Category Theory

- A Journey from Concretization to Abstraction -

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$$\begin{tikzcd}
A \arrow[loop left, "1_A"] \arrow[r, "f"] \arrow[dr, "g\circ f"] & B \arrow[loop right, "1_B"] \arrow[d, "g"] \\
& C \arrow[loop below, "1_C"]
\end{tikzcd}$$

A document presented for
the Category Theory

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April 23, 2024

Contents

Chapter 1

None

Example 1.1 (Determinant)**.**

[Two Categories]

1. **Category of Commutative Rings** (CRing)
   - **Objects**: Commutative rings $R, S, \ldots$
   - **Morphisms**: Ring homomorphisms $\phi : R \to S$
2. **Category of Groups** (Grp)
   - **Objects**: Groups
   - **Morphisms**: Group homomorphisms

[Two Functors] Note that $\mathbf{GL}_n$ represents the functor, while GL_n denotes the group.

1. **General Linear Group Functor** ($\mathbf{GL}_n : \mathsf{CRing} \to \mathsf{Grp}$)
   - **On Objects**:
   $$R \mapsto GL_n(R)$$
   Maps a ring R to the general linear group $GL_n(R)$, which consists of all $n \times n$ invertible matrices over R.
   - **On Morphisms**: Let $\mathbf{GL}_n(\phi) : GL_n(R) \to GL_n(S)$.
   $$\phi \mapsto \mathbf{GL}_n(\phi)$$
   It preserves invertibility.
2. **Unit Functor** ($\mathbf{U} : \mathsf{CRing} \to \mathsf{Grp}$)
   - **On Objects**:
   $$R \mapsto R^{\times}$$
   - **On Morphisms**: Let $\mathbf{U}(\phi) : R^{\times} \to S^{\times}$.
   $$\phi \mapsto \mathbf{U}(\phi)$$
   It preserves the unit property.

- **[Natural Transformation: Determinant]** The determinant

$$\det_R : GL_n(R) \to R^\times$$

is a natural transformation between two functors, $\mathbf{GL}_n$ and $\mathbf{U}$.

  - **On Objects**: $\eta_R : GL_n(R) \to R^\times$
  - **Naturality Condition**: The following diagram commutes:

$$\begin{tikzcd} GL_n(R) \arrow[r, "\eta_R"] \arrow[d, "\mathbf{GL_n}(\phi)"'] & R^\times \arrow[d, "\mathbf{U}(\phi)"] \\ GL_n(S) \arrow[r, "\eta_S"] & S^\times \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[d, "f"'] & F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ Y & F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

$$\begin{tikzcd} R \arrow[d, "\phi"'] & GL_n(R) \arrow[r, "\eta_R"] \arrow[d, "\mathbf{GL_n}(\phi)"'] & R^\times \arrow[d, "\mathbf{U}(\phi)"] \\ S & GL_n(S) \arrow[r, "\eta_S"] & S^\times \end{tikzcd}$$

Example 1.2. Consider $\mathbb{Z}$ and $GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc \neq 0 \right\}$.

[Two Categories]

1. **Category of Commutative Rings** (CRing)
   - **Object**: Commutative ring $(\mathbb{Z}, +, \times)$
   - **Morphism**: Ring homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}$
2. **Category of Groups** (Grp)
   - **Objects**: Groups $(GL_2(\mathbb{Z}), *)$ and $(\mathbb{Z}^\times, \times)$. Here '$*$' denotes matrix multiplication. Note that
   $$\mathbb{Z}^\times = \{1, -1\}.$$
   - **Morphisms**: Group homomorphisms
   $$\sigma : GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}) \quad \text{and} \quad \tau : \mathbb{Z}^\times \to \mathbb{Z}^\times.$$

   Note that $\tau : \{\pm 1\} \to \{\pm 1\}$

[Two Functors] Note that $\mathbf{GL}_2$ represents the functor, while GL_2 denotes the group.

1. **General Linear Group Functor** ($\mathbf{GL}_2 : \mathsf{CRing} \to \mathsf{Grp}$)
   - **On Object**:
   $$\mathbb{Z} \mapsto GL_2(\mathbb{Z})$$
   - **On Morphism**:
   $$\phi \mapsto \sigma = \mathbf{GL}_2(\phi)$$
   It preserves invertibility.
2. **Unit Functor** ($\mathbf{U} : \mathsf{CRing} \to \mathsf{Grp}$)
   - **On Object**:
   $$\mathbb{Z} \mapsto \mathbb{Z}^\times = \{\pm 1\}$$
   - **On Morphism**:
   $$\phi \mapsto \tau = \mathbf{U}(\phi)$$
   It preserves the unit property.

