Equivalence Relations, Equivalence Classes, Partitions, and Quotient Sets

Ji, Yong-Hyeon

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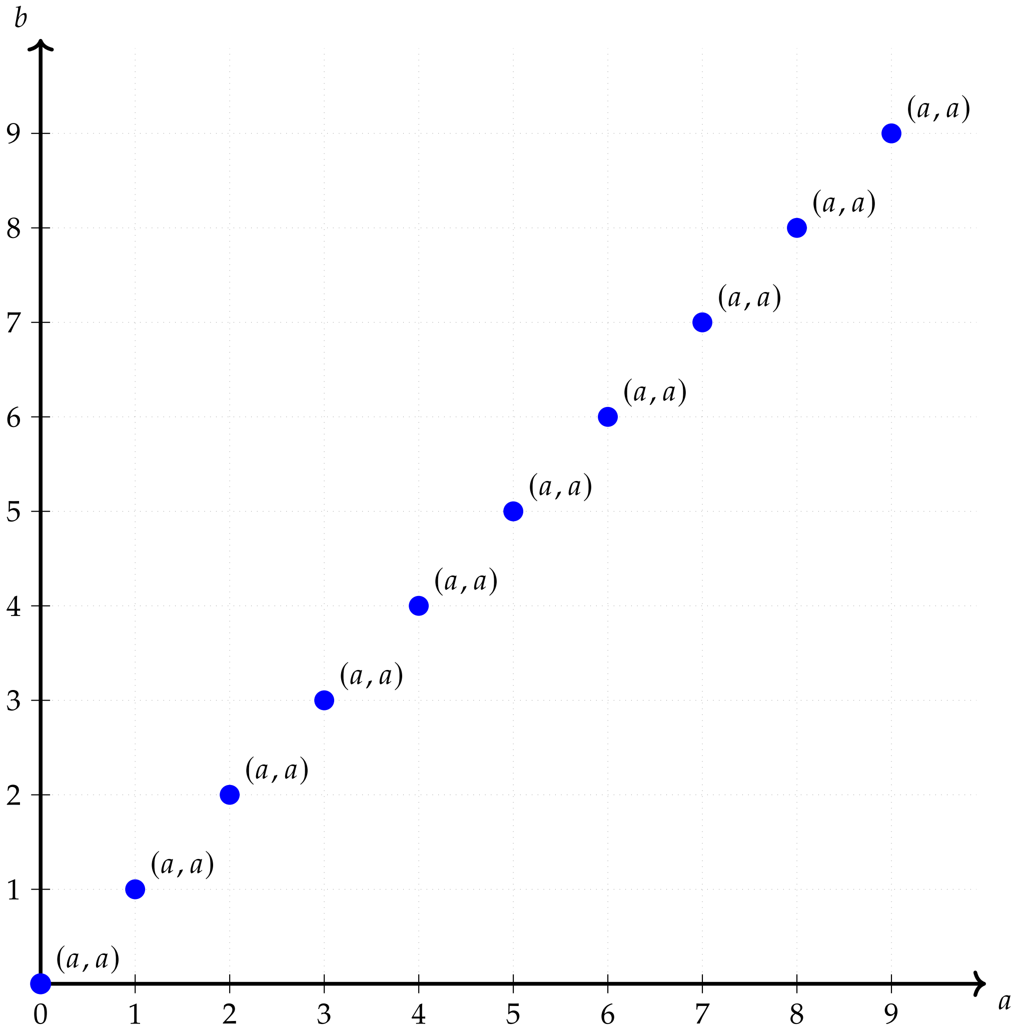
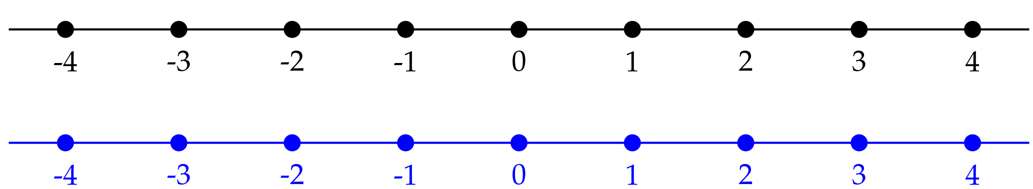


