Riemann; Complex Analysis

- HW1 -

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We cover the following topics in this note.

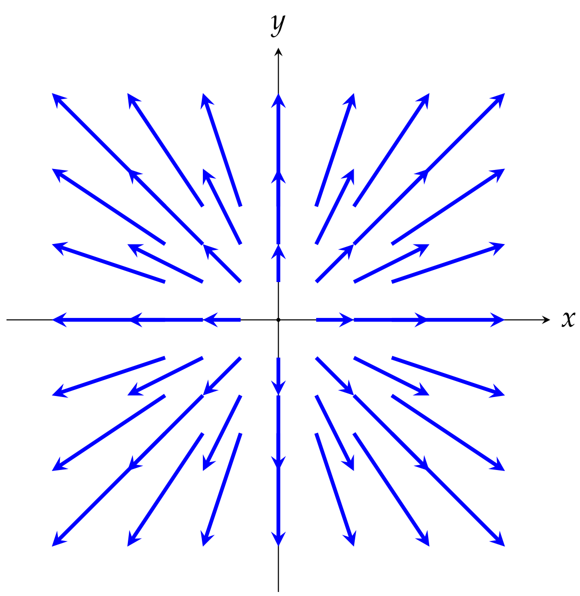
- Vector Fields
- Line Integrals for Vector Fields
- Surface Integrals for Vector Fields
- TBA

Scalar Function and Vector Fields

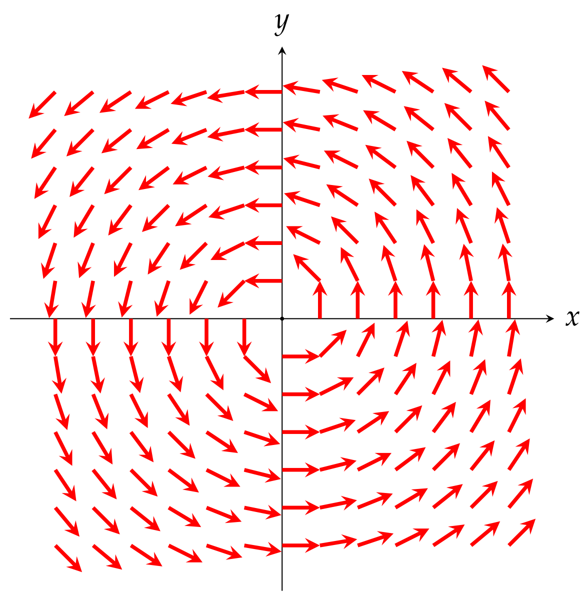
A **scalar function** on $\mathbb{R}^n$ is a real-valued function of an n-tuple; that is,

$$f : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n).$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $f(\mathbf{x}) \in \mathbb{R}$.



The radial field $\mathbf{F} = \langle x, y \rangle = x\mathbf{i} + y\mathbf{j}$



The spin field $\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)^{1/2}$

A **vector field** on $\mathbb{R}^n$ is a function

$$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n, \quad \mathbf{x} \mapsto \mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}), \ldots, F_n(\mathbf{x})),$$

where each component $F_i : \mathbb{R}^n \to \mathbb{R}$ is itself a scalar function.

Line Integral of Vector Fields

Definition (Line Integral of Scalar Function over Arc Length). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar function, and let C be a piecewise smooth curve in $\mathbb{R}^n$ given by a smooth parameterization:

$$\gamma : [a, b] \to \mathbb{R}^n, \quad t \mapsto \gamma(t) = (x_1(t), x_2(t), \ldots, x_n(t)).$$

Then the *arc length* of the curve C from $t = a$ to $t = b$ is defined by

$$L(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt} \right\| dt, \quad \text{where} \left\| \frac{d\gamma}{dt} \right\| = \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_1}{dt}\right)^2 + \cdots + \left(\frac{dx_1}{dt}\right)^2}.$$

The **line integral of the scalar function** f along the curve C with respect to arc length is defined by

$$\int_C f\, ds := \int_a^b f(\gamma(t)) \left\|\gamma'(t)\right\|\, dt.$$

Here, $ds = \|\gamma'(t)\|\, dt$ is the *infinitesimal arc length*.

Excellent — this is a key question for understanding how **vector calculus** connects with **differential forms** and notation. Let's walk through how to **rigorously deduce**:

$$\int_C F_1\, dx + F_2\, dy \quad \text{from} \quad \int_C \mathbf{F} \cdot d\mathbf{r}$$

for a vector field $\mathbf{F}(x, y) = (F_1(x, y), F_2(x, y))$.

