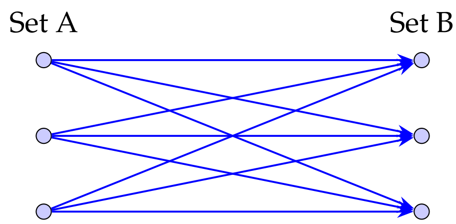
Equivalence Relations, Equivalence Classes, Partitions, and Quotient Sets

July 20, 2024

Cartesian Products / Set Products



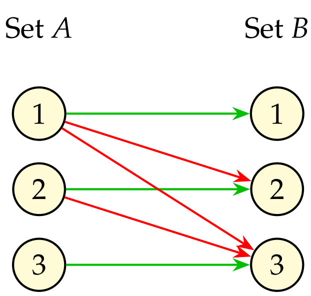
Example. If $A = \{1, 2\}$ and $B = \{x, y\}$, then:

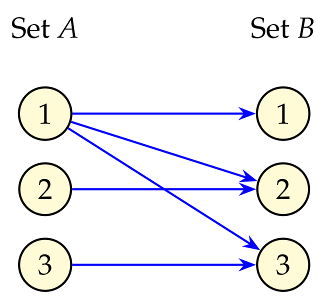
$$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$$

Definition. Let A and B are sets.

$$A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Relations





Green arrows: Equality (=)
Red arrows: Less than (<)
Blue arrows: Divisibility (|)

Definition. Let A and B are sets. A **relation** $\mathcal{R}$ from A to B is a *subset* of the Cartesian Product $A \times B$:

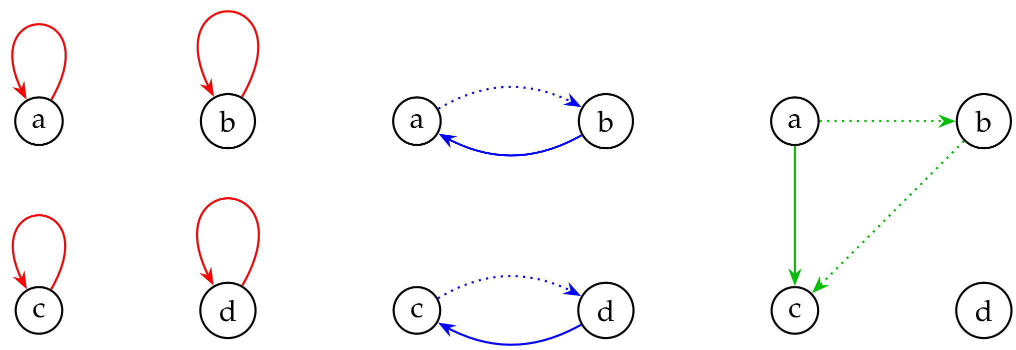
$$\mathcal{R} \subseteq A \times B$$

Notation.

$$(a, b) \in \mathcal{R} \subseteq A \times B \iff a \,\mathcal{R}\, b.$$

For example, $a = b \iff (a, b) \in =$

Equivalence Relations



Red arrows: Reflexivity
(each element is related to itself).
Blue arrows: Symmetry
(if a is related to b, then b is related to a).
Green arrows: Transitivity
(if a is related to b and b is related to c, then a is related to c).

Example. Equality on $\mathbb{Z}$:

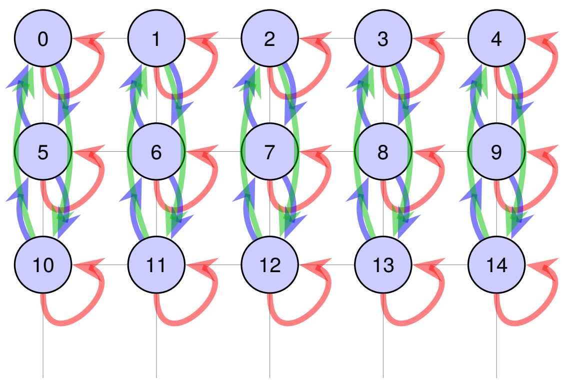
- $a \in \mathbb{Z} \implies a = a$;
- $a = b \implies b = a$;
- $a = b$ and $b = c \implies a = c$.

