Lecture Note: A 1-Form as Scalar Projection onto a Fixed Line

1 The Curve and Its Tangent Spaces

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a C^1-function and set

$$C \;=\; \{(x,y) \in \mathbb{R}^2 \mid y = f(x)\}.$$

Fix $a \in \mathbb{R}$ and let

$$p = (a, f(a)) \in C.$$

The inclusion $C \hookrightarrow \mathbb{R}^2$ may be written as

$$C \;\longrightarrow\; \mathbb{R}^2, \qquad p \;\mapsto\; \big(x(p),\, y(p)\big),$$

where $x, y\colon \mathbb{R}^2 \to \mathbb{R}$ are the standard coordinate functions, $x(x,y) = x,\ y(x,y) = y$.

The velocity of the parametrization $t \mapsto (t, f(t))$ at $t = a$ is

$$\Phi'(a) \;=\; \big(1,\ f'(a)\big) \;\in\; T_pC.$$

Thus

$$T_pC \;=\; \operatorname{span}\big\{(1, f'(a))\big\} \;\subset\; T_p\mathbb{R}^2 \cong \mathbb{R}^2.$$

Each tangent vector $v \in T_pC$ is uniquely

$$v \;=\; \tau\,(1, f'(a)), \qquad \tau \in \mathbb{R}.$$

We record the induced coordinate map on the fiber:

$$T_pC \;\longrightarrow\; \mathbb{R}^2, \qquad v \;\mapsto\; \begin{pmatrix} dx(v) \\ dy(v) \end{pmatrix}.$$

Since dx, dy are the dual basis on $T_p\mathbb{R}^2 \cong \mathbb{R}^2$, one has

$$dx\big(1, f'(a)\big) = 1, \quad dy\big(1, f'(a)\big) = f'(a).$$

2 A Fixed Line in the Plane

Choose a nonzero vector

$$w = (w_1, w_2) \in \mathbb{R}^2, \quad \|w\| \neq 0,$$

and let $L = \operatorname{span}\{w\} \subset \mathbb{R}^2$. Define the unit direction

$$\hat{w} = \frac{w}{\|w\|}, \quad \|\hat{w}\| = 1.$$

3 Definition of the 1-Form

Definition 1. *The* scalar–projection 1-form *onto the line L is*

$$\alpha : T\mathbb{R}^2 \longrightarrow \mathbb{R}, \qquad \alpha_p(v) = \langle \hat{w} , v \rangle,$$

for each $p \in \mathbb{R}^2$ and $v \in T_p\mathbb{R}^2$.

Since $\hat{w} = (\hat{w}_1, \hat{w}_2)$ and $v = (v^1, v^2)$, one has

$$\alpha_p(v) = \hat{w}_1 \, v^1 + \hat{w}_2 \, v^2 = \hat{w}_1 \, dx(v) + \hat{w}_2 \, dy(v).$$

Hence, in the usual notation for 1-forms,

$$\boxed{\alpha = \hat{w}_1 \, dx \; + \; \hat{w}_2 \, dy \; = \; \frac{w_1}{\sqrt{w_1^2 + w_2^2}} \, dx \; + \; \frac{w_2}{\sqrt{w_1^2 + w_2^2}} \, dy.}$$

4 Restriction to the Curve

Pulled back along the inclusion $i \colon C \hookrightarrow \mathbb{R}^2$, the 1-form α restricts to a well-defined 1-form on C:

$$i^*\alpha = \hat{w}_1 \, dx\big|_C \; + \; \hat{w}_2 \, dy\big|_C,$$

which on T_pC evaluates by

$$\big(i^*\alpha\big)_p\big(\tau(1, f'(a))\big) = \tau \, \big(\hat{w}_1 + \hat{w}_2 \, f'(a)\big).$$

Summary

- The map $p \mapsto (x(p), y(p))$ simply records the two ambient coordinates of each point in $C \subset \mathbb{R}^2$.
- The map $v \mapsto (dx(v), dy(v))$ records the two components of any tangent vector $v \in T_p\mathbb{R}^2$.
- The 1-form $\alpha = \hat{w}_1 \, dx + \hat{w}_2 \, dy$ is precisely the functional that takes each tangent vector and returns its scalar component in the fixed direction $\hat{w}$.