From Vector Calculus to Differential Forms

Ji, Yong-hyeon

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We cover the following topics in this note.

- Vector calculus (conservative fields, irrotational field)
- Differential forms (exact forms, closed forms)

Vector Calculus (in $\mathbb{R}^2$ or $\mathbb{R}^3$)		**Differential Forms**
Vector Field $\mathbf{F}$	$\Longleftrightarrow$	1-form ω
Conservative Vector Field ($\mathbf{F} = \nabla f$)	$\Longleftrightarrow$	Exact 1-form ($\omega = \mathrm{d}f$)
Irrotational Vector Field ($\nabla \times \mathbf{F} = \mathbf{0}$)	$\Longleftrightarrow$	Closed 1-form ($\mathrm{d}\omega = 0$)

The Fundamental Implication:

- **Conservative** $\implies$ **Irrotational**, i.e., **Exact** $\implies$ **Closed**
  (This is always true.)
- **Irrotational** $\implies$ **Conservative**, i.e, **Closed** $\implies$ **Exact**
  (This is only true on "nice" domains, e.g., simply connected ones.)

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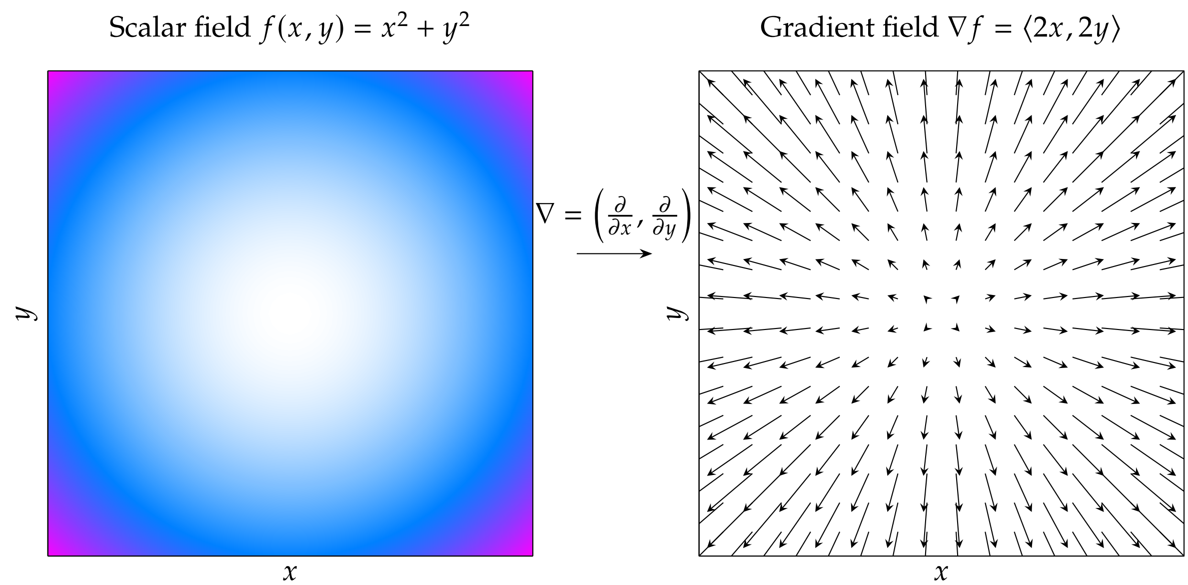
1 Conservative and Exact

Why do we think of the "Conservative Field (or Gradient Field)"?



A vector field **F** is **conservative** if it is the **gradient**[1] of some scalar potential function f:

$$\mathbf{F} = \nabla f = \begin{cases} \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle & \text{if } f : U(\subseteq \mathbb{R}^2) \to \mathbb{R} \\ \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle & \text{if } f : U(\subseteq \mathbb{R}^3) \to \mathbb{R} \\ \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle & \text{if } f : U(\subseteq \mathbb{R}^n) \to \mathbb{R} \end{cases}.$$

If **F** is conservative, line integrals only depend on the endpoints:

$$\int_\gamma \mathbf{F} \cdot d\mathbf{r} = \int_\gamma \nabla f \cdot d\mathbf{r} = f(\text{end}) - f(\text{start}),$$

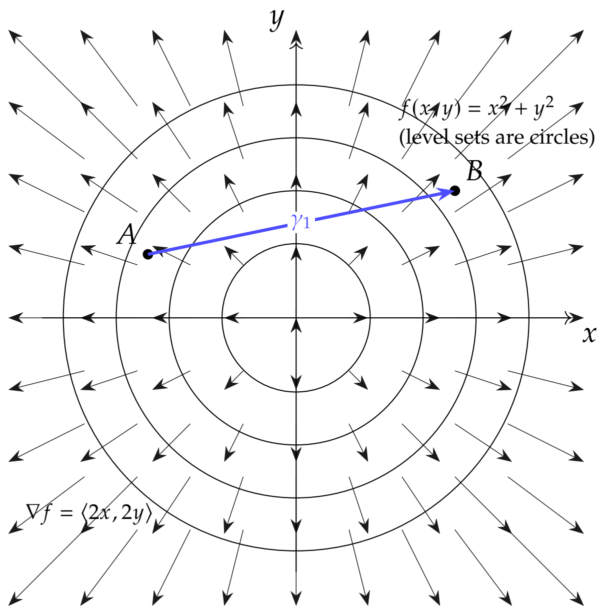
which turns hard line integrals into simple evaluations. Furthermore every closed loop (start = end) has integral 0.

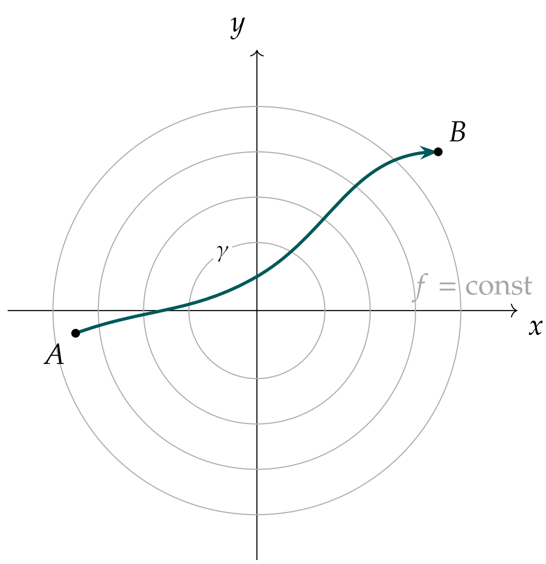
Equivalently (on a nice domain) every line integral of **F** is path-independent:

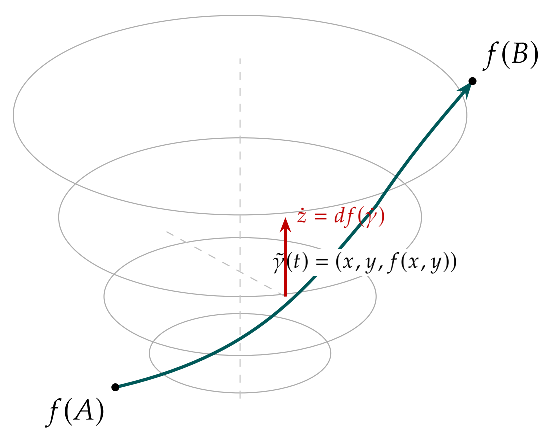
$$\int_{\gamma_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\gamma_2} \mathbf{F} \cdot d\mathbf{r}.$$

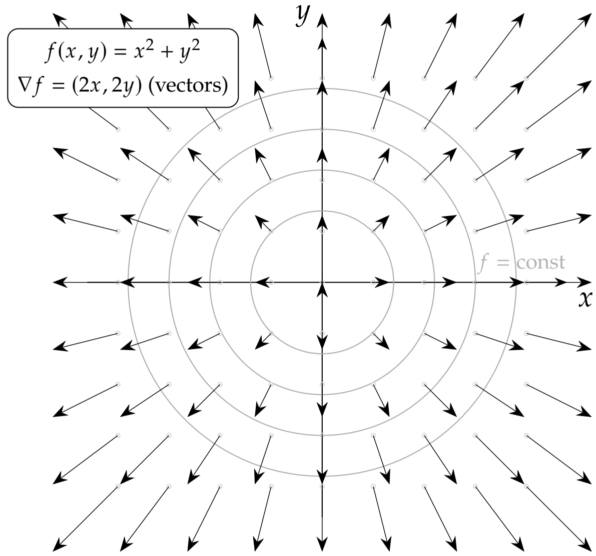
whenever γ_1 and γ_2 have the same endpoints.

