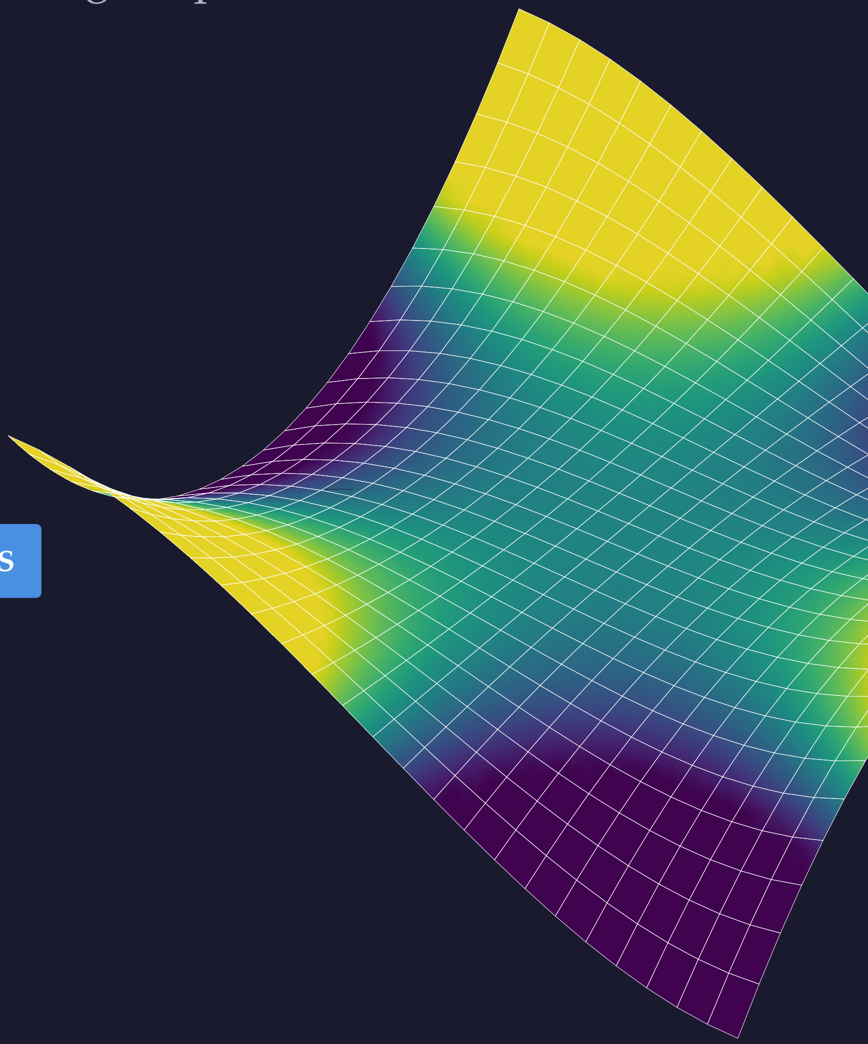
Riemann Surfaces and Algebraic Curves

A framework for understanding Elliptic Curves

Ji, Yonghyeon



PART I — MULTIVARIABLE CALCULUS

A Comprehensive Guide to Modern Algebraic Geometry

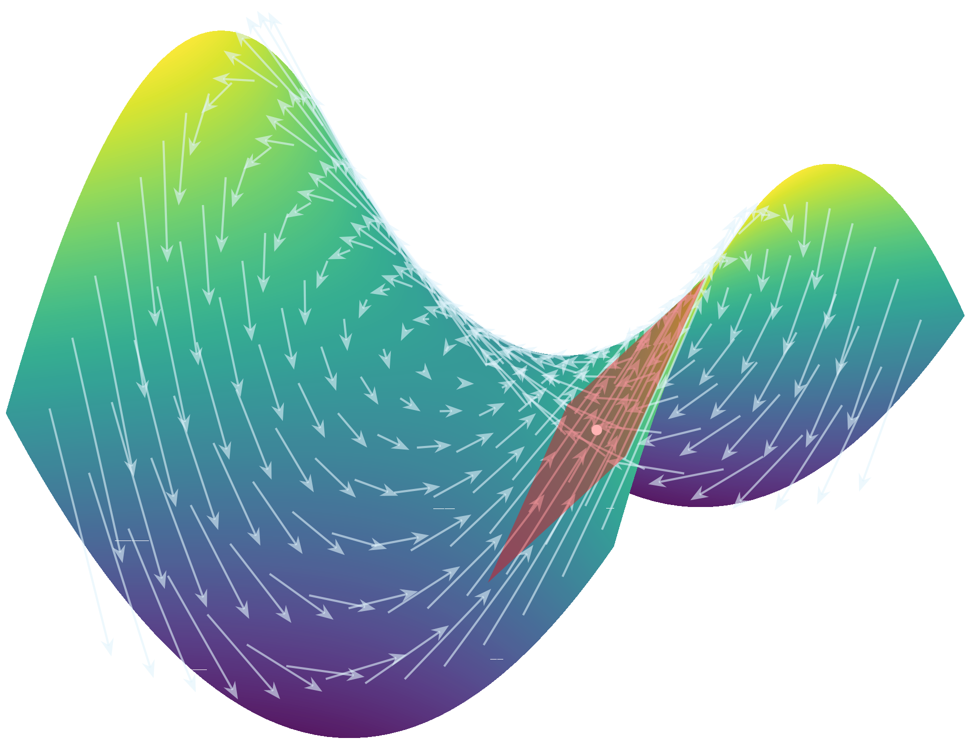
Riemann Surfaces and Algebraic Curves

A Framework for Understanding Elliptic Curves

Part I — Multivariable Calculus

Ji, Yonghyeon

January 28, 2026



WINTER 2026

The FTC hierarchy

Name	**Formula**
FTC I (Accumulation)	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\,\mathrm{d}t\right) = f(x).$
FTC II (Evaluation)	$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a).$
Fundamental Theorem of Line Integrals	$\int_C \nabla\phi \cdot \mathrm{d}\mathbf{r} = \phi(B) - \phi(A).$
Green's Theorem	$\oint_{\partial R} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_R \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\,\mathrm{d}A.$
Stokes' Theorem (3D)	$\oint_{\partial S} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F})\cdot\mathbf{n}\,\mathrm{d}S.$
Divergence Theorem	$\iint_{\partial V} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iiint_V (\nabla\cdot\mathbf{F})\,\mathrm{d}V.$
Generalized Stokes	$\int_{\partial\Omega} \omega = \int_\Omega \mathrm{d}\omega.$

Copyright

Changelog

v1.0	2025-12-29	Initial release.

Contents

1 Fundamental Theorem of Calculus

