(f,P_2) \\ &< L(f,P_1) + \frac{\epsilon}{2} + L(f,P_2) + \frac{\epsilon}{2} \\ &\le \int_a^c f(x)\,dx + \int_c^b f(x)\,dx + \varepsilon, \end{aligned}$$

and so

$$\int_a^b f(x)\,dx - \left(\int_a^c f(x)\,dx + \int_c^b f(x)\,dx\right) < \varepsilon \tag{*}$$

Since

$$\begin{aligned} \int_a^b f(x)\,dx = L(f) \ge L(f,P) &= L(f,P_1) + L(f,P_2) \\ &> U(f,P_1) - \frac{\epsilon}{2} + U(f,P_2) - \frac{\varepsilon}{2} \\ &\ge \int_a^c f(x)\,dx + \int_c^b f(x)\,dx - \varepsilon, \end{aligned}$$

we have

$$-\varepsilon < \int_a^b f(x)\,dx - \left(\int_a^c f(x)\,dx + \int_c^b f(x)\,dx\right). \tag{**}$$

Hence, by (*) and (**)

$$\left|\int_a^b f(x)\,dx - \left(\int_a^c f(x)\,dx + \int_c^b f(x)\,dx\right)\right| < \varepsilon \implies \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

□

Theorem 2.9. *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable function on $[a,b]$ and $g : [c,d] \to \mathbb{R}$ be a continuous function on $[c,d]$. If $f[I] \subseteq [c,d]$, then $g \circ f$ is Riemann integrable function.*

Proof. PASS. ◻

Corollary 2.9.1. *If $f : [a,b] \to \mathbb{R}$ be Riemann integrable function on $[a,b]$, then f^n is Riemann integrable.*

Corollary 2.9.2. *If $f : [a,b] \to \mathbb{R}$ be Riemann integrable function on $[a,b]$, then $|f|$ is Riemann integrable and*

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Proof. Let $x \in [a,b]$ then

$$-|f(x)| \le f(x) \le |f(x)| \implies -\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$$
$$\implies \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

◻

Intermediate Value Theorem for Integrals

Theorem 2.10. *Let f be a continuous function on $[a,b]$, then for at least one $x \in [a,b]$ we have*

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Proof. Since f is continuous on $[a,b]$,

$$\exists M = \max\left\{f(x) : x \in [a,b]\right\}, m = \min\left\{f(x) : x \in [a,b]\right\} \in \mathbb{R} : \forall t \in [a,b] : m \le f(t) \le M.$$

Then

$$m(b-a) = \int_a^b m\,dx \le \int_a^b f(t)\,dt \le \int_a^b M\,dt = M(b-a),$$

and so

$$m \le \frac{1}{b-a}\int_a^b f(t)\,dt \le M.$$

Then Bolzano's IVT,

$$\exists x \in [a,b] : f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

◻

2.3 The Fundamental Theorem of Calculus

⋆ Fundamental Theorem of Calculus: 1st form ⋆

Theorem 2.11. *Let $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and f' is Riemann integrable on $[a, b]$. Then*

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Proof. We want to show that

$$(\forall \varepsilon > 0) \quad \left| \int_a^b f'(x)\,dx - \big(f(b) - f(a)\big) \right| < \varepsilon.$$

Let $\varepsilon > 0$. Since f' is Riemann integrable on $[a, b]$,

$$\exists P = \{x_0, \ldots, x_n\} : \begin{cases} U(f', P) < U(f') + \varepsilon & \because U(f', P) > U(f') \\ L(f', P) < L(f') - \varepsilon & \because L(f', P) < L(f'). \end{cases}$$

Since f is differentiable on $[x_{i-1}, x_i]$, by Mean-Value Theorem, $\exists t_i \in [x_{i-1}, x_i]$ s.t.

$$f(x_i) - f(x_{i-1}) = f'(t_i)(x_i - x_{i-1}) \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Then

$$\sum_{i=1}^n f'(t_i)\Delta x_i = \sum_{i=1}^n \big[f(x_i) - f(x_{i-1})\big] = f(x_n) - f(x_0) = f(b) - f(a).$$

Since $m_i[f'] \le f'(t_i) \le M_i[f']$, we have

$$\begin{aligned}
&L(f', P) = \sum_{i=1}^n m_i[f']\Delta x_i \le \sum_{i=1}^n f'(t_i)\Delta x_i \le \sum_{i=1}^n M_i[f']\Delta x_i = U(f', P) \\
\Longrightarrow\ & L(f') - \varepsilon < L(f', P) \le f(b) - f(a) \le U(f', P) < U(f') + \varepsilon \\
\Longrightarrow\ & -\varepsilon < f(b) - f(a) - \int_a^b f'(x)\,dx < \varepsilon \quad \because U(f', P) = \int_a^b f'(x)\,dx = L(f', P) \\
\Longrightarrow\ & \left| f(b) - f(a) - \int_a^b f'(x)\,dx \right| < \varepsilon.
\end{aligned}$$

□

Example 2.4. If $g(x) = \tan^{-1} x$ for all $x \in [a, b]$ then $g'(x) = (x^2 + 1)^{-1}$ for all $x \in [a, b]$. Further, g' is continuous so it is Riemann integrable on $[a, b]$. Therefore, the fundamental theorem implies that

$$\int_a^b \frac{1}{x^2 + 1}\,dx = g(b) - g(a) = \tan^{-1}(b) - \tan^{-1}(a).$$

Example 2.5. If $h(x) = 2\sqrt{x}$ for all $x \in [0, b]$ then h is continuous on $[0, b]$ and $h(x) = (\sqrt{x})^{-1}$ for all $x \in (0, b]$. Since h' is not bounded on $(0, b]$, it is not Riemann integrable on $[0, b]$ no matter how we define $h(0)$. Therefore, the fundamental theorem cannot be applied. Note that

$$\int_0^b \frac{1}{\sqrt{x}}\,dx = \lim_{a \to 0+} \int_a^b \frac{1}{\sqrt{x}}\,dx.$$

Indefinite Integral

Definition 2.6. Let $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$. The function defined by

$$F(x) := \int_a^x f(t)\, dt \quad \text{for} \quad x \in [a, b]$$

is called **indefinite integral** of f with base-point a.

Lipschitz Function

Definition 2.7. A function $f : D \to \mathbb{R}$ is said to be a **Lipschitz function** or to satisfy a **Lipschitz condition** on D if

$$\exists K > 0 : \left| f(x) - f(y) \right| \leq K \left| x - y \right|.$$

Theorem 2.12. *If $f : D \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on D.*

Theorem 2.13. *If $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, then, indefinite integral F of is uniformly continuous on $[a, b]$.*

Proof. Let $x, y \in [a, b]$ with $y < x$:

Then

$$F(x) := \int_a^x f(t)\, dt = \int_a^y f(t)\, dt + \int_y^x f(t)\, dt \implies F(x) - F(y) = \int_y^x f(t)\, dt.$$

Since f is Riemann integrable on $[a, b]$ and is bounded on $[a, b]$, we have

$$\exists K > 0 : \forall t \in [a, b] : \left| f(t) \right| \leq K,$$

and so

$$\begin{aligned}
& -K \leq f(t) \leq K \\
\implies & \int_y^x (-K)\, dt \leq \int_y^x f(t)\, dt \leq \int_y^x K\, dt \\
\implies & -K(x - y) \leq F(x) - F(y) \leq K(x - y) \\
\implies & \left| F(x) - F(y) \right| \leq K \left| x - y \right|,
\end{aligned}$$

Thus F is a Lipschitz function on $[a, b]$, and so F is uniformly continuous on $[a, b]$. $\square$

⋆ Fundamental Theorem of Calculus: 2nd form ⋆

Theorem 2.14. *Let $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and continuous at a point $c \in [a, b]$. Then the indefinite integral F is differentiable at c and*

$$F'(c) = f(c).$$

Proof. We will show that $\lim_{h\to 0+} \frac{F(c+h)-F(c)}{h} = f(c)$, i.e.,

$$(\forall \varepsilon > 0)(\exists \delta > 0) : h \in (0, \delta) \implies \left| \frac{F(c+h) - F(c)}{h} - f(c) \right| < \varepsilon.$$

Let $\varepsilon > 0$ and $c \in [a, b)$. Consider the right-hand derivative. Since f is right-continuous at c,

$$\exists \delta > 0 : x \in [c, c+\delta) \implies |f(x) - f(c)| < \varepsilon.$$

Let $h \in \mathbb{R}$ satisfies $0 < h < \delta$, say, $h = x - c$. Then f is Riemann integrable on $[a, c+h]$, $[a, c]$ and $[c, c+h]$. Then

$$\begin{aligned} F(c+h) - F(c) &= \int_a^{c+h} f(t)\, dt - \int_a^c f(t)\, dt \\ &= \int_c^{c+h} f(t)\, dt. \end{aligned}$$

Since $c \le t \le c + h < c + \delta$, we know

$$|f(t) - f(c)| < \varepsilon, \quad \text{i.e.,} \quad f(c) - \varepsilon < f(t) < f(c) + \varepsilon.$$

