Mayer–Vietoris as “gluing” in grad–curl–div language (with explicit connecting map)

1. The de Rham complex and the vector-calculus complex

Let M be a smooth manifold. The de Rham complex is

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \xrightarrow{\mathrm{d}} \Omega^3(M) \longrightarrow \cdots$$

and its cohomology is

$$H^k_{\mathrm{dR}}(M) \;=\; \frac{\ker\bigl(\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)\bigr)}{\mathrm{im}\bigl(\mathrm{d} : \Omega^{k-1}(M) \to \Omega^k(M)\bigr)}.$$

On an oriented Riemannian 3-manifold (in particular on domains in $\mathbb{R}^3$), using the metric and Hodge star and the standard identifications

$$\Omega^0 \leftrightarrow \{\text{scalar fields}\}, \quad \Omega^1 \leftrightarrow \{\text{vector fields}\}, \quad \Omega^2 \leftrightarrow \{\text{vector fields}\}, \quad \Omega^3 \leftrightarrow \{\text{scalar densities}\},$$

the operator d corresponds (up to conventional sign/identification choices) to the familiar grad–curl–div chain

$$\Omega^0 \xrightarrow{\mathrm{d}\,\sim\,\nabla} \Omega^1 \xrightarrow{\mathrm{d}\,\sim\,\nabla\times} \Omega^2 \xrightarrow{\mathrm{d}\,\sim\,\nabla\cdot} \Omega^3,$$

so $\mathrm{d}^2 = 0$ corresponds to

$$\nabla \times (\nabla f) = 0, \qquad \nabla \cdot (\nabla \times A) = 0.$$

2. The gluing short exact sequence of complexes

Let $M = U \cup V$ be an open cover. For each $k \geq 0$ define maps

$$r : \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V), \qquad r(\omega) = (\omega|_U, \omega|_V),$$

and

$$\delta : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V), \qquad \delta(\alpha, \beta) = \alpha|_{U\cap V} - \beta|_{U\cap V}.$$

Proposition 1 (Gluing exact sequence at the cochain level)**.** *For each k there is a short exact sequence of vector spaces*

$$0 \longrightarrow \Omega^k(M) \xrightarrow{r} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\delta} \Omega^k(U \cap V) \longrightarrow 0,$$

and these assemble into a short exact sequence of cochain complexes

$$0 \to \Omega^\bullet(M) \xrightarrow{r} \Omega^\bullet(U) \oplus \Omega^\bullet(V) \xrightarrow{\delta} \Omega^\bullet(U \cap V) \to 0$$

because d *commutes with restriction and hence with r, δ.*

Remark 1 (Calculus interpretation: “agree on the overlap”)**.** A pair of local fields $(\alpha, \beta) \in \Omega^k(U) \oplus \Omega^k(V)$ lies in $\ker(\delta)$ iff α and β *agree on $U \cap V$*. Equivalently, they *glue* to a global k-form on M. Thus the short exact sequence above is the formal encoding of the basic principle: global objects are exactly compatible local objects on an open cover.

3. Mayer–Vietoris long exact sequence

Theorem 1 (Mayer–Vietoris for de Rham cohomology)**.** *The short exact sequence of cochain complexes in §2 induces a long exact sequence in cohomology:*

$$\cdots \to H^{k-1}_{\mathrm{dR}}(U\cap V) \xrightarrow{\partial} H^k_{\mathrm{dR}}(M) \xrightarrow{(r_U^*,r_V^*)} H^k_{\mathrm{dR}}(U)\oplus H^k_{\mathrm{dR}}(V) \xrightarrow{\delta^*} H^k_{\mathrm{dR}}(U\cap V) \xrightarrow{\partial} H^{k+1}_{\mathrm{dR}}(M) \to \cdots$$

where $\delta^*([\alpha],[\beta]) = [\alpha|_{U\cap V} - \beta|_{U\cap V}]$*.*

4. Explicit formula for the connecting morphism ∂ (the "overlap mismatch" map)

Fix a smooth partition of unity subordinate to the cover: choose $\rho_U, \rho_V \in C^\infty(M)$ such that

$$\rho_U + \rho_V = 1, \qquad \mathrm{supp}(\rho_U) \subset U, \qquad \mathrm{supp}(\rho_V) \subset V.$$

