Linear Algebra IV

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We cover the following topics in this note.

- Eigenvectors and Diagonalization.
- Characteristic polynomial.
- TBA.

Notation.

$\mathbb{F}$ a field.

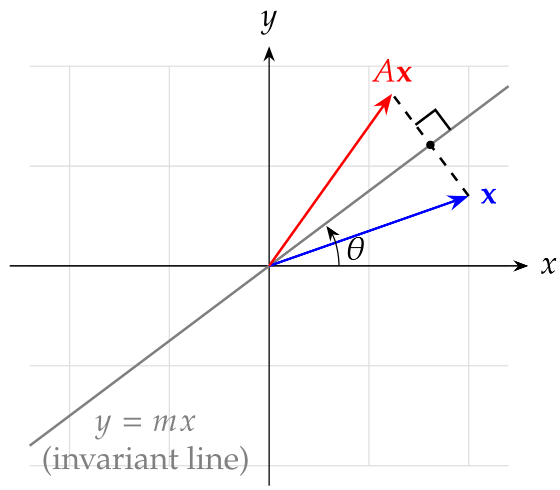
V a finite-dimensional $\mathbb{F}$-vector space.

Observation (Choosing a basis to simplify a linear map). Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of V such that $[T]_{\mathcal{B}}$ is a diagonal matrix:

$$[T]_{\mathcal{B}} = [T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}_{n\times n}, \quad \text{i.e.,} \quad T(\mathbf{v}_i) = d_i \mathbf{v}_i \text{ with } 1 \le i \le n.$$

Then T may be very complicated, but with respect to $[T]_{\mathcal{B}}$ it looks nice.

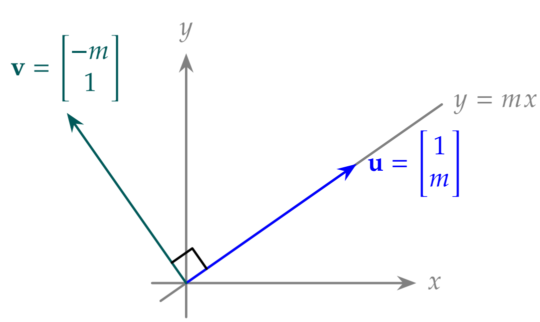
Example 1. Fix $m \in \mathbb{R}$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection across the line $y = mx$.



Step 1: Choose a basis adapted to the axis. Take

$$\mathbf{u} = \begin{bmatrix} 1 \\ m \end{bmatrix} \quad (\text{along } y = mx), \qquad \mathbf{v} = \begin{bmatrix} -m \\ 1 \end{bmatrix} \quad (\text{perpendicular to } y = mx),$$

since $\mathbf{u} \cdot \mathbf{v} = 1 \cdot (-m) + m \cdot 1 = 0$.



Step 2: Use the defining property of reflection. Reflection across the line

$$L = \operatorname{span}(\mathbf{u}) = \{(t, mt) : t \in \mathbb{R}\}$$

is defined by

- **vectors on the axis** stay fixed, and
- **vectors perpendicular to the axis** reverse direction.