Thus,

$$\begin{aligned} &\int_c^{c+h} \big(f(t) - \varepsilon\big)\, dt < \int_c^{c+h} f(t)\, dt < \int_c^{c+h} \big(f(t) + \varepsilon\big)\, dt \\ \implies &\big(f(c) - \varepsilon\big)\, h < F(c+h) - F(c) < \big(f(c) + \varepsilon\big)\, h \\ \implies &-\varepsilon < \frac{F(c+h) - F(c)}{h} - f(c) < \varepsilon \\ \implies &\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| < \varepsilon. \end{aligned}$$

□

Theorem 2.15. *If f is continuous on $[a, b]$, then the indefinite integral*

$$F(x) := \int_a^x f(t)\, dt \quad \textit{for} \quad x \in [a, b]$$

is differentiable on $[a, b]$ and

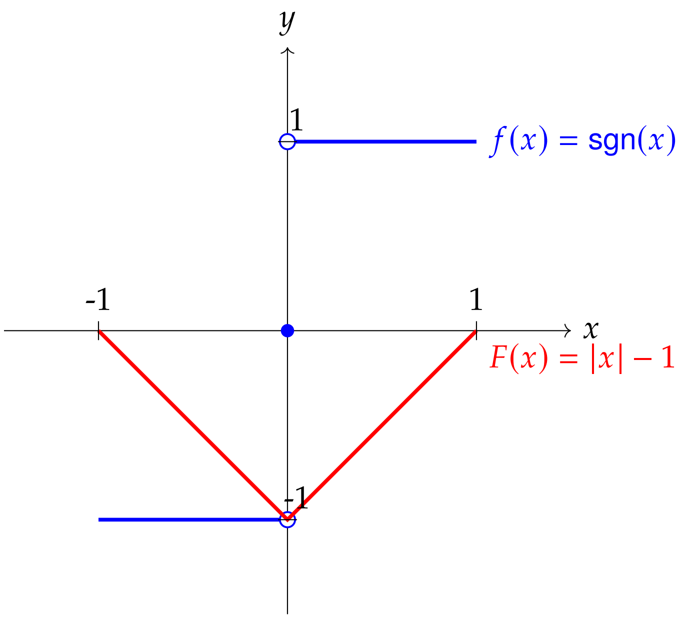
$$F'(x) = f(x)$$

for all $x \in [a, b]$.

Example 2.6. If $f(x) := \mathsf{sgn}(x)$ on $[-1,1]$, then f is Riemann integrable and has the indefinite integral

$$F(x) := |x| - 1$$

with the basepoint -1. However, since $F'(0)$ does not exist, F is not an anti-derivative of f on $[-1,1]$.



Example 2.7. For $x \in [0,3]$, if we define

$$F(x) := \int_0^x \lfloor t \rfloor \, dt$$

then although $f(x) = \lfloor x \rfloor$ is discontinuous on $[0,3]$, F is continuous on $[0,3]$.

Substitution Theorem

Theorem 2.16. *Let $J := [a, b]$ and let $g : J \to \mathbb{R}$ have a continuous derivative on J. If $f : I \to \mathbb{R}$ is continuous on an interval I containing $g(J)$ then*

$$\int_a^b f(g(t)) \cdot g'(t)\, dt = \int_{g(a)}^{g(b)} f(x)\, dx.$$

Proof. Since $g'(t)$ and $f(g(t))$ are both continuous on J, $f(g(t)) \cdot g'(t)$ is continuous on J. Thus $\int_a^b f(g(t)) \cdot g'(t)\, dt$ exists.

(1) Assume that g is constant. Since $g'(t) = 0$ and $g(a) = g(b)$,

$$\int_a^b f(g(t)) \cdot g'(t)\, dt = 0 = \int_{g(a)}^{g(b)} f(x)\, dx.$$

(2) Let g is not a constant. Then for $x \in g[J] \subseteq I$, define

$$F(x) := \int_{g(a)}^{x} f(s)\, ds.$$

By the Fundamental Theorem of Calculus: 2nd form,

$$\frac{d}{dx} F(x) = f(x).$$

and then

$$\frac{d}{dt}(F \circ g)(t) = \frac{d}{dt} F(g(t)) \frac{d}{dt} g(t) = f(g(t)) g'(t).$$

Thus

$$\begin{aligned}
\int_a^b f(g(t)) \cdot g'(t)\, dt &= \int_a^b (F \circ g)'(t)\, dt \\
&= (F \circ g)(b) - (F \circ g)(a) \\
&= F(g(b)) - F(g(a)) \\
&= \int_{g(a)}^{g(b)} f(x)\, dx - \int_{g(a)}^{g(a)} f(x)\, dx \\
&= \int_{g(a)}^{g(b)} f(x)\, dx.
\end{aligned}$$

$\square$

Example 2.8. Consider the integral

$$\int_1^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\, dt.$$

Let us substitution $g(t) := \sqrt{t}$ for $t \in [1,4]$ so that $g'(t)$ is continuous on $[1,4]$. If we let $f(x) := 2\sin x$ then the integrand has the form $f(g(t))g'(t)$. Then the integral equals

$$\int_1^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\, dt = \int_1^2 2\sin x\, dx = 2(\cos 1 - \cos 2).$$

However, if one consider the integral

$$\int_0^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\, dt,$$

the substitution theorem cannot be applicable since $g(t) := \sqrt{t}$ does not have a continuous derivative on $[0,4]$. Note that

$$\int_0^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\, dt = \lim_{a\to 0+} \int_a^4 \frac{a}{4} f(t)\, dt.$$

Integration by Parts

Theorem 2.17. *Let f, g be differentiable on $[a,b]$ and f', g' are Riemann integrable on $[a,b]$. Then*

$$\int_a^b f(x)g'(x)\, dx = \left[f(x)g(x)\right]_a^b - \int_a^b f'(x)g(x)\, dx.$$

Remark 2.2. $\int fg' = \int (fg)' - \int f'g$.

Taylor's Theorem with the Remainder

Theorem 2.18. *Suppose that $f', f'', \ldots, f^{(n)}, f^{(n+1)}$ exist on $[a,b]$ and that $f^{(n+1)}$ is Riemann integrable on $[a,b]$. Then we have*

$$f(b) = \sum_{i=0}^{n} \frac{f^{(n)}(a)}{n!}(b-a)^n + R_n$$

where the remainder R_n is given by

$$R_n = \frac{1}{n!}\int_a^b f^{(n+1)}(t)\cdot(b-t)^n\, dt.$$

2.4 Improper Integrals

Improper Integral

Definition 2.8. Let f be a function and $c \in (a, b)$.

(1) Let $f : [a, b) \to \mathbb{R}$ is Riemann integral on $[a, c]$. We say that f is **improper integrable** on $[a, b)$ if

$$\exists \lim_{c \to b-} \int_a^c f(x)\, dx \in \mathbb{R}.$$

(2) Let $f : (a, b] \to \mathbb{R}$ is Riemann integral on $[c, b]$. We say that f is also **improper integrable** on $(a, b]$ if

$$\exists \lim_{c \to a+} \int_c^b f(x)\, dx \in \mathbb{R}.$$

Example 2.9. Let $f(x) := x^{-\frac{1}{3}}$ for $x \in (0, 1]$. Since f is unbounded on $(0, 1]$, f is not Riemann integrable. However, for every $c \in (0, 1)$,

$$\lim_{c \to 0+} \int_c^1 x^{-\frac{1}{3}}\, dx = \lim_{c \to 0+} \frac{3}{2}(1 - c^{2/3}) = \frac{3}{2}.$$

Hence f is improper integrable on $(0, 1]$.

Example 2.10. Let $g(x) := x^{-1}$ for $x \in (0, 1]$. Then for every $c \in (0, 1)$,

$$\lim_{c \to 0+} \int_c^1 x^{-1}\, dx = \lim_{c \to 0+} (-\ln c) = \infty.$$

Hence g is not improper integrable on $(0, 1]$.

Definition 2.9. Let f be defined on $[a, \infty)$ and Riemann integrable on $[a, b]$ for every $b > a$. Then f is improper integrable on $[a, \infty)$ if

$$\exists \lim_{b \to \infty} \int_a^b f(x)\, dx$$

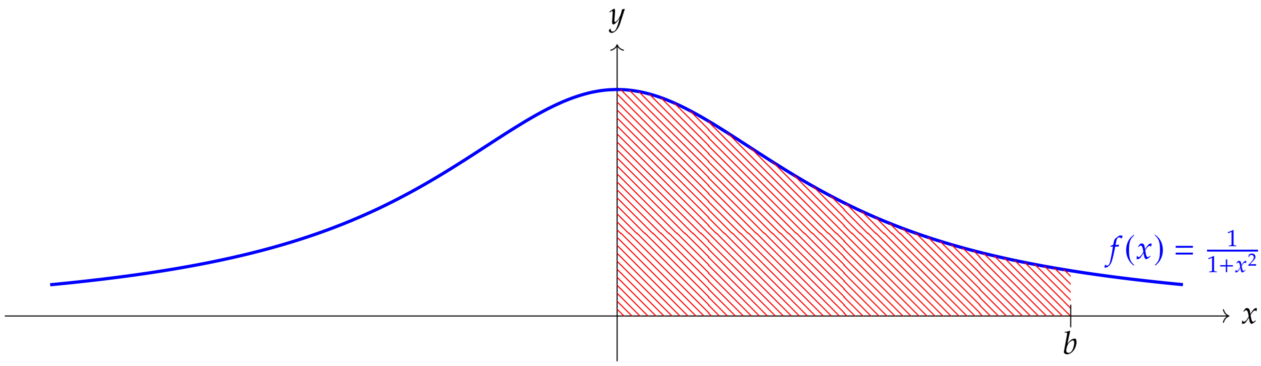
and

$$\int_a^\infty f(x)\, dx = \lim_{b \to \infty} \int_a^b f(x)\, dx.$$

Similarly, one can define

$$\int_{-\infty}^b f(x)\, dx = \lim_{a \to -\infty} \int_a^b f(x)\, dx.$$

Example 2.11. Let
$$f(x) := \frac{1}{1+x^2}.$$
Then f is well-defined and bound on $[0, \infty)$.



