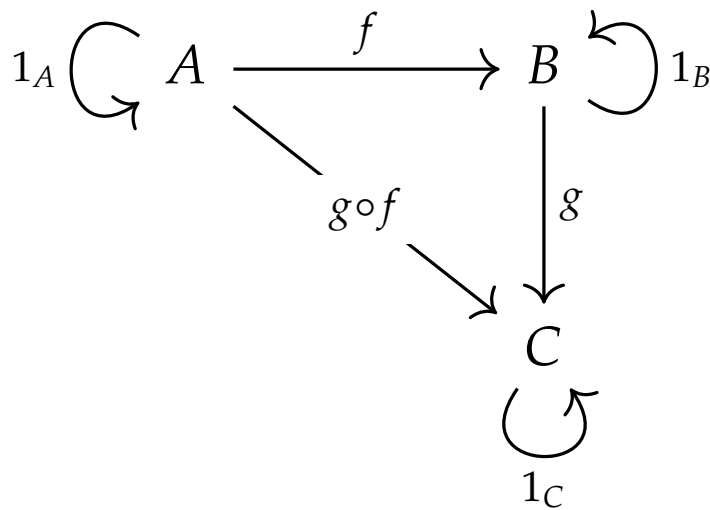


Category Theory

- A Journey from Concretization to Abstraction -

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A document presented for
the Category Theory

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Chapter 1

Categorical

Example 1.1 (Determinant).

[**Two Categories**]

1. **Category of Commutative Rings** (CRing)

- **Objects:** Commutative rings R, S, \dots
- **Morphisms:** Ring homomorphisms $\phi : R \rightarrow S$

2. **Category of Groups** (Grp)

- **Objects:** Groups
- **Morphisms:** Group homomorphisms

[**Two Functors**] Note that \mathbf{GL}_n represents the functor, while GL_n denotes the group.

1. **General Linear Group Functor** ($\mathbf{GL}_n : \mathbf{CRing} \rightarrow \mathbf{Grp}$)

- **On Objects:**

$$R \mapsto GL_n(R)$$

Maps a ring R to the general linear group $GL_n(R)$, which consists of all $n \times n$ invertible matrices over R .

- **On Morphisms:** Let $\mathbf{GL}_n(\phi) : GL_n(R) \rightarrow GL_n(S)$.

$$\phi \mapsto \mathbf{GL}_n(\phi)$$

It preserves invertibility.

2. **Unit Functor** ($\mathbf{U} : \mathbf{CRing} \rightarrow \mathbf{Grp}$)

- **On Objects:**

$$R \mapsto R^\times$$

- **On Morphisms:** Let $\mathbf{U}(\phi) : R^\times \rightarrow S^\times$.

$$\phi \mapsto \mathbf{U}(\phi)$$

It preserves the unit property.

- [**Natural Transformation: Determinant**] The determinant

$$\det_R : GL_n(R) \rightarrow R^\times$$

is a natural transformation between two functors, \mathbf{GL}_n and \mathbf{U} .

- **On Objects:** $\eta_R : GL_n(R) \rightarrow R^\times$
- **Naturality Condition:** The following diagram commutes:

$$\begin{array}{ccc}
 GL_n(R) & \xrightarrow{\eta_R} & R^\times \\
 \mathbf{GL}_n(\phi) \downarrow & & \downarrow \mathbf{U}(\phi) \\
 GL_n(S) & \xrightarrow{\eta_S} & S^\times
 \end{array}$$

$$\begin{array}{ccc}
 X & & F(X) \xrightarrow{\eta_X} G(X) \\
 f \downarrow & & \downarrow F(f) \quad \downarrow G(f) \\
 Y & & F(Y) \xrightarrow{\eta_Y} G(Y)
 \end{array}$$

$$\begin{array}{ccc}
 R & & GL_n(R) \xrightarrow{\eta_R} R^\times \\
 \phi \downarrow & & \downarrow \mathbf{GL}_n(\phi) \quad \downarrow \mathbf{U}(\phi) \\
 S & & GL_n(S) \xrightarrow{\eta_S} S^\times
 \end{array}$$

Example 1.2. Consider \mathbb{Z} and $GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc \neq 0 \right\}$.

