Rings

August 12, 2024

$$\begin{tikzcd}
 & & \text{Non-empty Set } S \neq \emptyset \arrow[d] & & \\
 & & \text{Binary Operation } (*) \arrow[d] & & \\
 & & \text{Magma} \arrow[dll, "\text{associativity}"'] \arrow[d] \arrow[drr, "\text{divisibility}"] & & \\
\text{Semi-group} \arrow[d, "\text{identity}"] \arrow[drr] & & \text{Unital Magma} \arrow[dll] \arrow[drr] & & \text{Quasi-group} \arrow[dll] \arrow[d, "\text{identity}"] \\
\text{Monoid} \arrow[drr, "\text{invertibility}"'] & & \text{Associative Quasi-group} \arrow[d] & & \text{Loop} \arrow[dll, "\text{associativity}"] \\
 & & \text{Group} & &
\end{tikzcd}$$

Product of Ideals

Product of Ideals (without finite sums)

Given two ideals I and J in a ring R, the product IJ without finite sums is defined as:

$$IJ \triangleq \{ab : a \in I, b \in J\}.$$

[Example in $\mathbb{Z}$] Consider $I = 2\mathbb{Z}$ and $J = 3\mathbb{Z}$ in $\mathbb{Z}$:

$$2\mathbb{Z} = \{2n : n \in \mathbb{Z}\},$$
$$3\mathbb{Z} = \{3m : m \in \mathbb{Z}\}.$$

By definition,

$$IJ = \{2n \cdot 3m : n, m \in \mathbb{Z}\} = \{6nm : n, m \in \mathbb{Z}\}.$$

Consider two general elements $6n_1m_1$ and $6n_2m_2$ in IJ. Their sum is

$$6n_1m_2 + 6n_2m_2 = 6(n_1m_1 + n_2m_2) = 6\sum_{k=1}^{2} n_km_k \stackrel{?}{\in} IJ = \{6nm : n, m \in \mathbb{Z}\}.$$

Without finite sums, we only have products $6ab$.

Product of Ideals (with finite sums)

$$IJ \triangleq \left\{\sum_{k=1}^{t} a_kb_k : a_k \in I, b_k \in J, t \in \mathbb{N}\right\}.$$

Consider $x = \sum_{k=1}^{s} 6a_kb_k$ and $y = \sum_{k=1}^{t} 6a_kb_k$. Then

$$\begin{aligned} x + y = \sum_{k=1}^{s} 6a_kb_k + \sum_{l=1}^{t} 6a_lb_l &= 6\left(\sum_{k=1}^{s} a_kb_k + \sum_{k=1}^{t} a_kb_k\right) \\ &= 6\sum_{k=1}^{s+t} a_kb_k \end{aligned}$$

[Example in $\mathbb{C}[x]$]

Product of Ideals (without finite sums)

Consider $I = (x)$ and $J = (x^2)$ in $\mathbb{C}[x]$:

$$(x) = \left\{x \cdot p(x) : p(x) \in \mathbb{C}[x]\right\},$$
$$(x^2) = \left\{x^2 \cdot q(x) : q \in \mathbb{C}[x]\right\}.$$

By definition,

$$IJ = \left\{x \cdot p(x) \cdot x^2 \cdot q(x) : p(x), q(x) \in \mathbb{C}[x]\right\} = \left\{x^3 p(x) q(x) : p(x), q(x) \in \mathbb{C}[x]\right\}.$$

Consider two general elements $x^3 p_1(x) q_1(x)$ and $x^3 p_2(x) q_2(x)$ in IJ. Their sum is

$$x^3 p_1(x) q_1(x) + x^3 p_2(x) q_2(x) = x^3 (p_1(x) q_1(x) + p_2(x) q_2(x)) = x^3 \sum_{i=1}^{2} p_i(x) q_i(x).$$

Product of Ideals (with finite sums)

$$IJ \triangleq \left\{\sum_{k=1}^{t} a_k b_k : a_k \in I, b_k \in J, t \in \mathbb{N}\right\}.$$