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Step 1: Parametrize the Curve C

Let $\gamma : [a, b] \to \mathbb{R}^2$ be a smooth parameterization of the curve C, given by:

$$\gamma(t) = (x(t), y(t)), \quad t \in [a, b].$$

Then the **derivative of the curve** is:

$$\gamma'(t) = (x'(t), y'(t)),$$

and the **differential displacement** is:

$$d\mathbf{r} = \gamma'(t)\, dt = (dx, dy) = (x'(t)\, dt, y'(t)\, dt).$$

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Step 2: Write the Line Integral of $\mathbf{F}$

The **vector line integral** is defined as:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b \left[F_1(x(t), y(t))\, x'(t) + F_2(x(t), y(t))\, y'(t)\right] dt.$$

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Step 3: Recognize the Total Differentials
From calculus, we know:

$$dx = x'(t)\, dt, \quad dy = y'(t)\, dt.$$

Substitute into the integral:

$$\int_a^b F_1(x(t), y(t))\, dx + F_2(x(t), y(t))\, dy.$$

So we have:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b F_1(x(t), y(t))\, dx + F_2(x(t), y(t))\, dy = \int_C F_1\, dx + F_2\, dy.$$

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Final Deduction
Thus:

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C F_1\, dx + F_2\, dy}$$

where: - $\mathbf{F} = (F_1, F_2)$, - $d\mathbf{r} = (dx, dy)$, - The dot product $\mathbf{F} \cdot d\mathbf{r} = F_1\, dx + F_2\, dy$.

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Interpretation
- The vector line integral becomes an integral of a **1-form**:

$$\mathbf{F} \cdot d\mathbf{r} \quad \leftrightarrow \quad F_1\, dx + F_2\, dy.$$

- The dot product turns into a sum of **components times differentials**.

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Would you like to go one step further and express this in terms of pullbacks or show how it generalizes to $\mathbb{R}^3$?

Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{F} : U \to \mathbb{R}^n$ a continuous vector field. Suppose $C \subset U$ is a smooth curve parametrized by

$$\mathbf{r}\colon [a,b] \longrightarrow \mathbb{R}^n, \quad t \mapsto \mathbf{r}(t),$$

with nonzero velocity $\mathbf{r}'(t)$. Then the *line integral* of $\mathbf{F}$ along C is defined by

$$\boxed{\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\,dt = \int_a^b \sum_{i=1}^n F_i\big(\mathbf{r}(t)\big)\, x_i'(t)\,dt,}$$

where $\mathbf{r}(t) = (x_1(t), \dots, x_n(t))$ and $\mathbf{F} = (F_1, \dots, F_n)$.

This integral "accumulates" at each infinitesimal step dt the projection of $\mathbf{F}$ onto the tangent vector $\mathbf{r}'(t)$, yielding a single real number that captures the *circulation* or *work* of $\mathbf{F}$ along C.

Example. Take $n = 2$ and $\mathbf{F}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ on $U = \mathbb{R}^2 \setminus \{(0,0)\}$. Let C be the unit circle $x^2+y^2 = 1$, counterclockwise. Parametrize $\mathbf{r}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Then

$$\mathbf{r}'(t) = (-\sin t, \cos t), \qquad \mathbf{F}\big(\mathbf{r}(t)\big) = \big(-\sin t, \cos t\big),$$

so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \big(-\sin t, \cos t\big)\cdot(-\sin t, \cos t)\,dt = \int_0^{2\pi} \big(\sin^2 t + \cos^2 t\big)\,dt = 2\pi.$$

Thus the total circulation (or "work") of $\mathbf{F}$ around the unit circle is 2π.

