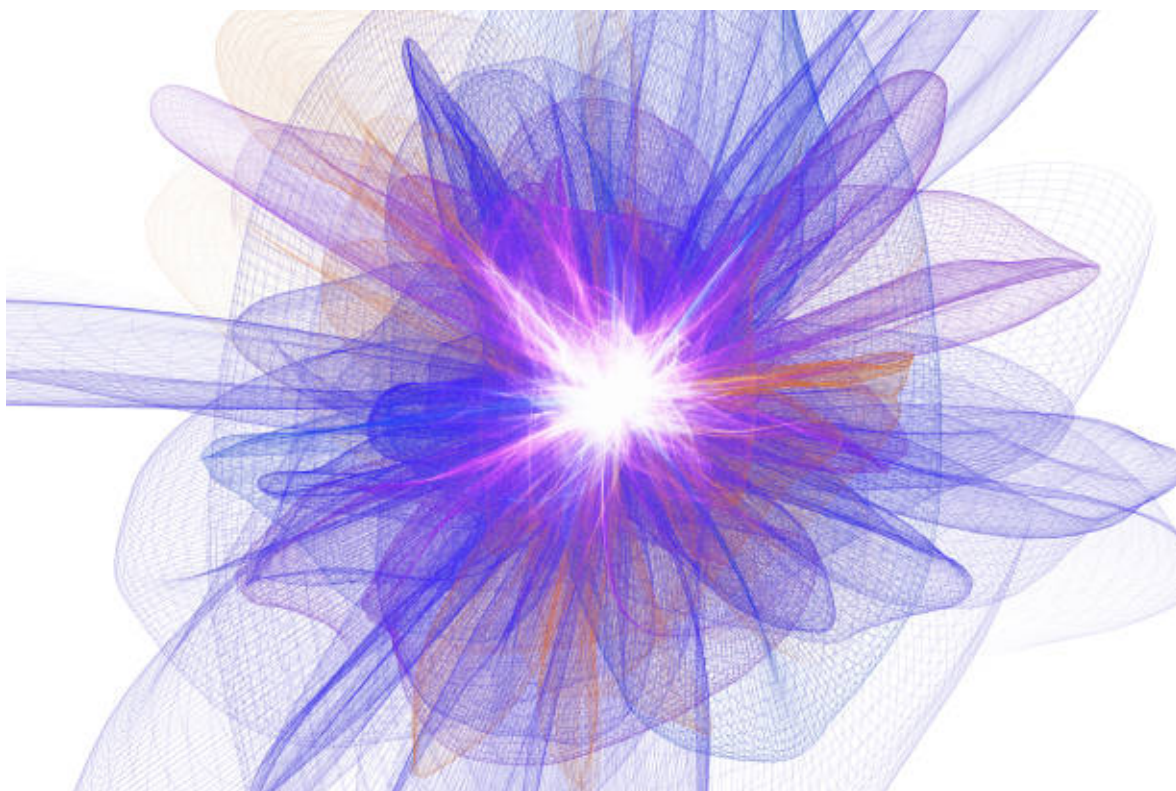


Introduction to Applied Mathematics

- Advance Calculus II -

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Chapter 1

Differentiation

1.1 Derivative and Carathéodory's Theorem

Derivative

Definition 1.1. Let $f : I \rightarrow \mathbb{R}$ and $a \in I$. We say that $L \in \mathbb{R}$ is the **derivative of f at a** if

$$\forall \epsilon > 0 : \exists \delta > 0 : x \in \mathcal{N}_\delta^*(a) \cap I \implies \left| \frac{f(x) - f(a)}{x - a} - L \right| < \epsilon.$$

Remark 1.1. We say that f is **differentiable** at a , and we write $L = f'(a)$. In other words,
$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

Proposition 1.1. If $f : I \rightarrow \mathbb{R}$ has a derivative at $a \in I$ then f is continuous at a . That is,

$$\exists f'(a) \implies f(a) = \lim_{x \rightarrow a} f(x).$$

Proof. Let $\exists f'(a)$. Then

$$\begin{aligned} \lim_{x \rightarrow a} [f(x) - f(a)] &= \lim_{x \rightarrow a} \left[\frac{f(x) - f(a)}{x - a} \cdot (x - a) \right] \quad \because x \in \mathcal{N}_\delta^*(a) \implies x \neq a \\ &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \lim_{x \rightarrow a} (x - a) \\ &= f'(a) \cdot 0 = 0. \end{aligned}$$

□

Remark 1.2. The continuity of $f : I \rightarrow \mathbb{R}$ at point does not assure the existence of the derivative at that point, e.g., $f(x) := |x|$ for $x \in \mathbb{R}$.

★ Carathéodory's Theorem ★

Theorem 1.2. Let f be defined on an interval I containing the point a . Then

$$\exists f'(a) \iff \exists \varphi \in \mathbb{R}^I \text{ such that } \begin{cases} \varphi \text{ is continuous on } I & \cdots (1) \\ f(x) - f(a) = \varphi(x)(x - a) & \cdots (2) \end{cases}$$

In this case, we have $\varphi(a) = f'(a)$.

Proof. (\Rightarrow) Assume that $\exists f'(a)$. Define a function $\varphi : I \rightarrow \mathbb{R}$ as following

$$\varphi(x) = \begin{cases} \frac{f(x) - f(a)}{x - a} & : x \neq a \\ f'(a) & : x = a. \end{cases}$$

Then

(i) φ is continuous on I , i.e., for all $a \in I$,

$$\lim_{x \rightarrow a} \varphi(x) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = f'(a) = \varphi(a).$$

(ii)

$$\begin{cases} f(x) - f(a) = \varphi(x)(x - a) & : x \neq a \\ 0 = \varphi(x) \cdot 0 & : x = a. \end{cases}$$

(\Leftarrow) Let $x \neq a$ and $x \rightarrow a$. The continuity of φ gives that

$$\exists \varphi(a) = \lim_{x \rightarrow a} \varphi(x) = \lim_{x \rightarrow a} \frac{\varphi(x)(x - a)}{(x - a)} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

That is, f is differentiable at a and $f'(a) = \varphi(a)$.

□

Example 1.1. Let us consider the function f defined by $f(x) := x^3$ for $x \in \mathbb{R}$. For any $a \in \mathbb{R}$, we see from the factorization

$$f(x) - f(a) = x^3 - a^3 = (x^2 + ax + a^2)(x - a)$$

that $\varphi(x) := x^2 + ax + a^2$ satisfies the condition of Carathéodory's Theorem. Therefore, we conclude that f is differentiable at $a \in \mathbb{R}$ and that $f'(a) = \varphi(a) = 3a^2$.

Chain Rule

Theorem 1.3. Let I, J be intervals in \mathbb{R} , let $g : J \rightarrow \mathbb{R}$ and $f : I \rightarrow \mathbb{R}$ be functions such that $f[I] \subseteq J$, and let $a \in I$. Then

$$\exists f'(a) \exists g'(f(a)) \implies \exists (g \circ f)'(a)$$

and $(g \circ f)'(a) = g'(f(a))f'(a)$.

Proof. We must show that there exists a continuous function $\varphi(x)$ s.t.

$$g(f(x)) - g(f(a)) = \varphi(x)(x - a).$$

(1) Since $\exists f'(a)$, by Carathéodory's Theorem, $\exists \sigma : I \rightarrow \mathbb{R}$ s.t.

- (i) σ is continuous at $a \in I$;
- (ii) $f(x) - f(a) = \sigma(x)(x - a)$;
- (iii) $f'(a) = \sigma(a)$.

(2) Since $\exists g'(f(a))$, by Carathéodory's Theorem, $\exists \tau : J \rightarrow \mathbb{R}$ s.t.

- (i) τ is continuous at $f(a) \in J$;
- (ii) $g(f(x)) - g(f(a)) = \tau(f(x))(f(x) - f(a))$;
- (iii) $g'(f(a)) = \tau(f(a))$.

Then

$$\begin{aligned} g(f(x)) - g(f(a)) &= \tau(f(x))(f(x) - f(a)) \quad \text{by (2)-(ii)} \\ &= \tau(f(x))\sigma(x)(x - a) \quad \text{by (1)-(ii)}. \end{aligned}$$

Let $\varphi(x) := \tau(f(x))\sigma(x)$. Then

- (i) $\varphi : I \rightarrow \mathbb{R}$ is continuous at a and
- (ii) $g(f(x)) - g(f(a)) = \varphi(x)(x - a)$,

and so, by Carathéodory's Theorem,

$$\exists (g \circ f)'(a) = \varphi(a) = \tau(f(a)) \cdot \sigma(a) = g'(f(a)) \cdot f'(a).$$

□

Remark 1.3. If f is a differentiable function, then the chain rule implies that the function $g \circ f = |f|$ is also differentiable at all points x where $f(x) \neq 0$, and its derivative is given by

$$|f(x)|'(x) = \text{sgn}(f(x)) \cdot f'(x) = \begin{cases} f'(x) & : f(x) > 0, \\ -f'(x) & : f(x) < 0. \end{cases}$$

Remark 1.4. A function f that is differentiable at every point of \mathbb{R} **need not** have a continuous derivative f' .

Differentiability of The Inverse Function

Theorem 1.4. Let $f : I \rightarrow \mathbb{R}$ be strictly monotone and continuous on I . Let $J := f[I]$ and $g : J \rightarrow \mathbb{R}$ be the strictly monotone and continuous function inverse to f . Then

$$\exists f'(a) \neq 0 \implies \exists g'(f(a)) = \frac{1}{f'(a)}.$$

Proof. Since $\exists f'(a)$, by Carathéodory's Theorem, $\exists \sigma : I \rightarrow \mathbb{R}$ s.t.

- (i) σ is continuous at $a \in I$;
- (ii) $f(x) - f(a) = \sigma(x)(x - a)$;
- (iii) $f'(a) = \sigma(a) \neq 0$.

Since $\sigma(a) \neq 0$, $\exists \delta > 0$ s.t. $\sigma(x) \neq 0$, $x \in \mathcal{N}_\delta(a) \cap I$. Let $\Omega := f[\mathcal{N}_\delta(a) \cap I]$. Since $g = f^{-1}$, we have

$$\begin{aligned} f(x) - f(a) &= f((g \circ f)(x)) - f((g \circ f)(a)) \quad \because f \circ g = \text{id} \\ &= \sigma((g \circ f)(x))((g \circ f)(x) - (g \circ f)(a)) \quad \text{by (ii)}. \end{aligned}$$

Since $f(x) \in \Omega \implies \sigma(x) \neq 0 \implies \sigma((g \circ f)(x)) \neq 0$,

$$g(f(x)) - g(f(a)) = \frac{1}{\sigma((g \circ f)(x))}(f(x) - f(a)).$$

Let $\varphi(x) := 1/\sigma((g \circ f)(x))$. Then φ is continuous at $f(a)$. By Carathéodory's Theorem,

$$g'(f(a)) = \varphi(a) = \frac{1}{\sigma((g \circ f)(a))} = \frac{1}{\sigma(a)} = \frac{1}{f'(a)}.$$

□

1.2 The Rolle's Theorem and the Mean Value Theorem

Absolute and Local Maxi/Mini-mum

Definition 1.2. Let $f : I \rightarrow \mathbb{R}$ be a function.

- f has an **absolute maximum** at $a \in I$ if $x \in I \implies f(x) \leq f(a)$.
- f has an **absolute minimum** at $a \in I$ if $x \in I \implies f(a) \leq f(x)$.
- f is said to have a **local (or relative) maximum** at $a \in I$ if

$$\exists \mathcal{N}_\delta(a) : f(x) \leq f(a), x \in \mathcal{N}_\delta(a) \cap I.$$

- f is said to have a **local (or relative) minimum** at $a \in I$ if

$$\exists \mathcal{N}_\delta(a) : f(a) \leq f(x), x \in \mathcal{N}_\delta(a) \cap I.$$

- f has a **local (or relative extremum)** at $a \in I$ either a relative maximum or a relative minimum at a .

Interior Extremum Theorem

Theorem 1.5. Let $f : (a, b) \rightarrow \mathbb{R}$ has a relative extremum and $c \in (a, b)$. Then

$$\exists f'(c) \implies f'(c) = 0.$$

Proof. Let f has a relative maximum at c , i.e.,

$$\exists \mathcal{N}_\delta(c) : x \in \mathcal{N}_\delta(c) \cap (a, b) \implies f(x) \leq f(c).$$

Assume that $f'(c) > 0$ then

$$\exists \mathcal{N}_\delta(c) \subseteq (a, b) : x \in \mathcal{N}_\delta^+(c) \implies \frac{f(x) - f(c)}{x - c} > 0.$$

If $c \in \mathcal{N}_\delta(c)$ and $x > c$, then we have

$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f(c)}{x - c} > 0.$$

But this contradicts the hypothesis that f has a relative maximum at c . Similarly if $f'(c) < 0$ then we have a contradiction. Hence $f'(c) = 0$. \square

Corollary 1.5.1. Let $f : (a, b) \rightarrow \mathbb{R}$ be continuous on (a, b) and suppose that f has a relative extremum at $c \in (a, b)$. Then either

$$\nexists f'(c) \quad \text{or} \quad f'(c) = 0.$$

★ Rolle's Theorem

Theorem 1.6. Let f be continuous on $I = [a, b]$, and let f be differentiable on (a, b) . Then

$$f(a) = 0 = f(b) \implies \exists c \in (a, b) : f'(c) = 0.$$

★ Mean Value Theorem of Differential Calculus ★

Theorem 1.7. Let f be continuous on $I = [a, b]$, and let f be differentiable on (a, b) . Then

$$\exists c \in (a, b) : f(b) - f(a) = f'(c)(b - a).$$

Proof. Consider the function whose graph is the line segment joining the points $(a, f(a))$ and $(b, f(b))$:

$$f(x) - f(a) = \frac{f(b) - f(a)}{b - a}(x - a).$$

Define a function $g : [a, b] \rightarrow \mathbb{R}$ s.t.

$$g(x) := f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then

- (i) g is continuous on $[a, b]$;
- (ii) g is differentiable on (a, b) ;
- (iii) $g(a) = 0 = g(b)$.

By Rolle's Theorem, $\exists c \in (a, b) : g'(c) = 0$. Then

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \implies g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \implies f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

Example 1.2. Prove that $e^x \geq 1 + x$ for $x \in \mathbb{R}$.

Solution. (1) $x = 0 \implies e^x = 1 + x$.

(2) Let $x > 0$ and $f(x) = e^x$. Then, by MVT,

$$\exists c \in (0, x) : f(x) - f(0) = f'(c)(x - 0),$$

and so

$$e^x - 1 = e^c x > x \implies e^x > 1 + x.$$

(3) Let $x < 0$ and $f(x) = e^x$. Then, by MVT,

$$\exists c \in (x, 0) : f(0) - f(x) = f'(c)(0 - x),$$

and so

$$1 - e^x = e^c(-x) < -x \implies 1 + x < e^x.$$

□

1.3 L'Hôpital's Rules

Theorem 1.8. Let f and g be defined on $[a, b]$, let $f(a) = 0 = g(a)$, and let $g(x) \neq 0$ for $x \in (a, b)$. If f and g are differentiable at a if $g'(a) \neq 0$, then the limit f/g at a exists and is equal to $f'(a)/g'(a)$. Thus

$$\lim_{x \rightarrow a+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof. Since $f(a) = 0 = g(a)$,

$$\lim_{x \rightarrow a+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a+} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \rightarrow a+} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}.$$

□

Remark 1.5. In L'Hôpital Rules, the hypothesis $f(a) = 0 = g(a)$ is essential. For example, it $f(x) := x + 17$ and $g(x) := 2x + 3$ for $x \in \mathbb{R}$ then,

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{17}{3} \quad \text{while} \quad \frac{f'(0)}{g'(0)} = \frac{1}{2}.$$

Cauchy's Mean Value Theorem of Differential Calculus

Theorem 1.9. Let f and g be continuous on $[a, b]$ and differentiable on (a, b) , and assume that $g'(x) \neq 0$ for all $x \in (a, b)$. Then

$$\exists c \in (a, b) : \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Since $g'(x) \neq 0$ for $x \in (a, b)$, $g(a) \neq g(b)$ by Rolle's Theorem. Define $h : [a, b] \rightarrow \mathbb{R}$ such that

$$h(x) := f(x) - g(a) - \frac{f(b) - f(a)}{g(b) - g(a)} (g(x) - g(a)).$$

Then

- (i) h is continuous on $[a, b]$;
- (ii) h is differentiable on (a, b) ;
- (iii) $h(a) = 0 = h(b)$.

By Rolle's Theorem,

$$\exists c \in (a, b) : h'(c) = 0.$$

Since $h'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(x)$, we have

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) \implies \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

□

Remark 1.6. Note that if $g(x) := x$ then the Cauchy's mean value theorem reduces to the mean value theorem.

Remark 1.7. By the Mean Value Theorem,

$$\exists \alpha, \beta \in (a, b) : \begin{cases} f(b) - f(a) = f'(\alpha)(b - a) \\ f(b) - f(a) = g'(\beta)(b - a) \end{cases}.$$

If $g'(x) \neq 0$ for $x \in (a, b)$, we have $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\alpha)}{g'(\beta)}$.