Consider $x = \sum_{k=1}^{s} 6a_k b_k$ and $y = \sum_{k=1}^{t} 6a_k b_k$. Then

$$\begin{aligned} x + y = \sum_{k=1}^{s} 6a_k b_k + \sum_{l=1}^{t} 6a_l b_l &= 6\left(\sum_{k=1}^{s} a_k b_k + \sum_{k=1}^{t} a_k b_k\right) \\ &= 6 \sum_{k=1}^{s+t} a_k b_k \end{aligned}$$

[Example in $\mathbb{Z}[x]$] Consider $I = (2, x)$ and $J = (3, x^2)$ in $\mathbb{Z}[x]$:

$$(2, x) = \left\{2f(x) + xg(x) : f, g \in \mathbb{Z}[x]\right\},$$
$$(3, x^2) = \left\{3h(x) + x^2k(x) : h, k \in \mathbb{C}[x]\right\}.$$

Let

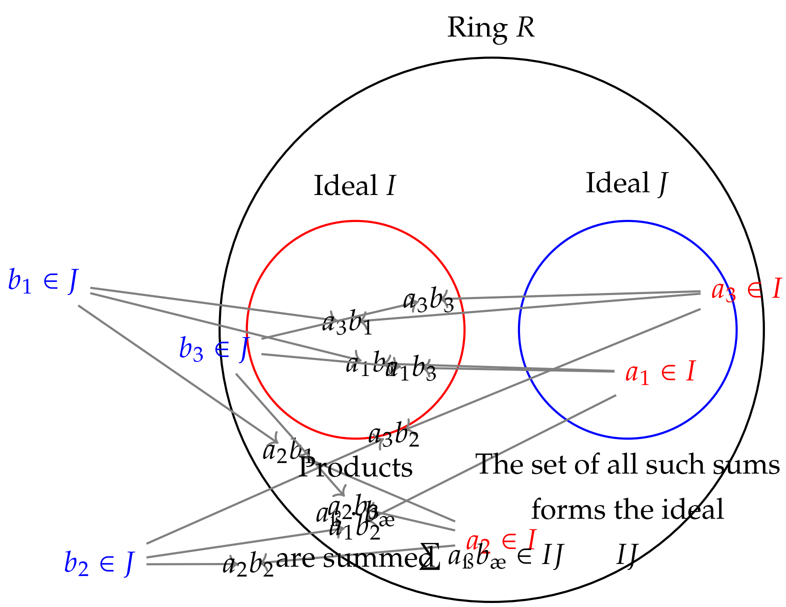
$$IJ \triangleq \left\{(2f + xg)(3h + x^2k) : f, g, h, k \in \mathbb{Z}[x]\right\}.$$

