The isomorphism $\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(x)$

We explain carefully why the field of meromorphic functions on $\mathbb{CP}^1$ is isomorphic to the field $\mathbb{C}(x)$ of rational functions in one variable.

1. Setup: charts on $\mathbb{CP}^1$

View $\mathbb{CP}^1$ as the Riemann sphere. Consider the standard affine chart

$$U_1 = \{[z_0 : z_1] \in \mathbb{CP}^1 \mid z_1 \neq 0\},$$

with coordinate map

$$\phi_1 : U_1 \longrightarrow \mathbb{C}, \qquad \phi_1([z_0 : z_1]) = \frac{z_0}{z_1}.$$

We write

$$x := \phi_1,$$

and think of x as the *coordinate function* on U_1. This function extends meromorphically to all of $\mathbb{CP}^1$, with a simple pole at $\infty = [1 : 0]$.

We define the field of meromorphic functions on $\mathbb{CP}^1$ as

$$\mathcal{M}(\mathbb{CP}^1) = \{F : \mathbb{CP}^1 \to \mathbb{CP}^1 \mid F \text{ holomorphic}\},$$

viewing a meromorphic function as a holomorphic map into $\mathbb{CP}^1$ (via the usual convention "finite value $\mapsto [f(p) : 1]$, pole $\mapsto [1 : 0]$").

On the other hand, the field $\mathbb{C}(x)$ is

$$\mathbb{C}(x) = \left\{ \frac{p(x)}{q(x)} \;\middle|\; p, q \in \mathbb{C}[x],\ q \not\equiv 0 \right\} \Big/ \sim,$$

where $\frac{p}{q} \sim \frac{p'}{q'}$ if $p(x)q'(x) = p'(x)q(x)$.

2. From $\mathbb{C}(x)$ to $\mathcal{M}(\mathbb{CP}^1)$

Lemma 1. *Every rational function $R(x) \in \mathbb{C}(x)$ defines a meromorphic function $F_R \in \mathcal{M}(\mathbb{CP}^1)$.*

Proof. Write

$$R(x) = \frac{p(x)}{q(x)}, \quad p, q \in \mathbb{C}[x],\ q \not\equiv 0.$$

On the affine chart U_1. For a point $[z_0 : z_1] \in U_1$, set $x([z_0 : z_1]) = z_0/z_1 =: z$. We *define* F_R on U_1 by

$$\phi_1\big(F_R([z_0 : z_1])\big) = R\big(\phi_1([z_0 : z_1])\big) = R(z),$$

i.e.

$$F_R|_{U_1} = \phi_1^{-1} \circ R \circ \phi_1.$$

Concretely, for $z_1 \neq 0$ and $R(z) \neq \infty$,

$$F_R([z_0 : z_1]) = [R(z_0/z_1) : 1],$$

and if $R(z) = \infty$ (i.e. $q(z) = 0$), we set

$$F_R([z_0 : z_1]) = [1 : 0].$$

Global description via homogeneous polynomials. Let

$$m = \max\{\deg p, \deg q\},$$

and define homogeneous polynomials of degree m by

$$P(z_0, z_1) = z_1^m\, p\left(\frac{z_0}{z_1}\right), \qquad Q(z_0, z_1) = z_1^m\, q\left(\frac{z_0}{z_1}\right).$$

Then we set, for $[z_0 : z_1] \in \mathbb{CP}^1$,

$$F_R([z_0 : z_1]) = \begin{cases} [P(z_0, z_1) : Q(z_0, z_1)], & Q(z_0, z_1) \neq 0, \\ [1 : 0], & Q(z_0, z_1) = 0. \end{cases}$$

This is well-defined on projective space and holomorphic everywhere (homogeneous polynomials define holomorphic maps on $\mathbb{CP}^1$).

On the chart U_1, this construction coincides with $\phi_1^{-1} \circ R \circ \phi_1$. Thus F_R is a meromorphic function on $\mathbb{CP}^1$ (equivalently, a holomorphic map $\mathbb{CP}^1 \to \mathbb{CP}^1$).