Fundamental Theorem for Gradient Fields

If $\mathbf{F} = \nabla f$ is a conservative vector field and C is a smooth curve from A to B, then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(B) - f(A).$$

Green's Theorem

For a positively oriented, simple closed curve C bounding a region R in the plane,

$$\oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A.$$

Divergence Theorem

Let $\mathbf{F}$ be a vector field defined on a region E with closed boundary surface S (outward-oriented). Then

$$\iiint_E \nabla \cdot \mathbf{F}\,\mathrm{d}V = \iint_S \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S.$$

Stokes' Theorem

Let S be an oriented surface with boundary curve C, and let $\mathbf{F}$ be a vector field. Then

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\mathrm{d}\mathbf{S}.$$

Triple Integral

To integrate a scalar function $f(x, y, z)$ over a region E in $\mathbb{R}^3$,

$$\iiint_E f(x, y, z)\,\mathrm{d}V.$$

1.1 Gradient Vector Fields

Scalar field

Definition 1.1. *Let $U \subseteq \mathbb{R}^n$ be an open set. A* scalar field *on U is a function*

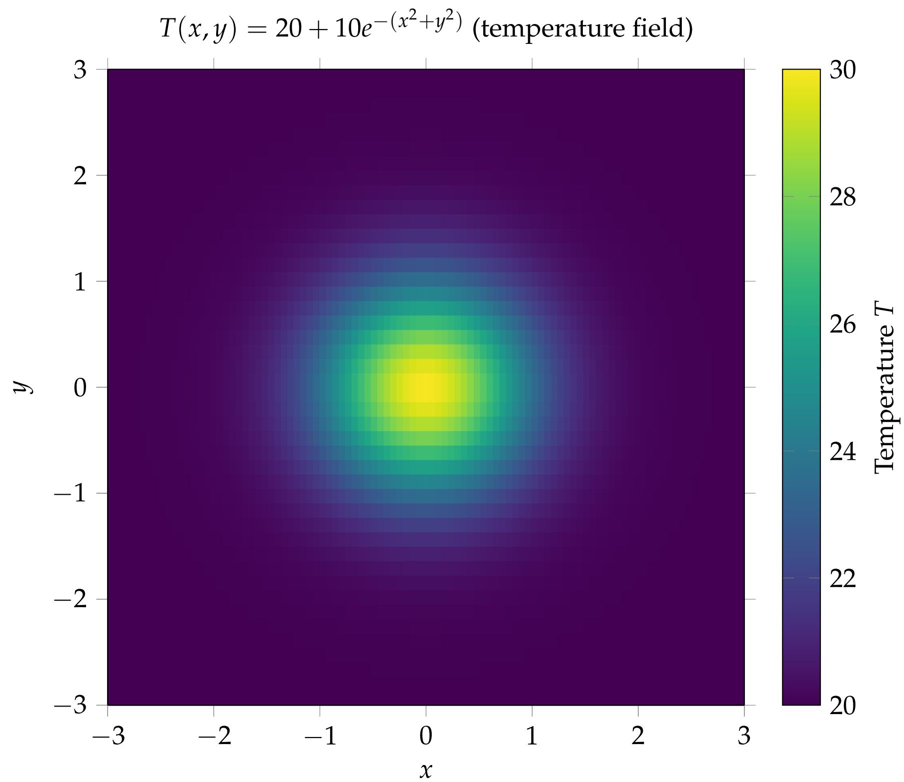
$$\begin{array}{rccc} f & : & U & \longrightarrow & \mathbb{R} \\ & & \mathbf{x} & \longmapsto & f(\mathbf{x}) \end{array}.$$

Equivalently, to each point $\mathbf{x} \in U$ the scalar field assigns a real number $f(\mathbf{x})$.