Proposition 2 (Concrete representative of ∂)**.** *Let* $[\eta] \in H^{k-1}_{\mathrm{dR}}(U \cap V)$ *with* $\mathrm{d}\eta = 0$ *on* $U \cap V$*. Extend* η *to forms* $\tilde\eta_U \in \Omega^{k-1}(U)$ *and* $\tilde\eta_V \in \Omega^{k-1}(V)$ *satisfying* $\tilde\eta_U|_{U\cap V} = \eta = \tilde\eta_V|_{U\cap V}$ *(existence holds by locality of forms). Define*

$$\omega_U := \rho_V \cdot \tilde\eta_U \in \Omega^{k-1}(U), \qquad \omega_V := -\rho_U \cdot \tilde\eta_V \in \Omega^{k-1}(V).$$

Then on $U \cap V$ *one has* $\omega_U - \omega_V = \eta$ *and hence* $\delta(\omega_U, \omega_V) = \eta$*. Moreover, the pair* $(\mathrm{d}\omega_U, \mathrm{d}\omega_V)$ *agrees on* $U \cap V$*, so it glues to a global closed* k*-form* $\theta \in \Omega^k(M)$*. The connecting map is*

$$\partial([\eta]) \;=\; [\theta] \in H^k_{\mathrm{dR}}(M),$$

and this class is independent of all choices (extensions, partition of unity).

Remark 2 (What ∂ *means*)**.** The element $[\eta] \in H^{k-1}(U\cap V)$ can be viewed as a "transition datum" on the overlap. The connecting map ∂ converts this overlap datum into a *global obstruction class* on M: it is precisely the obstruction to gluing primitives/potentials globally.

5. Translation to grad–curl–div: the case $k = 1$ (curl-free vs global gradient)

Assume we are in the 3-dimensional vector-calculus setting.

Take $k = 1$. Then H^1_{dR} detects the failure of a curl-free field to be a global gradient.

- A class in $H^0_{\mathrm{dR}}(U \cap V)$ is represented by a locally constant function on $U \cap V$ (when $U \cap V$ is disconnected, this is where "different constants on different components" appears).
- The connecting map
$$\partial : H^0_{\mathrm{dR}}(U \cap V) \longrightarrow H^1_{\mathrm{dR}}(M)$$
takes an overlap "constant mismatch" and outputs a global cohomology class of closed 1-forms, i.e. (under the identifications) a curl-free vector field modulo global gradients.

Concretely, let c be a locally constant function on $U \cap V$. Choose extensions $\tilde c_U \in C^\infty(U)$ and $\tilde c_V \in C^\infty(V)$ with $\tilde c_U|_{U\cap V} = c = \tilde c_V|_{U\cap V}$. Then

$$\theta \;=\; \mathrm{d}(\rho_V \tilde c_U) \;=\; -\mathrm{d}(\rho_U \tilde c_V) \quad \text{on } U \cap V$$

glues to a global closed 1-form on M, and $\partial([c]) = [\theta]$. In vector-calculus language, θ corresponds to a curl-free field F (since $\mathrm{d}\theta = 0$ means $\nabla \times F = 0$) whose failure to be a global gradient is encoded by the "jump" data c on $U \cap V$.

6. Worked example: $H^1(S^1) \cong \mathbb{R}$ from overlap-gluing (pure calculus intuition)

Let $M = S^1$. Cover S^1 by two open arcs U, V such that $U \cap V$ is a disjoint union of two open arcs, hence has two connected components:

$$U \cap V \;=\; W_1 \sqcup W_2.$$

Then

$$H^0_{\mathrm{dR}}(U) \cong \mathbb{R}, \quad H^0_{\mathrm{dR}}(V) \cong \mathbb{R}, \quad H^0_{\mathrm{dR}}(U \cap V) \cong \mathbb{R} \oplus \mathbb{R},$$

and (since $U, V, U \cap V$ are unions of contractible sets) one has

$$H^1_{\mathrm{dR}}(U) = H^1_{\mathrm{dR}}(V) = H^1_{\mathrm{dR}}(U \cap V) = 0.$$

The Mayer–Vietoris segment for degrees 0 and 1 becomes

$$0 \to H^0(S^1) \to H^0(U) \oplus H^0(V) \xrightarrow{\delta^*} H^0(U \cap V) \xrightarrow{\partial} H^1(S^1) \to 0.$$

