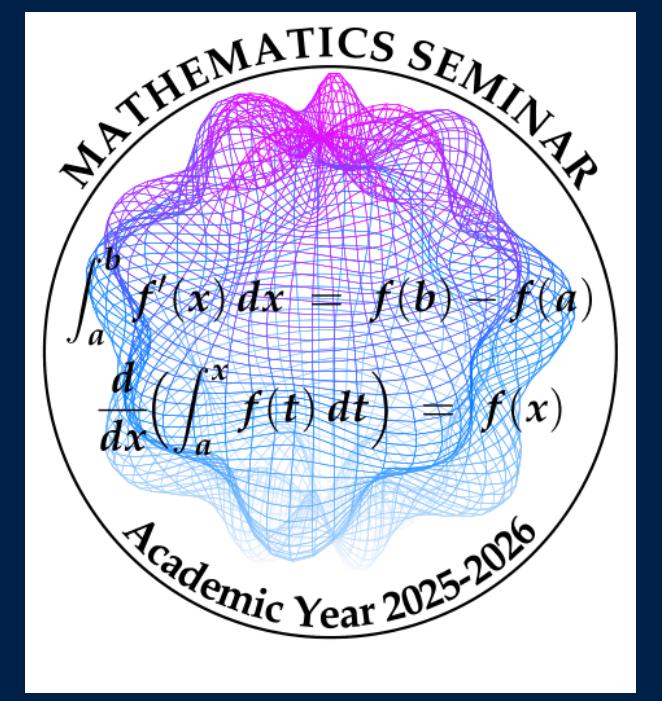


Riemann-Roch Theorem and its Application

- From de Rham to sheaf cohomology, Serre duality -



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Preliminaries I: Vector calculus as a cochain complex

On \mathbb{R}^3 , the familiar operators form a complex:

$$C^\infty \xrightarrow{\nabla} \Gamma(T^*) \xrightarrow{\nabla \times} \Gamma(\Lambda^2 T^*) \xrightarrow{\nabla \cdot} C^\infty,$$

with $(\nabla \times) \circ \nabla = 0$, $(\nabla \cdot) \circ (\nabla \times) = 0$. The de Rham complex is $(\Omega^\bullet(M), d)$:

$$0 \rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots$$

Cohomology measures **global obstructions** to solving $d\eta = \omega$.

Preliminaries II: Local-to-global glueing: Mayer–Vietoris

For $M = U \cup V$, we uses a partition of unity to build the short exact sequence

$$0 \rightarrow \Omega^k(M) \rightarrow \Omega^k(U) \oplus \Omega^k(V) \rightarrow \Omega^k(U \cap V) \rightarrow 0,$$

and then the long exact Mayer–Vietoris sequence in cohomology.

This means that “curl-free locally” does not imply “gradient globally” when topology obstructs global potentials.

Given $M = U \cup V$, we uses the short exact sequence of de Rham complexes

$$0 \rightarrow \Omega^k(M) \xrightarrow{\alpha^k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\beta^k} \Omega^k(U \cap V) \rightarrow 0$$

with $\beta^k(\omega_U, \omega_V) = \omega_U|_{U \cap V} - \omega_V|_{U \cap V}$, and surjectivity is built using a partition of unity $\{\rho_U, \rho_V\}$.

Why this belongs on a Riemann–Roch:

- Fine sheaves (built from partitions of unity) let we compute sheaf cohomology via resolutions.
- This is the conceptual bridge from grad/curl/div (de Rham) to $\bar{\partial}$ - and sheaf-cohomological formulas that culminate in Riemann–Roch.

Preliminaries III: From de Rham to Complex Geometry

On a compact Riemann surface M , complex structure refines $d = \partial + \bar{\partial}$. Holomorphic data live in sheaves:

\mathcal{O} (holomorphic functions), Ω (holomorphic 1-forms).

Divisors $D = \sum_p n_p p$ encode zeros/poles, and determine a line bundle $\mathcal{O}(D)$:

$$\mathcal{O}(D)(V) = \{TBA\}$$

Core I: Riemann–Roch (Curves)

Let M be a compact Riemann surface of genus g , and D a divisor. Define

$$\ell(D) = \dim H^0(M, \mathcal{O}(D)), \quad i(D) = \dim H^0(M, \Omega(-D)).$$

Riemann–Roch:

$$\ell(D) - i(D) = 1 - g + \deg D \quad \text{equivalently} \quad i(D) = \ell(K - D).$$

Core II: Cohomological mechanism: Euler characteristic + Serre duality

We derives Riemann–Roch from the Euler characteristic of $\mathcal{O}(D)$:

$$\chi(\mathcal{O}(D)) := \dim H^0(M, \mathcal{O}(D)) - \dim H^1(M, \mathcal{O}(D)) = \ell(D) - i(D),$$

using Serre duality to identify $H^1(M, \mathcal{O}(D))^* \simeq I(D)$.

Key principle:

$$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}_M) + \deg D, \quad \chi(\mathcal{O}_M) = 1 - g.$$

So χ is “topology + degree,” while $\ell(K - D)$ is the global obstruction.

