Lecture Notes: Differentials, Gradients, and Jacobians via Integrals

Contents

Overview

We explore the chain of operators and their inverses (integrals or anti-derivatives) in three contexts:

$$\underbrace{f}_{\Omega^0} \xrightarrow{d} \underbrace{df}_{\Omega^1} \xleftrightarrow{\nabla} \underbrace{\nabla f}_{\substack{\text{gradient} \\ \text{vector field}}} \longrightarrow \underbrace{\mathbf{F}}_{(\Omega^0)^m} \xrightarrow{d} \underbrace{d\mathbf{F}}_{\Omega^1 \otimes \mathbb{R}^m} \xleftrightarrow{D} \underbrace{D\mathbf{F}}_{\substack{\text{Jacobian} \\ \text{matrix}}} .$$

Here each double-arrow labeled I below denotes the appropriate integral inverse of d.

1 From Functions to 1-Forms

1.1 0-Forms and the Exterior Derivative

Definition 1. $\Omega^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$ *is the space of smooth real-valued functions (0-forms). The exterior derivative*

$$d : \Omega^0(\mathbb{R}^n) \longrightarrow \Omega^1(\mathbb{R}^n)$$

is defined by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\, dx^i.$$

1.2 Integral Inverse on 1-Forms

On a simply-connected region $U \subset \mathbb{R}^n$, the inverse of d on exact 1-forms is given by choosing a base point $x_0 \in U$ and integrating along any path:

$$I_{x_0} : \{\, \omega = df \,\} \longrightarrow \Omega^0(U), \qquad I_{x_0}(\omega)(x) = \int_{x_0}^{x} \omega.$$

By the Fundamental Theorem of Calculus,

$$I_{x_0}(df)(x) = \int_{x_0}^{x} df = f(x) - f(x_0).$$

2 Gradient as Metric Dual

Equipping $\mathbb{R}^n$ with the Euclidean metric identifies each 1-form $\omega = \sum_i g_i\, dx^i$ with the vector field $\sum_i g_i\, \partial/\partial x_i$.

Definition 2. *The* gradient *of $f \in \Omega^0(\mathbb{R}^n)$ is the vector field*

$$\nabla f(x) = \begin{pmatrix} \partial_{x_1} f(x) \\ \partial_{x_2} f(x) \\ \vdots \\ \partial_{x_n} f(x) \end{pmatrix}.$$

Remark 1. *One recovers f (up to constant) by integrating its gradient along any curve from x_0 to x:*

$$f(x) - f(x_0) = \int_{x_0}^{x} \nabla f \cdot d\mathbf{r}.$$

3 Vector Fields and Their Differentials

3.1 Vector Fields as $(\Omega^0)^m$

A smooth vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ is an m-tuple of scalar fields:

$$\mathbf{F}(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_m(x) \end{pmatrix} \in (\Omega^0(\mathbb{R}^n))^m.$$

3.2 Exterior Derivative on Vector Fields

Apply d componentwise:

$$d\mathbf{F} = \begin{pmatrix} dF_1 \\ dF_2 \\ \vdots \\ dF_m \end{pmatrix} \in \Omega^1(\mathbb{R}^n) \otimes \mathbb{R}^m,$$

where $dF_i = \sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j}(x)\, dx^j$.

3.3 Integral Inverse: Line Integrals of 1-Forms

If each dF_i is exact on U, define

$$I_{x_0}(d\mathbf{F}) = \begin{pmatrix} \int_{x_0}^{x} dF_1 \\ \int_{x_0}^{x} dF_2 \\ \vdots \\ \int_{x_0}^{x} dF_m \end{pmatrix} = \begin{pmatrix} F_1(x) - F_1(x_0) \\ F_2(x) - F_2(x_0) \\ \vdots \\ F_m(x) - F_m(x_0) \end{pmatrix}.$$

4 Jacobian Matrix

4.1 Identification with 1-Forms

Choosing the basis $\{dx^1, \dots, dx^n\}$ identifies $d\mathbf{F}$ with the $m \times n$ Jacobi an matrix:

$$D\mathbf{F}(x) = \begin{pmatrix} \partial_{x_1} F_1(x) & \cdots & \partial_{x_n} F_1(x) \\ \vdots & \ddots & \vdots \\ \partial_{x_1} F_m(x) & \cdots & \partial_{x_n} F_m(x) \end{pmatrix}.$$

This matrix gives the linear approximation:

$$\mathbf{F}(x+h) = \mathbf{F}(x) + D\mathbf{F}(x)\, h + o(\|h\|).$$

4.2 Recovering F via Component Integrals

If $D\mathbf{F}$ is integrable, then

$$\mathbf{F}(x) = \mathbf{F}(x_0) + \int_{x_0}^{x} D\mathbf{F}(x')\, dx',$$

with the integral taken componentwise.

Summary Diagram

$$\underbrace{f}_{\Omega^0} \xrightarrow{d} \underbrace{df}_{\Omega^1} \xleftrightarrow{I} \underbrace{\nabla f}_{\substack{\text{gradient}\\\text{vector field}}} \longrightarrow \underbrace{\mathbf{F}}_{(\Omega^0)^m} \xrightarrow{d} \underbrace{d\mathbf{F}}_{\Omega^1\otimes\mathbb{R}^m} \xleftrightarrow{I} \underbrace{D\mathbf{F}}_{\substack{\text{Jacobian}\\\text{matrix}}}.$$

Here d is the exterior derivative and I denotes the corresponding integral inverse.