$\square$

Hence we have a well-defined map

$$\Phi : \mathbb{C}(x) \longrightarrow \mathcal{M}(\mathbb{CP}^1), \qquad R \longmapsto F_R.$$

One checks directly that Φ respects addition and multiplication, so Φ is a field homomorphism.

3. Meromorphic functions on $\mathbb{CP}^1$ are rational

Lemma 2. *Every meromorphic function on $\mathbb{CP}^1$ (i.e. holomorphic map $\mathbb{CP}^1 \to \mathbb{CP}^1$ in the affine chart) is a rational function in the coordinate x.*

Sketch. Let $G \in \mathcal{M}(\mathbb{CP}^1)$. Consider the affine chart $U_1 \subset \mathbb{CP}^1$ with coordinate x, and restrict G to U_1. On the open set

$$V := G^{-1}(U_1) \subseteq \mathbb{CP}^1,$$

we can view the composition

$$g := \phi_1 \circ G : V \to \mathbb{C}$$

as a holomorphic function. The complement $\mathbb{CP}^1 \setminus V = G^{-1}(\infty)$ is finite, so g has only finitely many poles in the coordinate x, and possibly a pole at ∞.

Thus, via the coordinate x, g is a meromorphic function on the Riemann sphere $\mathbb{C} \cup \{\infty\}$. A standard result in complex analysis says that such a meromorphic function is a rational function:

$$g(x) = R(x) \in \mathbb{C}(x).$$

Concretely, one proves this by constructing a rational function with the same poles and principal parts as g, and then showing that their difference is entire and bounded on $\mathbb{C} \cup \{\infty\}$, hence constant. Therefore $g(x)$ is rational. $\square$

Applying this to G, we obtain a unique $R(x) \in \mathbb{C}(x)$ such that

$$\phi_1 \circ G = R \circ \phi_1 \quad \text{on } U_1.$$

4. From $\mathcal{M}(\mathbb{CP}^1)$ to $\mathbb{C}(x)$

Define

$$\Psi : \mathcal{M}(\mathbb{CP}^1) \longrightarrow \mathbb{C}(x)$$

by

$$\Psi(G) = R(x),$$

where $R(x)$ is the rational function given by the lemma, i.e. the unique function satisfying

$$\phi_1 \circ G = R \circ \phi_1 \quad \text{on } U_1.$$

This is a well-defined field homomorphism (composition of meromorphic maps corresponds to composition of rational functions).

5. Φ and Ψ are inverse isomorphisms

We now check that Φ and Ψ are inverses.

- For $R(x) \in \mathbb{C}(x)$,

  $$\Psi(\Phi(R)) = \Psi(F_R) = (\text{the rational function corresponding to } F_R).$$

  But on U_1,
  $$\phi_1 \circ F_R = R \circ \phi_1,$$
  by construction of F_R. Hence $\Psi(F_R) = R$, so

  $$\Psi \circ \Phi = \mathrm{id}_{\mathbb{C}(x)}.$$

- For $G \in \mathcal{M}(\mathbb{CP}^1)$, let $R = \Psi(G) \in \mathbb{C}(x)$. Then

  $$\phi_1 \circ G = R \circ \phi_1 \quad \text{on } U_1.$$

  But on U_1, we also have by definition

  $$F_R = \phi_1^{-1} \circ R \circ \phi_1.$$

  So G and F_R agree on the nonempty open set U_1. Since both are holomorphic maps $\mathbb{CP}^1 \to \mathbb{CP}^1$, the identity theorem implies $G \equiv F_R$ on all of $\mathbb{CP}^1$. Thus

  $$\Phi(\Psi(G)) = F_{\Psi(G)} = G,$$

  so
  $$\Phi \circ \Psi = \mathrm{id}_{\mathcal{M}(\mathbb{CP}^1)}.$$

Therefore Φ and Ψ are inverse field isomorphisms.

Theorem 1. *The field of meromorphic functions on $\mathbb{CP}^1$ is isomorphic to the field of rational functions in one variable:*

$$\boxed{\mathcal{M}(\mathbb{CP}^1) \;\cong\; \mathbb{C}(x).}$$