L'Hôpital's Rule - 1st

Theorem 1.10. Let $-\infty \leq a < b \leq \infty$ and let f, g be differentiable on (a, b) such that $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose that

$$\lim_{x \rightarrow a+} f(x) = 0 = \lim_{x \rightarrow a+} g(x).$$

Then

$$(1) \lim_{x \rightarrow a+} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \rightarrow a+} \frac{f(x)}{g(x)} = L.$$

$$(2) \lim_{x \rightarrow a+} \frac{f'(x)}{g'(x)} = \pm\infty \implies \lim_{x \rightarrow a+} \frac{f(x)}{g(x)} = \pm\infty.$$

Proof. We must show that $\lim_{x \rightarrow a+} \frac{f(x)}{g(x)} = L$, i.e.,

$$\begin{aligned} \forall \varepsilon > 0 : \exists \delta > 0 : x \in (a, a + \delta) &\implies \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon \\ \iff \forall \varepsilon > 0 : \exists c \in (a, b) : x \in (a, c) &\implies \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon. \end{aligned}$$

Since $g'(x) \neq 0$ for $x \in (a, b)$,

$$a < \alpha < x < b \implies g(x) - g(\alpha) \neq 0.$$

By Cauchy's Mean-Value Theorem,

$$\exists \gamma \in (\alpha, x) : \frac{f(x) - f(\alpha)}{g(x) - g(\alpha)} = \frac{f'(\gamma)}{g'(\gamma)}.$$

Let $\varepsilon > 0$. Then

$$\lim_{\gamma \rightarrow a+} \frac{f'(\gamma)}{g'(\gamma)} = L \implies \exists c \in (a, b) : \left[a < \gamma < x < c \implies \left| \frac{f'(\gamma)}{g'(\gamma)} - L \right| < \frac{\varepsilon}{2} \right]$$

Then

$$\begin{aligned}
& L - \frac{\varepsilon}{2} < \frac{f'(\gamma)}{g'(\gamma)} < L + \frac{\varepsilon}{2} \\
\Rightarrow & L - \frac{\varepsilon}{2} < \frac{f(x) - f(\alpha)}{g(x) - g(\alpha)} < L + \frac{\varepsilon}{2} \\
\Rightarrow & \lim_{\alpha \rightarrow a+} \left(L - \frac{\varepsilon}{2} \right) \leq \lim_{\alpha \rightarrow a+} \frac{f(x) - f(\alpha)}{g(x) - g(\alpha)} \leq \lim_{\alpha \rightarrow a+} \left(L + \frac{\varepsilon}{2} \right) \quad \because \lim_{\alpha \rightarrow a+} f(x) = 0 = \lim_{\alpha \rightarrow a+} g(x) \\
\Rightarrow & L - \frac{\varepsilon}{2} < L - \varepsilon \leq \frac{f(x)}{g(x)} \leq L + \frac{\varepsilon}{2} < L + \varepsilon \\
\Rightarrow & \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon.
\end{aligned}$$

Thus, $\lim_{x \rightarrow a+} \frac{f(x)}{g(x)} = L$. □

Example 1.3. Let $I := (0, \pi/2)$. Then evaluate

$$\lim_{x \rightarrow 0+} \left(\frac{1}{x} - \frac{1}{\sin x} \right),$$

which has the indeterminate form $\infty - \infty$.

Solution.

$$\lim_{x \rightarrow 0+} \left(\frac{1}{x} - \frac{1}{\sin x} \right) = \lim_{x \rightarrow 0+} \frac{\sin x - 1}{x \sin x} = \lim_{x \rightarrow 0+} \frac{\cos x - 1}{\sin x + x \cos x} = \lim_{x \rightarrow 0+} \frac{-\sin x}{2 \cos x - x \sin x} = 0.$$

□

Example 1.4. Let $I := (0, \infty)$. Then evaluate

$$\lim_{x \rightarrow 0+} x \ln x,$$

which has the indeterminate form $0 \times \infty$.

Solution.

$$\lim_{x \rightarrow 0+} x \ln x = \lim_{x \rightarrow 0+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0+} (-x) = 0.$$

□

Example 1.5. Let $I := (0, \infty)$ and consider

$$\lim_{x \rightarrow 0+} x^x$$

which has the indeterminate form 0^0 .

Solution. Let $f(x) := x^x$ then $\ln f(x) = x \ln x$. Then

$$\lim_{x \rightarrow 0+} (x \ln x) = \lim_{x \rightarrow 0+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0+} (-x) = 0.$$

Thus, $\lim_{x \rightarrow 0+} f(x) = \lim_{x \rightarrow 0+} e^{\ln f(x)} = e^0 = 1$. □

Example 1.6. Let $I := (0, \infty)$. Then evaluate

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x,$$

which has the indeterminate form 1^∞ .

Solution. Let $f(x) := \left(1 + \frac{1}{x}\right)^x$ then $\ln f(x) = x \ln \left(1 + \frac{1}{x}\right)$. Then

$$\lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x}\right) \stackrel{t=1/x}{=} \lim_{t \rightarrow 0+} \frac{\ln(1+t)}{t} = \lim_{t \rightarrow 0+} \frac{\frac{1}{1+t}}{1} = 1.$$

Thus, $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^1 = e$. □

Example 1.7. Let $I := (0, \infty)$. Then evaluate

$$\lim_{x \rightarrow \infty} (1+x)^{\frac{1}{x}},$$

which has the indeterminate form ∞^0 .

Solution. Let $f(x) := (1+x)^{1/x}$ then $\ln f(x) = \frac{\ln(1+x)}{x}$. Then

$$\lim_{x \rightarrow \infty} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{1+x}}{1} = 0.$$

Thus, $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^0 = 1$. □

1.4 Taylor's Theorem

★ Talyor's Theorem ★

Theorem 1.11. Let $n \in \mathbb{N}$ and $f : [a, b] \rightarrow \mathbb{R}$ be such that f and its derivatives $f', f'', \dots, f^{(n)}$ are continuous on $[a, b]$ and that $f^{(n+1)}$ exists on (a, b) . Then

$$t \in [a, b] \implies \forall x \in [a, b] : \exists c \in (t, x) : f(x) = \sum_{i=0}^n \frac{f^{(i)}(t)}{i!} (x-t)^i + \frac{f^{(n+1)}(c)}{(n+1)!} (x-t)^{n+1}.$$

Proof. Define a function $F : [a, b] \rightarrow \mathbb{R}$ such that

$$\begin{aligned} F(t) &= f(x) - \sum_{i=0}^n \frac{f^{(i)}(t)}{i!} (x-t)^i \\ &= f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!} (x-t)^2 - \dots - \frac{f^{(n)}(t)}{n!} (x-t)^n. \end{aligned}$$

We claim that

$$\exists c \in (a, x) : F(a) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}.$$

Define $G : [a, b] \rightarrow \mathbb{R}$ such that

$$G(t) = F(t) - \left(\frac{x-t}{x-a} \right)^{n+1} F(a).$$

Then

- (i) G is continuous on $[a, b]$;
- (ii) G is differentiable on $[a, b]$;
- (iii) $G(a) = 0 = G(b)$.

By Rolle's Theorem, $\exists c \in (a, x) : G'(c) = 0$. Then

$$G'(t) = F'(t) + \frac{(n+1)(x-t)^n}{(x-a)^{n+1}} F(a) \implies F(a) = -\frac{(x-a)^{n+1}}{(n+1)(x-c)^n} F'(c).$$

Since

$$\begin{aligned} F'(t) &= -f'(t) \\ &\quad - f''(t)(x-t) + f'(t) \\ &\quad - \frac{f'''(t)}{2!} (x-t)^2 + f''(t)(x-t) \\ &\quad - \dots \\ &\quad - \frac{f^{(n+1)}(t)}{n!} (x-t)^n + \frac{f^{(n)}(t)}{(n-1)!} (x-t)^{n-1}, \end{aligned}$$

we have

$$F'(c) = \frac{f^{(n+1)}(c)}{n!} (x-c)^n.$$

$$\text{Hence } F(a) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}.$$

□

Example 1.8 (Numerical Estimation). Approximate the number e with error less than 10^{-5} .

Solution. Let $f(x) = e^x$. Then

$$P_n(x) = \sum_{i=0}^n \frac{x^i}{i!} = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n.$$

By Taylor's theorem,

$$\exists c \in (0, x) : f(x) = P_n(x) + R_n(x), \text{ where } R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}.$$

For $c \in (0, 1)$

$$R_n(1) = \frac{e^c}{n+1} < \frac{3}{(n+1)!} < 10^{-5} \implies n = 8.$$

□

Example 1.9. For any $k \in \mathbb{N}$ and for all $x > 0$, prove that

$$x - \frac{1}{2}x^2 + \cdots - \frac{1}{2k}x^{2k} < \ln(1+x) < x - \frac{1}{2}x^2 + \cdots + \frac{1}{2k+1}x^{2k+1}.$$

Solution. Let $g(x) := \ln(1+x)$ for $x > 0$. Then

$$g'(x) = \frac{1}{1+x} \implies \begin{cases} P_n(x) = x - \frac{1}{2}x^2 + \cdots + \frac{(-1)^{n-1}}{n}x^n & \text{with } a = 0 \\ R_n(x) = \frac{(-1)^n c^{n+1}}{n+1}x^{n+1} & \text{for some } c \in (0, x) \end{cases}$$

Thus for any $x > 0$,

$$(1) \ n = 2k \implies R_{2k}(x) > 0,$$

$$(2) \ n = 2k+1 \implies R_{2k+1}(x) < 0.$$

□

1.5 Exercises

Exercise 1.1. Prove that

$$(\cos^{-1})'(x) = -\frac{1}{\sqrt{1-x^2}}$$

for $x \in (-1, 1)$.

Solution. Let $y := \cos^{-1}(x)$, i.e., $x = \cos y$. Then

$$\begin{aligned} \frac{d}{dx}x &= \frac{d}{dx}[\cos y] \implies 1 = -\sin y \cdot \frac{dy}{dx} \\ \implies -\frac{1}{\sin y} &= \frac{dy}{dx} \quad \because x \in (-1, 1) \implies y = \cos^{-1}(x) \in (0, \pi) \implies \sin y \neq 0. \end{aligned}$$

By Pythagorean identity,

$$\sin^2(y) + \cos^2(y) = 1 \implies \sin^2(y) = 1 - \cos^2(y) \implies \sin(y) = \sqrt{1-x^2}$$

and so

$$(\cos^{-1})'(x) = \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}.$$

□

Exercise 1.2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined as

$$f(x) := \begin{cases} x^2 \sin(x^{-2}) & : x \neq 0 \\ 0 & : x = 0. \end{cases}$$

Then, prove that f is differentiable on \mathbb{R} and f' is discontinuous on $[-1, 1]$.

Solution.

(1) **Differentiability of f on \mathbb{R} :** Let $x \neq 0$. Since $f(x) = x^2 \sin \frac{1}{x^2}$,

$$f'(x) = 2x \sin \frac{1}{x^2} + x^2 \cos \frac{1}{x^2} \cdot (-2) \frac{1}{x^3} = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}.$$

And

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \sin \frac{1}{h^2}}{h} = \lim_{h \rightarrow 0} h \sin \frac{1}{h^2} = 0$$

because $|\sin(h^{-2})| \leq 1 \implies 0 \leq |h \sin(h^{-2})| \leq |h|$. $\therefore \forall x \in \mathbb{R} : \exists f'(x)$.

(2) **Discontinuity of f' on $[-1, 1]$:** Let $n \in \mathbb{N}$. Then $\frac{1}{\sqrt{2n\pi}} \in [-1, 1] \setminus \{0\}$. Note that

$$f' \left(\frac{1}{\sqrt{2n\pi}} \right) = \frac{2}{\sqrt{2n\pi}} \sin(2n\pi) - 2\sqrt{2n\pi} \cos(2n\pi) = -2\sqrt{2n\pi} \neq 0.$$

Then

$$\lim_{n \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2n\pi}} = 0 \quad \text{and} \quad f' \left(\frac{1}{\sqrt{2n\pi}} \right) \neq 0 \quad \text{but} \quad f'(0) = 0.$$

□

Exercise 1.3. Let $f : I \rightarrow \mathbb{R}$ be differentiable at $c \in I$. Establish the **Straddle Lemma**: Given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $u, v \in I$ satisfy $c - \delta < u \leq c \leq v < c + \delta$, then

$$f(v) - f(u) - (v - u)f'(c) \leq \varepsilon(v - u).$$

[Hint: use the term $f(c) - cf'(c)$ and apply the Triangle Inequality.]

Solution. Let $\varepsilon > 0$. Since f is differentiable at c ,

$$\exists \delta > 0 : 0 < |x - c| < \delta \implies \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon.$$

Then

$$|f(x) - f(c) - (x - c)f'(c)| < \varepsilon|x - c|. \quad (*)$$

Let $u, v \in I$ satisfies $c - \delta < u \leq c \leq v < c + \delta$. Then

$$\begin{aligned} |f(v) - f(u) - (v - u)f'(c)| &= |f(v) - f(c) + f(c) - f(u) - (v - c + c - u)f'(c)| \\ &= |f(v) - f(c) - (v - c)f'(c) - (f(u) - f(c) - (u - c)f'(c))| \\ &\leq |f(v) - f(c) - (v - c)f'(c)| + |f(u) - f(c) - (u - c)f'(c)| \\ &< \varepsilon|v - c| + \varepsilon|u - c| \quad \text{by } (*) \\ &= \varepsilon(v - c) - \varepsilon(u - c) \quad \because u \leq c \leq v \\ &= \varepsilon(v - u). \end{aligned}$$

□

Exercise 1.4. Let $a > b > 0$ and $n \in \mathbb{N}$. Prove that

$$\sqrt[n]{a} - \sqrt[n]{b} < \sqrt[n]{a-b}$$

for $n \geq 2$.

Solution. Define $f : \mathbb{R}_{\geq 1} \rightarrow \mathbb{R}$ by

$$f(x) := \sqrt[n]{x} - \sqrt[n]{x-1}$$

for $n \geq 2$. Then

$$\begin{aligned} f'(x) &= \frac{1}{n} x^{\frac{1-n}{n}} - \frac{1}{n} (x-1)^{\frac{1-n}{n}} \\ &= \frac{1}{n} \left[\left(x^{\frac{n-1}{n}} \right)^{-1} - \left((x-1)^{\frac{n-1}{n}} \right)^{-1} \right] \\ &= \frac{1}{n} \left[\frac{1}{x^{\frac{n-1}{n}}} - \frac{1}{(x-1)^{\frac{n-1}{n}}} \right] \\ &= \frac{1}{n} \left[\frac{(x-1)^{\frac{n-1}{n}} - x^{\frac{n-1}{n}}}{x^{\frac{n-1}{n}} \cdot (x-1)^{\frac{n-1}{n}}} \right]. \end{aligned}$$

Note that

$$x > 1 \implies 0 < x-1 < x \implies (x-1)^{\frac{n-1}{n}} < x^{\frac{n-1}{n}}.$$

Thus, $f'(x) < 0$ for $x > 1$. That is, f is decreasing for $x \geq 1$. Then

$$a > b > 0 \implies 1 < \frac{a}{b} \implies f(a/b) < f(1) \implies \sqrt[n]{a/b} - \sqrt[n]{a/b-1} < 1.$$

Multiplying by $\sqrt[n]{b}$, we have

$$\sqrt[n]{a} - \sqrt[n]{a-b} < \sqrt[n]{b} \implies \sqrt[n]{a} - \sqrt[n]{b} < \sqrt[n]{a-b}.$$

□

Exercise 1.5. Use the Mean Value Theorem to show that

$$\frac{x-1}{x} < \ln x < x-1$$

for $x > 1$.

Solution.

(1) Let

$$f(x) := \ln x - \frac{x-1}{x} = \ln x - 1 + \frac{1}{x}.$$

Then $f'(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2}$. Since $x > 1$ and $f'(x) > 0$, by the Mean Value Theorem,

$$\exists c \in (1, x) : f(x) - f(1) = f'(c)(x-1),$$

i.e., $f(x) - f(1) > 0$. Thus

$$f(x) = \ln x - \frac{x-1}{x} > 0 = f(1) \implies \ln x > \frac{x-1}{x}.$$

(2) Let

$$g(x) := (x-1) - \ln x.$$

Then $g'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$. Since $x > 1$ and $g'(x) > 0$,

$$g(x) > g(1) = 0 \implies x-1 > \ln x.$$

□

Exercise 1.6. Prove or disprove: If f is differentiable and uniformly continuous on I then f is a Lipschitz function on I .

Solution. Counterexample: Let $f(x) := \sqrt{x}$ for $x \in (0, 1)$. Then f is uniformly continuous on $(0, 1)$ by continuous extension theorem. Then

$$\exists f^*(x) = \begin{cases} f(x) = \sqrt{x} & : x \in (0, 1) \\ 0 & : x = 0 \\ 1 & : x = 1. \end{cases}$$

But f is not a Lipschitz function on $(0, 1)$.

□

Exercise 1.7. Let f, g be differentiable on \mathbb{R} and suppose that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x > 0$. Show that $f(x) \leq g(x)$ for all $x > 0$.

Solution. Let $h(x) := g(x) - f(x)$. Since $h'(x) = g'(x) - f'(x) \geq 0$, h is an increasing function on $x > 0$. Thus, $g(x) \geq f(x)$ for all $x > 0$. \square

Exercise 1.8. Show that

$$\lim_{x \rightarrow c} \frac{x^c - c^x}{x^x - c^c} = \frac{1 - \ln c}{1 + \ln c}$$

for $c > 0$.

Solution. Note that

$$\begin{aligned} y := x^x &\implies \ln y = x \ln x \\ &\implies \frac{y'}{y} = \ln x + 1 \\ &\implies y' = x^x (\ln x + 1). \end{aligned} \qquad \begin{aligned} y := c^x &\implies \ln y = x \ln c \\ &\implies \frac{y'}{y} = \ln c \\ &\implies y' = c^x (\ln c). \end{aligned}$$

By L'Hôpital's rule, we have

$$\lim_{x \rightarrow c} \frac{cx^{c-1} - c^x \ln c}{x^x (\ln x + 1)} = \frac{c^c - c^c \ln c}{c^c (\ln c + 1)} = \frac{c^c (1 - \ln c)}{c^c (1 + \ln c)} = \frac{1 - \ln c}{1 + \ln c}$$

\square

Exercise 1.9. Let $f : (0, 1) \rightarrow \mathbb{R}$ be differentiable on $(0, \infty)$ and suppose that

$$\lim_{x \rightarrow \infty} (f(x) + f'(x)) = L.$$

Then prove that

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow \infty} f'(x) = 0.$$

[Hint: $f(x) = \frac{e^x f(x)}{e^x}$.]

