Elliptic Curve Cryptograph
- Learning ECC -

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Acknowledgements

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Chapter 1

Elliptic Curve Theory

1.1 A Puzzle of Squares and Pyramids

Consider the following question:

> "What is the number of balls that may be piled as a square pyramid and also re-arranged into a square array?"

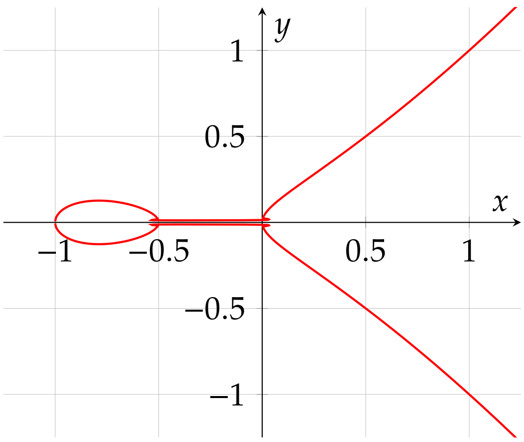
To address this, let x be the height of the pyramid. The number of balls in a pyramid of height x is given by:

$$1^2 + 2^2 + 3^2 + \ldots + x^2 = \frac{x(x+1)(2x+1)}{6}$$

We seek a configuration where this sum also forms a perfect square, i.e.,

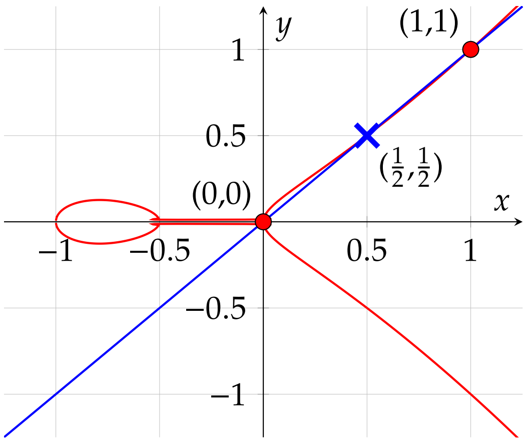
$$y^2 = \frac{x(x+1)(2x+1)}{6}$$

This equation forms the basis of our puzzle, intertwining the concepts of geometric and numeric squares.



1.1.1 Diophantus' Approach

We consider a set of known points to produce new points. The trivial solutions $(0, 0)$ and $(1, 1)$ fit the equation of the line $y = x$.



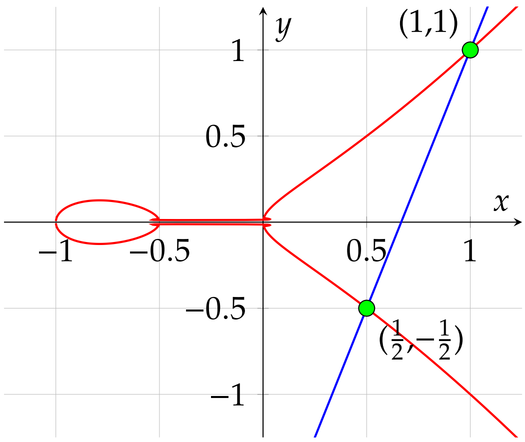
Intersecting this line with the curve described by our pyramid problem, we rearrange terms:

$$\begin{aligned}
\frac{x(x+1)(2x+1)}{6} &= x^2, \\
(x^2 + x)(2x+1) &= 6x^2, \\
2x^3 + x^2 + 2x^2 + x &= 6x^2, \\
x(2x^2 - 3x + 1) &= 0, \\
x(x-1)(2x-1) &= 0.
\end{aligned}$$

We find that $x = \frac{1}{2}$ is a solution, implying $y = \frac{1}{2}$. The symmetry of the curve also yields $\left(\frac{1}{2}, -\frac{1}{2}\right)$ as another solution.

1.1.2 Extending Diophantus' Method

Consider the line through $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and $(1, 1)$, which implies $y = 3x - 2$.



Intersecting this with our curve, we derive:

$$x^3 - \frac{51}{2}x^2 + \ldots = 0 \tag{1.1}$$

This leads to the solutions $x = 24$ and $y = 70$, demonstrating the power of algebraic manipulation and geometric insight.

