Category Theory

- A Journey from Concretization to Abstraction -

Ji, Yong-Hyeon

$$
\begin{tikzcd}
A \arrow[loop left, "1_A"] \arrow[r, "f"] \arrow[dr, "g \circ f" description] & B \arrow[loop right, "1_B"] \arrow[d, "g"] \\
& C \arrow[loop below, "1_C"]
\end{tikzcd}
$$

A document presented for
the Category Theory

Department of Information Security, Cryptology, and Mathematics
College of Science and Technology
Kookmin University

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Contents

Chapter 1

Category Theory

Category

Definition 1.1. A **category** $\mathcal{C}$ consists of the following components:

- a class of *objects*, denoted by $\mathsf{Obj}(\mathcal{C})$; and
- a class of *morphisms* (also called *arrows*) from A to B, for any objects A and B, is denoted by $\mathrm{Hom}_{\mathcal{C}}(A, B)$.

A category $\mathcal{C}$ satisfies the following three axioms:

1. (Composition of Morphisms) For any objects A, B and C and any morphisms $f : A \longrightarrow B$, $g : B \longrightarrow C$, there exists a **composite morphism** $g \circ f : A \longrightarrow C$ in the category $\mathcal{C}$.

$$\boxed{\forall A, B, C \in \mathsf{Obj}(\mathcal{C}) : \forall f \in \mathsf{Hom}_{\mathcal{C}}(A, B) : \forall g \in \mathsf{Hom}_{\mathcal{C}}(B, C) : \exists g \circ f \in \mathsf{Hom}_{\mathcal{C}}(A, C).}$$

2. (Identity Morphisms) For every object $A \in \mathsf{Obj}(\mathcal{C})$, there exists the **identity morphism** $\mathsf{id}_A \in \mathsf{Hom}_{\mathcal{C}}(A, A)$ such that for any morphism $f : A \longrightarrow B$ and $g : B \longrightarrow A$, $\mathsf{id}_A \circ f = f$ and $g \circ \mathsf{id}_A = g$.

$$\boxed{\forall A, B \in \mathsf{Obj}(\mathcal{C}) : \forall f \in \mathrm{Hom}_{\mathcal{C}}(A, B) : \forall g \in \mathrm{Hom}_{\mathcal{C}}(B, A) : \mathrm{id}_A \circ f = f \wedge g \circ \mathrm{id}_A = g}$$

3. (Associativity of Composition) The composition of morphisms must be associative. That is:

$$\boxed{\forall f \in \mathrm{Hom}_{\mathcal{C}}(A, B) : \forall g \in \mathrm{Hom}_{\mathcal{C}}(B, C) : \forall h \in \mathrm{Hom}_{\mathcal{C}}(C, D) : h \circ (g \circ f) = (h \circ g) \circ f}$$

Remark 1.1.

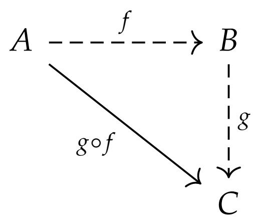

Figure 1.1: Composition of Morphisms

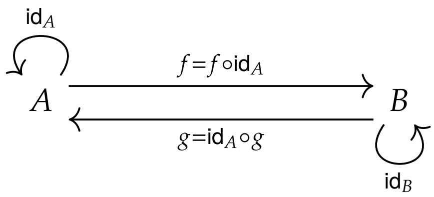

Figure 1.2: Identity Morphisms

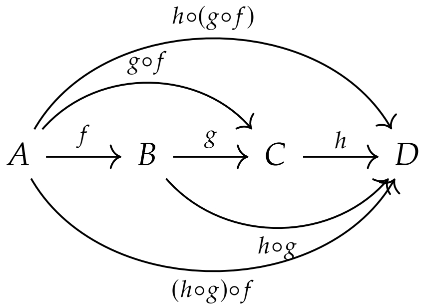

Figure 1.3: Associativity of Composition

Category

A **category** C consists of the following data:

