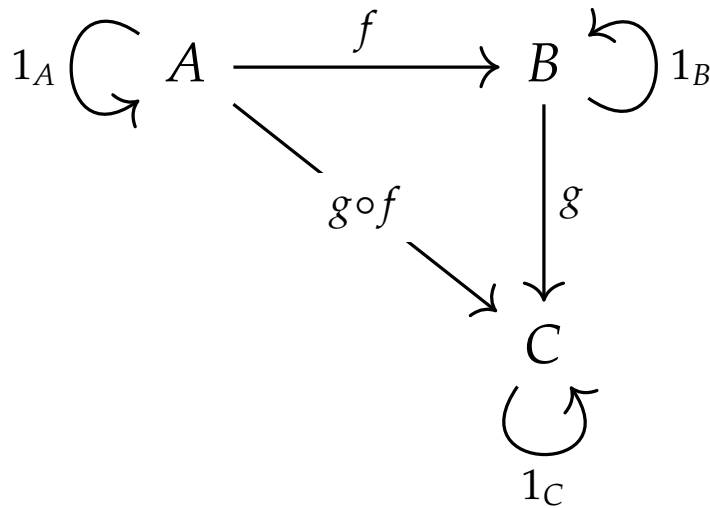


# Category Theory

- A Journey from Concretization to Abstraction -

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A document presented for  
the Category Theory

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April 24, 2024

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# Chapter 1

## None

**Example 1.1** (Determinant).

[ Two Categories ]

1. **Category of Commutative Rings** (CRing)
  - **Objects:** Commutative rings  $R, S, \dots$
  - **Morphisms:** Ring homomorphisms  $\phi : R \rightarrow S$
2. **Category of Groups** (Grp)
  - **Objects:** Groups
  - **Morphisms:** Group homomorphisms

[ Two Functors ] Note that  $\mathbf{GL}_n$  represents the functor, while  $GL_n$  denotes the group.

1. **General Linear Group Functor** ( $\mathbf{GL}_n : \mathbf{CRing} \rightarrow \mathbf{Grp}$ )
  - **On Objects:**

$$R \mapsto GL_n(R)$$

Maps a ring  $R$  to the general linear group  $GL_n(R)$ , which consists of all  $n \times n$  invertible matrices over  $R$ .

- **On Morphisms:** Let  $\mathbf{GL}_n(\phi) : GL_n(R) \rightarrow GL_n(S)$ .

$$\phi \mapsto \mathbf{GL}_n(\phi)$$

It preserves invertibility.

2. **Unit Functor** ( $\mathbf{U} : \mathbf{CRing} \rightarrow \mathbf{Grp}$ )

- **On Objects:**

$$R \mapsto R^\times$$

- **On Morphisms:** Let  $\mathbf{U}(\phi) : R^\times \rightarrow S^\times$ .

$$\phi \mapsto \mathbf{U}(\phi)$$

It preserves the unit property.

- [ **Natural Transformation: Determinant** ] The determinant

$$\det_R : GL_n(R) \rightarrow R^\times$$

is a natural transformation between two functors,  $\mathbf{GL}_n$  and  $\mathbf{U}$ .

- **On Objects:**  $\eta_R : GL_n(R) \rightarrow R^\times$
- **Naturality Condition:** The following diagram commutes:

$$\begin{array}{ccc}
 GL_n(R) & \xrightarrow{\eta_R} & R^\times \\
 \mathbf{GL}_n(\phi) \downarrow & & \downarrow \mathbf{U}(\phi) \\
 GL_n(S) & \xrightarrow{\eta_S} & S^\times
 \end{array}$$
  

$$\begin{array}{ccc}
 X & & F(X) \xrightarrow{\eta_X} G(X) \\
 f \downarrow & & \downarrow F(f) \quad \downarrow G(f) \\
 Y & & F(Y) \xrightarrow{\eta_Y} G(Y)
 \end{array}$$
  

$$\begin{array}{ccc}
 R & & GL_n(R) \xrightarrow{\eta_R} R^\times \\
 \phi \downarrow & & \downarrow \mathbf{GL}_n(\phi) \quad \downarrow \mathbf{U}(\phi) \\
 S & & GL_n(S) \xrightarrow{\eta_S} S^\times
 \end{array}$$

**Example 1.2.** Consider  $\mathbb{Z}$  and  $GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc \neq 0 \right\}$ .