1.2 Why is it called an Elliptic Curve?

The term "elliptic curve" has its roots in the quest to measure the circumference of an ellipse. Consider the trigonometric function $y = \sin w$. The inverse function, $w(y) = \sin^{-1} y$, is expressed as an integral:

$$w(y) = \sin^{-1} y = \int_0^y \frac{1}{\sqrt{1-t^2}}\, dt$$

This integral is foundational in understanding the link between elliptic curves and elliptic integrals.

1.2.1 Abel's Insight

Niels Henrik Abel, a prominent mathematician, extended this concept. Starting with $y = \sin w$, Abel explored the inverse functions of elliptic integrals, uncovering their double periodicity. He defined the function:

$$F(w) = \int_0^w \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

Abel's work laid the groundwork for understanding the complex nature of elliptic curves.

1.2.2 The Geometry of an Ellipse

An ellipse is defined by the equation $x^2/a^2 + y^2/b^2 = 1$. This simple equation belies the complexity of calculating its arc length.

1.2.3 The Arc Length of an Ellipse

Defining $k^2 = 1 - \frac{b^2}{a^2}$ and changing variables $x \to ax$, we express the arc length of an ellipse as:

$$a \int_{-1}^{1} \sqrt{\frac{1-k^2x^2}{1-x^2}}\, dx = a \int_{-1}^{1} \frac{1-k^2x^2}{\sqrt{(1-x^2)(1-k^2x^2)}}\, dx$$

This leads to the following representation of the arc length:

$$\textbf{Arc Length} = a \int_{-1}^{1} \frac{1-k^2x^2}{y}\, dx \quad \text{with} \quad y^2 = (1-x^2)(1-k^2x).$$

1.2.4 Connecting to an Elliptic Curve

The ellipse's arc length calculation brings us to a critical realization. An elliptic integral is generally expressed as:

$$\int R(x,y)\, dx$$

This integral, deeply connected to the geometry of ellipses, underpins the theory of elliptic curves.

Double Periodicity in Elliptic Curves

Elliptic curves are intimately connected with the study of complex tori, which can be represented through the use of doubly periodic functions. A fundamental example of such a function is the Weierstrass $\wp$ function, defined by a lattice Λ in the complex plane.

The Weierstrass $\wp$ Function

Given a lattice $\Lambda \subset \mathbb{C}$, the Weierstrass $\wp$ function is defined as:

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right). \tag{1.2}$$

This function is even, $\wp(-z) = \wp(z)$, and exhibits double periodicity with respect to the lattice Λ, meaning:

$$\wp(z+\omega) = \wp(z) \quad \text{for all } \omega \in \Lambda. \tag{1.3}$$

Elliptic Curves and the $\wp$ Function

An elliptic curve can be associated with the Weierstrass $\wp$ function. Specifically, an elliptic curve over $\mathbb{C}$ can be described in the Weierstrass form:

$$y^2 = 4x^3 - g_2 x - g_3, \tag{1.4}$$

where g_2 and g_3 are constants derived from the lattice Λ. The coordinates (x, y) on the elliptic curve correspond to the values of the Weierstrass $\wp$ function and its derivative:

$$x = \wp(z;\Lambda), \quad y = \wp'(z;\Lambda). \tag{1.5}$$

Double Periodicity

Two linearly independent periods.