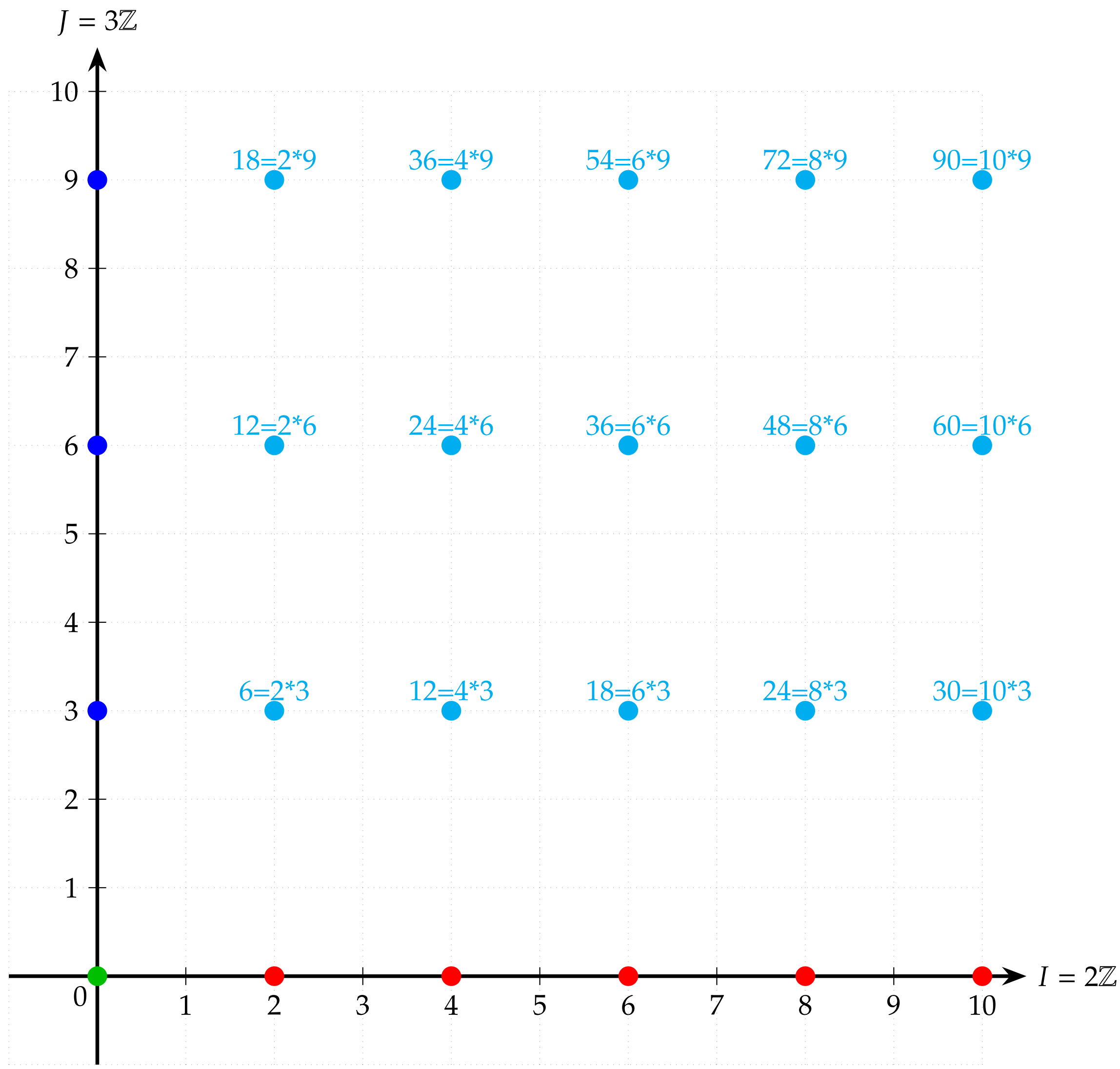
Consider

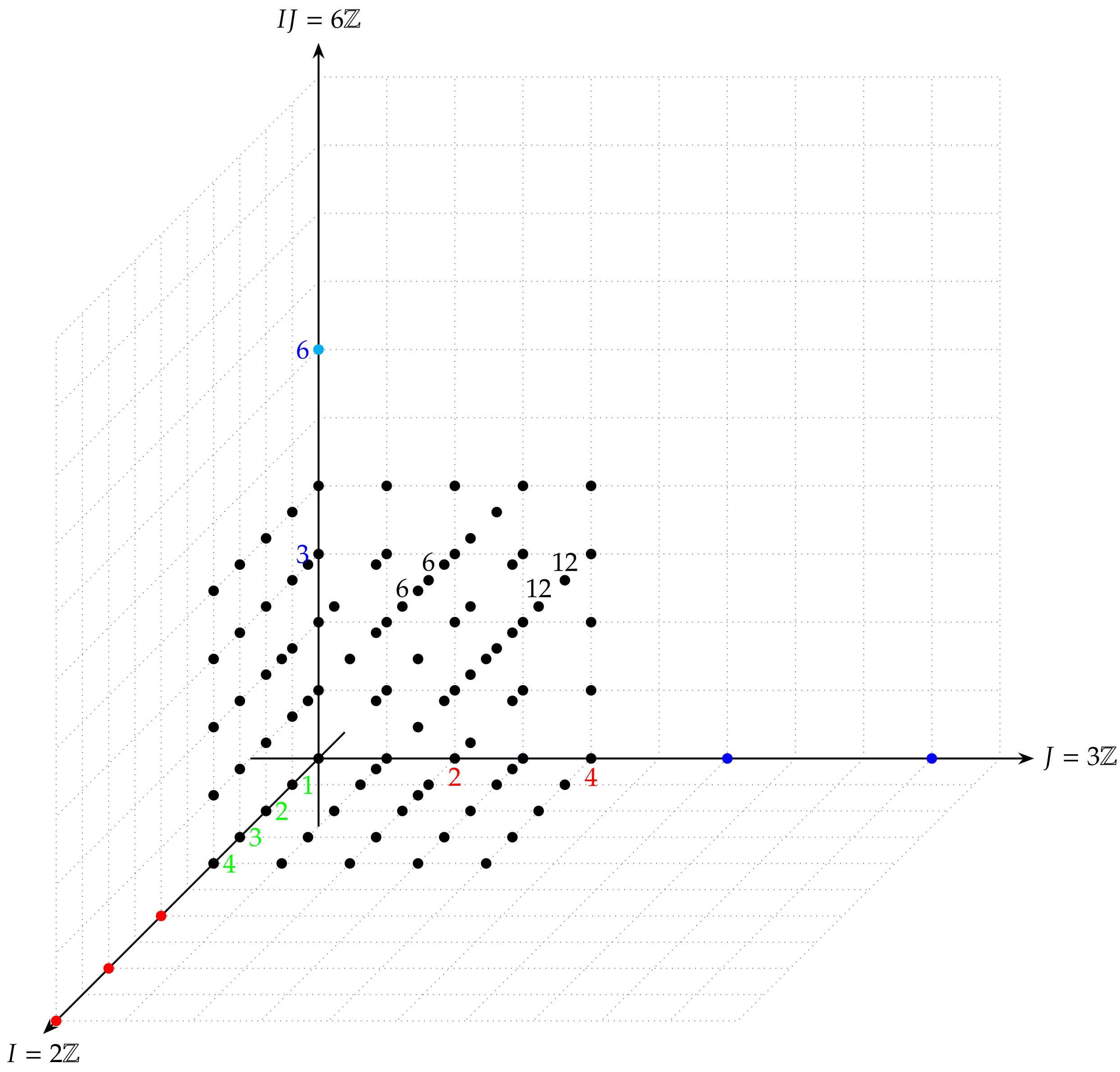
$$x_1 = (2 + x)(3 + x^2) \in IJ,$$
$$x_2 = (2x + x^3)(3 + x^2) \in IJ.$$

