

Concrete, calculus-style examples of $\mathcal{M}(X)$ for
 $X = \mathbb{CP}^1$ and $X = \mathbb{C}/\Lambda$

1 Case 1: $X = \mathbb{CP}^1$ (Riemann sphere)

1.1 Setup: coordinate and function field

View \mathbb{CP}^1 as the Riemann sphere:

$$\mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}.$$

Use the standard affine coordinate

$$z = \frac{z_0}{z_1}$$

on the chart $U_1 = \{[z_0 : z_1] \mid z_1 \neq 0\} \cong \mathbb{C}$. The point at infinity is

$$\infty = [1 : 0].$$

A meromorphic function on \mathbb{CP}^1 is the same as a rational function in z , i.e.

$$\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z).$$

We will *explicitly* analyze one such function and its differential.

1.2 A concrete meromorphic function f on \mathbb{CP}^1

Let

$$f(z) = \frac{(z-1)^2}{z(z-2)}.$$

Step 1: Zeros and poles on \mathbb{C}

- Zeros (where numerator vanishes):

$$(z - 1)^2 = 0 \Rightarrow z = 1 \text{ (double zero).}$$

So $\text{ord}_{z=1}(f) = +2$.

- Poles (where denominator vanishes):

$$z(z - 2) = 0 \Rightarrow z = 0, z = 2.$$

We check they are simple poles:

Near $z = 0$: write z as local coordinate. Then

$$f(z) = \frac{(z - 1)^2}{z(z - 2)} = \frac{z^2 - 2z + 1}{z(z - 2)} \sim \frac{1}{z(-2)} = -\frac{1}{2z} \text{ as } z \rightarrow 0.$$

So

$$\text{ord}_{z=0}(f) = -1.$$

Near $z = 2$: set $u = z - 2$, so $z = 2 + u$. Then

$$f(z) = f(2 + u) = \frac{(2 + u - 1)^2}{(2 + u)(2 + u - 2)} = \frac{(1 + u)^2}{(2 + u)u} \sim \frac{1}{2u} \text{ as } u \rightarrow 0.$$

So

$$\text{ord}_{z=2}(f) = -1.$$

So on the finite plane:

$$\text{ord}_1(f) = +2, \quad \text{ord}_0(f) = -1, \quad \text{ord}_2(f) = -1.$$

Step 2: Behavior at infinity via differential-calculus change of variable

To study $z = \infty$, use $w = 1/z$ as local coordinate near ∞ .

Then $z = 1/w$, and

$$f(z) = f\left(\frac{1}{w}\right).$$

Compute explicitly:

$$f\left(\frac{1}{w}\right) = \frac{(1/w - 1)^2}{(1/w)(1/w - 2)} = \frac{(1 - w)^2}{\frac{1}{w^2}(1 - 2w)} = (1 - w)^2 \cdot \frac{w^2}{1 - 2w}.$$

Now expand near $w = 0$. First,

$$(1 - w)^2 = 1 - 2w + w^2,$$

and

$$\frac{1}{1 - 2w} = 1 + 2w + 4w^2 + O(w^3).$$

So

$$f\left(\frac{1}{w}\right) = w^2(1 - 2w + w^2)(1 + 2w + 4w^2 + O(w^3)).$$

Multiply out just enough terms to see the leading behavior:

$$(1 - 2w + w^2)(1 + 2w + 4w^2) = 1 + (2w - 2w) + (\dots) = 1 + O(w^2),$$

so overall

$$f\left(\frac{1}{w}\right) = w^2(1 + O(w^2)) = w^2 + O(w^4).$$

Thus near $w = 0$, $f(1/w)$ has a *zero of order 2* as a function of w . But remember: the order at ∞ as a point of \mathbb{CP}^1 is the order of this function viewed on the sphere. More directly: in terms of the original coordinate z , we see that as $|z| \rightarrow \infty$,

$$f(z) = 1 + O\left(\frac{1}{z^2}\right),$$

so f is holomorphic and nonzero at ∞ . Hence

$$\text{ord}_\infty(f) = 0.$$

Step 3: The divisor of f

The divisor of f on \mathbb{CP}^1 is

$$\text{Div}(f) = 2 \cdot (1) - (0) - (2) + 0 \cdot (\infty).$$

Sum of coefficients:

$$2 - 1 - 1 + 0 = 0,$$

which matches the general fact that the sum of orders of a meromorphic function on a compact Riemann surface is zero.

1.3 Differential: computing $df = f'(z) dz$ in detail

Now do calculus: compute the derivative.