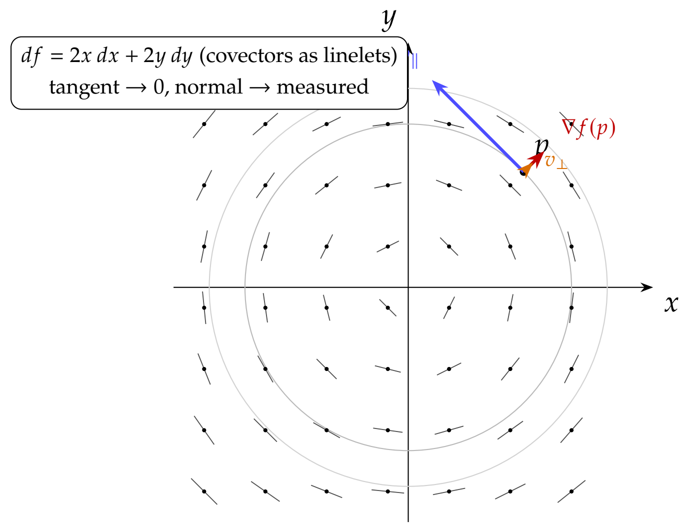
[1] Gradient is a measure of change in a scalar field


y
x
$f(x,y) = x^2 + y^2$
(level sets are circles)
A
B
γ_1
$\nabla f = \langle 2x, 2y \rangle$


y
x
B
A
γ
f = const


$f(B)$
$\dot{z} = df(\dot{\gamma})$
$\tilde{\gamma}(t) = (x, y, f(x,y))$
$f(A)$





From Gradients to Curl

Given a vector field **F**, we wish to determine whether **F** is conservative (i.e., $\mathbf{F} = \nabla f$ for some scalar field f). Trying to guess the potential function f is hard.

We already know that if a field **F** is conservative, it must be the gradient of some potential function f:

$$\mathbf{F} = \langle P(x, y), Q(x, y)\rangle = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \qquad \text{(in 2D)}$$

What happens if we differentiate $P(x, y) = \frac{\partial f}{\partial x}$ with respect to y and $Q(x, y) = \frac{\partial f}{\partial y}$ with respect to x?

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}, \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}.$$

Theorem 1 (Equality of mixed partials; Clairaut's Theorem). *If the partial derivatives $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ exist and are continuous at a point (a, b), then $\frac{\partial^2 f}{\partial y \partial x}(a, b) = \frac{\partial^2 f}{\partial x \partial y}(a, b)$, i.e., second order partial derivatives commute if f is C^2.*

If a vector field $\mathbf{F} = \langle P(x, y), Q(x, y)\rangle$ is a gradient, it **must** satisfy the condition

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

The quantity $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ represents the curl of $\mathbf{F} = \langle P, Q\rangle$ and encodes its local rotational behavior. Hence the condition $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ (i.e., $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$) means that the field is **irrotational** (has **zero curl**).

Remark 1. Consider a small rectangle centered at (x_0, y_0) with side lengths $\Delta x, \Delta y$.

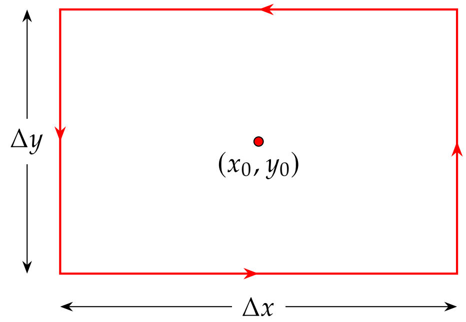


Figure 1: Circulation around an infinitesimal rectangle.

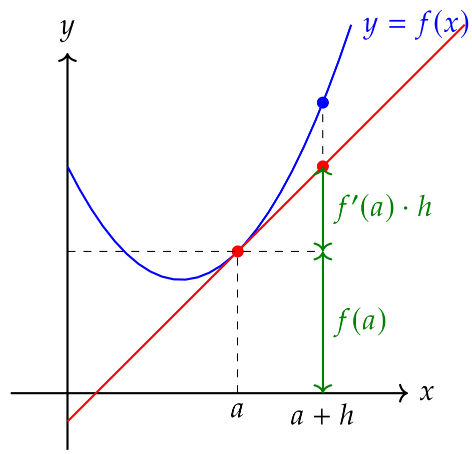
The total counterclockwise circulation is the sum of the line integrals along the four edges:

$$\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \int_{\text{bottom}} P\,dx + \int_{\text{right}} Q\,dy + \int_{\text{top}} P\,dx + \int_{\text{left}} Q\,dy.$$

We will approximate the value of P or Q along each edge as being constant, equal to its value at the midpoint of that edge. We find this value using a first-order Taylor expansion from the center point (x_0, y_0).

For a simple function of one variable, $f(x)$, if we know its value at a point a, then we can estimate its value at a nearly point $a + h$ using the tangent line at a:

$$f(a+h) \approx f(a) + f'(a) \cdot h \quad \text{or}$$
$$f(x) \approx f(a) + f'(a) \cdot (x - a).$$



In words, “New Value ≈ Old Value + (Rate of Change) × (Small Step)”.

For a function of two variables like $P(x, y)$, the idea is identical, but the "rate of change" now has two components (one for each direction), and the "tangent line" becomes a "tangent plane". The general first-order Taylor expansion for $P(x, y)$ around a center point (x_0, y_0) is

$$P(x_0 + a, y_0 + b) \approx P(x_0, y_0) + \frac{\partial P}{\partial x}(x_0, y_0) \cdot a + \frac{\partial P}{\partial y}(x_0, y_0) \cdot b$$

Here, a is the small step in the x-direction, and b is the small step in the y-direction.

1. The Horizontal Paths These integrals involve the horizontal component of $P(x, y)$.

- **Bottom Path (⟶):**

$$P\left(x, y_0 - \frac{\Delta y}{2}\right) \approx P(x_0, y_0) - \frac{\partial P}{\partial y}\frac{\Delta y}{2} \implies \int_{\text{bottom}} P \, \mathrm{d}x \approx \left(P(x_0, y_0) - \frac{\partial P}{\partial y}\frac{\Delta y}{2}\right)(\Delta x)$$

- **Top Path (⟵):**

$$P\left(x, y_0 + \frac{\Delta y}{2}\right) \approx P(x_0, y_0) + \frac{\partial P}{\partial y}\frac{\Delta y}{2} \implies \int_{\text{top}} P \, \mathrm{d}x \approx -\left(P(x_0, y_0) + \frac{\partial P}{\partial y}\frac{\Delta y}{2}\right)(\Delta x)$$

Here, we are left with only the parts that describe the *change* in P with respect to y.

$$\int_{\text{bottom}} P \, \mathrm{d}x + \int_{\text{top}} P \, \mathrm{d}x \approx \left(-\frac{\partial P}{\partial y}\frac{\Delta y}{2}\right)\Delta x - \left(\frac{\partial P}{\partial y}\frac{\Delta y}{2}\right)\Delta x = -\frac{\partial P}{\partial y}\Delta x \Delta y$$

2. The Vertical Paths These integrals involve the vertical component of $Q(x, y)$.

- **Right Path (↑):**

$$Q\left(x_0 + \frac{\Delta x}{2}, y\right) \approx Q(x_0, y_0) + \frac{\partial Q}{\partial x}\frac{\Delta x}{2} \implies \int_{\text{right}} Q \, \mathrm{d}y \approx \left(Q(x_0, y_0) + \frac{\partial Q}{\partial x}\frac{\Delta x}{2}\right)(\Delta y)$$

- **Left Path (↓):**

$$Q\left(x_0 - \frac{\Delta x}{2}, y\right) \approx Q(x_0, y_0) - \frac{\partial Q}{\partial x}\frac{\Delta x}{2} \implies \int_{\text{left}} Q \, \mathrm{d}y \approx -\left(Q(x_0, y_0) - \frac{\partial Q}{\partial x}\frac{\Delta x}{2}\right)(\Delta y)$$

Here, we are left with only the parts that describe the *change* in Q with respect to x.

$$\int_{\text{right}} Q \, \mathrm{d}y + \int_{\text{left}} Q \, \mathrm{d}y \approx \left(\frac{\partial Q}{\partial x}\frac{\Delta x}{2}\right)\Delta y + \left(\frac{\partial Q}{\partial x}\frac{\Delta x}{2}\right)\Delta y = \frac{\partial Q}{\partial x}\Delta x \Delta y$$

Now we sum the results from the horizontal and vertical pairs:

$$\begin{aligned}\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} &\approx \left(-\frac{\partial P}{\partial y}\Delta x \Delta y\right) + \left(\frac{\partial Q}{\partial x}\Delta x \Delta y\right) \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\Delta x \Delta y\end{aligned}$$

This shows that the total circulation around the tiny loop is approximately the quantity $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$ multiplied by the area of the loop ($\Delta A = \Delta x \Delta y$).