Example 1 (Temperature distribution)**.** Let $U = \mathbb{R}^2$. Define

$$T(x, y) = 20 + 10e^{-(x^2+y^2)}.$$

Then $T : U \to \mathbb{R}$ is a scalar field. One may interpret $T(x, y)$ as the temperature (in degrees) at the point (x, y). Notice that $T(0, 0) = 30$ and $T(x, y) \to 20$ as $x^2 + y^2 \to \infty$, so the temperature is highest at the origin and decays outward.



Vector field

Definition 1.2. *Let $U \subseteq \mathbb{R}^n$ be an open set. A* vector field *on U is a function*

$$\begin{array}{rccc} \mathbf{F} & : & U & \longrightarrow & \mathbb{R}^n \\ & & \mathbf{x} & \longmapsto & \mathbf{F}(\mathbf{x}) \end{array}.$$

Equivalently, to each point $\mathbf{x} \in U$ the vector field assigns a vector $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^n$. In coordinates, one often writes

$$\mathbf{F}(\mathbf{x}) = \langle F_1(\mathbf{x}), F_2(\mathbf{x}), \dots, F_n(\mathbf{x})\rangle,$$

where each component function $F_i : U \to \mathbb{R}$ is a scalar field on U.

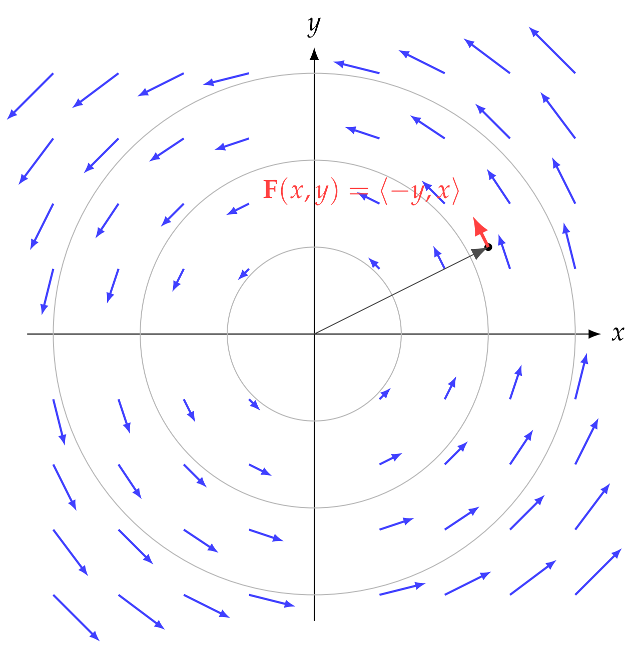
Example 2 (Rotation field)**.** Let $U = \mathbb{R}^2$. Define

$$\mathbf{F}(x, y) = \langle -y,\ x\rangle.$$

Then $\mathbf{F} : U \to \mathbb{R}^2$ is a vector field. At each point (x, y) it assigns the vector $(-y, x)$, which is perpendicular to $\langle x, y\rangle$ and tangent to the circle $x^2 + y^2 =$ constant. Moreover,

$$\|\mathbf{F}(x, y)\| = \sqrt{x^2 + y^2},$$

so the magnitude increases linearly with the distance from the origin. This field models a rigid counterclockwise rotational flow about the origin.



Remark.

Definition 1.3 (Conservative vector field). *Let $U \subseteq \mathbb{R}^n$ be an open set and let $\mathbf{F} : U \to \mathbb{R}^n$ be a vector field. We say that $\mathbf{F}$ is* conservative *on U if there exists a scalar field $f : U \to \mathbb{R}$ of class C^1 such that*

$$\mathbf{F} = \nabla f \quad \text{on } U.$$

In this case, f is called a potential function *for* $\mathbf{F}$.

Remark (Equivalent characterization). A vector field $\mathbf{F}$ on U is conservative if and only if for every piecewise C^1 curve C in U with endpoints $\mathbf{A}, \mathbf{B}$, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on $\mathbf{A}$ and $\mathbf{B}$ (path independence). Equivalently,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

for every piecewise C^1 closed curve C in U.

