

Automated Reasoning

- A Comprehensive Collection -

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Introduction

Welcome to the seminar on Automated Reasoning. This document is a compilation of various seminar materials designed to provide a comprehensive overview of the field. Here, we explore the fundamental concepts, methodologies, and applications of automated reasoning in computer science and logic.

This is an example of referencing Section

Chapter 1

Preliminaries

Truth Function

Definition 1.1. Let $\mathbb{B} = \{T, F\}$ be the *boolean domain*. Let $k \in \mathbb{N}$. A mapping

$$f : \mathbb{B}^k \rightarrow \mathbb{B}$$

is called a **truth function**.

Remark 1.1 (Truth Functions of Connectives). The logical connectives are assumed to be **truth-functional**. Hence, they are represented by certain **truth functions**.

Logical Negation The *logical not connective* defines the truth function f^\neg as follows:

$$\begin{aligned} f^\neg(F) &= T \\ f^\neg(T) &= F \end{aligned}$$

Logical Conjunction The *conjunction connective* defines the truth function f^\wedge as follows:

$$\begin{aligned} f^\wedge(T, T) &= T \\ f^\wedge(T, F) &= F \\ f^\wedge(F, T) &= F \\ f^\wedge(F, F) &= F \end{aligned}$$

Logical Disjunction The *disjunction connective* defines the truth function f^\vee as follows:

$$\begin{aligned} f^\vee(T, T) &= T \\ f^\vee(T, F) &= T \\ f^\vee(F, T) &= T \\ f^\vee(F, F) &= F \end{aligned}$$

Count of Truth Functions

Proposition 1.1. There are $2^{(2^k)}$ distinct truth functions on k variables.

Proof. Let $f : \mathbb{B}^k \rightarrow \mathbb{B}$ be a truth function for $k \in \mathbb{N}$. Then

(Cardinality of Cartesian Product of Finite Sets)

$$\#(\mathbb{B}^k) = \#(\overbrace{\mathbb{B} \times \cdots \times \mathbb{B}}^{k \text{ times}}) = \overbrace{\#\mathbb{B}\#\mathbb{B} \cdots \#\mathbb{B}}^{k \text{ times}} = \overbrace{2 \cdot 2 \cdots 2}^{k \text{ times}} = 2^k.$$

(Cardinality of Set of All Mappings.)

$$\#(T^S) := \{f \subseteq S \times T : f \text{ is a mapping}\} = (\#T)^{(\#S)} \implies \#(\mathbb{B}^{\mathbb{B}^k}) = 2^{(2^k)}$$

□

Unary Truth Functions

Corollary 1.1.1. *There are 4 distinct unary truth functions:*

- The constant function $f(p) = \text{F}$
- The constant function $f(p) = \text{T}$
- The identity function $f(p) = p$
- The logical not function $f(p) = \neg p$

Proof. From Count of Truth Functions there are $2^{(2^1)} = 4$ distinct truth functions on 1 variable. These can be depicted in a truth table as follows:

p	\circ_1	\circ_2	\circ_3	\circ_4
T	T	T	F	F
F	F	T	F	F

\circ_1 : Whether $p = \text{T}$ or $p = \text{F}$, $\circ_1(p) = \text{T}$. Thus \circ_1 is the *constant function* $\circ_1(p) = \text{T}$.

\circ_2 : We have

$$(1) \ p = \text{T} \implies \circ_2(p) = \text{T}$$

$$(2) \ p = \text{F} \implies \circ_2(p) = \text{F}$$

Thus \circ_2 is the *identity function* $\circ_2(p) = p$.

\circ_3 : We have

$$(1) \ p = \text{T} \implies \circ_3(p) = \text{F}$$

$$(2) \ p = \text{F} \implies \circ_3(p) = \text{T}$$

Thus \circ_3 is the *logical not function* $\circ_3(p) = \neg p$.

\circ_4 : Whether $p = \text{T}$ or $p = \text{F}$, $\circ_4(p) = \text{F}$. Thus \circ_4 is the *constant function* $\circ_4(p) = \text{F}$.

