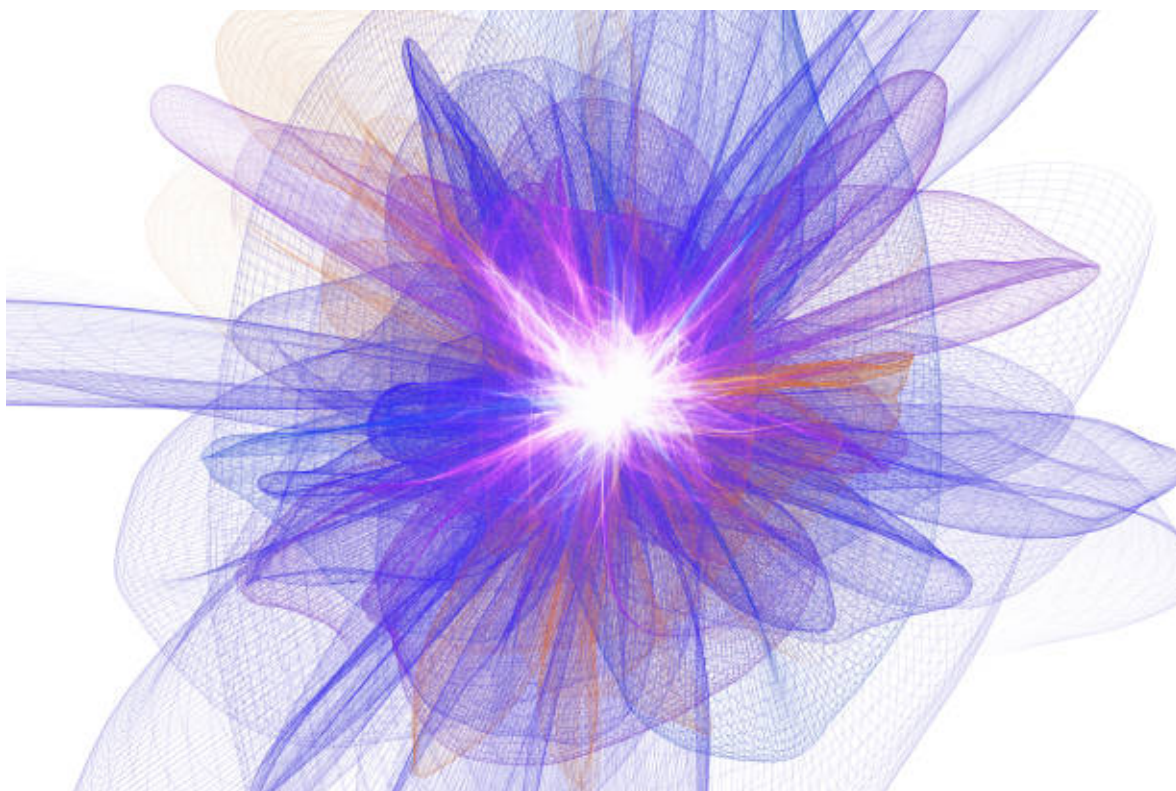


Introduction to Applied Mathematics

- Advance Calculus II -

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Chapter 1

Differentiation

1.1 Derivative and Carathéodory's Theorem

Derivative

Definition 1.1. Let $f : I \rightarrow \mathbb{R}$ and $a \in I$. We say that $L \in \mathbb{R}$ is the **derivative of f at a** if

$$\forall \epsilon > 0 : \exists \delta > 0 : x \in \mathcal{N}_\delta^*(a) \cap I \implies \left| \frac{f(x) - f(a)}{x - a} - L \right| < \epsilon.$$

Remark 1.1. We say that f is **differentiable** at a , and we write $L = f'(a)$. In other words,
$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

Proposition 1.1. If $f : I \rightarrow \mathbb{R}$ has a derivative at $a \in I$ then f is continuous at a . That is,

$$\exists f'(a) \implies f(a) = \lim_{x \rightarrow a} f(x).$$

Proof. Let $\exists f'(a)$. Then

$$\begin{aligned} \lim_{x \rightarrow a} [f(x) - f(a)] &= \lim_{x \rightarrow a} \left[\frac{f(x) - f(a)}{x - a} \cdot (x - a) \right] \quad \because x \in \mathcal{N}_\delta^*(a) \implies x \neq a \\ &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \lim_{x \rightarrow a} (x - a) \\ &= f'(a) \cdot 0 = 0. \end{aligned}$$

□

Remark 1.2. The continuity of $f : I \rightarrow \mathbb{R}$ at point does not assure the existence of the derivative at that point, e.g., $f(x) := |x|$ for $x \in \mathbb{R}$.

★ Carathéodory's Theorem ★

Theorem 1.2. Let f be defined on an interval I containing the point a . Then

$$\exists f'(a) \iff \exists \varphi \in \mathbb{R}^I \text{ such that } \begin{cases} \varphi \text{ is continuous on } I & \cdots (1) \\ f(x) - f(a) = \varphi(x)(x - a) & \cdots (2) \end{cases}$$

In this case, we have $\varphi(a) = f'(a)$.

Proof. (\Rightarrow) Assume that $\exists f'(a)$. Define a function $\varphi : I \rightarrow \mathbb{R}$ as following

$$\varphi(x) = \begin{cases} \frac{f(x) - f(a)}{x - a} & : x \neq a \\ f'(a) & : x = a. \end{cases}$$

Then

(i) φ is continuous on I , i.e., for all $a \in I$,

$$\lim_{x \rightarrow a} \varphi(x) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = f'(a) = \varphi(a).$$

(ii)

$$\begin{cases} f(x) - f(a) = \varphi(x)(x - a) & : x \neq a \\ 0 = \varphi(x) \cdot 0 & : x = a. \end{cases}$$

(\Leftarrow) Let $x \neq a$ and $x \rightarrow a$. The continuity of φ gives that

$$\exists \varphi(a) = \lim_{x \rightarrow a} \varphi(x) = \lim_{x \rightarrow a} \frac{\varphi(x)(x - a)}{(x - a)} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

That is, f is differentiable at a and $f'(a) = \varphi(a)$.

□

Example 1.1. Let us consider the function f defined by $f(x) := x^3$ for $x \in \mathbb{R}$. For any $a \in \mathbb{R}$, we see from the factorization

$$f(x) - f(a) = x^3 - a^3 = (x^2 + ax + a^2)(x - a)$$

that $\varphi(x) := x^2 + ax + a^2$ satisfies the condition of Carathéodory's Theorem. Therefore, we conclude that f is differentiable at $a \in \mathbb{R}$ and that $f'(a) = \varphi(a) = 3a^2$.

Chain Rule

Theorem 1.3. Let I, J be intervals in \mathbb{R} , let $g : J \rightarrow \mathbb{R}$ and $f : I \rightarrow \mathbb{R}$ be functions such that $f[I] \subseteq J$, and let $a \in I$. Then

$$\exists f'(a) \exists g'(f(a)) \implies \exists (g \circ f)'(a)$$

and $(g \circ f)'(a) = g'(f(a))f'(a)$.

Proof. We must show that there exists a continuous function $\varphi(x)$ s.t.

$$g(f(x)) - g(f(a)) = \varphi(x)(x - a).$$

(1) Since $\exists f'(a)$, by Carathéodory's Theorem, $\exists \sigma : I \rightarrow \mathbb{R}$ s.t.

- (i) σ is continuous at $a \in I$;
- (ii) $f(x) - f(a) = \sigma(x)(x - a)$;
- (iii) $f'(a) = \sigma(a)$.

(2) Since $\exists g'(f(a))$, by Carathéodory's Theorem, $\exists \tau : J \rightarrow \mathbb{R}$ s.t.

- (i) τ is continuous at $f(a) \in J$;
- (ii) $g(f(x)) - g(f(a)) = \tau(f(x))(f(x) - f(a))$;
- (iii) $g'(f(a)) = \tau(f(a))$.

Then

$$\begin{aligned} g(f(x)) - g(f(a)) &= \tau(f(x))(f(x) - f(a)) \quad \text{by (2)-(ii)} \\ &= \tau(f(x))\sigma(x)(x - a) \quad \text{by (1)-(ii)}. \end{aligned}$$

Let $\varphi(x) := \tau(f(x))\sigma(x)$. Then

- (i) $\varphi : I \rightarrow \mathbb{R}$ is continuous at a and
- (ii) $g(f(x)) - g(f(a)) = \varphi(x)(x - a)$,

and so, by Carathéodory's Theorem,

$$\exists (g \circ f)'(a) = \varphi(a) = \tau(f(a)) \cdot \sigma(a) = g'(f(a)) \cdot f'(a).$$

□

Remark 1.3. If f is a differentiable function, then the chain rule implies that the function $g \circ f = |f|$ is also differentiable at all points x where $f(x) \neq 0$, and its derivative is given by

$$|f(x)|'(x) = \text{sgn}(f(x)) \cdot f'(x) = \begin{cases} f'(x) & : f(x) > 0, \\ -f'(x) & : f(x) < 0. \end{cases}$$

Remark 1.4. A function f that is differentiable at every point of \mathbb{R} **need not** have a continuous derivative f' .

Differentiability of The Inverse Function

Theorem 1.4. Let $f : I \rightarrow \mathbb{R}$ be strictly monotone and continuous on I . Let $J := f[I]$ and $g : J \rightarrow \mathbb{R}$ be the strictly monotone and continuous function inverse to f . Then

$$\exists f'(a) \neq 0 \implies \exists g'(f(a)) = \frac{1}{f'(a)}.$$

Proof. Since $\exists f'(a)$, by Carathéodory's Theorem, $\exists \sigma : I \rightarrow \mathbb{R}$ s.t.

- (i) σ is continuous at $a \in I$;
- (ii) $f(x) - f(a) = \sigma(x)(x - a)$;
- (iii) $f'(a) = \sigma(a) \neq 0$.

Since $\sigma(a) \neq 0$, $\exists \delta > 0$ s.t. $\sigma(x) \neq 0$, $x \in \mathcal{N}_\delta(a) \cap I$. Let $\Omega := f[\mathcal{N}_\delta(a) \cap I]$. Since $g = f^{-1}$, we have

$$\begin{aligned} f(x) - f(a) &= f((g \circ f)(x)) - f((g \circ f)(a)) \quad \because f \circ g = \text{id} \\ &= \sigma((g \circ f)(x))((g \circ f)(x) - (g \circ f)(a)) \quad \text{by (ii)}. \end{aligned}$$

Since $f(x) \in \Omega \implies \sigma(x) \neq 0 \implies \sigma((g \circ f)(x)) \neq 0$,

$$g(f(x)) - g(f(a)) = \frac{1}{\sigma((g \circ f)(x))}(f(x) - f(a)).$$

Let $\varphi(x) := 1/\sigma((g \circ f)(x))$. Then φ is continuous at $f(a)$. By Carathéodory's Theorem,

$$g'(f(a)) = \varphi(a) = \frac{1}{\sigma((g \circ f)(a))} = \frac{1}{\sigma(a)} = \frac{1}{f'(a)}.$$

□

1.2 The Rolle's Theorem and the Mean Value Theorem

Absolute and Local Maxi/Mini-mum

Definition 1.2. Let $f : I \rightarrow \mathbb{R}$ be a function.

- f has an **absolute maximum** at $a \in I$ if $x \in I \implies f(x) \leq f(a)$.
- f has an **absolute minimum** at $a \in I$ if $x \in I \implies f(a) \leq f(x)$.
- f is said to have a **local (or relative) maximum** at $a \in I$ if

$$\exists \mathcal{N}_\delta(a) : f(x) \leq f(a), x \in \mathcal{N}_\delta(a) \cap I.$$

- f is said to have a **local (or relative) minimum** at $a \in I$ if

$$\exists \mathcal{N}_\delta(a) : f(a) \leq f(x), x \in \mathcal{N}_\delta(a) \cap I.$$

- f has a **local (or relative extremum)** at $a \in I$ either a relative maximum or a relative minimum at a .

Interior Extremum Theorem

Theorem 1.5. Let $f : (a, b) \rightarrow \mathbb{R}$ has a relative extremum and $c \in (a, b)$. Then

$$\exists f'(c) \implies f'(c) = 0.$$

Proof. Let f has a relative maximum at c , i.e.,

$$\exists \mathcal{N}_\delta(c) : x \in \mathcal{N}_\delta(c) \cap (a, b) \implies f(x) \leq f(c).$$

Assume that $f'(c) > 0$ then

$$\exists \mathcal{N}_\delta(c) \subseteq (a, b) : x \in \mathcal{N}_\delta^+(c) \implies \frac{f(x) - f(c)}{x - c} > 0.$$

If $c \in \mathcal{N}_\delta(c)$ and $x > c$, then we have

$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f(c)}{x - c} > 0.$$

But this contradicts the hypothesis that f has a relative maximum at c . Similarly if $f'(c) < 0$ then we have a contradiction. Hence $f'(c) = 0$. \square

Corollary 1.5.1. Let $f : (a, b) \rightarrow \mathbb{R}$ be continuous on (a, b) and suppose that f has a relative extremum at $c \in (a, b)$. Then either

$$\nexists f'(c) \quad \text{or} \quad f'(c) = 0.$$

★ Rolle's Theorem

Theorem 1.6. Let f be continuous on $I = [a, b]$, and let f be differentiable on (a, b) . Then

$$f(a) = 0 = f(b) \implies \exists c \in (a, b) : f'(c) = 0.$$

★ Mean Value Theorem of Differential Calculus ★

Theorem 1.7. Let f be continuous on $I = [a, b]$, and let f be differentiable on (a, b) . Then

$$\exists c \in (a, b) : f(b) - f(a) = f'(c)(b - a).$$

Proof. Consider the function whose graph is the line segment joining the points $(a, f(a))$ and $(b, f(b))$:

$$f(x) - f(a) = \frac{f(b) - f(a)}{b - a}(x - a).$$

Define a function $g : [a, b] \rightarrow \mathbb{R}$ s.t.

$$g(x) := f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then

- (i) g is continuous on $[a, b]$;
- (ii) g is differentiable on (a, b) ;
- (iii) $g(a) = 0 = g(b)$.