Then

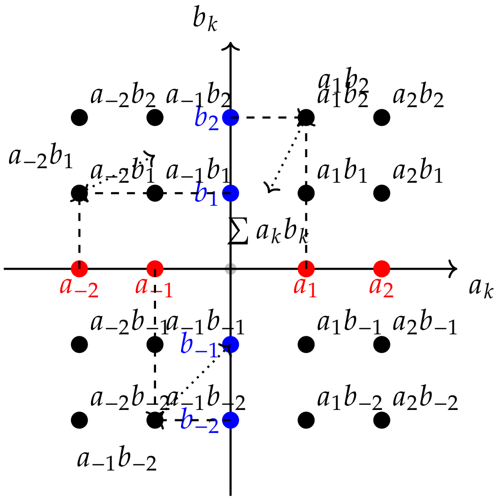
$$x_1 = (2 + x)(3 + x^2) = 6 + 2x^2 + 3x + x^3,$$
$$x_2 = 6x + 2x^3 + 3x^3 + x^5 = 6x + 5x^3 + x^5.$$



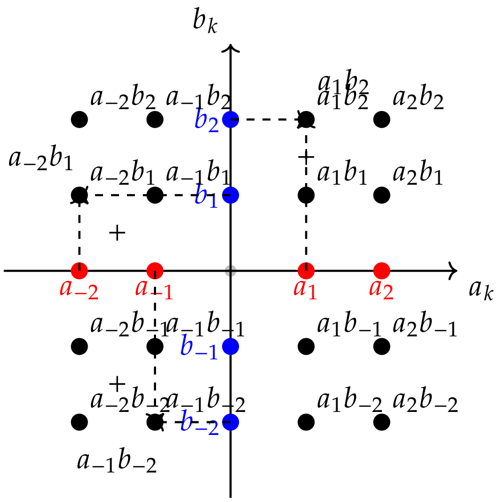

$J = 3\mathbb{Z}$
$I = 2\mathbb{Z}$
18=2*9
36=4*9
54=6*9
72=8*9
90=10*9
12=2*6
24=4*6
36=6*6
48=8*6
60=10*6
6=2*3
12=4*3
18=6*3
24=8*3
30=10*3
0
1
2
3
4
5
6
7
8
9
10



