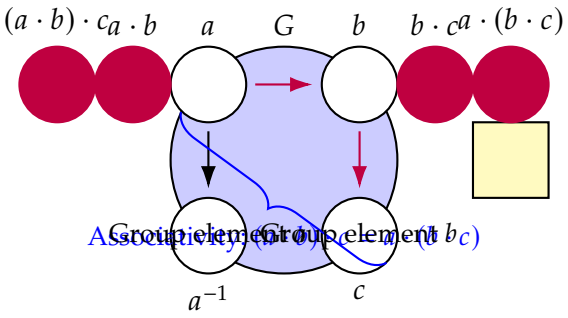
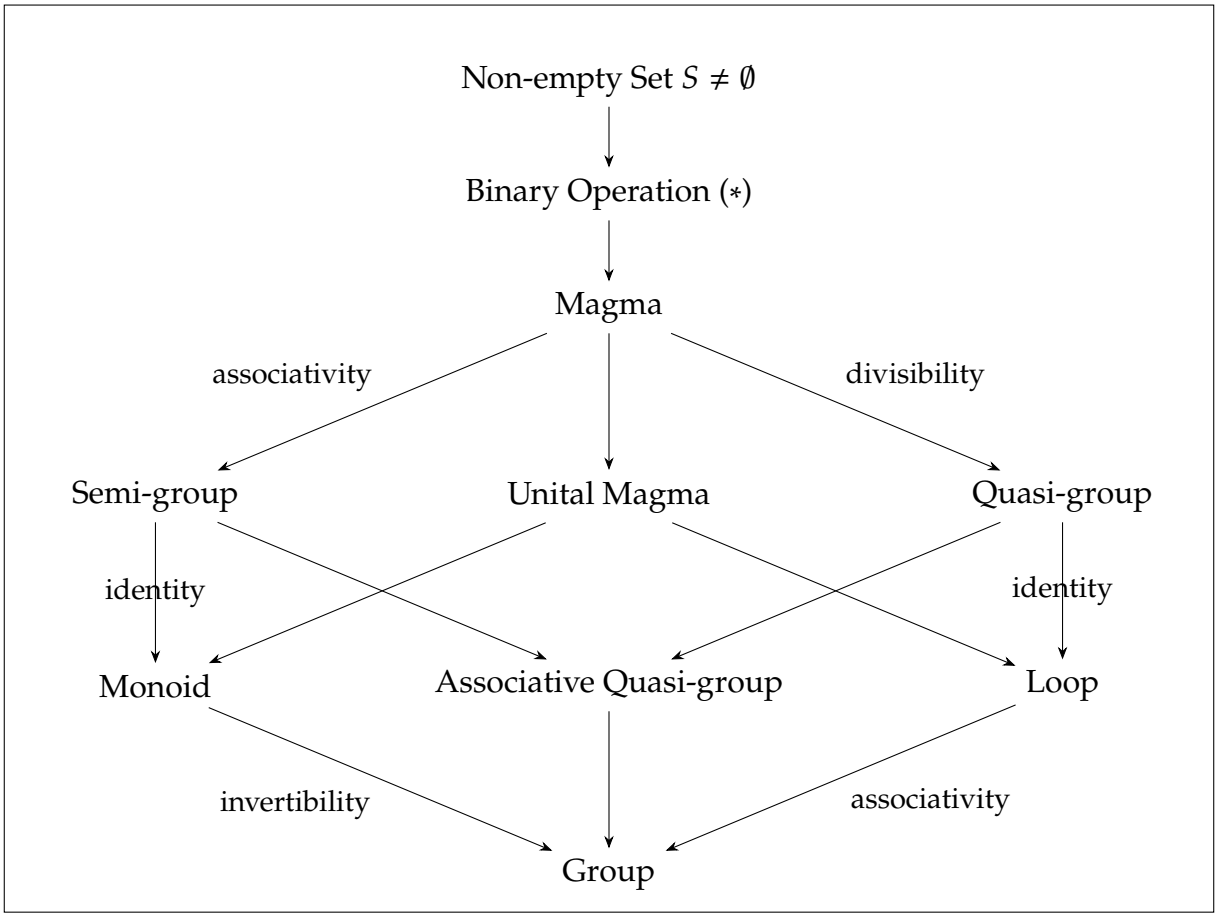
Groups

