Classifying Complex Elliptic Curves via the Period Map

Joy of Mathematics

Contents

1 Complex Elliptic Curves and Complex Tori

The goal of this note is to define the periods of a complex elliptic curve as explicit integrals and, using these, to describe the period map into the upper half-plane $\mathbb{H}$ with concrete computations.

Definition 1.1 (Complex Elliptic Curve)**.** *A* complex elliptic curve *(or elliptic curve over $\mathbb{C}$) is a pair $(E, 0)$ satisfying:*

1. *E is a one-dimensional complex analytic manifold (that is, a compact Riemann surface), and*
2. *E is equipped with the structure of a complex Lie group, with distinguished identity element $0 \in E$.*

Usually one also regards E as a smooth projective algebraic curve over $\mathbb{C}$.

A classical fact in the theory of elliptic curves is that every complex elliptic curve is (analytically) isomorphic to a complex torus $\mathbb{C}/\Lambda$ for a suitable lattice $\Lambda \subset \mathbb{C}$.

Theorem 1.2 (Classification of Complex Elliptic Curves)**.** *For any complex elliptic curve E there exists a lattice*

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \subset \mathbb{C}$$

such that E is analytically isomorphic to the complex torus

$$E \cong \mathbb{C}/\Lambda.$$

Conversely, for any lattice $\Lambda \subset \mathbb{C}$ of rank 2 (i.e. $\omega_1, \omega_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent), the complex torus $\mathbb{C}/\Lambda$ carries a natural structure of an elliptic curve.

Here $\omega_1, \omega_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent so that Λ is a discrete subgroup of rank 2 in $\mathbb{C}$.

2 Holomorphic 1-Forms and the Definition of Periods

To define periods, we use holomorphic 1-forms on an elliptic curve.

Proposition 2.1. *Let E be a complex elliptic curve. Then there exists a nonzero holomorphic 1-form ω on E. Moreover, the space*

$$H^0(E, \Omega^1_E)$$

of holomorphic 1-forms on E is a one-dimensional complex vector space.

In particular, any nonzero holomorphic 1-form ω spans this space, and every other holomorphic 1-form is a complex scalar multiple of ω.

Next we recall that, topologically, a complex elliptic curve is a real two-dimensional torus T^2. Thus its first (singular) homology group with integer coefficients is

$$H_1(E, \mathbb{Z}) \cong \mathbb{Z}^2.$$

Definition 2.2 (Marking)**.** *A* marking *of a complex elliptic curve E is the choice of a basis*

$$\alpha, \beta \in H_1(E, \mathbb{Z})$$

of the first homology group together with an orientation convention such that the intersection number satisfies

$$\langle \alpha, \beta \rangle = 1.$$

Such a basis (α, β) is also called a symplectic basis *of $H_1(E, \mathbb{Z})$.*

Definition 2.3 (Periods)**.** *Let ω be a nonzero holomorphic 1-form on E, and let (α, β) be a marking as above. Define the* periods *of ω by*

$$\tau_1 := \int_\alpha \omega, \qquad \tau_2 := \int_\beta \omega.$$

These are complex numbers, and the $\mathbb{Z}$-span

$$\Lambda = \langle \tau_1, \tau_2 \rangle_{\mathbb{Z}}$$

is a lattice in $\mathbb{C}$.

A crucial fact is that τ_1 and τ_2 are $\mathbb{R}$-linearly independent. In particular, the ratio

$$\frac{\tau_2}{\tau_1}$$

is not real; in fact, it lies in the upper half-plane

$$\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\}.$$

We now verify this by a standard computation.

3 Why the Period Lies in the Upper Half-Plane $\mathbb{H}$: Detailed Computation

Let ω be a nonzero holomorphic 1-form on E. Locally we can write

$$\omega = f(z)\, dz$$

for some holomorphic function f in a local coordinate z. Consider the 2-form

$$i\,\omega \wedge \overline{\omega}.$$

Since E is a compact Riemann surface, this provides a positive volume form on E, and hence

$$\int_E i\,\omega \wedge \overline{\omega} > 0.$$

This integral is essentially the area of E with respect to the metric induced by ω.

