Category Theory

May 7, 2024

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1 Definition

Category

A **category** $\mathcal{C}$ consists of

- A class $\mathsf{obj}(\mathcal{C})$ of **objects** $A, B, C, \ldots$ in $\mathcal{C}$
- For all (ordered) pairs of objects $X, Y \in \mathsf{obj}(\mathcal{C})$, a class $\mathsf{hom}(A, B)$ of **maps** or **arrow** called **morphisms** from A to B, called the **homo-set** of morphisms from A to B. If $f \in \mathsf{hom}(A, B)$, we write $f : A \longrightarrow B$.
  - for every morphism f, an object $\mathsf{Dom}(f)$ (called its **source** or **domain**), and an object $\mathsf{Cdm}(f)$ (called its **target** or **codomain**)
  - for every pair of morphisms f and g, where $\mathsf{Cdm}(f) = \mathsf{Dom}(g)$, a morphism $g \circ f$, called **composite** (also written gf or some times $f; g$)
  - for every object X, a morphism id_X (or 1_X), called the **identity morphism** on X;
- For any three objects $A, B, C \in \mathsf{obj}(\mathcal{C})$, a binary operation,

$$\begin{array}{cccc} \circ & : & \mathsf{hom}(A, B) \times \mathsf{hom}(B, C) & \longrightarrow & \mathsf{hom}(A, C) \\ & & (f, g) & \longmapsto & g \circ f \end{array},$$

  called **composition**, such that,
  - (*associativity*)

$$f \in \mathsf{hom}(A, B), g \in \mathsf{hom}(B, C), h \in \mathsf{hom}(C, D) \implies h \circ (g \circ f) = (h \circ g) \circ f;$$

  - (*identity*)

$$\begin{aligned} X \in \mathsf{obj}(\mathcal{C}) \implies & \exists \mathsf{id}_X \in \mathsf{hom}(X, X) \text{ such that} \\ & [f \in \mathsf{hom}(A, X), g \in \mathsf{hom}(X, B) \\ & \implies \mathsf{id}_X \circ f = f, g \circ \mathsf{id}_X = g] \end{aligned}$$

Remark 1. To describe a acategory it is necessary to specify:

- (Objects) $\mathrm{obj}\,(\mathcal{C}) = \{A, B, C, D, E \dots\}$

$$\mathcal{C}$$

$$\begin{tikzcd}
A & B & D \\
 & C & E
\end{tikzcd}$$

- (Morphisms) $\mathrm{hom}\,(A, B) = \{f, f', \dots\}$; $\mathrm{hom}\,(A, B) \neq \mathrm{hom}\,(B, A)$

$$\mathcal{C}$$

$$\begin{tikzcd}
A \arrow[loop left, "\mathrm{id}_A"] \arrow[r, "f"] \arrow[r, bend left, "f'"] \arrow[dr, "h"] \arrow[dr, dashed, bend right] \arrow[dr, dashed, bend right=60] & B \arrow[loop right, "\mathrm{id}_B"] \arrow[d, "g"] \arrow[d, bend left, "g'"] & D \arrow[loop above, "\mathrm{id}_D"] \arrow[d, dashed] \arrow[d, dashed, bend left] \arrow[d, dashed, bend right] \\
 & C \arrow[loop below, "\mathrm{id}_C"] & E \arrow[loop below, "\mathrm{id}_E"]
\end{tikzcd}$$

- (Composition)

$$\begin{tikzcd}
 & B \arrow[dr, "g"] & \\
A \arrow[ur, "f"] \arrow[rr, "g \circ f"] & & C
\end{tikzcd}$$

- (Identity)

$$\begin{tikzcd}
A \arrow[loop above, "\mathrm{id}_A"] \arrow[r, "\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A"] & B \arrow[loop above, "\mathrm{id}_B"]
\end{tikzcd}$$

- (Associativity)

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] \arrow[rrr, bend left=50, "h \circ (g \circ h)"] \arrow[rrr, bend right=50, "(h \circ g) \circ f"'] & B \arrow[r, "g"] \arrow[rr, bend right, "h \circ g"'] & C \arrow[r, "h"] & D
\end{tikzcd}$$

2 Examples

Example 1 (Trivial Category).

- $\mathsf{obj}\,(C) = \{A\}$
- $\mathsf{hom}\,(A, A) = \{\mathsf{id}_A\}$

$$
\begin{tikzcd}
A \arrow[loop right, "\mathsf{id}_A"]
\end{tikzcd}
$$

Example 2.