- **Objects**: A collection of objects, denoted $\mathrm{Ob}(C)$.
- **Morphisms**: For each pair of objects $A, B \in \mathrm{Ob}(C)$, there is a set $\mathrm{Hom}_C(A, B)$ of morphisms from A to B. If $f \in \mathrm{Hom}_C(A, B)$, we write $f : A \to B$.
- **Composition**: For any three objects $A, B, C \in \mathrm{Ob}(C)$, there is a binary operation

$$\circ : \mathrm{Hom}_C(B, C) \times \mathrm{Hom}_C(A, B) \to \mathrm{Hom}_C(A, C),$$

  which assigns to each pair of morphisms $f : A \to B$ and $g : B \to C$ a morphism $g \circ f : A \to C$, called their composition.
- **Identity Morphisms**: For each object $A \in \mathrm{Ob}(C)$, there exists a morphism $\mathrm{id}_A \in \mathrm{Hom}_C(A, A)$, such that for any morphism $f : A \to B$,

$$\mathrm{id}_B \circ f = f \quad \text{and} \quad f \circ \mathrm{id}_A = f.$$

- **Associativity**: For all morphisms $f : A \to B$, $g : B \to C$, and $h : C \to D$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Symbolically, a category C is written as:

$$C = \big(\mathrm{Ob}(C), \mathrm{Hom}_C, \circ, \mathrm{id}\big)$$

where $\circ$ is the composition operation, and id represents the identity morphisms.

Functor

Given two categories $\mathcal{C}$ and $\mathcal{D}$, a **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of:

- **Object mapping**: For each object $A \in \mathrm{Ob}(\mathcal{C})$, there is an object $F(A) \in \mathrm{Ob}(\mathcal{D})$.
- **Morphisms mapping**: For each morphism $f : A \to B$ in $\mathcal{C}$, there is a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$.

These assignments must satisfy the following properties:

- **Preservation of Identity**: For each object $A \in \mathcal{C}$,

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}.$$

- **Preservation of Composition**: For any pair of morphisms $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$,

$$F(g \circ f) = F(g) \circ F(f).$$

Symbolically, a functor F from $\mathcal{C}$ to $\mathcal{D}$ can be written as:

$$F : \mathcal{C} \to \mathcal{D}, \quad \big(A \mapsto F(A), \quad f \mapsto F(f)\big)$$

with the conditions $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(g \circ f) = F(g) \circ F(f)$.

Natural Transformation

Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a **natural transformation** $\eta : F \Rightarrow G$ is a collection of morphisms $\eta_A : F(A) \to G(A)$ in $\mathcal{D}$, one for each object $A \in \mathcal{C}$, such that for every morphism $f : A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow \eta_A & & \downarrow \eta_B \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

In other words, for every morphism $f : A \to B$ in $\mathcal{C}$, the following relation holds in $\mathcal{D}$:

$$\eta_B \circ F(f) = G(f) \circ \eta_A.$$

Symbolically, a natural transformation $\eta : F \Rightarrow G$ is a family of morphisms $\{\eta_A\}_{A \in \mathcal{C}}$, such that for all $f : A \to B$,

$$\eta_B \circ F(f) = G(f) \circ \eta_A.$$

$$
\begin{tikzcd}
R \arrow[d, "\phi"'] & & GL_n(R) \arrow[r, "\eta_R = \det_R"] \arrow[d, "\mathbf{GL}_n(\phi)"] & R^{\times} \arrow[d, "\mathbf{U}(\phi)"] \\
S & & GL_n(S) \arrow[r, "\eta_S = \det_S"] & S^{\times}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{Z} \arrow[d, "\phi"'] & & GL_2(\mathbb{Z}) \arrow[r, "\det_{\mathbb{Z}}"] \arrow[d, "\sigma = \mathbf{GL}_2(\phi)"] & \mathbb{Z}^{\times} \arrow[d, "\tau = \mathbf{U}(\phi)"] \\
\mathbb{Z} & & GL_2(\mathbb{Z}) \arrow[r, "\det_{\mathbb{Z}}"] & \mathbb{Z}^{\times}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{Z} \arrow[d, "\phi"', ""{name=P}] \arrow[r, dotted, bend left, "\mathbf{GL}_2"] \arrow[rr, dotted, bend left=60, "\mathbf{U}"] & GL_2(\mathbb{Z}) \arrow[r, "\det_{\mathbb{Z}}"] \arrow[d, "\sigma = \mathbf{GL}_2(\phi)", ""{name=S}] & \mathbb{Z}^{\times} \arrow[d, "\tau = \mathbf{U}(\phi)", ""'{name=T}] \\
\mathbb{Z} \arrow[r, dotted, bend right, "\mathbf{GL}_2"'] \arrow[rr, dotted, bend right=60, "\mathbf{U}"'] & GL_2(\mathbb{Z}) \arrow[r, "\det_{\mathbb{Z}}"] & \mathbb{Z}^{\times}
\arrow[from=P, to=S, dotted, "\mathbf{GL}_2"]
\arrow[from=P, to=T, dotted, "\mathbf{U}"]
\end{tikzcd}
$$