Category Theory

- A Journey from Concretization to Abstraction -

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$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] \arrow[loop left, "1_A"] & B \arrow[d, "g"] \arrow[loop right, "1_B"] \\
& C \arrow[loop below, "1_C"]
\end{tikzcd}
$$

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the Category Theory

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Contents

Chapter 1

None

Example 1.1 (Determinant)**.**

[Two Categories]

1. **Category of Commutative Rings** (CRing)
   - **Objects**: Commutative rings $R, S, \ldots$
   - **Morphisms**: Ring homomorphisms $\phi : R \to S$
2. **Category of Groups** (Grp)
   - **Objects**: Groups
   - **Morphisms**: Group homomorphisms

[Two Functors] Note that $\mathbf{GL}_n$ represents the functor, while GL_n denotes the group.

1. **General Linear Group Functor** ($\mathbf{GL}_n : \mathsf{CRing} \to \mathsf{Grp}$)
   - **On Objects**:
     $$R \mapsto GL_n(R)$$
     Maps a ring R to the general linear group $GL_n(R)$, which consists of all $n \times n$ invertible matrices over R.
   - **On Morphisms**: Let $\mathbf{GL}_n(\phi) : GL_n(R) \to GL_n(S)$.
     $$\phi \mapsto \mathbf{GL}_n(\phi)$$
     It preserves invertibility.
2. **Unit Functor** ($\mathbf{U} : \mathsf{CRing} \to \mathsf{Grp}$)
   - **On Objects**:
     $$R \mapsto R^\times$$
   - **On Morphisms**: Let $\mathbf{U}(\phi) : R^\times \to S^\times$.
     $$\phi \mapsto \mathbf{U}(\phi)$$
     It preserves the unit property.

- **[Natural Transformation: Determinant]** The determinant

$$\det_R : GL_n(R) \to R^\times$$

is a natural transformation between two functors, $\mathbf{GL}_n$ and $\mathbf{U}$.

  - **On Objects**: $\eta_R : GL_n(R) \to R^\times$
  - **Naturality Condition**: The following diagram commutes:

$$\begin{tikzcd} GL_n(R) \arrow[r, "\eta_R"] \arrow[d, "\mathbf{GL_n}(\phi)"'] & R^\times \arrow[d, "\mathbf{U}(\phi)"] \\ GL_n(S) \arrow[r, "\eta_S"] & S^\times \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[d, "f"'] & F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ Y & F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

$$\begin{tikzcd} R \arrow[d, "\phi"'] & GL_n(R) \arrow[r, "\eta_R"] \arrow[d, "\mathbf{GL_n}(\phi)"'] & R^\times \arrow[d, "\mathbf{U}(\phi)"] \\ S & GL_n(S) \arrow[r, "\eta_S"] & S^\times \end{tikzcd}$$

Example 1.2. Consider $\mathbb{Z}$ and $GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc \neq 0 \right\}$.

[Two Categories]

1. **Category of Commutative Rings** (CRing)
   - **Object**: Commutative ring $(\mathbb{Z}, +, \times)$
   - **Morphism**: Ring homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}$
2. **Category of Groups** (Grp)
   - **Objects**: Groups $(GL_2(\mathbb{Z}), *)$ and $(\mathbb{Z}^\times, \times)$. Here '$*$' denotes matrix multiplication. Note that
   $$\mathbb{Z}^\times = \{1, -1\}.$$
   - **Morphisms**: Group homomorphisms
   $$\sigma : GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}) \quad \text{and} \quad \tau : \mathbb{Z}^\times \to \mathbb{Z}^\times.$$
   Note that $\tau : \{\pm 1\} \to \{\pm 1\}$

[Two Functors] Note that $\mathbf{GL}_2$ represents the functor, while GL_2 denotes the group.

1. **General Linear Group Functor** ($\mathbf{GL}_2$: $\mathsf{CRing} \longrightarrow \mathsf{Grp}$)
   - **On Object**:
   $$\mathbb{Z} \mapsto GL_2(\mathbb{Z})$$
   - **On Morphism**:
   $$\phi \mapsto \sigma = \mathbf{GL}_2(\phi)$$
   It preserves invertibility.
2. **Unit Functor** ($\mathbf{U}$: $\mathsf{CRing} \longrightarrow \mathsf{Grp}$)
   - **On Object**:
   $$\mathbb{Z} \mapsto \mathbb{Z}^\times = \{\pm 1\}$$
   - **On Morphism**:
   $$\phi \mapsto \tau = \mathbf{U}(\phi)$$
   It preserves the unit property.

- **[Natural Transformation: Determinant]** The determinant

$$\begin{array}{rcccl} \det_{\mathbb{Z}} & : & GL_2(\mathbb{Z}) & \longrightarrow & \mathbb{Z}^\times = \{\pm 1\} \\ & & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & ad - bc = \det_{\mathbb{Z}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

is a natural transformation between two functors, $\mathbf{GL}_2$ and $\mathbf{U}$.