Identify $H^0(U) \oplus H^0(V) \cong \mathbb{R}^2$ and $H^0(U \cap V) \cong \mathbb{R}^2$ by sending a class to its constant value on each connected component. Then

$$\delta^*(a, b) \;=\; (a - b,\; a - b),$$

so $\mathrm{im}(\delta^*) = \{(t, t) : t \in \mathbb{R}\}$ is the diagonal in $\mathbb{R}^2$. Hence

$$H^1_{\mathrm{dR}}(S^1) \;\cong\; \frac{\mathbb{R}^2}{\{(t, t)\}} \;\cong\; \mathbb{R},$$

and the isomorphism is concretely given by the *difference of overlap constants*:

$$[(c_1, c_2)] \longmapsto c_1 - c_2.$$

Calculus meaning. Locally on U and V, a curl-free 1-field is a gradient of a potential (angle function). On $U \cap V$, the two local potentials differ by constants; because $U \cap V$ has *two* components, there can be *two* constants. If those constants disagree between the components, you cannot choose potentials that match globally; this is exactly the nontrivial class in $H^1(S^1)$, i.e. the global obstruction.

Extremely detailed computations of $H^1_{\mathrm{dR}}(S^1)$ and $H^1_{\mathrm{dR}}(S^2)$

0. Preliminaries: de Rham cohomology and Mayer–Vietoris

Definition 1 (de Rham cohomology)**.** For a smooth manifold M, define

$$H^k_{\mathrm{dR}}(M) \;=\; \frac{\ker(\mathrm{d}:\Omega^k(M)\to\Omega^{k+1}(M))}{\mathrm{im}(\mathrm{d}:\Omega^{k-1}(M)\to\Omega^k(M))}.$$

Elements of $\ker(\mathrm{d})$ are called *closed* k-forms; elements of $\mathrm{im}(\mathrm{d})$ are *exact.*

Proposition 3 (Gluing exact sequence for an open cover)**.** *If $M = U\cup V$ is an open cover, then for each k there is a short exact sequence*

$$0\to\Omega^k(M)\xrightarrow{r}\Omega^k(U)\oplus\Omega^k(V)\xrightarrow{\delta}\Omega^k(U\cap V)\to 0$$

where $r(\omega)=(\omega|_U,\omega|_V)$ and $\delta(\alpha,\beta)=\alpha|_{U\cap V}-\beta|_{U\cap V}$. Moreover, this is a short exact sequence of cochain complexes *because* d *commutes with restriction.*

Theorem 2 (Mayer–Vietoris long exact sequence in de Rham cohomology)**.** *From the short exact sequence of complexes above, one obtains a long exact sequence*

$$\cdots\to H^{k-1}(U\cap V)\xrightarrow{\partial}H^k(M)\xrightarrow{(r_U^*,r_V^*)}H^k(U)\oplus H^k(V)\xrightarrow{\delta^*}H^k(U\cap V)\xrightarrow{\partial}H^{k+1}(M)\to\cdots$$

where $\delta^*([\alpha],[\beta])=[\alpha|_{U\cap V}-\beta|_{U\cap V}]$.

Remark 3 (Concrete formula for ∂ via partition of unity)**.** Fix $\rho_U,\rho_V\in C^\infty(M)$ with $\rho_U+\rho_V=1$ and $\mathrm{supp}(\rho_U)\subset U$, $\mathrm{supp}(\rho_V)\subset V$. If $[\eta]\in H^{k-1}(U\cap V)$ is represented by a closed form η, choose extensions $\tilde\eta_U\in\Omega^{k-1}(U)$ and $\tilde\eta_V\in\Omega^{k-1}(V)$ with $\tilde\eta_U|_{U\cap V}=\eta=\tilde\eta_V|_{U\cap V}$. Define

$$\omega_U := \rho_V\,\tilde\eta_U,\qquad \omega_V := -\rho_U\,\tilde\eta_V.$$

Then $\delta(\omega_U,\omega_V)=\eta$, and $(\mathrm{d}\omega_U,\mathrm{d}\omega_V)$ agrees on $U\cap V$, hence glues to a global closed k-form θ on M. One sets $\partial([\eta]) := [\theta]\in H^k(M)$.

