Finding a Potential Function for the FTC

A Penetrating Example Using the Three Tests

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The entire purpose of these tests is to determine if we can find a potential function f for a vector field $\mathbf{F}$, allowing us to use the **Fundamental Theorem of Calculus for Line Integrals**. This lets us replace a potentially difficult line integral with a simple evaluation: $\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$.

Let's investigate the vector field $\mathbf{F}(x, y) = \langle e^y + y\cos(x), xe^y + \sin(x) \rangle$. We want to find its potential function $f(x, y)$.

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1 Test 1: Equality of Mixed Partials (Local Test)

This is the **fast check** to see if a potential function might even exist. If a field $\mathbf{F} = \langle P, Q \rangle$ is the gradient of f, it's necessary that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

- $P = e^y + y\cos(x)$
- $Q = xe^y + \sin(x)$

Let's compute the partials:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(e^y + y\cos(x)) = \mathbf{e^y} + \cos(\mathbf{x})$$
$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(xe^y + \sin(x)) = \mathbf{e^y} + \cos(\mathbf{x})$$

The mixed partials are **equal**. This tells us the field is **closed** (curl-free), so it's worth proceeding to find the potential function.

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2 Test 2: Path Independence (Global Test)

This test demonstrates the **physical consequence** of having a potential function: the work done between two points is the same regardless of the path. Let's calculate the line integral from $A = (0, 0)$ to $B = (\frac{\pi}{2}, 1)$ along two different paths.

2.1 Path 1 (Along the axes): $(0,0) \to (\frac{\pi}{2}, 0) \to (\frac{\pi}{2}, 1)$

1. **Segment 1** $(y = 0, dy = 0)$:
$$\int_0^{\pi/2} (e^0 + 0\cos(x))\,dx = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$$
2. **Segment 2** $(x = \frac{\pi}{2}, dx = 0)$:
$$\int_0^1 \left(\frac{\pi}{2}e^y + \sin\left(\frac{\pi}{2}\right)\right)\,dy = \int_0^1 \left(\frac{\pi}{2}e^y + 1\right)\,dy = \left[\frac{\pi}{2}e^y + y\right]_0^1 = \left(\frac{\pi}{2}e + 1\right) - \left(\frac{\pi}{2}\right) = \frac{\pi e}{2} + 1 - \frac{\pi}{2}$$

Total for Path 1: $\frac{\pi}{2} + \left(\frac{\pi e}{2} + 1 - \frac{\pi}{2}\right) = \boldsymbol{\frac{\pi e}{2} + 1}$.

2.2 Path 2 (Along the axes, different order): $(0,0) \to (0,1) \to (\frac{\pi}{2}, 1)$

1. **Segment 1** $(x = 0, dx = 0)$:
$$\int_0^1 (0 \cdot e^y + \sin(0))\,dy = \int_0^1 0\,dy = 0$$
2. **Segment 2** $(y = 1, dy = 0)$:
$$\int_0^{\pi/2} (e^1 + 1\cos(x))\,dx = [ex + \sin(x)]_0^{\pi/2} = \left(e\frac{\pi}{2} + \sin\left(\frac{\pi}{2}\right)\right) - (0 + 0) = \frac{\pi e}{2} + 1$$

Total for Path 2: $0 + \left(\frac{\pi e}{2} + 1\right) = \boldsymbol{\frac{\pi e}{2} + 1}$.

Since both paths yield the same result, the integral is **path-independent**, confirming the field is **conservative** (exact).

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3 Test 3: Potential Recovery (Constructive Test)

This is the **direct method** for finding the potential function $f(x, y)$ that we need for the FTC.

1. **Integrate P with respect to x**: We assume $\frac{\partial f}{\partial x} = P = e^y + y\cos(x)$.
$$f(x, y) = \int (e^y + y\cos(x))\,dx = xe^y + y\sin(x) + h(y)$$
The "constant" of integration, $h(y)$, is an unknown function of y.
2. **Differentiate with respect to y and match to Q**: Now, take the partial derivative of our candidate f and set it equal to Q.
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xe^y + y\sin(x) + h(y)) = xe^y + \sin(x) + h'(y)$$
We know this must equal $Q = xe^y + \sin(x)$.
$$xe^y + \sin(x) + h'(y) = xe^y + \sin(x)$$

3. **Solve for h(y)**:

$$h'(y) = 0 \implies h(y) = C$$

The function $h(y)$ is just a constant (which we can set to 0).

We have successfully recovered the potential function:

$$\mathbf{f(x, y) = xe^y + y\sin(x)}$$

Using the FTC

Now, we can compute the line integral from the global test instantly using our found potential function and the FTC:

$$\begin{aligned}
\int_{(0,0)}^{(\pi/2,1)} \mathbf{F} \cdot d\mathbf{r} &= f\left(\frac{\pi}{2}, 1\right) - f(0,0) \\
&= \left(\frac{\pi}{2}e^1 + 1\sin\left(\frac{\pi}{2}\right)\right) - (0 \cdot e^0 + 0\sin(0)) \\
&= \frac{\pi\mathbf{e}}{\mathbf{2}} + \mathbf{1}
\end{aligned}$$

This matches the path integral results and was far easier to compute.