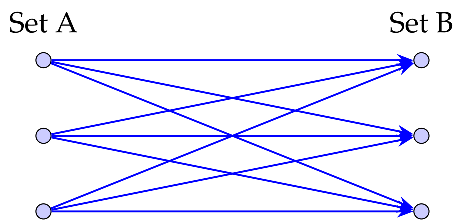
Equivalence Relations, Equivalence Classes, Partitions, and Quotient Sets

July 14, 2024

Cartesian Products / Set Products



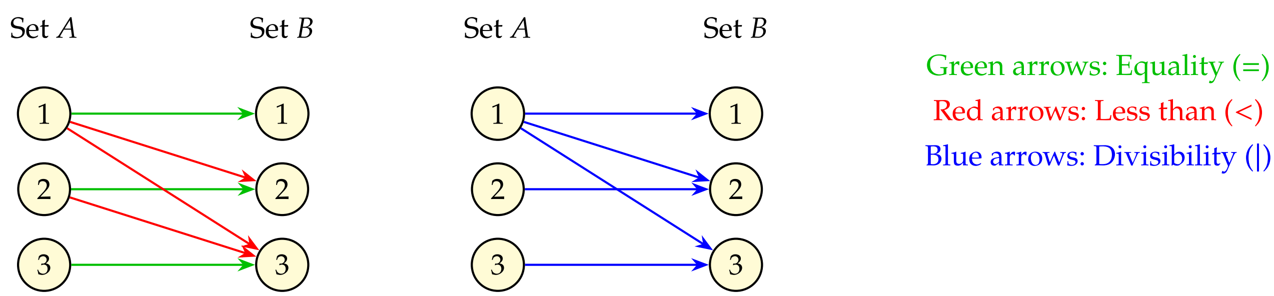
Example. If $A = \{1, 2\}$ and $B = \{x, y\}$, then:

$$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$$

Definition. Let A and B are sets.

$$A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Relations



Definition. Let A and B are sets. A **relation** $\mathcal{R}$ from A to B is a *subset* of the Cartesian Product $A \times B$:

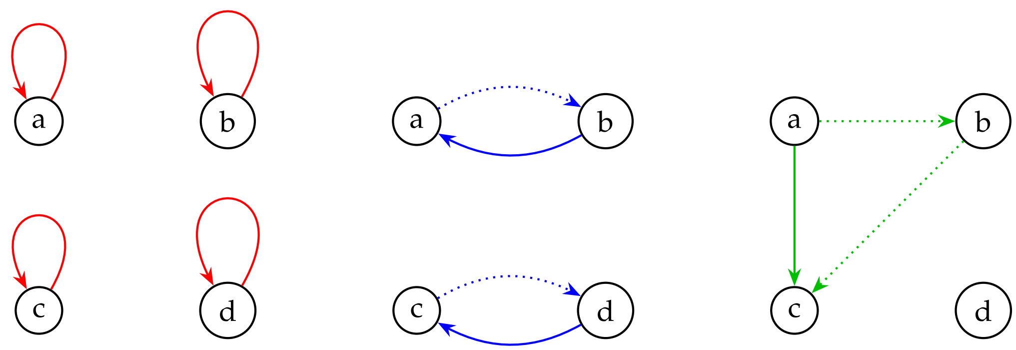
$$\mathcal{R} \subseteq A \times B$$

Notation.

$$(a, b) \in \mathcal{R} \subseteq A \times B \iff a\,\mathcal{R}\,b.$$

For example, $a = b \iff (a, b) \in =$

Equivalence Relations



Red arrows: Reflexivity
(each element is related to itself).
Blue arrows: Symmetry
(if a is related to b, then b is related to a).
Green arrows: Transitivity
(if a is related to b and b is related to c, then a is related to c).

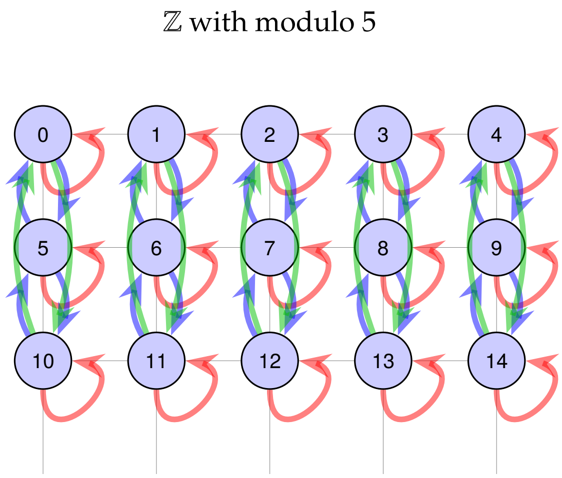
Example. Equality on $\mathbb{Z}$:

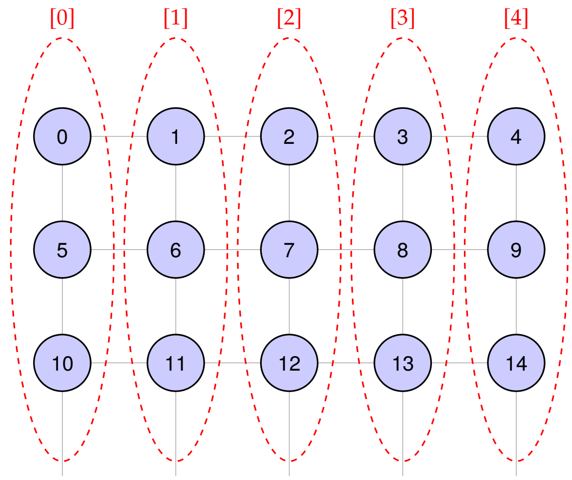
- $a \in \mathbb{Z} \implies a = a$;
- $a = b \implies b = a$;
- $a = b$ and $b = c \implies a = c$.