Figure 1: $a = a$ in $\mathbb{Z} \times \mathbb{Z}$



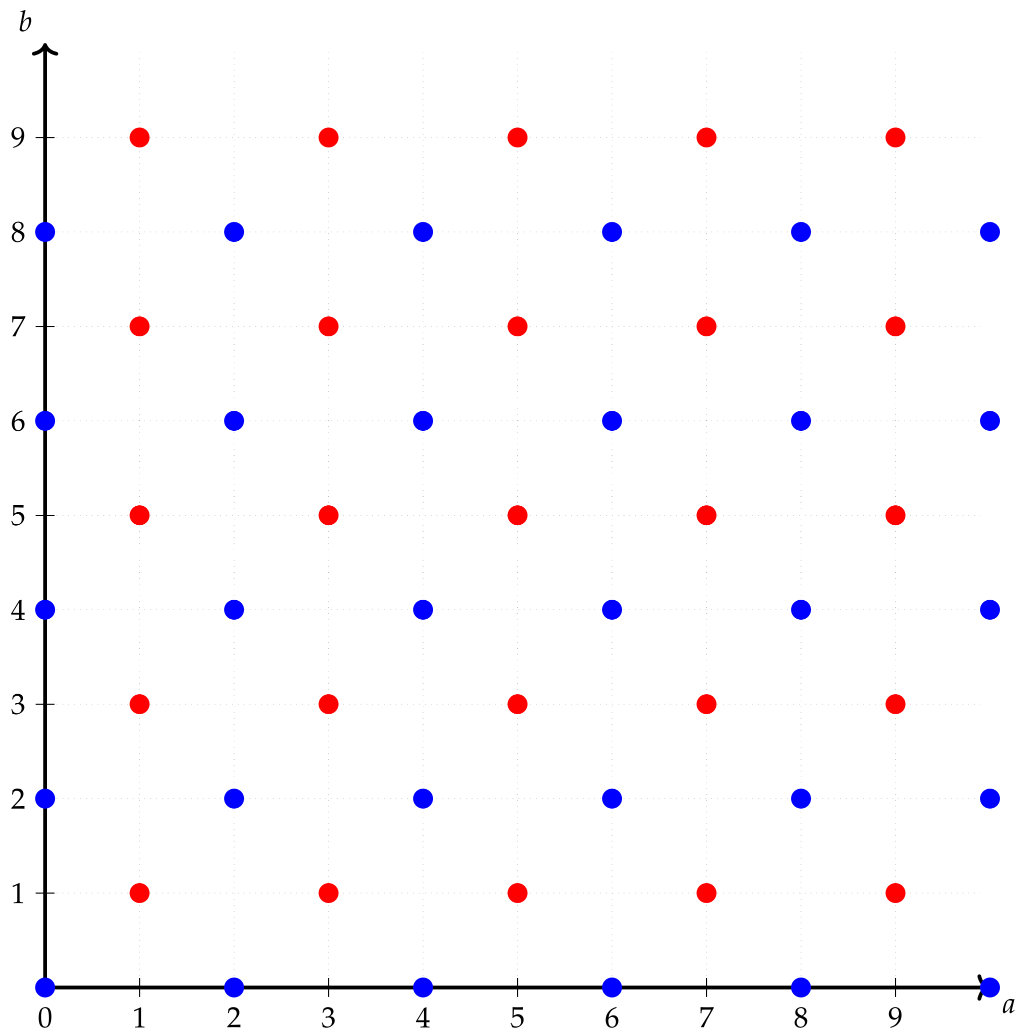
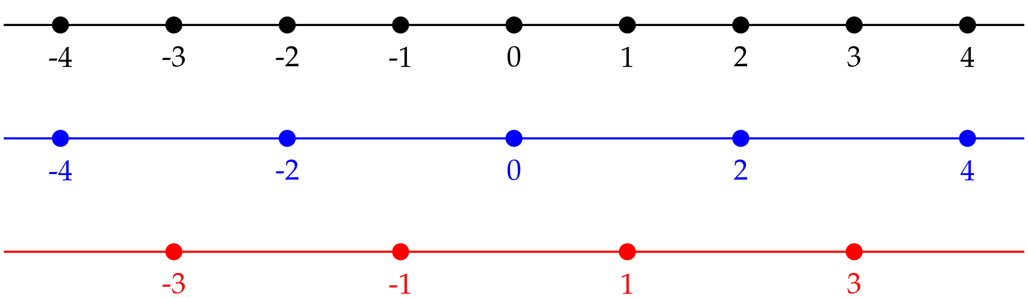