Solution. Since $f(x) = \frac{e^x f(x)}{e^x}$,

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{e^x f(x)}{e^x} \stackrel{\text{L'Hôpital's rule}}{=} \lim_{x \rightarrow \infty} \frac{e^x f(x) + e^x f'(x)}{e^x} = \lim_{x \rightarrow \infty} (f(x) + f'(x)) = L$$

and so $\lim_{x \rightarrow \infty} f'(x) = 0$. \square

Exercise 1.10. Let $I \subseteq \mathbb{R}$ be an open interval, let $f : I \rightarrow \mathbb{R}$ be differentiable on I , and suppose $f''(a)$ exists at $a \in I$. Show that

$$f''(a) = \lim_{h \rightarrow 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2}.$$

Solution.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} &\stackrel{\text{L'Hôpital's rule}}{=} \lim_{h \rightarrow 0} \frac{f'(a+h) - f'(a-h)}{2h} \\ &= \lim_{h \rightarrow 0} \left(\frac{1}{2} \cdot \frac{f'(a+h) - f(a) + f(a) - f'(a-h)}{h} \right) \\ &= \frac{1}{2} \left(\lim_{h \rightarrow 0} \frac{f'(a+h) - f(a)}{h} + \lim_{h \rightarrow 0} \frac{f'(a-h) - f(a)}{-h} \right) \\ &= \frac{1}{2} (f''(a) + f''(a)) \\ &= f''(a). \end{aligned}$$

□

Chapter 2

The Riemann Integral

2.1 Introduction to Riemann Integral

Partition

Definition 2.1. Consider a closed bounded interval $[a, b] \subseteq \mathbb{R}$. A **partition** of $[a, b]$ is a finite ordered set

$$P := \{a = x_0, x_1, \dots, x_{n-1}, x_n = b\} \text{ s.t. } x_0 < x_1 < \dots < x_{n-1} < x_n.$$

Upper and Lower Sum

Definition 2.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded on $[a, b]$ and $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ be a partition of $[a, b]$.

(1) The **upper sum** of f for the partition P is the sum

$$U(f, P) := \sum_{i=1}^n M_i[f](x_i - x_{i-1}), \quad M_i[f] := \sup \{f(x) : x \in [x_{i-1}, x_i]\}$$

for $i = 1, 2, \dots, n$.

(2) The **lower sum** of f for the partition P is the sum

$$L(f, P) := \sum_{i=1}^n m_i[f](x_i - x_{i-1}), \quad m_i[f] := \inf \{f(x) : x \in [x_{i-1}, x_i]\}$$

for $i = 1, 2, \dots, n$.

Proposition 2.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded on $[a, b]$ and P be a partition of $[a, b]$. Then

$$L(f, P) \leq U(f, P).$$

Proof. $M_i[f] \geq m_i[f] \implies L(f, P) \leq U(f, P)$.

□

Refinement

Definition 2.3. Let Q and P be partitions of $[a, b]$ and $P \subseteq Q$. We say that Q is a **refinement** of P .

Theorem 2.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded on $[a, b]$ and $P = \{a = x_0, x_1, \dots, x_n = b\}$ be a partition of $[a, b]$. Let Q is a refinement of P . Then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Proof. Assume that $Q := P \cup \{x^*\}$ s.t.

$$\begin{aligned} Q &= \{a = x_0, x_1, x_2, \dots, x_n = b\} \cup \{x^*\} \\ &= \{a, x_1, x_2, \dots, x_{j-1}, x^*, x_j, x_{j+1}, \dots, x_{n-1}, b\}. \end{aligned}$$

Let $M_j^L = \sup \{f(x) : [x_{j-1}, x^*]\}$ and $M_j^R = \sup \{f(x) : [x^*, x_j]\}$ then

$$M_j^L[f] \leq M_j[f] \quad \text{and} \quad M_j^R[f] \leq M_j[f],$$

and we have

$$\begin{aligned} U(f, Q) &= \left(\sum_{i=1}^{j-1} M_i[f] \Delta x_i \right) + \left(M_j^L[f](x^* - x_{j-1}) \right) + \left(M_j^R[f](x_j - x^*) \right) + \left(\sum_{i=j+1}^n M_i[f] \Delta x_i \right) \\ &\leq \left(\sum_{i=1}^{j-1} M_i[f] \Delta x_i \right) + M_j[f](x^* - x_{j-1}) + M_j[f](x_j - x^*) + \left(\sum_{i=j+1}^n M_i[f] \Delta x_i \right) \\ &= \sum_{i=1}^n M_i[f] \Delta x_i = U(f, P). \end{aligned}$$

Similarly, we have $L(f, P) \leq L(f, Q)$. □

Corollary 2.2.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded on $[a, b]$ and P and Q are partitions of $[a, b]$ then

$$L(f, Q) \leq U(f, P).$$

Proof. Let $R = P \cup Q$. By **Theorem 2.2**, we have

$$L(f, Q) \leq L(f, R) \leq U(f, R) \leq U(f, P)$$

since R is a refinement of both P and Q . □

Remark 2.1. By the completeness property of real number, there exist the followings:

$$L(f) := \sup \{L(f, P) : P \text{ is a partition of } [a, b]\},$$

$$U(f) := \inf \{U(f, P) : P \text{ is a partition of } [a, b]\}.$$

Moreover, $L(f) \leq U(f)$.

Upper and Lower Integral

Definition 2.4. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded on $[a, b]$.

(1) The **upper integral** of f on $[a, b]$ is defined by

$$\overline{\int_a^b} f(x)dx := U(f) = \inf \{U(f, P) : P \text{ is a partition of } [a, b]\}.$$

(2) The **lower integral** of f on $[a, b]$ is defined by

$$\underline{\int_a^b} f(x)dx := L(f) = \sup \{L(f, P) : P \text{ is a partition of } [a, b]\}.$$

Riemann Integral

Definition 2.5. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded on $[a, b]$. We say that f is **Riemann integrable** (or **integrable**) on $[a, b]$ if $L(f) = U(f)$. We define the **Riemann integral** of f on $[a, b]$ as follow:

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

Example 2.1. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a Dirichlet's discontinuous function defined by

$$f(x) := \begin{cases} 1 & : x \in \mathbb{Q}, \\ 0 & : x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

We claim that f is not Riemann integrable.

Solution. Let $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ be a partition of $[0, 1]$. Note that

$$M_i[f] \equiv 1 \quad \text{and} \quad m_i[f] \equiv 0$$

for $i = 1, 2, \dots, n$. Then

$$L(f, P) = \sum_{i=1}^n m_i[f] \Delta x_i = \sum_{i=1}^n (0 \cdot \Delta x_i) = 0,$$

$$U(f, P) = \sum_{i=1}^n M_i[f] \Delta x_i = \sum_{i=1}^n (1 \cdot \Delta x_i) = 1.$$

Therefore $L(f) = 0 \neq 1 = U(f)$, and so f is not Riemann integrable on $[0, 1]$. □

★ Riemann's Condition ★

Theorem 2.3. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded on $[a, b]$. Then

$$\overline{\int_a^b f(x)dx} = \underline{\int_a^b f(x)dx} \iff \forall \varepsilon > 0 : \exists P : U(f, P) - L(f, P) < \varepsilon.$$

Proof. (\Rightarrow) Let $\varepsilon > 0$. Then $\exists P_1, P_2$ such that

$$L(f) - \frac{\varepsilon}{2} < L(f, P_1) \quad \text{and} \quad U(f, P_2) < U(f) + \frac{\varepsilon}{2}.$$

Let $P := P_1 \cup P_2$. Since $L(f) = U(f)$, we have

$$\begin{aligned} U(f, P) - L(f, P) &\leq U(f, P_2) - L(f, P_1) \\ &< U(f) + \frac{\varepsilon}{2} - \left(L(f) - \frac{\varepsilon}{2} \right) \\ &= \varepsilon. \end{aligned}$$

(\Leftarrow) Let P be a partition of $[a, b]$. Since $U(f) \leq U(f, P)$ and $L(f, P) \leq L(f)$, for $\varepsilon > 0$,

$$0 \leq U(f) - L(f) \leq U(f, P) - L(f, P) < \varepsilon.$$

That is, $L(f) = U(f)$.

□

2.2 Properties of Riemann Integral

Theorem 2.4. If $f : [a, b] \rightarrow \mathbb{R}$ is monotone on $[a, b]$ then f is Riemann integrable on $[a, b]$.

Proof. Suppose that f is increasing on $[a, b]$. Let $\varepsilon > 0$. By the completeness property of \mathbb{R} ,

$$\exists N \in \mathbb{N} : [f(b) - f(a)] \frac{b-a}{N} < \varepsilon.$$

Correspondingly, there exists a partition $P_N = \{x_0, x_1, \dots, x_{N-1}, x_N\}$ such that

$$\Delta x_i = x_i - x_{i-1} = \frac{b-a}{N}$$

for $i = 1, 2, \dots, N$. Since $\begin{cases} M_i[f] = \sup \{f(x) : x \in [x_{i-1}, x_i]\} = f(x_i) \\ m_i[f] = \inf \{f(x) : x \in [x_{i-1}, x_i]\} = f(x_{i-1}) \end{cases}$,

$$\begin{aligned} U(f, P_N) - L(f, P_N) &= \sum_{i=1}^N M_i[f] \Delta x_i - \sum_{i=1}^N m_i[f] \Delta x_i \\ &= \sum_{i=1}^N [f(x_i) - f(x_{i-1})] \Delta x_i \\ &= [f(b) - f(a)] \frac{b-a}{N} < \varepsilon. \end{aligned}$$

By Riemann's Condition, f is Riemann integrable. Similarly a decreasing function on $[a, b]$ is also Riemann integrable on $[a, b]$. □

Uniform Continuity Theorem

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ then f is uniformly continuous on $[a, b]$.

Maximum-Minimum Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function on $[a, b]$. Then

$$\exists p, q \in [a, b] : f(p) \leq f(x) \leq f(q).$$

Theorem 2.5. If $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ then f is Riemann integrable on $[a, b]$.

Proof. Let $\varepsilon > 0$. Since f is continuous on $[a, b]$, f is uniformly continuous on $[a, b]$. Then

$$\exists \delta : \forall x, y \in [a, b] : |x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

Let $P = \{a = x_0, x_1, \dots, x_n = b\}$ be a partition of $[a, b]$ such that

$$\Delta_i = x_i - x_{i-1} < \delta \quad \text{for } i = 1, 2, \dots, n.$$

By Maximum-Minimum Theorem,

$$\exists s_i, t_i \in [x_{i-1}, x_i] : m_i[f] = f(s_i) \wedge M_i[f] = f(t_i) \quad \text{for } i = 1, 2, \dots, n.$$

Since $|s_i - t_i| < \delta$, we have

$$0 \leq M_i[f] - m_i[f] = f(t_i) - f(s_i) < \frac{\varepsilon}{b - a} \quad \text{for } i = 1, 2, \dots, n.$$

Therefore,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n (M_i[f] - m_i[f]) \Delta x_i \\ &< \sum_{i=1}^n \left(\frac{\varepsilon}{b - a} \right) \Delta x_i = \frac{\varepsilon}{b - a} (b - a) = \varepsilon. \end{aligned}$$

□

Example 2.2. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a function defined as

$$f(x) = \begin{cases} x \sin \frac{1}{x} & : x \neq 0 \\ 0 & : x = 0. \end{cases}$$

Since f is continuous on $[0, 1]$, f is Riemann integrable on $[a, b]$.

Linearity of Riemann Integral

Theorem 2.6. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable functions.

(1) For $\alpha \in \mathbb{R}$, αf is Riemann integrable and

$$\int_a^b \alpha f(x) dx = \alpha \int_a^b f(x) dx.$$

(2) $f + g$ is Riemann integrable and

$$\int_a^b (f + g)(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

Proof. (1) We must show that $U(\alpha f) = L(\alpha f)$. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$.

(i) ($\alpha = 0$) $U(\alpha f) = 0 = L(\alpha f)$.

(ii) ($\alpha > 0$) Since

$$\begin{aligned} M_i[\alpha f] &= \sup \{ \alpha f(x) : x \in [x_{i-1}, x_i] \} \\ &= \alpha \sup \{ f(x) : x \in [x_{i-1}, x_i] \} \\ &= \alpha M_i[f], \end{aligned}$$

we have

$$\begin{aligned} U(\alpha f) &= \inf \{ U(\alpha f, P) : P \text{ be a partition of } [a, b] \} \\ &= \inf \{ \alpha U(f, P) : P \text{ be a partition of } [a, b] \} \quad \because \sum_{i=1}^n M_i[\alpha f] \Delta x_i = \alpha \sum_{i=1}^n M_i[f] \Delta x_i \\ &= \alpha \inf \{ U(f, P) : P \text{ be a partition of } [a, b] \} \\ &= \alpha U(f). \end{aligned}$$

Similarly, $L(\alpha f) = \alpha L(f)$. Since f is Riemann integrable, i.e., $L(f) = U(f)$, thus,

$$U(\alpha f) = \alpha U(f) = \alpha L(f) = L(\alpha f).$$

(iii) ($\alpha < 0$) Similarly, it holds.

Moreover,

$$\int_a^b \alpha f(x) dx = U(\alpha f) = \alpha U(f) = \alpha \int_a^b f(x) dx.$$

(2) We must show that

$$\forall \varepsilon > 0 : \exists P : U(f + g, P) - L(f + g, P) < \varepsilon.$$

Let $\varepsilon > 0$. Since f, g are Riemann integrable on $[a, b]$, $\exists P_1, P_2$ such that

$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2} \quad \text{and} \quad U(g, P_2) - L(g, P_2) < \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2 = \{x_0, x_1, \dots, x_n\}$. Then P is a partition of $[a, b]$, and is a refinement of P_1 and P_2 . Since

$$m_i[f] + m_i[g] \leq m_i[f + g] \leq M_i[f + g] \leq M_i[f] + M_i[g],$$

we have

$$L(f, P) + L(g, P) \leq L(f + g, P) \leq U(f + g, P) \leq U(f, P) + U(g, P).$$

Hence

$$\begin{aligned} U(f + g, P) - L(f + g, P) &\leq U(f, P) + U(g, P) - [L(f, P) + L(g, P)] \\ &\leq U(f, P_1) + U(g, P_2) - [L(f, P_1) + L(g, P_2)] \\ &= [U(f, P_1) - L(f, P_2)] + [U(g, P_2) - L(g, P_2)] \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

We want to show that

$$\forall \varepsilon > 0 : \left| \int_a^b (f + g)(x) dx - \int_a^b f(x) dx - \int_a^b g(x) dx \right| < \varepsilon.$$

□

Corollary 2.6.1. *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable functions. Then for $\alpha, \beta \in \mathbb{R}$,*

$$\int_a^b (\alpha f + \beta g)(x) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx.$$

Theorem 2.7. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable function.

(1)

$$(\forall x \in [a, b] : f(x) \geq 0) \implies \int_a^b f(x) dx \geq 0.$$

(2)

$$(\forall x \in [a, b] : f(x) \leq g(x)) \implies \int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

Proof. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$.

(1) Since $f(x) \geq 0$ for all $x \in [a, b]$ and $m_i[f] \geq 0$ for $i = 1, \dots, n$, we have

$$\int_a^b f(x) dx = L(f) \geq L(f, P) = \sum_{i=1}^n m_i[f] \Delta x_i \geq 0.$$

(2) Since $g(x) - f(x) \geq 0$, by (1),

$$0 \leq \int_a^b (g - f)(x) dx = \int_a^b g(x) dx - \int_a^b f(x) dx \implies \int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

□

Example 2.3.