July 19, 2024



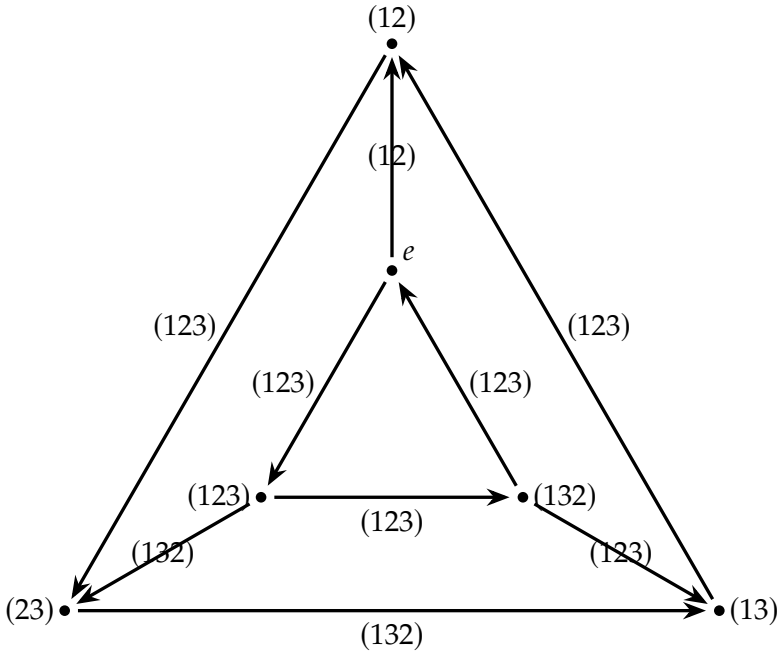


Name	Group (Set)	Operation	Form of Element	Identity	Inverse	Abelian
Integers	$\mathbb{Z}$	Addition	k	0	$-k$	✔
Integers modulo n	$\mathbb{Z}_n$	Addition mod n	k	0	$n-k$	✔
Rational Numbers	$\mathbb{Q}$	Multiplication	m/n	1	n/m	✔
Real Numbers	$\mathbb{R}^*$	Multiplication	x	1	$1/x$	✔
Complex Numbers	$\mathbb{C}^*$	Multiplication	$a+bi$	1	$\frac{1}{a^2+b^2}a - \frac{1}{a^2+b^2}bi$	✔
Vector Space	$\mathbb{R}^n$	Componentwise addition	$(a_1, a_2, \ldots, a_n)$	$(0, 0, \ldots, 0)$	$(-a_1, -a_2, \ldots, -a_n)$	✔
General Linear Group	$GL(2, F)$	Matrix multiplication	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$	✗
Special Linear Group	$SL(2, F)$	Matrix multiplication	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc = 1$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$	✗
Symmetric Group	S_n	Composition of permutations	Permutations of n element: $\sigma \in S_n$	Identity permutation e	Inverse permutation σ^{-1}	✗
Dihedral Group	D_n	Composition of symmetries	Rotations r^k and reflections sr^k, $k \in \{0, 1, \ldots, n-1\}$	Identity symmetry e	$r^{-k} = r^{n-k}$, $(sr^k)^{-1} = sr^{-k}$	✗
Quaternion Group	Q_8	Quaternion multiplication	Quaternions: $\pm 1, \pm i, \pm j, \pm k$	1	$-a$ for $a \in \{i, j, k\}$, $-1 = 1^{-1}, -i = i^{-1}, -j = j^{-1}, -k = k^{-1}$	✗