□

Binary Truth Functions

Corollary 1.1.2. *There are 16 distinct unary truth functions:*

- *Two constant operations:*
 - $f_T(p, q) = T$
 - $f_F(p, q) = F$
- *Two projections:*
 - $\text{Proj}_1(p, q) = p$
 - $\text{Proj}_2(p, q) = q$
- *Two negated projections:*
 - $\overline{\text{Proj}_1}(p, q) = \neg p$
 - $\overline{\text{Proj}_2}(p, q) = \neg q$
- *The conjunction: $p \wedge q$*
- *The disjunction: $p \vee q$*
- *Two conditionals:*
 - $p \implies q$
 - $q \implies p$
- *The biconditional (iff): $p \iff q$*
- *The exclusive or (xor): $\neg(p \iff q)$*
- *Two negated conditionals:*
 - $\neg(p \implies q)$
 - $\neg(q \implies p)$
- *The NAND $p \uparrow q$*
- *The NOR $p \downarrow q$*

Proof. From Count of Truth Functions there are $2^{(2^2)} = 16$ distinct truth functions on 2 variable. These can be depicted in a truth table as follows: □

p	T	T	F	F
q	T	F	T	F
$f_F(p, q)$	F	F	F	F
$p \downarrow q$	F	F	F	T
$\neg(p \Leftarrow q)$	F	F	T	F
$\text{Proj}_1(p, q)$	F	F	T	T
$\neg(p \Rightarrow q)$	F	T	F	F
$\text{Proj}_2(p, q)$	F	T	F	T
$\neg(p \iff q)$	F	T	T	F
$p \uparrow q$	F	T	T	T
$p \wedge q$	T	F	F	F
$p \iff q$	T	F	F	T
$\text{Proj}_2(p, q)$	T	F	T	F
$p \Rightarrow q$	T	F	T	T
$\text{Proj}_1(p, q)$	T	T	F	F
$p \Leftarrow q$	T	T	F	T
$p \vee q$	T	T	T	F
$f_T(p, q)$	T	T	T	T

Formal Grammar

Definition 1.2. The formal grammar of the language of propositional logic (and hence its WFFs) can be defined in the following ways.

- **Backus-Naur Form** In Backus-Naur form, the formal grammar of the language of propositional logic takes the following form:

$$\begin{aligned}
 \langle \text{formula} \rangle &::= p \mid \top \mid \perp && \text{where } p \in \mathcal{P}_0 \text{ is a letter} \\
 \langle \text{formula} \rangle &::= \neg \langle \text{formula} \rangle \\
 \langle \text{formula} \rangle &::= (\langle \text{formula} \rangle \langle \text{op} \rangle \langle \text{formula} \rangle) \\
 \langle \text{op} \rangle &::= \wedge \mid \vee \mid \Rightarrow \mid \iff
 \end{aligned}$$

Note that this is a top-down grammar: we start with a metasympol $\langle \text{formula} \rangle$ progressively replace it with constructs containing other metasympols and/or primitive symbols until finally we are left with a well-formed formula of L_0 consisting of nothing but primitive symbols.

Chapter 2

Conflict Driven Clause Learning (CDCL)

2.1 The Boolean Satisfiability Problem

Propositional Function (Formula)

Definition 2.1. A **propositional function** (or **formula**) is defined inductively as follows:

(Basic Step) Any propositional variable $x \in V$ is a formula.

(Inductive Step) If F and G are formulas, then the following are also formulas:

- $\neg F$
- $F \wedge G$
- $F \vee G$
- $F \rightarrow G$
- $F \leftrightarrow G$

Formally, the set of all Φ is the smallest set satisfying:

$$\Phi = V \cup \{\neg F : F \in \Phi\} \cup \{\circ(F, G) : F, G \in \Phi \text{ and } \circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}\}$$

Boolean Satisfiability Problem (SAT)

Definition 2.2. Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of propositional variables. Let L be a set of one or more propositional formulas constructed using only:

- $x_i \in X$ for $i = 1, \dots, n$;
- the $2^{(2^1)} = 4$ unary logical connectives;
- the $2^{(2^2)} = 16$ binary logical connectives;

The problem is to find truth values for all $x \in X$ such that all the formulas in L are true. Such a problem is a boolean satisfiability problem.

