1.

수업시간에 설명한 'Minus 1 Trick'을 이용하여 다음 방정식의 일반해(general solution)를 구하라.

$$\begin{cases} x_1 \quad\quad\quad - x_3 + 2x_4 = -1, \\ x_1 + x_2 + x_3 - x_4 = 2, \\ \quad\quad - x_2 - 2x_3 + 3x_4 = -3, \\ 5x_1 + 2x_2 - x_3 + 4x_4 = 1. \end{cases}$$

Sol. We have

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 0 & -1 & -2 & 3 & -3 \\ 5 & 2 & -1 & 4 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & -1 \\ 0 & 1 & 2 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & -6 & 6 \end{array}\right] \quad \begin{array}{l} R_2' \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 + R_2' \\ R_4 \leftarrow R_4 - 2R_2' \end{array}$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & -1 \\ 0 & 1 & 2 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_4 \leftarrow R_4 - 2R_2$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & -1 \\ 0 & 1 & 2 & -3 & 3 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right] \quad \text{Minus-1 Trick,}$$

and so

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \lambda_1 \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ -3 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda_1, \lambda_2 \in \mathbb{R}$. Thus, all solutions are given by

$$\left\{ \mathbf{x} \in \mathbb{R}^4 : \mathbf{x} = \lambda_1 \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ -3 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R} \right\}.$$

□

2.

예제와 같이 가우스 소거법(Gaussian Elimination)으로 행렬 B 역행렬을 구하라.

[예제] $A = \begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ 일 때, 역행렬은 $A^{-1} = \begin{pmatrix} 1/4 & 1/4 & 3/4 \\ 1/4 & 1/4 & -1/4 \\ -1/4 & 3/4 & -3/4 \end{pmatrix}$.

$$\left[\begin{array}{ccc|ccc} 0 & 3 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 1/4 & 3/4 \\ 0 & 1 & 0 & 1/4 & 1/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & 3/4 & -3/4 \end{array}\right].$$

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Sol. Since

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Swap } R_1 \text{ and } R_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_1 \leftarrow R_1/2$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 - R_3,$$

we have

$$B^{-1} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

3.

벡터공간 $\mathbb{R}^2$의 기저(ordered basis) $(\mathbf{v_1}, \mathbf{v_2})$가

$$\mathbf{v_1} = (1, -1), \quad \mathbf{v_2} = (2, 1)$$

일 때 표준기저로 다음과 같이 표현된 선형사상 T를

$$(x_1, x_2) \xrightarrow{T} (y_1, y_2) = (3x_1 + 2x_2, x_1 + 2x_2), \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

기저 $(\mathbf{v_1}, \mathbf{v_2})$로 나타내는 행렬 B를 구하면?

$$(\alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2}) \xrightarrow{T} (\beta_1 \mathbf{v_1} + \beta_2 \mathbf{v_2}), \quad \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = B \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

Sol. Note that

$$\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$$

$$\begin{tikzcd}
\mathscr{B} \arrow[r, "A_T"] \arrow[d, "T=S^{-1}", phantom] & \mathscr{B} \arrow[d, "T=S^{-1}"] \\
\tilde{\mathscr{B}} \arrow[u, "S"] \arrow[r, "B"] & \tilde{\mathscr{B}}
\end{tikzcd}$$

where $A_T = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$, $\mathscr{B} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\tilde{\mathscr{B}} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$. Since

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & 2/3 & 2/3 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1/3 & -2/3 \\ 0 & 2 & 2/3 & 2/3 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 1/3 & -2/3 \\ 0 & 1 & 1/3 & 1/3, \end{array}\right]$$

$$\begin{aligned} B = S^{-1} A_T S &= \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 6 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{bmatrix} \\ &= \begin{bmatrix} 11/3 & -4/3 \\ 2/3 & -4/3 \end{bmatrix}. \end{aligned}$$

□

4.

$\mathbb{R}^3$의 두 벡터 $\mathbf{b_1} = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $\mathbf{b_2} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$ 로 생성되는 부분공간 $W = \text{span}\langle \mathbf{b_1}, \mathbf{b_2} \rangle$ 위로 내린 벡터 $\mathbf{v} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ 의 정사영(projection) 벡터 $\pi(\mathbf{v})$를 구하면? 이 때, 정사영을 나타내는 행렬 P_π를 구하면?

$$\pi(\mathbf{v}) = P_\pi \mathbf{v}.$$

Sol. Let $\mathbf{B} = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & -1\\0 & 0\end{bmatrix}$. Since $\mathbf{B}^T\mathbf{B} = \begin{bmatrix}1 & 1 & 0\\1 & -1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\\0 & 0\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$, we have

$$\left[\begin{array}{cc|cc}2 & 0 & 1 & 0\\0 & 2 & 0 & 1\end{array}\right] \to \left[\begin{array}{cc|cc}1 & 0 & 1/2 & 0\\0 & 1 & 0 & 1/2\end{array}\right] \Longrightarrow (\mathbf{B}^T\mathbf{B})^{-1} = \begin{bmatrix}1/2 & 0\\0 & 1/2\end{bmatrix}.$$

Then

$$\boldsymbol{\lambda} = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{v} = \begin{bmatrix}1/2 & 0\\0 & 1/2\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\1 & -1 & 0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1/2 & -1/2 & 0\\1/2 & 1/2 & 0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}.$$

Thus,

$$\pi_W(\mathbf{v}) = \mathbf{B}\boldsymbol{\lambda} = \begin{bmatrix}1 & 1\\1 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\0\end{bmatrix}$$

and

$$P_\pi = \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T = \begin{bmatrix}1 & 1\\1 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}1/2 & -1/2 & 0\\1/2 & 1/2 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 0\end{bmatrix}.$$

□

5.