[Two Categories]

1. Category of Commutative Rings (CRing)

- **Object:** Commutative ring $(\mathbb{Z}, +, \times)$
- **Morphism:** Ring homomorphism $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$

2. Category of Groups (Grp)

- **Objects:** Groups $(GL_2(\mathbb{Z}), *)$ and $(\mathbb{Z}^\times, \times)$. Here ‘ $*$ ’ denotes matrix multiplication. Note that

$$\mathbb{Z}^\times = \{1, -1\}.$$

- **Morphisms:** Group homomorphisms

$$\sigma : GL_2(\mathbb{Z}) \rightarrow GL_2(\mathbb{Z}) \quad \text{and} \quad \tau : \mathbb{Z}^\times \rightarrow \mathbb{Z}^\times.$$

Note that $\tau : \{\pm 1\} \rightarrow \{\pm 1\}$

[**Two Functors**] Note that \mathbf{GL}_2 represents the functor, while GL_2 denotes the group.

1. **General Linear Group Functor** ($\mathbf{GL}_2 : \mathbf{CRing} \rightarrow \mathbf{Grp}$)

– **On Object:**

$$\mathbb{Z} \mapsto GL_2(\mathbb{Z})$$

– **On Morphism:**

$$\phi \mapsto \sigma = \mathbf{GL}_2(\phi)$$

It preserves invertibility.

2. **Unit Functor** ($\mathbf{U} : \mathbf{CRing} \rightarrow \mathbf{Grp}$)

– **On Object:**

$$\mathbb{Z} \mapsto \mathbb{Z}^\times = \{\pm 1\}$$

– **On Morphism:**

$$\phi \mapsto \tau = \mathbf{U}(\phi)$$

It preserves the unit property.

- [**Natural Transformation: Determinant**] The determinant

$$\begin{aligned} \det_{\mathbb{Z}} : GL_2(\mathbb{Z}) &\longrightarrow \mathbb{Z}^\times = \{\pm 1\} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto ad - bc = \det_{\mathbb{Z}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{aligned}$$

is a natural transformation between two functors, \mathbf{GL}_2 and \mathbf{U} .

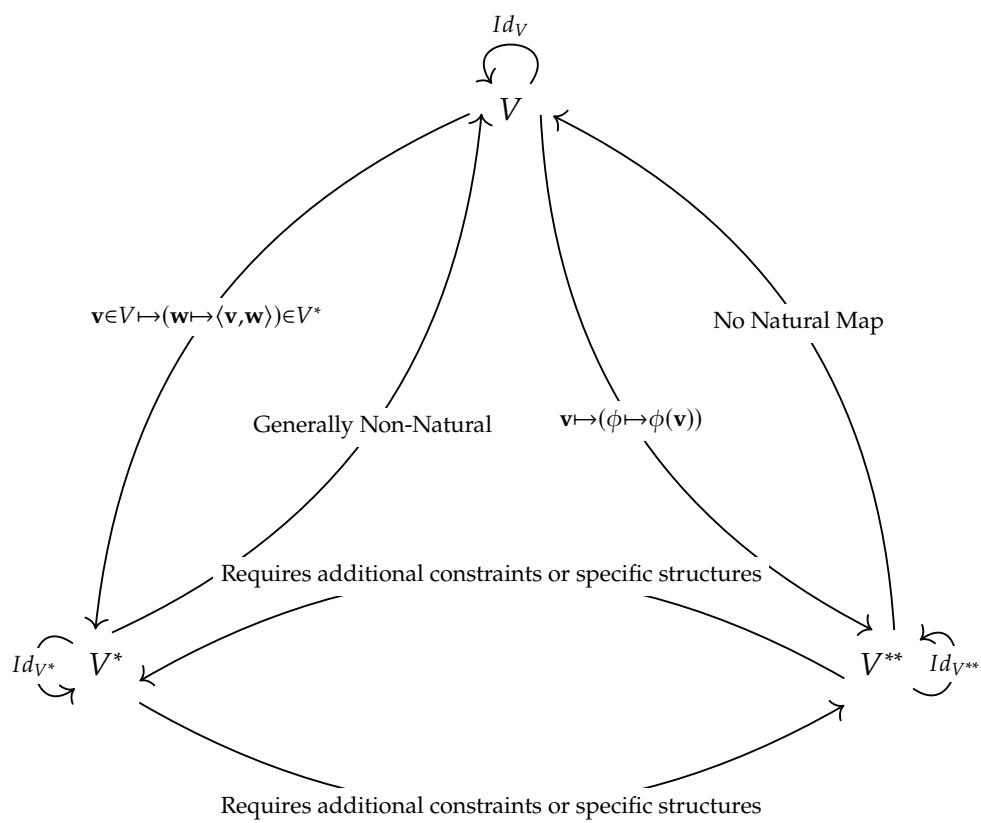
– **Naturality Condition:** The following diagram commutes:

