

# Category Theory

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## 1 Definition

### Category

A **category**  $C$  consists of

- A class  $\text{obj}(C)$  of **objects**  $A, B, C, \dots$  in  $C$
- For all (ordered) pairs of objects  $X, Y \in \text{obj}(C)$ , a class  $\text{hom}(A, B)$  of **maps** or **arrow** called **morphisms** from  $A$  to  $B$ , called the **homo-set** of morphisms from  $A$  to  $B$ . If  $f \in \text{hom}(A, B)$ , we write  $f : A \rightarrow B$ .
  - for every morphism  $f$ , an object  $\text{Dom}(f)$  (called its **source** or **domain**), and an object  $\text{Cdm}(f)$  (called its **target** or **codomain**)
  - for every pair of morphisms  $f$  and  $g$ , where  $\text{Cdm}(f) = \text{Dom}(g)$ , a morphism  $g \circ f$ , called **composite** (also written  $gf$  or some times  $f; g$ )
  - for every object  $X$ , a morphism  $\text{id}_X$  (or  $1_X$ ), called the **identity morphism** on  $X$ ;
- For any three objects  $A, B, C \in \text{obj}(C)$ , a binary operation,

$$\begin{aligned} \circ : \text{hom}(A, B) \times \text{hom}(B, C) &\longrightarrow \text{hom}(A, C) \\ (f, g) &\longmapsto g \circ f \end{aligned},$$

called **composition**, such that,

- (*associativity*)

$$f \in \text{hom}(A, B), g \in \text{hom}(B, C), h \in \text{hom}(C, D) \implies h \circ (g \circ f) = (h \circ g) \circ f;$$

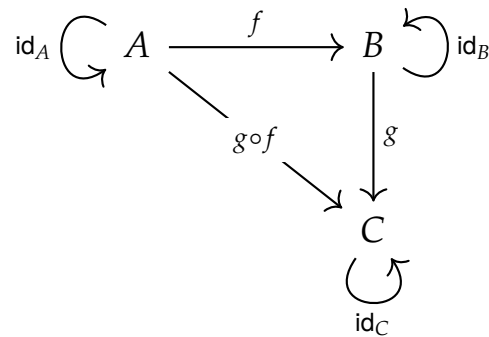
- (*identity*)

$$X \in \text{obj}(C) \implies \exists \text{id}_X \in \text{hom}(X, X) \text{ such that}$$

$$[f \in \text{hom}(A, X), g \in \text{hom}(X, B)$$

$$\implies \text{id}_X \circ f = f, g \circ \text{id}_X = g]$$

**Remark 1.**



To describe a category it is necessary to specify:

- (Objects)  $\text{obj}(C) = \{A, B, C, \dots\}$
- (Morphisms)  $\text{hom}(A, B) = \{f, g, \dots\}$ ;  $\text{hom}(A, B) \neq \text{hom}(B, A)$
- (Composition)

$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$A \xrightarrow{h=g \circ f} C$$

- (Identity)

$$\text{id}_A : A \rightarrow A$$

$$- f \circ \text{id}_A = f$$

$$- \text{id}_A \circ f = f$$

## 2 Examples

**Example 1** (Trivial Category).

- $\text{obj}(C) = \{A\}$
- $\text{hom}(A, A) = \{\text{id}_A\}$

$$A \xrightarrow{\text{id}_A} A$$

**Example 2.**

- $\text{obj}(C) = \{A, B\}$
- $\text{hom}(A, B) = \{f\}$
- $\text{hom}(B, A) = \emptyset$

$$A \xrightarrow{f} B$$

**Example 3.** Let  $(G, *)$  be a group.

- $\text{obj}(C) = \{X\}$
- $\text{hom}(X, X) = \{G\}$
- Define  $g \circ f := g * f$

**Example 4.**

- **Set**;

$$\text{Set} \xrightarrow{\text{Function}} \text{Set}$$

- **Grp**;

$$\text{Group} \xrightarrow{\text{Homomorphism}} \text{Group}$$

- **Top**;

$$\text{Topological Space} \xrightarrow{\text{Continuous Map}} \text{Topological Space}$$

- **Vect<sub>K</sub>**;

$$\text{Vector Space} \xrightarrow{\text{Linear Transformation}} \text{Vector Space}$$

**Example 5.**

- $f : x \rightarrow y$  if and only if  $x \leq y$

$$x \xrightarrow{f} y \xrightarrow{g} z$$

$$x \xrightarrow{h} z$$

- $\text{id}_x : x \rightarrow x$  if and only if  $x \leq x$

$$\begin{array}{ccc} & (\mathbb{R}, \leq) & \\ \text{Real Number} & \xrightarrow{\text{Ordering}} & \text{Real Number} \end{array}$$

