Coordinate and Tangent-Space Charts for a Plane Curve

1 The Curve as a 1–Dimensional Submanifold

Definition 1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a C^1–function. Its* graph

$$C \;=\; \bigl\{(x,y) \in \mathbb{R}^2 \;\big|\; y \;=\; f(x)\bigr\}$$

is a 1*–dimensional embedded submanifold of* $\mathbb{R}^2$ *.*

Fix $a \in \mathbb{R}$ and set

$$p \;=\; (a, f(a)) \;\in\; C.$$

2 Global Coordinate Chart on C

6.1. Chart via a Parametrization

Define

$$\Phi \;:\; \underbrace{\mathbb{R}}_{U} \;\longrightarrow\; C, \qquad \Phi(t) \;=\; \bigl(t,\, f(t)\bigr).$$

Proposition 1. Φ *is a diffeomorphism from $U = \mathbb{R}$ onto C, with inverse*

$$\Phi^{-1} \;:\; C \;\longrightarrow\; \mathbb{R}, \qquad (x,y) \;\longmapsto\; x.$$

Hence $t \in \mathbb{R}$ is a global coordinate on C, and every point $p \in C$ admits the unique representation $p = \Phi(t)$.

Proof. Φ is C^1 with Jacobian determinant $1 \neq 0$, hence a local diffeo; injectivity and surjectivity are immediate from its formula. The inverse $\Phi^{-1}(x,y) = x$ is C^1. □

6.2. Ambient Coordinate Restriction

Equivalently, let

$$\pi_i \;:\; \mathbb{R}^2 \;\longrightarrow\; \mathbb{R}, \quad \pi_1(x,y) = x,\; \pi_2(x,y) = y.$$

Then the pair

$$(x,y)\big|_C = \;(\pi_1,\pi_2)\big|_C :\; C \;\longrightarrow\; \mathbb{R}^2$$

serves as the inclusion of C into its ambient coordinate system, with

$$(x,y)\bigl(t, f(t)\bigr) \;=\; (t, f(t)).$$

3 Coordinate Chart on the Tangent Spaces

7.1. The Tangent Bundle and Its Trivialization

The tangent bundle of C may be presented via the push–forward

$$d\Phi \;:\; TU \;=\; U \times \underbrace{\mathbb{R}}_{\text{model fiber}} \;\longrightarrow\; TC \subset T\mathbb{R}^2,$$

where

$$d\Phi_t(\dot{t}) \;=\; \frac{d}{dt}\Big(t, f(t)\Big)\Big|_t \,\dot{t} \;=\; \dot{t}\,\big(1, f'(t)\big) \;\in\; T_{\Phi(t)}C.$$

Since $d\Phi_t$ is a vector-space isomorphism $\mathbb{R} \to T_{\Phi(t)}C$, this gives a trivialization

$$TC \;\cong\; U \times \mathbb{R},$$

and in particular, at $t = a$,

$$d\Phi_a \;:\; \mathbb{R} \;\xrightarrow{\sim}\; T_pC, \qquad \tau \;\longmapsto\; \tau\,(1, f'(a)).$$

Thus $\tau \in \mathbb{R}$ is a *local coordinate* on the fibre T_pC.

7.2. Cotangent-Bundle Coordinates

Dually, the ambient projections π_i induce

$$d\pi_i \;:\; T_p\mathbb{R}^2 \;\longrightarrow\; \mathbb{R}, \qquad (v_1, v_2) \;\longmapsto\; v_i.$$

Restricting to the line $T_pC = \operatorname{span}\{\,(1, f'(a))\} \subset T_p\mathbb{R}^2$ yields a map

$$(d\pi_1, d\pi_2)\big|_{T_pC} :\; T_pC \;\longrightarrow\; \underbrace{\mathbb{R}^2}_{\text{model cotangent fiber}},$$

$$v \;\longmapsto\; \big(dx(v),\, dy(v)\big).$$

Again this is injective onto the one-dimensional subspace $\{(\tau,\; f'(a)\,\tau)\} \subset \mathbb{R}^2$, and projecting to the first factor recovers the scalar coordinate τ.

4 Coordinate Functions on C

$$x \;=\; \pi_1\big|_C :\; C \longrightarrow \mathbb{R}, \qquad y \;=\; \pi_2\big|_C :\; C \longrightarrow \mathbb{R}.$$

Explicitly,

$$x\big(\Phi(t)\big) = t, \quad y\big(\Phi(t)\big) = f(t),$$

and at $p = \Phi(a)$, $x(p) = a$ and $y(p) = f(a)$.

5 Differentials on the Tangent Space

$$dx \;=\; d\pi_1\big|_{T_p\mathbb{R}^2} :\; T_p\mathbb{R}^2 \to \mathbb{R}, \quad dy \;=\; d\pi_2\big|_{T_p\mathbb{R}^2} :\; T_p\mathbb{R}^2 \to \mathbb{R},$$

$$dx(v_1, v_2) = v_1, \quad dy(v_1, v_2) = v_2.$$

Restricted to $T_pC = \mathrm{span}\{(1, f'(a))\}$ one has

$$dx\big(1, f'(a)\big) = 1, \qquad dy\big(1, f'(a)\big) = f'(a).$$

In particular $dy/dx\big|_p = f'(a)$ as expected.