$$\begin{array}{ccc} GL_2(\mathbb{Z}) & \xrightarrow{\eta_{\mathbb{Z}} = \det_{\mathbb{Z}}} & \mathbb{Z}^\times \\ \downarrow \sigma = \mathbf{GL}_2(\phi) & & \downarrow \tau = \mathbf{U}(\phi) \\ GL_n(\mathbb{Z}) & \xrightarrow{\eta_{\mathbb{Z}} = \det_{\mathbb{Z}}} & \mathbb{Z}^\times \end{array}$$

$$\begin{array}{ccc}
 R & GL_n(R) & \xrightarrow{\eta_R = \det_R} R^\times \\
 \phi \downarrow & \downarrow \mathbf{GL}_n(\phi) & \downarrow \mathbf{U}(\phi) \\
 S & GL_n(S) & \xrightarrow{\eta_S = \det_S} S^\times
 \end{array}$$

$$\begin{array}{ccc}
 \mathbb{Z} & GL_2(\mathbb{Z}) & \xrightarrow{\det_{\mathbb{Z}}} \mathbb{Z}^\times \\
 \phi \downarrow & \downarrow \sigma = \mathbf{GL}_2(\phi) & \downarrow \tau = \mathbf{U}(\phi) \\
 \mathbb{Z} & GL_2(\mathbb{Z}) & \xrightarrow{\det_{\mathbb{Z}}} \mathbb{Z}^\times
 \end{array}$$

$$\begin{array}{ccccc}
 & & \mathbf{U} & & \\
 & \text{GL}_2 & \curvearrowright & & \\
 \mathbb{Z} & \cdots & GL_2(\mathbb{Z}) & \xrightarrow{\det_{\mathbb{Z}}} & \mathbb{Z}^\times \\
 \downarrow \phi & \cdots & \downarrow \sigma = \mathbf{GL}_2(\phi) & & \downarrow \tau = \mathbf{U}(\phi) \\
 \mathbb{Z} & \cdots & GL_2(\mathbb{Z}) & \xrightarrow{\det_{\mathbb{Z}}} & \mathbb{Z}^\times \\
 & \text{GL}_2 & \curvearrowright & & \\
 & & \mathbf{U} & &
 \end{array}$$



Example 1.3 (Evaluation Mapping as Natural Transformation).**Definition** (*Evaluation Map*)

- Let V, K be sets, and
- let K^V be the set of all mappings from V to K .

The **evaluation mapping** for K^V is the mapping $\text{ev} : K^V \times V \rightarrow K$ defined by:

$$\text{ev}(f, v) := f(v)$$

for $f \in K^V$ and $v \in V$.