Step 1: Simplify f via partial fractions

We try to write

$$f(z) = A + \frac{B}{z} + \frac{C}{z-2}.$$

Compute directly:

$$\frac{(z-1)^2}{z(z-2)} = \frac{z^2 - 2z + 1}{z(z-2)}.$$

We look for A, B, C such that

$$\frac{z^2 - 2z + 1}{z(z-2)} = A + \frac{B}{z} + \frac{C}{z-2}.$$

Multiply both sides by $z(z-2)$:

$$z^2 - 2z + 1 = Az(z-2) + B(z-2) + Cz.$$

Expand the right-hand side:

$$Az(z-2) = A(z^2 - 2z) = Az^2 - 2Az,$$

$$B(z-2) = Bz - 2B,$$

$$Cz = Cz.$$

So

$$Az^2 - 2Az + Bz - 2B + Cz = Az^2 + (-2A + B + C)z - 2B.$$

Equate coefficients with the left side $z^2 - 2z + 1$:

$$\begin{cases} A = 1, \\ -2A + B + C = -2, \\ -2B = 1. \end{cases}$$

From $-2B = 1$, we get $B = -\frac{1}{2}$. Then

$$-2A + B + C = -2 \Rightarrow -2(1) - \frac{1}{2} + C = -2 \Rightarrow -2.5 + C = -2 \Rightarrow C = \frac{1}{2}.$$

So

$$f(z) = 1 - \frac{1}{2z} + \frac{1}{2(z-2)}.$$

Step 2: Differentiate term by term

Now

$$f(z) = 1 - \frac{1}{2z} + \frac{1}{2(z-2)}.$$

Differentiate:

$$f'(z) = 0 + \frac{1}{2z^2} - \frac{1}{2(z-2)^2}.$$

So the meromorphic 1-form

$$df = f'(z) dz = \left(\frac{1}{2z^2} - \frac{1}{2(z-2)^2} \right) dz.$$

Poles of df . Clearly:

df has poles of order 2 at $z = 0$ and $z = 2$.

No simple pole terms appear in the Laurent expansions; hence all residues are 0.

Check residues explicitly via Laurent series. Near $z = 0$:

$$\frac{1}{2z^2} - \frac{1}{2(z-2)^2} = \frac{1}{2z^2} - \frac{1}{2(4-4z+z^2)} = \frac{1}{2z^2} - \frac{1}{8} \cdot \frac{1}{1-z+z^2/4}.$$

Expand $\frac{1}{1-z+z^2/4}$ as a power series in z (no negative powers), so near $z = 0$ this second term has no negative-power part. Thus the only negative-power part is $\frac{1}{2z^2}$, which has no $(z-0)^{-1}$ term. So $\text{Res}_{z=0}(df) = 0$.

Similarly near $z = 2$, set $u = z - 2$. Then

$$df = \left(\frac{1}{2(2+u)^2} - \frac{1}{2u^2} \right) d(2+u) = \left(\frac{1}{8} \cdot \frac{1}{(1+u/2)^2} - \frac{1}{2u^2} \right) du.$$

Again the first term has only nonnegative powers in u , the second has u^{-2} but no u^{-1} . So $\text{Res}_{z=2}(df) = 0$.

At ∞ , we could check via $w = 1/z$. A general fact: for any meromorphic function f on a compact Riemann surface,

$$\sum_p \text{Res}_p(df) = 0.$$

Since the residues at 0 and 2 are zero, the residue at ∞ is also zero.

1.4 Moral for \mathbb{CP}^1 : function field via differential forms

In general, if f is *any* meromorphic function on \mathbb{CP}^1 , then

- df is a meromorphic 1-form whose residues all vanish.
- By analyzing the principal parts and using Liouville, one shows that f must be a rational function in the coordinate z .

Thus the function field is

$$\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z).$$

Our explicit f shows concretely how poles/zeros and the differential look, and how partial fractions naturally appear from calculus.

2 Case 2: $X = \mathbb{C}/\Lambda$ (complex torus)

Now we move to a more subtle example and use differential forms very concretely.

2.1 The torus and the basic holomorphic 1-form

Let $\Lambda \subset \mathbb{C}$ be a lattice:

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2, \quad \Im\left(\frac{\omega_2}{\omega_1}\right) > 0.$$

Define the complex torus

$$X = \mathbb{C}/\Lambda.$$

The projection $\pi : \mathbb{C} \rightarrow X$ is holomorphic and locally biholomorphic. The 1-form dz on \mathbb{C} is invariant under translations by Λ , so it descends to a global holomorphic 1-form on X . In fact,

$$H^0(X, \Omega_X^1) = \mathbb{C} \cdot dz,$$

i.e. there is a one-dimensional space of holomorphic 1-forms.