- $\mathsf{obj}\,(C) = \{A, B\}$
- $\mathsf{hom}\,(A, B) = \{f\}$
- $\mathsf{hom}\,(B, A) = \emptyset$

$$A \xrightarrow{f} B$$

Example 3. Let $(G, *)$ be a group.

- $\mathsf{obj}\,(C) = \{X\}$
- $\mathsf{hom}\,(X, X) = \{G\}$
- Define $g \circ f := g * f$

Example 4.

- **Set**;

$$\text{Set} \xrightarrow[\text{Function}]{} \text{Set}$$

- **Grp**;

$$\text{Group} \xrightarrow[\text{Homomorphism}]{} \text{Group}$$

- **Top**;

$$\text{Topological Space} \xrightarrow[\text{Continuous Map}]{} \text{Topological Space}$$

- $\textbf{Vect}_K$;

$$\text{Vector Space} \xrightarrow[\text{Linear Transformation}]{} \text{Vector Space}$$

Example 5.

- $f : x \longrightarrow y$ if and only if $x \leq y$

$$x \xrightarrow{\quad f \quad} y \xrightarrow{\quad g \quad} z$$

$$x \xrightarrow{\qquad\qquad h \qquad\qquad} z$$

- $\mathsf{id}_x : x \longrightarrow x$ if and only if $x \leq x$

$$\text{Real Number} \xrightarrow[\text{Ordering}]{(\mathbb{R}, \leq)} \text{Real Number}$$

3 Product and Dual Categories

3.1 Product Categories

$$\mathcal{C} \times \mathcal{D}$$

$$\mathsf{obj}\left((\mathcal{C} \times \mathcal{D})\right) = \mathsf{obj}\,(\mathcal{C}) \times \mathsf{obj}\,(\mathcal{D})$$

$$\mathsf{hom}_{\mathcal{C}\times\mathcal{D}}((A, B), (A', B')) = \mathsf{hom}_{\mathcal{C}}(A, A') \times \mathsf{hom}_{\mathcal{D}}(B, B')$$

$$\begin{array}{cc} \mathcal{C} & \mathcal{D} \\ A \xrightarrow{f} A' & B \xrightarrow{g} B' \end{array}$$

$$\begin{array}{c} \mathcal{C} \times \mathcal{D} \\ (A, B) \xrightarrow{(f,g)} (A', B') \end{array}$$

3.2 Dual Categories

$$\begin{array}{cc} \mathcal{C} & \mathcal{C}^{\mathrm{op}} \\ A \longrightarrow B & A \longleftarrow B \end{array}$$

4 Functors

$$
\begin{aligned}
F &: \mathcal{C} \to \mathcal{D} \\
F &: \mathsf{obj}\,(\mathcal{C}) \to \mathsf{obj}\,(\mathcal{D}) \\
F &: \mathsf{hom}\,(\mathcal{C}) \to \mathsf{hom}\,(\mathcal{D})
\end{aligned}
$$

$$
\begin{array}{rcccc}
F & : & \mathcal{C} & \longrightarrow & \mathcal{D} \\
 & & A & \longmapsto & F(A)
\end{array}
$$

$$
\begin{tikzcd}
A \arrow[rr, "f"] & {} \arrow[d, mapsto] & B \\
F(A) \arrow[rr, "F(f)"'] & {} & F(B)
\end{tikzcd}
$$

5 Natural Transformation

- Let

$$\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}$$

  be categories and functors.

- A map

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \end{tikzcd}$$

  is a natural transformation

$$\begin{tikzcd} & F(A) \in \mathsf{obj}\,(\mathcal{D}) \arrow[dd, "\eta_A"] \\ A \in \mathsf{obj}\,(\mathcal{C}) \arrow[ur, dotted, "F"] \arrow[dr, dotted, "G"'] \arrow[r, Rightarrow, "\eta"] & {} \\ & G(A) \in \mathsf{obj}\,(\mathcal{D}) \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, dotted, "F"] \arrow[d, "f"] \arrow[rr, dotted, bend left=60, "G"] \arrow[rr, Rightarrow, bend left=40, "\eta"] & F(A) \arrow[r, "\eta_A"'] \arrow[d, "F(f)"] \arrow[dr, dashed, "(\exists!)?"] & G(A) \arrow[d, "G(f)"] \\ B \arrow[r, dotted, "F"] \arrow[rr, dotted, bend right=60, "G"'] \arrow[rr, Rightarrow, bend right=40, "\eta"'] & F(B) \arrow[r, "\eta_B"] & G(B) \end{tikzcd}$$

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References

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