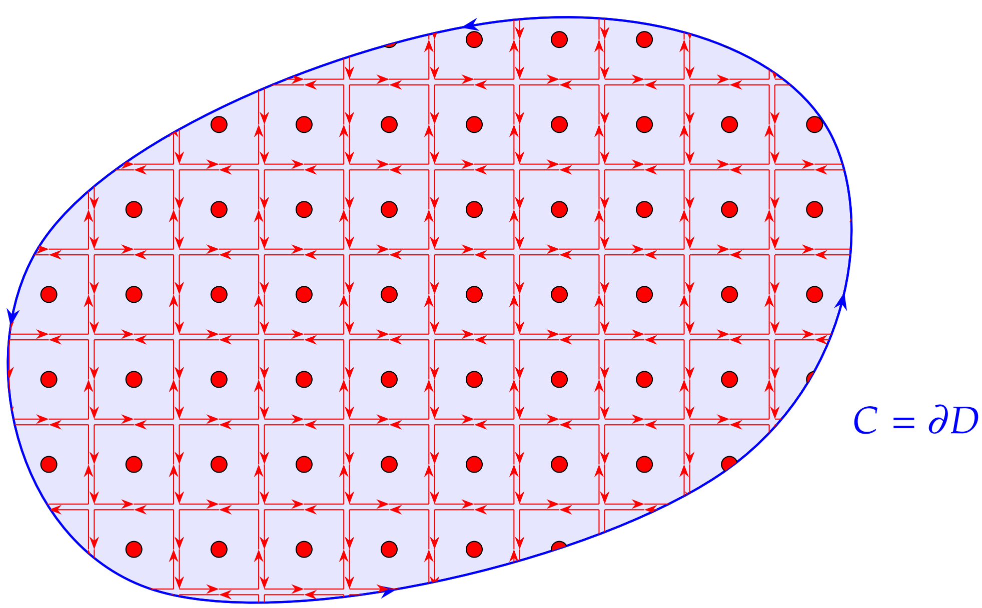
To get the property *at the point* (x_0, y_0), we find the circulation **density**. We divide by the area and take the limit as the rectangle shrinks to zero.

$$\lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint_{\partial R} \mathbf{F} \cdot \mathbf{dr} = \frac{\partial Q}{\partial x}(x_0, y_0) - \frac{\partial P}{\partial y}(x_0, y_0)$$

This is why we call the scalar quantity $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ the **curl**: it is the circulation per unit area at a point, which measures the local rotational tendency of the field.

Remark 2. If $C = \partial D$ is a positively oriented simple closed curve enclosing a region D, Green's theorem states

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{dr}}_{\substack{\textbf{Line Integral} \\ \textbf{(Total Circulation)}}} = \underbrace{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A}_{\substack{\textbf{Double Integral} \\ \textbf{(Sum of Local Curls)}}}$$



Example 1: Rigid rotation and angular velocity

Consider the rigid rotation field with angular speed ω:

$$\mathbf{F}(x, y) = \langle -\omega y,\ \omega x \rangle.$$

Then

$$\frac{\partial Q}{\partial x} = \omega, \qquad \frac{\partial P}{\partial y} = -\omega \quad \Rightarrow \quad \operatorname{curl} \mathbf{F} = Q_x - P_y = 2\omega.$$

This shows curl equals twice the angular velocity. For a circle of radius R, parametrize $r(t) = (R\cos t, R\sin t)$, $dr = (-R\sin t, R\cos t)\,dt$. Then

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \omega R^2\, dt = 2\pi\omega R^2.$$

Meanwhile, $\iint_D (2\omega)\, dA = 2\omega \cdot \pi R^2 = 2\pi\omega R^2$, agreeing with Green's theorem.

Example 2: Curl-free but not conservative (topology matters)

On $\mathbb{R}^2 \setminus \{(0,0)\}$, define

$$\mathbf{F}(x, y) = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

A direct calculation shows $Q_x - P_y = 0$ wherever defined (curl-free). However, the circulation around the unit circle is

$$\oint \mathbf{F} \cdot d\mathbf{r} = 2\pi \neq 0.$$

Hence there is no global potential function; the puncture creates a topological obstruction. This illustrates that $\operatorname{curl} \mathbf{F} = 0$ captures **local** rotation, while global circulation can persist in domains with holes.

Summary checklist

- $Q_x - P_y$ is the infinitesimal (per-area) circulation density.
- Green's theorem sums local curl to give total circulation.
- Rigid rotation: curl $= 2\omega$ (twice angular velocity).
- Curl $= 0$ can still have nonzero loop integrals if the domain has holes.

Test 1: Equality of Mixed Partials

Test 2: Path Independence

Test 3: Potential Recovery

2 Del to Differential ($\nabla \to \mathrm{d}$)

2.1 What is a covector?

Observation. For $\alpha = \begin{bmatrix} 2 & 1 \end{bmatrix} \in (\mathbb{R}^2)^*$ and any $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$,

$$\alpha(\mathbf{v}) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (2)(x) + (1)(y) = 2x + y.$$

Definition 1. Let $V = \mathbb{R}^n$. A **covector** (or **linear functional**) is a linear map

$$\alpha : V \to \mathbb{R}, \qquad \mathbf{v} \mapsto \alpha(\mathbf{v}) = \begin{bmatrix} \alpha_1 & \cdots & \alpha_n \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \sum_{i=0}^{n} \alpha_i v^i.$$

The set of all covectors is the dual space $V^* = \mathrm{Hom}(V, \mathbb{R})$.

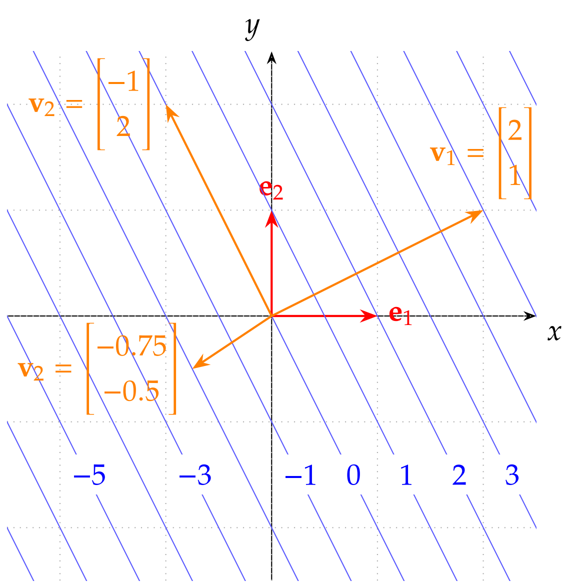
Remark 3. For covectors $\alpha, \beta \in V^*$ and $c \in \mathbb{R}$,

$$(\alpha + \beta)(\mathbf{v}) = \alpha(\mathbf{v}) + \beta(\mathbf{v}), \qquad (c\alpha)(\mathbf{v}) = c\alpha(\mathbf{v}) \quad \text{for all } \mathbf{v} \in V.$$

So V^* is a vector space with these operations.