Symmetric Groups

	e	(12)	(23)	(13)	(123)	(132)
e	e	(12)	(23)	(13)	(123)	(132)
(12)	(12)	e	(123)	(132)	(23)	(13)
(23)	(23)	(132)	e	(123)	(13)	(12)
(13)	(13)	(123)	(132)	e	(12)	(23)
(123)	(123)	(13)	(12)	(23)	(132)	e
(132)	(132)	(23)	(13)	(12)	e	(123)

Table 1: Symmetric Group S_3



Ring (Set)	Addition Operation	Multiplication Operation	Additive Identity	Multiplicative Identity	Form of Element	Commutative
Integers $\mathbb{Z}$	Standard addition +	Standard multiplication ·	0	1	Integers	Yes
Real Numbers $\mathbb{R}$	Standard addition +	Standard multiplication ·	0	1	Real numbers	Yes
Complex Numbers $\mathbb{C}$	Standard addition +	Standard multiplication ·	0	1	$a + bi$ where $a, b \in \mathbb{R}$	Yes
Polynomials with Real Coefficients $\mathbb{R}[x]$	Polynomial addition	Polynomial multiplication	0 (zero polynomial)	1 (constant polynomial)	$a_n x^n + \cdots + a_1 x + a_0$	Yes
Matrices $M_n(\mathbb{R})$	Matrix addition	Matrix multiplication	Zero matrix	Identity matrix	$n \times n$ matrices	No
Integers Modulo n $\mathbb{Z}/n\mathbb{Z}$	Addition modulo n	Multiplication modulo n	0	1	$\{0, 1, \ldots, n-1\}$	Yes

Group (Set)	Operation	Identity	Form of Element	Normal Subgroups	Abelian
Symmetric Group S_3	Composition of permutations	Identity permutation e	Permutations of 3 elements	$\{e, (123), (132)\}$, $\{e, (12), (13), (23), (123), (132)\}$	No
Dihedral Group D_4	Composition of symmetries	Identity symmetry e	Rotations r^k and reflections sr^k, $k \in \{0, 1, 2, 3\}$	$\{e, r, r^2, r^3\}$, $\{e, r^2, s, sr^2\}$	No
Quaternion Group Q_8	Quaternion multiplication	1	Quaternions $\pm 1, \pm i, \pm j, \pm k$	$\{1, -1\}$, $\{1, -1, i, -i\}$, $\{1, -1, j, -j\}$, $\{1, -1, k, -k\}$	No
Integers Modulo n $\mathbb{Z}/n\mathbb{Z}$	Addition modulo n	0	$\{0, 1, \ldots, n-1\}$	All subgroups are normal	Yes

Ring (Set)	Addition Operation	Multiplication Operation	Additive Identity	Multiplicative Identity	Form of Element
Matrices $M_n(\mathbb{R})$	Matrix addition	Matrix multiplication	Zero matrix	Identity matrix	$n \times n$ matrices
Quaternions $\mathbb{H}$	Quaternion addition	Quaternion multiplication	0	1	$a + bi + cj + dk$ where $a, b, c, d \in \mathbb{R}$
Differential Operators	Operator addition	Operator composition	Zero operator	Identity operator	$\sum a_i \frac{d^i}{dx^i}$
Group Rings $\mathbb{R}[G]$	Group ring addition	Group ring multiplication	Zero element	Identity element	$\sum a_g g$ where $a_g \in \mathbb{R}$ and $g \in G$
Endomorphism Rings End(V)	Function addition	Function composition	Zero map	Identity map	Linear transformations on vector space V
Octonions $\mathbb{O}$	Octonion addition	Octonion multiplication	0	1	$a + be_1 + ce_2 + de_3 + ee_4 + fe_5 + ge_6 + he_7$ where $a, b, c, d, e, f, g, h \in \mathbb{R}$