### [ Two Categories ]

#### 1. Category of Commutative Rings (CRing)

- **Object:** Commutative ring  $(\mathbb{Z}, +, \times)$
- **Morphism:** Ring homomorphism  $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$

#### 2. Category of Groups (Grp)

- **Objects:** Groups  $(GL_2(\mathbb{Z}), *)$  and  $(\mathbb{Z}^\times, \times)$ . Here ‘ $*$ ’ denotes matrix multiplication. Note that

$$\mathbb{Z}^\times = \{1, -1\}.$$

- **Morphisms:** Group homomorphisms

$$\sigma : GL_2(\mathbb{Z}) \rightarrow GL_2(\mathbb{Z}) \quad \text{and} \quad \tau : \mathbb{Z}^\times \rightarrow \mathbb{Z}^\times.$$

Note that  $\tau : \{\pm 1\} \rightarrow \{\pm 1\}$

[ **Two Functors** ] Note that  $\mathbf{GL}_2$  represents the functor, while  $GL_2$  denotes the group.

1. **General Linear Group Functor** ( $\mathbf{GL}_2 : \mathbf{CRing} \rightarrow \mathbf{Grp}$ )

– **On Object:**

$$\mathbb{Z} \mapsto GL_2(\mathbb{Z})$$

– **On Morphism:**

$$\phi \mapsto \sigma = \mathbf{GL}_2(\phi)$$

It preserves invertibility.

2. **Unit Functor** ( $\mathbf{U} : \mathbf{CRing} \rightarrow \mathbf{Grp}$ )

– **On Object:**

$$\mathbb{Z} \mapsto \mathbb{Z}^\times = \{\pm 1\}$$

– **On Morphism:**

$$\phi \mapsto \tau = \mathbf{U}(\phi)$$

It preserves the unit property.

- [ **Natural Transformation: Determinant** ] The determinant

$$\begin{aligned} \det_{\mathbb{Z}} : GL_2(\mathbb{Z}) &\longrightarrow \mathbb{Z}^\times = \{\pm 1\} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto ad - bc = \det_{\mathbb{Z}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{aligned}$$

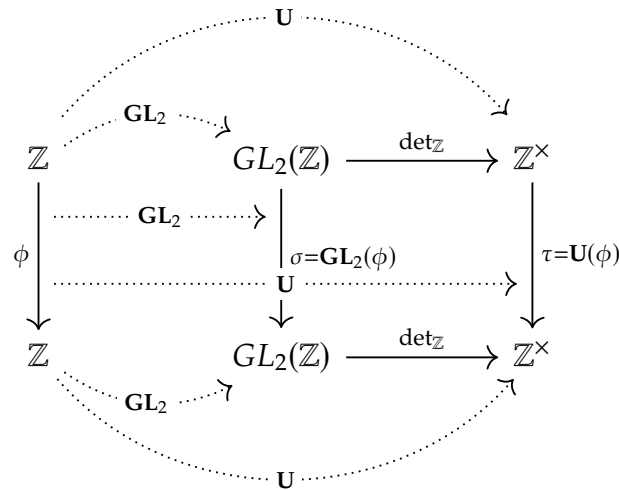
is a natural transformation between two functors,  $\mathbf{GL}_2$  and  $\mathbf{U}$ .

– **Naturality Condition:** The following diagram commutes:

$$\begin{array}{ccc} GL_2(\mathbb{Z}) & \xrightarrow{\eta_{\mathbb{Z}} = \det_{\mathbb{Z}}} & \mathbb{Z}^\times \\ \downarrow \sigma = \mathbf{GL}_2(\phi) & & \downarrow \tau = \mathbf{U}(\phi) \\ GL_n(\mathbb{Z}) & \xrightarrow{\eta_{\mathbb{Z}} = \det_{\mathbb{Z}}} & \mathbb{Z}^\times \end{array}$$

$$\begin{array}{ccc} R & & GL_n(R) \xrightarrow{\eta_R = \det_R} R^\times \\ \downarrow \phi & & \downarrow \mathbf{GL}_n(\phi) \quad \downarrow \mathbf{U}(\phi) \\ S & & GL_n(S) \xrightarrow{\eta_S = \det_S} S^\times \end{array}$$

$$\begin{array}{ccc} \mathbb{Z} & & GL_2(\mathbb{Z}) \xrightarrow{\det_{\mathbb{Z}}} \mathbb{Z}^\times \\ \downarrow \phi & & \downarrow \sigma = \mathbf{GL}_2(\phi) \quad \downarrow \tau = \mathbf{U}(\phi) \\ \mathbb{Z} & & GL_2(\mathbb{Z}) \xrightarrow{\det_{\mathbb{Z}}} \mathbb{Z}^\times \end{array}$$

**Example 1.3.****[ Category ]****1. Category of Finite Vector Spaces ( $\text{FinVect}_K$ )****– Objects:**

- \* Vector space  $V = K^n$  over a field  $K$
- \* Dual Space  $V^* = \{\phi \in K^V : \phi \text{ is linear transformation}\}$ .