Moreover f is Riemann integrable on $[0, b]$ for every $b > 0$ since f is continuous on $[0, \infty)$. Since
$$\lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\, dx = \lim_{b\to\infty} \left(\tan^{-1}(b) - \tan^{-1}(0)\right) = \lim_{b\to\infty} \tan^{-1}(b) = \frac{\pi}{2},$$
we obtain
$$\int_0^\infty \frac{1}{1+x^2}\, dx = \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\, dx = \frac{\pi}{2}.$$
Note that
$$\int_{-\infty}^\infty \frac{1}{1+x^2}\, dx = \pi \implies \frac{1}{\pi}\int_{-\infty}^\infty \frac{1}{1+x^2}\, dx = 1,$$
and so $g(x) := \dfrac{1}{\pi(1+x^2)}$ be a p.d.f

Example 2.12. Since
$$\int_0^\infty f(x)\, dx = \int_0^1 \frac{1}{\sqrt{x}}\, dx + \int_1^\infty \frac{1}{\sqrt{x}}\, dx \quad \text{and} \quad \int_1^\infty \frac{1}{\sqrt{x}}\, dx = \infty,$$
$f = x^{-1/2}$ is not improper integrable on $(0, \infty)$.

Comparison Test

Theorem 2.19. *Let $f, g : [a, \infty) \to \mathbb{R}$. For every $b > a$, f and g are Riemann integrable on $[a, b]$. Then if for $\geq a$, $f(x) \in \big[0, g(x)\big]$ and g is improper integrable on $[a, \infty)$, then f is improper integrable on $[a, \infty)$ and*

$$\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx.$$

Proof. For $b > a$, define

$$F(b) := \int_a^b f(x)\,dx \quad \text{and} \quad G(b) := \int_a^b g(x)\,dx.$$

Since $0 \leq f(x) \leq g(x)$ and $\exists \lim\limits_{b\to\infty} G(b)$,

$$0 \leq F(b) \leq G(b) \leq \lim_{b\to\infty} G(b).$$

Let

$$A := \left\{\int_a^c f(x)\,dx : a \leq c\right\}$$

then

(i) $\exists \int_a^b f(x)\,dx \implies A \neq \emptyset$ and

(ii) A has an upper bound $\lim\limits_{b\to\infty} G(b)$.

By the completeness axiom of real number,

$$\exists \sup A = \lim_{b\to\infty} F(b) = \int_a^\infty f(x)\,dx,$$

i.e., f is improper integrable on $[a, \infty)$. Moreover,

$$\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx.$$

□

Theorem 2.20. *Let $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ for every $b > a$. Then*

$$\exists M \in \mathbb{R}^+ : \int_a^\infty \big|f(x)\big|\,dx \leq M \implies \exists \int_a^\infty f(x)\,dx \,\exists \int_a^\infty \big|f(x)\big|\,dx.$$

2.5 Exercises

Exercise 2.1. Generate a function f which is bounded but isn't integrable on $[a,b]$.

Solution. Let $f:[a,b]\longrightarrow\mathbb{R}$ be a Dirichlet's discontinuous function defined by

$$f(x) := \begin{cases} 1 & : x \in \mathbb{Q} \\ 0 & : x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then, f is bounded on $[a,b]$ but f is not Riemann integrable. □

Exercise 2.2. Give an example of an integrable function f that $f(x_0) > 0$ for $x_0 \in [a,b]$ but such that $\displaystyle\int_a^b f(x)\,dx = 0$.

Solution. Let $f:[a,b]\longrightarrow\mathbb{R}$ be a function defined by

$$f(x) := \begin{cases} 1 & : x = x_0 \\ 0 & : x \in [a,b] \setminus \{x_0\}\,. \end{cases}$$

Then $f(x_0) > 0$ but $\displaystyle\int_a^b f(x)\,dx = 0$. □

Exercise 2.3. Given an example of a function $f:[0,1]\longrightarrow\mathbb{R}$ that isn't Riemann integrable but such that $|f|$ is Riemann integrable on $[0,1]$.

Solution. Let $f:[0,1]\longrightarrow\mathbb{R}$ be a function defined by

$$f(x) := \begin{cases} 1 & : x \in \mathbb{Q} \\ -1 & : x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then, f is not Riemann integrable on $[a,b]$ but $|f|$ is Riemann integrable on $[0,1]$. □

Exercise 2.4. Assume that $f:[a,b]\longrightarrow\mathbb{R}$ is Riemann integrable on $[a,b]$. For $x\in[a,b]$, let

$$F(x) = \int_a^x f(t)\,dt$$

then show that F is **Lipschitz** function on $[a,b]$.

Solution. **Theorem 2.13** □

Exercise 2.5. If f and g are continuous on $[a,b]$ and if

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx,$$

prove that there exists $c \in [a,b]$ such that $f(c) = g(c)$.

Solution. Since f and g are continuous on $[a,b]$, $f-g$ is also continuous on $[a,b]$. By the intermediate value theorem for integrals,

$$\exists c \in [a,b] : (f-g)(c) = \frac{1}{b-a}\int_a^b \big(f(x) - g(x)\big)\ dx.$$

Since $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$,

$$(f-g)(c) = 0 \implies f(c) = g(c).$$

□

Exercise 2.6. If f is continuous on $[-a,a]$, show that $\displaystyle\int_{-a}^a f(x^2)\,dx = 2\int_0^a f(x^2)\,dx$.

Solution. Since

$$\int_{-a}^a f(x^2)\,dx = \int_{-a}^0 f(x^2)\,dx + \int_0^a f(x^2)\,dx,$$

and by the substitution theorem yields

$$\int_{-a}^0 f(x^2)\,dx \overset{x=-t}{=} \int_a^0 f(t^2)(-\,dt) = \int_0^a f(t^2)\,dt.$$

Hence $\int_{-a}^a f(x^2)\,dx = \int_0^a f(t^2)\,dt + \int_0^a f(x^2)\,dx = 2\int_0^a f(x^2)\,dx$. □

Exercise 2.7. Prove that $f(x) = \dfrac{e^{-x}}{1+x^2}$ is improper integrable on $[0,\infty)$.

Solution. Let

$$g(x) := \frac{1}{1+x^2}$$

for $x \in [0,\infty)$. Note that g is improper integrable on $[0,\infty)$ and $\int_0^\infty g(x)\,dx = \frac{\pi}{2}$. Since $e^{-x} \le 1$ on $[0,\infty)$,

$$0 \le f(x) \le g(x).$$

Therefore, $f(x)$ is improper integrable on $[0,\infty)$ and $\int_0^\infty \frac{e^{-x}}{1+x^2} \le \frac{\pi}{2}$. □

Exercise 2.8. Prove that $\int_1^\infty \frac{1}{x^p}\,dx$ diverges when $p \le 1$ and converges when $p > 1$.

Solution. Since

$$\begin{aligned}\int_1^b \frac{1}{x}\,dx &= \ln b - \ln 1,\\ \int_1^b \frac{1}{x^p}\,dx &= \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right) \quad \text{for} \quad p \neq 1.\end{aligned}$$

We can see that the improper integral converges if $p > 1$ and diverges if $p \le 1$. □

Exercise 2.9. Prove that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

[Hint: use the polar coordinate system.]

Solution. Let $I = \int_0^\infty e^{-x^2}\,dx$. Then

$$\begin{aligned}I^2 &= \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-x^2}\,dx\right)\\ &= \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)\\ &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx dy\\ &= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\,r\,dr\,d\theta\\ &= \int_0^{\frac{\pi}{2}} \left[-\frac{1}{2}\cdot e^{-r^2}\right]_0^\infty d\theta\\ &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\right) d\theta\\ &= \left[\frac{1}{2}x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.\end{aligned}$$

Since $e^{-x^2} \ge 0$, we have

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-x^2}\,dx\right) = \frac{\pi}{4} \implies I = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

□

Exercise 2.10. Suppose that f is continuous on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$. Show that if $\int_a^b f(x)\, dx = 0$ then $f(x) = 0$ for all $x \in [a, b]$.

Solution. Assume that $f(x_0) \neq 0$. □

Exercise 2.11.

