Linear Algebra III

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We cover the following topics in this note.

- Determinant and Inverse matrix

Contents

1 Determinant

1.1 Matrix equation and Group structure

Let $\mathbb{F}$ be a field and let $\mathrm{Mat}_{m\times n}(\mathbb{F})$ denote the set of $m \times n$ matrices with entries in $\mathbb{F}$. Given matrices

$$A = (a_{ij})_{1\le i\le m,\, 1\le j\le n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in \mathrm{Mat}_{m\times n}(\mathbb{F}),$$

$$X = (x_{jk})_{1\le j\le n,\, 1\le k\le \ell} = \begin{bmatrix} x_{11} & \cdots & x_{1\ell} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{n\ell} \end{bmatrix} \in \mathrm{Mat}_{n\times \ell}(\mathbb{F}),$$

$$B = (b_{ik})_{1\le i\le m,\, 1\le k\le \ell} = \begin{bmatrix} b_{11} & \cdots & b_{1\ell} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{m\ell} \end{bmatrix} \in \mathrm{Mat}_{m\times \ell}(\mathbb{F}),$$

the matrix equation

$$AX = B \quad \left(\Leftrightarrow \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_{11} & \cdots & x_{1\ell} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{n\ell} \end{bmatrix} = \begin{bmatrix} b_{11} & \cdots & b_{1\ell} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{m\ell} \end{bmatrix} \right)$$

means the entries satisfy

$$b_{ik} = \sum_{j=1}^{n} a_{ij}x_{jk}, \qquad 1 \le i \le m,\ 1 \le k \le \ell.$$

To solve $AX = B$ by "dividing by A" from the left, we need an inverse matrix for A. Thus we restrict to square matrices and look at the multiplicative structure. Define

$$G = \mathrm{GL}_n(\mathbb{F}) = \left\{ A \in \mathrm{Mat}_{n\times n}(\mathbb{F}) : \exists\, A^{-1} \text{s.t.} AA^{-1} = A^{-1}A = I_n \right\}.$$

Then G is a group under matrix multiplication.

1.2 Computing A^{-1} by cofactors

Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{F})$ and write

$$A = (a_{ij})_{1\le i,j\le n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}.$$

To find A^{-1} we look for an $n \times n$ matrix

$$X = (x_{kj})_{1\le k,j\le n}$$

such that

$$AX = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = I_n.$$

In entries this means

$$\delta_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \qquad 1 \le i,j \le n,$$

where δ_{ij} is the *Kronecker delta*:

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$$

Recall the Laplace expansion (expansion along the j-th column):

$$\det A = \sum_{k=1}^{n} a_{kj} C_{kj},$$

where

$$C_{kj} = (-1)^{k+j} M_{kj}$$

is the *cofactor* of the entry a_{kj}, and M_{kj} is the determinant of the $(n-1)\times(n-1)$ minor obtained by deleting the k-th row and j-th column of A.

Fix a column index j. Set

$$b_{kj} = \frac{C_{jk}}{\det A}, \qquad k = 1, \dots, n.$$

Then

$$\sum_{k=1}^{n} a_{ik} b_{kj} = \frac{1}{\det A} \sum_{k=1}^{n} a_{ik} C_{jk} = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$$

Hence $AB = I_n$.

Define the *adjugate matrix* of A by

$$\operatorname{adj}(A) = (C_{ij})_{1 \le i,j \le n}.$$

In matrix form we have

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)^{\mathsf{T}}.$$

Therefore A^{-1} exists only when $\det A \neq 0$.

1.3 Determinant

Let $A \in \mathrm{Mat}_{n\times n}(F)$, $A = (a_{ij})$. The *determinant* of A is

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)},$$

where S_n is the symmetric group on $\{1, \dots, n\}$ and $\operatorname{sgn}(\sigma) \in \{\pm 1\}$ is the sign of the permutation σ.

Example ($n = 2$). For

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we have

$$\det A = ad - bc.$$

Geometrically, if A is regarded as having rows (or columns) the vectors (a, b) and (c, d) in $\mathbb{R}^2$, then $\det A$ is the signed area of the parallelogram spanned by these vectors.

More generally, for $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, $\det A$ is the signed volume of the parallelotope spanned by the row vectors of A.

Exercise. Show that

$$\det A = \det A^{\mathsf{T}}.$$

Thus $\det A$ is also the signed volume spanned by the column vectors of A.

Exercise. For $A, B \in \mathrm{Mat}_{n\times n}(F)$, prove that

$$\det(AB) = (\det A)(\det B).$$

Equivalent conditions for invertibility

Proposition 1. *Let $A \in \mathrm{Mat}_{n\times n}(F)$. The following conditions are equivalent:*

1. *A is invertible, i.e. there exists A^{-1} such that $AA^{-1} = A^{-1}A = I_n$.*

2. *The homogeneous system $AX = 0$ has only the trivial solution $X = 0$.*

3. *A is row-equivalent to the identity matrix I_n.*

4. *For every $B \in \mathrm{Mat}_{n\times\ell}(F)$ (any number of columns ℓ), the system $AX = B$ has a solution X.*

5. $\det A \neq 0$.

6. $\operatorname{rank} A = n$ *(i.e. A has full rank).*

7. *The linear map*
$$T_A : F^n \to F^n, \qquad T_A(x) = Ax,$$
is onto, i.e. $\operatorname{Im} T_A = F^n$ *and hence* $\dim \operatorname{Im} T_A = \operatorname{rank} A = n$.

8. *The row vectors of A are linearly independent.*

9. *The column vectors of A are linearly independent.*

Proof. As one example, we show (1) $\Rightarrow$ (2). Suppose $AX = 0$ and A is invertible. Then
$$X = A^{-1}(AX) = A^{-1}0 = 0,$$
so the only solution is $X = 0$. The remaining implications are proved by standard arguments from linear algebra. □

Elementary row operations

Definition 1. Given any matrix, an *elementary row operation* is one of:

1. Multiply a row by a nonzero scalar.
2. Interchange two rows.
3. Replace a row by itself plus a scalar multiple of another row.

These operations are used to bring a matrix to row-echelon form and, in the square case, to compute inverses.

Computing the inverse by row reduction

To compute the inverse of an invertible matrix $A \in \mathrm{Mat}_{n\times n}(F)$ by row operations, form the augmented matrix
$$\left[\, A \mid I_n \,\right]$$
and apply elementary row operations until the left block becomes I_n:
$$\left[\, A \mid I_n \,\right] \sim \left[\, I_n \mid A^{-1} \,\right].$$

Then the right block is A^{-1}.

2 Coordinate Change Matrix and Similarity

References

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