Definition 1.4 (Gradient operator)**.** *Let $U \subseteq \mathbb{R}^n$ be open. The* gradient operator *(or* nabla operator*) is the map*

$$\nabla : C^1(U) \longrightarrow C^0(U, \mathbb{R}^n)$$

defined by

$$(\nabla f)(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial f}{\partial x_1}(\mathbf{x}) \\ \dfrac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \dfrac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix}, \qquad \mathbf{x} = (x_1, \dots, x_n) \in U.$$

Remark (Directional derivative characterization). For $f \in C^1(U)$ and $\mathbf{x} \in U$, the vector $\nabla f(\mathbf{x}) \in \mathbb{R}^n$ is uniquely characterized by the property that

$$D_{\mathbf{v}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{v} \qquad \text{for all } \mathbf{v} \in \mathbb{R}^n,$$

where $D_{\mathbf{v}} f(\mathbf{x})$ denotes the directional derivative of f at $\mathbf{x}$ in the direction $\mathbf{v}$.

[Linearity of ∇] Let $f, g \in C^1(U)$ and $a, b \in \mathbb{R}$. Then

$$\nabla(af + bg) = a\,\nabla f + b\,\nabla g.$$

Remark (Jacobian/transpose viewpoint). For $f \in C^1(U)$, the total derivative at $\mathbf{x} \in U$ is the linear map

$$Df(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}, \qquad Df(\mathbf{x})\,\mathbf{v} = D_{\mathbf{v}} f(\mathbf{x}).$$

In coordinates, $Df(\mathbf{x})$ is represented by the $1 \times n$ Jacobian row matrix

$$Df(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}) \quad \cdots \quad \frac{\partial f}{\partial x_n}(\mathbf{x}) \right),$$

and the gradient is its transpose:

$$\nabla f(\mathbf{x}) = \Big(Df(\mathbf{x})\Big)^{\mathsf{T}}.$$

Consequently, for any $\mathbf{v} \in \mathbb{R}^n$,

$$D_{\mathbf{v}} f(\mathbf{x}) = Df(\mathbf{x})\,\mathbf{v} = \Big(\nabla f(\mathbf{x})\Big)^{\mathsf{T}} \mathbf{v} = \nabla f(\mathbf{x}) \cdot \mathbf{v}.$$

Fundamental Theorem for Gradient Fields

Theorem 1.5. *Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}$ be continuously differentiable ($f \in C^1(U)$). Let C be a piecewise C^1 curve in U with a piecewise C^1 parametrization*

$$\mathbf{r} : [a,b] \to U.$$

If $\mathbf{r}(a) = \mathbf{A}$ and $\mathbf{r}(b) = \mathbf{B}$, then

$$\int_C \nabla f \cdot d\mathbf{r} \;=\; f(\mathbf{B}) - f(\mathbf{A}).$$

In particular, the line integral of a gradient field depends only on the endpoints of the curve.

Proof. By definition of the line integral of a vector field along a parametrized curve,

$$\int_C \nabla f \cdot d\mathbf{r} \;=\; \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

Define a scalar-valued function $g : [a,b] \to \mathbb{R}$ by $g(t) = f(\mathbf{r}(t))$. Since $f \in C^1(U)$ and $\mathbf{r}$ is piecewise C^1, the composition g is piecewise C^1. On any subinterval where $\mathbf{r}$ is C^1, the multivariable chain rule gives

$$g'(t) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

Hence on each such subinterval we have $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = g'(t)$, and summing over the finitely many smooth pieces yields

$$\int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_a^b g'(t)\, dt.$$

By the (single-variable) Fundamental Theorem of Calculus,

$$\int_a^b g'(t)\, dt = g(b) - g(a).$$

Substituting back $g(t) = f(\mathbf{r}(t))$ gives

$$g(b) - g(a) = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(\mathbf{B}) - f(\mathbf{A}).$$

Therefore,

$$\int_C \nabla f \cdot d\mathbf{r} \;=\; f(\mathbf{B}) - f(\mathbf{A}),$$

as claimed. □

Consider a small rectangle centered at (x_0, y_0) with side lengths $\Delta x, \Delta y$.

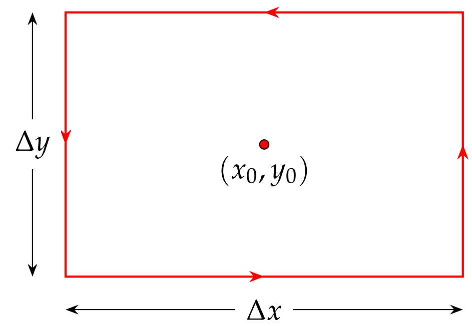


Figure 1: Circulation around an infinitesimal rectangle.