$$\phi(z+w_1) = \phi(z+w_2) = \phi(z) \text{ for all complex number } z.$$

It satisfies

$$[\phi'(z)]^2 = 4\phi(z)^3 - 60G_4\phi(z) - 140G_6$$

- So for $x = \phi(z)$ and $y = \phi'(z)$
- $y^2 = 4x^3 - 60G_3 x - 140G_6$

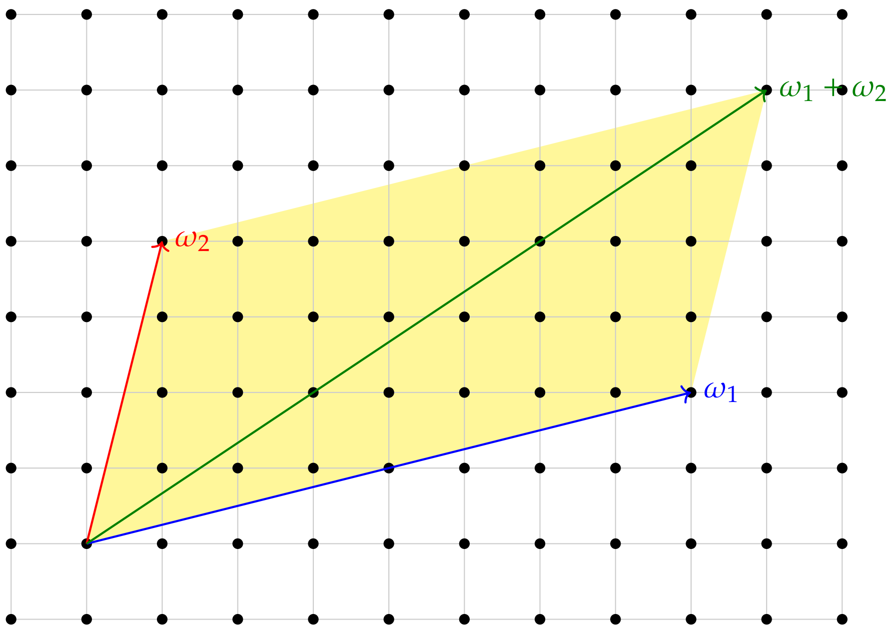
Elliptic Functions and Elliptic Curves

The $\wp$-function and its derivative satisfy an algebraic relation

$$\wp'(z)^2 = \wp(z)^3 + A\wp(z) + B$$

The double periodicity means that it is a function on the quotient space $\mathbb{C}/\Lambda$, where Λ is the lattice

$$\Lambda = \{n_1\omega_1 + n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\}.$$



The lattice L is generated by ω_1 and ω_2
in the quotient space $\mathbb{C}/L$.

Elliptic Functions and Elliptic Curves

Elliptic functions and elliptic curves are fundamental objects in complex analysis and algebraic geometry, respectively. They are interconnected through the Weierstrass $\wp$ function and its properties.

Weierstrass Elliptic Functions

The Weierstrass elliptic functions are defined with respect to a lattice $\Lambda \subset \mathbb{C}$. The Weierstrass $\wp$ function, a key example of an elliptic function, is defined as:

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right). \tag{1.6}$$

This function is doubly periodic and meromorphic with poles of order two at lattice points.

Elliptic Curves and the Weierstrass $\wp$ Function

An elliptic curve can be described as a set of points satisfying a cubic equation in two variables. Over the complex numbers, this curve can be associated with the Weierstrass $\wp$ function.

An elliptic curve in Weierstrass form is given by:

$$y^2 = 4x^3 - g_2 x - g_3, \tag{1.7}$$

where g_2 and g_3 are constants determined by the lattice Λ. The function $\wp$ and its derivative relate to the curve as follows:

$$x = \wp(z; \Lambda), \tag{1.8}$$
$$y = \wp'(z; \Lambda). \tag{1.9}$$

This establishes a correspondence between points on the complex torus $\mathbb{C}/\Lambda$ and points on the elliptic curve.

Properties of Elliptic Curves

Elliptic curves have several important properties:

- They form a group under a geometrically defined addition operation.
- The addition operation on the curve corresponds to the addition of points in the complex plane modulo the lattice Λ.
- Elliptic curves over finite fields have applications in number theory and cryptography.

The Complex Points on an Elliptic Curve

The ϕ-function gives a complex analytic isomorphism

$$\frac{\mathbb{C}}{L} = \big(\phi(z), \phi'(z)\big) \longrightarrow E(\mathbb{C})$$

with the notation that $\mathbb{C}$ is the complex numbers, L is a lattice, and $E(\mathbb{C})$ is the set of complex points on an elliptic curve.

Thus the points of E with coordinates in the complex numbers $\mathbb{C}$ form a *torus*, that is, the surface of a donut.

