Category Theory

- A Journey from Concretization to Abstraction -

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$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] \arrow[loop left, "1_A"] & B \arrow[d, "g"] \arrow[loop right, "1_B"] \\
 & C \arrow[loop below, "1_C"]
\end{tikzcd}
$$

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Chapter 1

Categorical

Example 1.1 (Determinant)**.**

[Two Categories]

1. **Category of Commutative Rings** (CRing)
   - **Objects**: Commutative rings $R, S, \ldots$
   - **Morphisms**: Ring homomorphisms $\phi : R \to S$
2. **Category of Groups** (Grp)
   - **Objects**: Groups
   - **Morphisms**: Group homomorphisms

[Two Functors] Note that $\mathbf{GL}_n$ represents the functor, while GL_n denotes the group.

1. **General Linear Group Functor** ($\mathbf{GL}_n : \mathsf{CRing} \to \mathsf{Grp}$)
   - **On Objects**:
     $$R \mapsto GL_n(R)$$
     Maps a ring R to the general linear group $GL_n(R)$, which consists of all $n \times n$ invertible matrices over R.
   - **On Morphisms**: Let $\mathbf{GL}_n(\phi) : GL_n(R) \to GL_n(S)$.
     $$\phi \mapsto \mathbf{GL}_n(\phi)$$
     It preserves invertibility.
2. **Unit Functor** ($\mathbf{U} : \mathsf{CRing} \to \mathsf{Grp}$)
   - **On Objects**:
     $$R \mapsto R^\times$$
   - **On Morphisms**: Let $\mathbf{U}(\phi) : R^\times \to S^\times$.
     $$\phi \mapsto \mathbf{U}(\phi)$$
     It preserves the unit property.

- **[Natural Transformation: Determinant]** The determinant

$$\det_R : GL_n(R) \to R^\times$$

is a natural transformation between two functors, $\mathbf{GL}_n$ and $\mathbf{U}$.

  - **On Objects**: $\eta_R : GL_n(R) \to R^\times$
  - **Naturality Condition**: The following diagram commutes:

$$\begin{tikzcd} GL_n(R) \arrow[r, "\eta_R"] \arrow[d, "\mathbf{GL_n}(\phi)"'] & R^\times \arrow[d, "\mathbf{U}(\phi)"] \\ GL_n(S) \arrow[r, "\eta_S"] & S^\times \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

$$\begin{tikzcd} R \arrow[d, "\phi"'] \\ S \end{tikzcd} \qquad \begin{tikzcd} GL_n(R) \arrow[r, "\eta_R"] \arrow[d, "\mathbf{GL_n}(\phi)"'] & R^\times \arrow[d, "\mathbf{U}(\phi)"] \\ GL_n(S) \arrow[r, "\eta_S"] & S^\times \end{tikzcd}$$

Example 1.2. Consider $\mathbb{Z}$ and $GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc \neq 0 \right\}$.

[Two Categories]

1. **Category of Commutative Rings** (CRing)
   - **Object**: Commutative ring $(\mathbb{Z}, +, \times)$
   - **Morphism**: Ring homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}$
2. **Category of Groups** (Grp)
   - **Objects**: Groups $(GL_2(\mathbb{Z}), *)$ and $(\mathbb{Z}^\times, \times)$. Here '$*$' denotes matrix multiplication. Note that
   $$\mathbb{Z}^\times = \{1, -1\}.$$
   - **Morphisms**: Group homomorphisms
   $$\sigma : GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}) \quad \text{and} \quad \tau : \mathbb{Z}^\times \to \mathbb{Z}^\times.$$
   Note that $\tau : \{\pm 1\} \to \{\pm 1\}$

[Two Functors] Note that $\mathbf{GL}_2$ represents the functor, while GL_2 denotes the group.