1. $H^1_{\mathrm{dR}}(S^1)$: two complementary computations

1A. Direct calculation using the angle coordinate

View $S^1\subset\mathbb{R}^2$ with standard angle coordinate $\theta\in\mathbb{R}/2\pi\mathbb{Z}$ and parametrization

$$\gamma:\mathbb{R}/2\pi\mathbb{Z}\to S^1,\qquad \gamma(\theta)=(\cos\theta,\sin\theta).$$

Then $\Omega^1(S^1)$ is a rank-1 $C^\infty(S^1)$-module generated by $\mathrm{d}\theta$ in the sense that any 1-form $\omega\in\Omega^1(S^1)$ can be uniquely written as

$$\omega = f(\theta)\,\mathrm{d}\theta,\qquad f\in C^\infty(S^1).$$

Lemma 1 (All 1-forms on S^1 are closed)**.** *If $\omega\in\Omega^1(S^1)$, then* $\mathrm{d}\omega=0$.

Proof. On any 1-manifold, $\Omega^2=0$ identically (there are no nonzero 2-forms), so $\mathrm{d}:\Omega^1\to\Omega^2$ is the zero map. Hence every 1-form is closed. □

Thus

$$H^1_{\mathrm{dR}}(S^1) = \frac{\Omega^1(S^1)}{\mathrm{d}\Omega^0(S^1)}.$$

So we must characterize which $f(\theta)\mathrm{d}\theta$ are exact.

Lemma 2 (Exactness criterion by period)**.** *Let $\omega = f(\theta)\mathrm{d}\theta \in \Omega^1(S^1)$. Then ω is exact iff*

$$\int_{S^1} \omega \; = \; \int_0^{2\pi} f(\theta)\,\mathrm{d}\theta \; = \; 0.$$

Proof. ($\Rightarrow$) If $\omega = \mathrm{d}g$ for some $g \in C^\infty(S^1)$, then by the fundamental theorem of calculus,

$$\int_{S^1} \omega = \int_{S^1} \mathrm{d}g = 0,$$

because the integral of an exact 1-form over a closed loop is zero.

($\Leftarrow$) Assume $\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta = 0$. Define

$$G(\theta) := \int_0^\theta f(t)\,\mathrm{d}t.$$

Then G is smooth on $\mathbb{R}$, and $G'(\theta) = f(\theta)$. Moreover,

$$G(\theta + 2\pi) - G(\theta) = \int_\theta^{\theta+2\pi} f(t)\,\mathrm{d}t = \int_0^{2\pi} f(t)\,\mathrm{d}t = 0,$$

so G is 2π-periodic and thus descends to a smooth function $g \in C^\infty(S^1)$ satisfying $\mathrm{d}g = f(\theta)\mathrm{d}\theta = \omega$. $\square$

Proposition 4 (The period map identifies $H^1_{\mathrm{dR}}(S^1) \cong \mathbb{R}$)**.** *Define the linear map ("period" or "circulation")*

$$\mathcal{P} : \Omega^1(S^1) \to \mathbb{R}, \qquad \mathcal{P}(\omega) = \int_{S^1} \omega.$$

Then $\mathcal{P}$ vanishes on exact forms and hence induces a well-defined linear map

$$\overline{\mathcal{P}} : H^1_{\mathrm{dR}}(S^1) \to \mathbb{R}, \qquad \overline{\mathcal{P}}([\omega]) = \int_{S^1} \omega.$$

Moreover, $\overline{\mathcal{P}}$ is an isomorphism. In particular,

$$H^1_{\mathrm{dR}}(S^1) \cong \mathbb{R}, \qquad [\mathrm{d}\theta] \textit{ is a generator with } \int_{S^1} \mathrm{d}\theta = 2\pi.$$

Proof. Well-definedness is immediate because $\int_{S^1} \mathrm{d}g = 0$ for all g.

Injectivity: If $\overline{\mathcal{P}}([\omega]) = 0$, then $\int_{S^1} \omega = 0$, so by the exactness criterion ω is exact, hence $[\omega] = 0$ in cohomology.

Surjectivity: Given $c \in \mathbb{R}$, take $\omega = \frac{c}{2\pi}\mathrm{d}\theta$. Then

$$\overline{\mathcal{P}}([\omega]) = \int_{S^1} \frac{c}{2\pi}\mathrm{d}\theta = \frac{c}{2\pi} \cdot 2\pi = c.$$

$\square$

Remark 4 (Vector-calculus reading in 1D)**.** On a 1-dimensional manifold, "curl" is vacuous and every 1-form is locally a gradient (Poincaré lemma in degree 1). The obstruction to a *global* potential is exactly the nonzero circulation $\int_{S^1} \omega$, which is the H^1 class.