**3 Product and Dual Categories****3.1 Product Categories**

$$C \times \mathcal{D}$$

$$\text{obj}((C \times \mathcal{D})) = \text{obj}(C) \times \text{obj}(\mathcal{D})$$

$$\text{hom}_{C \times \mathcal{D}}((A, B), (A', B')) = \text{hom}_C(A, A') \times \text{hom}_{\mathcal{D}}(B, B')$$

$$\begin{array}{ccc} C & & \mathcal{D} \\ A \xrightarrow{f} A' & B \xrightarrow{g} & B' \end{array}$$

$$\begin{array}{ccc} C \times \mathcal{D} & & \\ (A, B) \xrightarrow{(f, g)} & (A', B') \end{array}$$

**3.2 Dual Categories**

$$\begin{array}{ccc} C & & C^{\text{op}} \\ A \rightarrow B & A \leftarrow & B \end{array}$$

## 4 Functors

$$F : \mathcal{C} \rightarrow \mathcal{D}$$

$$F : \text{obj}(\mathcal{C}) \rightarrow \text{obj}(\mathcal{D})$$

$$F : \text{hom}(\mathcal{C}) \rightarrow \text{hom}(\mathcal{D})$$

$$F : \mathcal{C} \rightarrow \mathcal{D}$$

$$A \mapsto F(A)$$

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & \downarrow & \\ F(A) & \xrightarrow{\quad F(f) \quad} & F(B) \end{array}$$

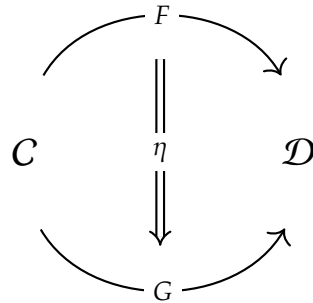
## 5 Natural Transformation

- Let

$$C \begin{matrix} \xrightarrow{F} \\ \xleftarrow{G} \end{matrix} \mathcal{D}$$

be categories and functors.

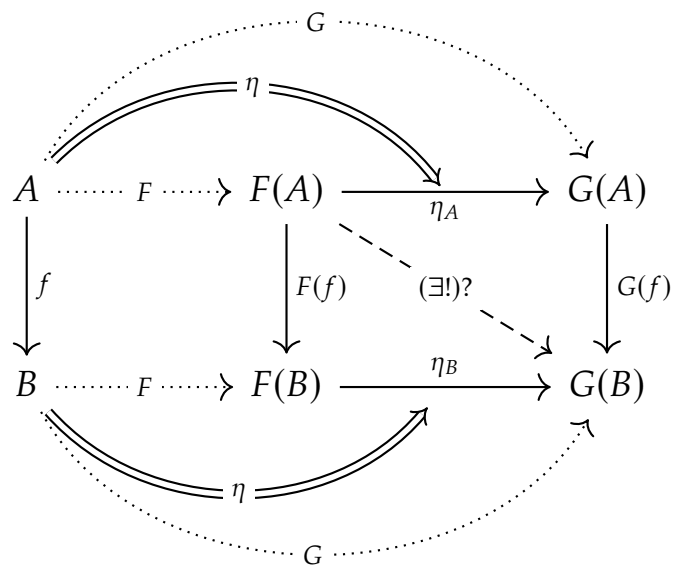
- A map



is a natural transformation

$$\begin{array}{ccc} & F(A) \in \text{obj}(\mathcal{D}) & \\ & \downarrow \eta_A & \\ A \in \text{obj}(\mathcal{C}) & \xrightarrow{\eta} & \\ & \downarrow \eta_A & \\ & G(A) \in \text{obj}(\mathcal{D}) & \end{array}$$

(Dotted arrows from  $A$  to  $F(A)$  and  $G(A)$  are labeled  $F$  and  $G$  respectively.)



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## References

- [1] “Intro to Category Theory” YouTube, uploaded by Warwick Mathematics Exchange, 1 Feb 2023, <https://www.youtube.com/watch?v=AUD2Rpoy604>
- [2] ProofWiki. “Definition:Metacategory” Accessed on [May 05, 2024]. <https://proofwiki.org/wiki/Definition:Metacategory>.
- [3] nLab. “category” Accessed on [May 05, 2024]. <https://ncatlab.org/nlab/show/category#Grothendieck61>.