$IJ = 6\mathbb{Z}$
6
3
6
6
12
12
$J = 3\mathbb{Z}$
1
2
3
4
2
4
$I = 2\mathbb{Z}$

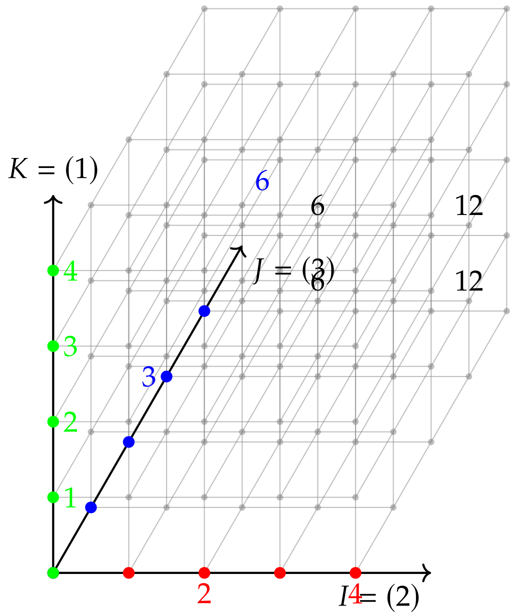


$IJ = \left\{\sum_{k=1}^{t}(a_kb_k) \mid a_k, b_k \in \mathbb{Z}, t \in \mathbb{N}\right\}$



$IJ = \left\{\sum_{k=1}^{t}(a_kb_k) \mid a_k, b_k \in \mathbb{Z}, t \in \mathbb{N}\right\}$

$$\mathbb{Z}\mathbb{Z} = \left\{\sum_{k=1}^{t}(a_k)(b_k) : a_k, b_k \in \mathbb{Z}, t \in \mathbb{N}\right\}$$



K = (1)
4
3
2
1
3
6
6
12
J = (3)
6
12
2
4
I = (2)

Ring (Set)	Addition Operation	Multiplication Operation	Additive Identity	Multiplicative Identity	Form of Element	Commutative
Integers $\mathbb{Z}$	Standard addition +	Standard multiplication $\cdot$	0	1	Integers	Yes
Real Numbers $\mathbb{R}$	Standard addition +	Standard multiplication $\cdot$	0	1	Real numbers	Yes
Complex Numbers $\mathbb{C}$	Standard addition +	Standard multiplication $\cdot$	0	1	$a + bi$ where $a, b \in \mathbb{R}$	Yes
Polynomials with Real Coefficients $\mathbb{R}[x]$	Polynomial addition	Polynomial multiplication	0 (zero polynomial)	1 (constant polynomial)	$a_n x^n + \cdots + a_1 x + a_0$	Yes
Matrices $M_n(\mathbb{R})$	Matrix addition	Matrix multiplication	Zero matrix	Identity matrix	$n \times n$ matrices	No
Integers Modulo n $\mathbb{Z}/n\mathbb{Z}$	Addition modulo n	Multiplication modulo n	0	1	$\{0, 1, \ldots, n-1\}$	Yes

Group (Set)	Operation	Identity	Form of Element	Normal Subgroups	Abelian
Symmetric Group S_3	Composition of permutations	Identity permutation e	Permutations of 3 elements	$\{e, (123), (132)\}$, $\{e, (12), (13), (23), (123), (132)\}$	No
Dihedral Group D_4	Composition of symmetries	Identity symmetry e	Rotations r^k and reflections sr^k, $k \in \{0, 1, 2, 3\}$	$\{e, r, r^2, r^3\}$, $\{e, r^2, s, sr^2\}$	No
Quaternion Group Q_8	Quaternion multiplication	1	Quaternions $\pm 1, \pm i, \pm j, \pm k$	$\{1, -1\}$, $\{1, -1, i, -i\}$, $\{1, -1, j, -j\}$, $\{1, -1, k, -k\}$	No
Integers Modulo n $\mathbb{Z}/n\mathbb{Z}$	Addition modulo n	0	$\{0, 1, \ldots, n-1\}$	All subgroups are normal	Yes