2.2 The Weierstrass \wp and its derivative \wp'

Define the Weierstrass \wp -function:

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right).$$

Facts:

- $\wp(z + \lambda) = \wp(z)$ for all $\lambda \in \Lambda$ (Λ -periodic).
- \wp is even: $\wp(-z) = \wp(z)$.
- \wp is meromorphic on \mathbb{C} with poles of order 2 at each lattice point.

Differentiate term by term (justified by uniform convergence on compact sets away from poles):

$$\wp'(z) = -\frac{2}{z^3} - 2 \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{(z - \lambda)^3}.$$

Then:

- \wp' is odd: $\wp'(-z) = -\wp'(z)$.
- $\wp'(z)$ has poles of order 3 at each lattice point.

Both \wp and \wp' are Λ -periodic, so they descend to meromorphic functions on $X = \mathbb{C}/\Lambda$:

$$\wp_X([z]) = \wp(z), \quad (\wp_X)'([z]) = \wp'(z).$$

2.3 The meromorphic 1-form $\wp'(z) dz = d\wp(z)$

Consider the 1-form

$$\omega = \wp'(z) dz.$$

This is clearly

$$\omega = d\wp(z),$$

as in calculus: derivative times dz .

Local expansion near a pole

Focus on the point $[0] \in X$ (the image of $0 \in \mathbb{C}$).

Near $z = 0$,

$$\wp(z) = \frac{1}{z^2} + c_2 z^2 + c_4 z^4 + \dots$$

for some complex coefficients c_k (coming from the lattice). Then

$$\wp'(z) = -\frac{2}{z^3} + 2c_2 z + 4c_4 z^3 + \dots$$

So near $z = 0$,

$$\omega = \left(-\frac{2}{z^3} + 2c_2 z + 4c_4 z^3 + \dots \right) dz.$$

In Laurent series form:

$$\omega = -2z^{-3} dz + 2c_2 z dz + 4c_4 z^3 dz + \dots$$

Note there is *no* term of the form $c_{-1} z^{-1} dz$, so

$$\text{Res}_{z=0}(\omega) = 0.$$

Similarly, at any other lattice point $\lambda \in \Lambda$, shifting $z \mapsto z - \lambda$ gives the same type of expansion: pole of order 3, no $1/(z - \lambda)$ term. Thus

$$\text{Res}_{[0]}(\omega) = 0, \quad \text{Res}_{[\lambda]}(\omega) = 0 \quad \text{in the quotient } X.$$

Exactness and periods

Since $\omega = d\wp$, it is an exact differential. On the torus, this implies for any closed loop γ in X ,

$$\oint_{\gamma} \omega = \oint_{\gamma} d\wp = 0.$$

This is the global version of the fact that an exact differential has zero integral over closed paths.

2.4 A very concrete elliptic function built from \wp

Fix some $a \in \mathbb{C}$ with $[a] \neq [0]$ in X , and also $[a] \neq [-a]$ (i.e. $2a \notin \Lambda$). Consider the function

$$g(z) = \wp(z) - \wp(a),$$

which is Λ -periodic and meromorphic.

Define:

$$f(z) = \frac{\wp'(z)}{\wp(z) - \wp(a)}.$$

This is a classical elliptic function: it is the derivative of the logarithm of $g(z)$:

$$f(z) = \frac{d}{dz} \log(\wp(z) - \wp(a)).$$

Poles of $f(z)$ on the torus X

At $z = a$. Near $z = a$, write $\zeta = z - a$. Then expand:

$$\wp(z) = \wp(a) + \wp'(a)\zeta + \frac{1}{2}\wp''(a)\zeta^2 + \dots$$

So

$$\wp(z) - \wp(a) = \wp'(a)\zeta + \frac{1}{2}\wp''(a)\zeta^2 + \dots$$

Similarly

$$\wp'(z) = \wp'(a) + \wp''(a)\zeta + \dots$$

Thus

$$f(z) = \frac{\wp'(z)}{\wp(z) - \wp(a)} = \frac{\wp'(a) + \wp''(a)\zeta + \dots}{\wp'(a)\zeta + \frac{1}{2}\wp''(a)\zeta^2 + \dots}.$$

Pull out $\wp'(a)\zeta$ from the denominator:

$$f(z) = \frac{\wp'(a) + \wp''(a)\zeta + \dots}{\wp'(a)\zeta \left(1 + \frac{\wp''(a)}{2\wp'(a)}\zeta + \dots\right)}.$$

Now expand:

$$\frac{1}{1 + \alpha\zeta + \dots} = 1 - \alpha\zeta + \dots,$$

so near $\zeta = 0$,

$$f(z) \sim \frac{1}{\zeta} \cdot \frac{\wp'(a)}{\wp'(a)} = \frac{1}{\zeta} = \frac{1}{z - a}.$$

