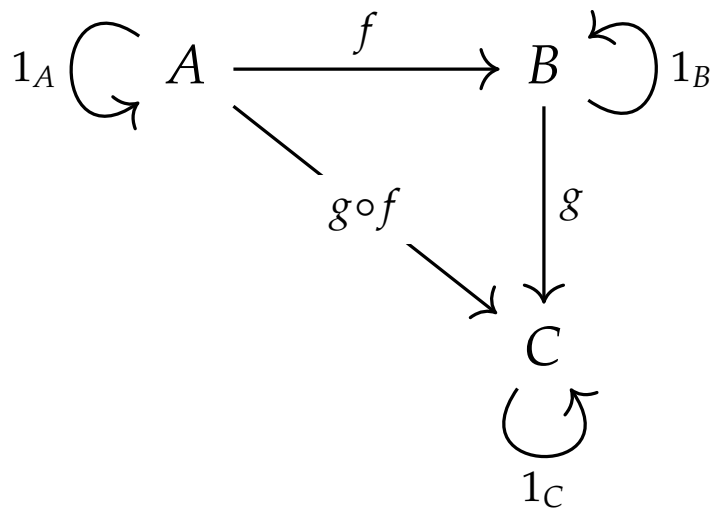# Category Theory

- A Journey from Concretization to Abstraction -

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A document presented for
the Category Theory

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Chapter 1

Categorical

Example 1.1 (Determinant)**.**

    [Two Categories]

      1. **Category of Commutative Rings** (CRing)
   - **Objects**: Commutative rings $R, S, \ldots$
   - **Morphisms**: Ring homomorphisms $\phi : R \to S$

      2. **Category of Groups** (Grp)
   - **Objects**: Groups
   - **Morphisms**: Group homomorphisms

    [Two Functors] Note that $\mathbf{GL}_n$ represents the functor, while GL_n denotes the group.

      1. **General Linear Group Functor** ($\mathbf{GL}_n$: CRing $\to$ Grp)
   - **On Objects**:
$$R \mapsto GL_n(R)$$
   Maps a ring R to the general linear group $GL_n(R)$, which consists of all $n \times n$ invertible matrices over R.
   - **On Morphisms**: Let $\mathbf{GL}_n(\phi) : GL_n(R) \to GL_n(S)$.
$$\phi \mapsto \mathbf{GL}_n(\phi)$$
   It preserves invertibility.

      2. **Unit Functor** ($\mathbf{U}$: CRing $\to$ Grp)
   - **On Objects**:
$$R \mapsto R^\times$$
   - **On Morphisms**: Let $\mathbf{U}(\phi) : R^\times \to S^\times$.
$$\phi \mapsto \mathbf{U}(\phi)$$
   It preserves the unit property.

- [**Natural Transformation: Determinant**] The determinant

$$\det_R : GL_n(R) \to R^\times$$

  is a natural transformation between two functors, $\mathbf{GL}_n$ and $\mathbf{U}$.

  - **On Objects**: $\eta_R : GL_n(R) \to R^\times$
  - **Naturality Condition**: The following diagram commutes:

$$
\begin{array}{ccc}
GL_n(R) & \xrightarrow{\ \eta_R\ } & R^\times \\
\downarrow{\scriptstyle \mathbf{GL_n}(\phi)} & & \downarrow{\scriptstyle \mathbf{U}(\phi)} \\
GL_n(S) & \xrightarrow{\ \eta_S\ } & S^\times
\end{array}
$$

$$
\begin{array}{ccccc}
X & & F(X) & \xrightarrow{\ \eta_X\ } & G(X) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\
Y & & F(Y) & \xrightarrow{\ \eta_Y\ } & G(Y)
\end{array}
$$

$$
\begin{array}{ccccc}
R & & GL_n(R) & \xrightarrow{\ \eta_R\ } & R^\times \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathbf{GL_n}(\phi)} & & \downarrow{\scriptstyle \mathbf{U}(\phi)} \\
S & & GL_n(S) & \xrightarrow{\ \eta_S\ } & S^\times
\end{array}
$$

Example 1.2. Consider $\mathbb{Z}$ and $GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc \neq 0 \right\}$.