Observation (Level sets). The value of α is constant on the **level sets**

$$2x + y = c, \qquad c \in \mathbb{R}.$$

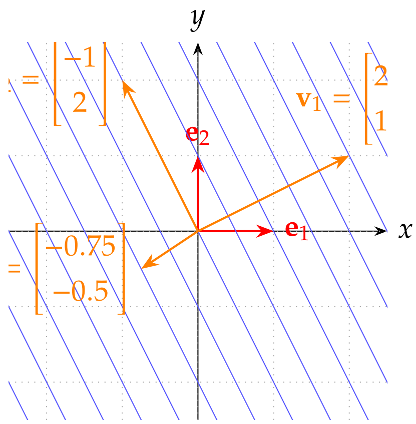


$2x + y = c \Leftrightarrow y = -2x + c$ (level sets of α)

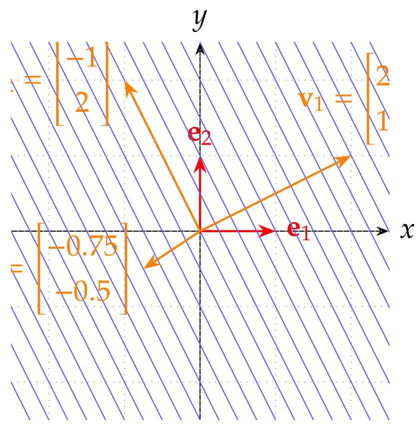
$$\alpha(\mathbf{v}_1) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5$$

$$\alpha(\mathbf{v}_2) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0$$

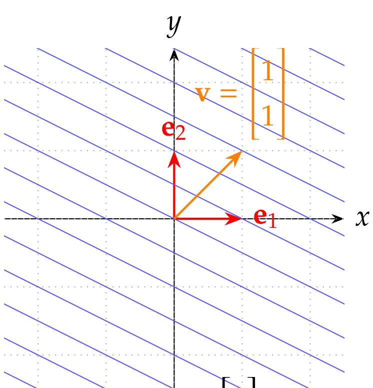
$$\alpha(\mathbf{v}_3) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} -0.75 \\ 0.5 \end{bmatrix} = -2$$



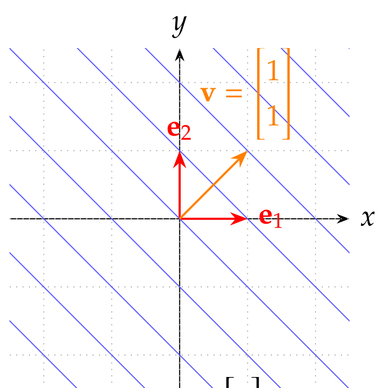
$\alpha = \begin{bmatrix}2 & 1\end{bmatrix}$ with $2x + y = c$



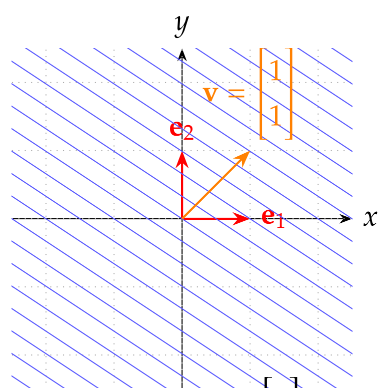
$2\alpha = \begin{bmatrix}4 & 2\end{bmatrix}$ with $4x + 2y = c$



$\alpha(\mathbf{v}) = \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 3$

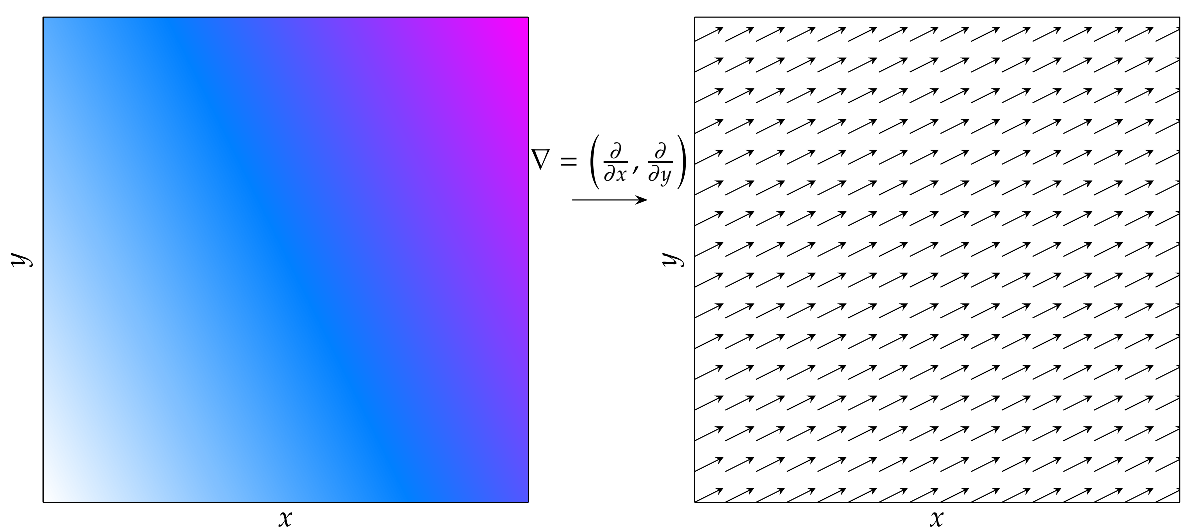


$\beta(\mathbf{v}) = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 2$



$(\alpha + \beta)(\mathbf{v}) = \begin{bmatrix}2 & 3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 5$

Scalar field $f(x, y) = 2x + y$ Gradient field $\nabla f = \langle 2, 1\rangle$



2.2 Dual basis

Let V be a $\mathbb{R}$-vector space with basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. A **covector** is a linear functional

$$\alpha : V \to \mathbb{R}, \qquad \alpha(a\mathbf{v}_1 + b\mathbf{v}_2) = a\,\alpha(\mathbf{v}_1) + b\,\alpha(\mathbf{v}_2).$$

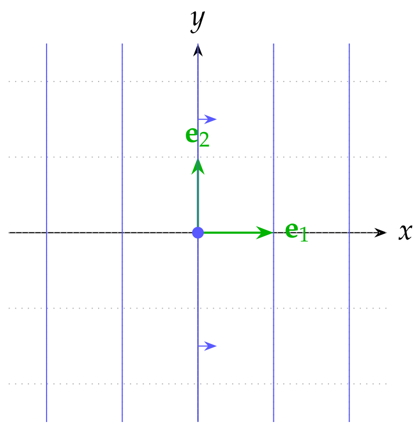
The collection of all covectors is the **dual space** $V^* = \mathrm{Hom}(V, \mathbb{R})$.

We cannot use a basis of V to measure covectors directly. To measure covectors, we use the dual basis of V^*. Given a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of V, the **dual basis** $\{\varepsilon^1, \varepsilon^2\} \subset V^*$ is defined by

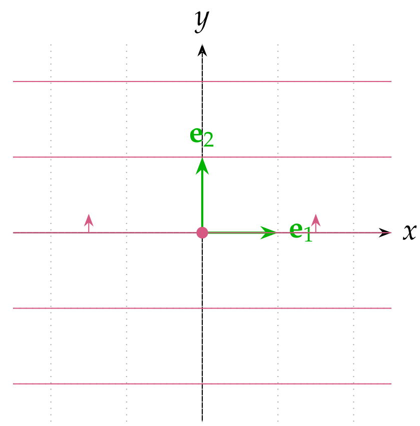
$$\varepsilon^i(\mathbf{e}_j) = \delta_{ij} = \begin{cases} 1 & : i = j \\ 0 & : i \neq j \end{cases} \qquad (i, j \in \{1, 2\}),$$

where δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$ if $i = j$, and 0 otherwise). Concretely,

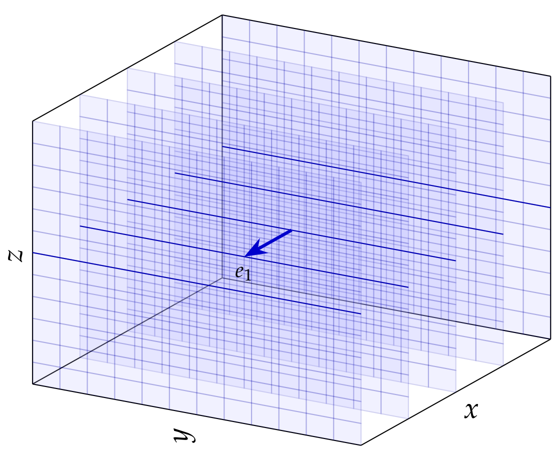
$$\varepsilon^1(\mathbf{e}_1) = 1, \quad \varepsilon^1(\mathbf{e}_2) = 0, \qquad \varepsilon^2(\mathbf{e}_1) = 0, \quad \varepsilon^2(\mathbf{e}_2) = 1.$$