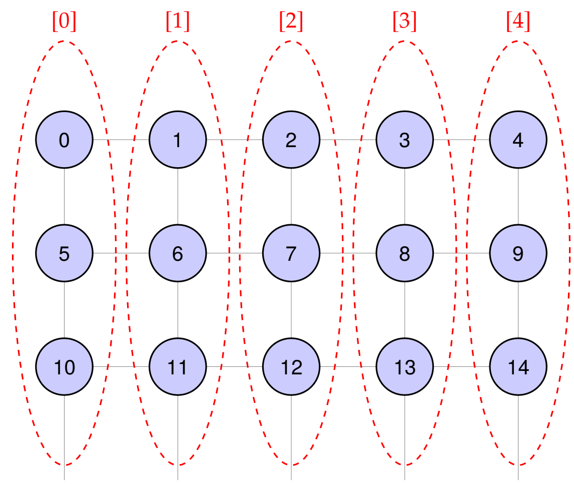
Definition. Given a set A, a relation $\mathcal{R} \subseteq A \times A$ is called an **equivalence relation** if it satisfies following properties:

- (Reflexivity) $a \in A \implies (a, a) \in \mathcal{R}$;
- (Symmetry) $(a, b) \in \mathcal{R} \implies (b, a) \in \mathcal{R}$;
- (Transitivity) $(a, b) \in \mathcal{R} \wedge (b, c) \in \mathcal{R} \implies (a, c) \in \mathcal{R}$.