[**Two Categories**]

1. **Category of Commutative Rings** (CRing)
   - **Object**: Commutative ring $(\mathbb{Z}, +, \times)$
   - **Morphism**: Ring homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}$
2. **Category of Groups** (Grp)
   - **Objects**: Groups $(GL_2(\mathbb{Z}), *)$ and $(\mathbb{Z}^\times, \times)$. Here '$*$' denotes matrix multiplication. Note that

$$\mathbb{Z}^\times = \{1, -1\}.$$

   - **Morphisms**: Group homomorphisms

$$\sigma : GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}) \quad \text{and} \quad \tau : \mathbb{Z}^\times \to \mathbb{Z}^\times.$$

     Note that $\tau : \{\pm 1\} \to \{\pm 1\}$

[**Two Functors**] Note that $\mathbf{GL}_2$ represents the functor, while GL_2 denotes the group.

1. **General Linear Group Functor** ($\mathbf{GL}_2 : \mathsf{CRing} \to \mathsf{Grp}$)
   - **On Object**:
     $$\mathbb{Z} \mapsto GL_2(\mathbb{Z})$$
   - **On Morphism**:
     $$\phi \mapsto \sigma = \mathbf{GL}_2(\phi)$$

   It preserves invertibility.

2. **Unit Functor** ($\mathbf{U} : \mathsf{CRing} \to \mathsf{Grp}$)
   - **On Object**:
     $$\mathbb{Z} \mapsto \mathbb{Z}^{\times} = \{\pm 1\}$$
   - **On Morphism**:
     $$\phi \mapsto \tau = \mathbf{U}(\phi)$$

   It preserves the unit property.

- [**Natural Transformation: Determinant**] The determinant

$$\det_{\mathbb{Z}} \; : \; GL_2(\mathbb{Z}) \longrightarrow \mathbb{Z}^{\times} = \{\pm 1\}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto ad - bc = \det_{\mathbb{Z}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

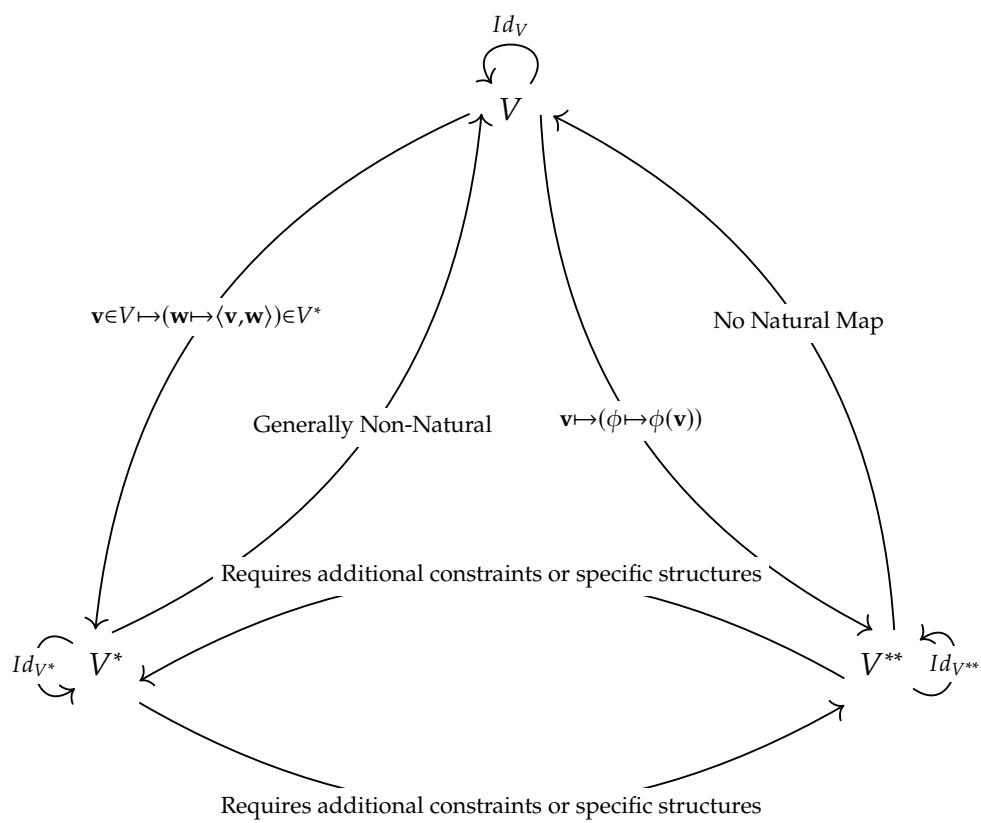is a natural transformation between two functors, $\mathbf{GL}_2$ and $\mathbf{U}$.