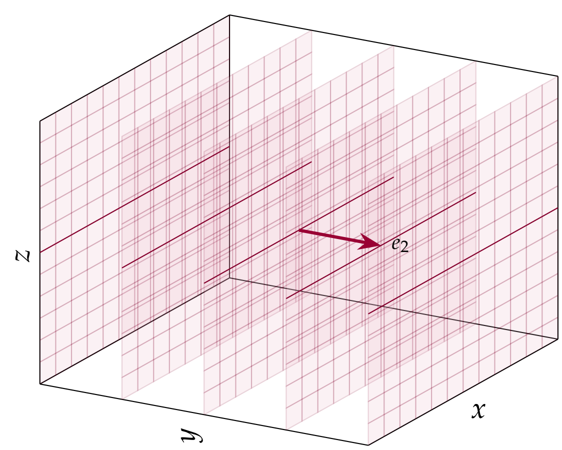
Covector $\varepsilon^1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ with lines $x = c$



Covector $\varepsilon^2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ with lines $y = c$



Covector $\varepsilon^1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ with planes $x = c$



Covector $\varepsilon^2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ with planes $y = c$

Every vector $\mathbf{v} \in V$ has a unique expansion $v = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = v^1\mathbf{e}_1 + v^2\mathbf{e}_2$. The dual basis **reads off** these coordinates:

$$\varepsilon^1(\mathbf{v}) = \varepsilon^1(v^1\mathbf{e}_1 + v^2\mathbf{e}_2) = v^1\varepsilon^1(\mathbf{e}_1) + v^2\varepsilon^1(\mathbf{e}_2) = v_1,$$
$$\varepsilon^2(\mathbf{v}) = \varepsilon^2(v^1\mathbf{e}_1 + v^2\mathbf{e}_2) = v^1\varepsilon^2(\mathbf{e}_1) + v^2\varepsilon^2(\mathbf{e}_2) = v_2.$$

In matrix form,

$$\varepsilon^1(\mathbf{v}) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = v^1, \qquad \varepsilon^2(\mathbf{v}) = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = v^2.$$

For any $\alpha \in V^*$ and $\mathbf{v} = v^1\mathbf{e}_1 + v^2\mathbf{e}_2$,

$$\alpha(\mathbf{v}) = \alpha(v^1\mathbf{e}_1 + v^2\mathbf{e}_2) = v^1\,\alpha(\mathbf{e}_1) + v^2\,\alpha(\mathbf{e}_2) = \varepsilon^1(\mathbf{v})\,\alpha(\mathbf{e}_1) + \varepsilon^2(\mathbf{v})\,\alpha(\mathbf{e}_2).$$

Set

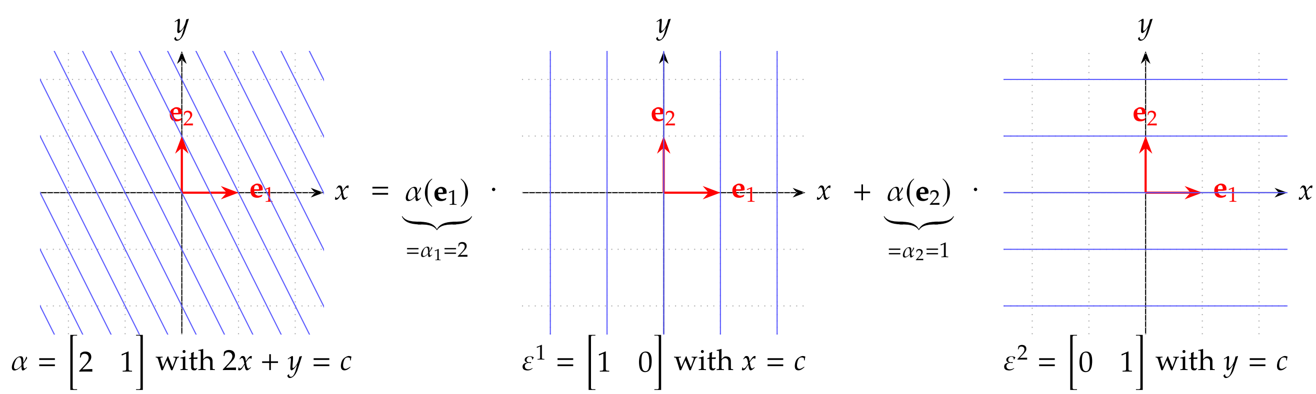
$$\alpha(\mathbf{e}_1) = \alpha_1 \in \mathbb{R}, \qquad \alpha(\mathbf{e}_2) = \alpha_2 \in \mathbb{R}.$$

Then for every $\mathbf{v} \in V$,

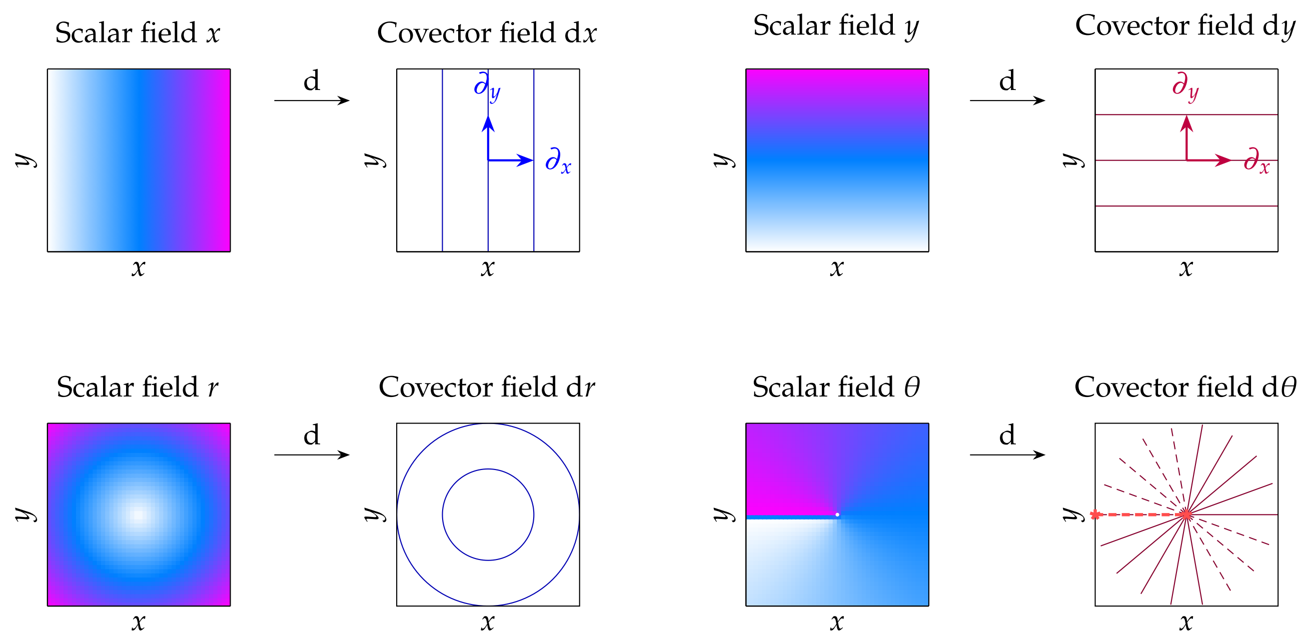
$$\alpha(\mathbf{v}) = \alpha_1\,\varepsilon^1(\mathbf{v}) + \alpha_2\,\varepsilon^2(\mathbf{v}).$$

Since this holds for all $\mathbf{v}$, we identify α as the linear combination

$$\boxed{\alpha = \alpha_1\,\varepsilon^1 + \alpha_2\,\varepsilon^2} \quad \in V^*.$$



2.3 $\mathrm{d}x, \mathrm{d}y$ and $\mathrm{d}r, \mathrm{d}\theta$



Remark 4 (Parametrization for scalar field $\theta(x, y)$)**.** Let

$$x = r\cos t, \qquad y = r\sin t, \qquad r > 0,\ t \in (-\pi, \pi].$$

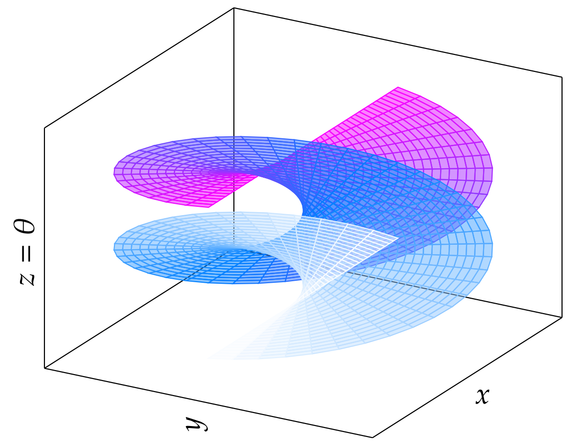
The scalar field $z = \theta(x, y)$ becomes the **helicoid** parametrized by

$$(r, t) \longmapsto (x, y, z = \theta(x, y)) = \left(r\cos t,\ r\sin t,\ \theta(r\cos t, r\sin t) = \arctan\left(\frac{\sin t}{\cos t}\right) = \arctan\big(\tan(t)\big) = t\right).$$