Solution. □

Exercise 2.12. Let f and g be Riemann integrable on $[a, b]$. Then show that fg is Riemann integrable on $[a, b]$.

Solution. Since $(f+g)^2$ and $(f-g)^2$ are Riemann integrable on $[a, b]$,

$$fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$$

is Riemann integrable on $[a, b]$. □

Chapter 3

Sequence of Functions

3.1 Pointwise and Uniform Convergence

3.2 Interchange of Limits

3.3 Series of Functions

3.4 Power Series

3.5 Exercises

Exercise 3.1. For $x \in [0, 1]$,

$$f_n(x) = \frac{1}{n^2x^2 + 1}.$$

Then show that $\{f_n\}$ converges pointwise on [0, 1] but not converges uniformly.

Solution. □

Exercise 3.2. For $x \in [0, 1]$,

$$f_n(x) = nx(1 - x)^n.$$

Then show that $\{f_n\}$ converges pointwise on [0, 1] but not converges uniformly.

Solution. □

Exercise 3.3. For $x \in \mathbb{R}$, let

$$f_n(x) = \frac{nx}{1 + n^2x^2}.$$

Then show that for $a > 0$, $\{f_n\}$ converges uniformly on $[a, \infty)$ but does not converge uniformly on $[0, \infty)$.

Solution. □

Exercise 3.4. Let $I = (-\infty, \infty)$. Then for $f_n(x) = n^3e^{-n}(1 - 2\sin^2 nx)$, the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly.

Solution. □

Exercise 3.5. Let $I = [0, 1]$. Then, if $f_n(x) = x^n$, $n \in \mathbb{N}$ show that the series

$$\sum_{n=1}^{\infty} f_n(x)$$

is a pointwise convergent but not a uniformly convergent series.

Solution. □

Exercise 3.6. Evaluate the radius of convergence of following power series

1. $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$,
2. $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) x^n$,
3. $\sum_{n=0}^{\infty} n^2(x-4)^n$.
4. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 3^n}$,
5. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$,
6. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n$.

Solution. □

Exercise 3.7. Evaluate the radius of convergence of following power series

1. $\sum_{n=0}^{\infty} \frac{x^n}{3^n + 5^n}$,
2. $\sum_{n=0}^{\infty} \frac{n^3 x^n}{n!}$.

Solution. □

Exercise 3.8. Let the radius of convergence of following power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is R then evaluate the radius of convergence of

$$\sum_{n=0}^{\infty} a_n x^{-n}.$$

Solution. □

Exercise 3.9. Suppose that the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is R. What is the radius of convergence of the following power series?

$$\sum_{n=0}^{\infty} a_n x^{2n}.$$

Solution. □

Chapter 4

Introduction to Fourier Series and Transform

4.1 Periodic Functions and Trigonometric Series

Periodic Functions

Definition 4.1. A function $f(x)$ is called **periodic** if

(1) it is defined for all $x \in \mathbb{R}$ and

(2) if $\exists p > 0$ such that

$$f(x+p) = f(x).$$

This number p is called a **period** of $f(x)$.

Trigonometric Series

Definition 4.2. The series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called a **trigonometric series**, and the a_n and b_n are called the coefficients of the series,where $a_0, a_1, a_2, \cdots, b_1, b_2, \cdots$ are real constants.

Remark 4.1.

- Fourier series arise from the practical task of representing a given periodic function $f(x)$ in terms of cosine and sine functions.
- These series are trigonometric series whose coefficients are determined from $f(x)$ by the Euler formulas, which we shall derive first.
- Afterwards we shall take a look at the theory of Fourier series.

4.2 Fourier Series

Fourier Series of a Periodic Function of Period 2π

Theorem 4.1. *Assume that $f(x)$ is a periodic function of period 2π and is integrable over a period, that is,*

$$f(x+2\pi) = f(x) \quad \textit{and} \quad \exists \int_x^{x+2\pi} f(t)\,dt = \int_{-\pi}^{\pi} f(x)\,dx.$$

Then, $f(x)$ can be represented by a trigonometric series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right),$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

for all $n \in \mathbb{N}$.

Proof. (1) Since

$$\int_{-\pi}^{\pi} \cos nx\,dx = 0 = \int_{-\pi}^{\pi} \sin nx\,dx,$$

we have

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\,dx &= \int_{-\pi}^{\pi} a_0\,dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)\,dx \\
&= \int_{-\pi}^{\pi} a_0\,dx + \cancelto{0}{\sum_{n=1}^{\infty} \int_{-\pi}^{\pi} \left(a_n \cos nx + b_n \sin nx\right)\,dx} \\
&= \int_{-\pi}^{\pi} a_0\,dx = 2\pi a_0.
\end{aligned}$$

and so $a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx$.

(2) Let $m \in \mathbb{N}$. Then

$$f(x)\cos mx = a_0 \cos mx + \cos mx \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right),$$

$$\int_{-\pi}^{\pi} f(x)\cos mx = \cancelto{0}{\int_{-\pi}^{\pi} a_0 \cos mx} + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} \left(a_n \cos nx \cos mx + b_n \sin nx \cos mx\right).$$

Note that

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \cancelto{0}{\frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x \, dx} + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x \, dx,$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \cancelto{0}{\frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)x \, dx} + \cancelto{0}{\frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)x \, dx},$$

and

$$\frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x \, dx = \begin{cases} 2\pi & : n = m, \\ 0 & : n \neq m. \end{cases}$$

Thus

$$\int_{-\pi}^{\pi} f(x) \cos mx = \frac{1}{2} \cdot 2\pi a_m = \pi a_m \overset{n=m}{\Longrightarrow} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx.$$

(3) Similarly, we have

$$\int_{-\pi}^{\pi} f(x) \sin mx = \pi b_m \overset{n=m}{\Longrightarrow} b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx.$$

□

Example 4.1 (Rectangular Wave)**.** Find the Fourier series of the periodic function $f(x)$ defined by

$$f(x) := \begin{cases} -k & : -\pi < x < 0 \\ k & : 0 < x < \pi \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x).$$

Solution. $a_0 = 0$, $a_n = 0$ and

$$b_n = \begin{cases} \dfrac{4k}{(2k+1)\pi} & : n = 2k+1, \\ 0 & : n = 2k. \end{cases}$$

□

Remark 4.2 (**The Gibbs' phenomenon**)**.** Its sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left-and right-hand limits of $f(x)$ at x_0. In other words, if f is not continuous at x_0 then

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx_0 + b_n \sin nx_0) = \frac{1}{2}\left(f(x_0+) + f(x_0-)\right).$$

Representation by a Fourier series

Theorem 4.2. *If a periodic function $f(x)$ with period 2π is*

(1) having continuous first and second derivatives,

(2) piecewise continuous in the interval $[-\pi, \pi]$,

(3) having a left-hand derivative and right-hand derivative at each point of that interval,

then the Fourier series of $f(x)$ is convergent.

Solution. Since

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \cancelto{0}{\frac{1}{\pi}\left[f(x)\cdot\frac{1}{n}\sin nx\right]_{-\pi}^{\pi}} - \frac{1}{n\pi} f'(x)\sin nx\,dx \\
&= -\frac{1}{n\pi}\left[f'(x)\cdot\frac{-1}{n}\cos nx\right]_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f''(x)\left(-\frac{1}{n}\cos nx\right)\,dx \\
&= \cancelto{0}{\frac{1}{n^2\pi}\left[f'(x)\cos nx\right]_{-\pi}^{\pi}} + \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\cos nx\,dx
\end{aligned}$$

and f'' is continuous on $[-\pi, \pi]$, we have $\exists M > 0$ s.t. $\left|f''(x)\right| \le M$. It follow that

$$|a_n| = \frac{1}{n^2\pi}\left|\int_{-\pi}^{\pi} f''(x)\cos nx\,dx\right| < \frac{1}{n^2\pi}\int_{-\pi}^{\pi} M\,dx = \frac{2M}{n^2}.$$

Similarly, $|b_n| < \dfrac{2M}{n^2}$. Thus,

$$\begin{aligned}
\left|f(x)\right| = \left|a_0 + \sum_{i=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right| &\le |a_0| + \sum_{n=1}^{\infty}\left(|a_n| + |b_n|\right) \\
&\le |a_0| + \sum_{n=1}^{\infty}\frac{4M}{n^2}.
\end{aligned}$$

Let $M_n := \dfrac{4M}{n^2}$ for $n \in \mathbb{N}$. Since $\exists |a_0| + \sum_{n=1}^{\infty} M_n$, by Weierstrass M-test,

$$\left|f(x)\right| \text{ converges} \implies f(x) \text{ converges uniformly on } [-\pi, \pi].$$

$\square$

Note (Review)**.** For $\mathbf{a} = (1, 2, 3) \in \mathbb{R}^3$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a orthonormal basis for $\mathbb{R}^3$. Then

$$\begin{cases} \mathbf{e}_1 = (1,0,0) \\ \mathbf{e}_2 = (0,1,0) \\ \mathbf{e}_3 = (0,0,1) \end{cases} \implies \mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 = (\mathbf{a}\cdot\mathbf{e}_1)\mathbf{e}_1 + (\mathbf{a}\cdot\mathbf{e}_2)\mathbf{e}_2 + (\mathbf{a}\cdot\mathbf{e}_3)\mathbf{e}_3 = \sum_{n=1}^{3}(\mathbf{a}\cdot\mathbf{e}_n)\mathbf{e}_n.$$

Note (Orthogonality Property of the Trigonometric System)**.** Let us define an inner product on the interval $[-\pi, \pi]$ such that

$$\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x)g(x)\, dx.$$

Here, we have

(1) $\langle 1, 1 \rangle = 2\pi$.