Figure 2: $a \equiv b \pmod 2$ in $\mathbb{Z} \times \mathbb{Z}$



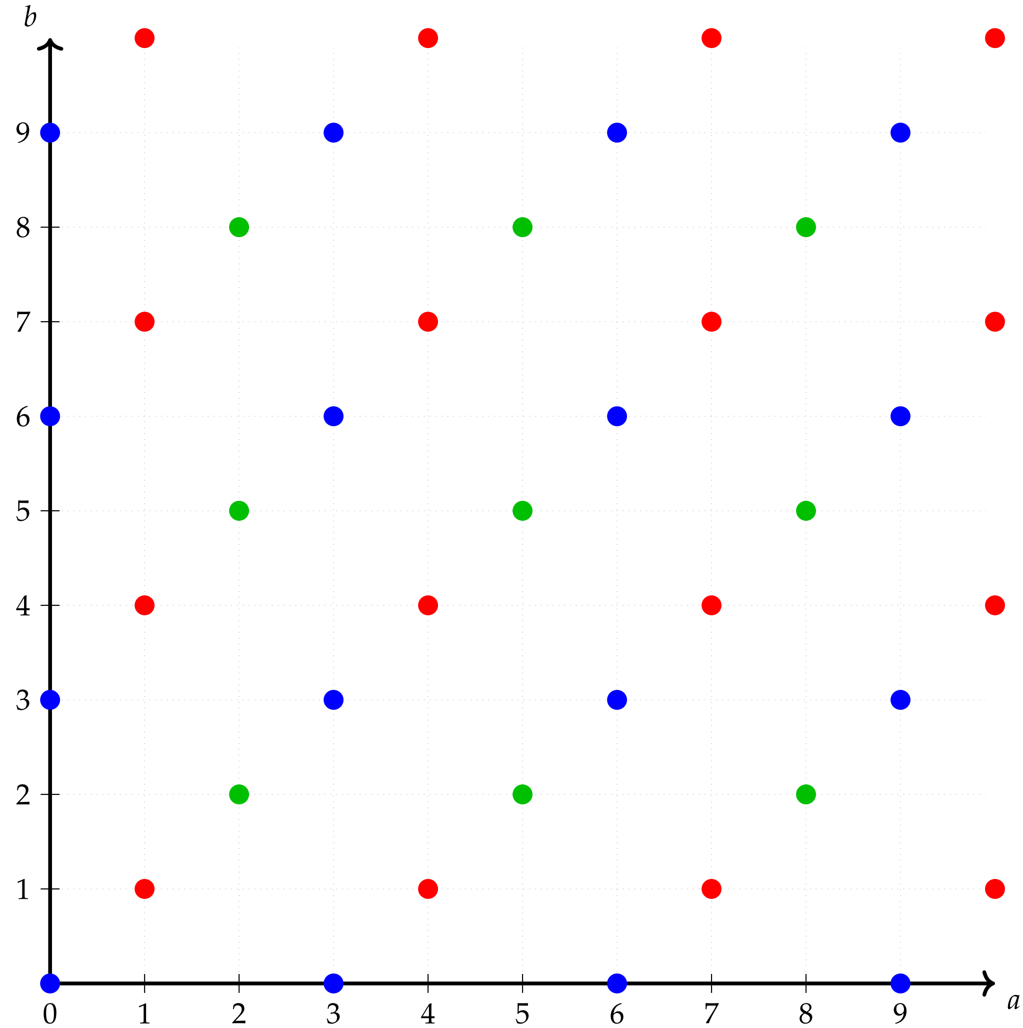