On the other hand, we can compute the same integral in terms of the periods of ω and the marking (α, β). This is given by a special case of the Riemann bilinear relations.

Proposition 3.1. *Let (α, β) be a marking on E, and let τ_1, τ_2 be the periods*

$$\tau_1 = \int_\alpha \omega, \qquad \tau_2 = \int_\beta \omega.$$

Then

$$\int_E i\,\omega \wedge \overline{\omega} = 2\,\mathrm{Im}(\tau_1 \overline{\tau_2}).$$

In particular,

$$\mathrm{Im}\left(\frac{\tau_2}{\tau_1}\right) > 0.$$

Idea of the proof. The general Riemann bilinear relation for a compact Riemann surface of genus g says that for holomorphic 1-forms ω, η one has

$$\int_E \omega \wedge \eta = \sum_{j=1}^{g} \left(\int_{\alpha_j} \omega \int_{\beta_j} \eta - \int_{\beta_j} \omega \int_{\alpha_j} \eta \right),$$

where $(\alpha_j, \beta_j)_{j=1}^{g}$ is a symplectic basis of $H_1(E, \mathbb{Z})$.

In our case, the genus is $g = 1$, so we choose $\alpha_1 = \alpha$ and $\beta_1 = \beta$. Taking $\eta = \overline{\omega}$ and using that $\overline{\int_\gamma \omega} = \int_\gamma \overline{\omega}$, we obtain

$$\int_E \omega \wedge \overline{\omega} = \left(\int_\alpha \omega \int_\beta \overline{\omega} - \int_\beta \omega \int_\alpha \overline{\omega} \right) = \tau_1 \overline{\tau_2} - \tau_2 \overline{\tau_1} = 2i\,\mathrm{Im}(\tau_1 \overline{\tau_2}).$$

Multiplying by i yields

$$\int_E i\,\omega \wedge \overline{\omega} = 2\,\mathrm{Im}(\tau_1 \overline{\tau_2}) > 0.$$

Thus $\mathrm{Im}(\tau_1 \overline{\tau_2}) > 0$, and

$$\mathrm{Im}\left(\frac{\tau_2}{\tau_1}\right) = \frac{\mathrm{Im}(\tau_1 \overline{\tau_2})}{|\tau_1|^2} > 0.$$

Hence the ratio

$$\tau := \frac{\tau_2}{\tau_1}$$

lies in the upper half-plane $\mathbb{H}$. □

We call this complex number τ (depending on the data $(E, \omega, \alpha, \beta)$) the *period* of the elliptic curve (more precisely, the period with respect to the given marking and 1-form).

4 Normalization and the Complex Torus $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$

From the discussion above, τ_1 is a nonzero complex number. We may thus *normalize* the holomorphic 1-form by dividing by τ_1.

Define a new holomorphic 1-form

$$\omega' := \frac{\omega}{\tau_1}.$$

Then the periods of ω' with respect to the same marking (α, β) satisfy

$$\int_\alpha \omega' = 1, \qquad \int_\beta \omega' = \tau,$$

where

$$\tau = \frac{\tau_2}{\tau_1} \in \mathbb{H}.$$

Definition 4.1 (Normalized Period and Standard Lattice)**.** *Choose a holomorphic 1-form ω' on E such that*

$$\int_\alpha \omega' = 1.$$

Let

$$\tau := \int_\beta \omega' \in \mathbb{H}.$$

We call this τ the normalized period *of E (relative to the marking), and we define the associated standard lattice*

$$\Lambda_\tau = \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot \tau \subset \mathbb{C}.$$

A classical theorem then states that the elliptic curve E (with the marking and normalized 1-form) is analytically isomorphic to the complex torus $\mathbb{C}/\Lambda_\tau$.