$X^2 + Y^2 = C$

- Let $x = a + b\sqrt{-1}$, $y = c + d\sqrt{-1}$.
- The solution over complex numbers is a surface, in fact topologically sphere.
- If unbelievable, check out level curves.
- Furthermore, it has group structure.

$$(a + b\sqrt{-1})(c + d\sqrt{-1}) \text{ becomes } ac - bd + (ad + bc)\sqrt{-1}$$

Why is it called Torus?

- Complex Tori

$$y^2 = x(x^2 - 1)$$

- If we introduce *points at infinity* and the *complex numbers*, we can argue that the graph is a torus.

Why Elliptic Curve?

- Discrete Logarithm Problem
- Given a finite group G with two of its elements a and b.
- Find an integer x such that, $a^x = b$ if it exists.
- Example: Non-zero elements of some finite field.

Better groups?

For a finite field F,

$$GL_2(F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc \neq 0,\ a, b, c, d \in F \right\}$$

- The Times(London) Jan. 1999 An Irish schoolgirl Sarah Flannery used matrices as an alternative to RSA. Her algorithm is far faster than the RSA and equally secure.

- The Art of Computer Programming by Donald Knuth

How about this group?

- $F = \mathbb{Z}/17\mathbb{Z} = \mathbb{Z} \pmod{17}$
- $6^2 = 36 \equiv 2 \bmod 17$
- 6 behaves like $\sqrt{2}$

$X^2 - 2Y^2 = 1$
$(3 + 2\sqrt{2})(3 - 2\sqrt{2}) = 1$
$(3 + 12)(3 - 12) = -36 \equiv 1 \bmod 17$

- Let $G = \{(x, y) \mid x^2 - 2y^2 = 1 \text{ over } \mathbb{F}\}$ The operation on G is defined as:

$$(x_1, y_1) \cdot (x_2, y_2) =$$

$$\left(x_1 + \sqrt{2}y_1\right)\left(x_2 + \sqrt{2}y_2\right) =$$

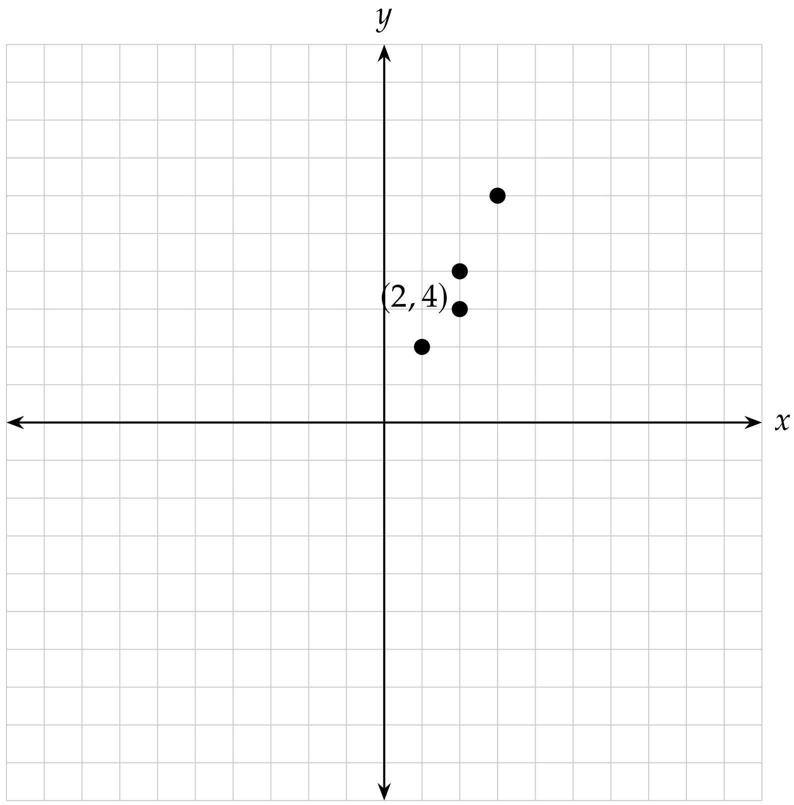
$$= \left(x_1x_2 + 2y_1y_2\right) + \sqrt{2}\left(x_1y_2 + x_2y_1\right)$$

$$(x_1, y_1) \cdot (x_2, y_2) =$$

$$= \left(x_1x_2 + 2y_1y_2\right) x_1y_2 + x_2y_1$$

Why Elliptic Curve?