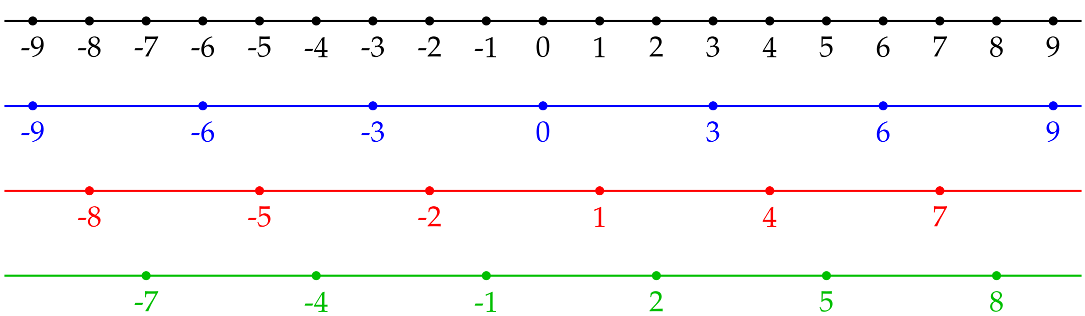
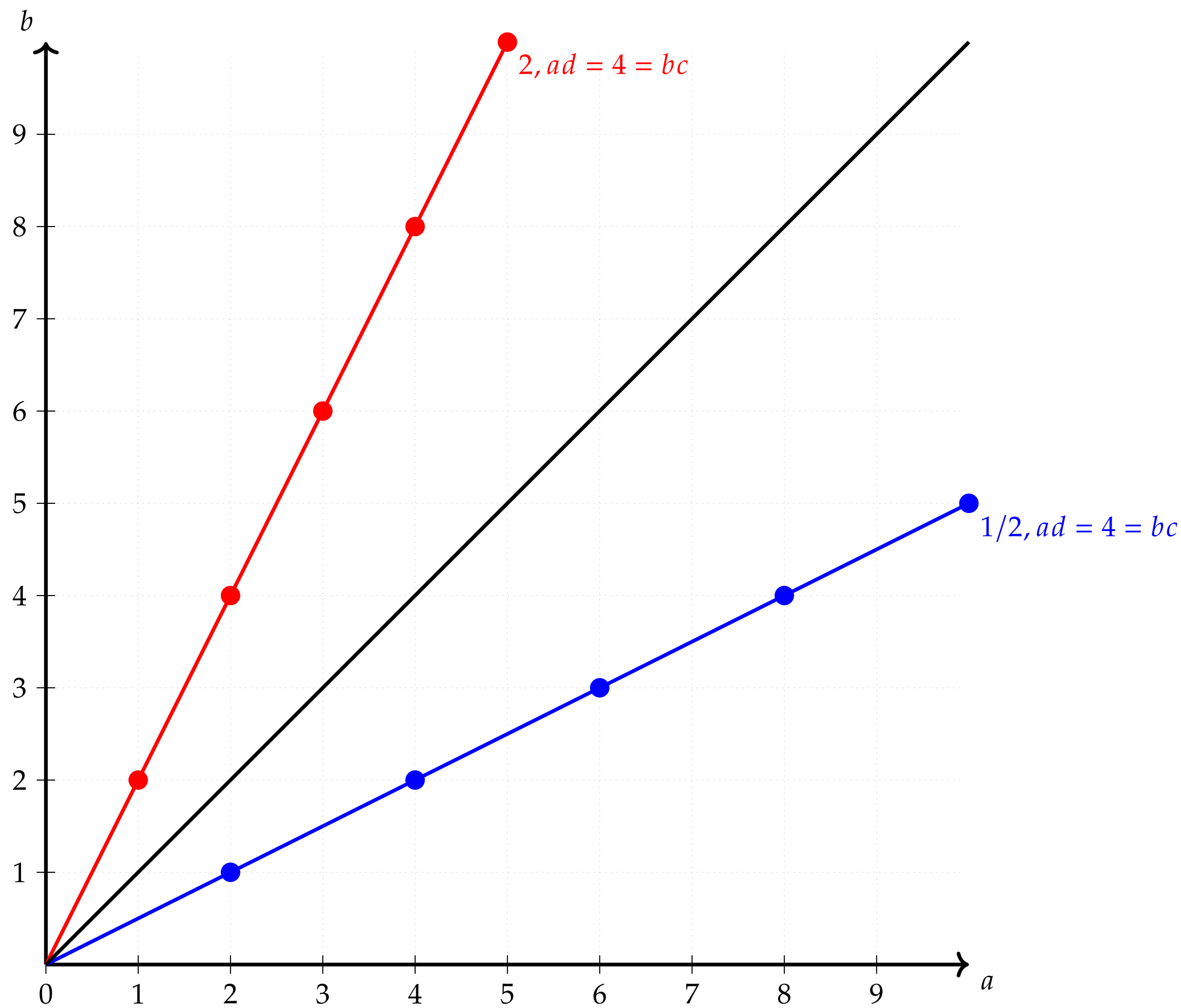


