Category Theory

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1 Definition

Category

A **category** $\mathcal{C}$ consists of

- A class $\mathsf{obj}(\mathcal{C})$ of **objects** $A, B, C, \ldots$ in $\mathcal{C}$
- For all (ordered) pairs of objects $X, Y \in \mathsf{obj}(\mathcal{C})$, a class $\mathsf{hom}(A, B)$ of **maps** or **arrow** called **morphisms** from A to B, called the **homo-set** of morphisms from A to B. If $f \in \mathsf{hom}(A, B)$, we write $f : A \to B$.
  - for every morphism f, an object $\mathsf{Dom}(f)$ (called its **source** or **domain**), and an object $\mathsf{Cdm}(f)$ (called its **target** or **codomain**)
  - for every pair of morphisms f and g, where $\mathsf{Cdm}(f) = \mathsf{Dom}(g)$, a morphism $g \circ f$, called **composite** (also written gf or some times $f; g$)
  - for every object X, a morphism id_X (or 1_X), called the **identity morphism** on X;
- For any three objects $A, B, C \in \mathsf{obj}(\mathcal{C})$, a binary operation,

$$\begin{array}{cccc} \circ & : & \mathsf{hom}(A,B) \times \mathsf{hom}(B,C) & \longrightarrow & \mathsf{hom}(A,C) \\ & & (f,g) & \longmapsto & g \circ f \end{array},$$

  called **composition**, such that,

  - (*associativity*)

$$f \in \mathsf{hom}(A,B), g \in \mathsf{hom}(B,C), h \in \mathsf{hom}(C,D) \implies h \circ (g \circ f) = (h \circ g) \circ f;$$

  - (*identity*)

$$\begin{aligned} X \in \mathsf{obj}(\mathcal{C}) \implies & \exists \mathsf{id}_X \in \mathsf{hom}(X,X) \text{ such that} \\ & [f \in \mathsf{hom}(A,X), g \in \mathsf{hom}(X,B) \\ & \implies \mathsf{id}_X \circ f = f, g \circ \mathsf{id}_X = g] \end{aligned}$$

Remark 1.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[loop left, "\mathrm{id}_A"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \arrow[loop right, "\mathrm{id}_B"] \\
& C \arrow[loop below, "\mathrm{id}_C"]
\end{tikzcd}$$

To describe a acategory it is necessary to specify:

- (Objects) $\mathrm{obj}\,(\mathcal{C}) = \{A, B, C, \dots\}$
- (Morphisms) $\mathrm{hom}\,(A, B) = \left\{f, g, \dots\right\}$; $\mathrm{hom}\,(A, B) \neq \mathrm{hom}\,(B, A)$
- (Composition)

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C$$

$$A \xrightarrow{\qquad h = g\circ f \qquad} C$$

- (Identity)

$$\mathrm{id}_A : A \longrightarrow A$$

  - $f \circ \mathrm{id}_A = f$
  - $\mathrm{id}_A \circ f = f$

2 Examples

Example 1 (Trivial Category).

- $\mathrm{obj}(\mathcal{C}) = \{A\}$
- $\hom(A, A) = \{\mathrm{id}_A\}$

$$A \circlearrowright \mathrm{id}_A$$

Example 2.

- $\mathrm{obj}(\mathcal{C}) = \{A, B\}$
- $\hom(A, B) = \{f\}$
- $\hom(B, A) = \emptyset$

$$A \xrightarrow{f} B$$

Example 3. Let $(G, *)$ be a group.

- $\mathrm{obj}(\mathcal{C}) = \{X\}$
- $\hom(X, X) = \{G\}$
- Define $g \circ f := g * f$

Example 4.

- **Set**;

$$\text{Set} \xrightarrow[\text{Function}]{} \text{Set}$$

- **Grp**;

$$\text{Group} \xrightarrow[\text{Homomorphism}]{} \text{Group}$$

- **Top**;

$$\text{Topological Space} \xrightarrow[\text{Continuous Map}]{} \text{Topological Space}$$

- $\mathbf{Vect}_K$;

$$\text{Vector Space} \xrightarrow[\text{Linear Transformation}]{} \text{Vector Space}$$

Example 5.