Ring (Set)	Addition Operation	Multiplication Operation	Additive Identity	Multiplicative Identity	Form of Element
Matrices $M_n(\mathbb{R})$	Matrix addition	Matrix multiplication	Zero matrix	Identity matrix	$n \times n$ matrices
Quaternions $\mathbb{H}$	Quaternion addition	Quaternion multiplication	0	1	$a + bi + cj + dk$ where $a, b, c, d \in \mathbb{R}$
Differential Operators	Operator addition	Operator composition	Zero operator	Identity operator	$\sum a_i \frac{d^i}{dx^i}$
Group Rings $\mathbb{R}[G]$	Group ring addition	Group ring multiplication	Zero element	Identity element	$\sum a_g g$ where $a_g \in \mathbb{R}$ and $g \in G$
Endomorphism Rings $\mathrm{End}(V)$	Function addition	Function composition	Zero map	Identity map	Linear transformations on vector space V
Octonions $\mathbb{O}$	Octonion addition	Octonion multiplication	0	1	$a + be_1 + ce_2 + de_3 + ee_4 + fe_5 + ge_6 + he_7$ where $a, b, c, d, e, f, g, h \in \mathbb{R}$

Type	**Ideal**	**Principal Ideal**	**Prime Ideal**	**Maximal Ideal**
Definition	A subset closed under addition and multiplication by any ring element	An ideal generated by a single element	An ideal where if $ab \in I$, then $a \in I$ or $b \in I$	An ideal such that there are no larger ideals except the ring itself
Is an Ideal	O	O	O	O
Can be Principal	O	O	O	O
Is Prime	X	X	O	O
Is Maximal	X	X	X	O

Ring (Set)	**Principal Ideals**	**Prime Ideals**	**Maximal Ideals**
$\mathbb{Z}$	$(2), (3), (5)$	$(2), (3), (5)$	$(2), (3), (5)$
$\mathbb{R}[x]$	$(x), (x-1), (x^2+1)$	(x)	$(x-1)$
$\mathbb{C}[x]$	$(x), (x-i), (x+i)$	$(x-i), (x+i)$	$(x-i), (x+i)$
$\mathbb{Z}/6\mathbb{Z}$	$(2), (3)$	None	$(2), (3)$
$\mathbb{Z}[i]$	$(1+i), (2)$	$(1+i)$	$(1+i)$
$\mathbb{Z}/p\mathbb{Z}$ where p is prime	$(0), (1)$	(0)	(0)
$\mathbb{R}$	(0)	None	(0)
$\mathbb{C}$	(0)	None	(0)
$\mathbb{Z}[x]$	$(2), (3), (x)$	$(x), (2)$	None
$M_n(\mathbb{R})$	(E_{11})	None	None
$\mathbb{H}$ (Quaternions)	$(1+i)$	None	None