(2) $\langle 1, \cos nx \rangle = \displaystyle\int_{-\pi}^{\pi} \cos nx\, dx = 0.$

(3) $\langle 1, \sin nx \rangle = \displaystyle\int_{-\pi}^{\pi} \sin nx\, dx = 0.$

(4) $\langle \cos n, \sin nx \rangle = \pi = \langle \sin nx, \sin nx \rangle$.

(5) $\langle \cos mx, \cos nx \rangle = \displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = 0$ for $n \neq m$.

(6) $\langle \sin mx, \sin nx \rangle = \displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = 0$ for $n \neq m$.

(7) $\langle \cos mx, \sin nx \rangle = \displaystyle\int_{-\pi}^{\pi} \cos mx \sin nx\, dx = 0$ for any n, m.

Then the trigonometric system

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots, \cos nx, \sin nx, \cdots$$

is **orthogonal** on the interval $[-\pi, \pi]$. Moreover,

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \cdots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \cdots$$

is orthonormal. Note that

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \\ &= \frac{1}{2\pi}\langle f(x), 1 \rangle 1 + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\langle f(x), \cos x \rangle \cos x + \frac{1}{\pi}\langle f(x), \sin nx \rangle \sin nx\right) \end{aligned}$$

and that

$$f(x) = \left\langle f(x), \frac{1}{\sqrt{2\pi}} \right\rangle \cdot \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left(\left\langle f(x), \frac{\cos nx}{\sqrt{\pi}} \right\rangle \frac{\cos nx}{\sqrt{\pi}} + \left\langle f(x), \frac{\sin nx}{\sqrt{\pi}} \right\rangle \frac{\sin nx}{\sqrt{\pi}}\right).$$

4.3 Functions of Any Period $p = 2L$, Even and Odd Functions

Fourier Series of a Periodic Function of Period $2L$)

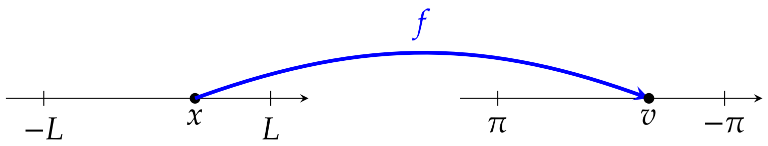
Theorem 4.3. *A function $f(x)$ of period $p = 2L$ has a* ***Fourier series****. This series can be written:*

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

with the ***Fourier coefficients*** *of $f(x)$ given by the* ***Euler formulas****, for $n = 1, 2, \cdots$,*

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \qquad \begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \end{aligned}$$

Proof.



Let $v = \frac{\pi}{L}x$. Then a function $g(v)$ defined by

$$f(x) = f\left(\frac{L}{\pi}v\right) =: g(v)$$

has period of 2π. Then $g(v)$ has the Fourier series

$$g(v) = a_0 + \sum_{i=0}^{\infty}(a_n \cos nv + b_n \sin nv) \quad \text{with} \quad \begin{cases} a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} g(v)\,dv, \\ a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} g(v)\cos nv\,dv, \\ b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} g(v)\sin nv\,dv. \end{cases}$$

Since $dv = \frac{\pi}{L}dx$, we have

$$f(x) = a_0 + \sum_{i=0}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \quad \text{with} \quad \begin{cases} a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx, \\ a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \\ b_n = \dfrac{1}{L}\displaystyle\int_{-\pi}^{\pi} f(x)\sin\frac{n\pi x}{L}\,dx. \end{cases}$$

$\square$

Example 4.2 (Example (Half-Wave Rectifier))**.** A sinusoidal voltage $E\sin\omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave. Let

$$u(t) = \begin{cases} 0 & \text{if } -L < t < 0, \\ E\sin\omega t & \text{if } 0 < t < L, \end{cases} \qquad p = 2L = \frac{2\pi}{\omega}.$$

Then, find the Fourier series of the periodic function $u(t)$.

Solution. The Fourier series of the $u(t)$ is

$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos 4\omega t + \cdots\right).$$

□

Even and Odd Functions

Definition 4.3. (1) A function $y = f(x)$ is **even** if

$$f(-x) = f(x), \quad x \in \mathbb{R}.$$

The graph of such a function is symmetric with respect to the y-axis.

(2) A function $g(x)$ is **odd** if

$$g(-x) = -g(x), \quad x \in \mathbb{R}.$$

The graph of such a function is symmetric with respect to the origin.

Remark 4.3. $f(x)$ and $g(x)$ satisfy $\displaystyle\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx$ and $\displaystyle\int_{-L}^{L} g(x)\,dx = 0$.

Fourier Cosine and Since Series

Theorem 4.4. *(1) The Fourier series of an even function of period 2L is a* ***Fourier cosine series***

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

with coefficients

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \ldots.$$

(2) The Fourier series of an odd function of period 2L is a ***Fourier sine series***

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

with coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \ldots.$$

Proof. (1) Since f is even,

$$\begin{cases} f(x)\cos x \text{ is even} \\ f(x)\sin x \text{ is odd} \end{cases} \Longrightarrow \begin{cases} a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx = \dfrac{1}{2L}\cdot 2\int_0^L f(x)\,dx = \dfrac{1}{L}\int_0^L f(x)\,dx, \\ a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \dfrac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \\ b_n = \dfrac{1}{L}\displaystyle\int_{-\pi}^{\pi} f(x)\sin\frac{n\pi x}{L}\,dx = 0. \end{cases}$$

(2) Similarly, it holds.

□

Example 4.3 (Sawtooth Wave)**.** Find the Fourier series of the function

$$f(x) = x + \pi$$

with $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$.

Solution. Let $f_1(x) := x$ then

$$f(x) = \pi + x = \pi + f_1(x).$$

Then since $f_1(x) = x$ is odd, $a_n = 0$ for $n = 0, 1, 2, \ldots$, and

$$b_n = \frac{2}{\pi} \int_0^\pi f_1(x) \sin \frac{n\pi x}{\pi}\, dx = \frac{2}{\pi} \int_0^\pi x \sin nx\, dx = -\frac{2}{n} \cos n\pi = (-1)^n \frac{2}{n}.$$

Hence, the Fourier series of $f(x)$ is

$$f(x) = \pi + f_1(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right).$$

$\square$

4.4 Introduction to Complex Fourier Series

Complex Fourier Series

Theorem 4.5. *For a function of period* $2L$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

the complex Fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(i\frac{n\pi x}{L} \right), \quad \text{where} \quad c_n = \frac{1}{2L} \int_{-L}^{L} f(x) \exp\left(-i\frac{n\pi x}{L} \right) dx$$

for $n = 0, \pm 1, \pm 2, \cdots$.

Proof. Recall the Euler formula

$$e^{inx} = \cos nx + i \sin nx \quad \text{and} \quad e^{-inx} = \cos nx - i \sin nx.$$

Then, we can examine that

$$\cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx} \right) \quad \text{and} \quad \sin nx = \frac{1}{2i}\left(e^{inx} - e^{-inx} \right).$$

Hence,

$$\begin{aligned} a_n \cos nx + b_n \sin nx &= a_n \frac{1}{2}\left(e^{inx} + e^{-inx} \right) + b_n \frac{1}{2i}\left(e^{inx} - e^{-inx} \right) \\ &= \frac{1}{2}(a_n - ib_n)e^{inx} + \frac{1}{2}(a_n + ib_n)e^{-inx}. \end{aligned}$$

Let us write

$$c_n = \frac{1}{2}(a_n - ib_n) \quad \text{and} \quad d_n = \frac{1}{2}(a_n + ib_n).$$

Then, we can write

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{inx} + d_n e^{-inx} \right),$$

where

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) (\cos nx - i \sin nx)\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx,$$

$$d_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) (\cos nx + i \sin nx)\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} dx.$$

Note that $d_n = c_{-n}$ for all n. □

Remark 4.4. The c_n are called the **complex Fourier coefficients** of $f(x)$.