1. **General Linear Group Functor** ($\mathbf{GL}_2 : \mathsf{CRing} \to \mathsf{Grp}$)
   - **On Object**:
   $$\mathbb{Z} \mapsto GL_2(\mathbb{Z})$$
   - **On Morphism**:
   $$\phi \mapsto \sigma = \mathbf{GL}_2(\phi)$$
   It preserves invertibility.
2. **Unit Functor** ($\mathbf{U} : \mathsf{CRing} \to \mathsf{Grp}$)
   - **On Object**:
   $$\mathbb{Z} \mapsto \mathbb{Z}^\times = \{\pm 1\}$$
   - **On Morphism**:
   $$\phi \mapsto \tau = \mathbf{U}(\phi)$$
   It preserves the unit property.

- **[Natural Transformation: Determinant]** The determinant

$$\begin{aligned} \det_{\mathbb{Z}} \quad : \quad GL_2(\mathbb{Z}) &\longrightarrow \mathbb{Z}^\times = \{\pm 1\} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto ad - bc = \det_{\mathbb{Z}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{aligned}$$

is a natural transformation between two functors, $\mathbf{GL}_2$ and $\mathbf{U}$.

- **Naturality Condition**: The following diagram commutes:

$$\begin{tikzcd} GL_2(\mathbb{Z}) \arrow[r, "\eta_{\mathbb{Z}}=\det_{\mathbb{Z}}"] \arrow[d, "\sigma=\mathbf{GL}_2(\phi)"] & \mathbb{Z}^\times \arrow[d, "\tau=\mathbf{U}(\phi)"] \\ GL_n(\mathbb{Z}) \arrow[r, "\eta_{\mathbb{Z}}=\det_{\mathbb{Z}}"] & \mathbb{Z}^\times \end{tikzcd}$$

$$
\begin{tikzcd}
R \arrow[d, "\phi"'] & GL_n(R) \arrow[r, "\eta_R=\det_R"] \arrow[d, "\mathbf{GL}_n(\phi)"] & R^{\times} \arrow[d, "\mathbf{U}(\phi)"] \\
S & GL_n(S) \arrow[r, "\eta_S=\det_S"] & S^{\times}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{Z} \arrow[d, "\phi"'] & GL_2(\mathbb{Z}) \arrow[r, "\det_{\mathbb{Z}}"] \arrow[d, "\sigma=\mathbf{GL}_2(\phi)"] & \mathbb{Z}^{\times} \arrow[d, "\tau=\mathbf{U}(\phi)"] \\
\mathbb{Z} & GL_2(\mathbb{Z}) \arrow[r, "\det_{\mathbb{Z}}"] & \mathbb{Z}^{\times}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{Z} \arrow[d, "\phi"', ""{name=P}] \arrow[r, dotted, bend left, "\mathbf{GL}_2"] \arrow[rr, dotted, bend left=50, "\mathbf{U}"] & GL_2(\mathbb{Z}) \arrow[r, "\det_{\mathbb{Z}}"] \arrow[d, "\sigma=\mathbf{GL}_2(\phi)", ""'{name=S}] & \mathbb{Z}^{\times} \arrow[d, "\tau=\mathbf{U}(\phi)", ""'{name=T}] \\
\mathbb{Z} \arrow[r, dotted, bend right, "\mathbf{GL}_2"'] \arrow[rr, dotted, bend right=50, "\mathbf{U}"'] & GL_2(\mathbb{Z}) \arrow[r, "\det_{\mathbb{Z}}"] & \mathbb{Z}^{\times}
\arrow[from=P, to=S, dotted, "\mathbf{GL}_2"]
\arrow[from=P, to=T, dotted, "\mathbf{U}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
& V \arrow[loop above, "Id_V"] \arrow[dl, bend right, "\mathbf{v}\in V\mapsto(\mathbf{w}\mapsto\langle\mathbf{v},\mathbf{w}\rangle)\in V^*"'] \arrow[dr, bend right, "\mathbf{v}\mapsto(\phi\mapsto\phi(\mathbf{v}))"'] & \\
V^* \arrow[loop left, "Id_{V^*}"] \arrow[ur, bend right, "\text{Generally Non-Natural}"'] \arrow[rr, bend right, "\text{Requires additional constraints or specific structures}"'] & & V^{**} \arrow[loop right, "Id_{V^{**}}"] \arrow[ul, bend right, "\text{No Natural Map}"'] \arrow[ll, bend right, "\text{Requires additional constraints or specific structures}"']
\end{tikzcd}
$$