Ring (Set)	Principal Ideals	Prime Ideals	Maximal Ideals
$\mathbb{Z}$	$(2) = \{2k \mid k \in \mathbb{Z}\}$ $(3) = \{3k \mid k \in \mathbb{Z}\}$ $(5) = \{5k \mid k \in \mathbb{Z}\}$	$(2) = \{2k \mid k \in \mathbb{Z}\}$ $(3) = \{3k \mid k \in \mathbb{Z}\}$ $(5) = \{5k \mid k \in \mathbb{Z}\}$	$(2) = \{2k \mid k \in \mathbb{Z}\}$ $(3) = \{3k \mid k \in \mathbb{Z}\}$ $(5) = \{5k \mid k \in \mathbb{Z}\}$
$\mathbb{R}[x]$	$(x) = \{xf(x) \mid f(x) \in \mathbb{R}[x]\}$ $(x-1) = \{(x-1)f(x) \mid f(x) \in \mathbb{R}[x]\}$ $(x^2+1) = \{(x^2+1)f(x) \mid f(x) \in \mathbb{R}[x]\}$	$(x) = \{xf(x) \mid f(x) \in \mathbb{R}[x]\}$	$(x-1) = \{(x-1)f(x) \mid f(x) \in \mathbb{R}[x]\}$
$\mathbb{C}[x]$	$(x) = \{xf(x) \mid f(x) \in \mathbb{C}[x]\}$ $(x-i) = \{(x-i)f(x) \mid f(x) \in \mathbb{C}[x]\}$ $(x+i) = \{(x+i)f(x) \mid f(x) \in \mathbb{C}[x]\}$	$(x-i) = \{(x-i)f(x) \mid f(x) \in \mathbb{C}[x]\}$ $(x+i) = \{(x+i)f(x) \mid f(x) \in \mathbb{C}[x]\}$	$(x-i) = \{(x-i)f(x) \mid f(x) \in \mathbb{C}[x]\}$ $(x+i) = \{(x+i)f(x) \mid f(x) \in \mathbb{C}[x]\}$
$\mathbb{Z}/6\mathbb{Z}$	$(2) = \{0, 2, 4\}$ $(3) = \{0, 3\}$	None	$(2) = \{0, 2, 4\}$ $(3) = \{0, 3\}$
$\mathbb{Z}[i]$	$(1+i) = \{(1+i)z \mid z \in \mathbb{Z}[i]\}$ $(2) = \{2z \mid z \in \mathbb{Z}[i]\}$	$(1+i) = \{(1+i)z \mid z \in \mathbb{Z}[i]\}$	$(1+i) = \{(1+i)z \mid z \in \mathbb{Z}[i]\}$
$\mathbb{Z}/p\mathbb{Z}$ where p is prime	$(0) = \{0\}$ $(1) = \mathbb{Z}/p\mathbb{Z}$	$(0) = \{0\}$	$(0) = \{0\}$
$\mathbb{R}$	$(0) = \{0\}$	None	$(0) = \{0\}$
$\mathbb{C}$	$(0) = \{0\}$	None	$(0) = \{0\}$
$\mathbb{Z}[x]$	$(2) = \{2f(x) \mid f(x) \in \mathbb{Z}[x]\}$ $(3) = \{3f(x) \mid f(x) \in \mathbb{Z}[x]\}$ $(x) = \{xf(x) \mid f(x) \in \mathbb{Z}[x]\}$	$(2) = \{2f(x) \mid f(x) \in \mathbb{Z}[x]\}$ $(x) = \{xf(x) \mid f(x) \in \mathbb{Z}[x]\}$	None
$M_n(\mathbb{R})$	$(E_{11}) = \{AE_{11}B \mid A, B \in M_n(\mathbb{R})\}$	None	None
$\mathbb{H}$ (Quaternions)	$(1+i) = \{(1+i)q \mid q \in \mathbb{H}\}$	None	None

Ring (Set)	Principal Ideals	Prime Ideals	Maximal Ideals
$\mathbb{Z}$	$(2) = \{2k \mid k \in \mathbb{Z}\}$ $(3) = \{3k \mid k \in \mathbb{Z}\}$ $(5) = \{5k \mid k \in \mathbb{Z}\}$	$(2) = \{2k \mid k \in \mathbb{Z}\}$ $(3) = \{3k \mid k \in \mathbb{Z}\}$ $(5) = \{5k \mid k \in \mathbb{Z}\}$	$(2) = \{2k \mid k \in \mathbb{Z}\}$ $(3) = \{3k \mid k \in \mathbb{Z}\}$ $(5) = \{5k \mid k \in \mathbb{Z}\}$
$\mathbb{R}[x]$	$(x) = \{xf(x) \mid f(x) \in \mathbb{R}[x]\}$ $(x-1) = \{(x-1)f(x) \mid f(x) \in \mathbb{R}[x]\}$ $(x^2+1) = \{(x^2+1)f(x) \mid f(x) \in \mathbb{R}[x]\}$	(x)	$(x-1)$
$\mathbb{C}[x]$	$(x) = \{xf(x) \mid f(x) \in \mathbb{C}[x]\}$ $(x-i) = \{(x-i)f(x) \mid f(x) \in \mathbb{C}[x]\}$ $(x+i) = \{(x+i)f(x) \mid f(x) \in \mathbb{C}[x]\}$	$(x-i)$ $(x+i)$	$(x-i)$ $(x+i)$
$\mathbb{Z}/6\mathbb{Z}$	$(2) = \{0, 2, 4\}$ $(3) = \{0, 3\}$	None	$(2) = \{0, 2, 4\}$ $(3) = \{0, 3\}$
$\mathbb{Z}[i]$	$(1+i) = \{(1+i)z \mid z \in \mathbb{Z}[i]\}$ $(2) = \{2z \mid z \in \mathbb{Z}[i]\}$	$(1+i)$	$(1+i)$
$\mathbb{Z}/p\mathbb{Z}$ where p is prime	$(0) = \{0\}$ $(1) = \mathbb{Z}/p\mathbb{Z}$	(0)	(0)
$\mathbb{R}$	$(0) = \{0\}$	None	(0)
$\mathbb{C}$	$(0) = \{0\}$	None	(0)
$\mathbb{Z}[x]$	$(2) = \{2f(x) \mid f(x) \in \mathbb{Z}[x]\}$ $(3) = \{3f(x) \mid f(x) \in \mathbb{Z}[x]\}$ $(x) = \{xf(x) \mid f(x) \in \mathbb{Z}[x]\}$	(2) (x)	None
$M_n(\mathbb{R})$	$(E_{11}) = \{AE_{11}B \mid A, B \in M_n(\mathbb{R})\}$	None	None
$\mathbb{H}$ (Quaternions)	$(1+i) = \{(1+i)q \mid q \in \mathbb{H}\}$	None	None