2.4 The surface $z = \theta(x, y)$

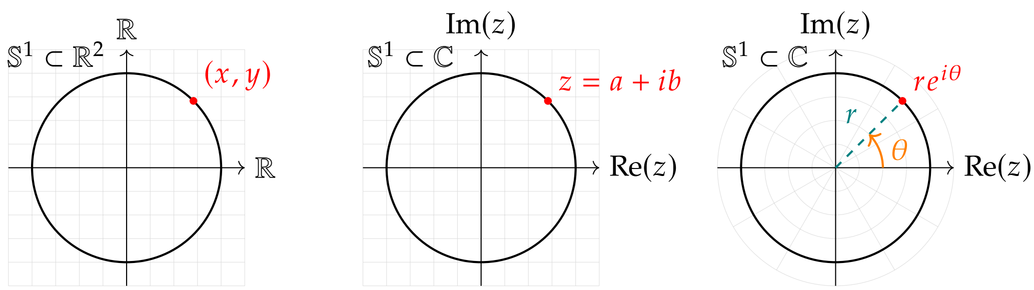
We visualize the scalar field $z = \theta(x, y)$ as a surface:

Surface $z = \theta(x, y)$ via $(r, t) \mapsto (r\cos t, r\sin t, t)$



We identify the plane $\mathbb{R}^2$ with the complex line $\mathbb{C}$ via

$$(x, y) \longleftrightarrow z := x + iy.$$



$x^2 + y^2 = 1$ $\qquad$ $|z| = 1$ $\qquad$ $|r| = 1$ and $\theta \in \mathbb{R}$

The one-argument arctangent

$$\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$$

is single-valued but **cannot** return angles outside that range. The correct globally defined angle (mod 2π), with a chosen principal branch, is given by the two-argument arctangent:

$$\theta = \operatorname{atan2}(y, x) \in (-\pi, \pi],$$

which is defined piecewise by

$$\theta = \begin{cases} \arctan\left(\frac{y}{x}\right), & x > 0, \\ \arctan\left(\frac{y}{x}\right) + \pi, & x < 0,\ y \geq 0, \\ \arctan\left(\frac{y}{x}\right) - \pi, & x < 0,\ y < 0, \\ \frac{\pi}{2}, & x = 0,\ y > 0, \\ -\frac{\pi}{2}, & x = 0,\ y < 0, \end{cases} \quad \text{(undefined at } x = y = 0\text{)}.$$

Hence:

$$\boxed{\text{Global angle (principal): } \theta(x, y) = \operatorname{atan2}(y, x).}$$

On the right half-plane $x > 0$ this reduces to the simple formula $\arctan(y/x)$.

3. Complex viewpoint: Arg of a complex number

Write $z = x + iy \in \mathbb{C}^{\times}$. The (principal) argument $\operatorname{Arg} z \in (-\pi, \pi]$ is defined by the **polar form**

$$z = |z|\, e^{i \operatorname{Arg} z}.$$

Equivalently,

$$\frac{z}{|z|} = \cos(\operatorname{Arg} z) + i \sin(\operatorname{Arg} z).$$

Thus the geometric angle $\theta(x, y)$ is the principal argument:

$$\boxed{\theta(x, y) = \operatorname{Arg}(x + iy) = \operatorname{atan2}(y, x).}$$

This is the cleanest global statement; it automatically handles all quadrants and the axes (except the origin).

4. When does $\arctan(y/x)$ equal the true angle?

Let $\theta = \mathrm{Arg}(x+iy) \in (-\pi, \pi]$. Then:

$$\arctan\left(\frac{y}{x}\right) = \theta \quad \Longleftrightarrow \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ (i.e. } x > 0\text{)}.$$

In general,

$$\arctan\left(\frac{y}{x}\right) = \theta - \pi N, \qquad N = \left\lfloor \frac{\theta}{\pi} + \frac{1}{2} \right\rfloor \in \{-1, 0, 1\}.$$

The subtracted multiple of π compensates for the quadrant.

5. Analytic formula via the complex logarithm

For a **complex** variable w, the complex arctangent is defined (on a log-branch domain) by

$$\arctan w \;=\; \frac{1}{2i}\Big(\log(1+iw) - \log(1-iw)\Big),$$

where log is a chosen branch of the complex logarithm. This identity is obtained by differentiating both sides and matching values at $w = 0$. Its branch cuts are typically placed so that $1 \pm iw$ avoid the logarithm's cut.

Specialization to $w = \dfrac{y}{x}$ (real ratio). When $x \neq 0$ and $w = y/x \in \mathbb{R}$, the above produces the real $\arctan(w) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. But to recover the **global** geometric angle you must still correct by $\pm\pi$ according to the sign of x (i.e. use atan2), or, equivalently, use $\mathrm{Arg}(x+iy)$ directly.

6. Branch cuts and continuity of the angle

Any single-valued selection of the angle on $\mathbb{R}^2 \setminus \{0\}$ must jump by 2π somewhere (a **branch cut**). The standard choice (principal branch) places the cut along the negative real axis:

$$\mathrm{Arg} : \; \mathbb{C} \setminus (-\infty, 0] \;\longrightarrow\; (-\pi, \pi].$$

Crossing the cut increases/decreases the angle by 2π. This is why a color plot of $\theta(x,y)$ shows a "seam" along $x < 0, y = 0$.

7. Differential of the angle and the 1-form $d\theta$

From

$$z = |z|e^{i\theta} \quad \Longrightarrow \quad d(\log z) = \frac{dz}{z} = d(\ln|z|) + i\,d\theta,$$

and writing $z = x + iy$, $dz = dx + i\,dy$, we get

$$\frac{dz}{z} = \frac{dx + i\,dy}{x + iy} = \frac{(x\,dx + y\,dy) + i(x\,dy - y\,dx)}{x^2 + y^2}.$$

Taking imaginary parts:

$$\boxed{d\theta = \frac{x\,dy - y\,dx}{x^2 + y^2} \qquad (x, y) \neq (0, 0).}$$

Geometrically:

- $d\theta$ vanishes on **radial** motion (along rays $\theta = \text{const}$).
- $d\theta$ measures **angular** motion (tangent to circles).

For a closed loop γ avoiding 0,

$$\int_\gamma d\theta = 2\pi \operatorname{Ind}(\gamma, 0),$$

the net number of turns around the origin (winding number).

8. Examples and edge cases

- $(x, y) = (1, 1)$: $\arctan\big(y/x\big) = \arctan(1) = \pi/4$; $\operatorname{Arg}(1 + i) = \pi/4$ (agree).
- $(x, y) = (-1, 1)$: $\arctan\big(y/x\big) = \arctan(-1) = -\pi/4$ but the true angle is $3\pi/4$. Here $\operatorname{atan2}(1, -1) = 3\pi/4 = \arctan(-1) + \pi$.
- $(x, y) = (0, -2)$: $\arctan\big(y/x\big)$ undefined; $\operatorname{atan2}(-2, 0) = -\pi/2$; $\operatorname{Arg}(-2i) = -\pi/2$.
- $(x, y) = (-3, 0)$: slope 0 so $\arctan(0) = 0$ (misleading); $\operatorname{atan2}(0, -3) = \pi$; $\operatorname{Arg}(-3) = \pi$ (principal).

9. A precise dictionary

- **Local/right half-plane formula:**

$$\theta = \arctan\big(y/x\big) \quad (\text{valid for } x > 0).$$

- **Global/principal angle:**

$$\theta = \operatorname{atan2}(y, x) = \operatorname{Arg}(x + iy) \in (-\pi, \pi].$$

- **Complex-analytic identity:** for $w \in \mathbb{C}$,

$$\arctan w = \frac{1}{2i}\Big(\log(1+iw) - \log(1-iw)\Big)$$

(on a suitable log-branch domain).