- **Naturality Condition**: The following diagram commutes:

$$
\begin{array}{ccc}
GL_2(\mathbb{Z}) & \xrightarrow{\;\eta_{\mathbb{Z}} = \det_{\mathbb{Z}}\;} & \mathbb{Z}^{\times} \\
\downarrow{\scriptstyle \sigma = \mathbf{GL}_2(\phi)} & & \downarrow{\scriptstyle \tau = \mathbf{U}(\phi)} \\
GL_n(\mathbb{Z}) & \xrightarrow{\;\eta_{\mathbb{Z}} = \det_{\mathbb{Z}}\;} & \mathbb{Z}^{\times}
\end{array}
$$

$$
\begin{array}{ccc}
R & \qquad GL_n(R) \xrightarrow{\ \eta_R = \det_R\ } R^\times \\[2em]
\downarrow{\scriptstyle\phi} & \qquad \downarrow{\scriptstyle\mathbf{GL}_n(\phi)} \qquad\qquad \downarrow{\scriptstyle\mathbf{U}(\phi)} \\[2em]
S & \qquad GL_n(S) \xrightarrow{\ \eta_S = \det_S\ } S^\times
\end{array}
$$

$$
\begin{array}{ccc}
\mathbb{Z} & \qquad GL_2(\mathbb{Z}) \xrightarrow{\ \det_{\mathbb{Z}}\ } \mathbb{Z}^\times \\[2em]
\downarrow{\scriptstyle\phi} & \qquad \downarrow{\scriptstyle\sigma = \mathbf{GL}_2(\phi)} \qquad\qquad \downarrow{\scriptstyle\tau = \mathbf{U}(\phi)} \\[2em]
\mathbb{Z} & \qquad GL_2(\mathbb{Z}) \xrightarrow{\ \det_{\mathbb{Z}}\ } \mathbb{Z}^\times
\end{array}
$$

Example 1.3 (Evaluation Mapping as Natural Transformation)**.**

> **Definition** (*Evaluation Map*)
>
> - Let V, K be sets, and
>
> - let K^V be the set of all mappings from V to K.
>
> The **evaluation mapping** for K^V is the mapping $\mathsf{ev} : K^V \times V \to K$ defined by:
>
> $$\mathsf{ev}(f, v) := f(v)$$
>
> for $f \in K^V$ and $v \in V$.

Consider a linear mapping

$$\phi : \mathbb{R}^2 \to \mathbb{R}, \quad \phi(\mathbf{v}) = \phi(v_1, v_2) = 3v_1 - 2v_2.$$

Then evaluation map $\mathsf{ev} : \mathbb{R}^{\mathbb{R}^2} \times \mathbb{R}^2 \to \mathbb{R}$ ensure

$$\mathsf{ev}(\phi, (1, 1)) = \phi(1, 1) = 3 \cdot 1 - 2 \cdot 1 = 1.$$

[Category]

1. **Category of Vector Space** (Vect_K)
   - **Object**:
     * Vector Spaces over field K: $V, W, \cdots$
     * Dual Spaces: $V^*, W^*, \cdots$. Note that

       $$V^* := \big\{ \phi : V \to K \mid \phi \text{ is linear functional} \big\}$$

     * Double dual Spaces: $V^{**}, W^{**}, \cdots$. Note that

       $$V^{**} := \big\{ \psi \in V^{**} \to K \mid \psi \text{ is linear functional} \big\}$$

   - **Morphism**: Ring homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}$
2. **Category of Groups** (Grp)
   - **Objects**: Groups $(GL_2(\mathbb{Z}), *)$ and $(\mathbb{Z}^\times, \times)$. Here '$*$' denotes matrix multiplication. Note that
     $$\mathbb{Z}^\times = \{1, -1\}.$$
   - **Morphisms**: Group homomorphisms

     $$\sigma : GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}) \quad \text{and} \quad \tau : \mathbb{Z}^\times \to \mathbb{Z}^\times.$$

     Note that $\tau : \{\pm 1\} \to \{\pm 1\}$

[**Two Functors**] Note that $\mathbf{GL}_2$ represents the functor, while GL_2 denotes the group.