More precisely, for the τ obtained from $(E, \omega', \alpha, \beta)$ as in Definition 4.1, there is an analytic isomorphism

$$E \cong \mathbb{C}/\Lambda_\tau$$

such that under the quotient map $\pi : \mathbb{C} \to \mathbb{C}/\Lambda_\tau \cong E$, the form ω' corresponds to the standard 1-form dz on $\mathbb{C}$, i.e.

$$\pi^*\omega' = dz.$$

Remark 4.2. *The normalization condition $\int_\alpha \omega' = 1$ fixes the scale of the 1-form and provides a standard reference. Under this normalization, the entire information of the marked elliptic curve is encoded in the single complex number $\tau \in \mathbb{H}$.*

5 Definition of the Period Map

We now define the moduli space of *marked* complex elliptic curves with a nonzero holomorphic 1-form.

Definition 5.1 (Moduli of Marked Elliptic Curves)**.** *Let $\mathcal{E}$ be the set of isomorphism classes of data*

$$(E, \omega, \alpha, \beta),$$

where:

- *E is a complex elliptic curve,*
- *$0 \neq \omega \in H^0(E, \Omega^1_E)$ is a nonzero holomorphic 1-form on E,*

- *(α, β) is a symplectic basis of $H_1(E, \mathbb{Z})$.*

Two such tuples $(E_1, \omega_1, \alpha_1, \beta_1)$ and $(E_2, \omega_2, \alpha_2, \beta_2)$ are considered equivalent if there exists a biholomorphism

$$f : E_1 \longrightarrow E_2$$

such that

$$f^*\omega_2 = \omega_1, \quad f_*(\alpha_1) = \alpha_2, \quad f_*(\beta_1) = \beta_2.$$

Given such a tuple $(E, \omega, \alpha, \beta)$, we can define its period as the normalized ratio

$$\tau = \frac{\displaystyle\int_\beta \omega}{\displaystyle\int_\alpha \omega} \in \mathbb{H}.$$

Definition 5.2 (Period Map)**.** *The* period map *is the map*

$$\mathrm{Per} : \mathcal{E} \longrightarrow \mathbb{H}$$

defined by

$$\mathrm{Per}(E, \omega, \alpha, \beta) = \frac{\displaystyle\int_\beta \omega}{\displaystyle\int_\alpha \omega}.$$

Theorem 5.3 (Biholomorphism of the Period Map)**.** *The period map*

$$\mathrm{Per} : \mathcal{E} \longrightarrow \mathbb{H}$$

is a biholomorphism. In particular, the moduli space $\mathcal{E}$ of marked elliptic curves with a holomorphic 1-form is analytically isomorphic to the upper half-plane $\mathbb{H}$.

Idea of the proof. For each $\tau \in \mathbb{H}$, consider the standard complex torus

$$E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau).$$

On $\mathbb{C}$ we have the standard 1-form dz, which descends to a holomorphic 1-form ω_τ on E_τ. Take the standard symplectic basis (α, β) of $H_1(E_\tau, \mathbb{Z})$ given by:

- α: the cycle corresponding to the direction of $1 \in \mathbb{C}$,
- β: the cycle corresponding to the direction of $\tau \in \mathbb{C}$.

Then by construction,

$$\int_\alpha \omega_\tau = 1, \qquad \int_\beta \omega_\tau = \tau,$$

so that

$$\mathrm{Per}(E_\tau, \omega_\tau, \alpha, \beta) = \tau.$$

This shows that Per is surjective.

Conversely, given an arbitrary $(E, \omega, \alpha, \beta)$, we can normalize ω so that $\int_\alpha \omega = 1$. The corresponding normalized period is

$$\tau = \int_\beta \omega \in \mathbb{H}.$$

A classical result states that E is analytically isomorphic to E_τ in such a way that ω corresponds to ω_τ up to scaling, and the marking (α, β) corresponds to the standard marking on E_τ. This shows that Per is injective. The dependence on $(E, \omega, \alpha, \beta)$ is analytic, so Per is a biholomorphism. $\square$

6 $\mathrm{SL}_2(\mathbb{Z})$-Action and the Unmarked Moduli Space

So far, we have kept track of a marking (α, β). If we "forget" the marking and keep only the underlying elliptic curve E, then there is an additional freedom: we may choose a different symplectic basis of $H_1(E, \mathbb{Z})$.