(1) Let $f(x) = 0$ and $g(x) = x$ for $x \in [-1, 3]$. Then

$$\int_{-1}^3 f(x) dx = 0 < 4 = \int_{-1}^3 g(x) dx \quad \text{but } f(x) > g(x) \text{ for } x \in [-1, 0).$$

(2) Let $f(x) = 0$ and $g(x) = \sin x$ for $x \in [0, 2\pi]$. Then

$$\int_0^{2\pi} f(x) dx = 0 = \int_0^{2\pi} g(x) dx \quad \text{but } f(x) \neq g(x) \text{ for } x \in (0, 2\pi) \setminus \{\pi\}.$$

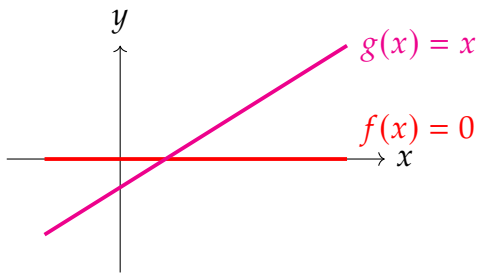


Figure 2.1: Example 2.3. - (1)

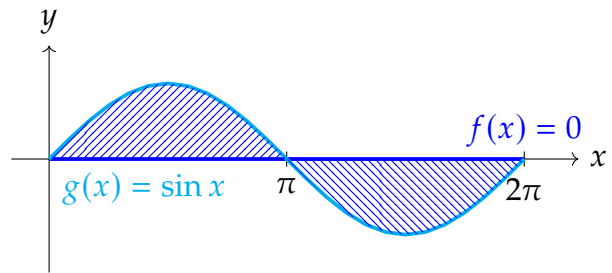


Figure 2.2: Example 2.3. - (2)

Theorem 2.8. Let $f : [a, b] \rightarrow \mathbb{R}$ be a function and $c \in (a, b)$. If f is Riemann integrable for closed sub-intervals $[a, c]$ and $[c, b]$ of $[a, b]$ then f is Riemann integrable on $[a, b]$. Moreover,

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

Proof. Let $\varepsilon > 0$. Since f is Riemann integrable on $[a, c]$,

$$\exists P_1, \text{ partition of } [a, c], \text{ such that } U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2}.$$

Since f is Riemann integrable on $[c, b]$,

$$\exists P_2, \text{ partition of } [c, b], \text{ such that } U(f, P_2) - L(f, P_2) < \frac{\varepsilon}{2}.$$

Let $P := P_1 \cup P_2$ be a partition of $[a, b]$. Then

$$\begin{aligned} U(f, P) - L(f, P) &= U(f, P_1) + U(f, P_2) - [L(f, P_1) + L(f, P_2)] \\ &= U(f, P_1) - L(f, P_1) + U(f, P_2) - L(f, P_2) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, f is Riemann integrable on $[a, b]$. By Riemann's condition,

$$\begin{aligned} \int_a^b f(x) dx &\leq U(f, P) = U(f, P_1) + U(f, P_2) \\ &< L(f, P_1) + \frac{\varepsilon}{2} + L(f, P_2) + \frac{\varepsilon}{2} \\ &\leq \int_a^c f(x) dx + \int_c^b f(x) dx + \varepsilon, \end{aligned}$$

and so

$$\int_a^b f(x) dx - \left(\int_a^c f(x) dx + \int_c^b f(x) dx \right) < \varepsilon \quad (*)$$

Since

$$\begin{aligned} \int_a^b f(x) dx &= L(f) \geq L(f, P) = L(f, P_1) + L(f, P_2) \\ &> U(f, P_1) - \frac{\varepsilon}{2} + U(f, P_2) - \frac{\varepsilon}{2} \\ &\geq \int_a^c f(x) dx + \int_c^b f(x) dx - \varepsilon, \end{aligned}$$

we have

$$-\varepsilon < \int_a^b f(x) dx - \left(\int_a^c f(x) dx + \int_c^b f(x) dx \right). \quad (**)$$

Hence, by (*) and (**)

$$\left| \int_a^b f(x) dx - \left(\int_a^c f(x) dx + \int_c^b f(x) dx \right) \right| < \varepsilon \implies \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

□

Theorem 2.9. Let $f : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable function on $[a, b]$ and $g : [c, d] \rightarrow \mathbb{R}$ be a continuous function on $[c, d]$. If $f[I] \subseteq [c, d]$, then $g \circ f$ is Riemann integrable function.

Proof. PASS. □

Corollary 2.9.1. If $f : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable function on $[a, b]$, then f^n is Riemann integrable.

Corollary 2.9.2. If $f : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable function on $[a, b]$, then $|f|$ is Riemann integrable and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

Proof. Let $x \in [a, b]$ then

$$\begin{aligned} -|f(x)| \leq f(x) \leq |f(x)| &\implies -\int_a^b |f(x)| dx \leq \int_a^b f(x) dx \leq \int_a^b |f(x)| dx \\ &\implies \left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx. \end{aligned}$$

□

Intermediate Value Theorem for Integrals

Theorem 2.10. Let f be a continuous function on $[a, b]$, then for at least one $x \in [a, b]$ we have

$$f(x) = \frac{1}{b-a} \int_a^b f(t) dt.$$

Proof. Since f is continuous on $[a, b]$,

$$\exists M = \max \{f(x) : x \in [a, b]\}, m = \min \{f(x) : x \in [a, b]\} \in \mathbb{R} : \forall t \in [a, b] : m \leq f(t) \leq M.$$

Then

$$m(b-a) = \int_a^b m dx \leq \int_a^b f(t) dt \leq \int_a^b M dt = M(b-a),$$

and so

$$m \leq \frac{1}{b-a} \int_a^b f(t) dt \leq M.$$

Then Bolzano's IVT,

$$\exists x \in [a, b] : f(x) = \frac{1}{b-a} \int_a^b f(t) dt.$$

□

2.3 The Fundamental Theorem of Calculus

★ Fundamental Theorem of Calculus: 1st form ★

Theorem 2.11. Let $f : [a, b] \rightarrow \mathbb{R}$ is differentiable on $[a, b]$ and f' is Riemann integrable on $[a, b]$. Then

$$\int_a^b f'(x) dx = f(b) - f(a).$$

Proof. We want to show that

$$(\forall \varepsilon > 0) \quad \left| \int_a^b f'(x) dx - (f(b) - f(a)) \right| < \varepsilon.$$

Let $\varepsilon > 0$. Since f' is Riemann integrable on $[a, b]$,

$$\exists P = \{x_0, \dots, x_n\} : \begin{cases} U(f', P) < U(f') + \varepsilon & \because U(f', P) > U(f') \\ L(f', P) < L(f') - \varepsilon & \because L(f', P) < L(f'). \end{cases}$$

Since f is differentiable on $[x_{i-1}, x_i]$, by Mean-Value Theorem, $\exists t_i \in [x_{i-1}, x_i]$ s.t.

$$f(x_i) - f(x_{i-1}) = f'(t_i)(x_i - x_{i-1}) \quad \text{for } i = 1, 2, \dots, n.$$

Then

$$\sum_{i=1}^n f'(t_i) \Delta x_i = \sum_{i=1}^n [f(x_i) - f(x_{i-1})] = f(x_n) - f(x_0) = f(b) - f(a).$$

Since $m_i[f'] \leq f'(t_i) \leq M_i[f']$, we have

$$\begin{aligned} L(f', P) &= \sum_{i=1}^n m_i[f'] \Delta x_i \leq \sum_{i=1}^n f'(t_i) \Delta x_i \leq \sum_{i=1}^n M_i[f'] \Delta x_i = U(f', P) \\ \Rightarrow L(f') - \varepsilon &< L(f', P) \leq f(b) - f(a) \leq U(f', P) < U(f') + \varepsilon \\ \Rightarrow -\varepsilon &< f(b) - f(a) - \int_a^b f'(x) dx < \varepsilon \quad \because U(f', P) = \int_a^b f'(x) dx = L(f', P) \\ \Rightarrow \left| f(b) - f(a) - \int_a^b f'(x) dx \right| &< \varepsilon. \end{aligned}$$

□

Example 2.4. If $g(x) = \tan^{-1} x$ for all $x \in [a, b]$ then $g'(x) = (x^2 + 1)^{-1}$ for all $x \in [a, b]$. Further, g' is continuous so it is Riemann integrable on $[a, b]$. Therefore, the fundamental theorem implies that

$$\int_a^b \frac{1}{x^2 + 1} dx = g(b) - g(a) = \tan^{-1}(b) - \tan^{-1}(a).$$

Example 2.5. If $h(x) = 2\sqrt{x}$ for all $x \in [0, b]$ then h is continuous on $[0, b]$ and $h(x) = (\sqrt{x})^{-1}$ for all $x \in (0, b]$. Since h' is not bounded on $(0, b]$, it is not Riemann integrable on $[0, b]$ no matter how we define $h(0)$. Therefore, the fundamental theorem cannot be applied. Note that

$$\int_0^b \frac{1}{\sqrt{x}} dx = \lim_{a \rightarrow 0^+} \int_a^b \frac{1}{\sqrt{x}} dx.$$

Indefinite Integral

Definition 2.6. Let $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$. The function defined by

$$F(x) := \int_a^x f(t) dt \quad \text{for } x \in [a, b]$$

is called **indefinite integral** of f with base-point a .

Lipschitz Function

Definition 2.7. A function $f : D \rightarrow \mathbb{R}$ is said to be a **Lipschitz function** or to satisfy a **Lipschitz condition** on D if

$$\exists K > 0 : |f(x) - f(y)| \leq K|x - y|.$$

Theorem 2.12. If $f : D \rightarrow \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on D .

Theorem 2.13. If $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$, then, indefinite integral F of f is uniformly continuous on $[a, b]$.

Proof. Let $x, y \in [a, b]$ with $y < x$:



Then

$$F(x) := \int_a^x f(t) dt = \int_a^y f(t) dt + \int_y^x f(t) dt \implies F(x) - F(y) = \int_y^x f(t) dt.$$

Since f is Riemann integrable on $[a, b]$ and is bounded on $[a, b]$, we have

$$\exists K > 0 : \forall t \in [a, b] : |f(t)| \leq K,$$

and so

$$\begin{aligned} & -K \leq f(t) \leq K \\ \implies & \int_y^x (-K) dt \leq \int_y^x f(t) dt \leq \int_y^x K dt \\ \implies & -K(x - y) \leq F(x) - F(y) \leq K(x - y) \\ \implies & |F(x) - F(y)| \leq K|x - y|, \end{aligned}$$

Thus F is a Lipschitz function on $[a, b]$, and so F is uniformly continuous on $[a, b]$. \square

★ Fundamental Theorem of Calculus: 2nd form ★

Theorem 2.14. Let $f : [a, b] \rightarrow \mathbb{R}$ is differentiable on $[a, b]$ and continuous at a point $c \in [a, b]$. Then the indefinite integral F is differentiable at c and

$$F'(c) = f(c).$$

Proof. We will show that $\lim_{h \rightarrow 0^+} \frac{F(c+h) - F(c)}{h} = f(c)$, i.e.,

$$(\forall \varepsilon > 0)(\exists \delta > 0) : h \in (0, \delta) \implies \left| \frac{F(c+h) - F(c)}{h} - f(c) \right| < \varepsilon.$$

Let $\varepsilon > 0$ and $c \in [a, b]$. Consider the right-hand derivative. Since f is right-continuous at c ,

$$\exists \delta > 0 : x \in [c, c + \delta) \implies |f(x) - f(c)| < \varepsilon.$$

Let $h \in \mathbb{R}$ satisfies $0 < h < \delta$, say, $h = x - c$. Then f is Riemann integrable on $[a, c + h]$, $[a, c]$ and $[c, c + h]$. Then

$$\begin{aligned} F(c+h) - F(c) &= \int_a^{c+h} f(t) dt - \int_a^c f(t) dt \\ &= \int_c^{c+h} f(t) dt. \end{aligned}$$

Since $c \leq t \leq c + h < c + \delta$, we know

$$|f(t) - f(c)| < \varepsilon, \quad \text{i.e.,} \quad f(c) - \varepsilon < f(t) < f(c) + \varepsilon.$$

Thus,

$$\begin{aligned} &\int_c^{c+h} (f(t) - \varepsilon) dt < \int_c^{c+h} f(t) dt < \int_c^{c+h} (f(t) + \varepsilon) dt \\ \implies &(f(c) - \varepsilon) h < F(c+h) - F(c) < (f(c) + \varepsilon) h \\ \implies &-\varepsilon < \frac{F(c+h) - F(c)}{h} - f(c) < \varepsilon \\ \implies &\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| < \varepsilon. \end{aligned}$$

□

Theorem 2.15. If f is continuous on $[a, b]$, then the indefinite integral

$$F(x) := \int_a^x f(t) dt \quad \text{for } x \in [a, b]$$

is differentiable on $[a, b]$ and

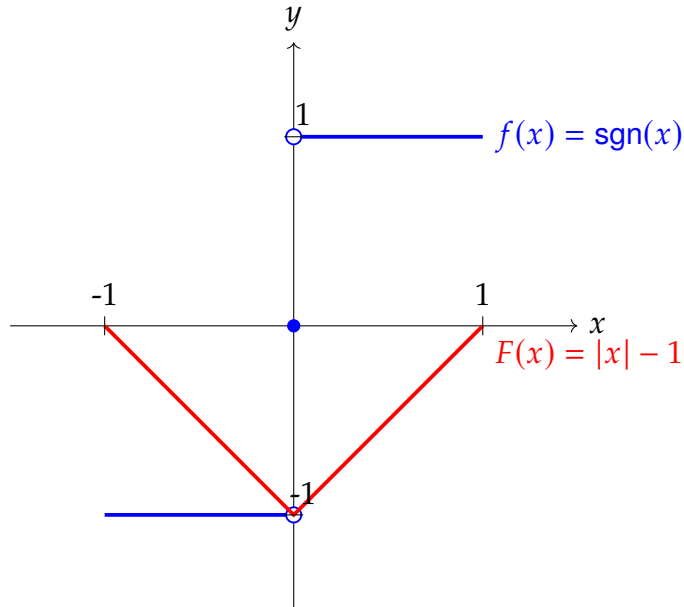
$$F'(x) = f(x)$$

for all $x \in [a, b]$.

Example 2.6. If $f(x) := \operatorname{sgn}(x)$ on $[-1, 1]$, then f is Riemann integrable and has the indefinite integral

$$F(x) := |x| - 1$$

with the basepoint -1 . However, since $F'(0)$ does not exist, F is not an anti-derivative of f on $[-1, 1]$.



Example 2.7. For $x \in [0, 3]$, if we define

$$F(x) := \int_0^x \lfloor t \rfloor dt$$

then although $f(x) = \lfloor x \rfloor$ is discontinuous on $[0, 3]$, F is continuous on $[0, 3]$.

Substitution Theorem

Theorem 2.16. Let $J := [a, b]$ and let $g : J \rightarrow \mathbb{R}$ have a continuous derivative on J . If $f : I \rightarrow \mathbb{R}$ is continuous on an interval I containing $g(J)$ then

$$\int_a^b f(g(t)) \cdot g'(t) dt = \int_{g(a)}^{g(b)} f(x) dx.$$

Proof. Since $g'(t)$ and $f(g(t))$ are both continuous on J , $f(g(t)) \cdot g'(t)$ is continuous on J . Thus $\int_a^b f(g(t)) \cdot g'(t) dt$ exists.

(1) Assume that g is constant. Since $g'(t) = 0$ and $g(a) = g(b)$,

$$\int_a^b f(g(t)) \cdot g'(t) dt = 0 = \int_{g(a)}^{g(b)} f(x) dx.$$

(2) Let g is not a constant. Then for $x \in g[J] \subseteq I$, define

$$F(x) := \int_{g(a)}^x f(s) ds.$$

By the Fundamental Theorem of Calculus: 2nd form,

$$\frac{d}{dx} F(x) = f(x).$$

and then

$$\frac{d}{dt} (F \circ g)(t) = \frac{d}{dt} F(g(t)) \frac{d}{dt} g(t) = f(g(t)) g'(t).$$

Thus

$$\begin{aligned} \int_a^b f(g(t)) \cdot g'(t) dt &= \int_a^b (F \circ g)'(t) dt \\ &= (F \circ g)(b) - (F \circ g)(a) \\ &= F(g(b)) - F(g(a)) \\ &= \int_{g(a)}^{g(b)} f(x) dx - \int_{g(a)}^{g(a)} f(x) dx \\ &= \int_{g(a)}^{g(b)} f(x) dx. \end{aligned}$$

□

Example 2.8. Consider the integral

$$\int_1^4 \frac{\sin \sqrt{t}}{\sqrt{t}} dt.$$

Let us substitution $g(t) := \sqrt{t}$ for $t \in [1, 4]$ so that $g'(t)$ is continuous on $[1, 4]$. If we let $f(x) := 2 \sin x$ then the integrand has the form $f(g(t))g'(t)$. Then the integral equals

$$\int_1^4 \frac{\sin \sqrt{t}}{\sqrt{t}} dt = \int_1^2 2 \sin x dx = 2(\cos 1 - \cos 2).$$

However, if one consider the integral

$$\int_0^4 \frac{\sin \sqrt{t}}{\sqrt{t}} dt,$$

the substitution theorem cannot be applicable since $g(t) := \sqrt{t}$ does not have a continuous derivative on $[0, 4]$. Note that

$$\int_0^4 \frac{\sin \sqrt{t}}{\sqrt{t}} dt = \lim_{a \rightarrow 0^+} \int_a^4 \frac{a}{4} f(t) dt.$$

Integration by Parts

Theorem 2.17. Let f, g be differentiable on $[a, b]$ and f', g' are Riemann integrable on $[a, b]$. Then

$$\int_a^b f(x)g'(x) dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x) dx.$$

Remark 2.2. $\int f g' = \int (f g)' - \int f' g$.

Taylor's Theorem with the Remainder

Theorem 2.18. Suppose that $f', f'', \dots, f^{(n)}, f^{(n+1)}$ exist on $[a, b]$ and that $f^{(n+1)}$ is Riemann integrable on $[a, b]$. Then we have

$$f(b) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (b-a)^i + R_n$$

where the remainder R_n is given by

$$R_n = \frac{1}{n!} \int_a^b f^{(n+1)}(t) \cdot (b-t)^n dt.$$

2.4 Improper Integrals

Improper Integral

Definition 2.8. Let f be a function and $c \in (a, b)$.

- (1) Let $f : [a, b) \rightarrow \mathbb{R}$ is Riemann integral on $[a, c]$. We say that f is **improper integrable** on $[a, b)$ if

$$\exists \lim_{c \rightarrow b-} \int_a^c f(x) dx \in \mathbb{R}.$$

- (2) Let $f : (a, b] \rightarrow \mathbb{R}$ is Riemann integral on $[c, b]$. We say that f is also **improper integrable** on $(a, b]$ if

$$\exists \lim_{c \rightarrow a+} \int_c^b f(x) dx \in \mathbb{R}.$$

Example 2.9. Let $f(x) := x^{-\frac{1}{3}}$ for $x \in (0, 1]$. Since f is unbounded on $(0, 1]$, f is not Riemann integrable. However, for every $c \in (0, 1)$,

$$\lim_{c \rightarrow 0+} \int_c^1 x^{-\frac{1}{3}} dx = \lim_{c \rightarrow 0+} \frac{3}{2}(1 - c^{2/3}) = \frac{3}{2}.$$

Hence f is improper integrable on $(0, 1]$.

Example 2.10. Let $g(x) := x^{-1}$ for $x \in (0, 1]$. Then for every $c \in (0, 1)$,

$$\lim_{c \rightarrow 0+} \int_c^1 x^{-1} dx = \lim_{c \rightarrow 0+} (-\ln c) = \infty.$$

Hence g is not improper integrable on $(0, 1]$.