- DLP (Discrete Logarithm Problem) on finite field can be solved faster than we thought!
- by “index calculus”
- To protect against this attack...
- Elliptic curves!



y
x
(2, 4)

Chapter 2

Elliptic Curves in Cryptography

- Elliptic Curve (EC) cryptography were first proposed in 1985 independently by Neal Koblitz and Victor Miller.
- The **discrete logarithm** problem on elliptic curve groups is believed to be more difficult than the corresponding problem in the multiplicative group of non-zero elements of the underlying finite field.

On finite fields

Consider $y^2 \equiv x^3 + 2x + 3 \pmod 5$

$$\begin{aligned}
x &= 0, y^2 = 3 & \text{no solution } (\text{mod } 5) \\
x &= 1, y^2 = 6 \equiv 1, & y = 1, 4 \ (\text{mod } 5) \\
x &= 2, y^2 = 15 \equiv 0, & y = 0 \ (\text{mod } 5) \\
x &= 3, y^2 = 36 \equiv 1, & y = 1, 4 \ (\text{mod } 5) \\
x &= 4, y^2 = 75 \equiv 0, & y = 0 \ (\text{mod } 5)
\end{aligned}$$

Then points on the elliptic curve are $(1,1)$, $(1,4)$, $(2,0)$, $(3,1)$, $(3,4)$, $(4,0)$ and the point at infinity. Denote it by $\mathcal{O}$.

Notation

- $GF(q)$ or $\mathbb{F}_q$: finite field with q elements, typically, $q = p$ where p is prime, or 2^m.
- $E(\mathbb{F}_q)$: elliptic curve over $\mathbb{F}_q$.
- (x, y): point on $E(\mathbb{F}_q)$.
- $\mathcal{O}$: point at infinity.

Definition of Elliptic curves

- An elliptic curve over a field K is a non-singular cubic curve in two variables, $f(x, y) = 0$ with a rational point (which may be a point at infinity).
- The field K is usually taken to be the complex numbers, reals, rationals, algebraic extensions of rationals, p-adic numbers, or a *finite field*.
- Elliptic curves groups for cryptography are examined with the underlying fields of $\mathbb{F}_p$ (where $p > 3$ is a prime) and $\mathbb{F}_{2^m}$ (a binary representation with 2^m elements).

EC

An *elliptic curve* is a plane curve defined by an equation of the form, when characteristic is neither 2 nor 3, and ... What the hell?

$$y^2 = x^3 + ax + b$$

Hmm...

- $x^3 + y^3 + 1 = 0$ is a cubic curve...?
- Let $x = u + v$, $y = u - v$.
- Then $(u + v)^3 + (u - v)^3 + 1 = 0$.
- This simplifies to $2u^3 + 6uv^2 + 1 = 0$.
- Which leads to $6(v/u)^2 = -(1/u)^3 - 2$.
- So $X = -6/u$, $Y = 36v/u$.
- Hence $Y^2 = X^3 - 432$.

Weierstrass Equation

A two-variable equation $F(x, y) = 0$, forms a curve in the plane.

The generalized Weierstrass Equation of elliptic curves:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

Quadratic Equation

- $x^2 + ax + b = 0$
- $x = t - \frac{a}{2}$
- $t^2 - \frac{a^2}{4} - 4b = 0$

Cubic Equation

- $x^3 + ax^2 + bx + c = 0$
- $x = t - \frac{a}{3}$
- $t^3 + pt + q = 0$
- $p = b - \frac{a^2}{3}$
- $q = c + \frac{2a^3}{27} - \frac{ab}{3}$