Examples

Sphere \mathbb{P}^1 ($g = 0, K = -2[\infty]$). For $D = n[\infty]$, $\ell(D) = n+1$ for $n \geq -1$ (polynomials of degree $\leq n$).

Elliptic curve ($g = 1, \deg K = 0$). Riemann–Roch becomes $\ell(D) - i(D) = \deg D$, and for $\deg D > 0$ one has $\ell(D) = \deg D$.

Topology link: Gauss–Bonnet parallel

We emphasizes the analogy with Gauss–Bonnet:

$$\int K_{Gauss} \omega = 2\pi \chi(M) = 2\pi(2 - 2g).$$

Applications I: Algebraic–geometric (AG) codes from Riemann–Roch

Let X/\mathbb{F}_q be a smooth projective curve of genus g . Choose:

$$D = P_1 + \dots + P_n \text{ (distinct rational points)}, \\ G \text{ (divisor, } \text{supp}(G) \cap \text{supp}(D) = \emptyset).$$

Define the evaluation code

$$C_L(D, G) = \{(f(P_1), \dots, f(P_n)) : f \in L(G)\} \subseteq \mathbb{F}_q^n, \\ L(G) = H^0(X, \mathcal{O}(G)).$$

Dimension: $k = \dim C_L(D, G) = \ell(G) - \ell(G - D)$, and for $\deg G$ sufficiently large, $\ell(G) = \deg G + 1 - g$,

as large-degree/embedding discussion.

Designed distance: $d \geq n - \deg G$. (Standard AG-code bound.)

Example for Applications I: Classic McEliece and FALOMA

Applications II: “Surface direction” via Riemann–Roch for surfaces

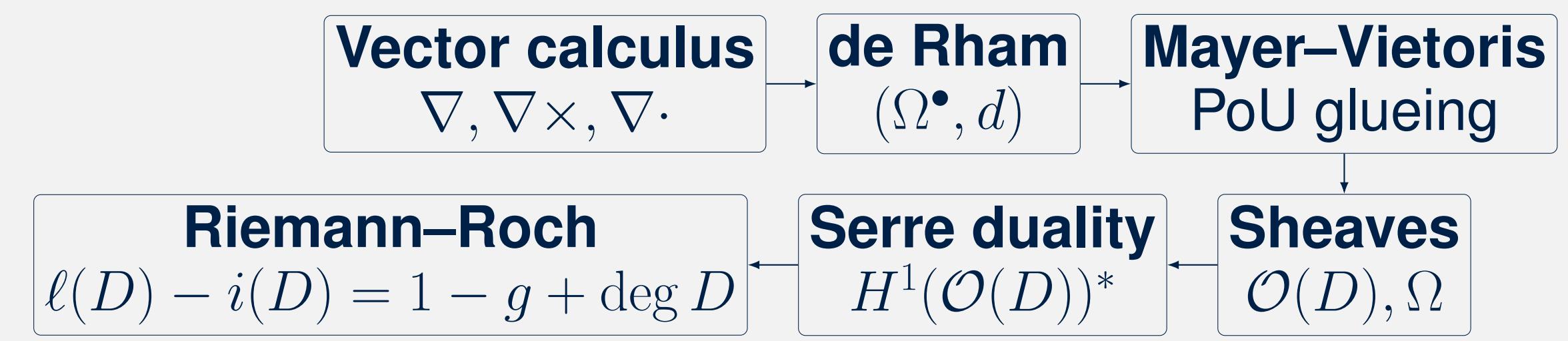
We states the surface analogue (for a smooth projective surface X and divisor D):

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2}(D \cdot D - D \cdot K_X),$$

where \cdot is the intersection pairing and K_X is the canonical divisor.

- **AG surface codes:** evaluate $H^0(X, \mathcal{O}_X(D))$ on rational points/curves; use the above χ plus vanishing theorems to estimate $\dim H^0$.
- **Topological/surface codes (quantum):** homological dimension depends on H_1 (genus); the same genus parameter appears in curve Riemann–Roch and in the obstruction term.

The pipeline: Calculus (grad/curl/div) → Riemann–Roch



Vector calculus as a cochain complex

On an oriented surface M , smooth differential forms package familiar operators:

$$0 \rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \rightarrow 0$$

- $d : \Omega^0 \rightarrow \Omega^1$ is the **gradient** operator in disguise.
- $d : \Omega^1 \rightarrow \Omega^2$ is the **curl** (scalar curl on a surface).
- The **divergence** is the codifferential d^* (adjoint of d via a metric).
- The Laplacian $\Delta = dd^* + d^*d$ yields **Hodge decomposition**:

$$\Omega^1(M) = \underbrace{d\Omega^0(M)}_{\text{grad part}} \oplus \underbrace{d^*\Omega^2(M)}_{\text{div part}} \oplus \underbrace{\mathcal{H}^1(M)}_{\text{harmonic}}$$

and $\mathcal{H}^1(M) \cong H_{\text{dR}}^1(M)$ (closed mod exact).

Index-counting slogan: “#(global solutions) – #(global obstructions)” is an Euler characteristic.

Riemann–Roch is precisely such an index formula for $\mathcal{O}(D)$.