Definition 2.9. Let f be defined on $[a, \infty)$ and Riemann integrable on $[a, b]$ for every $b > a$. Then f is improper integrable on $[a, \infty)$ if

$$\exists \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

and

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$$

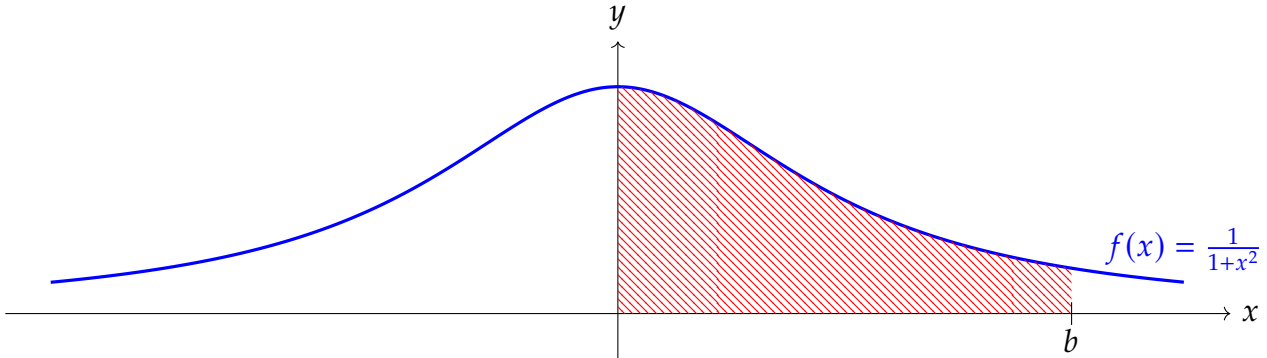
Similarly, one can define

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx.$$

Example 2.11. Let

$$f(x) := \frac{1}{1+x^2}.$$

Then f is well-defined and bound on $[0, \infty)$.



Moreover f is Riemann integrable on $[0, b]$ for every $b > 0$ since f is continuous on $[0, \infty)$. Since

$$\lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx = \lim_{b \rightarrow \infty} \left(\tan^{-1}(b) - \tan^{-1}(0) \right) = \lim_{b \rightarrow \infty} \tan^{-1}(b) = \frac{\pi}{2},$$

we obtain

$$\int_0^\infty \frac{1}{1+x^2} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx = \frac{\pi}{2}.$$

Note that

$$\int_{-\infty}^\infty \frac{1}{1+x^2} dx = \pi \implies \frac{1}{\pi} \int_{-\infty}^\infty \frac{1}{1+x^2} dx = 1,$$

and so $g(x) := \frac{1}{\pi(1+x^2)}$ be a p.d.f

Example 2.12. Since

$$\int_0^\infty f(x) dx = \int_0^1 \frac{1}{\sqrt{x}} dx + \int_1^\infty \frac{1}{\sqrt{x}} dx \quad \text{and} \quad \int_1^\infty \frac{1}{\sqrt{x}} dx = \infty,$$

$f = x^{-1/2}$ is not improper integrable on $(0, \infty)$.

Comparison Test

Theorem 2.19. Let $f, g : [a, \infty) \rightarrow \mathbb{R}$. For every $b > a$, f and g are Riemann integrable on $[a, b]$. Then if for $\geq a$, $f(x) \in [0, g(x)]$ and g is improper integrable on $[a, \infty)$, then f is improper integrable on $[a, \infty)$ and

$$\int_a^\infty f(x) dx \leq \int_a^\infty g(x) dx.$$

Proof. For $b > a$, define

$$F(b) := \int_a^b f(x) dx \quad \text{and} \quad G(b) := \int_a^b g(x) dx.$$

Since $0 \leq f(x) \leq g(x)$ and $\exists \lim_{b \rightarrow \infty} G(b)$,

$$0 \leq F(b) \leq G(b) \leq \lim_{b \rightarrow \infty} G(b).$$

Let

$$A := \left\{ \int_a^c f(x) dx : a \leq c \right\}$$

then

- (i) $\exists \int_a^b f(x) dx \implies A \neq \emptyset$ and
- (ii) A has an upper bound $\lim_{b \rightarrow \infty} G(b)$.

By the completeness axiom of real number,

$$\exists \sup A = \lim_{b \rightarrow \infty} F(b) = \int_a^\infty f(x) dx,$$

i.e., f is improper integrable on $[a, \infty)$. Moreover,

$$\int_a^\infty f(x) dx \leq \int_a^\infty g(x) dx.$$

□

Theorem 2.20. Let $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$ for every $b > a$. Then

$$\exists M \in \mathbb{R}^+ : \int_a^\infty |f(x)| dx \leq M \implies \exists \int_a^\infty f(x) dx \exists \int_a^\infty |f(x)| dx.$$

2.5 Exercises

Exercise 2.1. Generate a function f which is bounded but isn't integrable on $[a, b]$.

Solution. Let $f : [a, b] \rightarrow \mathbb{R}$ be a Dirichlet's discontinuous function defined by

$$f(x) := \begin{cases} 1 & : x \in \mathbb{Q} \\ 0 & : x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then, f is bounded on $[a, b]$ but f is not Riemann integrable. □

Exercise 2.2. Give an example of an integrable function f that $f(x_0) > 0$ for $x_0 \in [a, b]$ but such that $\int_a^b f(x) dx = 0$.

Solution. Let $f : [a, b] \rightarrow \mathbb{R}$ be a function defined by

$$f(x) := \begin{cases} 1 & : x = x_0 \\ 0 & : x \in [a, b] \setminus \{x_0\}. \end{cases}$$

Then $f(x_0) > 0$ but $\int_a^b f(x) dx = 0$. □

Exercise 2.3. Given an example of a function $f : [0, 1] \rightarrow \mathbb{R}$ that isn't Riemann integrable but such that $|f|$ is Riemann integrable on $[0, 1]$.

Solution. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a function defined by

$$f(x) := \begin{cases} 1 & : x \in \mathbb{Q} \\ -1 & : x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then, f is not Riemann integrable on $[a, b]$ but $|f|$ is Riemann integrable on $[0, 1]$. □

Exercise 2.4. Assume that $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$. For $x \in [a, b]$, let

$$F(x) = \int_a^x f(t) dt$$

then show that F is **Lipschitz** function on $[a, b]$.

Solution. Theorem 2.13 □

Exercise 2.5. If f and g are continuous on $[a, b]$ and if

$$\int_a^b f(x) dx = \int_a^b g(x) dx,$$

prove that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

Solution. Since f and g are continuous on $[a, b]$, $f - g$ is also continuous on $[a, b]$. By the intermediate value theorem for integrals,

$$\exists c \in [a, b] : (f - g)(c) = \frac{1}{b - a} \int_a^b (f(x) - g(x)) dx.$$

Since $\int_a^b f(x) dx = \int_a^b g(x) dx$,

$$(f - g)(c) = 0 \implies f(c) = g(c).$$

□

Exercise 2.6. If f is continuous on $[-a, a]$, show that $\int_{-a}^a f(x^2) dx = 2 \int_0^a f(x^2) dx$.

Solution. Since

$$\int_{-a}^a f(x^2) dx = \int_{-a}^0 f(x^2) dx + \int_0^a f(x^2) dx,$$

and by the substitution theorem yields

$$\int_{-a}^0 f(x^2) dx \stackrel{x=-t}{=} \int_a^0 f(t^2)(-dt) = \int_0^a f(t^2) dt.$$

Hence $\int_{-a}^a f(x^2) dx = \int_0^a f(t^2) dt + \int_0^a f(x^2) dx = 2 \int_0^a f(x^2) dx$.

□

Exercise 2.7. Prove that $f(x) = \frac{e^{-x}}{1+x^2}$ is improper integrable on $[0, \infty)$.

Solution. Let

$$g(x) := \frac{1}{1+x^2}$$

for $x \in [0, \infty)$. Note that g is improper integrable on $[0, \infty)$ and $\int_0^\infty g(x) dx = \frac{\pi}{2}$. Since $e^{-x} \leq 1$ on $[0, \infty)$,

$$0 \leq f(x) \leq g(x).$$

Therefore, $f(x)$ is improper integrable on $[0, \infty)$ and $\int_0^\infty \frac{e^{-x}}{1+x^2} \leq \frac{\pi}{2}$.

□

Exercise 2.8. Prove that $\int_1^\infty \frac{1}{x^p} dx$ diverges when $p \leq 1$ and converges when $p > 1$.

Solution. Since

$$\begin{aligned}\int_1^b \frac{1}{x} dx &= \ln b - \ln 1, \\ \int_1^b \frac{1}{x^p} dx &= \frac{1}{1-p} \left(\frac{1}{b^{p-1}} - 1 \right) \quad \text{for } p \neq 1.\end{aligned}$$

We can see that the improper integral converges if $p > 1$ and diverges if $p \leq 1$. □

Exercise 2.9. Prove that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

[Hint: use the polar coordinate system.]

Solution. Let $I = \int_0^\infty e^{-x^2} dx$. Then

$$\begin{aligned}I^2 &= \left(\int_0^\infty e^{-x^2} dx \right) \left(\int_0^\infty e^{-x^2} dx \right) \\ &= \left(\int_0^\infty e^{-x^2} dx \right) \left(\int_0^\infty e^{-y^2} dy \right) \\ &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy \\ &= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r dr d\theta \\ &= \int_0^{\frac{\pi}{2}} \left[-\frac{1}{2} \cdot e^{-r^2} \right]_0^\infty d\theta \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} \right) d\theta \\ &= \left[\frac{1}{2} \theta \right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.\end{aligned}$$

Since $e^{-x^2} \geq 0$, we have

$$I^2 = \left(\int_0^\infty e^{-x^2} dx \right) \left(\int_0^\infty e^{-x^2} dx \right) = \frac{\pi}{4} \implies I = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

□

Exercise 2.10. Suppose that f is continuous on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$. Show that if $\int_a^b f(x) dx = 0$ then $f(x) = 0$ for all $x \in [a, b]$.

Solution. Assume that $f(x_0) \neq 0$.

□

Exercise 2.11.

Solution.

□

Exercise 2.12. Let f and g be Riemann integrable on $[a, b]$. Then show that fg is Riemann integrable on $[a, b]$.

Solution. Since $(f + g)^2$ and $(f - g)^2$ are Riemann integrable on $[a, b]$,

$$fg = \frac{1}{4} \left((f + g)^2 - (f - g)^2 \right)$$

is Riemann integrable on $[a, b]$.

□

Chapter 3

Sequence of Functions

3.1 Pointwise and Uniform Convergence

3.2 Interchange of Limits

3.3 Series of Functions

3.4 Power Series

3.5 Exercises

Exercise 3.1. For $x \in [0, 1]$,

$$f_n(x) = \frac{1}{n^2 x^2 + 1}.$$

Then show that $\{f_n\}$ converges pointwise on $[0, 1]$ but not converges uniformly.

Solution. Note that $f_n(x)$ is continuous for $n \in \mathbb{N}$ and $x \in [0, 1]$. Then

$$\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 1 & : x = 0 \\ 0 & : x \in (0, 1] \end{cases}$$

and so $\{f_n\}$ converges pointwisely on $[0, 1]$. However, since f is not continuous on $[0, 1]$, $\{f_n\}$ does not converges uniformly to f on $[0, 1]$. \square

Exercise 3.2. For $x \in [0, 1]$,

$$f_n(x) = nx(1 - x)^n.$$

Then show that $\{f_n\}$ converges pointwise on $[0, 1]$ but not converges uniformly.

Solution. Note that

$$x \in \{0, 1\} \implies \lim_{n \rightarrow \infty} f_n(x) = 0.$$

Suppose that $x \in (0, 1)$. Since $0 < 1 - x < 1$,

$$\lim_{n \rightarrow \infty} nx(1 - x)^n = 0.$$

$\therefore \exists \lim_{n \rightarrow \infty} f_n(x) = 0$ on $[0, 1]$. Consider

$$\frac{d}{dx} (f_n(x) - f(x)) = n(1 - x)^n - n^2 x(1 - x)^{n-1} = n(1 - x)^{n-1}(1 - x - nx) = 0.$$

That is, $f_n(x)$ has local extremum at $x \in \left\{1, \frac{1}{n+1}\right\}$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup \{ |f_n(x) - f(x)| : x \in [0, 1] \} &= \lim_{n \rightarrow \infty} \sup \left\{ |f_n(1)|, \left| \frac{1}{n+1} \right| \right\} \\ &= \lim_{n \rightarrow \infty} \sup \left\{ 0, \frac{n}{n+1} \left(1 - \frac{1}{n+1} \right)^n \right\} \\ &= \lim_{n \rightarrow \infty} \frac{n}{n+1} \left(1 - \frac{1}{n+1} \right)^n \\ &= \frac{1}{e} > 0. \end{aligned}$$

Hence $\{f_n\}$ is not uniformly convergent to f . \square

Exercise 3.3. For $x \in \mathbb{R}$, let

$$f_n(x) = \frac{nx}{1 + n^2x^2}.$$

Then show that for $a > 0$, $\{f_n\}$ converges uniformly on $[a, \infty)$ but does not converge uniformly on $[0, \infty)$.

Solution. If $x = 0$ then $f_n(x) = 0$. Let $x \neq 0$. Let $x \neq 0$ and $\varepsilon > 0$. Then $\exists N > \frac{1}{|x|\varepsilon}$ s.t. if $n \geq N$,

$$|f_n(x) - 0| = \left| \frac{nx}{1 + n^2x^2} \right| \leq \left| \frac{nx}{n^2x^2} \right| = \left| \frac{1}{nx} \right| \leq \frac{1}{N|x|} < \varepsilon.$$

Thus $\exists \lim_{n \rightarrow \infty} f_n(x) = 0$.

(1) Let $a > 0$ and $x \in [a, \infty)$. Let $\varepsilon > 0$. Then $\exists N > \frac{1}{\varepsilon a}$ s.t. if $n \geq N$,

$$|f_n(x) - f(x)| = \left| \frac{nx}{1 + n^2x^2} - 0 \right| \leq \frac{1}{nx} \leq \frac{1}{na} \leq \frac{1}{Na} < \varepsilon.$$

$\therefore \{f_n\}$ converges uniformly to f on $[a, \infty)$.

(2) Let $x \in [0, \infty)$. Consider

$$\frac{d}{dx} (f_n(x) - f(x)) = \frac{n(1 + n^2x^2) - nx \cdot 2n^2x}{(1 + n^2x^2)^2} = \frac{n - n^3x^2}{(1 + n^2x^2)^2} = 0.$$

$\therefore x^2 = 1/n^2 \implies x = 1/n$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup \{ |f_n(x) - f(x)| : x \in [0, \infty) \} &= \lim_{n \rightarrow \infty} \sup \left\{ \left| f_n(0) \right|, \left| f_n\left(\frac{1}{n}\right) \right| \right\} \\ &= \lim_{n \rightarrow \infty} \sup \left\{ 0, \frac{1}{2} \right\} \\ &= \frac{1}{2} > 0. \end{aligned}$$

Hence $\{f_n\}$ does not converge uniformly to f on $[0, \infty)$.

□

Exercise 3.4. Let $I = (-\infty, \infty)$. Then for $f_n(x) = n^3 e^{-n^4} (1 - 2 \sin^2 nx)$, the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly.

Solution. For $x \in \mathbb{R}$, we have

$$|f_n(x)| = \left| \frac{n^3}{e^{n^4}} (1 - 2 \sin^2 nx) \right| \leq \frac{n^3}{e^{n^4}} \leq \frac{3n^3}{e^{n^4}} \quad \text{and} \quad \int_1^\infty \frac{3n^3}{e^{n^4}} dx = \frac{3}{4e}.$$

By Weierstrass M -test, $\exists \sum_{n=1}^\infty f_n(x)$ for $x \in \mathbb{R}$. □

Exercise 3.5. Let $I = [0, 1)$. Then, if $f_n(x) = x^n$, $n \in \mathbb{N}$ show that the series

$$\sum_{n=1}^\infty f_n(x)$$

is a pointwise convergent but not a uniformly convergent series.

Solution. Let $x \in [0, 1)$. Then

$$|x| < 1 \implies \sum_{i=1}^\infty f_n(x) = \sum_{n=1}^\infty x^n = \frac{x}{1-x}.$$

That is, the series converges pointwisely. Let

$$x_n := 1 - \frac{1}{n} \in [0, 1)$$

for $n \in \mathbb{N}$. Note that $\lim_{n \rightarrow \infty} x_n = 1$ and that $x_n \neq 1$ for $n \in \mathbb{N}$. Then

$$\lim_{n \rightarrow \infty} f_n(x_n) = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e} \neq \infty = f(\lim_{n \rightarrow \infty} x_n) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} \frac{\frac{n-1}{n}}{\frac{1}{n}} = \infty.$$

$\therefore \sum_{n=1}^\infty f_n(x)$ does not converges uniformly to $f(x) = \frac{x}{1-x}$, $x \in [0, 1)$. □

Exercise 3.6. Evaluate the radius of convergence of following power series

1. $\sum_{n=0}^{\infty} \frac{x^n}{2^n},$

2. $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) x^n,$

3. $\sum_{n=0}^{\infty} n2^n(x-4)^n.$

4. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 3^n},$

5. $\sum_{n=0}^{\infty} \frac{x^n}{n!},$

6. $\sum_{n=0}^{\infty} \frac{(n!)^2 x^n}{(2n)!}.$

Solution. 1. For our series, $a_n = \frac{x^n}{2^n}$. The limit L is calculated as follows:

$$L = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}/2^{n+1}}{x^n/2^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{2} \right| = \left| \frac{x}{2} \right|$$

For convergence, we set $L < 1$:

$$\left| \frac{x}{2} \right| < 1 \implies |x| < 2$$

Hence, the radius of convergence of the series is 2.