Definition. Given a set A, a relation $\mathcal{R} \subseteq A \times A$ is called an **equivalence relation** if it satisfies following properties:

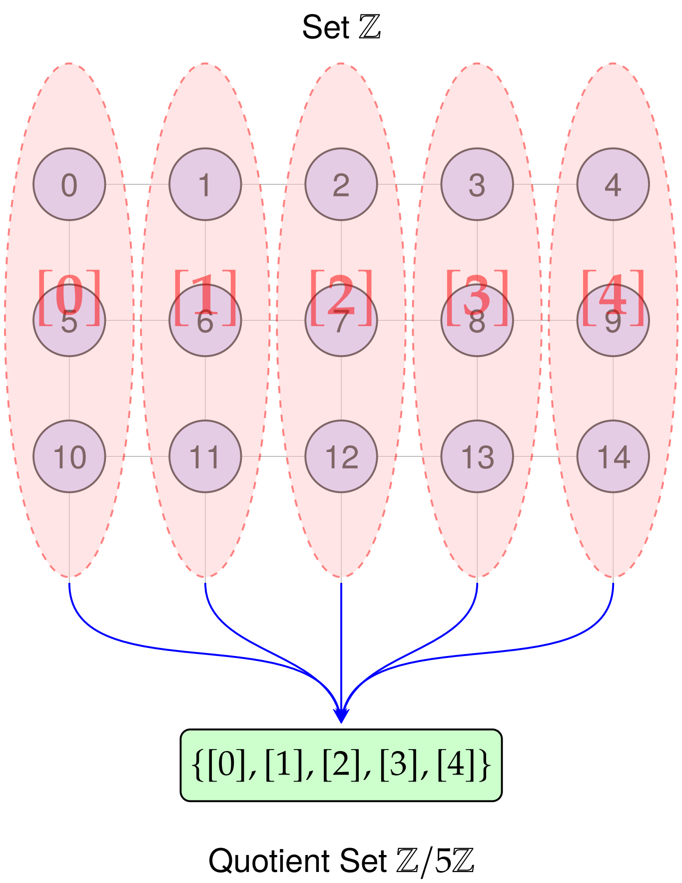
- (Reflexivity) $a \in A \implies (a, a) \in \mathcal{R}$;
- (Symmetry) $(a, b) \in \mathcal{R} \implies (b, a) \in \mathcal{R}$;
- (Transitivity) $(a, b) \in \mathcal{R} \wedge (b, c) \in \mathcal{R} \implies (a, c) \in \mathcal{R}$.

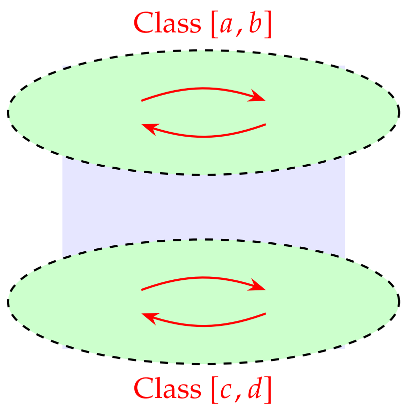
Equivalence Classes





Quotient Sets



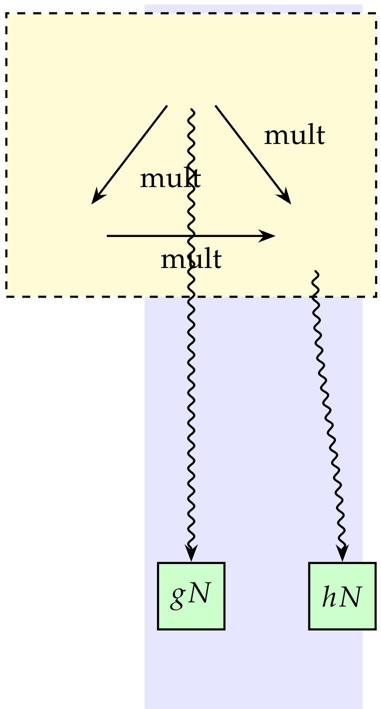


Visualization of Quotient Sets

This diagram represents the partition of a set into quotient sets (equivalence classes) determined by an equivalence relation. Each quotient set groups elements that are equivalent under the relation, as shown by the red arrows.

The quotient set $S/\sim$ is formed by the union of all distinct classes, each represented as a green ellipse.

Arrows within classes signify the symmetry and reflexivity properties inherent to the equivalence relation. Elements not directly connected by arrows are understood to be distinct within the quotient structure.



Visualization of Quotient Sets

The diagram represents a group G partitioned into quotient sets by a normal subgroup N. Each quotient set element, such as gN and hN, represents the set of all products of g with elements of N. This highlights how the quotient set G/N simplifies the structure of G by consolidating elements related by N.