

Category Theory

September 9, 2024

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1 Definition

Category

A **category** C consists of

- A class $\text{obj}(C)$ of **objects** A, B, C, \dots in C
- For all (ordered) pairs of objects $X, Y \in \text{obj}(C)$, a class $\text{hom}(A, B)$ of **maps** or **arrow** called **morphisms** from A to B , called the **homo-set** of morphisms from A to B . If $f \in \text{hom}(A, B)$, we write $f : A \rightarrow B$.
 - for every morphism f , an object $\text{Dom}(f)$ (called its **source** or **domain**), and an object $\text{Cdm}(f)$ (called its **target** or **codomain**)
 - for every pair of morphisms f and g , where $\text{Cdm}(f) = \text{Dom}(g)$, a morphism $g \circ f$, called **composite** (also written gf or some times $f; g$)
 - for every object X , a morphism id_X (or 1_X), called the **identity morphism** on X ;
- For any three objects $A, B, C \in \text{obj}(C)$, a binary operation,

$$\begin{aligned} \circ : \text{hom}(A, B) \times \text{hom}(B, C) &\longrightarrow \text{hom}(A, C) \\ (f, g) &\longmapsto g \circ f \end{aligned}$$

called **composition**, such that,

- (*associativity*)

$$f \in \text{hom}(A, B), g \in \text{hom}(B, C), h \in \text{hom}(C, D) \implies h \circ (g \circ f) = (h \circ g) \circ f;$$

- (*identity*)

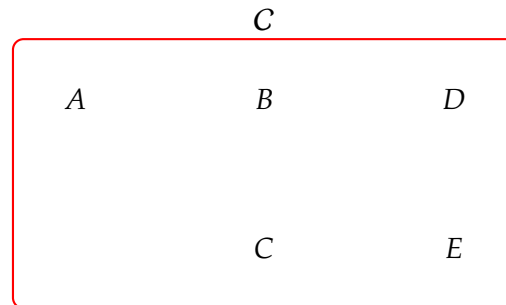
$$X \in \text{obj}(C) \implies \exists \text{id}_X \in \text{hom}(X, X) \text{ such that}$$

$$[f \in \text{hom}(A, X), g \in \text{hom}(X, B)$$

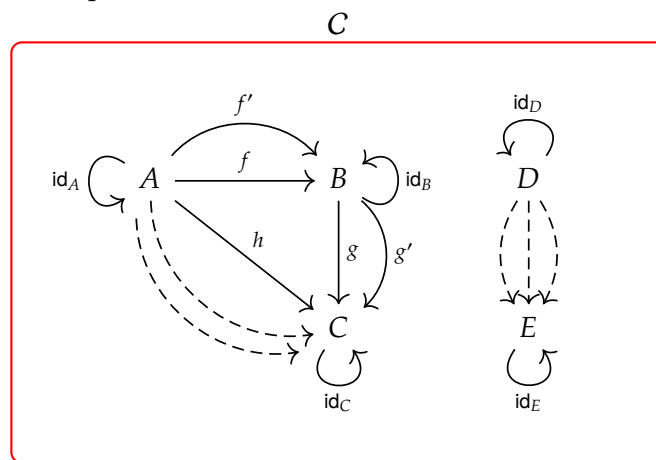
$$\implies \text{id}_X \circ f = f, g \circ \text{id}_X = g]$$

Remark 1. To describe a category it is necessary to specify:

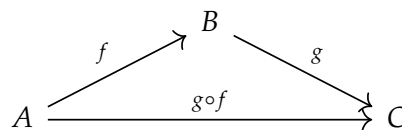
- (Objects) $\text{obj}(C)$



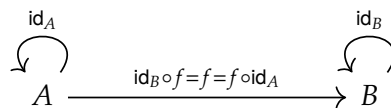
- (Morphisms; Arrows; Maps) $\text{hom}(A, B)$; $\text{hom}(A, B) \neq \text{hom}(B, A)$



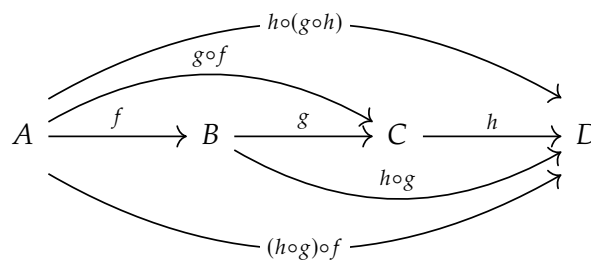
- (Composition)



- (Identity)



- (Associativity)



2 Examples

Example 1 (Trivial Category).

- $\text{obj}(C) = \{A\}$
- $\text{hom}(A, A) = \{\text{id}_A\}$

$$A \xrightarrow{\text{id}_A} A$$

Example 2.

- $\text{obj}(C) = \{A, B\}$
- $\text{hom}(A, B) = \{f\}$
- $\text{hom}(B, A) = \emptyset$

$$A \xrightarrow{f} B$$

Example 3. Let $(G, *)$ be a group.