By Rolle's Theorem, $\exists c \in (a, b) : g'(c) = 0$. Then

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \implies g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \implies f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

Example 1.2. Prove that $e^x \geq 1 + x$ for $x \in \mathbb{R}$.

Solution. (1) $x = 0 \implies e^x = 1 + x$.

(2) Let $x > 0$ and $f(x) = e^x$. Then, by MVT,

$$\exists c \in (0, x) : f(x) - f(0) = f'(c)(x - 0),$$

and so

$$e^x - 1 = e^c x > x \implies e^x > 1 + x.$$

(3) Let $x < 0$ and $f(x) = e^x$. Then, by MVT,

$$\exists c \in (x, 0) : f(0) - f(x) = f'(c)(0 - x),$$

and so

$$1 - e^x = e^c(-x) < -x \implies 1 + x < e^x.$$

□

1.3 L'Hôpital's Rules

Theorem 1.8. Let f and g be defined on $[a, b]$, let $f(a) = 0 = g(a)$, and let $g(x) \neq 0$ for $x \in (a, b)$. If f and g are differentiable at a if $g'(a) \neq 0$, then the limit f/g at a exists and is equal to $f'(a)/g'(a)$. Thus

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof. Since $f(a) = 0 = g(a)$,

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \rightarrow a^+} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}.$$

□

Remark 1.5. In L'Hôpital Rules, the hypothesis $f(a) = 0 = g(a)$ is essential. For example, it $f(x) := x + 17$ and $g(x) := 2x + 3$ for $x \in \mathbb{R}$ then,

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{17}{3} \quad \text{while} \quad \frac{f'(0)}{g'(0)} = \frac{1}{2}.$$

Cauchy's Mean Value Theorem of Differential Calculus

Theorem 1.9. Let f and g be continuous on $[a, b]$ and differentiable on (a, b) , and assume that $g'(x) \neq 0$ for all $x \in (a, b)$. Then

$$\exists c \in (a, b) : \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Since $g'(x) \neq 0$ for $x \in (a, b)$, $g(a) \neq g(b)$ by Rolle's Theorem. Define $h : [a, b] \rightarrow \mathbb{R}$ such that

$$h(x) := f(x) - g(a) - \frac{f(b) - f(a)}{g(b) - g(a)} (g(x) - g(a)).$$

Then

- (i) h is continuous on $[a, b]$;
- (ii) h is differentiable on (a, b) ;
- (iii) $h(a) = 0 = h(b)$.

By Rolle's Theorem,

$$\exists c \in (a, b) : h'(c) = 0.$$

Since $h'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(x)$, we have

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) \implies \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

□

Remark 1.6. Note that if $g(x) := x$ then the Cauchy's mean value theorem reduces to the mean value theorem.

Remark 1.7. By the Mean Value Theorem,

$$\exists \alpha, \beta \in (a, b) : \begin{cases} f(b) - f(a) = f'(\alpha)(b - a) \\ f(b) - f(a) = g'(\beta)(b - a) \end{cases}.$$

If $g'(x) \neq 0$ for $x \in (a, b)$, we have $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\alpha)}{g'(\beta)}$.

L'Hôpital's Rule - 1st

Theorem 1.10. Let $-\infty \leq a < b \leq \infty$ and let f, g be differentiable on (a, b) such that $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose that

$$\lim_{x \rightarrow a+} f(x) = 0 = \lim_{x \rightarrow a+} g(x).$$

Then

$$(1) \lim_{x \rightarrow a+} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \rightarrow a+} \frac{f(x)}{g(x)} = L.$$

$$(2) \lim_{x \rightarrow a+} \frac{f'(x)}{g'(x)} = \pm\infty \implies \lim_{x \rightarrow a+} \frac{f(x)}{g(x)} = \pm\infty.$$

Proof. We must show that $\lim_{x \rightarrow a+} \frac{f(x)}{g(x)} = L$, i.e.,

$$\begin{aligned} \forall \varepsilon > 0 : \exists \delta > 0 : x \in (a, a + \delta) &\implies \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon \\ \iff \forall \varepsilon > 0 : \exists c \in (a, b) : x \in (a, c) &\implies \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon. \end{aligned}$$

Since $g'(x) \neq 0$ for $x \in (a, b)$,

$$a < \alpha < x < b \implies g(x) - g(\alpha) \neq 0.$$

By Cauchy's Mean-Value Theorem,

$$\exists \gamma \in (\alpha, x) : \frac{f(x) - f(\alpha)}{g(x) - g(\alpha)} = \frac{f'(\gamma)}{g'(\gamma)}.$$

Let $\varepsilon > 0$. Then

$$\lim_{\gamma \rightarrow a+} \frac{f'(\gamma)}{g'(\gamma)} = L \implies \exists c \in (a, b) : \left[a < \gamma < x < c \implies \left| \frac{f'(\gamma)}{g'(\gamma)} - L \right| < \frac{\varepsilon}{2} \right]$$

Then

$$\begin{aligned}
 & L - \frac{\varepsilon}{2} < \frac{f'(\gamma)}{g'(\gamma)} < L + \frac{\varepsilon}{2} \\
 \Rightarrow & L - \frac{\varepsilon}{2} < \frac{f(x) - f(\alpha)}{g(x) - g(\alpha)} < L + \frac{\varepsilon}{2} \\
 \Rightarrow & \lim_{\alpha \rightarrow a+} \left(L - \frac{\varepsilon}{2} \right) \leq \lim_{\alpha \rightarrow a+} \frac{f(x) - f(\alpha)}{g(x) - g(\alpha)} \leq \lim_{\alpha \rightarrow a+} \left(L + \frac{\varepsilon}{2} \right) \quad \because \lim_{\alpha \rightarrow a+} f(x) = 0 = \lim_{\alpha \rightarrow a+} g(x) \\
 \Rightarrow & L - \frac{\varepsilon}{2} < L - \varepsilon \leq \frac{f(x)}{g(x)} \leq L + \frac{\varepsilon}{2} < L + \varepsilon \\
 \Rightarrow & \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon.
 \end{aligned}$$

Thus, $\lim_{x \rightarrow a+} \frac{f(x)}{g(x)} = L$. □

Example 1.3. Let $I := (0, \pi/2)$. Then evaluate

$$\lim_{x \rightarrow 0+} \left(\frac{1}{x} - \frac{1}{\sin x} \right),$$

which has the indeterminate form $\infty - \infty$.

Solution.

$$\lim_{x \rightarrow 0+} \left(\frac{1}{x} - \frac{1}{\sin x} \right) = \lim_{x \rightarrow 0+} \frac{\sin x - 1}{x \sin x} = \lim_{x \rightarrow 0+} \frac{\cos x - 1}{\sin x + x \cos x} = \lim_{x \rightarrow 0+} \frac{-\sin x}{2 \cos x - x \sin x} = 0.$$

□

Example 1.4. Let $I := (0, \infty)$. Then evaluate

$$\lim_{x \rightarrow 0+} x \ln x,$$

which has the indeterminate form $0 \times \infty$.

Solution.

$$\lim_{x \rightarrow 0+} x \ln x = \lim_{x \rightarrow 0+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0+} (-x) = 0.$$

□

Example 1.5. Let $I := (0, \infty)$ and consider

$$\lim_{x \rightarrow 0+} x^x$$

which has the indeterminate form 0^0 .

Solution. Let $f(x) := x^x$ then $\ln f(x) = x \ln x$. Then

$$\lim_{x \rightarrow 0+} (x \ln x) = \lim_{x \rightarrow 0+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0+} (-x) = 0.$$

Thus, $\lim_{x \rightarrow 0+} f(x) = \lim_{x \rightarrow 0+} e^{\ln f(x)} = e^0 = 1$. □

Example 1.6. Let $I := (0, \infty)$. Then evaluate

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x,$$

which has the indeterminate form 1^∞ .

Solution. Let $f(x) := \left(1 + \frac{1}{x}\right)^x$ then $\ln f(x) = x \ln \left(1 + \frac{1}{x}\right)$. Then

$$\lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x}\right) \stackrel{t=1/x}{=} \lim_{t \rightarrow 0+} \frac{\ln(1+t)}{t} = \lim_{t \rightarrow 0+} \frac{\frac{1}{1+t}}{1} = 1.$$

Thus, $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^1 = e$. □

Example 1.7. Let $I := (0, \infty)$. Then evaluate

$$\lim_{x \rightarrow \infty} (1+x)^{\frac{1}{x}},$$

which has the indeterminate form ∞^0 .

Solution. Let $f(x) := (1+x)^{1/x}$ then $\ln f(x) = \frac{\ln(1+x)}{x}$. Then

$$\lim_{x \rightarrow \infty} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{1+x}}{1} = 0.$$

Thus, $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^0 = 1$. □

1.4 Taylor's Theorem

★ Taylor's Theorem ★

Theorem 1.11. Let $n \in \mathbb{N}$ and $f : [a, b] \rightarrow \mathbb{R}$ be such that f and its derivatives $f', f'', \dots, f^{(n)}$ are continuous on $[a, b]$ and that $f^{(n+1)}$ exists on (a, b) . Then

$$t \in [a, b] \implies \forall x \in [a, b] : \exists c \in (t, x) : f(x) = \sum_{i=0}^n \frac{f^{(i)}(t)}{i!} (x-t)^i + \frac{f^{(n+1)}(c)}{(n+1)!} (x-t)^{n+1}.$$

Proof. Define a function $F : [a, b] \rightarrow \mathbb{R}$ such that

$$\begin{aligned} F(t) &= f(x) - \sum_{i=0}^n \frac{f^{(i)}(t)}{i!} (x-t)^i \\ &= f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!} (x-t)^2 - \dots - \frac{f^{(n)}(t)}{n!} (x-t)^n. \end{aligned}$$

We claim that

$$\exists c \in (a, x) : F(a) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}.$$

Define $G : [a, b] \rightarrow \mathbb{R}$ such that

$$G(t) = F(t) - \left(\frac{x-t}{x-a} \right)^{n+1} F(a).$$

Then

- (i) G is continuous on $[a, b]$;
- (ii) G is differentiable on $[a, b]$;
- (iii) $G(a) = 0 = G(b)$.

By Rolle's Theorem, $\exists c \in (a, x) : G'(c) = 0$. Then

$$G'(t) = F'(t) + \frac{(n+1)(x-t)^n}{(x-a)^{n+1}} F(a) \implies F(a) = -\frac{(x-a)^{n+1}}{(n+1)(x-c)^n} F'(c).$$

Since

$$\begin{aligned} F'(t) &= -f'(t) \\ &\quad - f''(t)(x-t) + f'(t) \\ &\quad - \frac{f'''(t)}{2!} (x-t)^2 + f''(t)(x-t) \\ &\quad - \dots \\ &\quad - \frac{f^{(n+1)}(t)}{n!} (x-t)^n + \frac{f^{(n)}(t)}{(n-1)!} (x-t)^{n-1}, \end{aligned}$$

we have

$$F'(c) = \frac{f^{(n+1)}(c)}{n!} (x-c)^n.$$

$$\text{Hence } F(a) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}.$$

□

Example 1.8 (Numerical Estimation). Approximate the number e with error less than 10^{-5} .

Solution. Let $f(x) = e^x$. Then

$$P_n(x) = \sum_{i=0}^n \frac{x^i}{i!} = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n.$$

By Taylor's theorem,

$$\exists c \in (0, x) : f(x) = P_n(x) + R_n(x), \text{ where } R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}.$$

For $c \in (0, 1)$

$$R_n(1) = \frac{e^c}{n+1!} < \frac{3}{(n+1)!} < 10^{-5} \implies n = 8.$$

□

Example 1.9. For any $k \in \mathbb{N}$ and for all $x > 0$, prove that

$$x - \frac{1}{2}x^2 + \cdots - \frac{1}{2k}x^{2k} < \ln(1+x) < x - \frac{1}{2}x^2 + \cdots + \frac{1}{2k+1}x^{2k+1}.$$

Solution. Let $g(x) := \ln(1+x)$ for $x > 0$. Then

$$g'(x) = \frac{1}{1+x} \implies \begin{cases} P_n(x) = x - \frac{1}{2}x^2 + \cdots + \frac{(-1)^{n-1}}{n}x^n & \text{with } a = 0 \\ R_n(x) = \frac{(-1)^n c^{n+1}}{n+1}x^{n+1} & \text{for some } c \in (0, x) \end{cases}$$

Thus for any $x > 0$,

$$(1) \ n = 2k \implies R_{2k}(x) > 0,$$

$$(2) \ n = 2k+1 \implies R_{2k+1}(x) < 0.$$

□

1.5 Exercises

Exercise 1.1. Prove that

$$(\cos^{-1})'(x) = -\frac{1}{\sqrt{1-x^2}}$$

for $x \in (-1, 1)$.

Solution. Let $y := \cos^{-1}(x)$, i.e., $x = \cos y$. Then

$$\begin{aligned} \frac{d}{dx}x &= \frac{d}{dx}[\cos y] \implies 1 = -\sin y \cdot \frac{dy}{dx} \\ \implies -\frac{1}{\sin y} &= \frac{dy}{dx} \quad \because x \in (-1, 1) \implies y = \cos^{-1}(x) \in (0, \pi) \implies \sin y \neq 0. \end{aligned}$$

By Pythagorean identity,

$$\sin^2(y) + \cos^2(y) = 1 \implies \sin^2(y) = 1 - \cos^2(y) \implies \sin(y) = \sqrt{1-x^2}$$

and so

$$(\cos^{-1})'(x) = \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}.$$

□

Exercise 1.2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined as

$$f(x) := \begin{cases} x^2 \sin(x^{-2}) & : x \neq 0 \\ 0 & : x = 0. \end{cases}$$

Then, prove that f is differentiable on \mathbb{R} and f' is discontinuous on $[-1, 1]$.

Solution.