1. **General Linear Group Functor** ($\mathbf{GL}_2$: CRing $\to$ Grp)
    - **On Object**:
$$\mathbb{Z} \mapsto GL_2(\mathbb{Z})$$
    - **On Morphism**:
$$\phi \mapsto \sigma = \mathbf{GL}_2(\phi)$$
    It preserves invertibility.

2. **Unit Functor** ($\mathbf{U}$: CRing $\to$ Grp)
    - **On Object**:
$$\mathbb{Z} \mapsto \mathbb{Z}^\times = \{\pm 1\}$$
    - **On Morphism**:
$$\phi \mapsto \tau = \mathbf{U}(\phi)$$
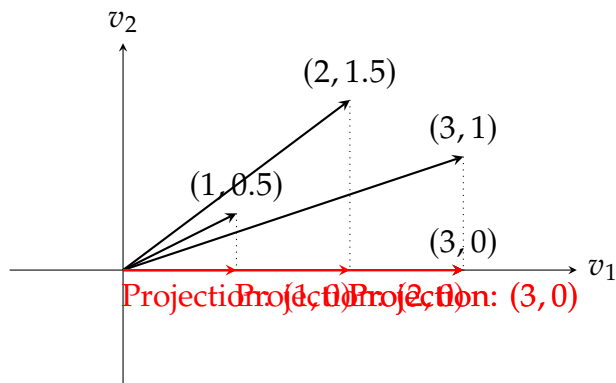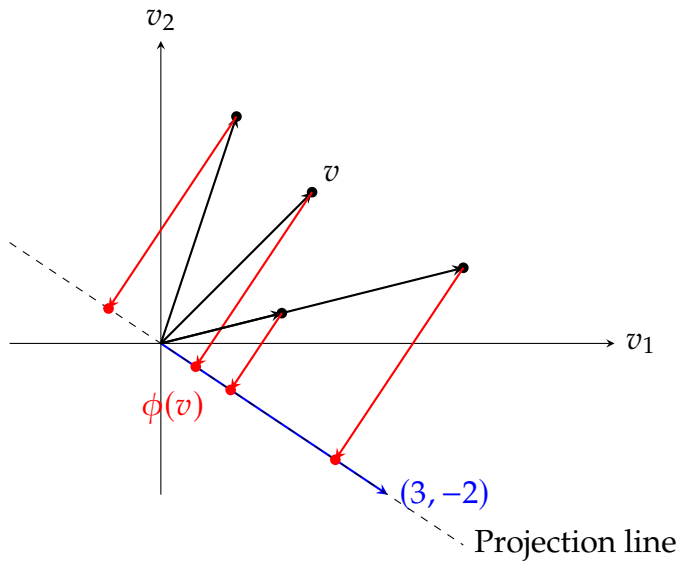   It preserves the unit property.

- [**Natural Transformation: Determinant**] The determinant

$$\begin{aligned} \det_{\mathbb{Z}} \quad &: \quad GL_2(\mathbb{Z}) \quad \longrightarrow \quad \mathbb{Z}^\times = \{\pm 1\} \\ &\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longmapsto \quad ad - bc = \det_{\mathbb{Z}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{aligned}$$

is a natural transformation between two functors, $\mathbf{GL}_2$ and $\mathbf{U}$.