Equivalence Classes

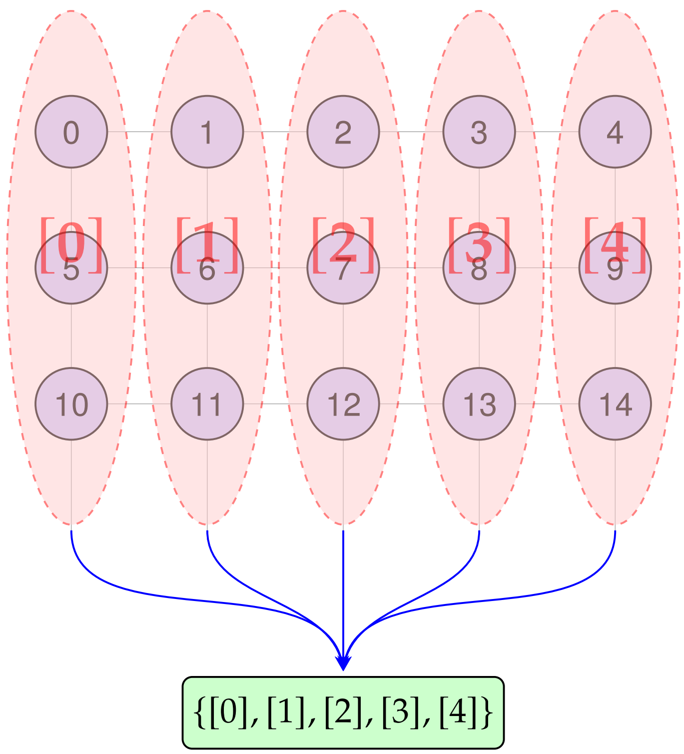
$\mathbb{Z}$ with modulo 5



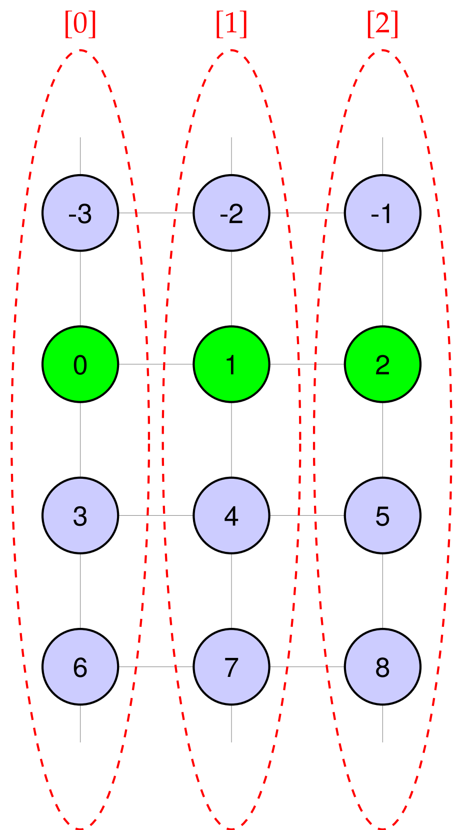


Quotient Sets

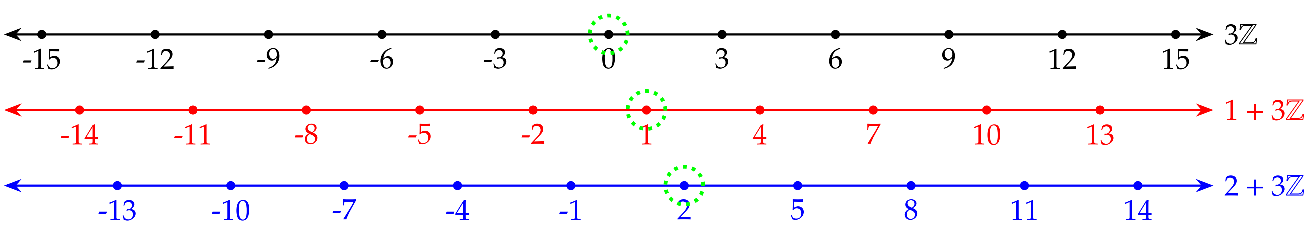
Set $\mathbb{Z}$