(1) **Differentiability of f on \mathbb{R} :** Let $x \neq 0$. Since $f(x) = x^2 \sin \frac{1}{x^2}$,

$$f'(x) = 2x \sin \frac{1}{x^2} + x^2 \cos \frac{1}{x^2} \cdot (-2) \frac{1}{x^3} = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}.$$

And

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \sin \frac{1}{h^2}}{h} = \lim_{h \rightarrow 0} h \sin \frac{1}{h^2} = 0$$

because $|\sin(h^{-2})| \leq 1 \implies 0 \leq |h \sin(h^{-2})| \leq |h|$. $\therefore \forall x \in \mathbb{R} : \exists f'(x)$.

(2) **Discontinuity of f' on $[-1, 1]$:** Let $n \in \mathbb{N}$. Then $\frac{1}{\sqrt{2n\pi}} \in [-1, 1] \setminus \{0\}$. Note that

$$f' \left(\frac{1}{\sqrt{2n\pi}} \right) = \frac{2}{\sqrt{2n\pi}} \sin(2n\pi) - 2\sqrt{2n\pi} \cos(2n\pi) = -2\sqrt{2n\pi} \neq 0.$$

Then

$$\lim_{n \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2n\pi}} = 0 \quad \text{and} \quad f' \left(\frac{1}{\sqrt{2n\pi}} \right) \neq 0 \quad \text{but} \quad f'(0) = 0.$$

□

Exercise 1.3. Let $f : I \rightarrow \mathbb{R}$ be differentiable at $c \in I$. Establish the **Straddle Lemma**: Given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $u, v \in I$ satisfy $c - \delta < u \leq c \leq v < c + \delta$, then

$$f(v) - f(u) - (v - u)f'(c) \leq \varepsilon(v - u).$$

[Hint: use the term $f(c) - cf'(c)$ and apply the Triangle Inequality.]

Solution. Let $\varepsilon > 0$. Since f is differentiable at c ,

$$\exists \delta > 0 : 0 < |x - c| < \delta \implies \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon.$$

Then

$$|f(x) - f(c) - (x - c)f'(c)| < \varepsilon|x - c|. \quad (*)$$

Let $u, v \in I$ satisfies $c - \delta < u \leq c \leq v < c + \delta$. Then

$$\begin{aligned} |f(v) - f(u) - (v - u)f'(c)| &= |f(v) - f(c) + f(c) - f(u) - (v - c + c - u)f'(c)| \\ &= |f(v) - f(c) - (v - c)f'(c) - (f(u) - f(c) - (u - c)f'(c))| \\ &\leq |f(v) - f(c) - (v - c)f'(c)| + |f(u) - f(c) - (u - c)f'(c)| \\ &< \varepsilon|v - c| + \varepsilon|u - c| \quad \text{by } (*) \\ &= \varepsilon(v - c) - \varepsilon(u - c) \quad \because u \leq c \leq v \\ &= \varepsilon(v - u). \end{aligned}$$

□

Exercise 1.4. Let $a > b > 0$ and $n \in \mathbb{N}$. Prove that

$$\sqrt[n]{a} - \sqrt[n]{b} < \sqrt[n]{a-b}$$

for $n \geq 2$.

Solution. Define $f : \mathbb{R}_{\geq 1} \rightarrow \mathbb{R}$ by

$$f(x) := \sqrt[n]{x} - \sqrt[n]{x-1}$$

for $n \geq 2$. Then

$$\begin{aligned} f'(x) &= \frac{1}{n} x^{\frac{1-n}{n}} - \frac{1}{n} (x-1)^{\frac{1-n}{n}} \\ &= \frac{1}{n} \left[\left(x^{\frac{n-1}{n}} \right)^{-1} - \left((x-1)^{\frac{n-1}{n}} \right)^{-1} \right] \\ &= \frac{1}{n} \left[\frac{1}{x^{\frac{n-1}{n}}} - \frac{1}{(x-1)^{\frac{n-1}{n}}} \right] \\ &= \frac{1}{n} \left[\frac{(x-1)^{\frac{n-1}{n}} - x^{\frac{n-1}{n}}}{x^{\frac{n-1}{n}} \cdot (x-1)^{\frac{n-1}{n}}} \right]. \end{aligned}$$

Note that

$$x > 1 \implies 0 < x-1 < x \implies (x-1)^{\frac{n-1}{n}} < x^{\frac{n-1}{n}}.$$

Thus, $f'(x) < 0$ for $x > 1$. That is, f is decreasing for $x \geq 1$. Then

$$a > b > 0 \implies 1 < \frac{a}{b} \implies f(a/b) < f(1) \implies \sqrt[n]{a/b} - \sqrt[n]{a/b-1} < 1.$$

Multiplying by $\sqrt[n]{b}$, we have

$$\sqrt[n]{a} - \sqrt[n]{a-b} < \sqrt[n]{b} \implies \sqrt[n]{a} - \sqrt[n]{b} < \sqrt[n]{a-b}.$$

□

Exercise 1.5. Use the Mean Value Theorem to show that

$$\frac{x-1}{x} < \ln x < x-1$$

for $x > 1$.

Solution.

(1) Let

$$f(x) := \ln x - \frac{x-1}{x} = \ln x - 1 + \frac{1}{x}.$$

Then $f'(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2}$. Since $x > 1$ and $f'(x) > 0$, by the Mean Value Theorem,

$$\exists c \in (1, x) : f(x) - f(1) = f'(c)(x-1),$$

i.e., $f(x) - f(1) > 0$. Thus

$$f(x) = \ln x - \frac{x-1}{x} > 0 = f(1) \implies \ln x > \frac{x-1}{x}.$$

(2) Let

$$g(x) := (x-1) - \ln x.$$

Then $g'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$. Since $x > 1$ and $g'(x) > 0$,

$$g(x) > g(1) = 0 \implies x-1 > \ln x.$$

□

Exercise 1.6. Prove or disprove: If f is differentiable and uniformly continuous on I then f is a Lipschitz function on I .

Solution. Counterexample: Let $f(x) := \sqrt{x}$ for $x \in (0, 1)$. Then f is uniformly continuous on $(0, 1)$ by continuous extension theorem. Then

$$\exists f^*(x) = \begin{cases} f(x) = \sqrt{x} & : x \in (0, 1) \\ 0 & : x = 0 \\ 1 & : x = 1. \end{cases}$$

But f is not a Lipschitz function on $(0, 1)$.

□

Exercise 1.7. Let f, g be differentiable on \mathbb{R} and suppose that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x > 0$. Show that $f(x) \leq g(x)$ for all $x > 0$.

Solution. Let $h(x) := g(x) - f(x)$. Since $h'(x) = g'(x) - f'(x) \geq 0$, h is an increasing function on $x > 0$. Thus, $g(x) \geq f(x)$ for all $x > 0$. \square

Exercise 1.8. Show that

$$\lim_{x \rightarrow c} \frac{x^c - c^x}{x^x - c^c} = \frac{1 - \ln c}{1 + \ln c}$$

for $c > 0$.

Solution. Note that

$$\begin{aligned} y := x^x &\implies \ln y = x \ln x \\ &\implies \frac{y'}{y} = \ln x + 1 \\ &\implies y' = x^x (\ln x + 1). \end{aligned} \qquad \begin{aligned} y := c^x &\implies \ln y = x \ln c \\ &\implies \frac{y'}{y} = \ln c \\ &\implies y' = c^x (\ln c). \end{aligned}$$

By L'Hôpital's rule, we have

$$\lim_{x \rightarrow c} \frac{cx^{c-1} - c^x \ln c}{x^x (\ln x + 1)} = \frac{c^c - c^c \ln c}{c^c (\ln c + 1)} = \frac{c^c (1 - \ln c)}{c^c (1 + \ln c)} = \frac{1 - \ln c}{1 + \ln c}$$

\square

Exercise 1.9. Let $f : (0, 1) \rightarrow \mathbb{R}$ be differentiable on $(0, \infty)$ and suppose that

$$\lim_{x \rightarrow \infty} (f(x) + f'(x)) = L.$$

Then prove that

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow \infty} f'(x) = 0.$$

[Hint: $f(x) = \frac{e^x f(x)}{e^x}$.]

Solution. Since $f(x) = \frac{e^x f(x)}{e^x}$,

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{e^x f(x)}{e^x} \stackrel{\text{L'Hôpital's rule}}{=} \lim_{x \rightarrow \infty} \frac{e^x f(x) + e^x f'(x)}{e^x} = \lim_{x \rightarrow \infty} (f(x) + f'(x)) = L$$

and so $\lim_{x \rightarrow \infty} f'(x) = 0$. \square

Exercise 1.10. Let $I \subseteq \mathbb{R}$ be an open interval, let $f : I \rightarrow \mathbb{R}$ be differentiable on I , and suppose $f''(a)$ exists at $a \in I$. Show that

$$f''(a) = \lim_{h \rightarrow 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2}.$$

Solution.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} &\stackrel{\text{L'Hôpital's rule}}{=} \lim_{h \rightarrow 0} \frac{f'(a+h) - f'(a-h)}{2h} \\ &= \lim_{h \rightarrow 0} \left(\frac{1}{2} \cdot \frac{f'(a+h) - f(a) + f(a) - f'(a-h)}{h} \right) \\ &= \frac{1}{2} \left(\lim_{h \rightarrow 0} \frac{f'(a+h) - f(a)}{h} + \lim_{h \rightarrow 0} \frac{f'(a-h) - f'(a)}{-h} \right) \\ &= \frac{1}{2} (f''(a) + f''(a)) \\ &= f''(a). \end{aligned}$$

□

Chapter 2

The Riemann Integral

2.1 Introduction to Riemann Integral

Partition

Definition 2.1. Consider a closed bounded interval $[a, b] \subseteq \mathbb{R}$. A **partition** of $[a, b]$ is a finite ordered set

$$P := \{a = x_0, x_1, \dots, x_{n-1}, x_n = b\} \text{ s.t. } x_0 < x_1 < \dots < x_{n-1} < x_n.$$

Upper and Lower Sum

Definition 2.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded on $[a, b]$ and $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ be a partition of $[a, b]$.

(1) The **upper sum** of f for the partition P is the sum

$$U(f, P) := \sum_{i=1}^n M_i[f](x_i - x_{i-1}), \quad M_i[f] := \sup \{f(x) : x \in [x_{i-1}, x_i]\}$$

for $i = 1, 2, \dots, n$.

(2) The **lower sum** of f for the partition P is the sum

$$L(f, P) := \sum_{i=1}^n m_i[f](x_i - x_{i-1}), \quad m_i[f] := \inf \{f(x) : x \in [x_{i-1}, x_i]\}$$

for $i = 1, 2, \dots, n$.

Proposition 2.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded on $[a, b]$ and P be a partition of $[a, b]$. Then

$$L(f, P) \leq U(f, P).$$

Proof. $M_i[f] \geq m_i[f] \implies L(f, P) \leq U(f, P)$.

□

Refinement

Definition 2.3. Let Q and P be partitions of $[a, b]$ and $P \subseteq Q$. We say that Q is a **refinement** of P .

Theorem 2.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded on $[a, b]$ and $P = \{a = x_0, x_1, \dots, x_n = b\}$ be a partition of $[a, b]$. Let Q is a refinement of P . Then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Proof. Assume that $Q := P \cup \{x^*\}$ s.t.

$$\begin{aligned} Q &= \{a = x_0, x_1, x_2, \dots, x_n = b\} \cup \{x^*\} \\ &= \{a, x_1, x_2, \dots, x_{j-1}, x^*, x_j, x_{j+1}, \dots, x_{n-1}, b\}. \end{aligned}$$

Let $M_j^L = \sup \{f(x) : [x_{j-1}, x^*]\}$ and $M_j^R = \sup \{f(x) : [x^*, x_j]\}$ then

$$M_j^L[f] \leq M_j[f] \quad \text{and} \quad M_j^R[f] \leq M_j[f],$$

and we have

$$\begin{aligned} U(f, Q) &= \left(\sum_{i=1}^{j-1} M_i[f] \Delta x_i \right) + \left(M_j^L[f](x^* - x_{j-1}) \right) + \left(M_j^R[f](x_j - x^*) \right) + \left(\sum_{i=j+1}^n M_i[f] \Delta x_i \right) \\ &\leq \left(\sum_{i=1}^{j-1} M_i[f] \Delta x_i \right) + M_j[f](x^* - x_{j-1}) + M_j[f](x_j - x^*) + \left(\sum_{i=j+1}^n M_i[f] \Delta x_i \right) \\ &= \sum_{i=1}^n M_i[f] \Delta x_i = U(f, P). \end{aligned}$$

Similarly, we have $L(f, P) \leq L(f, Q)$. □

Corollary 2.2.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded on $[a, b]$ and P and Q are partitions of $[a, b]$ then

$$L(f, Q) \leq U(f, P).$$

Proof. Let $R = P \cup Q$. By **Theorem 2.2**, we have

$$L(f, Q) \leq L(f, R) \leq U(f, R) \leq U(f, P)$$

since R is a refinement of both P and Q . □

Remark 2.1. By the completeness property of real number, there exist the followings:

$$L(f) := \sup \{L(f, P) : P \text{ is a partition of } [a, b]\},$$

$$U(f) := \inf \{U(f, P) : P \text{ is a partition of } [a, b]\}.$$

Moreover, $L(f) \leq U(f)$.

Upper and Lower Integral

Definition 2.4. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded on $[a, b]$.

(1) The **upper integral** of f on $[a, b]$ is defined by

$$\overline{\int_a^b} f(x)dx := U(f) = \inf \{U(f, P) : P \text{ is a partition of } [a, b]\}.$$

(2) The **lower integral** of f on $[a, b]$ is defined by

$$\underline{\int_a^b} f(x)dx := L(f) = \sup \{L(f, P) : P \text{ is a partition of } [a, b]\}.$$

Riemann Integral

Definition 2.5. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded on $[a, b]$. We say that f is **Riemann integrable** (or **integrable**) on $[a, b]$ if $L(f) = U(f)$. We define the **Riemann integral** of f on $[a, b]$ as follow:

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

Example 2.1. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a Dirichlet's discontinuous function defined by

$$f(x) := \begin{cases} 1 & : x \in \mathbb{Q}, \\ 0 & : x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

We claim that f is not Riemann integrable.