Note (literal). A *literal* is either a propositional variable (p) or its negation ($\neg p$).

Conjunctive Normal Form (CNF)

Definition 2.3. A formula F is in conjunctive normal form (CNF) if and only if it is a conjunction of disjunctions of literals, i.e. of the form

$$F = \bigvee_i \bigwedge_j l_{i,j}$$

where $l_{i,j}$ are the literals.

Remark 2.1. Usually, CNF formulae are represented in an alternative form called clauses using the commutativity, associativity and idempotence of the logical operators \vee and \wedge . A clause corresponds to a disjunction of literals and a CNF formula is a conjunction of clauses. We write a clause as a set of its literals and a formula as a set of clauses.

Example 2.1.

A propositional formula: $\neg(((x_0 \wedge \neg x_1) \vee x_2) \leftrightarrow x_1)$

An equivalent formula in CNF: $(x_0 \vee x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)$

In clause representation: $\{\{x_0, x_1, x_2\}, \{\neg x_1, \neg x_2\}\}$

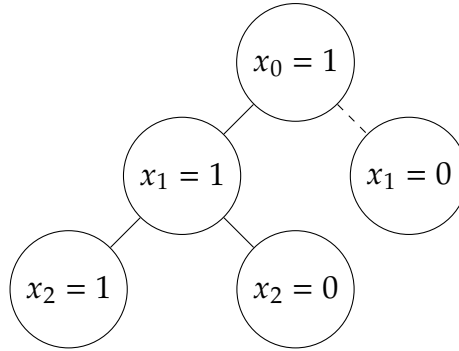
In reality, formulae are often not being transformed into equivalent CNF formulae but into equisatisfiable CNF formulae (formulae which are satisfiable if and only if the original formula is satisfiable) when preparing them for SAT, because this can be done efficiently.

2.2 Basics of SAT Solving

The Conflict Driven Clause Learning (CDCL) algorithm is a mix of two older approaches to SAT solving: Davis–Putnam–Logemann–Loveland (DPLL) and Resolution. It uses concepts from both and combines them in a new way.

2.2.1 Backtracking and Unit Propagation as in DPLL Solvers

Example 2.2. Example graph for $G = \{\{\neg x_0, \neg x_1, \neg x_2\}, \{\neg x_0, x_2\}, \{x_0, x_1, x_2\}\}$:



The backtracking algorithm has just discovered that setting $x_2 = 0$ makes the clause $\{\neg x_0, x_2\}$ false. So it goes back to explore the $x_1 = 0$ path and deletes the value of x_2 .

2.2.2 Resolution

If a formula F contains the clauses $\{\neg x\} \cup A$ and $\{x\} \cup A...$

2.3 Principles of CDCL

Now that we have seen how Backtracking and Resolution work we are ready to merge these approaches...

2.3.1 The trail

When applying CDCL rather than exploring a search tree of assignments...

2.3.2 Conflict clauses and backjumping

Consider our previous example again. When we want to continue building up our trail...

2.3.3 The implication graph

A nice way to illustrate the functionality of CDCL are implication graphs...

2.3.4 The algorithm

To get a clearer view Algorithm 4.1 shows the pseudocode for the CDCL algorithm...

2.4 Implementation

Let us now look at how the algorithm is implemented in real life...

2.4.1 Clauses

We use a monolithic array MEM to hold the original formula's clauses as well as the newly learned clauses...

2.4.2 Literals

Assume the variables are x_1, x_2, \dots, x_n . We represent x_k by k ...

2.5 Results

It is important to say that CDCL is a sound and complete algorithm for the propositional satisfiability problem...

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