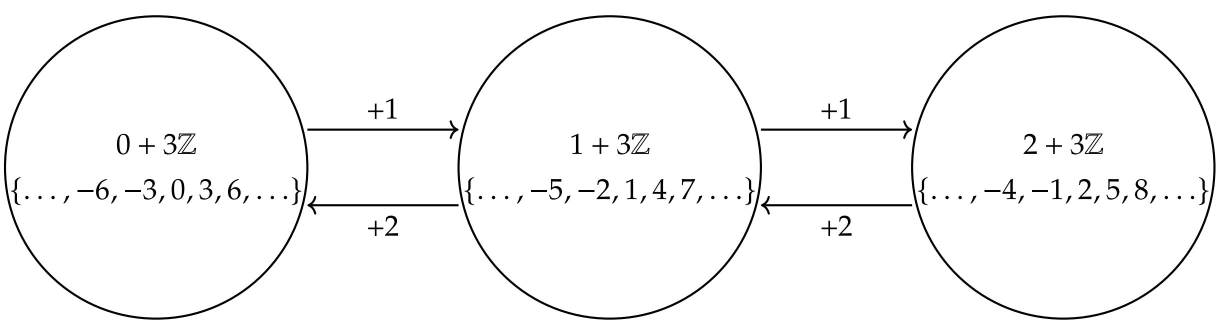
Quotient Set $\mathbb{Z}/5\mathbb{Z}$



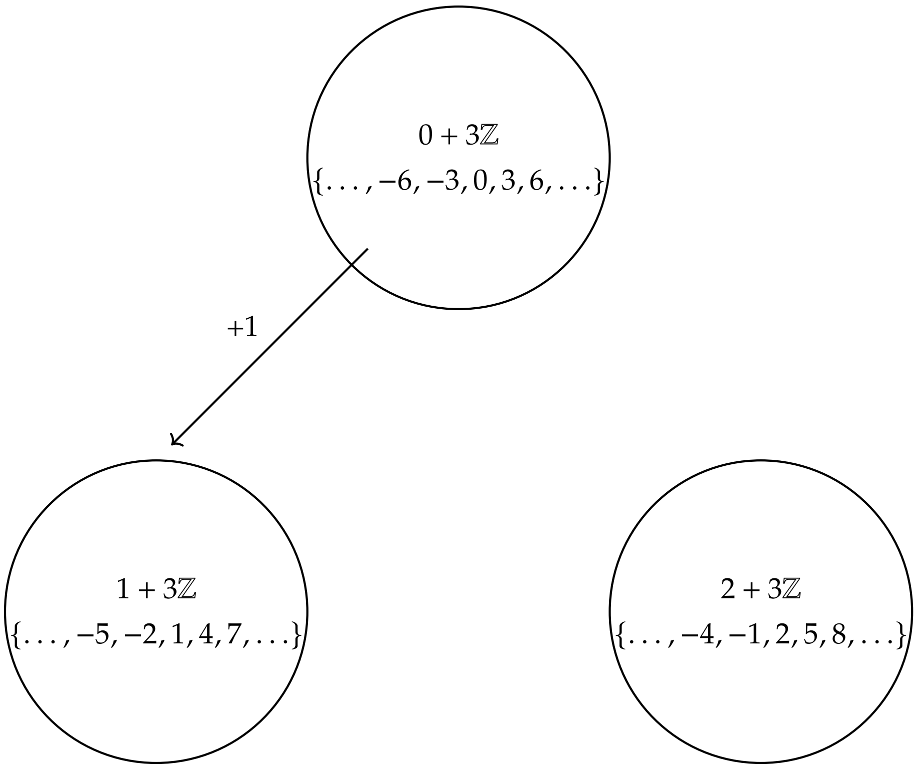
$$\mathbb{Z}/\!\sim \,= \big\{[0],[1],[2]\big\} \text{ with } a \sim b \overset{\text{def}}{\iff} a \equiv b \pmod 3$$