Solution. Let $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ be a partition of $[0, 1]$. Note that

$$M_i[f] \equiv 1 \quad \text{and} \quad m_i[f] \equiv 0$$

for $i = 1, 2, \dots, n$. Then

$$L(f, P) = \sum_{i=1}^n m_i[f] \Delta x_i = \sum_{i=1}^n (0 \cdot \Delta x_i) = 0,$$

$$U(f, P) = \sum_{i=1}^n M_i[f] \Delta x_i = \sum_{i=1}^n (1 \cdot \Delta x_i) = 1.$$

Therefore $L(f) = 0 \neq 1 = U(f)$, and so f is not Riemann integrable on $[0, 1]$. □

★ Riemann's Condition ★

Theorem 2.3. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded on $[a, b]$. Then

$$\overline{\int_a^b f(x)dx} = \underline{\int_a^b f(x)dx} \iff \forall \varepsilon > 0 : \exists P : U(f, P) - L(f, P) < \varepsilon.$$

Proof. (\Rightarrow) Let $\varepsilon > 0$. Then $\exists P_1, P_2$ such that

$$L(f) - \frac{\varepsilon}{2} < L(f, P_1) \quad \text{and} \quad U(f, P_2) < U(f) + \frac{\varepsilon}{2}.$$

Let $P := P_1 \cup P_2$. Since $L(f) = U(f)$, we have

$$\begin{aligned} U(f, P) - L(f, P) &\leq U(f, P_2) - L(f, P_1) \\ &< U(f) + \frac{\varepsilon}{2} - \left(L(f) - \frac{\varepsilon}{2} \right) \\ &= \varepsilon. \end{aligned}$$

(\Leftarrow) Let P be a partition of $[a, b]$. Since $U(f) \leq U(f, P)$ and $L(f, P) \leq L(f)$, for $\varepsilon > 0$,

$$0 \leq U(f) - L(f) \leq U(f, P) - L(f, P) < \varepsilon.$$

That is, $L(f) = U(f)$.

□

2.2 Properties of Riemann Integral

Theorem 2.4. If $f : [a, b] \rightarrow \mathbb{R}$ is monotone on $[a, b]$ then f is Riemann integrable on $[a, b]$.

Proof. Suppose that f is increasing on $[a, b]$. Let $\varepsilon > 0$. By the completeness property of \mathbb{R} ,

$$\exists N \in \mathbb{N} : [f(b) - f(a)] \frac{b-a}{N} < \varepsilon.$$

Correspondingly, there exists a partition $P_N = \{x_0, x_1, \dots, x_{N-1}, x_N\}$ such that

$$\Delta x_i = x_i - x_{i-1} = \frac{b-a}{N}$$

for $i = 1, 2, \dots, N$. Since $\begin{cases} M_i[f] = \sup \{f(x) : x \in [x_{i-1}, x_i]\} = f(x_i) \\ m_i[f] = \inf \{f(x) : x \in [x_{i-1}, x_i]\} = f(x_{i-1}) \end{cases}$,

$$\begin{aligned} U(f, P_N) - L(f, P_N) &= \sum_{i=1}^N M_i[f] \Delta x_i - \sum_{i=1}^N m_i[f] \Delta x_i \\ &= \sum_{i=1}^N [f(x_i) - f(x_{i-1})] \Delta x_i \\ &= [f(b) - f(a)] \frac{b-a}{N} < \varepsilon. \end{aligned}$$

By Riemann's Condition, f is Riemann integrable. Similarly a decreasing function on $[a, b]$ is also Riemann integrable on $[a, b]$. □

Uniform Continuity Theorem

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ then f is uniformly continuous on $[a, b]$.

Maximum-Minimum Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function on $[a, b]$. Then

$$\exists p, q \in [a, b] : f(p) \leq f(x) \leq f(q).$$

Theorem 2.5. If $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ then f is Riemann integrable on $[a, b]$.

Proof. Let $\varepsilon > 0$. Since f is continuous on $[a, b]$, f is uniformly continuous on $[a, b]$. Then

$$\exists \delta : \forall x, y \in [a, b] : |x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

Let $P = \{a = x_0, x_1, \dots, x_n = b\}$ be a partition of $[a, b]$ such that

$$\Delta_i = x_i - x_{i-1} < \delta \quad \text{for } i = 1, 2, \dots, n.$$

By Maximum-Minimum Theorem,

$$\exists s_i, t_i \in [x_{i-1}, x_i] : m_i[f] = f(s_i) \wedge M_i[f] = f(t_i) \quad \text{for } i = 1, 2, \dots, n.$$

Since $|s_i - t_i| < \delta$, we have

$$0 \leq M_i[f] - m_i[f] = f(t_i) - f(s_i) < \frac{\varepsilon}{b - a} \quad \text{for } i = 1, 2, \dots, n.$$

Therefore,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n (M_i[f] - m_i[f]) \Delta x_i \\ &< \sum_{i=1}^n \left(\frac{\varepsilon}{b - a} \right) \Delta x_i = \frac{\varepsilon}{b - a} (b - a) = \varepsilon. \end{aligned}$$

□

Example 2.2. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a function defined as

$$f(x) = \begin{cases} x \sin \frac{1}{x} & : x \neq 0 \\ 0 & : x = 0. \end{cases}$$

Since f is continuous on $[0, 1]$, f is Riemann integrable on $[a, b]$.

Linearity of Riemann Integral

Theorem 2.6. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable functions.

(1) For $\alpha \in \mathbb{R}$, αf is Riemann integrable and

$$\int_a^b \alpha f(x) dx = \alpha \int_a^b f(x) dx.$$

(2) $f + g$ is Riemann integrable and

$$\int_a^b (f + g)(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

Proof. (1) We must show that $U(\alpha f) = L(\alpha f)$. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$.

(i) ($\alpha = 0$) $U(\alpha f) = 0 = L(\alpha f)$.

(ii) ($\alpha > 0$) Since

$$\begin{aligned} M_i[\alpha f] &= \sup \{ \alpha f(x) : x \in [x_{i-1}, x_i] \} \\ &= \alpha \sup \{ f(x) : x \in [x_{i-1}, x_i] \} \\ &= \alpha M_i[f], \end{aligned}$$

we have

$$\begin{aligned} U(\alpha f) &= \inf \{ U(\alpha f, P) : P \text{ be a partition of } [a, b] \} \\ &= \inf \{ \alpha U(f, P) : P \text{ be a partition of } [a, b] \} \quad \because \sum_{i=1}^n M_i[\alpha f] \Delta x_i = \alpha \sum_{i=1}^n M_i[f] \Delta x_i \\ &= \alpha \inf \{ U(f, P) : P \text{ be a partition of } [a, b] \} \\ &= \alpha U(f). \end{aligned}$$

Similarly, $L(\alpha f) = \alpha L(f)$. Since f is Riemann integrable, i.e., $L(f) = U(f)$, thus,

$$U(\alpha f) = \alpha U(f) = \alpha L(f) = L(\alpha f).$$

(iii) ($\alpha < 0$) Similarly, it holds.

Moreover,

$$\int_a^b \alpha f(x) dx = U(\alpha f) = \alpha U(f) = \alpha \int_a^b f(x) dx.$$

(2) We must show that

$$\forall \varepsilon > 0 : \exists P : U(f + g, P) - L(f + g, P) < \varepsilon.$$

Let $\varepsilon > 0$. Since f, g are Riemann integrable on $[a, b]$, $\exists P_1, P_2$ such that

$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2} \quad \text{and} \quad U(g, P_2) - L(g, P_2) < \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2 = \{x_0, x_1, \dots, x_n\}$. Then P is a partition of $[a, b]$, and is a refinement of P_1 and P_2 . Since

$$m_i[f] + m_i[g] \leq m_i[f + g] \leq M_i[f + g] \leq M_i[f] + M_i[g],$$

we have

$$L(f, P) + L(g, P) \leq L(f + g, P) \leq U(f + g, P) \leq U(f, P) + U(g, P).$$

Hence

$$\begin{aligned} U(f + g, P) - L(f + g, P) &\leq U(f, P) + U(g, P) - [L(f, P) + L(g, P)] \\ &\leq U(f, P_1) + U(g, P_2) - [L(f, P_1) + L(g, P_2)] \\ &= [U(f, P_1) - L(f, P_2)] + [U(g, P_2) - L(g, P_2)] \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

We want to show that

$$\forall \varepsilon > 0 : \left| \int_a^b (f + g)(x) dx - \int_a^b f(x) dx - \int_a^b g(x) dx \right| < \varepsilon.$$

□

Corollary 2.6.1. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable functions. Then for $\alpha, \beta \in \mathbb{R}$,*

$$\int_a^b (\alpha f + \beta g)(x) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx.$$

Theorem 2.7. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable function.

(1)

$$(\forall x \in [a, b] : f(x) \geq 0) \implies \int_a^b f(x) dx \geq 0.$$

(2)

$$(\forall x \in [a, b] : f(x) \leq g(x)) \implies \int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

Proof. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$.

(1) Since $f(x) \geq 0$ for all $x \in [a, b]$ and $m_i[f] \geq 0$ for $i = 1, \dots, n$, we have

$$\int_a^b f(x) dx = L(f) \geq L(f, P) = \sum_{i=1}^n m_i[f] \Delta x_i \geq 0.$$

(2) Since $g(x) - f(x) \geq 0$, by (1),

$$0 \leq \int_a^b (g - f)(x) dx = \int_a^b g(x) dx - \int_a^b f(x) dx \implies \int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

□

Example 2.3.

(1) Let $f(x) = 0$ and $g(x) = x$ for $x \in [-1, 3]$. Then

$$\int_{-1}^3 f(x) dx = 0 < 4 = \int_{-1}^3 g(x) dx \quad \text{but } f(x) > g(x) \text{ for } x \in [-1, 0).$$

(2) Let $f(x) = 0$ and $g(x) = \sin x$ for $x \in [0, 2\pi]$. Then

$$\int_0^{2\pi} f(x) dx = 0 = \int_0^{2\pi} g(x) dx \quad \text{but } f(x) \neq g(x) \text{ for } x \in (0, 2\pi) \setminus \{\pi\}.$$

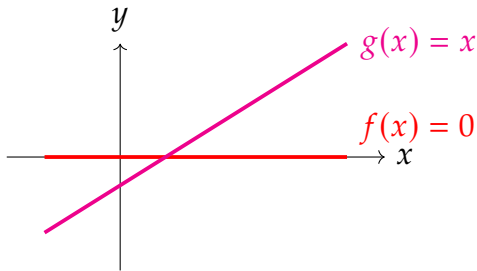


Figure 2.1: Example 2.3. - (1)

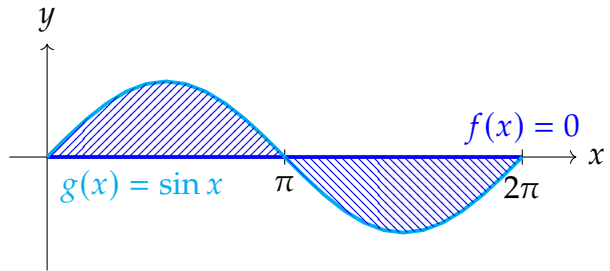


Figure 2.2: Example 2.3. - (2)

Theorem 2.8. Let $f : [a, b] \rightarrow \mathbb{R}$ be a function and $c \in (a, b)$. If f is Riemann integrable for closed sub-intervals $[a, c]$ and $[c, b]$ of $[a, b]$ then f is Riemann integrable on $[a, b]$. Moreover,

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

Proof. Let $\varepsilon > 0$. Since f is Riemann integrable on $[a, c]$,

$$\exists P_1, \text{ partition of } [a, c], \text{ such that } U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2}.$$

Since f is Riemann integrable on $[c, b]$,

$$\exists P_2, \text{ partition of } [c, b], \text{ such that } U(f, P_2) - L(f, P_2) < \frac{\varepsilon}{2}.$$

Let $P := P_1 \cup P_2$ be a partition of $[a, b]$. Then

$$\begin{aligned} U(f, P) - L(f, P) &= U(f, P_1) + U(f, P_2) - [L(f, P_1) + L(f, P_2)] \\ &= U(f, P_1) - L(f, P_1) + U(f, P_2) - L(f, P_2) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, f is Riemann integrable on $[a, b]$. By Riemann's condition,

$$\begin{aligned} \int_a^b f(x) dx &\leq U(f, P) = U(f, P_1) + U(f, P_2) \\ &< L(f, P_1) + \frac{\varepsilon}{2} + L(f, P_2) + \frac{\varepsilon}{2} \\ &\leq \int_a^c f(x) dx + \int_c^b f(x) dx + \varepsilon, \end{aligned}$$

and so

$$\int_a^b f(x) dx - \left(\int_a^c f(x) dx + \int_c^b f(x) dx \right) < \varepsilon \quad (*)$$

Since

$$\begin{aligned} \int_a^b f(x) dx &= L(f) \geq L(f, P) = L(f, P_1) + L(f, P_2) \\ &> U(f, P_1) - \frac{\varepsilon}{2} + U(f, P_2) - \frac{\varepsilon}{2} \\ &\geq \int_a^c f(x) dx + \int_c^b f(x) dx - \varepsilon, \end{aligned}$$

we have

$$-\varepsilon < \int_a^b f(x) dx - \left(\int_a^c f(x) dx + \int_c^b f(x) dx \right). \quad (**)$$

Hence, by (*) and (**)

$$\left| \int_a^b f(x) dx - \left(\int_a^c f(x) dx + \int_c^b f(x) dx \right) \right| < \varepsilon \implies \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

□

Theorem 2.9. Let $f : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable function on $[a, b]$ and $g : [c, d] \rightarrow \mathbb{R}$ be a continuous function on $[c, d]$. If $f[I] \subseteq [c, d]$, then $g \circ f$ is Riemann integrable function.