4.5 Fourier Integrals

Fourier Integral

Theorem 4.6. *Let f be a nonperiodic function. Then,*

$$f(x) = \int_0^\infty \left(A(\omega)\cos\omega x + B(\omega)\sin\omega x\right) d\omega,$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v \, dv \quad \text{and} \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v \, dv.$$

*This is called a representation of $f(x)$ by a **Fourier integral**.*

Proof. (1) We consider any periodic function $f_L(x)$ of period $2L$ that can be represented by a Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\omega_n x + b_n\sin\omega_n x\right), \quad \omega_n = \frac{n\pi}{L},$$

and find out what happens if we let $L \to \infty$. Let us write, for $\omega_n = \frac{n\pi}{L}$,

$$\begin{aligned} f_L(x) =& a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\omega_n x + b_n\sin\omega_n x\right) \\ =& \frac{1}{2L}\int_{-L}^{L} f(v)\, dv \\ &+ \frac{1}{L}\sum_{n=1}^{\infty}\left(\cos\omega_n x \int_{-L}^{L} f(v)\cos\omega_n v\, dv\right) \\ &+ \frac{1}{L}\sum_{n=1}^{\infty}\left(\sin\omega_n x \int_{-L}^{L} f(v)\sin\omega_n v\, dv\right). \end{aligned}$$

We now set

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L} \quad \text{implies} \quad \frac{1}{L} = \frac{\Delta\omega}{\pi}.$$

Then we can write the Fourier series in the form

$$\begin{aligned} f_L(x) =& \frac{1}{2L}\int_{-L}^{L} f(v)\, dv \\ &+ \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\cos\omega_n x\, \Delta\omega \int_{-L}^{L} f(v)\cos\omega_n v\, dv\right) \\ &+ \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\sin\omega_n x\, \Delta\omega \int_{-L}^{L} f(v)\sin\omega_n v\, dv\right). \end{aligned}$$

This representation is valid for any fixed L, arbitrary large, but finite.

(2) Let $L \to \infty$ and assume that the resulting nonperiodic function

$$f(x) = \lim_{L\to\infty} f_L(x)$$

is absolutely integrable on the x-axis. Then

$$\frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv \to 0 \quad \text{and} \quad \Delta\omega = \frac{\pi}{L} \to 0.$$

It seems plausible that the infinite series becomes an integral from 0 to ∞, which represents $f(x)$,

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\cos\omega x \int_{-\infty}^{\infty} f(v)\cos\omega v\,dv + \sin\omega x \int_{-\infty}^{\infty} f(v)\sin\omega v\,dv\right] d\omega.$$

□

Example 4.4 (Single Pulse, Sine Integral)**.** Find the Fourier integral representation of the function

$$f(x) := \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1 \end{cases}$$

Solution. Since

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv = \frac{1}{\pi}\int_{-1}^{1}\cos\omega v\,dv = \frac{2\sin\omega}{\pi\omega} \quad \text{and} \quad B(\omega) = 0,$$

the Fourier integral representation of $f(x)$ is

$$F(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x\sin\omega}{\omega}\,d\omega.$$

□

Remark 4.5. The average of the left- and right-hand limits of $f(x)$ at $x = 1$ is equal to $1/2$. Furthermore, we obtain

$$\int_0^\infty \frac{\cos\omega x\sin\omega}{\omega}\,d\omega = \begin{cases} \frac{\pi}{2} & \text{if } 0 \le x < 1, \\ \frac{\pi}{4} & \text{if } x = 1, \\ 0 & \text{if } x > 1. \end{cases}$$

This integral is called **Dirichlet's discontinuous factor**. If $x = 0$, then

$$\int_0^\infty \frac{\sin\omega}{\omega}\,d\omega = \frac{\pi}{2}.$$

We see that this integral is the limit of the so-called **sine integral**

$$\mathrm{Si}(x) = \int_0^x \frac{\sin\omega}{\omega}\,d\omega$$

as $x \to \infty$.

Fourier Sine and Cosine Integrals

Theorem 4.7. *(1) If $f(x)$ is an even function, then $B(\omega) = 0$ and*

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos\omega v\,dv.$$

The Fourier integral then reduces to the ***Fourier cosine integral***

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\,d\omega.$$

(2) If $f(x)$ is an odd function, then $A(\omega) = 0$ and

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin\omega v\,dv.$$

The Fourier integral then reduces to the ***Fourier sine integral***

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\,d\omega.$$

Proof. (1) If $f(x)$ is an even function, then

$$\begin{aligned} f(x) &= \int_0^\infty \left(\frac{2}{\pi}\int_0^\infty f(v)\cos\omega v\,dv\right)\cos\omega x\,d\omega \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\omega)\cos\omega x\,d\omega \quad \text{with} \quad \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\,dx. \end{aligned}$$

(2) If $f(x)$ is an odd function, then

$$\begin{aligned} f(x) &= \int_0^\infty \left(\frac{2}{\pi}\int_0^\infty f(v)\sin\omega v\,dv\right)\sin\omega x\,d\omega \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(\omega)\sin\omega x\,d\omega \quad \text{with} \quad \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\omega x\,dx. \end{aligned}$$

□

Definition (Fourier Cosine and Sine Transforms)

Definition 4.4.

- **Fourier cosine transform** for an even function:

$$\mathcal{F}_c(f) = \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx.$$

- **Inverse Fourier cosine transform** of $\hat{f}_c(\omega)$:

$$\mathcal{F}_c^{-1}(f) = \hat{f}_c^{-1}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos \omega x \, d\omega.$$

- **Fourier sine transform** for an odd function:

$$\mathcal{F}_s(f) = \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx.$$

- **Inverse Fourier sine transform** of $\hat{f}_s(\omega)$:

$$\mathcal{F}_s^{-1}(f) = \hat{f}_s^{-1}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega) \sin \omega x \, d\omega.$$

Example 4.5. Find Fourier cosine and sine transforms of the function

$$f(x) := \begin{cases} k & : 0 < x < a, \\ 0 & : x > a \end{cases}$$

Solution. Fourier cosine and sine transforms of $f(x)$ are

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^a k \cos \omega x \, dx = k\sqrt{\frac{2}{\pi}} \left(\frac{\sin a\omega}{\omega}\right), \; \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^a k \sin \omega x \, dx = k\sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos a\omega}{\omega}\right).$$

□

Linearity of Fourier Cosine and Sine Transforms

Theorem 4.8. *The Fourier cosine and sine transforms are* ***linear operators*** *i.e., for $a, b \in \mathbb{R}$,*

$$\mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g), \quad \mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g).$$

Proof. Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned} \mathcal{F}_c(af + bg) &= \sqrt{\frac{2}{\pi}} \left(af(x) + bg(x)\right) \cos \omega x \, dx \\ &= a\left(\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx\right) + b\left(\sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos \omega x \, dx\right) \\ &= a\mathcal{F}_c(f) + b\mathcal{F}_c(g). \end{aligned}$$

Similarly, $\mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g)$. □

Cosine and Sine Transforms of Derivatives

Theorem 4.9. *Let $f(x)$ be continuous and absolutely integrable on the x-axis, let $f'(x)$ be piecewise continuous on each finite interval, and let $f(x) \to 0$ as $x \to \infty$. Then*

$$\mathcal{F}_c\{f'(x)\} = \omega \mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0) \quad \textit{and} \quad \mathcal{F}_s\{f'(x)\} = -\omega \mathcal{F}_c\{f(x)\}.$$

Furthermore,

$$\mathcal{F}_c\{f''(x)\} = -\omega^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$$

$$\mathcal{F}_s\{f''(x)\} = -\omega^2 \mathcal{F}_s\{f(x)\} + \omega \sqrt{\frac{2}{\pi}} f(0).$$

Proof. Note that

$$\begin{aligned}
\mathcal{F}_c(f') &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos \omega x \, dx \\
&= \sqrt{\frac{2}{\pi}} \left[f(x) \cos \omega x \right]_0^\infty - \sqrt{\frac{2}{\pi}} \int_0^\infty (-\omega \sin \omega x) \, dx \\
&= -\sqrt{\frac{2}{\pi}} f(0) + \omega \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx \\
&= -\sqrt{\frac{2}{\pi}} f(0) + \omega \mathcal{F}_s(f).
\end{aligned}$$

and that

$$\begin{aligned}
\mathcal{F}_s(f') &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin \omega x \, dx \\
&= -\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \omega \cos \omega x \, dx \\
&= -\omega \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx \\
&= -\omega \mathcal{F}_c(f).
\end{aligned}$$

Then

$$\mathcal{F}_c(f'') = \omega \mathcal{F}_s(f') - \sqrt{\frac{2}{\pi}} f'(0) = -\omega^2 \mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}} f'(0),$$

$$\mathcal{F}_s(f'') = -\omega \mathcal{F}_c(f') = -\omega \left(\omega \mathcal{F}_s(f) - \sqrt{\frac{2}{\pi}} f(0) \right) = -\omega^2 \mathcal{F}_s(f) + \omega \sqrt{\frac{2}{\pi}} f(0).$$

□

Example 4.6. Find the Fourier cosine transform of $f(x) = e^{-ax}$, where $a > 0$.