Consider a linear mapping

$$\phi : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad \phi(\mathbf{v}) = \phi(v_1, v_2) = 3v_1 - 2v_2.$$

Then evaluation map $\text{ev} : \mathbb{R}^{\mathbb{R}^2} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ ensure

$$\text{ev}(\phi, (1, 1)) = \phi(1, 1) = 3 \cdot 1 - 2 \cdot 1 = 1.$$

[Category]**1. Category of Vector Space (Vect_K)****– Object:**

- * Vector Spaces over field K : V, W, \dots
- * Dual Spaces: V^*, W^*, \dots . Note that

$$V^* := \{ \phi : V \rightarrow K \mid \phi \text{ is linear functional} \}$$

- * Double dual Spaces: V^{**}, W^{**}, \dots . Note that

$$V^{**} := \{ \psi \in V^{**} \rightarrow K \mid \psi \text{ is linear functional} \}$$

– Morphism: Ring homomorphism $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ **2. Category of Groups (Grp)**

- **Objects:** Groups $(GL_2(\mathbb{Z}), *)$ and $(\mathbb{Z}^\times, \times)$. Here ‘ $*$ ’ denotes matrix multiplication. Note that

$$\mathbb{Z}^\times = \{1, -1\}.$$

- **Morphisms:** Group homomorphisms

$$\sigma : GL_2(\mathbb{Z}) \rightarrow GL_2(\mathbb{Z}) \quad \text{and} \quad \tau : \mathbb{Z}^\times \rightarrow \mathbb{Z}^\times.$$

Note that $\tau : \{\pm 1\} \rightarrow \{\pm 1\}$

[**Two Functors**] Note that \mathbf{GL}_2 represents the functor, while GL_2 denotes the group.

1. **General Linear Group Functor** ($\mathbf{GL}_2 : \mathbf{CRing} \rightarrow \mathbf{Grp}$)

– **On Object:**

$$\mathbb{Z} \mapsto GL_2(\mathbb{Z})$$

– **On Morphism:**

$$\phi \mapsto \sigma = \mathbf{GL}_2(\phi)$$

It preserves invertibility.

2. **Unit Functor** ($\mathbf{U} : \mathbf{CRing} \rightarrow \mathbf{Grp}$)

– **On Object:**

$$\mathbb{Z} \mapsto \mathbb{Z}^\times = \{\pm 1\}$$

– **On Morphism:**

$$\phi \mapsto \tau = \mathbf{U}(\phi)$$

It preserves the unit property.

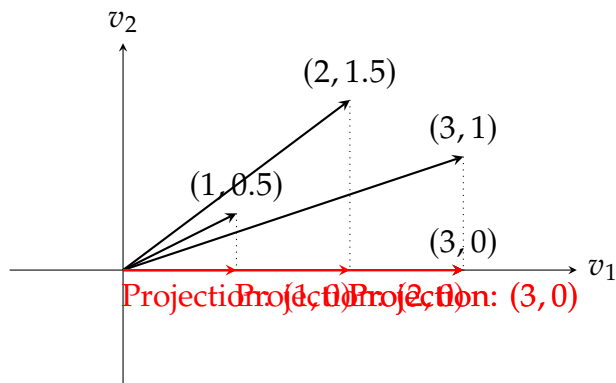
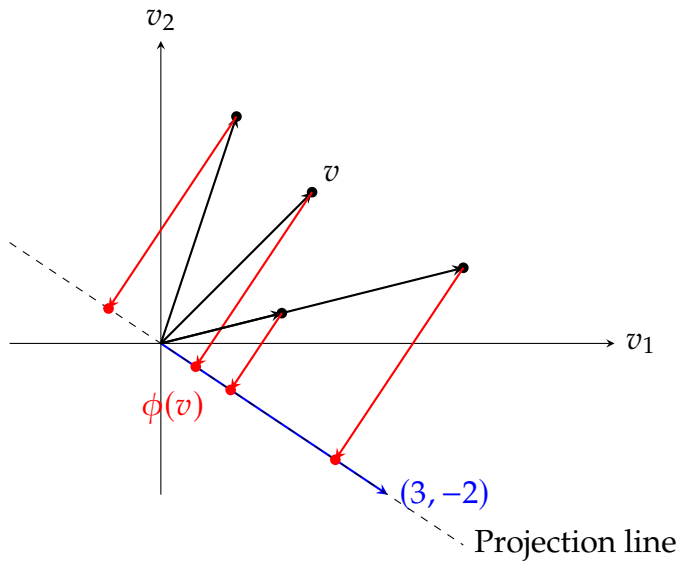
- [**Natural Transformation: Determinant**] The determinant