**– Morphisms:**

- \* Linear Transformation  $T : V(= K^n) \rightarrow W(= K^m)$
- \* Dual Transformation  $T^* : W^* \rightarrow V^*$  defined by

$$T^*f := f \circ T \quad \text{for all } f \in W^*.$$

Sometimes we denote  $V^* := \mathcal{L}(V, K)$ . Note that

- $(T^*f)\mathbf{v} = f(T\mathbf{v})$  for all  $\mathbf{v} \in V$ ;
- $f \in W^* \implies T^*f \in V^*$ ;
- $T^*$  is linear.

**[ Functor ]****1. Dual Space Functor ( $\mathcal{D} : \text{FinVect}_K \rightarrow \text{FinVect}_K$ )****– On Objects:**

$$V \mapsto V^* = \mathcal{D}(V)$$

Maps a vector space  $V$  to the dual space  $V^*$ , which consists of all linear functionals from  $V$  to  $K$ .

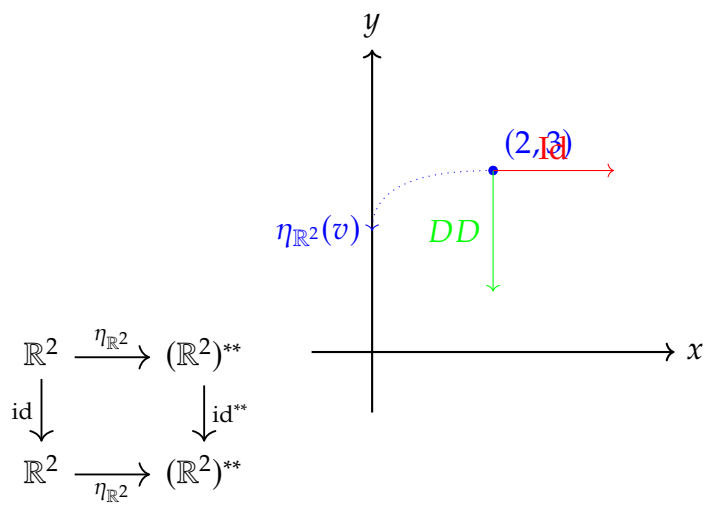
**– On Morphisms:**

$$T \mapsto T^* = \mathcal{D}(T)$$

- **[ Natural Transformation: Determinant ]** The determinant

$$\det_R : GL_n(R) \rightarrow R^\times$$

is a natural transformation between two functors,  $\mathbf{GL}_n$  and  $\mathbf{U}$ .



## 1.1 Category Definitions

### 1.1.1 Matrices over Commutative Rings

Define the category  $\mathbf{Mat}_n(\mathbf{CRing})$  where:

- Objects are  $n \times n$  matrices over any commutative ring  $R$ .
- Morphisms are ring homomorphisms applied element-wise to matrices.

### 1.1.2 Determinant Functor

The functor  $\mathbf{Det}: \mathbf{Mat}_n(\mathbf{CRing}) \rightarrow \mathbf{Grp}$  is defined by:

- On Objects: Mapping  $A$  to the group formed under multiplication by  $\det(A)$ .
- On Morphisms: If  $f: R \rightarrow S$  is a ring homomorphism, then  $\det(f(A)) = f(\det(A))$ , maintaining the structure of group morphisms.

## 1.2 Properties and Transformations

### 1.2.1 Functor Properties

The determinant functor preserves:

- Composition:  $\det(AB) = \det(A) \det(B)$
- Identities:  $\det(I) = 1$ , where  $I$  is the identity matrix.

### 1.2.2 Natural Transformations

Consider a transformation scaling each element of matrix  $A$  by a unit  $u$  in  $R$ . The determinant changes as:

$$\tau_A: \det(A) \mapsto u^n \det(A)$$

where  $n$  is the matrix dimension, reflecting the multiplicative scaling in the group.

## 1.3 Conclusion

Through the lens of category theory, the determinant integrates the algebraic structures of commutative rings and groups, enhancing our understanding of its invariant and multiplicative properties.