Figure 3: $a \equiv b \pmod{3}$ in $\mathbb{Z} \times \mathbb{Z}$




b
2, ad = 4 = bc
1/2, ad = 4 = bc
a
0
1
2
3
4
5
6
7
8
9

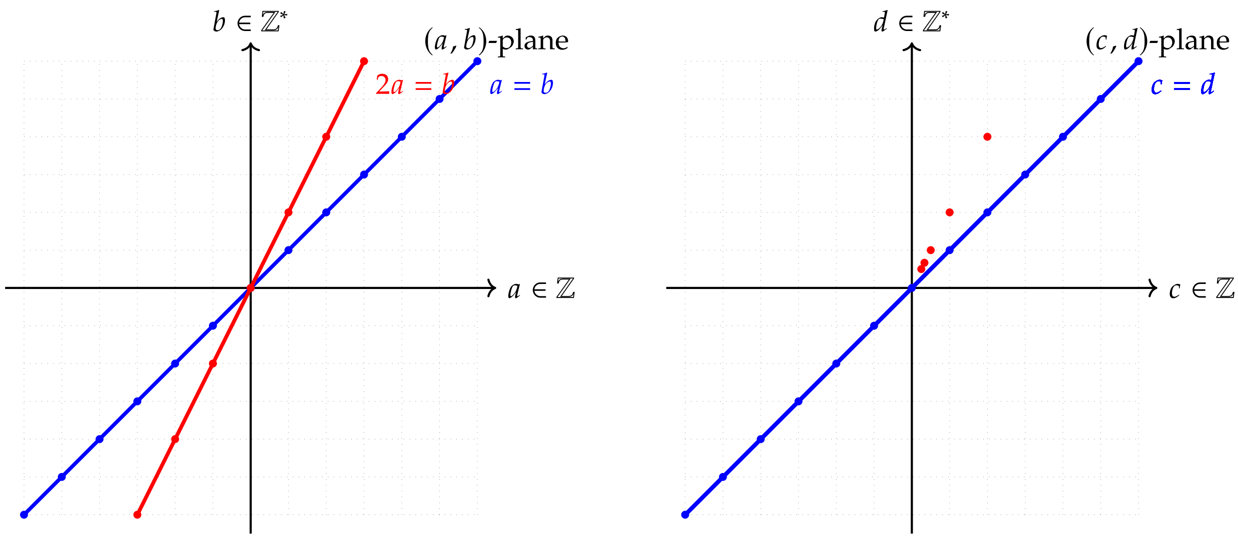


Figure 4: $ad = bc = 2$

1 Introduction

This document explores the concepts of equivalence relations, partitioning, and quotient sets by observing and analyzing patterns within the set of integers, $\mathbb{Z}$. We begin with intuitive observations and gradually formalize these into rigorous mathematical definitions.