Type	Ideal	Principal Ideal	Prime Ideal	Maximal Ideal
Definition	A subset closed under addition and multiplication by any ring element	An ideal generated by a single element	An ideal where if $ab \in I$, then $a \in I$ or $b \in I$	An ideal such that there are no larger ideals except the ring itself
Is an Ideal	O	O	O	O
Can be Principal	O	O	O	O
Is Prime	X	X	O	O
Is Maximal	X	X	X	O

Ring (Set)	Principal Ideals	Prime Ideals	Maximal Ideals
$\mathbb{Z}$	$(2), (3), (5)$	$(2), (3), (5)$	$(2), (3), (5)$
$\mathbb{R}[x]$	$(x), (x-1), (x^2+1)$	(x)	$(x-1)$
$\mathbb{C}[x]$	$(x), (x-i), (x+i)$	$(x-i), (x+i)$	$(x-i), (x+i)$
$\mathbb{Z}/6\mathbb{Z}$	$(2), (3)$	None	$(2), (3)$
$\mathbb{Z}[i]$	$(1+i), (2)$	$(1+i)$	$(1+i)$
$\mathbb{Z}/p\mathbb{Z}$ where p is prime	$(0), (1)$	(0)	(0)
$\mathbb{R}$	(0)	None	(0)
$\mathbb{C}$	(0)	None	(0)
$\mathbb{Z}[x]$	$(2), (3), (x)$	$(x), (2)$	None
$M_n(\mathbb{R})$	(E_{11})	None	None
$\mathbb{H}$ (Quaternions)	$(1+i)$	None	None

Ring (Set)	Principal Ideals	Prime Ideals	Maximal Ideals
$\mathbb{Z}$	$(2) = \{2k \mid k \in \mathbb{Z}\}$ $(3) = \{3k \mid k \in \mathbb{Z}\}$ $(5) = \{5k \mid k \in \mathbb{Z}\}$	$(2) = \{2k \mid k \in \mathbb{Z}\}$ $(3) = \{3k \mid k \in \mathbb{Z}\}$ $(5) = \{5k \mid k \in \mathbb{Z}\}$	$(2) = \{2k \mid k \in \mathbb{Z}\}$ $(3) = \{3k \mid k \in \mathbb{Z}\}$ $(5) = \{5k \mid k \in \mathbb{Z}\}$
$\mathbb{R}[x]$	$(x) = \{xf(x) \mid f(x) \in \mathbb{R}[x]\}$ $(x-1) = \{(x-1)f(x) \mid f(x) \in \mathbb{R}[x]\}$ $(x^2+1) = \{(x^2+1)f(x) \mid f(x) \in \mathbb{R}[x]\}$	$(x) = \{xf(x) \mid f(x) \in \mathbb{R}[x]\}$	$(x-1) = \{(x-1)f(x) \mid f(x) \in \mathbb{R}[x]\}$
$\mathbb{C}[x]$	$(x) = \{xf(x) \mid f(x) \in \mathbb{C}[x]\}$ $(x-i) = \{(x-i)f(x) \mid f(x) \in \mathbb{C}[x]\}$ $(x+i) = \{(x+i)f(x) \mid f(x) \in \mathbb{C}[x]\}$	$(x-i) = \{(x-i)f(x) \mid f(x) \in \mathbb{C}[x]\}$ $(x+i) = \{(x+i)f(x) \mid f(x) \in \mathbb{C}[x]\}$	$(x-i) = \{(x-i)f(x) \mid f(x) \in \mathbb{C}[x]\}$ $(x+i) = \{(x+i)f(x) \mid f(x) \in \mathbb{C}[x]\}$
$\mathbb{Z}/6\mathbb{Z}$	$(2) = \{0, 2, 4\}$ $(3) = \{0, 3\}$	None	$(2) = \{0, 2, 4\}$ $(3) = \{0, 3\}$
$\mathbb{Z}[i]$	$(1+i) = \{(1+i)z \mid z \in \mathbb{Z}[i]\}$ $(2) = \{2z \mid z \in \mathbb{Z}[i]\}$	$(1+i) = \{(1+i)z \mid z \in \mathbb{Z}[i]\}$	$(1+i) = \{(1+i)z \mid z \in \mathbb{Z}[i]\}$
$\mathbb{Z}/p\mathbb{Z}$ where p is prime	$(0) = \{0\}$ $(1) = \mathbb{Z}/p\mathbb{Z}$	$(0) = \{0\}$	$(0) = \{0\}$
$\mathbb{R}$	$(0) = \{0\}$	None	$(0) = \{0\}$
$\mathbb{C}$	$(0) = \{0\}$	None	$(0) = \{0\}$
$\mathbb{Z}[x]$	$(2) = \{2f(x) \mid f(x) \in \mathbb{Z}[x]\}$ $(3) = \{3f(x) \mid f(x) \in \mathbb{Z}[x]\}$ $(x) = \{xf(x) \mid f(x) \in \mathbb{Z}[x]\}$	$(2) = \{2f(x) \mid f(x) \in \mathbb{Z}[x]\}$ $(x) = \{xf(x) \mid f(x) \in \mathbb{Z}[x]\}$	None
$M_n(\mathbb{R})$	$(E_{11}) = \{AE_{11}B \mid A, B \in M_n(\mathbb{R})\}$	None	None
$\mathbb{H}$ (Quaternions)	$(1+i) = \{(1+i)q \mid q \in \mathbb{H}\}$	None	None