Every vector $\mathbf{w} \in \mathbb{R}^2$ decomposes uniquely as

$$\mathbf{w} = a\,\mathbf{u} + b\,\mathbf{v}.$$

A reflection keeps the parallel component and flips the perpendicular one, so

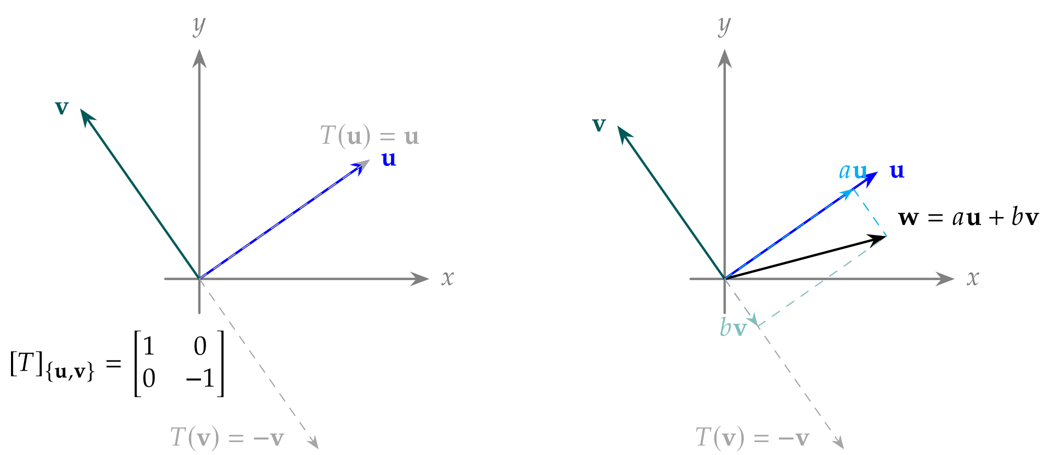
$$T(a\mathbf{u}) = a\mathbf{u} \qquad T(b\mathbf{v}) = -b\mathbf{v}.$$

Since T is linear,

$$T(a\mathbf{u} + b\mathbf{v}) = T(a\mathbf{u}) + T(b\mathbf{v}) = a\mathbf{u} - b\mathbf{v}.$$

In the basis $\mathcal{B} = \{\mathbf{u}, \mathbf{v}\}$,

$$[T]_{\mathcal{B}} = \begin{bmatrix} T(\mathbf{u}) & T(\mathbf{v}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$



Step 3: Convert back to the standard basis. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis and let $\mathcal{B} = \{\mathbf{u}, \mathbf{v}\}$ be another basis. Define the change-of-coordinates matrix

$$P := [\mathrm{Id}]_{\mathcal{B}}^{\mathcal{E}} = \begin{bmatrix} \mathrm{Id}(\mathbf{u}) & \mathrm{Id}(\mathbf{v}) \end{bmatrix} = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}.$$

Then for every $\mathbf{x} \in \mathbb{R}^2$ we have

$$[\mathbf{x}]_{\mathcal{E}} = P[\mathbf{x}]_{\mathcal{B}} \qquad \text{and} \qquad [\mathbf{x}]_{\mathcal{B}} = P^{-1}[\mathbf{x}]_{\mathcal{E}}.$$

If $D = [T]_{\mathcal{B}}^{\mathcal{B}}$ is the matrix of T in the basis $\mathcal{B}$, then

$$[T(\mathbf{x})]_{\mathcal{B}} = D[\mathbf{x}]_{\mathcal{B}}.$$

Converting the output back to standard coordinates gives

$$[T(\mathbf{x})]_{\mathcal{E}} = P[T(\mathbf{x})]_{\mathcal{B}} = P\,D\,[\mathbf{x}]_{\mathcal{B}} = P\,D\,P^{-1}[\mathbf{x}]_{\mathcal{E}}.$$

Therefore the standard-basis matrix of T is

$$\begin{aligned} A := [T]_{\mathcal{E}}^{\mathcal{E}} &= [\mathrm{Id}]_{\mathcal{B}}^{\mathcal{E}}\,[T]_{\mathcal{B}}^{\mathcal{B}}\,[\mathrm{Id}]_{\mathcal{E}}^{\mathcal{B}} \\ &= \underbrace{P}_{\text{Convert back to standard coordinates}} \quad \underbrace{D}_{\text{Apply } T \text{ in } \{\mathbf{u},\mathbf{v}\}\text{-coordinates}} \quad \underbrace{P^{-1}}_{\text{Convert to } \{\mathbf{u},\mathbf{v}\}\text{-coordinates}}. \end{aligned}$$

Compute P^{-1} (since $\det P = 1 + m^2$):