2. To find the radius of convergence of the power series $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) x^n$, we apply the ratio test. The general term of the series is $a_n = \ln\left(\frac{n+1}{n}\right) x^n$. The ratio test involves the limit

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

For our series, this becomes

$$L = \lim_{n \rightarrow \infty} \left| \frac{\ln\left(\frac{n+2}{n+1}\right) x^{n+1}}{\ln\left(\frac{n+1}{n}\right) x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\ln\left(\frac{n+2}{n+1}\right)}{\ln\left(\frac{n+1}{n}\right)} \cdot x \right|$$

Simplifying the limit:

1. The logarithmic ratio:

$$\lim_{n \rightarrow \infty} \frac{\ln\left(\frac{n+2}{n+1}\right)}{\ln\left(\frac{n+1}{n}\right)} = \lim_{n \rightarrow \infty} \frac{\ln(n+2) - \ln(n+1)}{\ln(n+1) - \ln(n)}$$

As $n \rightarrow \infty$, this ratio approaches 1.

2. Therefore, the limit L simplifies to:

$$L = \lim_{n \rightarrow \infty} |x|$$

For convergence, $L < 1$ gives:

$$|x| < 1$$

Thus, the radius of convergence of the series is 1.

3. Let $f_n(x) := n2^n(x-4)^n$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{f_{n+1}(x)}{f_n(x)} \right| < 1 &\implies \lim_{n \rightarrow \infty} \left| \frac{(n+1)2^{n+1}(x-4)^{n+1}}{n2^n(x-4)^n} \right| = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right) 2|x-4| = 2|x-4| < 1 \\ &\implies |x-4| < \frac{1}{2}. \end{aligned}$$

Thus $R = 1/2$.

(Another Solution) Let $a_n = n2^n$. Then $R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1/2$.

4. To evaluate the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 3^n}$, we use the root test. The root test involves the limit

$$L = \lim_{n \rightarrow \infty} \left(\left| \frac{(x-2)^n}{n^2 3^n} \right| \right)^{\frac{1}{n}}.$$

Calculating this limit, we get

$$L = \lim_{n \rightarrow \infty} \left| \frac{(x-2)^n}{n^2 3^n} \right|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left| \frac{(x-2)}{3} \right| \left(\frac{1}{n^2} \right)^{\frac{1}{n}}.$$

As n approaches infinity, $\left(\frac{1}{n^2} \right)^{\frac{1}{n}}$ approaches 1. Thus, the limit simplifies to

$$L = \left| \frac{x-2}{3} \right|.$$

For convergence, $L < 1$ yields

$$\left| \frac{x-2}{3} \right| < 1.$$

Solving this inequality:

$$-1 < \frac{x-2}{3} < 1 \implies -1 < x < 5.$$

Therefore, the radius of convergence of the series is the distance between the center (2) and the endpoints of the interval, which is 3.

5. To evaluate the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, we use the ratio test. The general term of the series is $a_n = \frac{x^n}{n!}$. The ratio test involves the limit $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

For our series, this limit is

$$L = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1} \cdot n!}{x^n \cdot (n+1)!} \right| = \lim_{n \rightarrow \infty} \left| \frac{x \cdot x^n \cdot n!}{x^n \cdot n! \cdot (n+1)} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right|$$

As n approaches infinity, $\frac{x}{n+1}$ approaches 0 for any finite value of x . Therefore, the limit L is 0 for all x .

Since $L < 1$ for all x , the radius of convergence is infinite. This means that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all real numbers x .

6. Let $f_n(x) = \frac{(n!)^2 x^n}{(2n)!}$. Then

$$\lim_{n \rightarrow \infty} \left| \frac{f_{n+1}(x)}{f_n(x)} \right| = \lim_{n \rightarrow \infty} \left| \frac{\{(n+1)!\}^2 x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2 x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{(2n+2)(2n+1)} x \right| = \frac{1}{4} |x|.$$

Thus, $\frac{1}{4} |x| < 1 \implies |x| < 4 = R$.

□

Exercise 3.7. Evaluate the radius of convergence of following power series

1. $\sum_{n=0}^{\infty} \frac{x^n}{3^n + 5^n},$

2. $\sum_{n=0}^{\infty} \frac{n^3 x^n}{n!}.$

Solution. 1. Let $f_n(x) = \frac{x^n}{3^n + 5^n}$ then

$$\lim_{n \rightarrow \infty} \left| \frac{f_{n+1}(x)}{f_n(x)} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{3^{n+1} + 5^{n+1}} \cdot \frac{3^n + 5^n}{x^n} \right| = \frac{1}{5} |x| < 1 \implies |x| < 5 = R.$$

2. Let $f_n(x) = \frac{n^3 x^n}{n!}$ then

$$\lim_{n \rightarrow \infty} \left| \frac{f_{n+1}(x)}{f_n(x)} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3 x^{n+1}}{(n+1)!} \cdot \frac{n!}{n^3 x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{(n+1)n^3} x \right| = 0 < 1 \implies R = \infty.$$

□

Exercise 3.8. Let the radius of convergence of following power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is R then evaluate the radius of convergence of

$$\sum_{n=0}^{\infty} a_n x^{-n}.$$

Solution. Let $f_n(x) := a_n x^{-n} = a_n/x^n$. Then

$$\lim_{n \rightarrow \infty} \left| \frac{f_{n+1}(x)}{f_n(x)} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{x^{n+1}} \cdot \frac{x^n}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \cdot \frac{1}{x} \right| < 1 \implies |x| > \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Since the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is R ,

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

Hence the radius of convergence of $\sum_{n=0}^{\infty} \frac{a_n}{x^n}$ is $1/R$. □

Exercise 3.9. Suppose that the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is R . What is the radius of convergence of the following power series?

$$\sum_{n=0}^{\infty} a_n x^{2n}.$$

Solution. Since the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is R ,

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

Let $f_n(x) := a_n x^{2n}$. Then

$$\lim_{n \rightarrow \infty} \left| \frac{f_{n+1}(x)}{f_n(x)} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1} x^{2n+2}}{a_n x^{2n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| |x|^2 < 1 \implies |x|^2 < \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = R \implies |x| < \sqrt{R}.$$

Hence the radius of convergence of $\sum_{n=0}^{\infty} a_n x^{2n}$ is \sqrt{R} . □

Chapter 4

Introduction to Fourier Series and Transform

4.1 Periodic Functions and Trigonometric Series

Periodic Functions

Definition 4.1. A function $f(x)$ is called **periodic** if

(1) it is defined for all $x \in \mathbb{R}$ and

(2) if $\exists p > 0$ such that

$$f(x + p) = f(x).$$

This number p is called a **period** of $f(x)$.

Trigonometric Series

Definition 4.2. The series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called a **trigonometric series**, and the a_n and b_n are called the coefficients of the series, where $a_0, a_1, a_2, \dots, b_1, b_2, \dots$ are real constants.

Remark 4.1.

- Fourier series arise from the practical task of representing a given periodic function $f(x)$ in terms of cosine and sine functions.
- These series are trigonometric series whose coefficients are determined from $f(x)$ by the Euler formulas, which we shall derive first.
- Afterwards we shall take a look at the theory of Fourier series.

4.2 Fourier Series

Fourier Series of a Periodic Function of Period 2π

Theorem 4.1. Assume that $f(x)$ is a periodic function of period 2π and is integrable over a period, that is,

$$f(x + 2\pi) = f(x) \quad \text{and} \quad \exists \int_x^{x+2\pi} f(t) dt = \int_{-\pi}^{\pi} f(x) dx.$$

Then, $f(x)$ can be represented by a trigonometric series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

for all $n \in \mathbb{N}$.

Proof. (1) Since

$$\int_{-\pi}^{\pi} \cos nx dx = 0 = \int_{-\pi}^{\pi} \sin nx dx,$$

we have

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) dx &= \int_{-\pi}^{\pi} a_0 dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) dx \\ &= \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos nx + b_n \sin nx) dx \\ &= \int_{-\pi}^{\pi} a_0 dx = 2\pi a_0. \end{aligned}$$

and so $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx.$

(2) Let $m \in \mathbb{N}$. Then

$$\begin{aligned} f(x) \cos mx &= a_0 \cos mx + \cos mx \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \\ \int_{-\pi}^{\pi} f(x) \cos mx &= \int_{-\pi}^{\pi} a_0 \cos mx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n \cos nx \cos mx + b_n \sin nx \cos mx). \end{aligned}$$

Note that

$$\begin{aligned}\int_{-\pi}^{\pi} \cos nx \cos mx \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} \cos(n+m)x \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \cos(n-m)x \, dx, \\ \int_{-\pi}^{\pi} \sin nx \cos mx \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} \sin(n+m)x \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \sin(n-m)x \, dx,\end{aligned}$$

and

$$\frac{1}{2} \int_{-\pi}^{\pi} \cos(n-m)x \, dx = \begin{cases} 2\pi & : n = m, \\ 0 & : n \neq m. \end{cases}$$

Thus

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \frac{1}{2} \cdot 2\pi a_m = \pi a_m \xrightarrow{n=m} a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx.$$

(3) Similarly, we have

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \pi b_m \xrightarrow{n=m} b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

□

Example 4.1 (Rectangular Wave). Find the Fourier series of the periodic function $f(x)$ defined by

$$f(x) := \begin{cases} -k & : -\pi < x < 0 \\ k & : 0 < x < \pi \end{cases} \quad \text{and} \quad f(x+2\pi) = f(x).$$

Solution. $a_0 = 0$, $a_n = 0$ and

$$b_n = \begin{cases} \frac{4k}{(2k+1)\pi} & : n = 2k+1, \\ 0 & : n = 2k. \end{cases}$$

□

Remark 4.2 (The Gibbs' phenomenon). Its sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left- and right-hand limits of $f(x)$ at x_0 . In other words, if f is not continuous at x_0 then

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx_0 + b_n \sin nx_0) = \frac{1}{2} (f(x_0+) + f(x_0-)).$$

Representation by a Fourier series

Theorem 4.2. If a periodic function $f(x)$ with period 2π is

- (1) having continuous first and second derivatives,
 - (2) piecewise continuous in the interval $[-\pi, \pi]$,
 - (3) having a left-hand derivative and right-hand derivative at each point of that interval,
- then the Fourier series of $f(x)$ is convergent.

Solution. Since

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\
 &= \frac{1}{\pi} \left[f(x) \cdot \frac{1}{n} \sin nx \right]_{-\pi}^{\pi} - \frac{1}{n\pi} \int_{-\pi}^{\pi} f'(x) \sin nx \, dx \\
 &= -\frac{1}{n\pi} \left[f'(x) \cdot \frac{-1}{n} \cos nx \right]_{-\pi}^{\pi} - \frac{1}{n\pi} \int_{-\pi}^{\pi} f''(x) \left(-\frac{1}{n} \cos nx \right) dx \\
 &= \frac{1}{n^2\pi} \left[f'(x) \cos nx \right]_{-\pi}^{\pi} + \frac{1}{n^2\pi} \int_{-\pi}^{\pi} f''(x) \cos nx \, dx
 \end{aligned}$$

and f'' is continuous on $[-\pi, \pi]$, we have $\exists M > 0$ s.t. $|f''(x)| \leq M$. It follows that

$$|a_n| = \frac{1}{n^2\pi} \left| \int_{-\pi}^{\pi} f''(x) \cos nx \, dx \right| < \frac{1}{n^2\pi} \int_{-\pi}^{\pi} M \, dx = \frac{2M}{n^2}.$$

Similarly, $|b_n| < \frac{2M}{n^2}$. Thus,

$$\begin{aligned}
 |f(x)| &= \left| a_0 + \sum_{i=1}^{\infty} (a_i \cos ix + b_i \sin ix) \right| \leq |a_0| + \sum_{n=1}^{\infty} (|a_n| + |b_n|) \\
 &\leq |a_0| + \sum_{n=1}^{\infty} \frac{4M}{n^2}.
 \end{aligned}$$

Let $M_n := \frac{4M}{n^2}$ for $n \in \mathbb{N}$. Since $\exists |a_0| + \sum_{n=1}^{\infty} M_n$, by Weierstrass M -test,

$$|f(x)| \text{ converges} \implies f(x) \text{ converges uniformly on } [-\pi, \pi].$$

□

Note (Review). For $\mathbf{a} = (1, 2, 3) \in \mathbb{R}^3$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a orthonormal basis for \mathbb{R}^3 . Then

$$\begin{cases} \mathbf{e}_1 = (1, 0, 0) \\ \mathbf{e}_2 = (0, 1, 0) \\ \mathbf{e}_3 = (0, 0, 1) \end{cases} \implies \mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 = (\mathbf{a} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{a} \cdot \mathbf{e}_2)\mathbf{e}_2 + (\mathbf{a} \cdot \mathbf{e}_3)\mathbf{e}_3 = \sum_{n=1}^3 (\mathbf{a} \cdot \mathbf{e}_n)\mathbf{e}_n.$$

Note (Orthogonality Property of the Trigonometric System). Let us define an inner product on the interval $[-\pi, \pi]$ such that

$$\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x)g(x) dx.$$

Here, we have

$$(1) \langle 1, 1 \rangle = 2\pi.$$

$$(2) \langle 1, \cos nx \rangle = \int_{-\pi}^{\pi} \cos nx dx = 0.$$

$$(3) \langle 1, \sin nx \rangle = \int_{-\pi}^{\pi} \sin nx dx = 0.$$

$$(4) \langle \cos nx, \sin nx \rangle = \pi = \langle \sin nx, \sin nx \rangle.$$

$$(5) \langle \cos mx, \cos nx \rangle = \int_{-\pi}^{\pi} \cos mx \cos nx dx = 0 \text{ for } n \neq m.$$

$$(6) \langle \sin mx, \sin nx \rangle = \int_{-\pi}^{\pi} \sin mx \sin nx dx = 0 \text{ for } n \neq m.$$

$$(7) \langle \cos mx, \sin nx \rangle = \int_{-\pi}^{\pi} \cos mx \sin nx dx = 0 \text{ for any } n, m.$$

Then the trigonometric system

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx, \dots$$

is **orthogonal** on the interval $[-\pi, \pi]$. Moreover,

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \dots$$

is orthonormal. Note that

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ &= \frac{1}{2\pi} \langle f(x), 1 \rangle 1 + \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \langle f(x), \cos x \rangle \cos x + \frac{1}{\pi} \langle f(x), \sin nx \rangle \sin nx \right) \end{aligned}$$

and that

$$f(x) = \left\langle f(x), \frac{1}{\sqrt{2\pi}} \right\rangle \cdot \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \left(\left\langle f(x), \frac{\cos nx}{\sqrt{\pi}} \right\rangle \frac{\cos nx}{\sqrt{\pi}} + \left\langle f(x), \frac{\sin nx}{\sqrt{\pi}} \right\rangle \frac{\sin nx}{\sqrt{\pi}} \right).$$

4.3 Functions of Any Period $p = 2L$, Even and Odd Functions

Fourier Series of a Periodic Function of Period $2L$

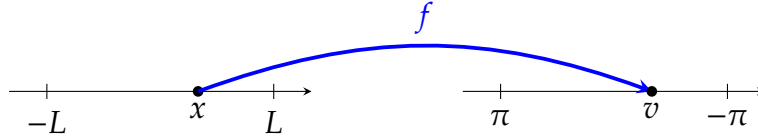
Theorem 4.3. A function $f(x)$ of period $p = 2L$ has a **Fourier series**. This series can be written:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas**, for $n = 1, 2, \dots$,

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx, \quad a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \\ b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx.$$

Proof.



Let $v = \frac{\pi}{L}x$. Then a function $g(v)$ defined by

$$f(x) = f\left(\frac{L}{\pi}v\right) =: g(v)$$

has period of 2π . Then $g(v)$ has the Fourier series

$$g(v) = a_0 + \sum_{i=0}^{\infty} (a_n \cos nv + b_n \sin nv) \quad \text{with} \quad \begin{cases} a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(v) dv, \\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \cos nv dv, \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(v) \sin nv dv. \end{cases}$$

Since $dv = \frac{\pi}{L}dx$, we have

$$f(x) = a_0 + \sum_{i=0}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \quad \text{with} \quad \begin{cases} a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx, \\ a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \\ b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx. \end{cases}$$

□

Example 4.2 (Example (Half-Wave Rectifier)). A sinusoidal voltage $E \sin \omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave. Let

$$u(t) = \begin{cases} 0 & \text{if } -L < t < 0, \\ E \sin \omega t & \text{if } 0 < t < L, \end{cases} \quad p = 2L = \frac{2\pi}{\omega}.$$

Then, find the Fourier series of the periodic function $u(t)$.

Solution. The Fourier series of the $u(t)$ is

$$u(t) = \frac{E}{\pi} + \frac{E}{2} \sin \omega t - \frac{2E}{\pi} \left(\frac{1}{1 \cdot 3} \cos 2\omega t + \frac{1}{3 \cdot 5} \cos 4\omega t + \cdots \right).$$

□

Even and Odd Functions

Definition 4.3. (1) A function $y = f(x)$ is **even** if

$$f(-x) = f(x), \quad x \in \mathbb{R}.$$

The graph of such a function is symmetric with respect to the y -axis.

(2) A function $g(x)$ is **odd** if

$$g(-x) = -g(x), \quad x \in \mathbb{R}.$$

The graph of such a function is symmetric with respect to the origin.

Remark 4.3. $f(x)$ and $g(x)$ satisfy $\int_{-L}^L f(x) dx = 2 \int_0^L f(x) dx$ and $\int_{-L}^L g(x) dx = 0$.

Fourier Cosine and Sine Series

Theorem 4.4. (1) The Fourier series of an even function of period $2L$ is a **Fourier cosine series**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

with coefficients

$$a_0 = \frac{1}{L} \int_0^L f(x) dx \quad \text{and} \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 1, 2, \dots$$

(2) The Fourier series of an odd function of period $2L$ is a **Fourier sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

with coefficients

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, \dots$$

Proof. (1) Since f is even,

$$\begin{cases} f(x) \cos x \text{ is even} \\ f(x) \sin x \text{ is odd} \end{cases} \implies \begin{cases} a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{2L} \cdot 2 \int_0^L f(x) dx = \frac{1}{L} \int_0^L f(x) dx, \\ a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \\ b_n = \frac{1}{L} \int_{-\pi}^{\pi} f(x) \sin \frac{n\pi x}{L} dx = 0. \end{cases}$$

(2) Similarly, it holds.