Example 1.3 (Evaluation Mapping as Natural Transformation)**.**

Definition (*Evaluation Map*)

- Let V, K be sets, and
- let K^V be the set of all mappings from V to K.

The **evaluation mapping** for K^V is the mapping $\mathsf{ev} : K^V \times V \to K$ defined by:

$$\mathsf{ev}(f, v) := f(v)$$

for $f \in K^V$ and $v \in V$.

Consider a linear mapping

$$\phi : \mathbb{R}^2 \to \mathbb{R}, \quad \phi(\mathbf{v}) = \phi(v_1, v_2) = 3v_1 - 2v_2.$$

Then evaluation map $\mathsf{ev} : \mathbb{R}^{\mathbb{R}^2} \times \mathbb{R}^2 \to \mathbb{R}$ ensure

$$\mathsf{ev}(\phi, (1, 1)) = \phi(1, 1) = 3 \cdot 1 - 2 \cdot 1 = 1.$$

[Category]

1. **Category of Vector Space** (Vect_K)
   - **Object**:
     * Vector Spaces over field K: $V, W, \cdots$
     * Dual Spaces: $V^*, W^*, \cdots$. Note that

       $$V^* := \left\{\phi : V \to K \mid \phi \text{ is linear functional}\right\}$$

     * Double dual Spaces: $V^{**}, W^{**}, \cdots$. Note that

       $$V^{**} := \left\{\psi \in V^{**} \to K \mid \psi \text{ is linear functional}\right\}$$

   - **Morphism**: Ring homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}$
2. **Category of Groups** (Grp)
   - **Objects**: Groups $(GL_2(\mathbb{Z}), *)$ and $(\mathbb{Z}^\times, \times)$. Here '$*$' denotes matrix multiplication. Note that

     $$\mathbb{Z}^\times = \{1, -1\}.$$

   - **Morphisms**: Group homomorphisms

     $$\sigma : GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}) \quad \text{and} \quad \tau : \mathbb{Z}^\times \to \mathbb{Z}^\times.$$

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   - **On Object**:
   $$\mathbb{Z} \mapsto GL_2(\mathbb{Z})$$
   - **On Morphism**:
   $$\phi \mapsto \sigma = \mathbf{GL}_2(\phi)$$
   It preserves invertibility.
2. **Unit Functor** ($\mathbf{U} : \mathsf{CRing} \to \mathsf{Grp}$)
   - **On Object**:
   $$\mathbb{Z} \mapsto \mathbb{Z}^\times = \{\pm 1\}$$
   - **On Morphism**:
   $$\phi \mapsto \tau = \mathbf{U}(\phi)$$
   It preserves the unit property.

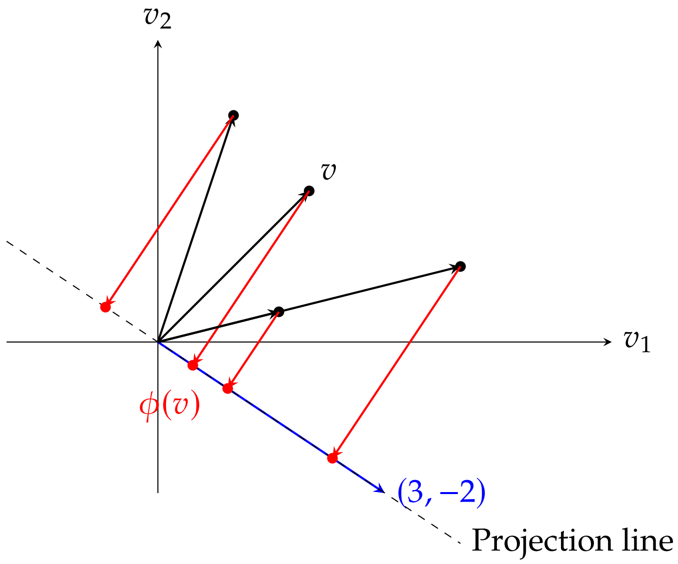
- **[Natural Transformation: Determinant]** The determinant