- **Differential:**

$$d\theta = \frac{x\,dy - y\,dx}{x^2+y^2} = \operatorname{Im}\left(\frac{dz}{z}\right).$$

10. Common pitfalls (and remedies)

- Using $\arctan(y/x)$ globally **without** correcting the quadrant. Remedy: use $\operatorname{atan2}(y,x)$ or $\operatorname{Arg}(x+iy)$.
- Forgetting the point $(0,0)$ is excluded; θ and $d\theta$ are not defined there.
- Expecting a continuous single-valued θ on the punctured plane **without** a branch cut. Any single-valued angle has a jump by 2π along some cut.
- Confusing degrees and radians: in plotting packages atan2 often returns **degrees**; multiply by $\pi/180$ to get radians.

1. Argument as a scalar field on $\mathbb{C}^\times$

For $z = x + iy \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, write $z = re^{i\theta}$ with

$$r = |z| = \sqrt{x^2 + y^2}, \qquad \theta = \arg z \in \mathbb{R}/2\pi\mathbb{Z}.$$

Any **single-valued** choice of θ requires a **branch cut**. The **principal argument** is

$$\operatorname{Arg} z \in (-\pi, \pi], \qquad \text{cut usually along } (-\infty, 0) \subset \mathbb{R}.$$

In coordinates, $\operatorname{Arg}(x+iy)$ coincides with the quadrant-aware angle $\operatorname{atan2}(y, x)$. (Using $\arctan(y/x)$ **without** quadrant care is ambiguous on $x < 0$.)

Level sets. For fixed θ_0, the set $\{z \neq 0 : \arg z = \theta_0\}$ is the **ray** $\{re^{i\theta_0} : r > 0\}$.

2. Relation to the complex logarithm

On a simply connected domain avoiding 0 and the cut, define a holomorphic branch of the log:

$$\operatorname{Log} z = \ln|z| + i\, \operatorname{Arg} z = u + iv, \qquad u = \ln r,\ v = \operatorname{Arg} z.$$

Then $\theta = \operatorname{Arg} z = \operatorname{Im}(\operatorname{Log} z)$ and

$$d(\operatorname{Log} z) = \frac{dz}{z}, \qquad \operatorname{Im}\big(d(\operatorname{Log} z)\big) = d\theta.$$

3. Differential and gradient

With $z = x + iy$ and $dz = dx + i\,dy$,

$$\frac{dz}{z} = \frac{dx + i\,dy}{x + iy} = \frac{(x\,dx + y\,dy) + i(x\,dy - y\,dx)}{x^2 + y^2}.$$

Hence

$$\boxed{d\theta = \operatorname{Im}\left(\frac{dz}{z}\right) = \frac{x\,dy - y\,dx}{x^2 + y^2}} \quad \text{on } \mathbb{C}^\times.$$

In the Euclidean metric, the gradient of θ is

$$\boxed{\nabla\theta(x, y) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right), \qquad \|\nabla\theta\| = \frac{1}{\sqrt{x^2 + y^2}} = \frac{1}{r}.}$$

Thus $\nabla\theta$ is **tangent to circles** and **orthogonal to the radius**.

4. Winding number via a line integral

For a closed C^1 curve γ avoiding 0,

$$\int_\gamma d\theta = \operatorname{Im}\int_\gamma \frac{dz}{z} = 2\pi \operatorname{Ind}(\gamma, 0),$$

the net angle (in radians) swept by the radius vector as you traverse γ. Equivalently, $\int_\gamma \frac{dz}{z} = 2\pi i \operatorname{Ind}(\gamma, 0)$.

Example (unit circle). Let $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. Then $dz/z = i\,dt$, so

$$\int_\gamma d\theta = \operatorname{Im}\int_0^{2\pi} i\,dt = \operatorname{Im}[2\pi i] = 2\pi,$$

which is the expected one full turn around the origin.

5. Harmonic conjugates and Cauchy–Riemann

On a branch domain, $u = \ln r$ and $v = \theta$ satisfy Cauchy–Riemann and are harmonic:

$$\Delta u = 0, \qquad \Delta v = 0 \quad \text{on } \mathbb{C}^\times, \qquad \nabla v = J\,\nabla u, \ \ J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

So $\nabla(\ln r) = \frac{1}{r^2}(x, y)$ is radial, while $\nabla\theta = \frac{1}{r^2}(-y, x)$ is the $+\pi/2$ rotation of it.

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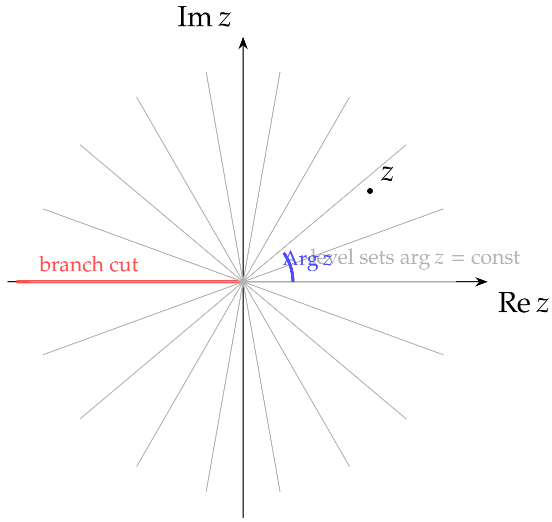
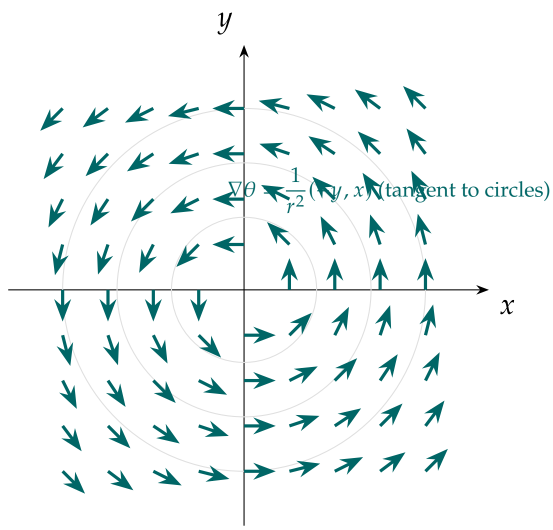
6. Two compact TikZ visuals

(A) Branch cut and level sets of θ (rays)



(B) Gradient field $\nabla\theta$ (tangent to circles)



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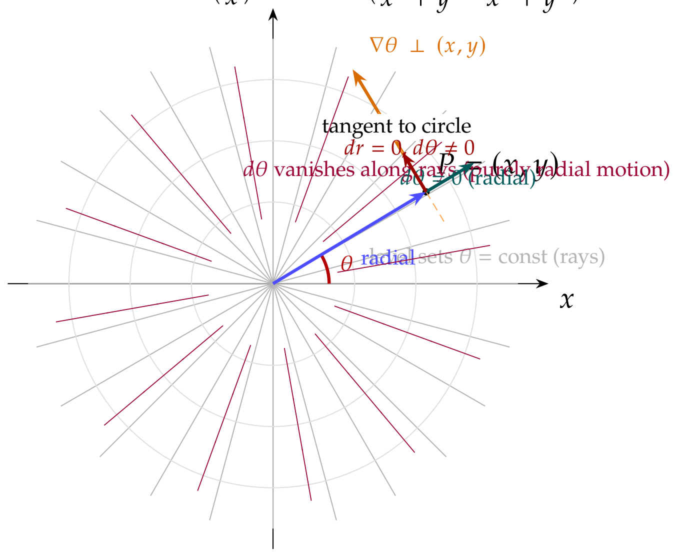
7. Quick dictionary

- $\arg z$: multi-valued angle; $\operatorname{Arg} z$ is a single-valued branch (needs a cut).
- $\operatorname{Log} z = \ln|z| + i\ \operatorname{Arg} z$ (holomorphic on a branch domain), $d\operatorname{Log} z = \dfrac{dz}{z}$.
- $d\theta = \operatorname{Im}\left(\dfrac{dz}{z}\right) = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$.