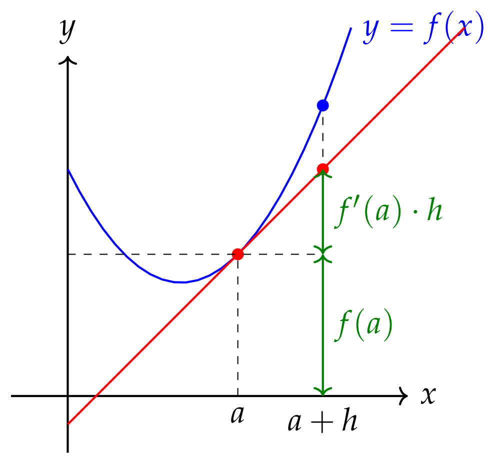
The total counterclockwise circulation is the sum of the line integrals along the four edges:

$$\oint_{\partial R} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\text{bottom}} P \, \mathrm{d}x + \int_{\text{right}} Q \, \mathrm{d}y + \int_{\text{top}} P \, \mathrm{d}x + \int_{\text{left}} Q \, \mathrm{d}y.$$

We will approximate the value of P or Q along each edge as being constant, equal to its value at the midpoint of that edge. We find this value using a first-order Taylor expansion from the center point (x_0, y_0).

For a simple function of one variable, $f(x)$, if we know its value at a point a, then we can estimate its value at a nearly point $a + h$ using the tangent line at a:

$$f(a+h) \approx f(a) + f'(a) \cdot h \quad \text{or}$$
$$f(x) \approx f(a) + f'(a) \cdot (x - a).$$



In words, "New Value $\approx$ Old Value + (Rate of Change) $\times$ (Small Step)".

For a function of two variables like $P(x,y)$, the idea is identical, but the "rate of change" now has two components (one for each direction), and the "tangent line" becomes a "tangent plane". The general first-order Taylor expansion for $P(x,y)$ around a center point (x_0,y_0) is

$$P(x_0+a,y_0+b) \approx P(x_0,y_0) + \frac{\partial P}{\partial x}(x_0,y_0)\cdot a + \frac{\partial P}{\partial y}(x_0,y_0)\cdot b$$

Here, a is the small step in the x-direction, and b is the small step in the y-direction.

1. The Horizontal Paths These integrals involve the horizontal component of $P(x,y)$.

- **Bottom Path ($\rightarrow$):**

$$\begin{aligned} &P\left(x, y_0 - \frac{\Delta y}{2}\right) \approx P(x_0,y_0) - \frac{\partial P}{\partial y}(x_0,y_0)\frac{\Delta y}{2} \\ &\implies \int_{\text{bottom}} P\,\mathrm{d}x \approx \int_{-\Delta x/2}^{\Delta x/2}\left(P_0 + P_x s - P_y\frac{\Delta y}{2}\right)\,ds \qquad (x = x_0 + s,\ dx = ds) \\ &\implies \int_{\text{bottom}} P\,\mathrm{d}x \approx \left(P(x_0,y_0) - \frac{\partial P}{\partial y}(x_0,y_0)\frac{\Delta y}{2}\right)(\Delta x) \end{aligned}$$

  Note that

$$\int_{-\Delta x/2}^{\Delta x/2} P_x s\,ds = P_x\left[\frac{s^2}{2}\right]_{-\Delta x/2}^{\Delta x/2} = P_x\left(\frac{(\Delta x/2)^2}{2} - \frac{(\Delta x/2)^2}{2}\right) = 0.$$

- **Top Path ($\leftarrow$):**

$$P\left(x, y_0 + \frac{\Delta y}{2}\right) \approx P(x_0,y_0) + \frac{\partial P}{\partial y}\frac{\Delta y}{2} \implies \int_{\text{top}} P\,\mathrm{d}x \approx -\left(P(x_0,y_0) + \frac{\partial P}{\partial y}\frac{\Delta y}{2}\right)(\Delta x)$$

Here, we are left with only the parts that describe the *change* in P with respect to y.

$$\int_{\text{bottom}} P\,\mathrm{d}x + \int_{\text{top}} P\,\mathrm{d}x \approx \left(-\frac{\partial P}{\partial y}\frac{\Delta y}{2}\right)\Delta x - \left(\frac{\partial P}{\partial y}\frac{\Delta y}{2}\right)\Delta x = -\frac{\partial P}{\partial y}\Delta x\Delta y$$

2. The Vertical Paths These integrals involve the vertical component of $Q(x,y)$.

- **Right Path ($\uparrow$):**

$$Q\left(x_0 + \frac{\Delta x}{2}, y\right) \approx Q(x_0,y_0) + \frac{\partial Q}{\partial x}\frac{\Delta x}{2} \implies \int_{\text{right}} Q\,\mathrm{d}y \approx \left(Q(x_0,y_0) + \frac{\partial Q}{\partial x}\frac{\Delta x}{2}\right)(\Delta y)$$

- **Left Path ($\downarrow$):**

$$Q\left(x_0 - \frac{\Delta x}{2}, y\right) \approx Q(x_0,y_0) - \frac{\partial Q}{\partial x}\frac{\Delta x}{2} \implies \int_{\text{left}} Q\,\mathrm{d}y \approx -\left(Q(x_0,y_0) - \frac{\partial Q}{\partial x}\frac{\Delta x}{2}\right)(\Delta y)$$

Here, we are left with only the parts that describe the *change* in Q with respect to x.