- **Naturality Condition**: The following diagram commutes:

$$
\begin{array}{ccc}
GL_2(\mathbb{Z}) & \xrightarrow{\;\eta_{\mathbb{Z}}=\det_{\mathbb{Z}}\;} & \mathbb{Z}^\times \\
\Big\downarrow{\scriptstyle \sigma=\mathbf{GL}_2(\phi)} & & \Big\downarrow{\scriptstyle \tau=\mathbf{U}(\phi)} \\
GL_n(\mathbb{Z}) & \xrightarrow{\;\eta_{\mathbb{Z}}=\det_{\mathbb{Z}}\;} & \mathbb{Z}^\times
\end{array}
$$

Definition 1.1 (Vector Space and Dual Space).

- Let V be a finite-dimensional *vector space* over field $\mathbb{K}$. That is, $V = K^n$.

- The *dual space V^** of V consists of all linear functionals on V. That is,

$$V^* = \{\phi : V \to \mathbb{R} \mid \phi \text{ is linear}\}.$$

Definition 1.2 (Double Dual Space). The *double dual space V^{**}* of a vector space V is defined as the dual of the dual space, i.e., $V^{**} = (V^*)^*$. For each $\psi \in V^{**}$, ψ is a functional acting on elements of V^*.

Natural Isomorphism Between V and V^{**}

Theorem 1.1 (Canonical Isomorphism). *For every finite-dimensional vector space V, there is a canonical isomorphism $\eta_V : V \to V^{**}$ defined by*

$$\eta_V(v)(\phi) = \phi(v),$$

for every $v \in V$ and $\phi \in V^*$. This map is a natural isomorphism, meaning it commutes with all linear transformations.

Definition 1.3 (Induced Map on Dual and Double Dual Spaces). For a linear transformation $T : V \to W$ between vector spaces, the induced map on the dual space, $T^* : W^* \to V^*$, is defined by

$$T^*(\phi) = \phi \circ T,$$

where $\phi \in W^*$. Similarly, the induced map on the double dual space, $T^{**} : V^{**} \to W^{**}$, is defined by

$$T^{**}(\psi) = \psi \circ T^*,$$

where $\psi \in V^{**}$.

1.1   Example: Application in $\mathbb{R}^2$

Example 1.4 (Matrix Transformation and Double Dual). Consider the vector space $V = \mathbb{R}^2$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be represented by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

We compute the induced maps T^* and T^{**} and verify the natural transformation property with η_V.

Action of T and η_V: If $\mathbf{v} = (1, 1)$, then

$$T(\mathbf{v}) = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

The action of $\eta_{\mathbb{R}^2}$ and ϕ defined by $\phi(\mathbf{v}) = 3v_1 - 2v_2$ gives

$$\eta_{\mathbb{R}^2}((1, 1))(\phi) = 1 \quad \text{and} \quad \eta_{\mathbb{R}^2}((3, 7))(\phi) = 3 \times 3 - 2 \times 7 = -5.$$

Verification of Commutativity: To confirm the natural transformation property, we need to show

$$T^{**}(\eta_{\mathbb{R}^2}(\mathbf{v})) = \eta_{\mathbb{R}^2}(T(\mathbf{v})).$$

For $\mathbf{v} = (1, 1)$, the calculations show that both sides of the equation represent consistent transformations via the isomorphism $\eta_{\mathbb{R}^2}$, proving the natural transformation.

Example 1.5.

[Category]

Category of Finite Vector Spaces (Vect_K)
- **Objects**:
  * Vector space V over a field K
  * Dual Space