$$\begin{aligned} \det_{\mathbb{Z}} : GL_2(\mathbb{Z}) &\longrightarrow \mathbb{Z}^\times = \{\pm 1\} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto ad - bc = \det_{\mathbb{Z}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{aligned}$$

is a natural transformation between two functors, \mathbf{GL}_2 and \mathbf{U} .

– **Naturality Condition:** The following diagram commutes:

$$\begin{array}{ccc} GL_2(\mathbb{Z}) & \xrightarrow{\eta_{\mathbb{Z}} = \det_{\mathbb{Z}}} & \mathbb{Z}^\times \\ \downarrow \sigma = \mathbf{GL}_2(\phi) & & \downarrow \tau = \mathbf{U}(\phi) \\ GL_n(\mathbb{Z}) & \xrightarrow{\eta_{\mathbb{Z}} = \det_{\mathbb{Z}}} & \mathbb{Z}^\times \end{array}$$



$$\begin{array}{ccc}
 V & \xrightarrow{T} & W \\
 (\cdot)^* \downarrow & & \downarrow (\cdot)^* \\
 V^* & \xrightarrow{T^*} & W^* \\
 (\cdot)^* \uparrow & & \uparrow (\cdot)^* \\
 V^{**} & \xrightarrow{T^{**}} & W^{**}
 \end{array}$$

Definition 1.1 (Vector Space and Dual Space).

- Let V be a finite-dimensional *vector space* over field \mathbb{K} . That is, $V = K^n$.
- The *dual space* V^* of V consists of all linear functionals on V . That is,

$$V^* = \{\phi : V \rightarrow \mathbb{R} \mid \phi \text{ is linear}\}.$$

Definition 1.2 (Double Dual Space). The *double dual space* V^{**} of a vector space V is defined as the dual of the dual space, i.e., $V^{**} = (V^*)^*$. For each $\psi \in V^{**}$, ψ is a functional acting on elements of V^* .

Natural Isomorphism Between V and V^{**}

Theorem 1.1 (Canonical Isomorphism). *For every finite-dimensional vector space V , there is a canonical isomorphism $\eta_V : V \rightarrow V^{**}$ defined by*

$$\eta_V(v)(\phi) = \phi(v),$$

for every $v \in V$ and $\phi \in V^$. This map is a natural isomorphism, meaning it commutes with all linear transformations.*

Definition 1.3 (Induced Map on Dual and Double Dual Spaces). For a linear transformation $T : V \rightarrow W$ between vector spaces, the induced map on the dual space, $T^* : W^* \rightarrow V^*$, is defined by

$$T^*(\phi) = \phi \circ T,$$

where $\phi \in W^*$. Similarly, the induced map on the double dual space, $T^{**} : V^{**} \rightarrow W^{**}$, is defined by

$$T^{**}(\psi) = \psi \circ T^*,$$

where $\psi \in V^{**}$.

1.1 Example: Application in \mathbb{R}^2

Example 1.4 (Matrix Transformation and Double Dual). Consider the vector space $V = \mathbb{R}^2$, and let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be represented by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

We compute the induced maps T^* and T^{**} and verify the natural transformation property with η_V .