- $\text{obj}(C) = \{X\}$
- $\text{hom}(X, X) = \{G\}$
- Define $g \circ f := g * f$

Example 4.

- **Set;**

$$\text{Set} \xrightarrow{\text{Function}} \text{Set}$$

- **Grp;**

$$\text{Group} \xrightarrow{\text{Homomorphism}} \text{Group}$$

- **Top;**

$$\text{Topological Space} \xrightarrow{\text{Continuous Map}} \text{Topological Space}$$

- **Vect_K;**

$$\text{Vector Space} \xrightarrow{\text{Linear Transformation}} \text{Vector Space}$$

Example 5.

- $f : x \rightarrow y$ if and only if $x \leq y$

$$x \xrightarrow{f} y \xrightarrow{g} z$$

$$x \xrightarrow{h} z$$

- $\text{id}_x : x \rightarrow x$ if and only if $x \leq x$

$$\begin{array}{ccc} & (\mathbb{R}, \leq) & \\ \text{Real Number} & \xrightarrow{\text{Ordering}} & \text{Real Number} \end{array}$$

3 Product and Dual Categories**3.1 Product Categories**

$$C \times \mathcal{D}$$

$$\text{obj}((C \times \mathcal{D})) = \text{obj}(C) \times \text{obj}(\mathcal{D})$$

$$\text{hom}_{C \times \mathcal{D}}((A, B), (A', B')) = \text{hom}_C(A, A') \times \text{hom}_{\mathcal{D}}(B, B')$$

$$\begin{array}{ccc} C & & \mathcal{D} \\ A \xrightarrow{f} A' & B \xrightarrow{g} B' \end{array}$$

$$\begin{array}{ccc} C \times \mathcal{D} & & \\ (A, B) \xrightarrow{(f, g)} & (A', B') \end{array}$$

3.2 Dual Categories

$$\begin{array}{ccc} C & & C^{\text{op}} \\ A \rightarrow B & A \leftarrow B \end{array}$$

4 Functors

$$F : \mathcal{C} \rightarrow \mathcal{D}$$

$$F : \text{obj}(\mathcal{C}) \rightarrow \text{obj}(\mathcal{D})$$

$$F : \text{hom}(\mathcal{C}) \rightarrow \text{hom}(\mathcal{D})$$

$$F : \mathcal{C} \rightarrow \mathcal{D}$$

$$A \mapsto F(A)$$

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow & & \\ F(A) & \xrightarrow{\quad F(f) \quad} & F(B) \end{array}$$

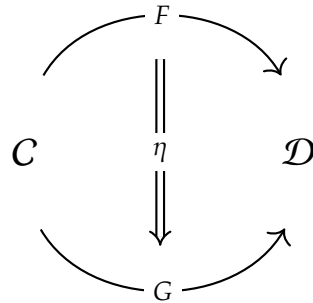
5 Natural Transformation

- Let

$$C \begin{matrix} \xrightarrow{F} \\ \xrightarrow{G} \end{matrix} \mathcal{D}$$

be categories and functors.

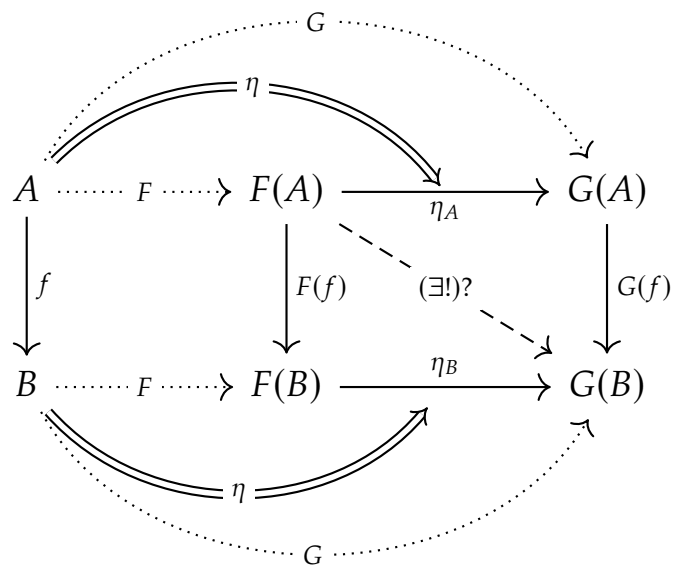
- A map



is a natural transformation

$$\begin{array}{ccc} & F(A) \in \text{obj}(\mathcal{D}) & \\ & \downarrow \eta_A & \\ A \in \text{obj}(\mathcal{C}) & \xrightarrow{\eta} & \\ & \downarrow \eta_A & \\ & G(A) \in \text{obj}(\mathcal{D}) & \end{array}$$

(Dotted arrows from A to $F(A)$ and $G(A)$ are labeled F and G respectively.)



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References

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