Ring (Set)	Principal Ideals	Prime Ideals	Maximal Ideals
$\mathbb{Z}$	$(2) = \{2k \mid k \in \mathbb{Z}\}$ $(3) = \{3k \mid k \in \mathbb{Z}\}$ $(5) = \{5k \mid k \in \mathbb{Z}\}$	$(2) = \{2k \mid k \in \mathbb{Z}\}$ $(3) = \{3k \mid k \in \mathbb{Z}\}$ $(5) = \{5k \mid k \in \mathbb{Z}\}$	$(2) = \{2k \mid k \in \mathbb{Z}\}$ $(3) = \{3k \mid k \in \mathbb{Z}\}$ $(5) = \{5k \mid k \in \mathbb{Z}\}$
$\mathbb{R}[x]$	$(x) = \{xf(x) \mid f(x) \in \mathbb{R}[x]\}$ $(x-1) = \{(x-1)f(x) \mid f(x) \in \mathbb{R}[x]\}$ $(x^2+1) = \{(x^2+1)f(x) \mid f(x) \in \mathbb{R}[x]\}$	(x)	$(x-1)$
$\mathbb{C}[x]$	$(x) = \{xf(x) \mid f(x) \in \mathbb{C}[x]\}$ $(x-i) = \{(x-i)f(x) \mid f(x) \in \mathbb{C}[x]\}$ $(x+i) = \{(x+i)f(x) \mid f(x) \in \mathbb{C}[x]\}$	$(x-i)$ $(x+i)$	$(x-i)$ $(x+i)$
$\mathbb{Z}/6\mathbb{Z}$	$(2) = \{0, 2, 4\}$ $(3) = \{0, 3\}$	None	$(2) = \{0, 2, 4\}$ $(3) = \{0, 3\}$
$\mathbb{Z}[i]$	$(1+i) = \{(1+i)z \mid z \in \mathbb{Z}[i]\}$ $(2) = \{2z \mid z \in \mathbb{Z}[i]\}$	$(1+i)$	$(1+i)$
$\mathbb{Z}/p\mathbb{Z}$ where p is prime	$(0) = \{0\}$ $(1) = \mathbb{Z}/p\mathbb{Z}$	(0)	(0)
$\mathbb{R}$	$(0) = \{0\}$	None	(0)
$\mathbb{C}$	$(0) = \{0\}$	None	(0)
$\mathbb{Z}[x]$	$(2) = \{2f(x) \mid f(x) \in \mathbb{Z}[x]\}$ $(3) = \{3f(x) \mid f(x) \in \mathbb{Z}[x]\}$ $(x) = \{xf(x) \mid f(x) \in \mathbb{Z}[x]\}$	(2) (x)	None
$M_n(\mathbb{R})$	$(E_{11}) = \{AE_{11}B \mid A, B \in M_n(\mathbb{R})\}$	None	None
$\mathbb{H}$ (Quaternions)	$(1+i) = \{(1+i)q \mid q \in \mathbb{H}\}$	None	None