  - **Naturality Condition**: The following diagram commutes:

$$\begin{tikzcd} GL_2(\mathbb{Z}) \arrow[r, "\eta_{\mathbb{Z}}=\det_{\mathbb{Z}}"] \arrow[d, "\sigma=\mathbf{GL}_2(\phi)"] & \mathbb{Z}^\times \arrow[d, "\tau=\mathbf{U}(\phi)"] \\ GL_n(\mathbb{Z}) \arrow[r, "\eta_{\mathbb{Z}}=\det_{\mathbb{Z}}"] & \mathbb{Z}^\times \end{tikzcd}$$

$$\begin{tikzcd} R \arrow[d, "\phi"'] & GL_n(R) \arrow[r, "\eta_R=\det_R"] \arrow[d, "\mathbf{GL}_n(\phi)"] & R^\times \arrow[d, "\mathbf{U}(\phi)"] \\ S & GL_n(S) \arrow[r, "\eta_S=\det_S"] & S^\times \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z} \arrow[d, "\phi"'] & GL_2(\mathbb{Z}) \arrow[r, "\det_{\mathbb{Z}}"] \arrow[d, "\sigma=\mathbf{GL}_2(\phi)"] & \mathbb{Z}^\times \arrow[d, "\tau=\mathbf{U}(\phi)"] \\ \mathbb{Z} & GL_2(\mathbb{Z}) \arrow[r, "\det_{\mathbb{Z}}"] & \mathbb{Z}^\times \end{tikzcd}$$

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r, dotted, bend left, "\mathbf{GL}_2"] \arrow[rr, dotted, bend left=60, "\mathbf{U}"] \arrow[d, "\phi"'] \arrow[r, phantom, shift right=4, "\mathbf{GL}_2"] \arrow[rr, phantom, shift right=8, "\mathbf{U}"] & GL_2(\mathbb{Z}) \arrow[r, "\det_{\mathbb{Z}}"] \arrow[d, "\sigma=\mathbf{GL}_2(\phi)"] & \mathbb{Z}^{\times} \arrow[d, "\tau=\mathbf{U}(\phi)"] \\
\mathbb{Z} \arrow[r, dotted, bend right, "\mathbf{GL}_2"'] \arrow[rr, dotted, bend right=60, "\mathbf{U}"'] & GL_2(\mathbb{Z}) \arrow[r, "\det_{\mathbb{Z}}"] & \mathbb{Z}^{\times}
\end{tikzcd}
$$

Definition 1.1 (Vector Space and Dual Space)**.**

- Let V be a finite-dimensional *vector space* over field $\mathbb{K}$. That is, $V = K^n$.
- The *dual space* V^* of V consists of all linear functionals on V. That is,

$$V^* = \{\phi : V \to \mathbb{R} \mid \phi \text{ is linear}\}.$$

Definition 1.2 (Double Dual Space)**.** The *double dual space* V^{**} of a vector space V is defined as the dual of the dual space, i.e., $V^{**} = (V^*)^*$. For each $\psi \in V^{**}$, ψ is a functional acting on elements of V^*.

Natural Isomorphism Between V and V^{**}

Theorem 1.1 (Canonical Isomorphism)**.** *For every finite-dimensional vector space V, there is a canonical isomorphism $\eta_V : V \to V^{**}$ defined by*

$$\eta_V(v)(\phi) = \phi(v),$$

for every $v \in V$ and $\phi \in V^*$. This map is a natural isomorphism, meaning it commutes with all linear transformations.

Definition 1.3 (Induced Map on Dual and Double Dual Spaces)**.** For a linear transformation $T : V \to W$ between vector spaces, the induced map on the dual space, $T^* : W^* \to V^*$, is defined by

$$T^*(\phi) = \phi \circ T,$$

where $\phi \in W^*$. Similarly, the induced map on the double dual space, $T^{**} : V^{**} \to W^{**}$, is defined by

$$T^{**}(\psi) = \psi \circ T^*,$$

where $\psi \in V^{**}$.

1.1 Example: Application in $\mathbb{R}^2$

Example 1.3 (Matrix Transformation and Double Dual)**.** Consider the vector space $V = \mathbb{R}^2$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be represented by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

We compute the induced maps T^* and T^{**} and verify the natural transformation property with η_V.

Action of T and η_V: If $\mathbf{v} = (1, 1)$, then

$$T(\mathbf{v}) = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

The action of $\eta_{\mathbb{R}^2}$ and ϕ defined by $\phi(\mathbf{v}) = 3v_1 - 2v_2$ gives

$$\eta_{\mathbb{R}^2}((1, 1))(\phi) = 1 \quad \text{and} \quad \eta_{\mathbb{R}^2}((3, 7))(\phi) = 3 \times 3 - 2 \times 7 = -5.$$

Verification of Commutativity: To confirm the natural transformation property, we need to show

$$T^{**}(\eta_{\mathbb{R}^2}(\mathbf{v})) = \eta_{\mathbb{R}^2}(T(\mathbf{v})).$$

For $\mathbf{v} = (1, 1)$, the calculations show that both sides of the equation represent consistent transformations via the isomorphism $\eta_{\mathbb{R}^2}$, proving the natural transformation.

Example 1.4.