$$\int_{\text{right}} Q\,\mathrm{d}y + \int_{\text{left}} Q\,\mathrm{d}y \approx \left(\frac{\partial Q}{\partial x}\frac{\Delta x}{2}\right)\Delta y + \left(\frac{\partial Q}{\partial x}\frac{\Delta x}{2}\right)\Delta y = \frac{\partial Q}{\partial x}\Delta x \Delta y$$

Now we sum the results from the horizontal and vertical pairs:

$$\begin{aligned}\oint_{\partial R} \mathbf{F}\cdot d\mathbf{r} &\approx \left(-\frac{\partial P}{\partial y}\Delta x \Delta y\right) + \left(\frac{\partial Q}{\partial x}\Delta x \Delta y\right)\\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\Delta x \Delta y\end{aligned}$$

This shows that the total circulation around the tiny loop is approximately the quantity $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$ multiplied by the area of the loop ($\Delta A = \Delta x \Delta y$).

To get the property *at the point* (x_0, y_0), we find the circulation **density**. We divide by the area and take the limit as the rectangle shrinks to zero.

$$\lim_{\Delta A \to 0} \frac{1}{\Delta A}\oint_{\partial R} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \frac{\partial Q}{\partial x}(x_0, y_0) - \frac{\partial P}{\partial y}(x_0, y_0)$$

This is why we call the scalar quantity $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ the **curl**: it is the circulation per unit area at a point, which measures the local rotational tendency of the field.

If $C = \partial D$ is a positively oriented simple closed curve enclosing a region D, Green's theorem states

$$\underbrace{\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}}_{\substack{\textbf{Line Integral}\\ \textbf{(Total Circulation)}}} = \underbrace{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A}_{\substack{\textbf{Double Integral}\\ \textbf{(Sum of Local Curls)}}}$$

1. Let $\mathbf{F} = \langle 2x, 2y \rangle$. Show that $\mathbf{F}$ is conservative and compute

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where C is any path from $(0,0)$ to $(1,1)$.

Sol. Let $\mathbf{F} = \langle P, Q \rangle = \langle 2x, 2y \rangle$. Since

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2x) = 0 \qquad \text{and} \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(2y) = 0,$$

we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere in $\mathbb{R}^2$, a simply connected domain. Therefore $\mathbf{F}$ is conservative.

To find a potential function f with $\nabla f = \mathbf{F}$, we solve

$$f_x = 2x \quad \Rightarrow \quad f(x,y) = \int 2x\, dx = x^2 + g(y),$$

for some function $g(y)$. Then

$$f_y = g'(y) = 2y \quad \Rightarrow \quad g(y) = y^2 + C.$$

Hence a potential function is

$$f(x,y) = x^2 + y^2 \quad \text{(constant irrelevant)}.$$

By the Fundamental Theorem for Line Integrals, for any path C from $(0,0)$ to $(1,1)$,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1) - f(0,0) = \left(1^2 + 1^2\right) - \left(0^2 + 0^2\right) = 2.$$

□

2. Determine whether the vector field $\mathbf{F} = \langle y, x \rangle$ is conservative. If so, find a potential function.

Sol. Let $\mathbf{F} = \langle P, Q \rangle = \langle y, x \rangle$. Compute the mixed partials:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y) = 1, \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x) = 1.$$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere in $\mathbb{R}^2$ (a simply connected domain), $\mathbf{F}$ is conservative.
To find a potential function f such that $\nabla f = \mathbf{F}$, we solve

$$f_x = P = y.$$

Integrating with respect to x gives

$$f(x, y) = \int y \, dx = xy + g(y),$$

where g is a function of y only. Differentiate with respect to y:

$$f_y = x + g'(y).$$

But f_y must equal $Q = x$, so $g'(y) = 0$, hence $g(y) = C$.
Therefore, a potential function is

$$f(x, y) = xy \quad \text{(up to an additive constant)}.$$

Let $\mathbf{F} = \langle P, Q \rangle = \langle y, x \rangle$ on an open set $U \subseteq \mathbb{R}^2$. Since $P, Q \in C^1(U)$, $\mathbf{F}$ is conservative on U if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ on U.

Compute:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y) = 1, \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x) = 1.$$

Thus $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere on U, so $\mathbf{F}$ is conservative.