1B. Mayer–Vietoris computation for S^1 (extremely explicit)

Choose an open cover $S^1 = U \cup V$ where U, V are open arcs, each diffeomorphic to an interval, such that $U \cap V$ is the disjoint union of *two* open arcs:

$$U \cap V = W_1 \sqcup W_2, \qquad W_1 \cap W_2 = \varnothing.$$

(Geometrically: take U and V as two large overlapping arcs whose overlap occurs in two separated regions.)

Lemma 3 (Cohomology of the pieces)**.** *Because U, V, W_1, W_2 are each diffeomorphic to an open interval, they are contractible. Hence:*

$$H^0(U) \cong \mathbb{R}, \quad H^0(V) \cong \mathbb{R}, \quad H^0(W_i) \cong \mathbb{R}, \qquad H^1(U) = H^1(V) = H^1(W_i) = 0.$$

Therefore

$$H^0(U \cap V) \cong H^0(W_1) \oplus H^0(W_2) \cong \mathbb{R} \oplus \mathbb{R}, \qquad H^1(U \cap V) = 0.$$

Now write out the Mayer–Vietoris long exact sequence in low degrees. The relevant segment is

$$0 \to H^0(S^1) \xrightarrow{r^*} H^0(U) \oplus H^0(V) \xrightarrow{\delta^*} H^0(U \cap V) \xrightarrow{\partial} H^1(S^1) \xrightarrow{r^*} H^1(U) \oplus H^1(V) = 0.$$

Exactness at the last term implies ∂ is *surjective*:

$$\operatorname{im}(\partial) = H^1(S^1).$$

Exactness at $H^0(U \cap V)$ gives

$$\ker(\partial) = \operatorname{im}(\delta^*).$$

Therefore

$$H^1(S^1) \;\cong\; \frac{H^0(U \cap V)}{\operatorname{im}(\delta^*)}.$$

It remains to compute δ^* explicitly on constants.

Lemma 4 (Explicit form of δ^* on H^0)**.** *Identify*

$$H^0(U) \oplus H^0(V) \cong \mathbb{R} \oplus \mathbb{R}$$

by sending a class to its constant value on each connected set. Similarly, identify

$$H^0(U \cap V) \cong \mathbb{R} \oplus \mathbb{R}$$

by sending a class to its pair of constants on (W_1, W_2). Then

$$\delta^*(a, b) = (a - b,\ a - b).$$

Proof. A class in $H^0(U)$ is represented by a locally constant function; since U is connected, it is constant a. Similarly b on V. On each overlap component $W_i \subset U \cap V$, the difference

$$a|_{W_i} - b|_{W_i} = a - b$$

is the constant value of the class in $H^0(W_i)$. Hence the pair is $(a - b, a - b)$. □

Thus

$$\operatorname{im}(\delta^*) = \{(t,t) : t \in \mathbb{R}\} \subset \mathbb{R}^2,$$

the diagonal. Consequently

$$H^1(S^1) \cong \frac{\mathbb{R}^2}{\{(t,t)\}} \cong \mathbb{R},$$

and a concrete isomorphism is given by the *difference of overlap constants*:

$$[(c_1, c_2)] \longmapsto c_1 - c_2.$$

This exhibits the slogan:

> local potentials exist on U and V, but their constants of integration on W_1 and W_2 need not match; the mismatch $c_1 - c_2$ is the H^1 obstruction.

2. $H^1_{\mathrm{dR}}(S^2) = 0$ (with Mayer–Vietoris and explicit maps)

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. Let $N = (0,0,1)$ (north pole) and $S = (0,0,-1)$ (south pole). Define the standard two-chart cover

$$U := S^2 \setminus \{S\}, \qquad V := S^2 \setminus \{N\}.$$

Then U and V are each diffeomorphic to $\mathbb{R}^2$ via stereographic projection. Their intersection is

$$U \cap V = S^2 \setminus \{N, S\}.$$

2A. Cohomology of U, V, and $U \cap V$

Lemma 5 (Cohomology of U and V)**.** *Since $U \cong \mathbb{R}^2$ and $V \cong \mathbb{R}^2$ are contractible,*

$$H^0(U) \cong \mathbb{R}, \quad H^0(V) \cong \mathbb{R}, \qquad H^1(U) = H^1(V) = 0.$$