- **[Natural Transformation: Determinant]** The determinant

$$\begin{array}{rccc} \det_{\mathbb{Z}} & : & GL_2(\mathbb{Z}) & \longrightarrow & \mathbb{Z}^\times = \{\pm 1\} \\ & & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & ad - bc = \det_{\mathbb{Z}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

is a natural transformation between two functors, $\mathbf{GL}_2$ and $\mathbf{U}$.

- **Naturality Condition**: The following diagram commutes:

$$\begin{tikzcd} GL_2(\mathbb{Z}) \arrow[r, "\eta_{\mathbb{Z}}=\det_{\mathbb{Z}}"] \arrow[d, "\sigma=\mathbf{GL}_2(\phi)"] & \mathbb{Z}^\times \arrow[d, "\tau=\mathbf{U}(\phi)"] \\ GL_n(\mathbb{Z}) \arrow[r, "\eta_{\mathbb{Z}}=\det_{\mathbb{Z}}"] & \mathbb{Z}^\times \end{tikzcd}$$

$$\begin{tikzcd} R \arrow[d, "\phi"'] & GL_n(R) \arrow[r, "\eta_R=\det_R"] \arrow[d, "\mathbf{GL}_n(\phi)"] & R^\times \arrow[d, "\mathbf{U}(\phi)"] \\ S & GL_n(S) \arrow[r, "\eta_S=\det_S"] & S^\times \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z} \arrow[d, "\phi"'] & GL_2(\mathbb{Z}) \arrow[r, "\det_{\mathbb{Z}}"] \arrow[d, "\sigma=\mathbf{GL}_2(\phi)"] & \mathbb{Z}^\times \arrow[d, "\tau=\mathbf{U}(\phi)"] \\ \mathbb{Z} & GL_2(\mathbb{Z}) \arrow[r, "\det_{\mathbb{Z}}"] & \mathbb{Z}^\times \end{tikzcd}$$

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r, dotted, bend left, "\mathbf{GL}_2"] \arrow[rr, dotted, bend left=60, "\mathbf{U}"] \arrow[d, "\phi"'] & GL_2(\mathbb{Z}) \arrow[r, "\det_{\mathbb{Z}}"] \arrow[d, "\sigma=\mathbf{GL}_2(\phi)"] & \mathbb{Z}^{\times} \arrow[d, "\tau=\mathbf{U}(\phi)"] \\
\mathbb{Z} \arrow[r, dotted, bend right, "\mathbf{GL}_2"'] \arrow[rr, dotted, bend right=60, "\mathbf{U}"'] & GL_2(\mathbb{Z}) \arrow[r, "\det_{\mathbb{Z}}"] & \mathbb{Z}^{\times}
\end{tikzcd}
$$

1.1 Category Definitions

1.1.1 Matrices over Commutative Rings

Define the category **Mat_n_(CRing)** where:

- Objects are $n \times n$ matrices over any commutative ring R.
- Morphisms are ring homomorphisms applied element-wise to matrices.

1.1.2 Determinant Functor

The functor **Det**: **Mat_n_(CRing)** $\to$ **Grp** is defined by:

- On Objects: Mapping A to the group formed under multiplication by $\det(A)$.
- On Morphisms: If $f : R \to S$ is a ring homomorphism, then $\det(f(A)) = f(\det(A))$, maintaining the structure of group morphisms.

1.2 Properties and Transformations

1.2.1 Functor Properties

The determinant functor preserves:

- Composition: $\det(AB) = \det(A)\det(B)$
- Identities: $\det(I) = 1$, where I is the identity matrix.

1.2.2 Natural Transformations

Consider a transformation scaling each element of matrix A by a unit u in R. The determinant changes as:

$$\tau_A : \det(A) \mapsto u^n \det(A)$$

where n is the matrix dimension, reflecting the multiplicative scaling in the group.

1.3 Conclusion

Through the lens of category theory, the determinant integrates the algebraic structures of commutative rings and groups, enhancing our understanding of its invariant and multiplicative properties.