□

Example 4.3 (Sawtooth Wave). Find the Fourier series of the function

$$f(x) = x + \pi$$

with $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$.

Solution. Let $f_1(x) := x$ then

$$f(x) = \pi + x = \pi + f_1(x).$$

Then since $f_1(x) = x$ is odd, $a_n = 0$ for $n = 0, 1, 2, \dots$, and

$$b_n = \frac{2}{\pi} \int_0^\pi f_1(x) \sin \frac{n\pi x}{\pi} dx = \frac{2}{\pi} \int_0^\pi x \sin nx dx = -\frac{2}{n} \cos n\pi = (-1)^n \frac{2}{n}.$$

Hence, the Fourier series of $f(x)$ is

$$f(x) = \pi + f_1(x) = \pi + 2 \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots \right).$$

□

4.4 Introduction to Complex Fourier Series

Complex Fourier Series

Theorem 4.5. For a function of period $2L$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

the complex Fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp \left(i \frac{n\pi x}{L} \right), \quad \text{where} \quad c_n = \frac{1}{2L} \int_{-L}^L f(x) \exp \left(-i \frac{n\pi x}{L} \right) dx$$

for $n = 0, \pm 1, \pm 2, \dots$.

Proof. Recall the Euler formula

$$e^{inx} = \cos nx + i \sin nx \quad \text{and} \quad e^{-inx} = \cos nx - i \sin nx.$$

Then, we can examine that

$$\cos nx = \frac{1}{2} (e^{inx} + e^{-inx}) \quad \text{and} \quad \sin nx = \frac{1}{2i} (e^{inx} - e^{-inx}).$$

Hence,

$$\begin{aligned} a_n \cos nx + b_n \sin nx &= a_n \frac{1}{2} (e^{inx} + e^{-inx}) + b_n \frac{1}{2i} (e^{inx} - e^{-inx}) \\ &= \frac{1}{2} (a_n - ib_n) e^{inx} + \frac{1}{2} (a_n + ib_n) e^{-inx}. \end{aligned}$$

Let us write

$$c_n = \frac{1}{2} (a_n - ib_n) \quad \text{and} \quad d_n = \frac{1}{2} (a_n + ib_n).$$

Then, we can write

$$f(x) = c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + d_n e^{-inx}),$$

where

$$c_n = \frac{1}{2} (a_n - ib_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) (\cos nx - i \sin nx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx,$$

$$d_n = \frac{1}{2} (a_n + ib_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) (\cos nx + i \sin nx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} dx.$$

Note that $d_n = c_{-n}$ for all n . □

Remark 4.4. The c_n are called the **complex Fourier coefficients** of $f(x)$.

4.5 Fourier Integrals

Fourier Integral

Theorem 4.6. Let f be a nonperiodic function. Then,

$$f(x) = \int_0^{\infty} (A(\omega) \cos \omega x + B(\omega) \sin \omega x) d\omega,$$

where

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos \omega v dv \quad \text{and} \quad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin \omega v dv.$$

This is called a representation of $f(x)$ by a **Fourier integral**.

Proof. (1) We consider any periodic function $f_L(x)$ of period $2L$ that can be represented by a Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \omega_n x + b_n \sin \omega_n x), \quad \omega_n = \frac{n\pi}{L},$$

and find out what happens if we let $L \rightarrow \infty$. Let us write, for $\omega_n = \frac{n\pi}{L}$,

$$\begin{aligned} f_L(x) &= a_0 + \sum_{n=1}^{\infty} (a_n \cos \omega_n x + b_n \sin \omega_n x) \\ &= \frac{1}{2L} \int_{-L}^L f(v) dv \\ &\quad + \frac{1}{L} \sum_{n=1}^{\infty} \left(\cos \omega_n x \int_{-L}^L f(v) \cos \omega_n v dv \right) \\ &\quad + \frac{1}{L} \sum_{n=1}^{\infty} \left(\sin \omega_n x \int_{-L}^L f(v) \sin \omega_n v dv \right). \end{aligned}$$

We now set

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L} \quad \text{implies} \quad \frac{1}{L} = \frac{\Delta\omega}{\pi}.$$

Then we can write the Fourier series in the form

$$\begin{aligned} f_L(x) &= \frac{1}{2L} \int_{-L}^L f(v) dv \\ &\quad + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\cos \omega_n x \Delta\omega \int_{-L}^L f(v) \cos \omega_n v dv \right) \\ &\quad + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\sin \omega_n x \Delta\omega \int_{-L}^L f(v) \sin \omega_n v dv \right). \end{aligned}$$

This representation is valid for any fixed L , arbitrary large, but finite.

(2) Let $L \rightarrow \infty$ and assume that the resulting nonperiodic function

$$f(x) = \lim_{L \rightarrow \infty} f_L(x)$$

is absolutely integrable on the x -axis. Then

$$\frac{1}{2L} \int_{-L}^L f_L(v) dv \rightarrow 0 \quad \text{and} \quad \Delta\omega = \frac{\pi}{L} \rightarrow 0.$$

It seems plausible that the infinite series becomes an integral from 0 to ∞ , which represents $f(x)$,

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[\cos \omega x \int_{-\infty}^\infty f(v) \cos \omega v dv + \sin \omega x \int_{-\infty}^\infty f(v) \sin \omega v dv \right] d\omega.$$

□

Example 4.4 (Single Pulse, Sine Integral). Find the Fourier integral representation of the function

$$f(x) := \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1 \end{cases}$$

Solution. Since

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \cos \omega v dv = \frac{1}{\pi} \int_{-1}^1 \cos \omega v dv = \frac{2 \sin \omega}{\pi \omega} \quad \text{and} \quad B(\omega) = 0,$$

the Fourier integral representation of $f(x)$ is

$$F(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos \omega x \sin \omega}{\omega} d\omega.$$

□

Remark 4.5. The average of the left- and right-hand limits of $f(x)$ at $x = 1$ is equal to $1/2$. Furthermore, we obtain

$$\int_0^\infty \frac{\cos \omega x \sin \omega}{\omega} d\omega = \begin{cases} \frac{\pi}{2} & \text{if } 0 \leq x < 1, \\ \frac{\pi}{4} & \text{if } x = 1, \\ 0 & \text{if } x > 1. \end{cases}$$

This integral is called **Dirichlet's discontinuous factor**. If $x = 0$, then

$$\int_0^\infty \frac{\sin \omega}{\omega} d\omega = \frac{\pi}{2}.$$

We see that this integral is the limit of the so-called **sine integral**

$$\text{Si}(x) = \int_0^x \frac{\sin \omega}{\omega} d\omega$$

as $x \rightarrow \infty$.

Fourier Sine and Cosine Integrals

Theorem 4.7. (1) If $f(x)$ is an even function, then $B(\omega) = 0$ and

$$A(\omega) = \frac{2}{\pi} \int_0^{\infty} f(v) \cos \omega v \, dv.$$

The Fourier integral then reduces to the **Fourier cosine integral**

$$f(x) = \int_0^{\infty} A(\omega) \cos \omega x \, d\omega.$$

(2) If $f(x)$ is an odd function, then $A(\omega) = 0$ and

$$B(\omega) = \frac{2}{\pi} \int_0^{\infty} f(v) \sin \omega v \, dv.$$

The Fourier integral then reduces to the **Fourier sine integral**

$$f(x) = \int_0^{\infty} B(\omega) \sin \omega x \, d\omega.$$

Proof. (1) If $f(x)$ is an even function, then

$$\begin{aligned} f(x) &= \int_0^{\infty} \left(\frac{2}{\pi} \int_0^{\infty} f(v) \cos \omega v \, dv \right) \cos \omega x \, d\omega \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c(\omega) \cos \omega x \, d\omega \quad \text{with} \quad \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x \, dx. \end{aligned}$$

(2) If $f(x)$ is an odd function, then

$$\begin{aligned} f(x) &= \int_0^{\infty} \left(\frac{2}{\pi} \int_0^{\infty} f(v) \sin \omega v \, dv \right) \sin \omega x \, d\omega \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_s(\omega) \sin \omega x \, d\omega \quad \text{with} \quad \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x \, dx. \end{aligned}$$

□

Definition (Fourier Cosine and Sine Transforms)

Definition 4.4.

- **Fourier cosine transform** for an even function:

$$\mathcal{F}_c(f) = \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx.$$

- **Inverse Fourier cosine transform** of $\hat{f}_c(\omega)$:

$$\mathcal{F}_c^{-1}(f) = \hat{f}_c^{-1}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos \omega x \, d\omega.$$

- **Fourier sine transform** for an odd function:

$$\mathcal{F}_s(f) = \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx.$$

- **Inverse Fourier sine transform** of $\hat{f}_s(\omega)$:

$$\mathcal{F}_s^{-1}(f) = \hat{f}_s^{-1}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(\omega) \sin \omega x \, d\omega.$$

Example 4.5. Find Fourier cosine and sine transforms of the function

$$f(x) := \begin{cases} k & : 0 < x < a, \\ 0 & : x > a \end{cases}$$

Solution. Fourier cosine and sine transforms of $f(x)$ are

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^a k \cos \omega x \, dx = k \sqrt{\frac{2}{\pi}} \left(\frac{\sin a\omega}{\omega} \right), \quad \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^a k \sin \omega x \, dx = k \sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos a\omega}{\omega} \right).$$

□

Linearity of Fourier Cosine and Sine Transforms

Theorem 4.8. The Fourier cosine and sine transforms are **linear operators** i.e., for $a, b \in \mathbb{R}$,

$$\mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g), \quad \mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g).$$

Proof. Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned} \mathcal{F}_c(af + bg) &= \sqrt{\frac{2}{\pi}} (af(x) + bg(x)) \cos \omega x \, dx \\ &= a \left(\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx \right) + b \left(\sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos \omega x \, dx \right) \\ &= a\mathcal{F}_c(f) + b\mathcal{F}_c(g). \end{aligned}$$

Similarly, $\mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g)$.

□

Cosine and Sine Transforms of Derivatives

Theorem 4.9. Let $f(x)$ be continuous and absolutely integrable on the x -axis, let $f'(x)$ be piecewise continuous on each finite interval, and let $f(x) \rightarrow 0$ as $x \rightarrow \infty$. Then

$$\mathcal{F}_c\{f'(x)\} = \omega \mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0) \quad \text{and} \quad \mathcal{F}_s\{f'(x)\} = -\omega \mathcal{F}_c\{f(x)\}.$$

Furthermore,

$$\mathcal{F}_c\{f''(x)\} = -\omega^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$$

$$\mathcal{F}_s\{f''(x)\} = -\omega^2 \mathcal{F}_s\{f(x)\} + \omega \sqrt{\frac{2}{\pi}} f(0).$$

Proof. Note that

$$\begin{aligned} \mathcal{F}_c(f') &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos \omega x \, dx \\ &= \sqrt{\frac{2}{\pi}} [f(x) \cos \omega x]_0^\infty - \sqrt{\frac{2}{\pi}} \int_0^\infty (-\omega \sin \omega x) \, dx \\ &= -\sqrt{\frac{2}{\pi}} f(0) + \omega \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx \\ &= -\sqrt{\frac{2}{\pi}} f(0) + \omega \mathcal{F}_s(f). \end{aligned}$$

and that

$$\begin{aligned} \mathcal{F}_s(f') &= \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin \omega x \, dx \\ &= -\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \omega \cos \omega x \, dx \\ &= -\omega \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx \\ &= -\omega \mathcal{F}_c(f). \end{aligned}$$

Then

$$\begin{aligned} \mathcal{F}_c(f'') &= \omega \mathcal{F}_s(f') - \sqrt{\frac{2}{\pi}} f'(0) = -\omega^2 \mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}} f'(0), \\ \mathcal{F}_s(f'') &= -\omega \mathcal{F}_c(f') = -\omega \left(\omega \mathcal{F}_s(f) - \sqrt{\frac{2}{\pi}} f(0) \right) = -\omega^2 \mathcal{F}_s(f) + \omega \sqrt{\frac{2}{\pi}} f(0). \end{aligned}$$

□

Example 4.6. Find the Fourier cosine transform of $f(x) = e^{-ax}$, where $a > 0$.

Solution. Let $f(x) = e^{-ax}$. Since $f''(x) = a^2 e^{-ax} = a^2 f(x)$,

$$a^2 \mathcal{F}_c(f) = \mathcal{F}_c(a^2 f) = \mathcal{F}_c(f'') = -\omega^2 \mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}} f'(0) = -\omega^2 \mathcal{F}_c(f) + a \sqrt{\frac{2}{\pi}}.$$

Therefore

$$(a^2 + \omega^2) \mathcal{F}_c(f) = a \sqrt{\frac{2}{\pi}} \implies \mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + \omega^2} \right).$$

□

Note (Fourier Transform). The Fourier integral is

$$f(x) = \int_0^\infty [A(\omega) \cos \omega x + B(\omega) \sin \omega x] d\omega, \quad \text{where} \quad \begin{cases} A(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \cos \omega v dv, \\ B(\omega) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \sin \omega v dv. \end{cases}$$

Then, we have

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty f(v) (\cos \omega v \cos \omega x + \sin \omega v \sin \omega x) dv \right] d\omega \\ &= \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^\infty f(v) \cos(\omega x - \omega v) dv \right] d\omega \\ &= \frac{1}{\pi} \int_0^\infty F(\omega) d\omega \quad \text{with} \quad F(\omega) := \int_{-\infty}^\infty f(v) \cos(\omega x - \omega v) dv. \end{aligned}$$

Since $\cos(\omega x - \omega v)$ is an even function of ω , $F(\omega)$ is an even function. Thus $f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty F(\omega) d\omega$.

Let

$$G(\omega) := \int_{-\infty}^\infty f(v) \sin(\omega x - \omega v) dv.$$

Since $\sin(\omega x - \omega v)$ is an odd function of ω , $G(\omega)$ is an odd function. Thus $\frac{1}{2\pi} \int_{-\infty}^\infty G(\omega) d\omega = 0$.

Therefore,

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^\infty F(\omega) d\omega + i \cdot \frac{1}{2\pi} \int_{-\infty}^\infty G(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(v) \cos(\omega x - \omega v) dv \right] d\omega \\ &\quad + \frac{i}{2\pi} \int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(v) \sin(\omega x - \omega v) dv \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(v) e^{i(\omega x - \omega v)} dv \right] d\omega, \end{aligned}$$

and so

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(v) e^{-i\omega v} dv \right] e^{i\omega x} d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \hat{f}(\omega) e^{i\omega x} d\omega \quad \text{with} \quad \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) e^{-i\omega v} dx. \end{aligned}$$

Fourier and Inverse Fourier Transforms

Definition 4.5.

- **Fourier transform of f :**

$$\mathcal{F}(f) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx.$$

- **Inverse Fourier transform of $\hat{f}(\omega)$:**

$$\mathcal{F}^{-1}(f) = \hat{f}^{-1}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

Existence of the Fourier Transform

Theorem 4.10. *Sufficient for the existence of the Fourier transform are the following two conditions, as we mention without proof.*

- (1) $f(x)$ is piecewise continuous on every finite interval.
- (2) $f(x)$ is absolutely integrable on the x -axis.

Linearity of the Fourier Transform

Theorem 4.11. *The Fourier transform is a linear operation; that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and any constants a and b ,*

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g).$$

Proof. Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned} \mathcal{F}(af + bg) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (af + bg)(x) e^{-i\omega x} dx \\ &= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx \\ &= a\mathcal{F}(f) + b\mathcal{F}(g). \end{aligned}$$

□

Fourier Transform of the Derivative of $f(x)$

Theorem 4.12. *Let $f(x)$ be continuous on the x -axis and $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Furthermore, let $f'(x)$ be absolutely integrable on the x -axis. Then*

$$\mathcal{F}\{f'(x)\} = i\omega \mathcal{F}\{f(x)\}$$

and

$$\mathcal{F}\{f''(x)\} = i\omega \mathcal{F}\{f'(x)\} = -\omega^2 \mathcal{F}\{f(x)\}.$$

Proof. Since

$$\begin{aligned}
 \mathcal{F}(f') &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-i\omega x} dx \\
 &= \frac{1}{\sqrt{2\pi}} \left[\cancel{f(x)e^{-i\omega x}} \right]_{-\infty}^{\infty} \overset{0}{=} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) (-i\omega e^{-i\omega x}) dx \\
 &= i\omega \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \right) \\
 &= i\omega \mathcal{F}(f),
 \end{aligned}$$

we have

$$\mathcal{F}(f'') = i\omega \mathcal{F}(f') = i\omega(i\omega \mathcal{F}(f)) = -\omega^2 \mathcal{F}(f).$$

□

Convolution

Definition 4.6. The **convolution** $f * g$ of functions f and g is defined by

$$(f * g)(x) := \int_{-\infty}^{\infty} f(t)g(x-t)dt = \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

Convolution Theorem

Theorem 4.13. Suppose that $f(x)$ and $g(x)$ are piecewise continuous, bounded, and absolutely integrable on the x -axis. Then

$$\mathcal{F}(f * g) = \sqrt{2\pi} \mathcal{F}\{f\} \mathcal{F}\{g\}.$$

Proof.

$$\begin{aligned}
 \mathcal{F}(f * g) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f * g)(x) e^{-i\omega x} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t)g(x-t) dt \right] e^{-i\omega x} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t)g(s) dt \right] e^{-i\omega(s+t)} ds \quad \text{with } s := x - t \\
 &= \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s) e^{-i\omega s} ds \\
 &= \sqrt{2\pi} \mathcal{F}(f) \mathcal{F}(g);
 \end{aligned}$$

□

Chapter 5

Introduction to PDEs

5.1 Basic Concepts

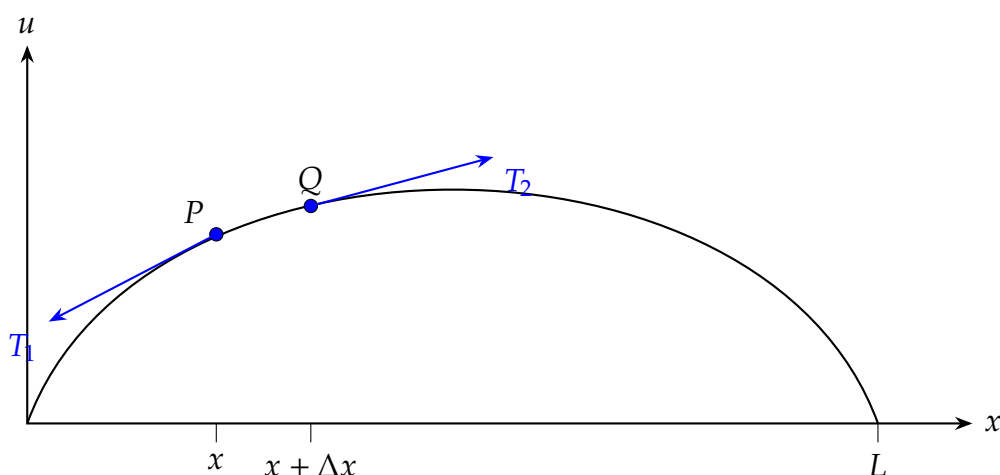
Partial Differential Equation

Definition 5.1. A partial differential equation (PDE) is an equation involving one or more partial derivatives of an (unknown) function, call it u , that depends on two or more variables, often time t and one or several variables in space.