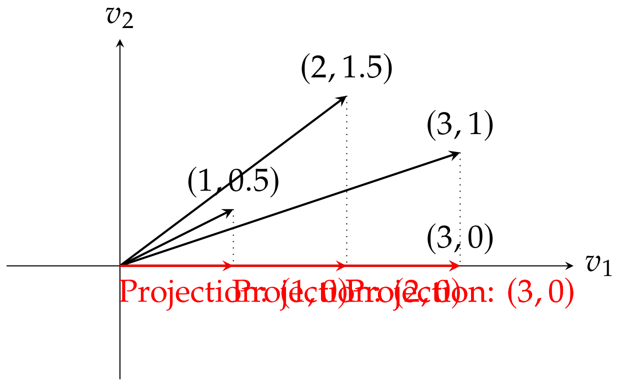
$$\begin{array}{rcccl} \det_{\mathbb{Z}} & : & GL_2(\mathbb{Z}) & \longrightarrow & \mathbb{Z}^\times = \{\pm 1\} \\ & & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & ad - bc = \det_{\mathbb{Z}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

is a natural transformation between two functors, $\mathbf{GL}_2$ and $\mathbf{U}$.

- **Naturality Condition**: The following diagram commutes:

$$\begin{tikzcd} GL_2(\mathbb{Z}) \arrow[r, "\eta_{\mathbb{Z}}=\det_{\mathbb{Z}}"] \arrow[d, "\sigma=\mathbf{GL}_2(\phi)"] & \mathbb{Z}^\times \arrow[d, "\tau=\mathbf{U}(\phi)"] \\ GL_n(\mathbb{Z}) \arrow[r, "\eta_{\mathbb{Z}}=\det_{\mathbb{Z}}"] & \mathbb{Z}^\times \end{tikzcd}$$





$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "(\cdot)^*"'] & W \arrow[d, "(\cdot)^*"] \\
V^* \arrow[r, "T^*"'] & W^* \\
V^{**} \arrow[u, "(\cdot)^*"] \arrow[r, "T^{**}"] & W^{**} \arrow[u, "(\cdot)^*"']
\end{tikzcd}$$

Definition 1.1 (Vector Space and Dual Space)**.**

- Let V be a finite-dimensional *vector space* over field $\mathbb{K}$. That is, $V = K^n$.
- The *dual space* V^* of V consists of all linear functionals on V. That is,

$$V^* = \{\phi : V \to \mathbb{R} \mid \phi \text{ is linear}\}.$$

Definition 1.2 (Double Dual Space)**.** The *double dual space* V^{**} of a vector space V is defined as the dual of the dual space, i.e., $V^{**} = (V^*)^*$. For each $\psi \in V^{**}$, ψ is a functional acting on elements of V^*.

Natural Isomorphism Between V and V^{**}

Theorem 1.1 (Canonical Isomorphism). *For every finite-dimensional vector space V, there is a canonical isomorphism $\eta_V : V \to V^{**}$ defined by*

$$\eta_V(v)(\phi) = \phi(v),$$

for every $v \in V$ and $\phi \in V^*$. This map is a natural isomorphism, meaning it commutes with all linear transformations.

Definition 1.3 (Induced Map on Dual and Double Dual Spaces). For a linear transformation $T : V \to W$ between vector spaces, the induced map on the dual space, $T^* : W^* \to V^*$, is defined by

$$T^*(\phi) = \phi \circ T,$$

where $\phi \in W^*$. Similarly, the induced map on the double dual space, $T^{**} : V^{**} \to W^{**}$, is defined by

$$T^{**}(\psi) = \psi \circ T^*,$$

where $\psi \in V^{**}$.