내적(inner product)이 $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$로 정의된 실수 벡터공간 $\mathbb{R}^3$의 세 벡터

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v_3} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

에 그람-슈미트 직교화 과정(Gram-Schmidt orthogonalization process)을 적용하여 $\mathbb{R}^3$의 정규직교기저(orthonormal basis)를 구하면?

(정규직교기저는 서로 수직이며 크기가 1인 벡터로 구성된 기저(basis)를 의미한다)

Sol. (i) (1st vector) Let $\mathbf{u}_1 := \mathbf{v}_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$. Since $||\mathbf{u}_1|| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$, we have

$$\mathbf{w}_1 := \frac{\mathbf{u}_1}{||\mathbf{u}||} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(ii) (2nd vector)

$$\mathbf{u}_2 := \mathbf{v}_2 - \pi_{\text{span}\langle \mathbf{u}_1 \rangle}(\mathbf{v}_2) = \mathbf{v}_2 - \frac{\mathbf{u}_1 \mathbf{u}_1^T}{||\mathbf{u}_1||^2} \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus $\mathbf{w}_2 := \frac{\mathbf{u}_2}{||\mathbf{u}_2||} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(iii) (3rd vector) Let $\mathbf{B} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $\mathbf{B}^T \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and so

$$\pi_{\text{span}\langle \mathbf{u}_1, \mathbf{u}_2 \rangle} = \mathbf{P}_\pi = \mathbf{B}(\mathbf{B}^T \mathbf{B})^{-1} \mathbf{B}^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus,

$$\mathbf{u}_3 := \mathbf{v}_3 - \pi_{\mathrm{span}\langle \mathbf{u}_1, \mathbf{u}_2 \rangle}(\mathbf{v}_3) = \mathbf{v}_3 - \mathbf{P}_\pi \mathbf{v}_3 = (\mathbf{I}_3 - \mathbf{P}_\pi)\mathbf{v}_3 = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

and then $\mathbf{w}_3 := \dfrac{\mathbf{u}_3}{||\mathbf{u}_3||} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

By (i),(ii) and (iii), we have orthonormal basis

$$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

□

6.

다음 행렬의 행렬식(determinant)을 구하라. 단, x_1, x_2, x_3는 서로 다른 실수이다.

$$\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}$$

Sol.

$$\begin{aligned}
\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} &= \begin{vmatrix} x_2 & x_2^2 \\ x_3 & x_3^2 \end{vmatrix} - x_1 \begin{vmatrix} 1 & x_2^2 \\ 1 & x_3^2 \end{vmatrix} + x_1^2 \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} \\
&= (x_2 x_3^2 - x_2^2 x_3) - x_1(x_3^2 - x_2^2) + x_1^2(x_3 - x_2) \\
&= x_2 x_3 (x_3 - x_2) - x_1(x_2 + x_3)(x_2 - x_3) + x_1^2(x_3 - x_2) \\
&= x_2 x_3 (x_3 - x_2) + x_1(x_2 + x_3)(x_3 - x_2) + x_1^2(x_3 - x_2) \\
&= x_2 x_3 (x_3 - x_2) + x_1(x_3 - x_2)(x_1 + x_2 + x_3) \\
&= (x_3 - x_2)(x_2 x_3 + x_1(x_1 + x_2 + x_3)).
\end{aligned}$$

□

7.

다음 행렬의 스펙트럴 분해(Spectral Decomposition)를 구하라.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = PDP^T.$$

Sol.

(Step 1) **Find the Eigenvalues:**

$$\det(A - \lambda I_2) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0.$$

Thus, $\lambda_1 := 3$ and $\lambda_2 := 1$.

(Step 2) **Find the Eigenvectors:**

(i) For $\lambda_1 = 3$,

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \mathbf{v}_1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(ii) For $\lambda_2 = 1$,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \mathbf{v}_2 := \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(Step 3) **Normalize the Eigenvectors:**

$$\hat{\mathbf{v}}_1 := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \hat{\mathbf{v}}_2 := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

(Step 4) **Construct P and D:**

$$P := \begin{bmatrix} \hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix},$$

$$D := \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

By Step 1-4, hence,

$$A = PDP^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

□

8.