$$\mathbb{Z}/3\mathbb{Z} = \{0+3\mathbb{Z}, 1+3\mathbb{Z}, 2+3\mathbb{Z}\}$$



$$\mathbb{Z}/3\mathbb{Z} = \{0+3\mathbb{Z}, 1+3\mathbb{Z}, 2+3\mathbb{Z}\}$$

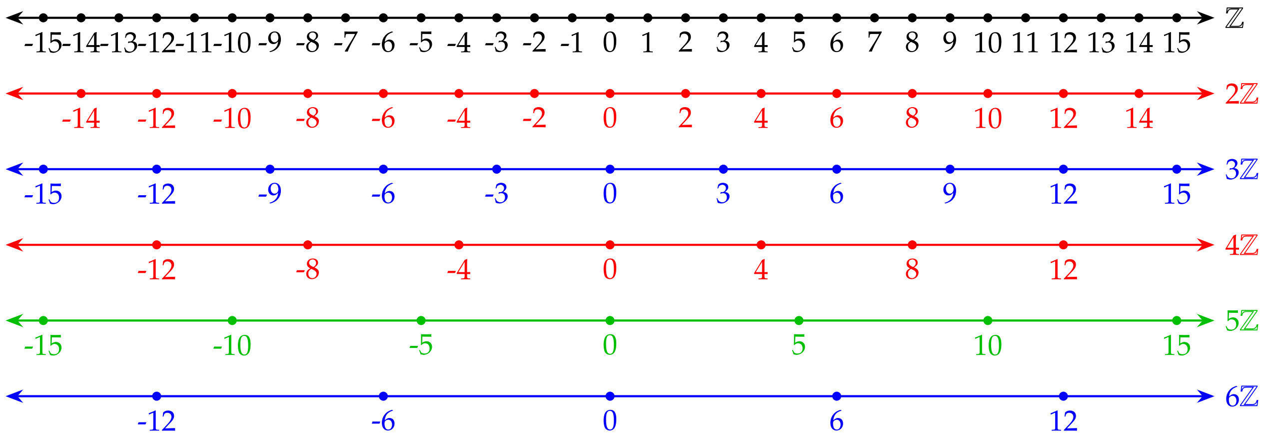


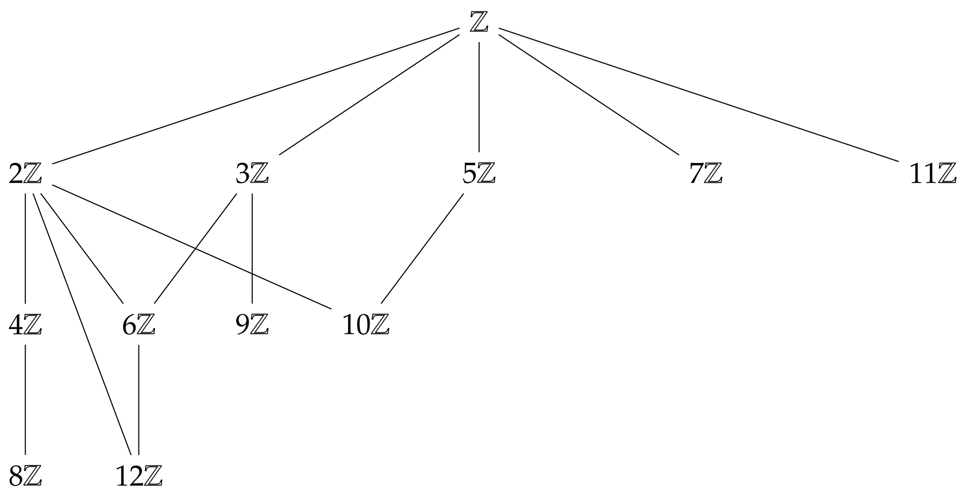
Groups

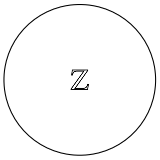
Subgroups

Cyclic Subgroups

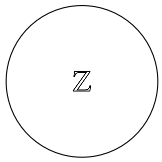
Normal Subgroups


-15 -14 -13 -12 -11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 ℤ
-14 -12 -10 -8 -6 -4 -2 0 2 4 6 8 10 12 14 2ℤ
-15 -12 -9 -6 -3 0 3 6 9 12 15 3ℤ
-12 -8 -4 0 4 8 12 4ℤ
-15 -10 -5 0 5 10 15 5ℤ
-12 -6 0 6 12 6ℤ


ℤ
2ℤ 3ℤ 5ℤ 7ℤ 11ℤ
4ℤ 6ℤ 9ℤ 10ℤ
8ℤ 12ℤ

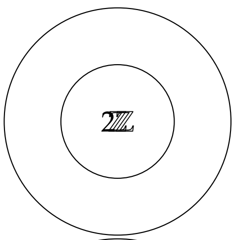


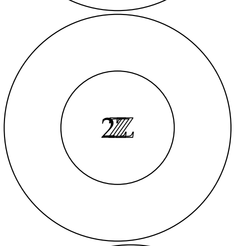
Operation: $+$
Identity: 0
Inverses: $\forall a \in \mathbb{Z}, -a \in \mathbb{Z}$

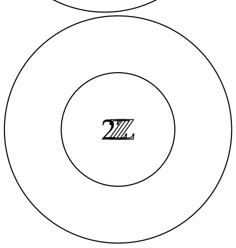


Addition: $+$
Additive Identity: 0
Additive Inverses: $\forall a \in \mathbb{Z}, -a \in \mathbb{Z}$

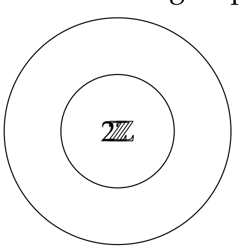
Multiplication: $\cdot$
Multiplicative Identity: 1



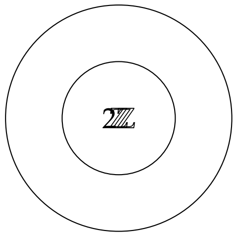




Normal Subgroup

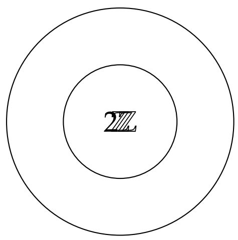


Ideal



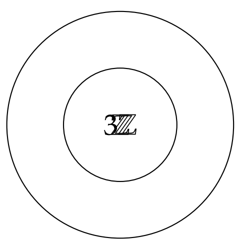
Quotient Group $\mathbb{Z}/2\mathbb{Z}$

Cosets: $\{0, 1\}$ modulo 2

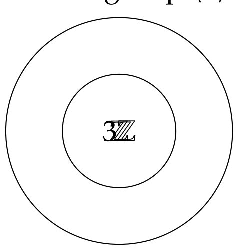


Quotient Ring $\mathbb{Z}/2\mathbb{Z}$

Elements: $\{0, 1\}$ modulo 2



Cyclic Subgroup $\langle 3 \rangle = 3\mathbb{Z}$



Principal Ideal $(3) = 3\mathbb{Z}$