2 Natural Observations

Consider the set of all integers $\mathbb{Z}$. An initial observation might note that integers can be grouped based on their remainders when divided by a certain number, say n. This leads to an intuitive grouping based on shared properties.

2.1 Grouping by Remainders

When we divide integers by 2, for instance, every integer falls into one of two categories: even or odd. This division is based on the remainder of the division:

- Even integers $(0, 2, 4, -2, -4, \ldots)$ have a remainder of 0.
- Odd integers $(1, 3, 5, -1, -3, \ldots)$ have a remainder of 1.

3 Formalizing Observations into Equivalence Relations

The observed grouping suggests a relation among integers based on their remainders when divided by a number n. We define an equivalence relation $\sim$ on $\mathbb{Z}$ as follows:

3.1 Definition of Equivalence Relation

A relation $\sim$ on $\mathbb{Z}$ is called an equivalence relation if it satisfies the following properties:

- **Reflexivity**: $a \sim a$
- **Symmetry**: If $a \sim b$ then $b \sim a$
- **Transitivity**: If $a \sim b$ and $b \sim c$ then $a \sim c$

For our example, define $a \sim b$ iff $a \equiv b \pmod{n}$.

4 Partitioning of Integers

Given our equivalence relation, we can partition $\mathbb{Z}$ into disjoint subsets, where each subset contains integers that are equivalent under $\sim$.

4.1 Equivalence Classes

Each subset, known as an equivalence class, includes all integers sharing the same remainder when divided by n. The set of all equivalence classes is given by:

$$\{[a] \mid a \in \mathbb{Z}\}, \text{ where } [a] = \{b \in \mathbb{Z} \mid b \equiv a \pmod{n}\}$$

5 Quotient Set

The collection of all such equivalence classes forms a quotient set, denoted as $\mathbb{Z}/n\mathbb{Z}$, which simplifies the study of integers by focusing on the structure of these classes rather than individual elements.

6 Conclusion

Through the example of integers, we have seen how natural observations can lead to the formal mathematical concepts of equivalence relations, partitioning, and quotient sets, providing a structured way to analyze and simplify complex sets.

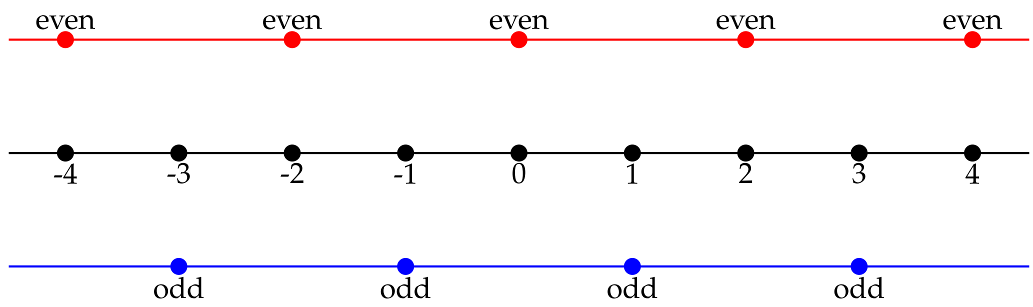


Figure 5: Illustration of Equivalence Classes and Partitioning of Integers Modulo 2

7 Introduction

In this seminar, we explore foundational concepts of group theory and ring theory through intuitive examples within the set of integers, $\mathbb{Z}$. We focus on naturally deriving the definitions of normal subgroups and various types of ideals.

8 Group-Theoretic Concepts

We begin with basic observations about the integers under addition and their subgroup structures.