To find a potential function f with $\nabla f = \mathbf{F}$, we require

$$f_x = y, \qquad f_y = x.$$

Integrate $f_x = y$ with respect to x:

$$f(x, y) = xy + g(y),$$

for some function g of y alone. Differentiate with respect to y:

$$f_y(x, y) = x + g'(y).$$

Set this equal to x (since $f_y = x$):

$$x + g'(y) = x \;\Rightarrow\; g'(y) = 0 \;\Rightarrow\; g(y) = C.$$

Therefore a potential function is

$$f(x, y) = xy + C.$$

(Any two potential functions differ by an additive constant.) □

Theorem 1.6 (Curl test for conservativeness in $\mathbb{R}^2$). *Let $U \subseteq \mathbb{R}^2$ be an open and simply connected set, and let*

$$\mathbf{F} = \langle P, Q \rangle : U \to \mathbb{R}^2$$

be a C^1 vector field. Then the following are equivalent:

(a) $\mathbf{F}$ *is conservative on U, i.e. there exists a C^1 function $f : U \to \mathbb{R}$ such that $\nabla f = \mathbf{F}$.*

(b) $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ *everywhere on U.*

Proof. **(1)⇒(2).** Assume $\mathbf{F}$ is conservative, so $\mathbf{F} = \nabla f$ for some $f \in C^1(U)$. In coordinates,

$$P = f_x, \qquad Q = f_y.$$

If moreover $f \in C^2(U)$ (which holds, for instance, if $P, Q \in C^1$ and f is constructed as in the converse direction), then

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(f_x) = f_{xy}, \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(f_y) = f_{yx}.$$

By Clairaut's theorem (equality of mixed partials for C^2 functions), $f_{xy} = f_{yx}$, hence

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{on } U.$$

(2)⇒(1). Assume $P_y = Q_x$ on U. Let C be any positively oriented, piecewise C^1, simple closed curve in U bounding a region $D \subseteq U$. By Green's Theorem,

$$\oint_C P\,dx + Q\,dy = \iint_D \left(Q_x - P_y\right) dA.$$

Since $Q_x - P_y = 0$ on U, it follows that

$$\oint_C P\,dx + Q\,dy = 0$$

for every such curve C.

Path independence. Fix two piecewise C^1 curves C_1 and C_2 in U with the same endpoints A and B. Consider the closed curve $C = C_1 \cup (-C_2)$ obtained by traversing C_1 from A to B and then C_2 from B back to A. Because U is simply connected, this closed curve can be decomposed into finitely many simple closed curves each bounding a region contained in U. Since the integral around each such simple closed curve is 0, additivity of line integrals yields

$$\oint_C P\,dx + Q\,dy = 0.$$

Therefore,

$$\int_{C_1} P\,dx + Q\,dy = \int_{C_2} P\,dx + Q\,dy,$$

so the line integral depends only on endpoints (path independence).

Construction of a potential. Fix a base point $A_0 \in U$. Define $f : U \to \mathbb{R}$ by

$$f(x,y) = \int_{C_{A_0 \to (x,y)}} P\,dx + Q\,dy,$$

where $C_{A_0 \to (x,y)}$ is any piecewise C^1 curve in U from A_0 to (x,y). By path independence, f is well-defined.

Verification that $\nabla f = \mathbf{F}$. Let $(x,y) \in U$ and choose h small so that the horizontal segment from (x,y) to $(x+h,y)$ lies in U. Using the definition of f and path independence,

$$f(x+h,y) - f(x,y) = \int_x^{x+h} P(s,y)\,ds.$$

Divide by h and let $h \to 0$ to obtain

$$f_x(x,y) = P(x,y).$$

Similarly, using a vertical segment,

$$f(x,y+h) - f(x,y) = \int_y^{y+h} Q(x,t)\,dt,$$

so

$$f_y(x,y) = Q(x,y).$$

Hence $\nabla f = \langle f_x, f_y \rangle = \langle P, Q \rangle = \mathbf{F}$, and $\mathbf{F}$ is conservative on U. $\square$

3. Let $f(x,y,z) = xyz$. Compute ∇f and evaluate the line integral of ∇f over the path from $(1,0,0)$ to $(1,2,3)$.

Sol. Given $f(x,y,z) = xyz$, its gradient is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle yz,\ xz,\ xy \rangle.$$

To evaluate the line integral of ∇f over any path C from $(1,0,0)$ to $(1,2,3)$, use the Fundamental Theorem for Line Integrals:

$$\int_C \nabla f \cdot d\mathbf{r} = f(1,2,3) - f(1,0,0).$$

Compute the endpoint values:

$$f(1,2,3) = (1)(2)(3) = 6, \qquad f(1,0,0) = (1)(0)(0) = 0.$$

Hence,

$$\int_C \nabla f \cdot d\mathbf{r} = 6 - 0 = 6.$$

$\square$

4. Let $\mathbf{F} = \nabla f$ for $f(x,y) = x^2 + y^2$. Compute the line integral over a circular path from $(1,0)$ to $(0,1)$ and explain the result.