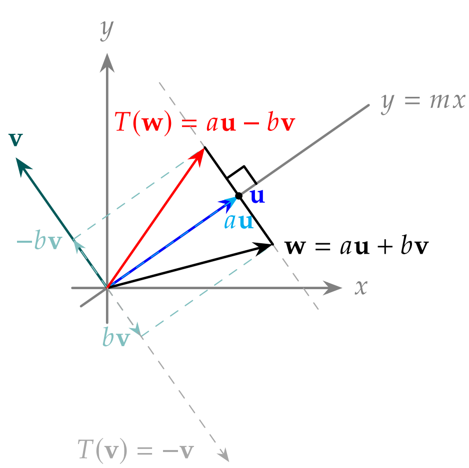
$$P^{-1} = \frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ -m & 1 \end{bmatrix}.$$

Since

$$PD = \begin{bmatrix} 1 & -m \\ m & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & m \\ m & -1 \end{bmatrix},$$

we obtain

$$\begin{aligned} A &= (PD)P^{-1} \\ &= \frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & -1 \end{bmatrix}\begin{bmatrix} 1 & m \\ -m & 1 \end{bmatrix} \\ &= \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}. \end{aligned}$$



Eigenvector & Eigenvalue

Definition. Let $T : V \to V$ be $\mathbb{F}$-linear. A nonzero vector $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ is an **eigenvector** of T if $\exists \lambda \in \mathbb{F}$ such that

$$T(\mathbf{v}) = \lambda \mathbf{v} \in V.$$

The scalar λ is called the **eigenvalue** corresponding to $\mathbf{v}$.

Remark 1. If $\mathbf{v} \neq 0$ and $T(\mathbf{v}) = \lambda\mathbf{v}$, then the one-dimensional subspace $\mathbb{F}\mathbf{v}$ satisfying

$$\begin{array}{rccc} T|_{\mathbb{F}\mathbf{v}} : & \mathbb{F}\mathbf{v} & \longrightarrow & \mathbb{F}\mathbf{v} \\ & c\mathbf{v} & \longmapsto & \lambda(c\mathbf{v}) \end{array} \qquad (\because T(c\mathbf{v}) = cT(\mathbf{v}) = c\lambda\mathbf{v} = \lambda(c\mathbf{v})),$$

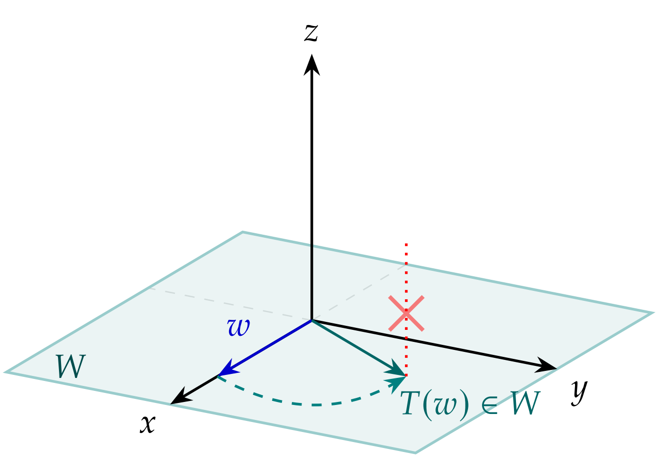
Equivalently, the restriction $T|_{\mathbb{F}\mathbf{v}} : \mathbb{F}\mathbf{v} \to \mathbb{F}\mathbf{v}$ acts as scalar multiplication by λ.

Remark 2 (T-invariant)**.** Let $T : V \to V$ be $\mathbb{F}$-linear. Let a subspace $W \leq V$ satisfy

$$T[W] \subseteq W \qquad (\iff \forall \mathbf{w} \in W,\ T(\mathbf{w}) \in W).$$

1. The restriction map
$$T|_W : W \to W, \qquad \mathbf{w} \mapsto T(\mathbf{w})$$
is a well-defined linear operator on W.