Proof. PASS. □

Corollary 2.9.1. If $f : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable function on $[a, b]$, then f^n is Riemann integrable.

Corollary 2.9.2. If $f : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable function on $[a, b]$, then $|f|$ is Riemann integrable and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

Proof. Let $x \in [a, b]$ then

$$\begin{aligned} -|f(x)| \leq f(x) \leq |f(x)| &\implies -\int_a^b |f(x)| dx \leq \int_a^b f(x) dx \leq \int_a^b |f(x)| dx \\ &\implies \left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx. \end{aligned}$$

□

Intermediate Value Theorem for Integrals

Theorem 2.10. Let f be a continuous function on $[a, b]$, then for at least one $x \in [a, b]$ we have

$$f(x) = \frac{1}{b-a} \int_a^b f(t) dt.$$

Proof. Since f is continuous on $[a, b]$,

$$\exists M = \max \{f(x) : x \in [a, b]\}, m = \min \{f(x) : x \in [a, b]\} \in \mathbb{R} : \forall t \in [a, b] : m \leq f(t) \leq M.$$

Then

$$m(b-a) = \int_a^b m dx \leq \int_a^b f(t) dt \leq \int_a^b M dt = M(b-a),$$

and so

$$m \leq \frac{1}{b-a} \int_a^b f(t) dt \leq M.$$

Then Bolzano's IVT,

$$\exists x \in [a, b] : f(x) = \frac{1}{b-a} \int_a^b f(t) dt.$$

□

2.3 The Fundamental Theorem of Calculus

★ Fundamental Theorem of Calculus: 1st form ★

Theorem 2.11. Let $f : [a, b] \rightarrow \mathbb{R}$ is differentiable on $[a, b]$ and f' is Riemann integrable on $[a, b]$. Then

$$\int_a^b f'(x) dx = f(b) - f(a).$$

Proof. We want to show that

$$(\forall \varepsilon > 0) \quad \left| \int_a^b f'(x) dx - (f(b) - f(a)) \right| < \varepsilon.$$

Let $\varepsilon > 0$. Since f' is Riemann integrable on $[a, b]$,

$$\exists P = \{x_0, \dots, x_n\} : \begin{cases} U(f', P) < U(f') + \varepsilon & \because U(f', P) > U(f') \\ L(f', P) < L(f') - \varepsilon & \because L(f', P) < L(f'). \end{cases}$$

Since f is differentiable on $[x_{i-1}, x_i]$, by Mean-Value Theorem, $\exists t_i \in [x_{i-1}, x_i]$ s.t.

$$f(x_i) - f(x_{i-1}) = f'(t_i)(x_i - x_{i-1}) \quad \text{for } i = 1, 2, \dots, n.$$

Then

$$\sum_{i=1}^n f'(t_i) \Delta x_i = \sum_{i=1}^n [f(x_i) - f(x_{i-1})] = f(x_n) - f(x_0) = f(b) - f(a).$$

Since $m_i[f'] \leq f'(t_i) \leq M_i[f']$, we have

$$\begin{aligned} L(f', P) &= \sum_{i=1}^n m_i[f'] \Delta x_i \leq \sum_{i=1}^n f'(t_i) \Delta x_i \leq \sum_{i=1}^n M_i[f'] \Delta x_i = U(f', P) \\ \implies L(f') - \varepsilon &< L(f', P) \leq f(b) - f(a) \leq U(f', P) < U(f') + \varepsilon \\ \implies -\varepsilon &< f(b) - f(a) - \int_a^b f'(x) dx < \varepsilon \quad \because U(f', P) = \int_a^b f'(x) dx = L(f', P) \\ \implies \left| f(b) - f(a) - \int_a^b f'(x) dx \right| &< \varepsilon. \end{aligned}$$

□

Example 2.4. If $g(x) = \tan^{-1} x$ for all $x \in [a, b]$ then $g'(x) = (x^2 + 1)^{-1}$ for all $x \in [a, b]$. Further, g' is continuous so it is Riemann integrable on $[a, b]$. Therefore, the fundamental theorem implies that

$$\int_a^b \frac{1}{x^2 + 1} dx = g(b) - g(a) = \tan^{-1}(b) - \tan^{-1}(a).$$

Example 2.5. If $h(x) = 2\sqrt{x}$ for all $x \in [0, b]$ then h is continuous on $[0, b]$ and $h(x) = (\sqrt{x})^{-1}$ for all $x \in (0, b]$. Since h' is not bounded on $(0, b]$, it is not Riemann integrable on $[0, b]$ no matter how we define $h(0)$. Therefore, the fundamental theorem cannot be applied. Note that

$$\int_0^b \frac{1}{\sqrt{x}} dx = \lim_{a \rightarrow 0^+} \int_a^b \frac{1}{\sqrt{x}} dx.$$

Indefinite Integral

Definition 2.6. Let $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$. The function defined by

$$F(x) := \int_a^x f(t) dt \quad \text{for } x \in [a, b]$$

is called **indefinite integral** of f with base-point a .

Lipschitz Function

Definition 2.7. A function $f : D \rightarrow \mathbb{R}$ is said to be a **Lipschitz function** or to satisfy a **Lipschitz condition** on D if

$$\exists K > 0 : |f(x) - f(y)| \leq K|x - y|.$$

Theorem 2.12. If $f : D \rightarrow \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on D .

Theorem 2.13. If $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$, then, indefinite integral F of f is uniformly continuous on $[a, b]$.

Proof. Let $x, y \in [a, b]$ with $y < x$:



Then

$$F(x) := \int_a^x f(t) dt = \int_a^y f(t) dt + \int_y^x f(t) dt \implies F(x) - F(y) = \int_y^x f(t) dt.$$

Since f is Riemann integrable on $[a, b]$ and is bounded on $[a, b]$, we have

$$\exists K > 0 : \forall t \in [a, b] : |f(t)| \leq K,$$

and so

$$\begin{aligned} & -K \leq f(t) \leq K \\ \implies & \int_y^x (-K) dt \leq \int_y^x f(t) dt \leq \int_y^x K dt \\ \implies & -K(x - y) \leq F(x) - F(y) \leq K(x - y) \\ \implies & |F(x) - F(y)| \leq K|x - y|, \end{aligned}$$

Thus F is a Lipschitz function on $[a, b]$, and so F is uniformly continuous on $[a, b]$. \square

★ Fundamental Theorem of Calculus: 2nd form ★

Theorem 2.14. Let $f : [a, b] \rightarrow \mathbb{R}$ is differentiable on $[a, b]$ and continuous at a point $c \in [a, b]$. Then the indefinite integral F is differentiable at c and

$$F'(c) = f(c).$$

Proof. We will show that $\lim_{h \rightarrow 0^+} \frac{F(c+h) - F(c)}{h} = f(c)$, i.e.,

$$(\forall \varepsilon > 0)(\exists \delta > 0) : h \in (0, \delta) \implies \left| \frac{F(c+h) - F(c)}{h} - f(c) \right| < \varepsilon.$$

Let $\varepsilon > 0$ and $c \in [a, b]$. Consider the right-hand derivative. Since f is right-continuous at c ,

$$\exists \delta > 0 : x \in [c, c + \delta) \implies |f(x) - f(c)| < \varepsilon.$$

Let $h \in \mathbb{R}$ satisfies $0 < h < \delta$, say, $h = x - c$. Then f is Riemann integrable on $[a, c + h]$, $[a, c]$ and $[c, c + h]$. Then

$$\begin{aligned} F(c+h) - F(c) &= \int_a^{c+h} f(t) dt - \int_a^c f(t) dt \\ &= \int_c^{c+h} f(t) dt. \end{aligned}$$

Since $c \leq t \leq c + h < c + \delta$, we know

$$|f(t) - f(c)| < \varepsilon, \quad \text{i.e.,} \quad f(c) - \varepsilon < f(t) < f(c) + \varepsilon.$$

Thus,

$$\begin{aligned} &\int_c^{c+h} (f(t) - \varepsilon) dt < \int_c^{c+h} f(t) dt < \int_c^{c+h} (f(t) + \varepsilon) dt \\ \implies &(f(c) - \varepsilon)h < F(c+h) - F(c) < (f(c) + \varepsilon)h \\ \implies &-\varepsilon < \frac{F(c+h) - F(c)}{h} - f(c) < \varepsilon \\ \implies &\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| < \varepsilon. \end{aligned}$$

□

Theorem 2.15. If f is continuous on $[a, b]$, then the indefinite integral

$$F(x) := \int_a^x f(t) dt \quad \text{for } x \in [a, b]$$

is differentiable on $[a, b]$ and

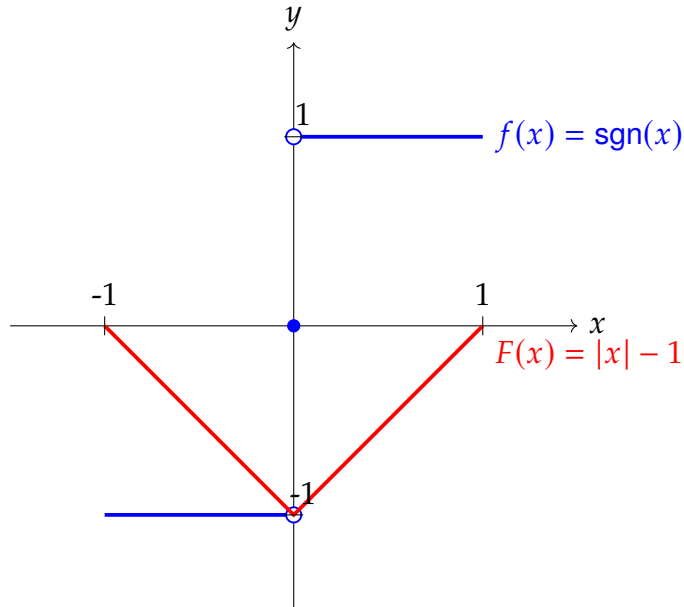
$$F'(x) = f(x)$$

for all $x \in [a, b]$.

Example 2.6. If $f(x) := \operatorname{sgn}(x)$ on $[-1, 1]$, then f is Riemann integrable and has the indefinite integral

$$F(x) := |x| - 1$$

with the basepoint -1 . However, since $F'(0)$ does not exist, F is not an anti-derivative of f on $[-1, 1]$.



Example 2.7. For $x \in [0, 3]$, if we define

$$F(x) := \int_0^x \lfloor t \rfloor dt$$

then although $f(x) = \lfloor x \rfloor$ is discontinuous on $[0, 3]$, F is continuous on $[0, 3]$.

Substitution Theorem

Theorem 2.16. Let $J := [a, b]$ and let $g : J \rightarrow \mathbb{R}$ have a continuous derivative on J . If $f : I \rightarrow \mathbb{R}$ is continuous on an interval I containing $g(J)$ then

$$\int_a^b f(g(t)) \cdot g'(t) dt = \int_{g(a)}^{g(b)} f(x) dx.$$

Proof. Since $g'(t)$ and $f(g(t))$ are both continuous on J , $f(g(t)) \cdot g'(t)$ is continuous on J . Thus $\int_a^b f(g(t)) \cdot g'(t) dt$ exists.

(1) Assume that g is constant. Since $g'(t) = 0$ and $g(a) = g(b)$,

$$\int_a^b f(g(t)) \cdot g'(t) dt = 0 = \int_{g(a)}^{g(b)} f(x) dx.$$

(2) Let g is not a constant. Then for $x \in g[J] \subseteq I$, define

$$F(x) := \int_{g(a)}^x f(s) ds.$$

By the Fundamental Theorem of Calculus: 2nd form,

$$\frac{d}{dx} F(x) = f(x).$$

and then

$$\frac{d}{dt} (F \circ g)(t) = \frac{d}{dt} F(g(t)) \frac{d}{dt} g(t) = f(g(t)) g'(t).$$

Thus

$$\begin{aligned} \int_a^b f(g(t)) \cdot g'(t) dt &= \int_a^b (F \circ g)'(t) dt \\ &= (F \circ g)(b) - (F \circ g)(a) \\ &= F(g(b)) - F(g(a)) \\ &= \int_{g(a)}^{g(b)} f(x) dx - \int_{g(a)}^{g(a)} f(x) dx \\ &= \int_{g(a)}^{g(b)} f(x) dx. \end{aligned}$$

□

Example 2.8. Consider the integral

$$\int_1^4 \frac{\sin \sqrt{t}}{\sqrt{t}} dt.$$

Let us substitution $g(t) := \sqrt{t}$ for $t \in [1, 4]$ so that $g'(t)$ is continuous on $[1, 4]$. If we let $f(x) := 2 \sin x$ then the integrand has the form $f(g(t))g'(t)$. Then the integral equals

$$\int_1^4 \frac{\sin \sqrt{t}}{\sqrt{t}} dt = \int_1^2 2 \sin x dx = 2(\cos 1 - \cos 2).$$

However, if one consider the integral

$$\int_0^4 \frac{\sin \sqrt{t}}{\sqrt{t}} dt,$$

the substitution theorem cannot be applicable since $g(t) := \sqrt{t}$ does not have a continuous derivative on $[0, 4]$. Note that

$$\int_0^4 \frac{\sin \sqrt{t}}{\sqrt{t}} dt = \lim_{a \rightarrow 0^+} \int_a^4 \frac{a}{4} f(t) dt.$$

Integration by Parts

Theorem 2.17. Let f, g be differentiable on $[a, b]$ and f', g' are Riemann integrable on $[a, b]$. Then

$$\int_a^b f(x)g'(x) dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x) dx.$$

Remark 2.2. $\int f g' = \int (f g)' - \int f' g$.