Order of PDE

Definition 5.2. The order of the highest derivative is called the order of the PDE. As for ODEs, second-order PDEs will be the most important ones in applications.

5.2 Vibrating String, One-Dimensional Wave Equation



One-Dimensional Wave Equation

Definition 5.3. One-dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Here $c^2 = \frac{T}{\rho}$, where T denotes the tension and ρ is the mass of the undeflected string per unit length.

Remark 5.1. Note that

- (1) No motion in the horizontal direction. $T_1 \cos \alpha = T_2 \cos \beta = T = \text{constant}$.
- (2) The force in the vertical direction is given by $T_2 \sin \beta - T_1 \sin \alpha$. By Newton's second law $\mathbf{F} = m\mathbf{a}$, we have $T_2 \sin \beta - T_1 \sin \alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$.

Then

$$\frac{T_2 \sin \beta}{T_2 \cos \beta} - \frac{T_1 \sin \alpha}{T_1 \cos \alpha} = \tan \beta - \tan \alpha = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}.$$

Since $\tan \alpha = \left. \frac{\partial u}{\partial x} \right|_x$ and $\tan \beta = \left. \frac{\partial u}{\partial x} \right|_{x+\Delta x}$, we have

$$\begin{aligned} \left. \frac{\partial u}{\partial x} \right|_{x+\Delta x} - \left. \frac{\partial u}{\partial x} \right|_x &= \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}, \\ \frac{u_x(x + \Delta x) - u_x(x)}{\Delta x} &= \frac{\rho}{T} u_{tt}, \\ \lim_{\Delta x \rightarrow 0} \frac{u_x(x + \Delta x) - u_x(x)}{\Delta x} &= \frac{\rho}{T} u_{tt}, \\ u_{xx} &= \frac{\rho}{T} u_{tt}. \end{aligned}$$

By letting $c^2 := T/\rho$, we have the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Note. The two-dimensional wave equation is $u_{tt} = c^2(u_{xx} + u_{yy})$.

5.3 Separation of Variables, Use of Fourier Series

Note (Solving Wave Equation). Find **solutions** $u(x, t)$, which satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with **boundary conditions**

$$u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0 \quad \text{for all } t$$

and **initial conditions**

$$u(x, 0) = f(x) \quad \text{and} \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x).$$

Note (Method of Separating Variables). Let

$$u(x, t) = F(x)G(t) \neq 0.$$

We denote $F'(x) = \frac{dF}{dx}$ and $\dot{G}(t) = \frac{dG}{dt}$. Then

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = F''G.$$

We then have

$$F\ddot{G} = c^2 F''G \implies \frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k.$$

for constant k . This yields immediately two ODEs, namely

$$F'' - kF = 0 \quad \text{and} \quad \ddot{G} - c^2 kG = 0.$$

(1) Assume that $k = 0$, i.e., $F'' = 0$ and so $F(x) = ax + b$ for $a, b \in \mathbb{R}$. By the boundary conditions, we have

$$(i) \quad u(0, t) = F(0)G(t) = 0 \implies b = 0 \text{ and}$$

$$(ii) \quad u(L, t) = F(L)G(t) = 0 \implies a = 0.$$

$\therefore F(x) = 0$. It is contradiction.

(2) For positive $k = \mu^2 > 0$ the general solution is

$$F(x) = Ae^{\mu x} + Be^{-\mu x}.$$

Since $u(0, t) = F(0)G(t) = (A + B)G(t) = 0$, we have $B = -A$. The boundary conditions give

$$(i) \quad u(0, t) = F(0)G(t) = (A + B)G(t) = 0 \implies B = -A \text{ and}$$

$$(ii) \quad u(L, t) = A(e^{\mu L} - e^{-\mu L})G(t) = 0 \implies A = 0.$$

$\therefore F(x) = 0$. It is contradiction.

Hence we choose negative k , say, $k = -p^2 < 0$. Then F satisfies

$$F'' + p^2 F = 0.$$

Its general solution is

$$F(x) = A \cos px + B \sin px.$$

By the boundary conditions, we have

$$(i) \quad u(0, t) = F(0)G(t) = 0 \implies A = 0 \text{ and}$$

$$(ii) \quad u(L, t) = F(L)G(t) = 0 \implies F(L) = B \sin pL = 0 \xrightarrow{B \neq 0} \sin pL = 0.$$

Hence

$$pL = n\pi \quad \text{so that} \quad p = \frac{n\pi}{L}$$

We thus obtain infinitely many solutions $F(x) = F_n(x)$, where

$$F_n(x) = B \sin \frac{n\pi x}{L}, \quad n = 1, 2, \dots$$

Let $\lambda_n = \frac{cn\pi}{L}$. Since $\ddot{G} + \lambda_n^2 G = 0$, a general solution is

$$G_n(t) = C \cos \lambda_n t + D \sin \lambda_n t.$$

Hence, solutions of the wave equation with given boundary and initial conditions are

$$u_n(x, t) = F_n(x)G_n(t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi x}{L}, \quad B_n = BC, B_n^* = CD, n = 1, 2, \dots$$

Note (Solution of Wave Equation). In order to obtain a solution that satisfies the initial conditions, we consider the infinite series

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi x}{L}.$$

From initial conditions, we obtain

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x).$$

Hence, based on the Fourier sine series of $f(x)$,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, \dots$$

Then

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = \left[\sum_{n=1}^{\infty} (-B_n \lambda_n \sin \lambda_n t + B_n^* \lambda_n \cos \lambda_n t) \sin \frac{n\pi x}{L} \right]_{t=0} = \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi x}{L} = g(x).$$

Therefore

$$B_n^* \lambda_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx \implies B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} dx$$

for $n = 1, 2, \dots$. In summary,

Solutions of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions

$$u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0 \quad \text{for all } t$$

and initial conditions

$$u(x, 0) = f(x) \quad \text{and} \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x).$$

are given by

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi x}{L}, \quad n = 1, 2, \dots,$$

where

$$\lambda_n = \frac{cn\pi}{L}, \quad B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad \text{and} \quad B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} dx.$$

5.4 D'Alembert's Solution of the Wave Equation

Observation (D'Alembert's Solution)

- Introduce the new independent variables $v = x + ct$ and $z = x - ct$.
- Then, applying chain rule $u_x = u_v v_x + u_z z_x = u_v + u_z$.
- Assume that all the partial derivatives involved are continuous, so that

$$u_{xv} = u_{vx} \quad \text{and correspondingly} \quad u_{xx} = (u_v + u_z)_x = u_{vv} + 2u_{vz} + u_{zz}.$$

- Similarly, $u_{tt} = c^2(u_{vv} - 2u_{vz} + u_{zz})$. By inserting these two results in the wave equation, we get $u_{vx} = \frac{\partial^2 u}{\partial z \partial v} = 0$. By integrating above identity with respect to z , we find $\frac{\partial u}{\partial v} = h(v)$, where $h(v)$ is an arbitrary function of v . Integrating this with respect to v gives

$$u = \int h(v) dv + \psi(z),$$

where $\psi(z)$ is an arbitrary function of z . Since the integral is a function of v , say, $\phi(v)$, the solution u is of the form

$$u(x, t) = \phi(v) + \psi(z) = \phi(x + ct) + \psi(x - ct).$$

This is known as d'Alembert's solution of the wave equation.

- Consider the two initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$.
- Then, we have

$$u(x, 0) = \phi(x) + \psi(x) = f(x) \quad \text{and} \quad u_t(x, 0) = \phi'(x) - \psi'(x) = g(x).$$

- Thus, we have

$$\phi(x) - \psi(x) = k(x_0) \quad \text{where} \quad k(x_0) = \phi(x_0) - \psi(x_0).$$

- Therefore,

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c} \int_{x_0}^x g(s) ds + \frac{1}{2}k(x_0)$$

and similarly,

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c} \int_{x_0}^x g(s) ds - \frac{1}{2}k(x_0).$$

Theorem (D'Alembert's Solution of the Wave Equation)

The solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with two initial conditions

$$u(x, 0) = f(x) \quad \text{and} \quad u_t(x, 0) = g(x)$$

is given by

$$u(x, t) = \frac{1}{2}(f(x + ct) + f(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds.$$

5.5 One-Dimensional Heat Equation: Solution by Fourier Series

Heat Equation

Definition 5.4. Let $u(\mathbf{x}, t)$ be the temperature at the time t . Then, the **heat equation** is given by

$$\frac{\partial u}{\partial t} = c^2 \Delta u(\mathbf{x}, t) = c^2(u_{xx} + u_{yy} + u_{zz}),$$

where c^2 is the **thermal diffusivity** of the material and $\Delta u(\mathbf{x}, t)$ is the **Laplacian** of $u(\mathbf{x}, t)$.

Remark 5.2. Let Ω be a region in the body and let $\partial\Omega$ be its (smooth) boundary surface. The rate of flow is proportional to the gradient of the temperature

$$\mathbf{v} = -K\nabla u(\mathbf{x}, t),$$

where $u(\mathbf{x}, t)$ is temperature, t is time, and K (constant) is called the thermal conductivity of the body. The amount of heat leaving Ω per unit of time is

$$\int_{\partial\Omega} (\mathbf{v} \cdot \mathbf{N}) dS,$$

where $\mathbf{v} \cdot \mathbf{N}$ is the component of v in the direction of the outer unit normal vector \mathbf{N} of $\partial\Omega$. Applying divergence theorem

$$\int_{\partial\Omega} (\mathbf{v} \cdot \mathbf{N}) dS = - \int_{\Omega} \nabla \cdot (K\nabla u(\mathbf{x}, t)) dx = -K \int_{\Omega} \nabla^2 u(\mathbf{x}, t) dx,$$

where $\nabla^2 u(\mathbf{x}, t) = \Delta u(\mathbf{x}, t) = u_{xx} + u_{yy} + u_{zz}$ is the Laplacian of u . Total amount of heat H in Ω is

$$H = \int_{\Omega} \sigma \rho u(\mathbf{x}, t) dx,$$

where the constant σ is the specific heat of the material of the body and ρ is the density (mass per unit volume) of the material. The time rate of decrease of H is

$$-\frac{\partial H}{\partial t} = - \int_{\Omega} \sigma \rho \frac{\partial u}{\partial t} dx$$

and this must be equal to the above amount of heat leaving Ω . Therefore,

$$- \int_{\Omega} \sigma \rho \frac{\partial u}{\partial t} dx = -K \int_{\Omega} \Delta u(\mathbf{x}, t) dx \implies \int_{\Omega} \left(\sigma \rho \frac{\partial u}{\partial t} - K \Delta u(\mathbf{x}, t) \right) dx = 0.$$

Since this holds for any region Ω in the body, the integrand (if continuous) must be zero everywhere; that is,

$$\frac{\partial u}{\partial t} = c^2 \Delta u, \quad c^2 = \frac{K}{\sigma \rho}$$

where c^2 is called the thermal diffusivity of the material. This PDE is called the heat equation.

Note (Solving One-Dimensional Heat Equation). Find **solutions** $u(x, t)$, which satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with (Dirichlet) **boundary conditions** $u(0, t) = 0$ and $u(L, t) = 0$ for all t and **initial conditions**

$$u(x, 0) = f(x) \quad \text{and} \quad f(L) = 0.$$

Note (Method of Separating Variables). Let $u(x, t) = F(x)G(t) \neq 0$. Then

$$\frac{\partial u}{\partial t} = F\dot{G} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = F''G.$$

We then have

$$F\dot{G} = c^2 F''G \implies \frac{\dot{G}}{c^2 G} = \frac{F''}{F} = k.$$

for negative constant $k = -p^2 < 0$. Then $F'' + p^2 F = 0$ and $\dot{G} + c^2 p^2 G = 0$. A general solution is

$$F(x) = A \cos px + B \sin px.$$

By the boundary conditions, we have

$$(i) \quad u(0, t) = F(0)G(t) = 0 \implies A = 0 \text{ and}$$

$$(ii) \quad u(L, t) = F(L)G(t) = 0 \implies F(L) = B \sin pL = 0 \xrightarrow{B \neq 0} \sin pL = 0.$$

Hence

$$pL = n\pi \quad \text{so that} \quad p = \frac{n\pi}{L}$$

We thus obtain infinitely many solutions $F(x) = F_n(x)$, where

$$F_n(x) = B \sin \frac{n\pi x}{L}, \quad n = 1, 2, \dots$$

Let $\lambda_n = \frac{cn\pi}{L}$. Since $\dot{G} + \lambda_n^2 G = 0$, a general solution is

$$G_n(t) = C_n e^{-\lambda_n^2 t}.$$

Hence, solutions of the wave equation with given boundary and initial conditions are

$$u_n(x, t) = F_n(x)G_n(t) = B_n \sin \frac{n\pi x}{L} e^{-\lambda_n^2 t}, \quad B_n = BC_n, n = 1, 2, \dots$$

Solution of the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions

$$u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0 \quad \text{for all } t$$

and initial condition

$$u(x, 0) = f(x)$$

is

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t),$$

where

$$u_n(x, t) = F_n(x)G_n(t) = B_n \sin \frac{n\pi x}{L} e^{-\lambda_n^2 t}, \quad n = 1, 2, \dots$$

Here,

$$\lambda_n = \frac{cn\pi}{L}, \quad B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

Laplace Equation

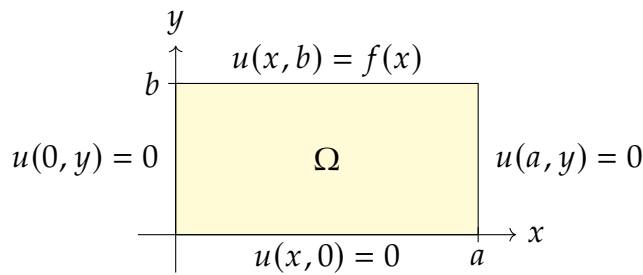
Definition 5.5. The two-dimensional **Laplace equation** is given by

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Note (Solving Laplace Equation). Find a solution $u(x, y)$, which satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with Dirichlet problem in a rectangle Ω , assuming that the temperature $u(x, y)$ equals a given function $f(x)$ on the upper side and 0 on the other three sides of the rectangle.



Let $u(x, y) = F(x)G(y)$. Then, we have

$$\frac{1}{F} \frac{d^2 F}{dx^2} = -\frac{1}{G} \frac{d^2 G}{dy^2} = -k.$$

From this and the left and right boundary conditions,

$$\frac{d^2 F}{dx^2} + kF = 0, \quad F(0) = F(a) = 0.$$

This gives $k = \left(\frac{n\pi}{a}\right)^2$ and corresponding nonzero solutions $F(x) = F_n(x) = \sin \frac{n\pi x}{a}$, $n = 1, 2, \dots$. The differential equation for G then becomes

$$\frac{d^2 G}{dy^2} - \left(\frac{n\pi}{a}\right)^2 G = 0$$

and solutions are

$$G(y) = G_n(y) = A_n e^{\frac{n\pi y}{a}} + B_n e^{-\frac{n\pi y}{a}}.$$

The boundary condition $u = 0$ on the lower side of Ω implies that $G_n(0) = A_n + B_n = 0$ or $B_n = -A_n$. This gives

$$G_n(y) = A_n e^{\frac{n\pi y}{a}} - A_n e^{-\frac{n\pi y}{a}} = 2A_n \sinh \frac{n\pi y}{a}.$$

By letting $2A_n = A_n^*$, we obtain the eigenfunctions

$$u_n(x, y) = F_n(x)G_n(y) = A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}.$$

These satisfy the boundary condition $u = 0$ on the left, right, and lower sides. Let

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}.$$

From the boundary condition on the upper side $u(x, b) = f(x)$, we have

$$u(x, b) = f(x) = \sum_{n=1}^{\infty} \left(A_n^* \sin \frac{n\pi x}{a} \right) \sinh \frac{n\pi b}{a}.$$

This shows that

$$A_n^* \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx.$$

Therefore, the solution is

$$u(x, y) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a},$$

where

$$A_n^* = \frac{2}{a \sinh \left(\frac{n\pi b}{a} \right)} \int_0^a f(x) \sin \frac{n\pi x}{a} dx.$$

Fourier Cosine and Sine Series**Theorem 5.1.** *content...***Fourier Cosine and Sine Series****Definition 5.6.** *content...*