1.1 Example: Application in $\mathbb{R}^2$

Example 1.4 (Matrix Transformation and Double Dual). Consider the vector space $V = \mathbb{R}^2$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be represented by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

We compute the induced maps T^* and T^{**} and verify the natural transformation property with η_V.

Action of T and η_V: If $\mathbf{v} = (1, 1)$, then

$$T(\mathbf{v}) = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

The action of $\eta_{\mathbb{R}^2}$ and ϕ defined by $\phi(\mathbf{v}) = 3v_1 - 2v_2$ gives

$$\eta_{\mathbb{R}^2}((1, 1))(\phi) = 1 \quad \text{and} \quad \eta_{\mathbb{R}^2}((3, 7))(\phi) = 3 \times 3 - 2 \times 7 = -5.$$

Verification of Commutativity: To confirm the natural transformation property, we need to show

$$T^{**}(\eta_{\mathbb{R}^2}(\mathbf{v})) = \eta_{\mathbb{R}^2}(T(\mathbf{v})).$$

For $\mathbf{v} = (1, 1)$, the calculations show that both sides of the equation represent consistent transformations via the isomorphism $\eta_{\mathbb{R}^2}$, proving the natural transformation.

Example 1.5.

[Category]

1. **Category of Finite Vector Spaces** ($\mathsf{FinVect}_K$)
   - **Objects**:
     * Vector space $V = K^n$ over a field K
     * Dual Space $V^* = \left\{\phi \in K^V : \phi \text{ is linear transformation}\right\}$.
   - **Morphisms**:
     * Linear Transformation $T : V(= K^n) \to W(= K^m)$
     * Dual Transformation $T^* : W^* \to V^*$ defined by

$$T^* f := f \circ T \quad \text{for all} \quad f \in W * .$$

       Sometimes we denote $V^* := \mathcal{L}(V, K)$. Note that
       · $(T^* f)\mathbf{v} = f(T\mathbf{v})$ for all $\mathbf{v} \in V$;
       · $f \in W^* \implies T^* f \in V^*$;
       · T^* is linear.

[Functor]

1. **Dual Space Functor** ($\mathcal{D} : \mathsf{FinVect}_K \to \mathsf{FinVect}_K$)
   - **On Objects**:

$$V \mapsto V^* = \mathcal{D}(V)$$

     Maps a vector space V to the dual space V^*, which which consists of all linear functionals from V to K.
   - **On Morphisms**:

$$T \mapsto T^* = \mathcal{D}(T)$$

- **[Natural Transformation: Determinant]** The determinant

$$\det_R : GL_n(R) \to R^{\times}$$

is a natural transformation between two functors, $\mathbf{GL}_n$ and $\mathbf{U}$.

1.2 Currying, Kleisli Category, Grothendieck Construction

1.2.1 Currying

Note.

$$\begin{array}{rcl} A^{B\times C} & \simeq & (A^C)^B \\ (f : B\times C \to A) & \rightleftarrows & \left(\begin{array}{ccccc} g & : & B & \longrightarrow & [C \to A] \\ & & b\in B & \longmapsto & g_b \in A^C \end{array}\right). \end{array}$$

Consider $f : X\times Y \to Z$. Then

$$\begin{aligned} f : X\times Y \to Z &\implies f \in Z^{X\times Y} \\ &\implies f \in (Z^Y)^X, \text{i.e., } x\in X \mapsto f_x \in Z^Y \\ &\implies f : X \to [Y \to Z]. \end{aligned}$$

```
def unit(x):
        return (x, "Ops:")
```

Bibliography

[1] ProofWiki. “Definition:Set of All Mappings” Accessed on [April 29, 2024]. https://proofwiki.org/wiki/Definition:Set_of_All_Mappings.