Changing the symplectic basis (α, β) is equivalent to acting by an element of the group $\mathrm{SL}_2(\mathbb{Z})$. Concretely, let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

and define a new basis (α', β') by

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Then (α', β') is again a symplectic basis of $H_1(E, \mathbb{Z})$.

Let ω be a holomorphic 1-form on E, and let

$$\tau_1 = \int_\alpha \omega, \quad \tau_2 = \int_\beta \omega$$

be its periods. The periods with respect to (α', β') are then

$$\int_{\alpha'} \omega = a\tau_1 + b\tau_2, \qquad \int_{\beta'} \omega = c\tau_1 + d\tau_2.$$

If we normalize ω' so that $\int_{\alpha'} \omega' = 1$, the new normalized period is

$$\tau' = \frac{\displaystyle\int_{\beta'} \omega'}{\displaystyle\int_{\alpha'} \omega'} = \frac{c\tau_1 + d\tau_2}{a\tau_1 + b\tau_2} = \frac{c + d\tau}{a + b\tau},$$

where $\tau = \tau_2/\tau_1$ is the previous normalized period.

Thus we obtain the familiar fractional linear (Möbius) transformation

$$\tau' = \gamma \cdot \tau := \frac{a\tau + b}{c\tau + d}.$$

Remark 6.1. *Changing the marking (α, β) corresponds exactly to the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half-plane $\mathbb{H}$ via fractional linear transformations*

$$\tau \longmapsto \frac{a\tau + b}{c\tau + d}.$$

Hence, when we forget the marking and consider elliptic curves up to isomorphism without *additional data, the moduli space is*

$$\{\textit{complex elliptic curves up to isomorphism}\} \;\cong\; \mathbb{H}/\mathrm{SL}_2(\mathbb{Z}),$$

which is the classical moduli space of elliptic curves.

7 Weierstrass Equation and a Concrete Example of Period Integrals

We conclude with a standard example of how the period is realized as a concrete integral for an elliptic curve in Weierstrass form.

Consider the elliptic curve given by the Weierstrass equation

$$E : \quad y^2 = 4x^3 - g_2 x - g_3,$$

where $g_2, g_3 \in \mathbb{C}$ are constants such that the cubic polynomial on the right-hand side has distinct roots (so that E is nonsingular).

Let $\wp(z)$ be the Weierstrass $\wp$-function associated to a lattice $\Lambda \subset \mathbb{C}$, and let $\wp'(z)$ denote its derivative. Then the corresponding elliptic curve $\mathbb{C}/\Lambda$ can be parametrized by

$$x = \wp(z), \qquad y = \wp'(z),$$

and this parametrization satisfies the Weierstrass equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

On this curve, consider the 1-form

$$\omega = \frac{dx}{y}.$$

Proposition 7.1. *With $x = \wp(z)$ and $y = \wp'(z)$ as above, we have*

$$\omega = \frac{dx}{y} = dz.$$

Therefore, the periods of ω along the basic cycles of $\mathbb{C}/\Lambda$ coincide exactly with the generators of the lattice Λ.

Proof. Since $y = \wp'(z) = \dfrac{d}{dz}\wp(z)$, we have

$$dx = d(\wp(z)) = \wp'(z)\,dz.$$

Substituting into $\omega = \dfrac{dx}{y}$, we get

$$\omega = \frac{\wp'(z)\,dz}{\wp'(z)} = dz.$$

Thus the pullback of ω to $\mathbb{C}$ via $z \mapsto (\wp(z), \wp'(z))$ is just dz. If α, β are the basic cycles corresponding to generators ω_1, ω_2 of the lattice Λ, then

$$\int_\alpha \omega = \int_\alpha dz = \omega_1, \qquad \int_\beta \omega = \int_\beta dz = \omega_2.$$

Hence the periods of ω are exactly the generators of Λ. $\square$

This example explicitly shows how the period lattice of the holomorphic 1-form $\omega = \dfrac{dx}{y}$ is identified with the lattice Λ defining the complex torus $\mathbb{C}/\Lambda$. It illustrates concretely the relationship between periods and the complex torus structure of an elliptic curve.