Solution. Let $f(x) = e^{-ax}$. Since $f''(x) = a^2 e^{-ax} = a^2 f(x)$,

$$a^2 \mathcal{F}_c(f) = \mathcal{F}_c(a^2 f) = \mathcal{F}_c(f'') = -\omega^2 \mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}} f'(0) = -\omega^2 \mathcal{F}_c(f) + a\sqrt{\frac{2}{\pi}}.$$

Therefore

$$(a^2 + \omega^2)\mathcal{F}_c(f) = a\sqrt{\frac{2}{\pi}} \implies \mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + \omega^2} \right).$$

□

Note (Fourier Transform). The Fourier integral is

$$f(x) = \int_0^\infty \left[A(\omega) \cos \omega x + B(\omega) \sin \omega x \right] d\omega, \quad \text{where} \quad \begin{cases} A(\omega) = \dfrac{1}{\pi} \displaystyle\int_{-\infty}^{\infty} f(v) \cos \omega v \, dv, \\ B(\omega) = \dfrac{1}{\pi} \displaystyle\int_{-\infty}^{\infty} f(v) \sin \omega v \, dv. \end{cases}$$

Then, we have

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^{\infty} f(v)(\cos \omega v \cos \omega x + \sin \omega v \sin \omega x) dv \right] d\omega \\ &= \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^{\infty} f(v) \cos(\omega x - \omega v) dv \right] d\omega \\ &= \frac{1}{\pi} \int_0^\infty F(w) d\omega \quad \text{with} \quad F(\omega) := \int_{-\infty}^{\infty} f(v) \cos(\omega x - \omega v) dv. \end{aligned}$$

Since $\cos(\omega x - \omega v)$ is an even function of ω, $F(\omega)$ is an even function. Thus $f(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} F(w) d\omega$.
Let

$$G(\omega) := \int_{-\infty}^{\infty} f(v) \sin(\omega x - \omega v) dv.$$

Since $\sin(\omega x - \omega v)$ is an odd function of ω, $G(\omega)$ is an odd function. Thus $\dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} G(\omega) d\omega = 0$.
Therefore,

$$\begin{aligned} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) d\omega + i \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} G(w) d\omega &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) \cos(\omega x - \omega v) dv \right] d\omega \\ &\quad + \frac{i}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) \sin(\omega x - \omega v) dv \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) e^{i(\omega x - \omega v)} dv \right] d\omega, \end{aligned}$$

and so

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} dv \right] e^{i\omega x} d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{i\omega x} d\omega \quad \text{with} \quad \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega v} dx. \end{aligned}$$

Fourier and Inverse Fourier Transforms

Definition 4.5.

- **Fourier transform** of f:

$$\mathcal{F}(f) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx.$$

- **Inverse Fourier transform** of $\hat{f}(\omega)$:

$$\mathcal{F}^{-1}(f) = \hat{f}^{-1}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega$$

Existence of the Fourier Transform

Theorem 4.10. *Sufficient for the existence of the Fourier transform are the following two conditions, as we mention without proof.*

(1) $f(x)$ is piecewise continuous on every finite interval.

(2) $f(x)$ is absolutely integrable on the x-axis.

Linearity of the Fourier Transform

Theorem 4.11. *The Fourier transform is a linear operation; that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and any constants a and b,*

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g).$$

Proof. Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{F}(af + bg) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (af + bg)(x) e^{-i\omega x}\, dx \\
&= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-i\omega x}\, dx \\
&= a\mathcal{F}(f) + b\mathcal{F}(g).
\end{aligned}$$

□

Fourier Transform of the Derivative of $f(x)$

Theorem 4.12. *Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$. Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then*

$$\mathcal{F}\{f'(x)\} = i\omega\mathcal{F}\{f(x)\}$$

and

$$\mathcal{F}\{f''(x)\} = i\omega\mathcal{F}\{f'(x)\} = -\omega^2\mathcal{F}\{f(x)\}.$$

Proof. Since

$$\begin{aligned}
\mathcal{F}(f') &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)e^{-i\omega x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \cancelto{0}{\left[f(x)e^{-i\omega x}\right]_{-\infty}^{\infty}} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)(-i\omega e^{-i\omega x})\, dx \\
&= i\omega \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dx \right) \\
&= i\omega \mathcal{F}(f),
\end{aligned}$$

we have

$$\mathcal{F}(f'') = i\omega\mathcal{F}(f') = i\omega(i\omega\mathcal{F}(f)) = -\omega^2\mathcal{F}(f).$$

□

Convolution

Definition 4.6. The **convolution** $f * g$ of functions f and g is defined by

$$(f * g)(x) := \int_{-\infty}^{\infty} f(t)g(x-t)dt = \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

Convolution Theorem

Theorem 4.13. *Suppose that $f(x)$ and $g(x)$ are piecewise continuous, bounded, and absolutely integrable on the x-axis. Then*

$$\mathcal{F}(f * g) = \sqrt{2\pi}\mathcal{F}\{f\}\mathcal{F}\{g\}.$$

Proof.

$$\begin{aligned}
\mathcal{F}(f * g) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f * g)(x)e^{-i\omega x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t)g(x-t)\, dt \right] e^{-i\omega x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t)g(s)\, dt \right] e^{-i\omega(s+t)}\, ds \quad \text{with} \quad s := x - t \\
&= \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s)e^{-i\omega s}\, ds \\
&= \sqrt{2\pi}\mathcal{F}(f)\mathcal{F}(g);
\end{aligned}$$

□

Chapter 5

Introduction to PDEs

5.1 Basic Concepts

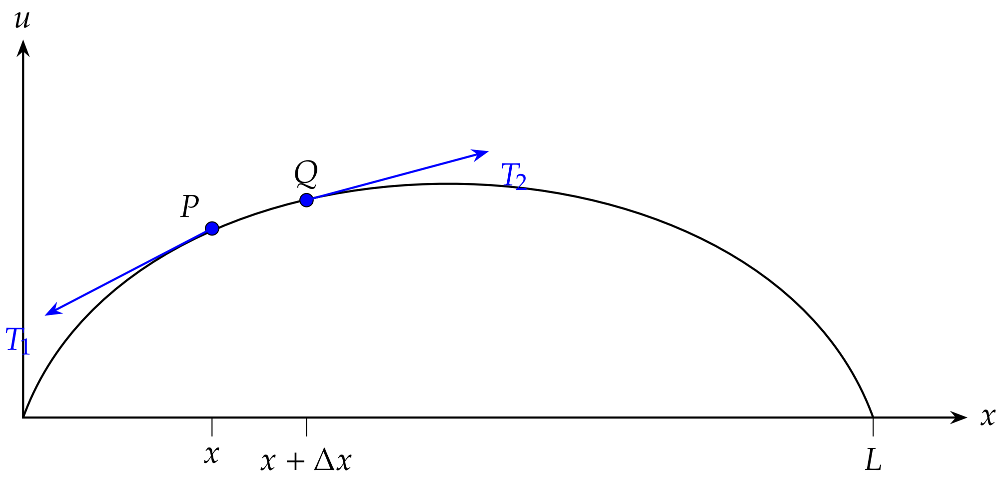
Partial Differential Equation

Definition 5.1. A partial differential equation (PDE) is an equation involving one or more partial derivatives of an (unknown) function, call it u, that depends on two or more variables, often time t and one or several variables in space.

Order of PDE

Definition 5.2. The order of the highest derivative is called the order of the PDE. As for ODEs, second-order PDEs will be the most important ones in applications.

5.2 Vibrating String, One-Dimensional Wave Equation



One-Dimensional Wave Equation

Definition 5.3. One-dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Here $c^2 = \frac{T}{\rho}$, where T denotes the tension and ρ is the mass of the undeflected string per unit length.

Remark 5.1. Note that

(1) No motion in the horizontal direction. $T_1 \cos\alpha = T_2 \cos\beta = T = \text{constant}$.

(2) The force in the vertical direction is given by $T_2 \sin\beta - T_1 \sin\alpha$. By Newton's second law $\mathbf{F} = m\mathbf{a}$, we have $T_2 \sin\beta - T_1 \sin\alpha = \rho\Delta x \dfrac{\partial^2 u}{\partial t^2}$.

Then

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \tan\beta - \tan\alpha = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}.$$

Since $\tan\alpha = \left.\dfrac{\partial u}{\partial x}\right|_x$ and $\tan\beta = \left.\frac{\partial u}{\partial x}\right|_{x+\Delta x}$, we have

$$\begin{aligned} \left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_x &= \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}, \\ \frac{u_x(x+\Delta x) - u_x(x)}{\Delta x} &= \frac{\rho}{T}u_{tt}, \\ \lim_{\Delta x \to 0}\frac{u_x(x+\Delta x) - u_x(x)}{\Delta x} &= \frac{\rho}{T}u_{tt}, \\ u_{xx} &= \frac{\rho}{T}u_{tt}. \end{aligned}$$

By letting $c^2 := T/\rho$, we have the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Note. The two-dimensional wave equation is $u_{tt} = c^2(u_{xx} + u_{yy})$.

5.3 Separation of Variables, Use of Fourier Series

Note (Solving Wave Equation)**.** Find **solutions** $u(x,t)$, which satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with **boundary conditions**

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0 \quad \text{for all} \quad t$$

and **initial conditions**

$$u(x,0) = f(x) \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x).$$

Note (**Method of Separating Variables).** Let

$$u(x,t) = F(x)G(t) \neq 0.$$

We denote $F'(x) = \frac{\mathrm{d}F}{\mathrm{d}x}$ and $\dot{G}(t) = \frac{\mathrm{d}G}{\mathrm{d}t}$. Then

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = F''G.$$

We then have

$$F\ddot{G} = c^2F''G \implies \frac{\ddot{G}}{c^2G} = \frac{F''}{F} = k.$$

for constant k. This yields immediately two ODEs, namely

$$F'' - kF = 0 \quad \text{and} \quad \ddot{G} - c^2kG = 0.$$

(1) Assume that $k = 0$, i.e., $F'' = 0$ and so $F(x) = ax+b$ for $a, b \in \mathbb{R}$. By the boundary conditions, we have

 (i) $u(0,t) = F(0)G(t) = 0 \implies b = 0$ and
 (ii) $u(L,t) = F(L)G(t) = 0 \implies a = 0.$

 $\therefore F(x) = 0$. It is contradiction.