- $f : x \longrightarrow y$ if and only if $x \leq y$

$$x \xrightarrow{\quad f \quad} y \xrightarrow{\quad g \quad} z$$

$$x \xrightarrow{\qquad\qquad h \qquad\qquad} z$$

- $\mathsf{id}_x : x \longrightarrow x$ if and only if $x \leq x$

$$\text{Real Number} \xrightarrow[\text{Ordering}]{(\mathbb{R}, \leq)} \text{Real Number}$$

3 Product and Dual Categories

3.1 Product Categories

$$\mathcal{C} \times \mathcal{D}$$

$$\mathsf{obj}\left((\mathcal{C} \times \mathcal{D})\right) = \mathsf{obj}\,(\mathcal{C}) \times \mathsf{obj}\,(\mathcal{D})$$
$$\mathsf{hom}_{\mathcal{C}\times\mathcal{D}}((A, B), (A', B')) = \mathsf{hom}_{\mathcal{C}}(A, A') \times \mathsf{hom}_{\mathcal{D}}(B, B')$$

$$\begin{array}{cc} \mathcal{C} & \mathcal{D} \\ A \xrightarrow{f} A' & B \xrightarrow{g} B' \end{array}$$

$$\begin{array}{c} \mathcal{C} \times \mathcal{D} \\ (A, B) \xrightarrow{(f,g)} (A', B') \end{array}$$

3.2 Dual Categories

$$\begin{array}{cc} \mathcal{C} & \mathcal{C}^{\text{op}} \\ A \longrightarrow B & A \longleftarrow B \end{array}$$

4 Functors

$$
\begin{aligned}
F &: \mathcal{C} \to \mathcal{D} \\
F &: \mathsf{obj}\,(\mathcal{C}) \to \mathsf{obj}\,(\mathcal{D}) \\
F &: \mathsf{hom}\,(\mathcal{C}) \to \mathsf{hom}\,(\mathcal{D})
\end{aligned}
$$

$$
\begin{array}{cccc}
F : & \mathcal{C} & \longrightarrow & \mathcal{D} \\
& A & \longmapsto & F(A)
\end{array}
$$

$$
\begin{tikzcd}
A \arrow[rr, "f"] & {} \arrow[d, mapsto] & B \\
F(A) \arrow[rr, "F(f)"'] & {} & F(B)
\end{tikzcd}
$$

5 Natural Transformation

- Let

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathcal{D} \end{tikzcd}$$

  be categories and functors.

- A map

$$\begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=60, "F"{name=U}] \arrow[r, bend right=60, "G"'{name=D}] & \mathcal{D} \arrow[Rightarrow, from=U, to=D, "\eta"] \end{tikzcd}$$

  is a natural transformation

$$\begin{tikzcd} & F(A) \in \mathsf{obj}\,(\mathcal{D}) \arrow[d, "\eta_A"] \\ A \in \mathsf{obj}\,(\mathcal{C}) \arrow[ur, dotted, "F"] \arrow[dr, dotted, "G"'] \arrow[r, Rightarrow, "\eta"] & {} \\ & G(A) \in \mathsf{obj}\,(\mathcal{D}) \end{tikzcd}$$

$$\begin{tikzcd}[column sep=large] A \arrow[r, dotted, "F"] \arrow[d, "f"] \arrow[rr, bend left=60, dotted, "G"] \arrow[r, Rightarrow, bend left=45, "\eta", to path={}] & F(A) \arrow[r, "\eta_A"'] \arrow[d, "F(f)"] \arrow[dr, dashed, "(\exists!)?"] & G(A) \arrow[d, "G(f)"] \\ B \arrow[r, dotted, "F"] \arrow[rr, bend right=60, dotted, "G"'] & F(B) \arrow[r, "\eta_B"] & G(B) \end{tikzcd}$$

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References

[1] "Intro to Category Theory" YouTube, uploaded by Warwick Mathematics Exchange, 1 Feb 2023, https://www.youtube.com/watch?v=AUD2Rpoy604

[2] ProofWiki. "Definition:Metacategory" Accessed on [May 05, 2024]. https://proofwiki.org/wiki/Definition:Metacategory.

[3] nLab. "category" Accessed on [May 05, 2024]. https://ncatlab.org/nlab/show/category#Grothendieck61.