- $\displaystyle\int_\gamma d\theta = 2\pi \operatorname{Ind}(\gamma, 0)$ (net turning / winding number).
- $u = \ln r$ and $v = \theta$ are harmonic conjugates: $u + iv = \operatorname{Log} z$.

$$\theta(x,y) = \arctan\left(\frac{y}{x}\right), \quad \nabla\theta = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \quad d\theta = \frac{-y\,dx + x\,dy}{x^2+y^2}$$



1. 1-forms (covectors) and what they measure

A **1-form** at a point is a linear functional on tangent vectors. In coordinates:

$$dx, dy \in T_p^*\mathbb{R}^2 \quad \text{and} \quad dr, d\theta \in T_p^*(\mathbb{R}^2 \setminus \{0\})$$

act on an infinitesimal displacement v by returning the corresponding coordinate rate: if $v = v^x \partial_x + v^y \partial_y$ then

$$dx(v) = v^x, \qquad dy(v) = v^y.$$

In polar coordinates (r, θ) with $v = v^r \partial_r + v^\theta \partial_\theta$,

$$dr(v) = v^r, \qquad d\theta(v) = v^\theta.$$

(Be mindful: with the Euclidean metric, $\|\partial_\theta\| = r$, so $d\theta$ measures **angular** rate, not arclength; the arclength 1-form along circles is $r\,d\theta$.)

2. From a scalar field f to its differential df

A smooth scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ has differential

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = f_x\,dx + f_y\,dy.$$

In polar coordinates:

$$df = f_r\,dr + f_\theta\,d\theta, \qquad f_r = \frac{\partial f}{\partial r}, \quad f_\theta = \frac{\partial f}{\partial \theta}.$$

With the Euclidean metric, $df(\cdot) = \nabla f \cdot (\cdot)$, and

$$\nabla f = f_x\,\mathbf{e}_x + f_y\,\mathbf{e}_y = f_r\,\mathbf{e}_r + \frac{1}{r} f_\theta\,\mathbf{e}_\theta.$$

3. Level sets and what the basic 1-forms "kill"

A 1-form is naturally visualized by its **level sets** (where the associated coordinate is constant):

- dx: level sets $x = c$ are vertical lines. Motion **along** these lines is tangent (horizontal value 0 for dx); motion across them gives nonzero dx.
- dy: level sets $y = c$ are horizontal lines. dy kills horizontal motion, measures vertical motion.
- dr: level sets $r = c$ are circles. dr kills tangential (angular) motion along circles; it measures radial motion.

- $d\theta$: level sets $\theta = c$ are rays from the origin. $d\theta$ kills radial motion; it measures angular motion (rate). The arclength form along circles is $r\,d\theta$.

4. Cartesian $\leftrightarrow$ Polar (for $r > 0$)

Let $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\boxed{\begin{aligned} dx &= \cos\theta\,dr \;-\; r\sin\theta\,d\theta, \\ dy &= \sin\theta\,dr \;+\; r\cos\theta\,d\theta, \end{aligned}} \qquad \boxed{\begin{aligned} dr &= \cos\theta\,dx + \sin\theta\,dy, \\ d\theta &= \frac{-\sin\theta\,dx + \cos\theta\,dy}{r}. \end{aligned}}$$

Thus $df = f_x\,dx + f_y\,dy = f_r\,dr + f_\theta\,d\theta$ with $f_r = f_x\cos\theta + f_y\sin\theta$ and $f_\theta = -rf_x\sin\theta + rf_y\cos\theta$.

5. Example: $f(x,y) = x^2 + y^2 = r^2$

$$df = 2x\,dx + 2y\,dy \;=\; 2r\,dr, \qquad \text{level sets: } x^2 + y^2 = c \iff r = \sqrt{c}.$$

Here df kills the angular direction (since $df = 2r\,dr$) and measures only the radial component.

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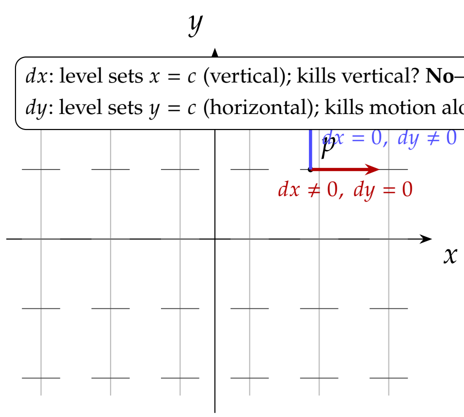
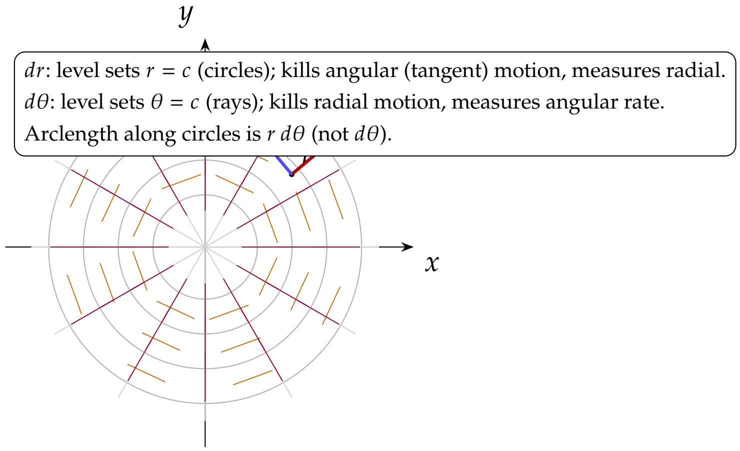
6. Tiny TikZ sketches (1-forms as linelets)

(a) Cartesian: dx and dy



(b) Polar: dr and $d\theta$



Big picture mappings

- **Variable** $x \longrightarrow dx$ (small change): If x is a coordinate on $\mathbb{R}$ (or part of a coordinate chart), then dx is its **differential**—a linear map that takes a tangent vector (an infinitesimal displacement) and returns the rate of change of x along that displacement. Formally, $dx \in T_p^*M$ is a covector.
- **Scalar field** $f \longrightarrow df$ (covector field): A smooth $f : M \to \mathbb{R}$ determines its **differential** df, a **1-form** (covector field) defined pointwise by

$$df_p(v) \;=\; \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)), \quad \gamma(0) = p,\ \dot{\gamma}(0) = v.$$

  In coordinates $x^1, \ldots, x^n$, $df = \sum_i \frac{\partial f}{\partial x^i}\, dx^i$.
- **Function** $f \longrightarrow$ **level sets**: For each constant $c \in \mathbb{R}$, the **level set** $f^{-1}(c) = \{p \in M : f(p) = c\}$ collects points where f has the same value. When $\nabla f \neq 0$, level sets are hypersurfaces orthogonal (via a metric) to ∇f.
- **0-form** $\longrightarrow$ **1-form**: A smooth function f is a **0-form**. Its exterior derivative d sends 0-forms to 1-forms: $d : \Omega^0(M) \to \Omega^1(M)$, namely $f \mapsto df$.

Concrete example on $\mathbb{R}^2$

Let $f(x, y) = x^2 + y^2$.

$$df \;=\; 2x\, dx + 2y\, dy, \qquad \nabla f \;=\; (2x, 2y) \quad \text{(Euclidean metric)}.$$

At a point $p = (x, y)$, for a small displacement $v = (v_x, v_y)$,

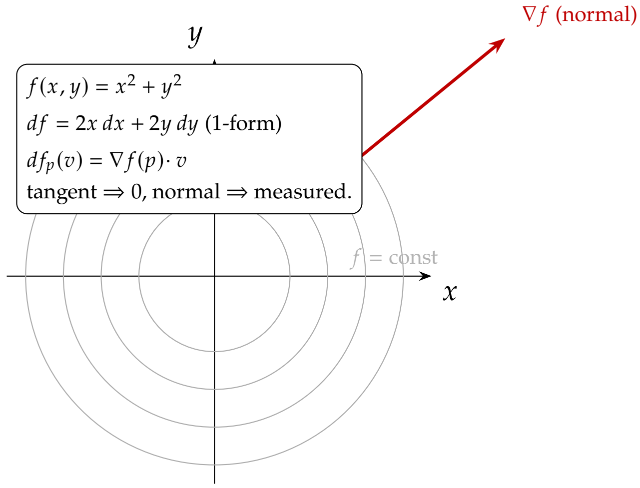
$$df_p(v) \;=\; 2x\, v_x + 2y\, v_y \;=\; \nabla f(p) \cdot v.$$

Level sets are circles $x^2 + y^2 = c$ (concentric about the origin). Tangent motion along a level set contributes nothing to df; only the component of motion across level sets (along the gradient) contributes.

One-line summary

$$\boxed{x \mapsto dx \text{ (basis 1-forms)}; \quad f \mapsto df \text{ (1-form)}; \quad f \mapsto \{f = c\} \text{ (level sets)}; \quad \text{0-form} \xrightarrow{d} \text{1-form}}.$$

Tiny TikZ sketch (level sets and df)



3 Zero Curl and Closed