Ring (Set)	Examples of Ideals	Examples of Principal Ideals	Examples of Prime Ideals	Examples of Maximal Ideals
$\mathbb{Z}$	$(0),(2),(3)$ $(4),(6)$ (0)	$(2)=\{2k \mid k\in\mathbb{Z}\}$ $(3)=\{3k \mid k\in\mathbb{Z}\}$ $(5)=\{5k \mid k\in\mathbb{Z}\}$	$(2)=\{2k \mid k\in\mathbb{Z}\}$ $(3)=\{3k \mid k\in\mathbb{Z}\}$ $(5)=\{5k \mid k\in\mathbb{Z}\}$	$(2)=\{2k \mid k\in\mathbb{Z}\}$ $(3)=\{3k \mid k\in\mathbb{Z}\}$ $(5)=\{5k \mid k\in\mathbb{Z}\}$
$\mathbb{R}[x]$	$(0),(x),(x-1)$ (x^2+1)	$(x)=\{xf(x) \mid f(x)\in\mathbb{R}[x]\}$ $(x-1)=\{(x-1)f(x) \mid f(x)\in\mathbb{R}[x]\}$ $(x^2+1)=\{(x^2+1)f(x) \mid f(x)\in\mathbb{R}[x]\}$	(x)	$(x-1)$
$\mathbb{C}[x]$	$(0),(x),(x-i)$ $(x+i)$	$(x)=\{xf(x) \mid f(x)\in\mathbb{C}[x]\}$ $(x-i)=\{(x-i)f(x) \mid f(x)\in\mathbb{C}[x]\}$ $(x+i)=\{(x+i)f(x) \mid f(x)\in\mathbb{C}[x]\}$	$(x-i)$ $(x+i)$	$(x-i)$ $(x+i)$
$\mathbb{Z}/6\mathbb{Z}$	$(0),(2),(3)$	$(2)=\{0,2,4\}$ $(3)=\{0,3\}$	None	$(2)=\{0,2,4\}$ $(3)=\{0,3\}$
$\mathbb{Z}[i]$	$(0),(1+i),(2)$	$(1+i)=\{(1+i)z \mid z\in\mathbb{Z}[i]\}$ $(2)=\{2z \mid z\in\mathbb{Z}[i]\}$	$(1+i)$	$(1+i)$
$\mathbb{Z}/p\mathbb{Z}$ where p is prime	$(0),(1)$	$(0)=\{0\}$ $(1)=\mathbb{Z}/p\mathbb{Z}$	(0)	(0)
$\mathbb{R}$	(0)	$(0)=\{0\}$	None	(0)
$\mathbb{C}$	(0)	$(0)=\{0\}$	None	(0)
$\mathbb{Z}[x]$	$(0),(2),(3)$	$(2)=\{2f(x) \mid f(x)\in\mathbb{Z}[x]\}$ $(3)=\{3f(x) \mid f(x)\in\mathbb{Z}[x]\}$ $(x)=\{xf(x) \mid f(x)\in\mathbb{Z}[x]\}$	(2) (x)	None
$M_n(\mathbb{R})$	$(0),(E_{11})$	$(E_{11})=\{AE_{11}B \mid A,B\in M_n(\mathbb{R})\}$	None	None
$\mathbb{H}$ (Quaternions)	$(0),(1+i)$	$(1+i)=\{(1+i)q \mid q\in\mathbb{H}\}$	None	None