Sol. Given $f(x,y) = x^2 + y^2$, we have

$$\mathbf{F} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2x,\, 2y \rangle.$$

Since $\mathbf{F}$ is a gradient field, it is conservative. Therefore, by the Fundamental Theorem for Line Integrals, for any smooth path C from $(1,0)$ to $(0,1)$ (including a circular arc),

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0,1) - f(1,0).$$

Compute:

$$f(0,1) = 0^2 + 1^2 = 1, \qquad f(1,0) = 1^2 + 0^2 = 1.$$

Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 1 - 1 = 0.$$

Explanation. The line integral depends only on the endpoints because $\mathbf{F} = \nabla f$ is conservative. Here both endpoints lie on the same level curve of f (indeed, on the circle $x^2 + y^2 = 1$), so f has the same value at $(1,0)$ and $(0,1)$; thus the net change in potential is zero, and the work done by $\mathbf{F}$ along the circular path is 0. □

1.2 Green's Theorem

1. Use Green's Theorem to evaluate

$$\oint_C x\,dy - y\,dx$$

where C is the unit circle oriented counterclockwise.

Sol. Write the integral in Green's Theorem form:

$$\oint_C P\,dx + Q\,dy,$$

where here $P(x,y) = -y$ and $Q(x,y) = x$, since

$$x\,dy - y\,dx = (-y)\,dx + x\,dy.$$

By Green's Theorem (with C positively oriented),

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

where D is the unit disk. Compute:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x) = 1, \qquad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(-y) = -1.$$

Thus

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2.$$

Therefore,

$$\oint_C x\,dy - y\,dx = \iint_D 2\,dA = 2 \cdot \mathrm{Area}(D) = 2 \cdot \pi(1)^2 = 2\pi.$$

□

2. Let $\mathbf{F} = \langle y^2, 2xy \rangle$. Use Green's Theorem to evaluate the line integral around the boundary of the square $[0,1] \times [0,1]$.

Sol. Let $\mathbf{F} = \langle P, Q \rangle = \langle y^2,\, 2xy \rangle$, and let C be the positively oriented (counterclockwise) boundary of the square

$$D = [0,1] \times [0,1].$$

By Green's Theorem,

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y, \qquad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y^2) = 2y.$$

Hence,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y - 2y = 0.$$

Therefore,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_D 0\,dA = 0.$$

□

3. Evaluate

$$\oint_C (x+y)\mathrm{d}x + (x-y)\mathrm{d}y$$

where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$ oriented counterclockwise.

Sol. Write the line integral as $\oint_C P\,dx + Q\,dy$ with

$$P(x,y) = x+y, \qquad Q(x,y) = x-y.$$

Since C is positively oriented (counterclockwise), Green's Theorem gives

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA,$$

where D is the triangular region with vertices $(0,0)$, $(1,0)$, $(1,1)$.
Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x-y) = 1, \qquad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x+y) = 1.$$

Thus,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 1 = 0.$$

Therefore,

$$\oint_C (x+y)\,dx + (x-y)\,dy = \iint_D 0\,dA = 0.$$

□

4. Determine if

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$$

for $\mathbf{F} = \langle y, -x \rangle$ around a circle of radius r centered at the origin.

Sol. Let $\mathbf{F} = \langle P, Q \rangle = \langle y, -x \rangle$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \oint_C y\,dx - x\,dy,$$

where C is the circle of radius r centered at the origin, oriented counterclockwise. Using Green's Theorem,

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

where D is the disk of radius r. Compute

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(-x) = -1, \qquad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y) = 1.$$

Hence

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -1 - 1 = -2.$$

Therefore,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (-2)\,dA = -2\,\mathrm{Area}(D) = -2 \cdot \pi r^2 = -2\pi r^2.$$

So the integral is *not* zero (except in the degenerate case $r = 0$). For clockwise orientation, the value would be $+2\pi r^2$. □

1.3 Divergence Theorem

1. Let $\mathbf{F} = \langle x, y, z \rangle$. Use the Divergence Theorem to compute the flux across the surface of the unit sphere.

   Sol. content... $\square$

2. Let $\mathbf{F} = \langle x^2, y^2, z^2 \rangle$. Compute both the divergence and the surface integral over the unit cube $[0,1]^3$.

   Sol. content... $\square$

3. Use the Divergence Theorem to find the outward flux of $\mathbf{F} = \langle yz, xz, xy \rangle$ through the unit cube.

    Sol. content... $\square$

4. Let $\mathbf{F} = \langle x, -y, z \rangle$. Verify the Divergence Theorem on the upper hemisphere of radius 1 centered at the origin.

 Sol. content... $\square$

1.4 Stokes' Theorem

1. Let $\mathbf{F} = \langle -y, x, 0 \rangle$. Use Stokes' Theorem to compute the circulation around the boundary of the disk $x^2 + y^2 \leq 1$ in the xy-plane.

   Sol. content... $\square$

2. Let $\mathbf{F} = \langle z, 0, x \rangle$. Use Stokes' Theorem on the triangular surface with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$.

   Sol. content... $\square$

3. Compute both sides of Stokes' Theorem for $\mathbf{F} = \langle y, z, x \rangle$ on the surface $z = 0$ bounded by the unit circle.

   Sol. content... $\square$

4. Use Stokes' Theorem to show that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

if $\mathbf{F}$ is the gradient of some scalar field f.

Sol. content... □

2 Differential Forms

TBA

3 Winding Numbers and Complexification

TBA