구간 $[a,b]$에서 정의된 연속함수 집합 $\mathcal{C}([a,b])$는 벡터공간이 된다. 두 벡터 f,g의 내적 $\langle f,g\rangle$과 노음(norm) $\|f\|$을

$$\langle f,g\rangle = \int_a^b f(x)g(x)dx, \quad \|f\| = \sqrt{\langle f,f\rangle}$$

로 정의할 때, 다음 코시-슈바르츠(Cauchy-Schwarz) 부등식을 아래 순서로 증명하라.

$$\langle f,g\rangle \le \|f\|^2\,\|g\|^2$$

(a) 모든 실수 t에 대하여 성립하는 부등식

$$\int_a^b [tf(x)-g(x)]^2\,dx \ge 0$$

을 t에 대한 2차부등식 $At^2+Bt+C\ge 0$로 정리하라.

(b) 정리한 부등식이 모든 실수에 대하여 성립하기 위해 $A>0$이고 판별식이 0보다 작거나 같다는 조건으로부터 다음을 보여라.

$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \left(\int_a^b [f(x)]^2\,dx\right)\left(\int_a^b [g(x)]^2\,dx\right).$$

Sol. (a)

$$\begin{aligned}
\int_a^b \left[tf(x)-g(x)\right]^2\,dx \ge 0 &\iff \int_a^b \left[t^2[f(x)]^2 - 2tf(x)g(x) + [g(x)]^2\right]\,dx \ge 0\\
&\iff t^2\int_a^b [f(x)]^2\,dx - 2t\int_a^b f(x)g(x)\,dx + \int_a^b [g(x)]^2\,dx \ge 0\\
&\iff At^2+Bt+C\ge 0 \quad \text{with} \begin{cases} A = \int_a^b [f(x)]^2\,dx,\\ B = -2\int_a^b f(x)g(x)\,dx,\\ C = \int_a^b [g(x)]^2\,dx.\end{cases}
\end{aligned}$$

(b)

$$\begin{aligned}
B^2-4AC = &\left(-2\int_a^b f(x)g(x)\,dx\right)^2 - 4\left(\int_a^b [f(x)]^2\,dx\right)\left(\int_a^b [g(x)]^2\,dx\right) \le 0\\
\implies &4\left(\int_a^b f(x)g(x)\right) \le 4\left(\int_a^b [f(x)]^2\,dx\right)\left(\int_a^b [g(x)]^2\,dx\right)\\
\implies &\left(\int_a^b f(x)g(x)\,dx\right)^2 \le \left(\int_a^b [f(x)]^2\,dx\right)\left(\int_a^b [g(x)]^2\,dx\right).
\end{aligned}$$

$\square$

9.

연속함수 집합 $\mathcal{C}([0,\pi])$에서

$$S = \left\{ \sqrt{\frac{2}{\pi}}\sin(mx) \,\middle|\, m = 1, 2, 3, \ldots \right\}$$

는 정규직교집합(orthonormal set)이 됨을 보여라.
즉, $f_m(x) = \sqrt{\frac{2}{\pi}}\sin(mx)$라 하면 $\langle f_i, f_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j \end{cases}$.

Sol. (i) **Normalization.** We want to show that

$$\|f_m\|^2 = \langle f_m, f_m \rangle = \int_0^\pi [f_m(x)]^2 \, dx = 1.$$

Now, we have

$$\begin{aligned}
\int_0^\pi [f_m(x)]^2 \, dx = \int_0^\pi \left(\sqrt{\frac{2}{\pi}}\sin(mx) \right)^2 dx &= \frac{2}{\pi} \int_0^\pi \sin^2(mx) \, dx \\
&= \frac{2}{\pi} \int_0^\pi \frac{1 - \cos(2mx)}{2} \, dx \\
&= \frac{2}{\pi} \int_0^\pi \left(\frac{1}{2} - \frac{1}{2}\cos(2mx) \right) dx \\
&= \frac{2}{\pi} \left[\frac{1}{2}x - \frac{1}{4m}\sin(2mx) \right]_0^\pi \\
&= \frac{2}{\pi} \left[\frac{\pi}{2} - 0 - (0 - 0) \right] \\
&= 1.
\end{aligned}$$

(ii) **Orthogonality.** Let $m \neq n$. We want to show that

$$\langle f_m, f_n \rangle = \int_0^\pi f_m(x) f_n(x) \, dx = 0.$$

Now, we have

$$\begin{aligned}
\int_0^\pi f_m(x) f_n(x) \, dx &= \int_0^\pi \sqrt{\frac{2}{\pi}}\sin(mx)\sqrt{\frac{2}{\pi}}\sin(nx) \, dx \\
&= \frac{2}{\pi} \int_0^\pi \sin(mx)\sin(nx) \, dx.
\end{aligned}$$

Since $\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$, we obtain

$$\int_0^\pi f_m(x) f_n(x)\, dx = \frac{1}{\pi}\int_0^\pi \cos((m-n)x)\, dx - \frac{1}{\pi}\int_0^\pi \cos((m+n)x)\, dx.$$

Since integrating cosine function over a whole period result in 0, thus,

$$\int_0^\pi f_m(x) f_n(x) = 0.$$

Therefore, any two different element in S are orthogonal each other.

$\square$

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