Field Characteristics

- If characteristic field is not 2:

$$\left(v + \frac{a_1 x}{2} + \frac{a_3}{2}\right)^2 = x^3 + \left(\frac{a_1^2}{4} + a_2\right) x^2 + a_4 x + \left(\frac{a_1 a_3}{4} + a_6\right)$$

$$\Rightarrow y_1^2 = x^3 + a_2' x^2 + a_4' x + a_6'$$

- If characteristics of field is neither 2 nor 3:

$$x_1 = x + \frac{a_2}{3}$$

$$\Rightarrow y_1^2 = x_1^3 + \Delta x + B$$

Discriminant

- Discriminant of $x^2 + bx + c$ is $b^2 - 4c$
- $b^2 - 4c$ is non-zero $\Leftrightarrow$ no double roots
- Discriminant of $x^3 + ax + b$ is $-4a^3 - 27b^2$
- $-4a^3 - 27b^2$ is non-zero $\Leftrightarrow$ no double roots

j-invariant

- Define j of this elliptic curve E as $j(E)/1728 = 4a^3/(4a^3 + 27b^2)$
- If we change $x = m^2 x, y = m^3 y$, get $\tilde{E}$:
- then $j(E) = j(\tilde{E})$
- j-value fixes E

$$y^2 = x^3 + ax + b$$

j-invariant

- If we change $x = m^2x, y = m^3y$, get $\tilde{E}$:
- then $j(E) = j(\tilde{E})$
- Why not something like $x = mx + ny^2 + s$?
- It has to keep the point at infinity and keep the form $y^2 = x^3 + ax + b$

Points on the Elliptic Curve (EC)

- Elliptic Curve over field L
- $E(L) = \{\infty\} \cup \{(x, y) \in L \times L \mid y^2 + \ldots = x^3 + \ldots\}$
- It is useful to add the point at infinity.

Group Law

- A group law may be defined where the sum of two points is the reflection across the x-axis of the third point on the same line
- Chords and tangents

The Abelian Group

Given two points P, Q on E, there is a third point, denoted by $P + Q$ on $\bar{E}$, and the following relations hold for all P, Q, R in E.

- $P + Q = Q + P$ (commutativity)
- $(P + Q) + R = P + (Q + R)$ (associativity)
- $P + O = O + P = P$ (existence of an identity element)
- there exists $(-P)$ such that $(-P) + P = O$ (existence of inverses)

Associativity

- $(P + Q) + R = P + (Q + R)$
- Associativity is non-trivial.
- It gives Pascal's theorem and Pappus's theorem.

Elliptic Curve Picture

- Consider elliptic curve $E : y^2 = x^3 - x + 1$
- If P_1 and P_2 are on E, we can define $P_3 = P_1 + P_2$ as shown in the picture.

Doubling of a point

- Let $P = Q$
- $2y_1\frac{dy}{dx} = 3x_1^2 + a$
- $m = \frac{dy}{dx} = \frac{3x_1^2+a}{2y_1}$
- If $y_1 \neq 0$ (since then $P_1 + P_2 = \infty$):
  - $0 = x^3 - m^2x^2 + \ldots$
  - $x_3 = m^2 - 2x_1, y_3 = m(x_1 - x_3) - y_1$
- What happens when $P_2 = \infty = O$?

Sum of two points

Define for two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the Elliptic curve:

$$\lambda = \begin{cases} \frac{y_2-y_1}{x_2-x_1} & \text{for } x_1 \neq x_2 \\ \frac{3x_1^2+a}{2y_1} & \text{for } x_1 = x_2 \end{cases}$$

Then $P + Q$ is given by $R(x_3, y_3)$:

$$x_3 = \lambda^2 - x_1 - x_2$$
$$y_3 = \lambda(x_3 - x_1) + y_1$$

What is -P?

- $y^2 = x^3 + ax + b$
- $P = (x_1, y_1)$
- What is -P? Is -P = $(x_1, -y_1)$?
- Yes. But this works only for $y^2 = x^3 + ax + b$.
- For $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$
- -P = $(x_1, -a_1x_1 - a_3 - y_1)$

Motivation

- over $\mathbb{F}_3$
- $Y^2Z + 2XYZ + YZ^2 = X^3 - XZ^2 + 7Z^3$ has a solution (1,2,1).
- Note that (0,1,0) is a solution.
- Important Point 1: We do not say (0,0,0) is a solution of the Weierstrass equation.

Homogeneous vs Affine

- Important Point 2: We treat $(1,2,1) \sim (2,1,2)$, i.e., consider them to be identical and call it a point of the curve given by the Weierstrass equation.