2. If $\dim V < \infty$ and $\mathcal{B} = (\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n)$ is a basis of V such that $(\mathbf{w}_1, \ldots, \mathbf{w}_k)$ is a basis of W, then
$$[T]_{\mathcal{B}} = \begin{pmatrix} A & * \\ 0 & B \end{pmatrix},$$
where A represents $T|_W$ and B represents the induced map on V/W.



$$\begin{array}{c|ccc} & e_1 \in W & e_2 \in W & \\ \hline x & \cos\theta & -\sin\theta & 0 \\ y & \sin\theta & \cos\theta & 0 \\ z & 0 & 0 & 1 \end{array}$$

Diagonalizability of Linear Operator

Definition 1. We say $T : V \to V$ is *diagonalizable* if $\exists$ a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is diagonal.

Remark 3. T is diagonalizable if and only if V has a basis consisting of eigenvectors of T.
A diagonal matrix

$$\begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

acts by scaling each basis vector $\mathbf{v}_i$ *independently*, and "scaling a nonzero vector" is the eigenvector condition $T(\mathbf{v}_i) = d_i \mathbf{v}_i$.

Note. Let $A \in \mathrm{Mat}_n(\mathbb{F})$. The associated linear map $L_A : \mathbb{F}^n \to \mathbb{F}^n$ is given by $L_A(\mathbf{x}) = A\mathbf{x}$.

Diagonalizability of Matrix

Definition 2. A matrix $A \in \mathrm{Mat}_n(\mathbb{F})$ is *diagonalizable over* $\mathbb{F}$ if $\exists P \in \mathrm{GL}_n(\mathbb{F})$ and a diagonal matrix D such that

$$D = P^{-1}AP.$$

Eigenbasis and Similarity

Proposition 1. *Let $A \in \mathrm{Mat}_n(\mathbb{F})$.*

(i) If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of $\mathbb{F}^n$ consisting of eigenvectors of A, and $P = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$, then $P^{-1}AP$ is diagonal with $P \in \mathrm{GL}_n(\mathbb{F})$.

(ii) Conversely, if $P^{-1}AP = D$ is diagonal, then the columns of P form an eigenbasis of A (with eigenvalues given by the diagonal entries of D).

Characteristic polynomial

Definition 3. For $A \in \mathrm{Mat}_n(\mathbb{F})$, the **characteristic polynomial** of A is

$$\chi_A(x) := \det(A - x\mathrm{Id}_n) \in \mathbb{F}[x].$$

Eigenvalues are roots

Proposition 2. *Let* $A \in \mathrm{Mat}_n(\mathbb{F})$ *and define the characteristic polynomial*

$$\chi_A(x) := \det(A - x\mathrm{Id}_n) \in \mathbb{F}[x].$$

A scalar $\lambda \in \mathbb{F}$ *is an eigenvalue of* A *if and only if* $\chi_A(x) = 0$.

Observation (Characteristic polynomial in 2×2). Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}_2(\mathbb{F})$. Then

$$
\begin{aligned}
\det(A - \lambda I_2) &= \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \\
&= (a - \lambda)(d - \lambda) - bc \\
&= ad - (a + d)\lambda + \lambda^2 - bc \\
&= \lambda^2 - (a + d)\lambda + (ad - bc) \\
&= \lambda^2 - \mathrm{tr}(A)\lambda + \det(A).
\end{aligned}
$$

$$\chi_A(\lambda) = \lambda^2 - (\mathrm{tr}A)\,\lambda + \det(A)$$

$$\begin{pmatrix} \boxed{a} & b \\ c & \boxed{d} \end{pmatrix} \qquad \boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

Observation (Characteristic polynomial in 3×3). Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \in \mathrm{Mat}_3(\mathbb{F})$. Then