Taylor's Theorem with the Remainder

Theorem 2.18. Suppose that $f', f'', \dots, f^{(n)}, f^{(n+1)}$ exist on $[a, b]$ and that $f^{(n+1)}$ is Riemann integrable on $[a, b]$. Then we have

$$f(b) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (b-a)^i + R_n$$

where the remainder R_n is given by

$$R_n = \frac{1}{n!} \int_a^b f^{(n+1)}(t) \cdot (b-t)^n dt.$$

2.4 Improper Integrals

Improper Integral

Definition 2.8. Let f be a function and $c \in (a, b)$.

- (1) Let $f : [a, b) \rightarrow \mathbb{R}$ is Riemann integral on $[a, c]$. We say that f is **improper integrable** on $[a, b)$ if

$$\exists \lim_{c \rightarrow b-} \int_a^c f(x) dx \in \mathbb{R}.$$

- (2) Let $f : (a, b] \rightarrow \mathbb{R}$ is Riemann integral on $[c, b]$. We say that f is also **improper integrable** on $(a, b]$ if

$$\exists \lim_{c \rightarrow a+} \int_c^b f(x) dx \in \mathbb{R}.$$

Example 2.9. Let $f(x) := x^{-\frac{1}{3}}$ for $x \in (0, 1]$. Since f is unbounded on $(0, 1]$, f is not Riemann integrable. However, for every $c \in (0, 1)$,

$$\lim_{c \rightarrow 0+} \int_c^1 x^{-\frac{1}{3}} dx = \lim_{c \rightarrow 0+} \frac{3}{2}(1 - c^{2/3}) = \frac{3}{2}.$$

Hence f is improper integrable on $(0, 1]$.

Example 2.10. Let $g(x) := x^{-1}$ for $x \in (0, 1]$. Then for every $c \in (0, 1)$,

$$\lim_{c \rightarrow 0+} \int_c^1 x^{-1} dx = \lim_{c \rightarrow 0+} (-\ln c) = \infty.$$

Hence g is not improper integrable on $(0, 1]$.

Definition 2.9. Let f be defined on $[a, \infty)$ and Riemann integrable on $[a, b]$ for every $b > a$. Then f is improper integrable on $[a, \infty)$ if

$$\exists \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

and

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$$

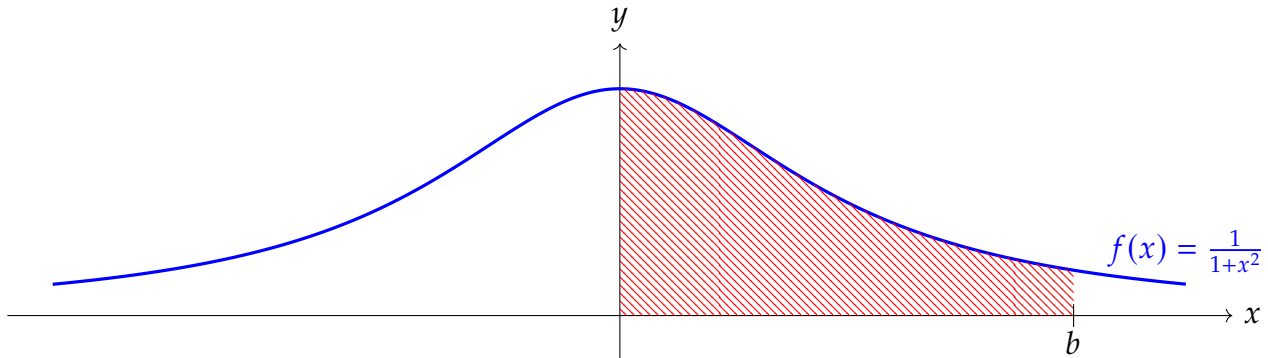
Similarly, one can define

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx.$$

Example 2.11. Let

$$f(x) := \frac{1}{1+x^2}.$$

Then f is well-defined and bound on $[0, \infty)$.



Moreover f is Riemann integrable on $[0, b]$ for every $b > 0$ since f is continuous on $[0, \infty)$. Since

$$\lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx = \lim_{b \rightarrow \infty} \left(\tan^{-1}(b) - \tan^{-1}(0) \right) = \lim_{b \rightarrow \infty} \tan^{-1}(b) = \frac{\pi}{2},$$

we obtain

$$\int_0^\infty \frac{1}{1+x^2} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx = \frac{\pi}{2}.$$

Note that

$$\int_{-\infty}^\infty \frac{1}{1+x^2} dx = \pi \implies \frac{1}{\pi} \int_{-\infty}^\infty \frac{1}{1+x^2} dx = 1,$$

and so $g(x) := \frac{1}{\pi(1+x^2)}$ be a p.d.f

Example 2.12. Since

$$\int_0^\infty f(x) dx = \int_0^1 \frac{1}{\sqrt{x}} dx + \int_1^\infty \frac{1}{\sqrt{x}} dx \quad \text{and} \quad \int_1^\infty \frac{1}{\sqrt{x}} dx = \infty,$$

$f = x^{-1/2}$ is not improper integrable on $(0, \infty)$.

Comparison Test

Theorem 2.19. Let $f, g : [a, \infty) \rightarrow \mathbb{R}$. For every $b > a$, f and g are Riemann integrable on $[a, b]$. Then if for $\geq a$, $f(x) \in [0, g(x)]$ and g is improper integrable on $[a, \infty)$, then f is improper integrable on $[a, \infty)$ and

$$\int_a^\infty f(x) dx \leq \int_a^\infty g(x) dx.$$

Proof. For $b > a$, define

$$F(b) := \int_a^b f(x) dx \quad \text{and} \quad G(b) := \int_a^b g(x) dx.$$

Since $0 \leq f(x) \leq g(x)$ and $\exists \lim_{b \rightarrow \infty} G(b)$,

$$0 \leq F(b) \leq G(b) \leq \lim_{b \rightarrow \infty} G(b).$$

Let

$$A := \left\{ \int_a^c f(x) dx : a \leq c \right\}$$

then

- (i) $\exists \int_a^b f(x) dx \implies A \neq \emptyset$ and
- (ii) A has an upper bound $\lim_{b \rightarrow \infty} G(b)$.

By the completeness axiom of real number,

$$\exists \sup A = \lim_{b \rightarrow \infty} F(b) = \int_a^\infty f(x) dx,$$

i.e., f is improper integrable on $[a, \infty)$. Moreover,

$$\int_a^\infty f(x) dx \leq \int_a^\infty g(x) dx.$$

□

Theorem 2.20. Let $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$ for every $b > a$. Then

$$\exists M \in \mathbb{R}^+ : \int_a^\infty |f(x)| dx \leq M \implies \exists \int_a^\infty f(x) dx \exists \int_a^\infty |f(x)| dx.$$

2.5 Exercises

Exercise 2.1. Generate a function f which is bounded but isn't integrable on $[a, b]$.

Solution. Let $f : [a, b] \rightarrow \mathbb{R}$ be a Dirichlet's discontinuous function defined by

$$f(x) := \begin{cases} 1 & : x \in \mathbb{Q} \\ 0 & : x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then, f is bounded on $[a, b]$ but f is not Riemann integrable. □

Exercise 2.2. Give an example of an integrable function f that $f(x_0) > 0$ for $x_0 \in [a, b]$ but such that $\int_a^b f(x) dx = 0$.

Solution. Let $f : [a, b] \rightarrow \mathbb{R}$ be a function defined by

$$f(x) := \begin{cases} 1 & : x = x_0 \\ 0 & : x \in [a, b] \setminus \{x_0\}. \end{cases}$$

Then $f(x_0) > 0$ but $\int_a^b f(x) dx = 0$. □

Exercise 2.3. Given an example of a function $f : [0, 1] \rightarrow \mathbb{R}$ that isn't Riemann integrable but such that $|f|$ is Riemann integrable on $[0, 1]$.

Solution. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a function defined by

$$f(x) := \begin{cases} 1 & : x \in \mathbb{Q} \\ -1 & : x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then, f is not Riemann integrable on $[a, b]$ but $|f|$ is Riemann integrable on $[0, 1]$. □

Exercise 2.4. Assume that $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$. For $x \in [a, b]$, let

$$F(x) = \int_a^x f(t) dt$$

then show that F is **Lipschitz** function on $[a, b]$.

Solution. Theorem 2.13 □

Exercise 2.5. If f and g are continuous on $[a, b]$ and if

$$\int_a^b f(x) dx = \int_a^b g(x) dx,$$

prove that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

Solution. Since f and g are continuous on $[a, b]$, $f - g$ is also continuous on $[a, b]$. By the intermediate value theorem for integrals,

$$\exists c \in [a, b] : (f - g)(c) = \frac{1}{b - a} \int_a^b (f(x) - g(x)) dx.$$

Since $\int_a^b f(x) dx = \int_a^b g(x) dx$,

$$(f - g)(c) = 0 \implies f(c) = g(c).$$

□

Exercise 2.6. If f is continuous on $[-a, a]$, show that $\int_{-a}^a f(x^2) dx = 2 \int_0^a f(x^2) dx$.

Solution. Since

$$\int_{-a}^a f(x^2) dx = \int_{-a}^0 f(x^2) dx + \int_0^a f(x^2) dx,$$

and by the substitution theorem yields

$$\int_{-a}^0 f(x^2) dx \stackrel{x=-t}{=} \int_a^0 f(t^2)(-dt) = \int_0^a f(t^2) dt.$$

Hence $\int_{-a}^a f(x^2) dx = \int_0^a f(t^2) dt + \int_0^a f(x^2) dx = 2 \int_0^a f(x^2) dx$.

□

Exercise 2.7. Prove that $f(x) = \frac{e^{-x}}{1+x^2}$ is improper integrable on $[0, \infty)$.

Solution. Let

$$g(x) := \frac{1}{1+x^2}$$

for $x \in [0, \infty)$. Note that g is improper integrable on $[0, \infty)$ and $\int_0^\infty g(x) dx = \frac{\pi}{2}$. Since $e^{-x} \leq 1$ on $[0, \infty)$,

$$0 \leq f(x) \leq g(x).$$

Therefore, $f(x)$ is improper integrable on $[0, \infty)$ and $\int_0^\infty \frac{e^{-x}}{1+x^2} \leq \frac{\pi}{2}$.

□

Exercise 2.8. Prove that $\int_1^\infty \frac{1}{x^p} dx$ diverges when $p \leq 1$ and converges when $p > 1$.

Solution. Since

$$\begin{aligned}\int_1^b \frac{1}{x} dx &= \ln b - \ln 1, \\ \int_1^b \frac{1}{x^p} dx &= \frac{1}{1-p} \left(\frac{1}{b^{p-1}} - 1 \right) \quad \text{for } p \neq 1.\end{aligned}$$

We can see that the improper integral converges if $p > 1$ and diverges if $p \leq 1$. \square

Exercise 2.9. Prove that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

[Hint: use the polar coordinate system.]

Solution. Let $I = \int_0^\infty e^{-x^2} dx$. Then

$$\begin{aligned}I^2 &= \left(\int_0^\infty e^{-x^2} dx \right) \left(\int_0^\infty e^{-x^2} dx \right) \\ &= \left(\int_0^\infty e^{-x^2} dx \right) \left(\int_0^\infty e^{-y^2} dy \right) \\ &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy \\ &= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r dr d\theta \\ &= \int_0^{\frac{\pi}{2}} \left[-\frac{1}{2} \cdot e^{-r^2} \right]_0^\infty d\theta \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} \right) d\theta \\ &= \left[\frac{1}{2} \theta \right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.\end{aligned}$$

Since $e^{-x^2} \geq 0$, we have

$$I^2 = \left(\int_0^\infty e^{-x^2} dx \right) \left(\int_0^\infty e^{-x^2} dx \right) = \frac{\pi}{4} \implies I = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

\square

Exercise 2.10. Suppose that f is continuous on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$. Show that if $\int_a^b f(x) dx = 0$ then $f(x) = 0$ for all $x \in [a, b]$.

Solution. Assume that $f(x_0) \neq 0$.

□

Exercise 2.11.

Solution.

□

Exercise 2.12. Let f and g be Riemann integrable on $[a, b]$. Then show that fg is Riemann integrable on $[a, b]$.

Solution. Since $(f + g)^2$ and $(f - g)^2$ are Riemann integrable on $[a, b]$,

$$fg = \frac{1}{4} \left((f + g)^2 - (f - g)^2 \right)$$

is Riemann integrable on $[a, b]$.

□

Chapter 3

Sequence of Functions

3.1 Pointwise and Uniform Convergence

3.2 Interchange of Limits

3.3 Series of Functions

3.4 Power Series

3.5 Exercises

Exercise 3.1. For $x \in [0, 1]$,

$$f_n(x) = \frac{1}{n^2 x^2 + 1}.$$

Then show that $\{f_n\}$ converges pointwise on $[0, 1]$ but not converges uniformly.

Solution.

□

Exercise 3.2. For $x \in [0, 1]$,

$$f_n(x) = nx(1 - x)^n.$$

Then show that $\{f_n\}$ converges pointwise on $[0, 1]$ but not converges uniformly.

Solution.

□

Exercise 3.3. For $x \in \mathbb{R}$, let

$$f_n(x) = \frac{nx}{1 + n^2 x^2}.$$

Then show that for $a > 0$, $\{f_n\}$ converges uniformly on $[a, \infty)$ but does not converge uniformly on $[0, \infty)$.

Solution.

□

Exercise 3.4. Let $I = (-\infty, \infty)$. Then for $f_n(x) = n^3 e^{-n}(1 - 2 \sin^2 nx)$, the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly.

Solution.

□

Exercise 3.5. Let $I = [0, 1]$. Then, if $f_n(x) = x^n$, $n \in \mathbb{N}$ show that the series

$$\sum_{n=1}^{\infty} f_n(x)$$

is a pointwise convergent but not a uniformly convergent series.

Solution.

□

Exercise 3.6. Evaluate the radius of convergence of following power series

1. $\sum_{n=0}^{\infty} \frac{x^n}{2^n},$

2. $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) x^n,$

3. $\sum_{n=0}^{\infty} n^2(x-4)^n.$

4. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 3^n},$

5. $\sum_{n=0}^{\infty} \frac{x^n}{n!},$

6. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n.$

Solution.