8.1 Subgroups and Normal Subgroups

Consider the set of integers $\mathbb{Z}$ under addition. Subgroups of $\mathbb{Z}$ are subsets that are closed under addition and have additive inverses. For example, the set of even integers, $2\mathbb{Z}$, forms a subgroup.

8.1.1 Observation of Normality

Notice that in $\mathbb{Z}$, not only is $2\mathbb{Z}$ a subgroup, but it is also normal because integer addition is commutative:

$$a + b = b + a, \quad \forall a, b \in \mathbb{Z}$$

Thus, any subgroup of $\mathbb{Z}$ is normal, since subgroup elements commute with all elements of $\mathbb{Z}$.

9 Ring-Theoretic Concepts

Next, we consider $\mathbb{Z}$ as a ring, and observe properties that lead to the definitions of ideals.

9.1 Ideals of a Ring

An ideal in $\mathbb{Z}$ is a subset closed under addition and absorption by multiplication from $\mathbb{Z}$. The set $n\mathbb{Z}$ for any integer n is an example of an ideal.

9.1.1 Prime and Maximal Ideals

Prime Ideals An ideal I in a ring R is prime if for any elements a, b in R, $ab \in I$ implies $a \in I$ or $b \in I$. In $\mathbb{Z}$, the ideal generated by a prime number p, $p\mathbb{Z}$, is a prime ideal. For instance:

$$3\mathbb{Z} \text{ is prime because if } ab \in 3\mathbb{Z}, \text{ then either } a \text{ or } b \text{ is divisible by } 3.$$

Maximal Ideals An ideal I is maximal if there is no other ideal J such that $I \subset J \subset R$. In $\mathbb{Z}$, $p\mathbb{Z}$ is also maximal if p is a prime number, because the quotient $\mathbb{Z}/p\mathbb{Z}$ is a field.

10 Conclusion

Through intuitive observations within the set of integers, we have derived significant algebraic concepts such as normal subgroups and different types of ideals. This approach not only simplifies understanding but also demonstrates the practical applications and implications of these concepts in abstract algebra.

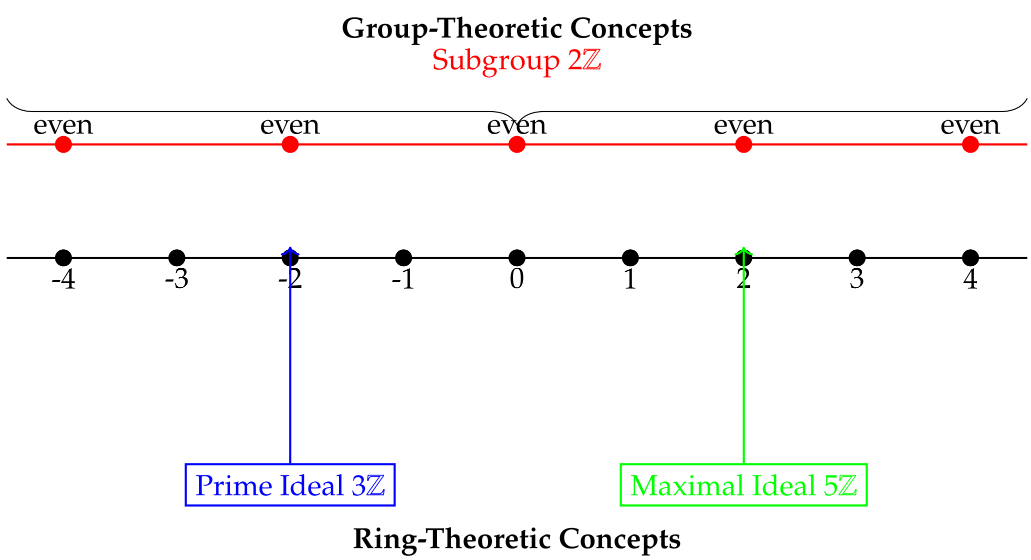


Figure 6: Illustration of Subgroups, Normal Subgroups, and Ideals within $\mathbb{Z}$