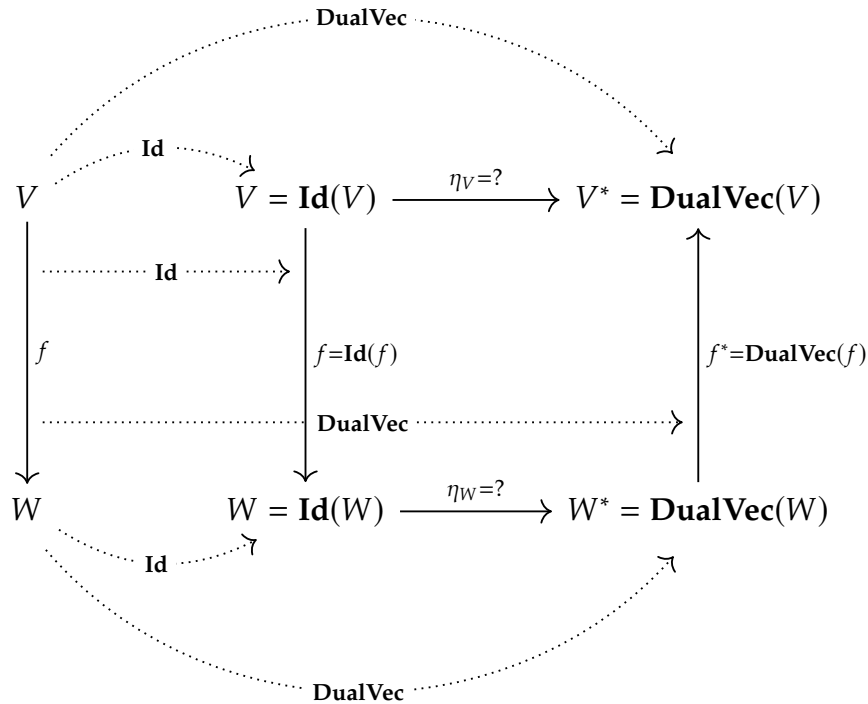$$V^* = \left\{ \phi \in K^V : \phi \text{ is linear transformation} \right\} \subseteq (V \times K).$$

- **Morphisms**:
  * $f : V \to W$
  * $f^* : W^* \to V^*$ defined by

$$f^*(\phi_W) = (\phi_W \circ f) \in V^*$$

  for $\phi_W \in W^*$.
  * Let $f : V \to W$ and $g : W \to U$.
  * $f^* : W^* \to V^*, g^* : U^* \to W^*$
  * **DualVec**$(g \circ f) = $ **DualVec**$(f) \circ $ **DualVec**$(g) = f^* \circ g^*$

[Functor]

1. **Dual Space Functor** ($\mathcal{D} : \mathsf{FinVect}_K \to \mathsf{FinVect}_K$)
   - **On Objects**:
$$V \mapsto V^* = \mathcal{D}(V)$$

   Maps a vector space V to the dual space V^*, which which consists of all linear functionals from V to K.

- **On Morphisms**:

$$T \mapsto T^* = \mathcal{D}(T)$$

- **[Natural Transformation: Determinant]** The determinant

$$\det_R : GL_n(R) \to R^\times$$

is a natural transformation between two functors, $\mathbf{GL}_n$ and $\mathbf{U}$.

1.2   Currying, Kleisli Category, Grothendieck Construction

1.2.1   Currying

Note.

$$A^{B \times C} \simeq (A^C)^B$$

$$(f : B \times C \to A) \; \underset{\leftarrow}{\to} \; \left(g \quad : \quad \begin{array}{rcl} B & \longrightarrow & [C \to A] \\ b \in B & \longmapsto & g_b \in A^C \end{array} \right) .$$

Consider $f : X \times Y \to Z$. Then

$$\begin{aligned} f : X \times Y \to Z & \implies f \in Z^{X \times Y} \\ & \implies f \in (Z^Y)^X, \text{i.e., } x \in X \mapsto f_x \in Z^Y \\ & \implies f : X \to [Y \to Z]. \end{aligned}$$

1.2.2 Kleisli Category

> **Kleisli Category**
>
> **Definition 1.4.** The Kleisli

[Category]

Category C_{Py} of Python integers with functions between them as morphisms.

* **Obj**

```python
def f1(x):
    return (x + 1, str(x) + "+1")

def f2(x):
    return (x + 2, str(x) + "+2")

def f3(x):
    return (x + 3, str(x) + "+3")

def unit(x):
    return (x, "Ops:")

def bind(t, f):
    res = f(t[0])
    return (res[0], t[1] + res[1] + ";")

print( bind(bind(bind(unit(x), f1), f2), f3) )
```

$$X \xrightarrow{\ f\ } TY \xrightarrow{\ Tg\ } T^2Z \xrightarrow{\ \mu_Z\ } TZ$$

with the arc labelled $g \circ f$ from X to T^2Z.

Bibliography

[1] ProofWiki. "Definition:Set of All Mappings" Accessed on [April 29, 2024]. `https://proofwiki.org/wiki/Definition:Set_of_All_Mappings`.