# Commutative Rings

August 15, 2024

## The Zero Set of Simultaneous Equations Forms a Topology

### Definitions and Preliminaries

- **Topological Space**: A set $X$ with a collection $\mathcal{T} \subseteq 2^X$ is a topological space if $\mathcal{T}$ satisfies the following:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$ (inclusion of the empty set and the entire space).
2. $\mathcal{T}$ is closed under arbitrary unions, i.e., if $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.
3. $\mathcal{T}$ is closed under finite intersections, i.e., if $U_1, U_2, \ldots, U_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$.

- $\mathbb{C}[x] = \left\{\sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{C}, n \in \mathbb{Z}_{\geq 0}\right\}$
- $\mathbb{C} = \{\alpha : \alpha \in \mathbb{C}\}$
- Note that $\mathbb{C} \subseteq \mathbb{C}[x]$ since $\mathbb{C} = \{\alpha : \alpha \in \mathbb{C}\} = \left\{0x^n + 0x^{n-1} + \cdots + \alpha x^0 : \alpha \in \mathbb{C}\right\}$
- Consider
$$\begin{array}{rccc} \phi_\alpha & : & \mathbb{C}[x] & \longrightarrow & \mathbb{C} \\ & & f & \longmapsto & f(\alpha) \end{array}$$
  and
- Consider $f(x) \in \mathbb{C}[x]$ s.t.
$$\begin{array}{rccc} f & : & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & & \alpha & \longmapsto & f(\alpha) \end{array}$$
- By the first isomorphism theorem, we have $\mathbb{C}[x]/\langle x - \alpha\rangle \simeq \mathbb{C}$
- $\langle x - \alpha\rangle = \left\{(x-\alpha)f(x) : f(x) \in \mathbb{C}[x]\right\} \subseteq \mathbb{C}[x]$
- $\mathbb{C}[x]/\langle x - \alpha\rangle$;
  - $p(x) = (x-\alpha)q(x) + r(x) = (x-\alpha)q(x) + r$
  - $\mathbb{C}[x]/\langle x - \alpha\rangle = \left\{r + \langle x - \alpha\rangle : r \in \mathbb{C}\right\}$

| **Mapping** | $\begin{array}{rcccl} \psi_p & : & \mathbb{Z} & \longrightarrow & \mathbb{Z}_p \\ & & n & \longmapsto & n \bmod p = \psi_p(n) \end{array}$ | $\begin{array}{rcccl} \phi_a & : & \mathbb{C}[x] & \longrightarrow & \mathbb{C} \\ & & f(x) & \longmapsto & f(a) = \phi_a(f(x)) \end{array}$ |
|---|---|---|
| Additive Homo. | $\psi_p(a+b) := (a+b) \bmod p$ | $\phi_a(f+g) := f(a) + g(a)$ |
| Multiplicative Homo. | $\psi_p(ab) := (ab) \bmod p$ | $\phi_a(fg) := f(a)g(a)$ |
| Kernel | $\ker(\psi_p) = p\mathbb{Z}$ | $\ker(\phi_a) = (x-a)\mathbb{C}[x]$ |
| Image | $\mathbb{Z}_p$ | $\mathbb{C}$ |
| Ideal | $p\mathbb{Z} = \langle p\rangle$ | $(x-a)\mathbb{C}[x] = \langle x - a\rangle$ |
| Prime Ideal | $\langle p\rangle$ is prime | $\langle x - a\rangle$ is prime |
| Maximal Ideal | $\langle p\rangle$ is maximal | $\langle x - a\rangle$ is maximal |
| Isomorphism | $\mathbb{Z}_p \simeq \mathbb{Z}/p\mathbb{Z}$ | $\mathbb{C} \simeq \mathbb{C}[x]/\langle x - a\rangle$ |
| Element of Domain | $\begin{array}{rcccl} n & : & arg2 & \longrightarrow & arg3 \\ & & arg4 & \longmapsto & arg5 \end{array}$ | $\begin{array}{rcccl} f & : & \left\{\langle x-\alpha\rangle : \alpha \in \mathbb{C}\right\} & \longrightarrow & \coprod_\alpha \mathbb{C}[x]/\langle x-\alpha\rangle \\ & & \langle x - \alpha\rangle & \longmapsto & arg5 \end{array}$ |

| | | 2 | 3 | 5 | 7 |
|---|---|---|---|---|---|
| $\varphi(2\mathbb{Z})$ | $\longrightarrow 2\mathbb{Z}$ | 1 | 0 | 0 | 0 |
| $\varphi(3\mathbb{Z})$ | $\longrightarrow 3\mathbb{Z}$ | 0 | 1 | 0 | 0 |
| $\varphi(5\mathbb{Z})$ | $\longrightarrow 5\mathbb{Z}$ | 0 | 0 | 1 | 0 |

Prime numbers $p$