Lemma 6 (Homotopy type and H^1 of the overlap)**.** *The manifold $U \cap V = S^2 \setminus \{N, S\}$ deformation retracts onto the equator $S^1 \subset S^2$. Hence*

$$H^0(U \cap V) \cong \mathbb{R}, \qquad H^1(U \cap V) \cong H^1(S^1) \cong \mathbb{R}.$$

Proof. Geometrically, remove the poles; every remaining point lies on a unique meridian segment crossing the equator. Define a deformation retraction by sliding points along meridians to the equator (keeping longitude fixed). Thus $U \cap V \simeq S^1$, and de Rham cohomology is homotopy invariant, giving the claims. $\square$

2B. Mayer–Vietoris in degrees 0 and 1

Write the Mayer–Vietoris long exact sequence in low degrees:

$$0 \to H^0(S^2) \xrightarrow{r^*} H^0(U) \oplus H^0(V) \xrightarrow{\delta^*} H^0(U \cap V) \xrightarrow{\partial} H^1(S^2) \xrightarrow{r^*} H^1(U) \oplus H^1(V).$$

Substitute the computed groups:

$$0 \to \mathbb{R} \xrightarrow{r^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta^*} \mathbb{R} \xrightarrow{\partial} H^1(S^2) \xrightarrow{r^*} 0 \oplus 0 = 0.$$

Thus ∂ is surjective and

$$H^1(S^2) \cong \frac{H^0(U \cap V)}{\operatorname{im}(\delta^*)} \cong \frac{\mathbb{R}}{\operatorname{im}(\delta^*)}.$$

Therefore it suffices to show $\delta^* : \mathbb{R}^2 \to \mathbb{R}$ is surjective.

Lemma 7 (Explicit computation of δ^* on H^0 for S^2 cover)**.** *Under the identifications $H^0(U) \cong \mathbb{R}$, $H^0(V) \cong \mathbb{R}$, $H^0(U \cap V) \cong \mathbb{R}$ via constants,*

$$\delta^*(a, b) = a - b.$$

In particular, δ^* is surjective.

Proof. As in the S^1 case, a class in $H^0(U)$ is represented by a constant a on connected U, similarly b on V. Since $U \cap V$ is connected, the difference $a|_{U\cap V} - b|_{U\cap V}$ is the constant $a - b$ in $H^0(U \cap V)$. Surjectivity: given $c \in \mathbb{R}$, choose $(a, b) = (c, 0)$ so $a - b = c$. $\square$

Corollary 1 ($H^1_{\mathrm{dR}}(S^2) = 0$)**.** *Because δ^* is surjective,* $\mathrm{im}(\delta^*) = \mathbb{R}$, *hence*

$$H^1_{\mathrm{dR}}(S^2) \cong \mathbb{R}/\mathbb{R} = 0.$$

Remark 5 (Calculus reading: every curl-free tangent field has a global potential on S^2)**.** Under the 1-form/vector-field identification (with a metric), a closed 1-form corresponds to a curl-free field. The statement $H^1(S^2) = 0$ says:

every closed 1-form on S^2 is exact, i.e. every curl-free field is a global gradient.

From the Mayer–Vietoris perspective: local potentials exist on U and V (since $H^1(U) = H^1(V) = 0$), and because the overlap $U \cap V$ is connected, the only ambiguity is an *additive constant*, which can be adjusted to make the two potentials agree globally. There is no "two-component mismatch" as in the S^1 case.

2C. (Optional but illuminating) What happens one degree higher: $H^2(S^2) \cong \mathbb{R}$

Although you asked for H^1, it is instructive to record the adjacent MV segment:

$$H^1(U) \oplus H^1(V) = 0 \longrightarrow H^1(U \cap V) \cong \mathbb{R} \xrightarrow{\partial} H^2(S^2) \longrightarrow H^2(U) \oplus H^2(V) = 0.$$

Exactness forces $\partial : \mathbb{R} \to H^2(S^2)$ to be an isomorphism, so $H^2(S^2) \cong \mathbb{R}$. This is the formal place where the "flux/area" class of the sphere lives.