Ring (Set)	Examples of Ideals	Examples of Principal Ideals	Examples of Prime Ideals	Examples of Maximal Ideals
$\mathbb{Z}$	$(0), (2), (3)$ $(4), (6)$ (0)	$(2) = \{2k \mid k \in \mathbb{Z}\}$ $(3) = \{3k \mid k \in \mathbb{Z}\}$ $(5) = \{5k \mid k \in \mathbb{Z}\}$	$(2) = \{2k \mid k \in \mathbb{Z}\}$ $(3) = \{3k \mid k \in \mathbb{Z}\}$ $(5) = \{5k \mid k \in \mathbb{Z}\}$	$(2) = \{2k \mid k \in \mathbb{Z}\}$ $(3) = \{3k \mid k \in \mathbb{Z}\}$ $(5) = \{5k \mid k \in \mathbb{Z}\}$
$\mathbb{R}[x]$	$(0), (x), (x-1)$ (x^2+1)	$(x) = \{xf(x) \mid f(x) \in \mathbb{R}[x]\}$ $(x-1) = \{(x-1)f(x) \mid f(x) \in \mathbb{R}[x]\}$ $(x^2+1) = \{(x^2+1)f(x) \mid f(x) \in \mathbb{R}[x]\}$	(x)	$(x-1)$
$\mathbb{C}[x]$	$(0), (x), (x-i)$ $(x+i)$	$(x) = \{xf(x) \mid f(x) \in \mathbb{C}[x]\}$ $(x-i) = \{(x-i)f(x) \mid f(x) \in \mathbb{C}[x]\}$ $(x+i) = \{(x+i)f(x) \mid f(x) \in \mathbb{C}[x]\}$	$(x-i)$ $(x+i)$	$(x-i)$ $(x+i)$
$\mathbb{Z}/6\mathbb{Z}$	$(0), (2), (3)$	$(2) = \{0, 2, 4\}$ $(3) = \{0, 3\}$	None	$(2) = \{0, 2, 4\}$ $(3) = \{0, 3\}$
$\mathbb{Z}[i]$	$(0), (1+i), (2)$	$(1+i) = \{(1+i)z \mid z \in \mathbb{Z}[i]\}$ $(2) = \{2z \mid z \in \mathbb{Z}[i]\}$	$(1+i)$	$(1+i)$
$\mathbb{Z}/p\mathbb{Z}$ where p is prime	$(0), (1)$	$(0) = \{0\}$ $(1) = \mathbb{Z}/p\mathbb{Z}$	(0)	(0)
$\mathbb{R}$	(0)	$(0) = \{0\}$	None	(0)
$\mathbb{C}$	(0)	$(0) = \{0\}$	None	(0)
$\mathbb{Z}[x]$	$(0), (2), (3)$	$(2) = \{2f(x) \mid f(x) \in \mathbb{Z}[x]\}$ $(3) = \{3f(x) \mid f(x) \in \mathbb{Z}[x]\}$ $(x) = \{xf(x) \mid f(x) \in \mathbb{Z}[x]\}$	(2) (x)	None
$M_n(\mathbb{R})$	$(0), (E_{11})$	$(E_{11}) = \{AE_{11}B \mid A, B \in M_n(\mathbb{R})\}$	None	None
$\mathbb{H}$ (Quaternions)	$(0), (1+i)$	$(1+i) = \{(1+i)q \mid q \in \mathbb{H}\}$	None	None