$$
\begin{aligned}
\det(A - \lambda I_3) &= \det \begin{bmatrix} a - \lambda & b & c \\ d & e - \lambda & f \\ g & h & i - \lambda \end{bmatrix} = (a - \lambda) \det \begin{bmatrix} e - \lambda & f \\ h & i - \lambda \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i - \lambda \end{bmatrix} + c \det \begin{bmatrix} d & e - \lambda \\ g & h \end{bmatrix} \\
&= (a - \lambda)\big((e - \lambda)(i - \lambda) - fh\big) - b\big(d(i - \lambda) - fg\big) + c\big(dh - ge + g\lambda\big) \\
&= (a - \lambda)\big(ei - (e + i)\lambda + \lambda^2 - fh\big) - b\big(di - d\lambda - fg\big) + c\big(dh - ge + g\lambda\big) \\
&= \big(aei - a(e + i)\lambda + a\lambda^2 - afh\big) - \big(ei\lambda - (e + i)\lambda^2 + \lambda^3 - fh\lambda\big) \\
&\quad - \big(bdi - bd\lambda - bfg\big) + \big(cdh - cge + cg\lambda\big) \\
&= -\lambda^3 + (a + e + i)\lambda^2 - (ae + ai + ei - fh - bd - cg)\lambda + (aei - afh - bdi - bfg + cdh - cge) \\
&= -\lambda^3 + \mathrm{tr}(A)\lambda^2 - \left(\det \begin{bmatrix} a & b \\ d & e \end{bmatrix} + \det \begin{bmatrix} a & c \\ g & i \end{bmatrix} + \det \begin{bmatrix} e & f \\ h & i \end{bmatrix}\right)\lambda + \det(A).
\end{aligned}
$$

$$\chi_A(\lambda) = -\lambda^3 + c_2\lambda^2 - c_1\lambda + c_0$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$a + e + i$ $\qquad$ $\det\begin{bmatrix} a & b \\ d & e \end{bmatrix} + \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} + \det\begin{bmatrix} a & c \\ g & i \end{bmatrix}$ $\qquad$ $\det(A)$

Note (Principal minors). A principal minor of a square matrix is the determinant of a principal submatrix, which is formed by selecting the same set of indices for both rows and columns.

Let $A = (a_{ij}) \in \mathrm{Mat}_n(\mathbb{F})$ and $\{i_1 < i_2 < \cdots < i_\ell\} \subseteq \{1, \ldots, n\}$. For each $\ell = 1, \ldots, n$, we define

$$E_\ell(A) := \sum_{1 \le i_1 < \cdots < i_k \le n} \det \begin{bmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \cdots & a_{i_1,i_k} \\ a_{i_2,i_1} & a_{i_2,i_2} & \cdots & a_{i_2,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k,i_1} & a_{i_k,i_2} & \cdots & a_{i_k,i_k} \end{bmatrix},$$

so

$$\begin{aligned} E_1(A) &:= \sum_{i=1}^n \det \begin{bmatrix} a_{ii} \end{bmatrix} = \sum_{i=1}^n a_{ii} = \mathrm{tr}(A), \\ E_2(A) &:= \sum_{1 \le i < j \le n} \det \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}, \\ E_3(A) &:= \sum_{1 \le i < j < k \le n} \det \begin{bmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{bmatrix}, \\ &\cdots, \\ E_n(A) &:= \det(A). \end{aligned}$$

Characteristic polynomial: Trace and Determinant as coefficients

Theorem 3. *Let $A = (a_{ij}) \in \mathrm{Mat}_n(\mathbb{F})$ and define*

$$\chi_A(x) := \det(A - x\mathrm{Id}_n) \in \mathbb{F}[x].$$

For $k = 1, \ldots, n$ define $E_k(A)$ by

$$E_k(A) \ := \sum_{1 \le i_1 < \cdots < i_k \le n} \det \left[a_{i_p i_q} \right]_{p,q=1}^k,$$

so $E_1(A) = \sum_{i=1}^n a_{ii} = \mathrm{tr}(A)$ and $E_n(A) = \det(A)$. Then

$$\chi_A(x) = (-1)^n \left(x^n - E_1(A)x^{n-1} + E_2(A)x^{n-2} - \cdots + (-1)^n E_n(A) \right).$$

Proof. □

References

[1] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 31. 선형대수학 (h) 고유벡터와 행렬의 대각화 -1” YouTube Video, 29:46. Published November 06, 2019. URL: https://www.youtube.com/watch?v=RSOxa1rI_Kk.