Thus f has a *simple pole* at $z = a$ with residue

$$\text{Res}_{z=a}(f(z) dz) = 1.$$

At $z = -a$. Because \wp is even and \wp' is odd,

$$\wp(-z) = \wp(z), \quad \wp'(-z) = -\wp'(z).$$

The equation $\wp(z) = \wp(a)$ has the two solutions $z = \pm a$ modulo Λ . So $\wp(z) - \wp(a)$ also vanishes at $z = -a$. Repeating the same expansion, or just using the symmetry, we find

$$\text{Res}_{z=-a}(f(z) dz) = 1.$$

At $z = 0$ (and other lattice points). Near $z = 0$,

$$\wp(z) \sim \frac{1}{z^2}, \quad \wp'(z) \sim -\frac{2}{z^3}.$$

Then

$$f(z) = \frac{\wp'(z)}{\wp(z) - \wp(a)} \sim \frac{-2z^{-3}}{z^{-2} - \wp(a)} = \frac{-2z^{-3}}{z^{-2}(1 - \wp(a)z^2)} = \frac{-2}{z} \cdot \frac{1}{1 - \wp(a)z^2}.$$

Expand $\frac{1}{1 - \wp(a)z^2} = 1 + \wp(a)z^2 + \dots$, so

$$f(z) \sim \frac{-2}{z} + (\text{holomorphic terms}).$$

Thus at $z = 0$ we have a simple pole with residue

$$\text{Res}_{z=0}(f(z) dz) = -2.$$

Similarly, at other lattice points $\lambda \neq 0$, the local behavior is like near 0 shifted by λ , giving the same residue pattern, but on the torus $X = \mathbb{C}/\Lambda$ all lattice points project to a finite set of points, and you can group residues within one fundamental domain.

Residue theorem on the torus

On the compact Riemann surface X ,

$$\sum_{p \in X} \text{Res}_p(f(z) dz) = 0.$$

From the computations:

- $\text{Res}_{[a]} = 1$,

- $\text{Res}_{[-a]} = 1$,
- $\text{Res}_{[0]} = -2$

(and no other poles in a fundamental parallelogram), so

$$1 + 1 - 2 = 0.$$

This is a very concrete check of the residue theorem on a torus for this particular elliptic function.

2.5 Function field $\mathcal{M}(\mathbb{C}/\Lambda)$ via \wp and \wp'

A fundamental theorem: \wp and \wp' satisfy

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

where g_2, g_3 are complex constants depending on Λ .

If we set

$$X := \wp(z), \quad Y := \wp'(z),$$

then

$$Y^2 = 4X^3 - g_2X - g_3.$$

From the differential-form viewpoint:

$$dX = d\wp(z) = \wp'(z) dz = Y dz \quad \Rightarrow \quad dz = \frac{dX}{Y}.$$

So the basic holomorphic 1-form dz on the torus becomes

$$dz = \frac{dX}{\sqrt{4X^3 - g_2X - g_3}},$$

when we view the torus as the algebraic curve $Y^2 = 4X^3 - g_2X - g_3$.

The function field is then

$$\mathcal{M}(\mathbb{C}/\Lambda) = \mathbb{C}(\wp, \wp') \cong \mathbb{C}(X, Y) / (Y^2 - 4X^3 + g_2X + g_3).$$

Concretely, any meromorphic function F on X is of the form

$$F([z]) = R(\wp(z), \wp'(z))$$

for some rational function $R(X, Y)$.

Our explicit example

$$f(z) = \frac{\wp'(z)}{\wp(z) - \wp(a)} = \frac{Y}{X - \wp(a)}$$

is exactly such a rational function in $(X, Y) = (\wp, \wp')$, and we computed its poles and residues using differential-form techniques in detail.

Conclusion

- For $X = \mathbb{CP}^1$, we picked a very concrete rational function $f(z) = (z - 1)^2/(z(z - 2))$, computed its divisor, wrote it via partial fractions, and computed df as a meromorphic 1-form, verifying the residue behavior. This illustrates $\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z)$.
- For $X = \mathbb{C}/\Lambda$, we used \wp, \wp' and a very explicit elliptic function $f(z) = \wp'(z)/(\wp(z) - \wp(a))$, expanded near poles, and computed residues of the meromorphic 1-form $f(z) dz$. This illustrates concretely how $\mathcal{M}(\mathbb{C}/\Lambda)$ is generated by \wp, \wp' , and how differential forms (like $\wp'(z) dz = d\wp(z)$) control the structure of the function field.