[Category]

1. **Category of Finite Vector Spaces** ($\mathsf{FinVect}_K$)
   - **Objects**:
     * Vector space $V = K^n$ over a field K
     * Dual Space $V^* = \{\phi \in K^V : \phi \text{ is linear transformation}\}$.
   - **Morphisms**:
     * Linear Transformation $T : V(= K^n) \to W(= K^m)$
     * Dual Transformation $T^* : W^* \to V^*$ defined by

$$T^* f := f \circ T \quad \text{for all} \quad f \in W* .$$

Sometimes we denote $V^* := \mathcal{L}(V, K)$. Note that

· $(T^* f)\mathbf{v} = f(T\mathbf{v})$ for all $\mathbf{v} \in V$;

· $f \in W^* \implies T^* f \in V^*$;

· T^* is linear.

[Functor]

1. **Dual Space Functor** ($\mathcal{D} : \mathsf{FinVect}_K \to \mathsf{FinVect}_K$)
   - **On Objects**:

$$V \mapsto V^* = \mathcal{D}(V)$$

Maps a vector space V to the dual space V^*, which which consists of all linear functionals from V to K.

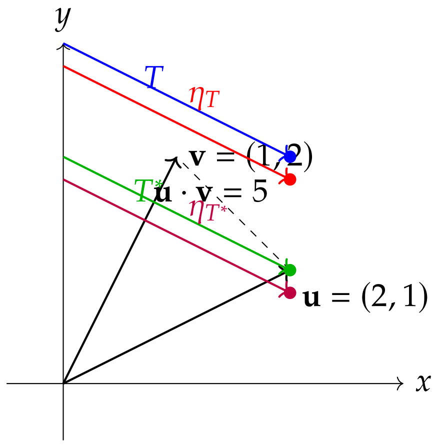
   - **On Morphisms**:

$$T \mapsto T^* = \mathcal{D}(T)$$

- **[Natural Transformation: Determinant]** The determinant

$$\det_R : GL_n(R) \to R^\times$$

is a natural transformation between two functors, $\mathbf{GL}_n$ and $\mathbf{U}$.



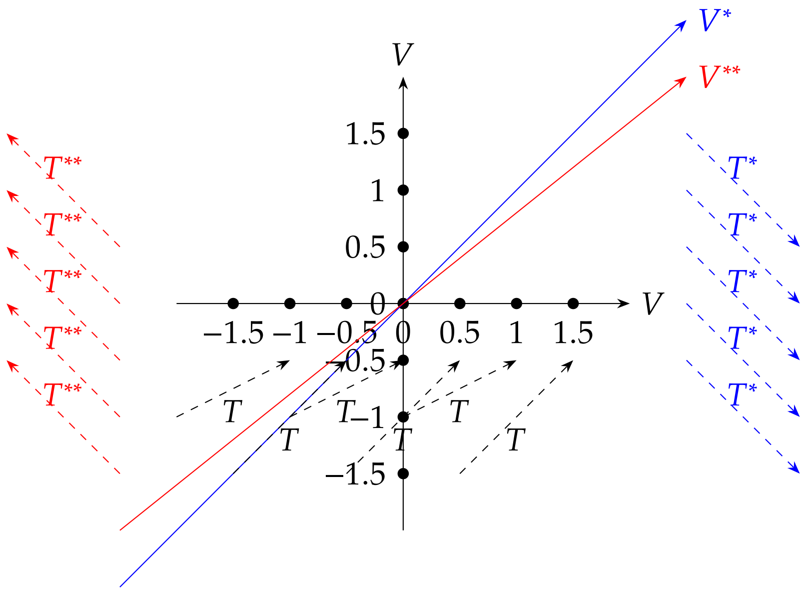


Figure 1.1: Visualization of Vector Space, Dual Space, Double Dual Space, Linear Transformation, Dual Transformation, and Double Transformation

1.2 Category Definitions

1.2.1 Matrices over Commutative Rings

Define the category **Mat_n_(CRing)** where:

- Objects are $n \times n$ matrices over any commutative ring R.
- Morphisms are ring homomorphisms applied element-wise to matrices.

1.2.2 Determinant Functor

The functor **Det**: **Mat_n_(CRing)** $\to$ **Grp** is defined by:

- On Objects: Mapping A to the group formed under multiplication by $\det(A)$.
- On Morphisms: If $f : R \to S$ is a ring homomorphism, then $\det(f(A)) = f(\det(A))$, maintaining the structure of group morphisms.

1.3 Properties and Transformations

1.3.1 Functor Properties

The determinant functor preserves:

- Composition: $\det(AB) = \det(A)\det(B)$
- Identities: $\det(I) = 1$, where I is the identity matrix.

1.3.2 Natural Transformations

Consider a transformation scaling each element of matrix A by a unit u in R. The determinant changes as:

$$\tau_A : \det(A) \mapsto u^n \det(A)$$

where n is the matrix dimension, reflecting the multiplicative scaling in the group.

1.4 Conclusion

Through the lens of category theory, the determinant integrates the algebraic structures of commutative rings and groups, enhancing our understanding of its invariant and multiplicative properties.