A Calculus-First, Step-by-Step Lecture on the Mayer–Vietoris Sequence for Ω^1 on S^2

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Abstract

We develop the Mayer–Vietoris argument for de Rham 1-forms on the unit sphere S^2 using only first-year calculus (single-variable and double integrals) and basic vector calculus. In particular:

- Review of antiderivatives: $\int \sin t\, dt$, $\int \cos t\, dt$.
- Gradient theorem (line integral around the unit circle equals 2π).
- Surface integral of curl via iterated integrals (value 27/4).
- Detailed Mayer–Vietoris: gluing local antiderivatives on overlapping hemispheres using two FTC steps in ϕ and θ.

1 1. Fundamental Theorem of Calculus (FTC) Revisited

Let $F(t)$ be differentiable with continuous derivative on $[a, b]$. The FTC states:

$$\int_a^b F'(t)\, dt = F(b) - F(a).$$

Thus any continuous $f(t)$ admits an antiderivative $F(t)$ with $F'(t) = f(t)$, giving

$$\int f(t)\, dt = F(t) + C.$$

Examples

1. $f(t) = \sin t$. Since $\frac{d}{dt}(-\cos t) = \sin t$, we have by FTC

   $$\int_a^b \sin t\, dt = [-\cos t]_a^b = -\cos b + \cos a,$$

   and thus

   $$\int \sin t\, dt = -\cos t + C, \quad d(-\cos t) = \sin t\, dt.$$

2. $f(t) = \cos t$. Since $\frac{d}{dt}(\sin t) = \cos t$, we have

   $$\int_a^b \cos t\, dt = [\sin t]_a^b = \sin b - \sin a,$$

   so

   $$\int \cos t\, dt = \sin t + C, \quad d(\sin t) = \cos t\, dt.$$

2 2. Line Integrals and the Gradient Theorem

Given a vector field $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ and a parameterized curve $C : \mathbf{r}(t) = (x(t), y(t))$, $t \in [\alpha, \beta]$, the line integral is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_\alpha^\beta \left(P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)\right) dt.$$

If $\mathbf{F} = \nabla F$ is a gradient field, then

$$\int_C \nabla F \cdot d\mathbf{r} = F\big(\mathbf{r}(\beta)\big) - F\big(\mathbf{r}(\alpha)\big).$$

Example: Circulation around the Unit Circle

Let C be the unit circle parameterized by

$$\mathbf{r}(t) = (\cos t, \sin t), \quad t \in [0, 2\pi],$$

and consider the field

$$\mathbf{F}(x,y) = \Big(-\tfrac{y}{x^2+y^2},\ \tfrac{x}{x^2+y^2}\Big).$$

On C, we have

$$x'(t) = -\sin t, \quad y'(t) = \cos t, \quad P = -\sin t, \quad Q = \cos t.$$

Hence the integrand simplifies to

$$P\,x' + Q\,y' = (-\sin t)(-\sin t) + (\cos t)(\cos t) = \sin^2 t + \cos^2 t = 1.$$

By the FTC,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 1\,dt = 2\pi.$$

This demonstrates that $\mathbf{F}$ is closed but not exact on $\mathbb{R}^2 \setminus \{0\}$.

3 3. Surface Integral of Curl (Stokes' Theorem)

Stokes' theorem states

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

We compute the left side directly by iterated integrals.

Example: Constant Curl on a Square

Let S be the unit square in the $z = 0$ plane, $(u,v) \in [0,1]^2$, and suppose

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = 27\,u\,v, \quad \mathbf{n} = (0,0,1),$$

so $dS = du\,dv$. Then

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_0^1 \int_0^1 27\,u\,v\,du\,dv.$$

Compute step by step using FTC:

$$\int_0^1 u\,du = \left[\tfrac{u^2}{2}\right]_0^1 = \tfrac{1}{2},$$
$$\int_0^1 v\,dv = \tfrac{1}{2}.$$

Thus

$$27 \times \tfrac{1}{2} \times \tfrac{1}{2} = \frac{27}{4}.$$

4 4. Mayer–Vietoris for Ω^1 on S^2

Cover S^2 by two open hemispheres:

$$U = \{(x,y,z) \in S^2 : z > 0\}, \quad V = \{(x,y,z) \in S^2 : z < 0\}.$$

Their overlap $U \cap V$ contains the equator S^1.

The de Rham complex begins

$$0 \longrightarrow \Omega^0(S^2) \xrightarrow{d} \Omega^1(S^2) \xrightarrow{d} \Omega^2(S^2) \longrightarrow \cdots .$$

Restriction gives a short exact sequence in degree 1:

$$0 \longrightarrow \Omega^1(S^2) \xrightarrow{r_1} \Omega^1(U) \oplus \Omega^1(V) \xrightarrow{s_1} \Omega^1(U \cap V) \longrightarrow 0.$$

Exactness at $\Omega^1(U) \oplus \Omega^1(V)$ means

$$\mathrm{Im}(r_1) = \ker(s_1) \quad \Longleftrightarrow \quad (\beta_U, \beta_V) \text{ glue to a global } \alpha \iff \beta_U|_{U\cap V} = \beta_V|_{U\cap V}.$$

Step-by-Step Gluing via FTC

1. Spherical Coordinates. On each hemisphere use coordinates (ϕ, θ):

$$x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi,$$

with $\phi \in (0,\pi)$, $\theta \in [0, 2\pi)$.

2. Local Potential on U (FTC in ϕ). Write

$$\beta_U = P_U(\phi,\theta)\,d\phi + Q_U(\phi,\theta)\,d\theta.$$

For fixed θ, define

$$F_U(\phi,\theta) = \int_{\phi_0}^{\phi} P_U(s,\theta)\,ds.$$

By FTC, $\partial_\phi F_U = P_U$, so

$$dF_U = P_U\,d\phi + \frac{\partial F_U}{\partial\theta}\,d\theta.$$

3. Overlap and Difference. Similarly define $F_V(\phi,\theta)$ on V. On $U \cap V$, $\beta_U = \beta_V$ implies

$$F_U - F_V = H(\theta)$$

for some function H of θ alone.

4. FTC in θ Forces H Constant. Differentiate:

$$H'(\theta) = \partial_\theta F_U - \partial_\theta F_V = Q_U - Q_V = 0$$

so H is constant. By adding this constant to F_V, we achieve $F_U = F_V$ on the overlap.

5. Gluing to a Global Potential. Define

$$F(p) = \begin{cases} F_U(p), & p \in U, \\ F_V(p), & p \in V, \end{cases}$$

which is smooth and satisfies $dF|_U = \beta_U$, $dF|_V = \beta_V$. Hence $(\beta_U, \beta_V) \in \mathrm{Im}(r_1)$.

This completes the Mayer–Vietoris gluing using only two one-dimensional FTC applications, fully accessible to first-year calculus students.