Action of T and η_V : If $\mathbf{v} = (1, 1)$, then

$$T(\mathbf{v}) = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

The action of $\eta_{\mathbb{R}^2}$ and ϕ defined by $\phi(\mathbf{v}) = 3v_1 - 2v_2$ gives

$$\eta_{\mathbb{R}^2}((1, 1))(\phi) = 1 \quad \text{and} \quad \eta_{\mathbb{R}^2}((3, 7))(\phi) = 3 \times 3 - 2 \times 7 = -5.$$

Verification of Commutativity: To confirm the natural transformation property, we need to show

$$T^{**}(\eta_{\mathbb{R}^2}(\mathbf{v})) = \eta_{\mathbb{R}^2}(T(\mathbf{v})).$$

For $\mathbf{v} = (1, 1)$, the calculations show that both sides of the equation represent consistent transformations via the isomorphism $\eta_{\mathbb{R}^2}$, proving the natural transformation.

Example 1.5.**[Category]****1. Category of Finite Vector Spaces (FinVect_K)****– Objects:**

- * Vector space $V = K^n$ over a field K
- * Dual Space $V^* = \{\phi \in K^V : \phi \text{ is linear transformation}\}$.

– Morphisms:

- * Linear Transformation $T : V(= K^n) \rightarrow W(= K^m)$
- * Dual Transformation $T^* : W^* \rightarrow V^*$ defined by

$$T^*f := f \circ T \quad \text{for all } f \in W^*.$$

Sometimes we denote $V^* := \mathcal{L}(V, K)$. Note that

- $(T^*f)\mathbf{v} = f(T\mathbf{v})$ for all $\mathbf{v} \in V$;
- $f \in W^* \implies T^*f \in V^*$;
- T^* is linear.

[Functor]**1. Dual Space Functor ($\mathcal{D} : \text{FinVect}_K \rightarrow \text{FinVect}_K$)****– On Objects:**

$$V \mapsto V^* = \mathcal{D}(V)$$

Maps a vector space V to the dual space V^* , which consists of all linear functionals from V to K .

– On Morphisms:

$$T \mapsto T^* = \mathcal{D}(T)$$

- **[Natural Transformation: Determinant]** The determinant

$$\det_R : GL_n(R) \rightarrow R^\times$$

is a natural transformation between two functors, \mathbf{GL}_n and \mathbf{U} .

1.2 Currying, Kleisli Category, Grothendieck Construction

1.2.1 Currying

Note.

$$A^{B \times C} \simeq (A^C)^B$$

$$(f : B \times C \rightarrow A) \mapsto \left(g : \begin{array}{ccc} B & \longrightarrow & [C \rightarrow A] \\ b \in B & \longmapsto & g_b \in A^C \end{array} \right).$$

Consider $f : X \times Y \rightarrow Z$. Then

$$\begin{aligned} f : X \times Y \rightarrow Z &\implies f \in Z^{X \times Y} \\ &\implies f \in (Z^Y)^X, \text{ i.e., } x \in X \mapsto f_x \in Z^Y \\ &\implies f : X \rightarrow [Y \rightarrow Z]. \end{aligned}$$

```
def f1(x):
    return (x + 1, str(x) + "+1")
```

```
def f2(x):
    return (x + 2, str(x) + "+2")
```

```
def f3(x):
    return (x + 3, str(x) + "+3")
```

```
def unit(x):
    return (x, "Ops:")
```

```
def bind(t, f):
    res = f(t[0])
    return (res[0], t[1] + res[1] + ";")
```

```
print( bind(bind(bind(unit(x), f1), f2), f3) )
```

$$\begin{array}{ccccccc} & & g \circ f & & & & \\ & \nearrow & & \searrow & & & \\ X & \xrightarrow{f} & TY & \xrightarrow{Tg} & T^2Z & \xrightarrow{\mu_Z} & TZ \end{array}$$

Bibliography

- [1] ProofWiki. "Definition:Set of All Mappings" Accessed on [April 29, 2024]. https://proofwiki.org/wiki/Definition:Set_of_All_Mappings.