- $\frac{5^2}{13^2} + \frac{12^2}{13^2} = \frac{13^2}{13^2}$

- $\frac{10^2}{26^2} + \frac{24^2}{26^2} = \frac{26^2}{26^2}$

- $X^2 + Y^2 = Z^2$ implies $\left(\frac{X}{Z}\right)^2 + \left(\frac{Y}{Z}\right)^2 = 1$

Projective Co-ordinates

- Two-dimensional projective space P_K^2 over K is given by the equivalence classes of triples (x, y, z) with x, y, z in K and at least one of x, y, z non-zero.

- Two triples (x_1, y_1, z_1) and (x_2, y_2, z_2) are said to be equivalent if there exists a non-zero element λ in K, such that:
$$(x_1, y_1, z_1) = (\lambda x_2, \lambda y_2, \lambda z_2)$$

- The equivalence class depends only on the ratios and hence is denoted by $(x : y : z)$.

Singularity

- For an elliptic curve $y^2 = f(x)$, define $F(x, y) = y^2 - F(x)$. A singularity of the EC is a point (x_0, y_0) such that:

  - $\frac{\partial F}{\partial x}(x_0, y_0) = \frac{\partial F}{\partial y}(x_0, y_0) = 0$
  - or, $2y_0 = -f'(x_0) = 0$
  - or, $f(x_0) = f'(x_0)$
  - Therefore, f has a double root.

Singularity

- $y^2 = x^2(x - 1)$ double roots $x = 0$

- Let $x - 1 = s^2$

- $y^2 = (s^2 + 1)^2(s^2)$

- Hence $x = s^2 + 1, y = s(s^2 + 1)$

If singular, then

- K = a field
- $K(x, y) = K(t)$
- For $y^2 = x^2(x-1)$, $x = s^2+1$, $y = s(s^2+1)$
- For $y^2 = x^3$, $y = t^3$, $x = t^2$
- For an elliptic curve, $K(x, y)$ is never $K(t)$.

Projective Form

- $E : Y^2Z + a_1XYZ + a_3Y^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$
- has a point $(0, 1, 0)$, point at infinity, denoted by $\mathcal{O}$.

Elliptic Curves in Characteristic 2

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

- If a_1 is not 0, this reduces to the form:

$$y^2 + xy = x^3 + Ax^2 + B$$

- If a_1 is 0, the reduced form is:

$$y^2 + a_3y = x^3 + Bx + C$$

- Note that the form cannot be:

$$y^2 = x^3 + Ax + B$$

EC over Finite Fields

- An elliptic curve may be defined over any finite field GF(q).
- For $GF(2^m)$, the curve has a different form:

$$y^2 + xy = x^3 + ax^2 + b$$

- where b is not 0.
- Addition formulae are similar to those over the reals.

Terminology

- Order of point P is the smallest integer r such that $[r]P = \mathcal{O}$.
- Order of the curve is the number of points of $E(\mathbb{F})$, denoted by $\#E(\mathbb{F})$.

Group Properties

- Let $\#E(\mathbb{F}_q)$ denote the number of points on an elliptic curve $E(\mathbb{F}_q)$, including $\mathcal{O}$.
- Hasse bound: $\#E(\mathbb{F}_q) = q + 1 - t$, where $|t| < 2\sqrt{q}$.
- The group of points is either cyclic or a product of two cyclic groups.

So it's an Abelian Group...

- Group homomorphism? Isogeny, isogenous.
- Endomorphism, isomorphic.
- Examples of endomorphisms are:
  - $[2] : E \longrightarrow E, P \mapsto [2]P$
  - $[n] : E \longrightarrow E, P \mapsto [n]P$

Non-trivial Isogeny

- $E : y^2 = x^3 - x$
- $[i = \sqrt{-1}] : (x, y) \mapsto (-x, iy)$
- $[i = \sqrt{-1}]^2 = [i][i] = (-1) : (x, y) \mapsto (x, -y), P \mapsto -P$
- here $i^2 = -1$
- $6^2 = -1 \mod 37$
- Called complex multiplication.

Frobenius Map

- GF(q), $q = p^k$
- $F : \mathrm{GF}(q) \longrightarrow \mathrm{GF}(q)$
- $F(x) = x^p$ for any x
- F is an isomorphism of GF(q). So F defines an isogeny for any elliptic curve over GF(q).