□

Exercise 3.7. Evaluate the radius of convergence of following power series

1. $\sum_{n=0}^{\infty} \frac{x^n}{3^n + 5^n},$

2. $\sum_{n=0}^{\infty} \frac{n^3 x^n}{n!}.$

Solution.

□

Exercise 3.8. Let the radius of convergence of following power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is R then evaluate the radius of convergence of

$$\sum_{n=0}^{\infty} a_n x^{-n}.$$

Solution.

□

Exercise 3.9. Suppose that the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is R . What is the radius of convergence of the following power series?

$$\sum_{n=0}^{\infty} a_n x^{2n}.$$

Solution.

□

Chapter 4

Introduction to Fourier Series and Transform

4.1 Periodic Functions and Trigonometric Series

Periodic Functions

Definition 4.1. A function $f(x)$ is called **periodic** if

(1) it is defined for all $x \in \mathbb{R}$ and

(2) if $\exists p > 0$ such that

$$f(x + p) = f(x).$$

This number p is called a **period** of $f(x)$.

Trigonometric Series

Definition 4.2. The series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called a **trigonometric series**, and the a_n and b_n are called the coefficients of the series, where $a_0, a_1, a_2, \dots, b_1, b_2, \dots$ are real constants.

Remark 4.1.

- Fourier series arise from the practical task of representing a given periodic function $f(x)$ in terms of cosine and sine functions.
- These series are trigonometric series whose coefficients are determined from $f(x)$ by the Euler formulas, which we shall derive first.
- Afterwards we shall take a look at the theory of Fourier series.

4.2 Fourier Series

Fourier Series of a Periodic Function of Period 2π

Theorem 4.1. Assume that $f(x)$ is a periodic function of period 2π and is integrable over a period, that is,

$$f(x + 2\pi) = f(x) \quad \text{and} \quad \exists \int_x^{x+2\pi} f(t) dt = \int_{-\pi}^{\pi} f(x) dx.$$

Then, $f(x)$ can be represented by a trigonometric series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

for all $n \in \mathbb{N}$.

Proof. (1) Since

$$\int_{-\pi}^{\pi} \cos nx dx = 0 = \int_{-\pi}^{\pi} \sin nx dx,$$

we have

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) dx &= \int_{-\pi}^{\pi} a_0 dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) dx \\ &= \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos nx + b_n \sin nx) dx \\ &= \int_{-\pi}^{\pi} a_0 dx = 2\pi a_0. \end{aligned}$$

$$\text{and so } a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx.$$

(2) Let $m \in \mathbb{N}$. Then

$$\begin{aligned} f(x) \cos mx &= a_0 \cos mx + \cos mx \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \\ \int_{-\pi}^{\pi} f(x) \cos mx &= \int_{-\pi}^{\pi} a_0 \cos mx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n \cos nx \cos mx + b_n \sin nx \cos mx). \end{aligned}$$

Note that

$$\begin{aligned}\int_{-\pi}^{\pi} \cos nx \cos mx \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} \cos(n+m)x \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \cos(n-m)x \, dx, \\ \int_{-\pi}^{\pi} \sin nx \cos mx \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} \sin(n+m)x \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \sin(n-m)x \, dx,\end{aligned}$$

and

$$\frac{1}{2} \int_{-\pi}^{\pi} \cos(n-m)x \, dx = \begin{cases} 2\pi & : n = m, \\ 0 & : n \neq m. \end{cases}$$

Thus

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \frac{1}{2} \cdot 2\pi a_m = \pi a_m \xrightarrow{n=m} a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx.$$

(3) Similarly, we have

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \pi b_m \xrightarrow{n=m} b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

□

Example 4.1 (Rectangular Wave). Find the Fourier series of the periodic function $f(x)$ defined by

$$f(x) := \begin{cases} -k & : -\pi < x < 0 \\ k & : 0 < x < \pi \end{cases} \quad \text{and} \quad f(x+2\pi) = f(x).$$

Solution. $a_0 = 0$, $a_n = 0$ and

$$b_n = \begin{cases} \frac{4k}{(2k+1)\pi} & : n = 2k+1, \\ 0 & : n = 2k. \end{cases}$$

□

Remark 4.2 (The Gibbs' phenomenon). Its sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left- and right-hand limits of $f(x)$ at x_0 . In other words, if f is not continuous at x_0 then

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx_0 + b_n \sin nx_0) = \frac{1}{2} (f(x_0+) + f(x_0-)).$$

Representation by a Fourier series

Theorem 4.2. If a periodic function $f(x)$ with period 2π is

- (1) having continuous first and second derivatives,
 - (2) piecewise continuous in the interval $[-\pi, \pi]$,
 - (3) having a left-hand derivative and right-hand derivative at each point of that interval,
- then the Fourier series of $f(x)$ is convergent.

Solution. Since

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\
 &= \frac{1}{\pi} \left[f(x) \cdot \frac{1}{n} \sin nx \right]_{-\pi}^{\pi} - \frac{1}{n\pi} \int_{-\pi}^{\pi} f'(x) \sin nx \, dx \\
 &= -\frac{1}{n\pi} \left[f'(x) \cdot \frac{-1}{n} \cos nx \right]_{-\pi}^{\pi} - \frac{1}{n\pi} \int_{-\pi}^{\pi} f''(x) \left(-\frac{1}{n} \cos nx \right) dx \\
 &= \frac{1}{n^2\pi} \left[f'(x) \cos nx \right]_{-\pi}^{\pi} + \frac{1}{n^2\pi} \int_{-\pi}^{\pi} f''(x) \cos nx \, dx
 \end{aligned}$$

and f'' is continuous on $[-\pi, \pi]$, we have $\exists M > 0$ s.t. $|f''(x)| \leq M$. It follows that

$$|a_n| = \frac{1}{n^2\pi} \left| \int_{-\pi}^{\pi} f''(x) \cos nx \, dx \right| < \frac{1}{n^2\pi} \int_{-\pi}^{\pi} M \, dx = \frac{2M}{n^2}.$$

Similarly, $|b_n| < \frac{2M}{n^2}$. Thus,

$$\begin{aligned}
 |f(x)| &= \left| a_0 + \sum_{i=1}^{\infty} (a_i \cos ix + b_i \sin ix) \right| \leq |a_0| + \sum_{n=1}^{\infty} (|a_n| + |b_n|) \\
 &\leq |a_0| + \sum_{n=1}^{\infty} \frac{4M}{n^2}.
 \end{aligned}$$

Let $M_n := \frac{4M}{n^2}$ for $n \in \mathbb{N}$. Since $\exists |a_0| + \sum_{n=1}^{\infty} M_n$, by Weierstrass M -test,

$$|f(x)| \text{ converges} \implies f(x) \text{ converges uniformly on } [-\pi, \pi].$$

□

Note (Review). For $\mathbf{a} = (1, 2, 3) \in \mathbb{R}^3$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a orthonormal basis for \mathbb{R}^3 . Then

$$\begin{cases} \mathbf{e}_1 = (1, 0, 0) \\ \mathbf{e}_2 = (0, 1, 0) \\ \mathbf{e}_3 = (0, 0, 1) \end{cases} \implies \mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 = (\mathbf{a} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{a} \cdot \mathbf{e}_2)\mathbf{e}_2 + (\mathbf{a} \cdot \mathbf{e}_3)\mathbf{e}_3 = \sum_{n=1}^3 (\mathbf{a} \cdot \mathbf{e}_n)\mathbf{e}_n.$$

Note (Orthogonality Property of the Trigonometric System). Let us define an inner product on the interval $[-\pi, \pi]$ such that

$$\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x)g(x) dx.$$

Here, we have

$$(1) \langle 1, 1 \rangle = 2\pi.$$

$$(2) \langle 1, \cos nx \rangle = \int_{-\pi}^{\pi} \cos nx dx = 0.$$

$$(3) \langle 1, \sin nx \rangle = \int_{-\pi}^{\pi} \sin nx dx = 0.$$

$$(4) \langle \cos nx, \sin nx \rangle = \pi = \langle \sin nx, \sin nx \rangle.$$

$$(5) \langle \cos mx, \cos nx \rangle = \int_{-\pi}^{\pi} \cos mx \cos nx dx = 0 \text{ for } n \neq m.$$

$$(6) \langle \sin mx, \sin nx \rangle = \int_{-\pi}^{\pi} \sin mx \sin nx dx = 0 \text{ for } n \neq m.$$

$$(7) \langle \cos mx, \sin nx \rangle = \int_{-\pi}^{\pi} \cos mx \sin nx dx = 0 \text{ for any } n, m.$$

Then the trigonometric system

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx, \dots$$

is **orthogonal** on the interval $[-\pi, \pi]$. Moreover,

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \dots$$

is orthonormal. Note that

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ &= \frac{1}{2\pi} \langle f(x), 1 \rangle 1 + \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \langle f(x), \cos x \rangle \cos x + \frac{1}{\pi} \langle f(x), \sin nx \rangle \sin nx \right) \end{aligned}$$

and that

$$f(x) = \left\langle f(x), \frac{1}{\sqrt{2\pi}} \right\rangle \cdot \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \left(\left\langle f(x), \frac{\cos nx}{\sqrt{\pi}} \right\rangle \frac{\cos nx}{\sqrt{\pi}} + \left\langle f(x), \frac{\sin nx}{\sqrt{\pi}} \right\rangle \frac{\sin nx}{\sqrt{\pi}} \right).$$

4.3 Functions of Any Period $p = 2L$, Even and Odd Functions

Fourier Series of a Periodic Function of Period $2L$

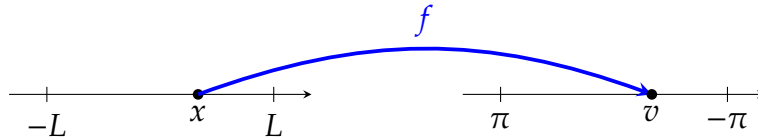
Theorem 4.3. A function $f(x)$ of period $p = 2L$ has a **Fourier series**. This series can be written:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas**, for $n = 1, 2, \dots$,

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx, \quad a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \\ b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx.$$

Proof.



Let $v = \frac{\pi}{L}x$. Then a function $g(v)$ defined by

$$f(x) = f\left(\frac{L}{\pi}v\right) =: g(v)$$

has period of 2π . Then $g(v)$ has the Fourier series

$$g(v) = a_0 + \sum_{i=0}^{\infty} (a_n \cos nv + b_n \sin nv) \quad \text{with} \quad \begin{cases} a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(v) dv, \\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \cos nv dv, \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \sin nv dv. \end{cases}$$

Since $dv = \frac{\pi}{L}dx$, we have

$$f(x) = a_0 + \sum_{i=0}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \quad \text{with} \quad \begin{cases} a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx, \\ a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \\ b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx. \end{cases}$$

□

Example 4.2 (Example (Half-Wave Rectifier)). A sinusoidal voltage $E \sin \omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave. Let

$$u(t) = \begin{cases} 0 & \text{if } -L < t < 0, \\ E \sin \omega t & \text{if } 0 < t < L, \end{cases} \quad p = 2L = \frac{2\pi}{\omega}.$$

Then, find the Fourier series of the periodic function $u(t)$.

Solution. The Fourier series of the $u(t)$ is

$$u(t) = \frac{E}{\pi} + \frac{E}{2} \sin \omega t - \frac{2E}{\pi} \left(\frac{1}{1 \cdot 3} \cos 2\omega t + \frac{1}{3 \cdot 5} \cos 4\omega t + \cdots \right).$$

□

Even and Odd Functions

Definition 4.3. (1) A function $y = f(x)$ is **even** if

$$f(-x) = f(x), \quad x \in \mathbb{R}.$$

The graph of such a function is symmetric with respect to the y -axis.

(2) A function $g(x)$ is **odd** if

$$g(-x) = -g(x), \quad x \in \mathbb{R}.$$

The graph of such a function is symmetric with respect to the origin.

Remark 4.3. $f(x)$ and $g(x)$ satisfy $\int_{-L}^L f(x) dx = 2 \int_0^L f(x) dx$ and $\int_{-L}^L g(x) dx = 0$.

Fourier Cosine and Sine Series

Theorem 4.4. (1) The Fourier series of an even function of period $2L$ is a **Fourier cosine series**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

with coefficients

$$a_0 = \frac{1}{L} \int_0^L f(x) dx \quad \text{and} \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 1, 2, \dots$$

(2) The Fourier series of an odd function of period $2L$ is a **Fourier sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

with coefficients

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, \dots$$

Proof. (1) Since f is even,

$$\begin{cases} f(x) \cos x \text{ is even} \\ f(x) \sin x \text{ is odd} \end{cases} \implies \begin{cases} a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{2L} \cdot 2 \int_0^L f(x) dx = \frac{1}{L} \int_0^L f(x) dx, \\ a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \\ b_n = \frac{1}{L} \int_{-\pi}^{\pi} f(x) \sin \frac{n\pi x}{L} dx = 0. \end{cases}$$

(2) Similarly, it holds.

□

Example 4.3 (Sawtooth Wave). Find the Fourier series of the function

$$f(x) = x + \pi$$

with $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$.

Solution. Let $f_1(x) := x$ then

$$f(x) = \pi + x = \pi + f_1(x).$$

Then since $f_1(x) = x$ is odd, $a_n = 0$ for $n = 0, 1, 2, \dots$, and

$$b_n = \frac{2}{\pi} \int_0^\pi f_1(x) \sin \frac{n\pi x}{\pi} dx = \frac{2}{\pi} \int_0^\pi x \sin nx dx = -\frac{2}{n} \cos n\pi = (-1)^n \frac{2}{n}.$$

Hence, the Fourier series of $f(x)$ is

$$f(x) = \pi + f_1(x) = \pi + 2 \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots \right).$$

□

4.4 Introduction to Complex Fourier Series

Complex Fourier Series

Theorem 4.5. For a function of period $2L$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

the complex Fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp \left(i \frac{n\pi x}{L} \right), \quad \text{where} \quad c_n = \frac{1}{2L} \int_{-L}^L f(x) \exp \left(-i \frac{n\pi x}{L} \right) dx$$

for $n = 0, \pm 1, \pm 2, \dots$.