(2) For positive $k = \mu^2 > 0$ the general solution is

$$F(x) = Ae^{\mu x} + Be^{-\mu x}.$$

 Since $u(0,t) = F(0)G(t) = (A+B)G(t) = 0$, we have $B = -A$. The boundary conditions give

 (i) $u(0,t) = F(0)G(t) = (A+B)G(t) = 0 \implies B = -A$ and
 (ii) $u(L,t) = A(e^{\mu L} - e^{-\mu L})G(t) = 0 \implies A = 0.$

 $\therefore F(x) = 0$. It is contradiction.

Hence we choose negative k, say, $k = -p^2 < 0$. Then F satisfies

$$F'' + p^2F = 0.$$

Its general solution is

$$F(x) = A\cos px + B\sin px.$$

By the boundary conditions, we have

(i) $u(0,t) = F(0)G(t) = 0 \implies A = 0$ and

(ii) $u(L,t) = F(L)G(t) = 0 \implies F(L) = B\sin pL = 0 \overset{B\neq 0}{\implies} \sin pL = 0.$

Hence

$$pL = n\pi \quad \text{so that} \quad p = \frac{n\pi}{L}$$

We thus obtain infinitely many solutions $F(x) = F_n(x)$, where

$$F_n(x) = B\sin\frac{n\pi x}{L}, \quad n = 1, 2, \cdots.$$

Let $\lambda_n = \frac{cn\pi}{L}$. Since $\ddot{G} + \lambda_n^2 G = 0$, a general solution is

$$G_n(t) = C\cos\lambda_n t + D\sin\lambda_n t.$$

Hence, solutions of the wave equation with given boundary and initial conditions are

$$u_n(x,t) = F_n(x)G_n(t) = (B_n\cos\lambda_n t + B_n^*\sin\lambda_n t)\sin\frac{n\pi x}{L}, \quad B_n = BC, B_n^* = CD, n = 1, 2, \cdots.$$

Note (**Solution of Wave Equation**)**.** In order to obtain a solution that satisfies the initial conditions, we consider the infinite series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi x}{L}.$$

From initial conditions, we obtain

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x).$$

Hence, based on the Fourier sine series of $f(x)$,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \quad n = 1, 2, \ldots.$$

Then

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty} (-B_n \lambda_n \sin \lambda_n t + B_n^* \lambda_n \cos \lambda_n t) \sin \frac{n\pi x}{L} \right]_{t=0} = \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi x}{L} = g(x).$$

Therefore

$$B_n^* \lambda_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx \implies B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$

for $n = 1, 2, \ldots$. In summary,

Solutions of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0 \quad \text{for all} \quad t$$

and initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x).$$

are given by

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi x}{L}, \quad n = 1, 2, \ldots,$$

where

$$\lambda_n = \frac{cn\pi}{L}, \quad B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \quad \text{and} \quad B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx.$$

5.4 D'Alembert's Solution of the Wave Equation

Observation (D'Alembert's Solution)

- Introduce the new independent variables $v = x + ct$ and $z = x - ct$.
- Then, applying chain rule $u_x = u_v v_x + u_z z_x = u_v + u_z$.
- Assume that all the partial derivatives involved are continuous, so that

$$u_{xv} = u_{vx} \quad \text{and correspondingly} \quad u_{xx} = (u_v + u_z)_x = u_{vv} + 2u_{vz} + u_{zz}.$$

- Similarly,$u_{tt} = c^2(u_{vv} - 2u_{vz} + u_{zz})$. By inserting these two results in the wave equation, we get $u_{vx} = \frac{\partial^2 u}{\partial z \partial v} = 0$. By integrating above identity with respect to z, we find $\frac{\partial u}{\partial v} = h(v)$, where $h(v)$ is an arbitrary function of v. Integrating this with respect to v gives

$$u = \int h(v)dv + \psi(z),$$

where $\psi(z)$ is an arbitrary function of z. Since the integral is a function of v, say, $\phi(v)$, the solution u is of the form

$$u(x,t) = \phi(v) + \psi(z) = \phi(x + ct) + \psi(x - ct).$$

This is known as d'Alembert's solution of the wave equation.

- Consider the two initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.
- Then, we have

$$u(x,0) = \phi(x) + \psi(x) = f(x) \quad \text{and} \quad u_t(x,0) = \phi'(x) - \psi'(x) = g(x).$$

- Thus, we have

$$\phi(x) - \psi(x) = k(x_0) \quad \text{where} \quad k(x_0) = \phi(x_0) - \psi(x_0).$$

- Therefore,

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)ds + \frac{1}{2}k(x_0)$$

and similarly,

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)ds - \frac{1}{2}k(x_0).$$

Theorem (D'Alembert's Solution of the Wave Equation)
The solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

with two initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x)$$

is given by

$$u(x,t) = \frac{1}{2}(f(x + ct) + f(x - ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

5.5 One-Dimensional Heat Equation: Solution by Fourier Series

Fourier Cosine and Since Series

Theorem 5.1. *content...*

Fourier Cosine and Since Series

Definition 5.4. content...

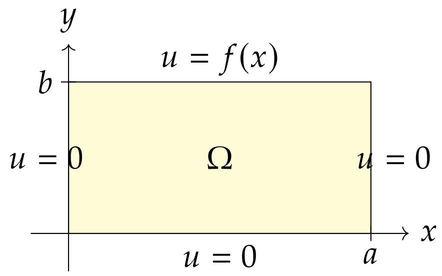
Observation
Definition (Fourier Cosine and Sine Transforms)

Laplace Equation

Definition 5.5. The two-dimensional **Laplace equation** is given by

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Remark 5.2 (Boundary conditions)**.** A heat problem then consists of this equation to be considered in some region Ω of the xy-plane and a given boundary condition on the boundary curve of Ω. This is called a boundary value problem.



One calls it:

1. Dirichlet problem if u is prescribed on $\partial\Omega$,
2. Neumann problem if the normal derivative $\nabla u \cdot N = \frac{\partial u}{\partial N}$ is prescribed on $\partial\Omega$,
3. Mixed problem if u is prescribed on a portion of $\partial\Omega$ and $\nabla u \cdot N$ on the rest of $\partial\Omega$.

Observation (Solving Laplace Equation) Find a solution $u(x, y)$, which satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with Dirichlet problem in a rectangle Ω, assuming that the temperature $u(x, y)$ equals a given function $f(x)$ on the upper side and 0 on the other three sides of the rectangle.

Observation (Method of Separating Variables)

- Let $u(x, y) = F(x)G(y)$.
- Then, we have
$$\frac{1}{F}\frac{d^2F}{dx^2} = -\frac{1}{G}\frac{d^2G}{dy^2} = -k.$$
- From this and the left and right boundary conditions,
$$\frac{d^2F}{dx^2} + kF = 0, \quad F(0) = F(a) = 0.$$
- This gives $k = \left(\frac{n\pi}{a}\right)^2$ and corresponding nonzero solutions $F(x) = F_n(x) = \sin\frac{n\pi x}{a}$, $n = 1, 2, \ldots$.

The differential equation for G then becomes

$$\frac{d^2G}{dy^2} - \left(\frac{n\pi}{a}\right)^2 G = 0$$

and solutions are

$$G(y) = G_n(y) = A_n e^{\frac{n\pi y}{a}} + B_n e^{-\frac{n\pi y}{a}}.$$

The boundary condition $u = 0$ on the lower side of Ω implies that $G_n(0) = A_n + B_n = 0$ or $B_n = -A_n$. This gives

$$G_n(y) = A_n e^{\frac{n\pi y}{a}} - A_n e^{-\frac{n\pi y}{a}} = 2A_n \sinh \frac{n\pi y}{a}.$$

By letting $2A_n = A_n^*$, we obtain the eigenfunctions

$$u_n(x, y) = F_n(x)G_n(y) = A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}.$$

These satisfy the boundary condition $u = 0$ on the left, right, and lower sides.

Observation (Method of Separating Variables)

- Let
$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}.$$
- From the boundary condition on the upper side $u(x, b) = f(x)$, we have
$$u(x, b) = f(x) = \sum_{n=1}^{\infty} \left(A_n^* \sin \frac{n\pi x}{a}\right) \sinh \frac{n\pi b}{a}.$$
- This shows that
$$A_n^* \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx.$$
- Therefore, the solution is
$$u(x, y) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a},$$
where
$$A_n^* = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx.$$

Fourier Cosine and Since Series

Theorem 5.2. *content...*

Fourier Cosine and Since Series

Definition 5.6. content...