Category Theory

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Category

A **category** $\mathcal{C}$ consists of

- a class $\mathcal{C}_0 = \mathsf{Obj}(\mathcal{C})$ of **objects** $X, Y, Z, \ldots$
- a class $\mathcal{C}_1 = \mathsf{Mor}(\mathcal{C})$ of **morphisms** (or **arrow**) $f, g, h, \ldots$ between its objects
  - for every morphism f, an object $\mathsf{Dom}(f)$ (called its **source** or **domain**), and an object $\mathsf{Cdm}(f)$ (called its **target** or **codomain**)
  - for every pair of morphisms f and g, where $\mathsf{Cdm}(f) = \mathsf{Dom}(g)$, a morphism $g \circ f$, called **composite** (also written gf or some times $f; g$)
  - for every object X, a morphism id_X (or 1_X), called the **identity morphism** on X;

The morphisms of $\mathcal{C}$ satisfy the following property:

(C1) (Composition)

$$\begin{aligned} X, Y, Z \in \mathsf{Obj}(\mathcal{C}) \implies & [f \in \mathsf{Hom}(X, Y), g \in \mathsf{Hom}(Y, Z), \mathsf{Cdm}(f) = \mathsf{Dom}(g) \\ & \implies \exists (g \circ f) \in \mathsf{Hom}(X, Z)] \end{aligned}$$

(C2) (Identity)

$$\begin{aligned} X \in \mathsf{Obj}(\mathcal{C}) \implies & \exists \mathsf{id}_X \in \mathsf{Hom}(X, X) : \\ & [Y \in \mathsf{Obj}(\mathcal{C}), f \in \mathsf{Hom}(X, Y), g \in \mathsf{Hom}(Y, Z) \\ & \implies f \circ \mathsf{id}_X = f, \mathsf{id}_X \circ g = g] \end{aligned}$$

(C3) (Associativity)

$$f, g, h \in \mathsf{Mor}(\mathcal{C}) \implies f \circ (g \circ h) = (f \circ g) \circ h.$$

Remark 1. To describe a metacategory it is necessary to specify:

- The collection $\mathsf{Obj}\,(C)$ of objects;
- The collection $\mathsf{Mor}(C)$ of morphisms;
- For each object X, an identity morphism $\mathsf{id}_X : X \to X$;
- For every appropriate pair of morphisms f, g, the composite morphism $g \circ f$.

$$
\begin{tikzcd}
X \arrow[loop left, "\mathsf{id}_X"] \arrow[r, "f"] \arrow[dr, "g\circ f" description] & Y \arrow[loop right, "\mathsf{id}_Y"] \arrow[d, "g"] \\
 & Z \arrow[loop below, "\mathsf{id}_Z"]
\end{tikzcd}
$$

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References

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[2] nLab. "category" Accessed on [May 05, 2024]. `https://ncatlab.org/nlab/show/category#Grothendieck61`.