Proof. Recall the Euler formula

$$e^{inx} = \cos nx + i \sin nx \quad \text{and} \quad e^{-inx} = \cos nx - i \sin nx.$$

Then, we can examine that

$$\cos nx = \frac{1}{2} (e^{inx} + e^{-inx}) \quad \text{and} \quad \sin nx = \frac{1}{2i} (e^{inx} - e^{-inx}).$$

Hence,

$$\begin{aligned} a_n \cos nx + b_n \sin nx &= a_n \frac{1}{2} (e^{inx} + e^{-inx}) + b_n \frac{1}{2i} (e^{inx} - e^{-inx}) \\ &= \frac{1}{2} (a_n - ib_n) e^{inx} + \frac{1}{2} (a_n + ib_n) e^{-inx}. \end{aligned}$$

Let us write

$$c_n = \frac{1}{2} (a_n - ib_n) \quad \text{and} \quad d_n = \frac{1}{2} (a_n + ib_n).$$

Then, we can write

$$f(x) = c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + d_n e^{-inx}),$$

where

$$c_n = \frac{1}{2} (a_n - ib_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) (\cos nx - i \sin nx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx,$$

$$d_n = \frac{1}{2} (a_n + ib_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) (\cos nx + i \sin nx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} dx.$$

Note that $d_n = c_{-n}$ for all n . □

Remark 4.4. The c_n are called the **complex Fourier coefficients** of $f(x)$.

4.5 Fourier Integrals

Fourier Integral

Theorem 4.6. Let f be a nonperiodic function. Then,

$$f(x) = \int_0^{\infty} (A(\omega) \cos \omega x + B(\omega) \sin \omega x) d\omega,$$

where

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos \omega v dv \quad \text{and} \quad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin \omega v dv.$$

This is called a representation of $f(x)$ by a **Fourier integral**.

Proof. (1) We consider any periodic function $f_L(x)$ of period $2L$ that can be represented by a Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \omega_n x + b_n \sin \omega_n x), \quad \omega_n = \frac{n\pi}{L},$$

and find out what happens if we let $L \rightarrow \infty$. Let us write, for $\omega_n = \frac{n\pi}{L}$,

$$\begin{aligned} f_L(x) &= a_0 + \sum_{n=1}^{\infty} (a_n \cos \omega_n x + b_n \sin \omega_n x) \\ &= \frac{1}{2L} \int_{-L}^L f(v) dv \\ &\quad + \frac{1}{L} \sum_{n=1}^{\infty} \left(\cos \omega_n x \int_{-L}^L f(v) \cos \omega_n v dv \right) \\ &\quad + \frac{1}{L} \sum_{n=1}^{\infty} \left(\sin \omega_n x \int_{-L}^L f(v) \sin \omega_n v dv \right). \end{aligned}$$

We now set

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L} \quad \text{implies} \quad \frac{1}{L} = \frac{\Delta\omega}{\pi}.$$

Then we can write the Fourier series in the form

$$\begin{aligned} f_L(x) &= \frac{1}{2L} \int_{-L}^L f(v) dv \\ &\quad + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\cos \omega_n x \Delta\omega \int_{-L}^L f(v) \cos \omega_n v dv \right) \\ &\quad + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\sin \omega_n x \Delta\omega \int_{-L}^L f(v) \sin \omega_n v dv \right). \end{aligned}$$

This representation is valid for any fixed L , arbitrary large, but finite.

(2) Let $L \rightarrow \infty$ and assume that the resulting nonperiodic function

$$f(x) = \lim_{L \rightarrow \infty} f_L(x)$$

is absolutely integrable on the x -axis. Then

$$\frac{1}{2L} \int_{-L}^L f_L(v) dv \rightarrow 0 \quad \text{and} \quad \Delta\omega = \frac{\pi}{L} \rightarrow 0.$$

It seems plausible that the infinite series becomes an integral from 0 to ∞ , which represents $f(x)$,

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[\cos \omega x \int_{-\infty}^\infty f(v) \cos \omega v dv + \sin \omega x \int_{-\infty}^\infty f(v) \sin \omega v dv \right] d\omega.$$

□

Example 4.4 (Single Pulse, Sine Integral). Find the Fourier integral representation of the function

$$f(x) := \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1 \end{cases}$$

Solution. Since

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \cos \omega v dv = \frac{1}{\pi} \int_{-1}^1 \cos \omega v dv = \frac{2 \sin \omega}{\pi \omega} \quad \text{and} \quad B(\omega) = 0,$$

the Fourier integral representation of $f(x)$ is

$$F(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos \omega x \sin \omega}{\omega} d\omega.$$

□

Remark 4.5. The average of the left- and right-hand limits of $f(x)$ at $x = 1$ is equal to $1/2$. Furthermore, we obtain

$$\int_0^\infty \frac{\cos \omega x \sin \omega}{\omega} d\omega = \begin{cases} \frac{\pi}{2} & \text{if } 0 \leq x < 1, \\ \frac{\pi}{4} & \text{if } x = 1, \\ 0 & \text{if } x > 1. \end{cases}$$

This integral is called **Dirichlet's discontinuous factor**. If $x = 0$, then

$$\int_0^\infty \frac{\sin \omega}{\omega} d\omega = \frac{\pi}{2}.$$

We see that this integral is the limit of the so-called **sine integral**

$$\text{Si}(x) = \int_0^x \frac{\sin \omega}{\omega} d\omega$$

as $x \rightarrow \infty$.

Fourier Sine and Cosine Integrals

Theorem 4.7. (1) If $f(x)$ is an even function, then $B(\omega) = 0$ and

$$A(\omega) = \frac{2}{\pi} \int_0^{\infty} f(v) \cos \omega v \, dv.$$

The Fourier integral then reduces to the **Fourier cosine integral**

$$f(x) = \int_0^{\infty} A(\omega) \cos \omega x \, d\omega.$$

(2) If $f(x)$ is an odd function, then $A(\omega) = 0$ and

$$B(\omega) = \frac{2}{\pi} \int_0^{\infty} f(v) \sin \omega v \, dv.$$

The Fourier integral then reduces to the **Fourier sine integral**

$$f(x) = \int_0^{\infty} B(\omega) \sin \omega x \, d\omega.$$

Proof. content...

□

Theorem ()**Fourier Cosine and Sine Series****Theorem 4.8.** *content...***Fourier Cosine and Sine Series****Definition 4.4.** *content...***Observation**

- If $f(x)$ is an even function, then

$$f(x) = \int_0^\infty \left(\frac{2}{\pi} \int_0^\infty f(v) \cos \omega v \, dv \right) \cos \omega x \, d\omega = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos \omega x \, d\omega.$$

- If $f(x)$ is an odd function, then

$$f(x) = \int_0^\infty \left(\frac{2}{\pi} \int_0^\infty f(v) \sin \omega v \, dv \right) \sin \omega x \, d\omega = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega) \sin \omega x \, d\omega.$$

Definition (Fourier Cosine and Sine Transforms)

- Fourier cosine transform for an even function:

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx.$$

- Inverse Fourier cosine transform of $\hat{f}_c(\omega)$:

$$\hat{f}_c^{-1}(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos \omega x \, d\omega.$$

- Fourier sine transform for an odd function:

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx.$$

- Inverse Fourier sine transform of $\hat{f}_s(\omega)$:

$$\hat{f}_s^{-1}(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega) \sin \omega x \, d\omega.$$

Example (Fourier cosine and sine transforms)

Find Fourier cosine and sine transforms of the function

$$f(x) := \begin{cases} k & \text{if } 0 < x < a, \\ 0 & \text{if } x > a \end{cases}$$

Answer.Fourier cosine and sine transforms of $f(x)$ are

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^a k \cos \omega x \, dx = \sqrt{\frac{2}{\pi}} \left(\frac{k \sin a\omega}{\omega} \right),$$

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^a k \sin \omega x \, dx = \sqrt{\frac{2}{\pi}} \left(\frac{k(1 - \cos a\omega)}{\omega} \right).$$

Observation (D'Alembert's Solution)

- Introduce the new independent variables

$$v = x + ct \quad \text{and} \quad z = x - ct.$$

- Then, applying chain rule

$$u_x = u_v v_x + u_z z_x = u_v + u_z.$$

- Assume that all the partial derivatives involved are continuous, so that

$$u_{xv} = u_{vx} \quad \text{and correspondingly} \quad u_{xx} = (u_v + u_z)_x = u_{vv} + 2u_{vz} + u_{zz}.$$

- Similarly,

$$u_{tt} = c^2(u_{vv} - 2u_{vz} + u_{zz}).$$

By inserting these two results in the wave equation, we get

$$u_{vx} = \frac{\partial^2 u}{\partial z \partial v} = 0.$$

By integrating above identity with respect to z , we find

$$\frac{\partial u}{\partial v} = h(v),$$

where $h(v)$ is an arbitrary function of v . Integrating this with respect to v gives

$$u = \int h(v) dv + \psi(z),$$

where $\psi(z)$ is an arbitrary function of z . Since the integral is a function of v , say, $\phi(v)$, the solution u is of the form

$$u(x, t) = \phi(v) + \psi(z) = \phi(x + ct) + \psi(x - ct).$$

This is known as d'Alembert's solution of the wave equation.

Observation (D'Alembert's Solution with Initial Conditions)

- Consider the two initial conditions

$$u(x, 0) = f(x) \quad \text{and} \quad u_t(x, 0) = g(x).$$

- Then, we have

$$u(x, 0) = \phi(x) + \psi(x) = f(x) \quad \text{and} \quad u_t(x, 0) = \phi'(x) - \psi'(x) = g(x).$$

- Thus, we have

$$\phi(x) - \psi(x) = k(x_0) \quad \text{where} \quad k(x_0) = \phi(x_0) - \psi(x_0).$$

- Therefore,

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c} \int_{x_0}^x g(s)ds + \frac{1}{2}k(x_0)$$

and similarly,

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c} \int_{x_0}^x g(s)ds - \frac{1}{2}k(x_0).$$

Theorem (D'Alembert's Solution of the Wave Equation)

The solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with two initial conditions

$$u(x, 0) = f(x) \quad \text{and} \quad u_t(x, 0) = g(x)$$

is given by

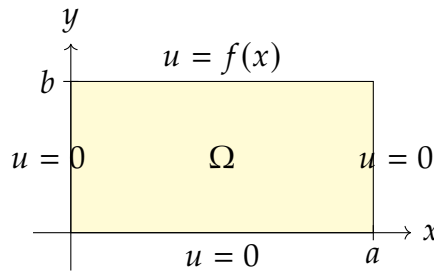
$$u(x, t) = \frac{1}{2}(f(x + ct) + f(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)ds.$$

Laplace Equation

Definition 4.5. The two-dimensional **Laplace equation** is given by

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Remark 4.6 (Boundary conditions). A heat problem then consists of this equation to be considered in some region Ω of the xy -plane and a given boundary condition on the boundary curve of Ω . This is called a boundary value problem.



One calls it:

1. Dirichlet problem if u is prescribed on $\partial\Omega$,
2. Neumann problem if the normal derivative $\nabla u \cdot N = \frac{\partial u}{\partial N}$ is prescribed on $\partial\Omega$,
3. Mixed problem if u is prescribed on a portion of $\partial\Omega$ and $\nabla u \cdot N$ on the rest of $\partial\Omega$.

Observation (Solving Laplace Equation) Find a solution $u(x, y)$, which satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with Dirichlet problem in a rectangle Ω , assuming that the temperature $u(x, y)$ equals a given function $f(x)$ on the upper side and 0 on the other three sides of the rectangle.

Observation (Method of Separating Variables)

- Let $u(x, y) = F(x)G(y)$.
- Then, we have

$$\frac{1}{F} \frac{d^2 F}{dx^2} = -\frac{1}{G} \frac{d^2 G}{dy^2} = -k.$$

- From this and the left and right boundary conditions,

$$\frac{d^2 F}{dx^2} + kF = 0, \quad F(0) = F(a) = 0.$$

- This gives $k = \left(\frac{n\pi}{a}\right)^2$ and corresponding nonzero solutions $F(x) = F_n(x) = \sin \frac{n\pi x}{a}$, $n = 1, 2, \dots$

The differential equation for G then becomes

$$\frac{d^2 G}{dy^2} - \left(\frac{n\pi}{a}\right)^2 G = 0$$

and solutions are

$$G(y) = G_n(y) = A_n e^{\frac{n\pi y}{a}} + B_n e^{-\frac{n\pi y}{a}}.$$

The boundary condition $u = 0$ on the lower side of Ω implies that $G_n(0) = A_n + B_n = 0$ or $B_n = -A_n$. This gives

$$G_n(y) = A_n e^{\frac{n\pi y}{a}} - A_n e^{-\frac{n\pi y}{a}} = 2A_n \sinh \frac{n\pi y}{a}.$$

By letting $2A_n = A_n^*$, we obtain the eigenfunctions

$$u_n(x, y) = F_n(x)G_n(y) = A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}.$$

These satisfy the boundary condition $u = 0$ on the left, right, and lower sides.

Observation (Method of Separating Variables)

- Let

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}.$$

- From the boundary condition on the upper side $u(x, b) = f(x)$, we have

$$u(x, b) = f(x) = \sum_{n=1}^{\infty} \left(A_n^* \sin \frac{n\pi x}{a} \right) \sinh \frac{n\pi b}{a}.$$

- This shows that

$$A_n^* \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx.$$

- Therefore, the solution is

$$u(x, y) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a},$$

where

$$A_n^* = \frac{2}{a \sinh \left(\frac{n\pi b}{a} \right)} \int_0^a f(x) \sin \frac{n\pi x}{a} dx.$$

Fourier Cosine and Sine Series**Theorem 4.9.** *content...***Fourier Cosine and Sine Series****Definition 4.6.** *content...*