Problem #1 (Line Integral around Unit Circle). Let $C \subset \mathbb{R}^2$ be the unit circle defined by

$$C : x^2 + y^2 = 1,$$

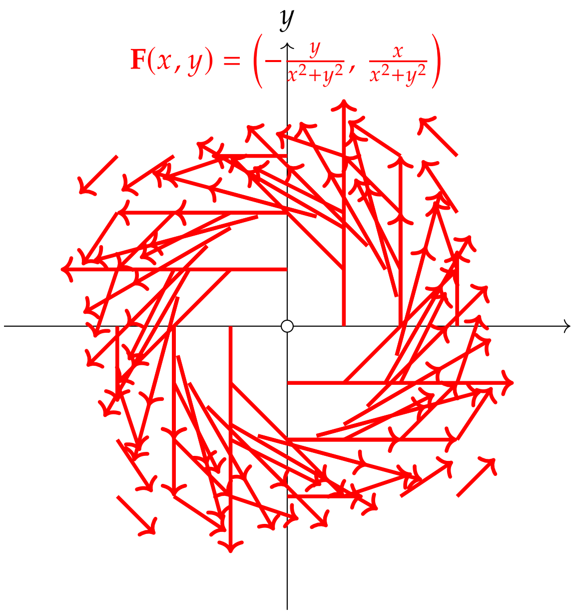
traversed in the *counterclockwise direction*. Let the vector field $\mathbf{F} : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ be defined by

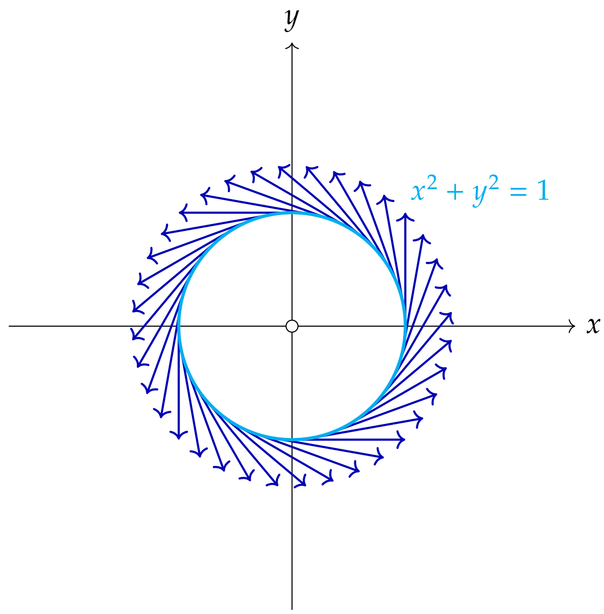
$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

Evaluate the *line integral* of $\mathbf{F}$ along C:

$$\oint_C \mathbf{F}\cdot d\mathbf{r}.$$

Sol.





Consider the vector field:

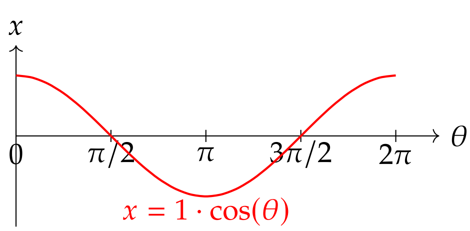
$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right),$$

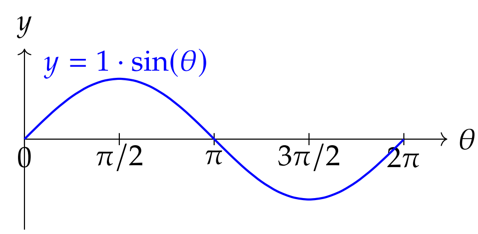
and the curve C is the unit circle $x^2+y^2=1$, traversed counterclockwise.

Step 1. (Parametrization) Define a function

$$\begin{array}{rcccl} \gamma & : & [0,2\pi] & \longrightarrow & \left\{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\right\} \\ & & \theta & \longmapsto & \gamma(\theta) = (\cos\theta, \sin\theta) \end{array}.$$

Here, $\frac{d\gamma}{d\theta} = (-\sin\theta, \cos\theta)$.





Step 2. (Evaluate $\mathbf{F}(\gamma(\theta))$ and the dot product) We have

$$\mathbf{F}(\gamma(\theta)) = \mathbf{F}(\cos\theta, \sin\theta) \overset{\sin^2\theta+\cos^2\theta=1}{=} \left(\frac{-\sin\theta}{1}, \frac{\cos\theta}{1}\right) = (-\sin\theta, \cos\theta).$$

and

$$\mathbf{F}(\gamma(\theta))\cdot\frac{d\gamma}{d\theta} = (-\sin\theta)(-\sin\theta) + (\cos\theta)(\cos\theta) = \sin^2\theta + \cos^2\theta = 1.$$

Step 3. (Integral)

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\gamma(\theta)) \cdot \frac{d\gamma}{d\theta}\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi.$$

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