E[n]

- For any group G, any natural number n, $G[n] = \{g | g^n = 1\}$.
- $E[n] = \{P | [n]P = \mathcal{O}\}$.

Appendix A

Additional Data A

A.1 Existence of an Additional Root in Cubic Functions via the Intermediate Value Theorem

Theorem: Let $f(x) = ax^3 + bx^2 + cx + d$ be a cubic function, where $a, b, c, d \in \mathbb{R}$ and $a \neq 0$. If x_1 and x_2 are two distinct roots of $f(x)$, there exists at least one other root x_3 of $f(x)$.

Proof:

1. *Cubic Function*: A cubic function is defined as $f(x) = ax^3 + bx^2 + cx + d$, which is a polynomial of degree 3, and thus continuous over $\mathbb{R}$.

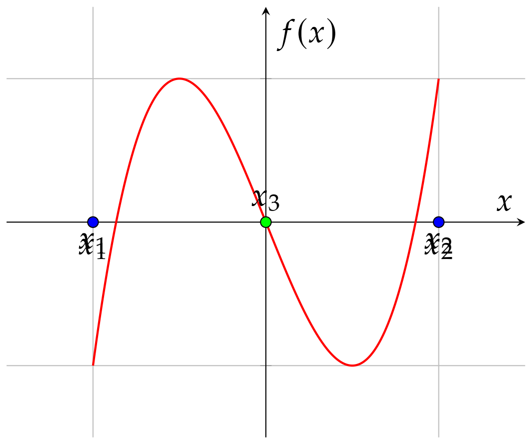
2. *Known Roots*: Assume x_1 and x_2 are two distinct roots of $f(x)$, i.e., $f(x_1) = f(x_2) = 0$.

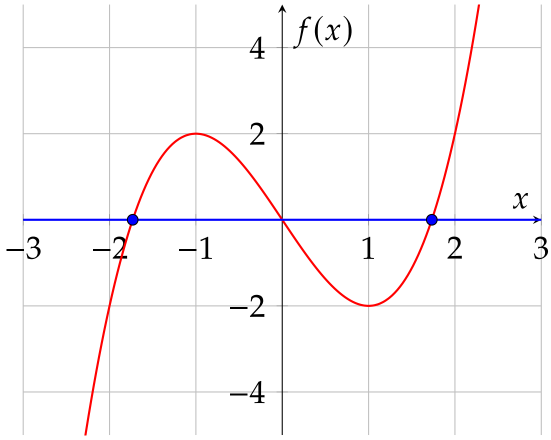
3. *Intermediate Value Theorem (IVT)*: The IVT states that for any continuous function g on an interval $[a, b]$, if $g(a)$ and $g(b)$ have opposite signs, there is at least one c in (a, b) such that $g(c) = 0$.

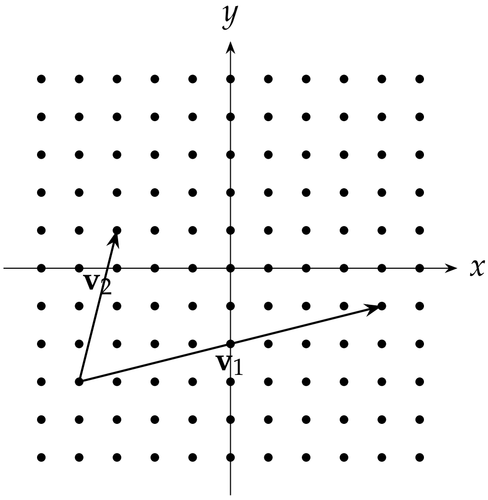
4. *Application to Cubic Function*: By the nature of cubic functions, they must change direction at least once between two roots. This implies the function will either attain a local maximum or minimum between x_1 and x_2.

5. *Existence of Third Root*: If the local extremum is above or below the x-axis, the function must cross the x-axis to change direction, implying the existence of another root x_3 in the interval (x_1, x_2).

6. *Conclusion*: Therefore, by drawing a straight line through $(x_1, 0)$ and $(x_2, 0)$, this line will intersect the graph of $f(x)$ at least at one other point, indicating the existence of another root x_3.




f(x)
4
2
x
−3
−2
−1
1
2
3
−2
−4


y
x
v2
v1