Advanced Calculus II

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We cover the following topics in this note.

- Convergence of Sequences
- Inequality Rule for Absolute Values
- Limit Theorem (Algebraic Property of Limit of Sequence)

Sequence

Definition. Let $X \subseteq \mathbb{R}$. A **sequence** is a function

$$f : \mathbb{N} \to X(\subseteq \mathbb{R}), \quad n \mapsto f(n) := a_n.$$

Instead of using function notation $f(n)$, the values of the sequence are denoted by $\{a_n\}_{n=1}^{\infty}$, where $a_n = f(n)$ is called n-th term of the sequence.

Remark. A sequence in $X \subseteq \mathbb{R}$ is a function

$$a : \mathbb{N} \to X, \quad n \mapsto a_n,$$

where $a_n \in X$ for all $n \in \mathbb{N}$. We sometimes write

$$\{a_n\}, \quad \{a_n\}_{n=1}^{\infty}, \quad \{a_n\}_{n\in\mathbb{N}}, \quad (a_n)_{n\in\mathbb{N}}, \quad \text{or} \quad \langle a_n \rangle_{n\in\mathbb{N}}.$$

Convergence of Sequence

Definition. A real sequence $\{a_n\}_{n=1}^{\infty}$ $(\subseteq \mathbb{R})$ is said to **converge** to $L \in \mathbb{R}$ if and only if

$$\forall \varepsilon > 0,\ \exists N_\epsilon \in \mathbb{N} \text{ such that } \left[n \geq N_\epsilon \implies |a_n - L| < \varepsilon \right].$$

Remark. The real number $L \in \mathbb{R}$ is called **the limit**[1]. When a sequence $\{a_n\}_{n=1}^{\infty}$ has the limit L, we will use the notation

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty.$$

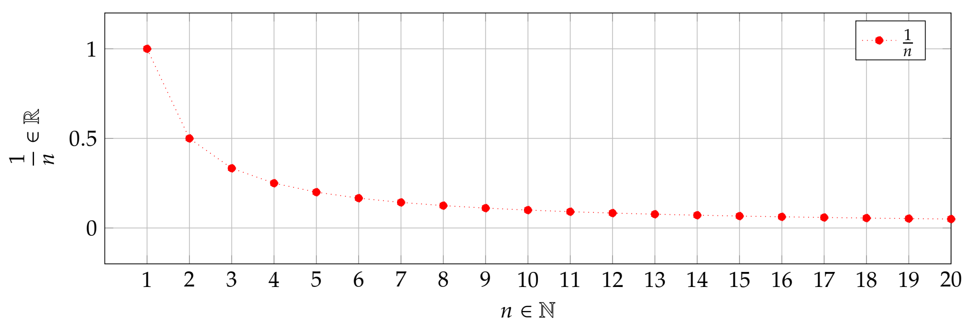
That is,

$$\lim_{n\to\infty} a_n = L \iff \forall \varepsilon > 0 : \exists N \in \mathbb{N} : \left[n \geq N \implies |a_n - L| < \varepsilon \right].$$

Note. If a sequence has a limit, we say that the sequence is **convergent**; if it has no limit, we say that the sequence is **divergent**.

Example. Consider the sequence defined by $a_n = 1/n$ for each $n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$



Proof. Let $\varepsilon > 0$. By the Archimedean property, we obtain

$$\exists N_\epsilon \in \mathbb{N} \quad \text{s.t.} \quad 1 < \varepsilon \cdot N_\epsilon, \text{ i.e., } \frac{1}{N_\epsilon} < \varepsilon.$$

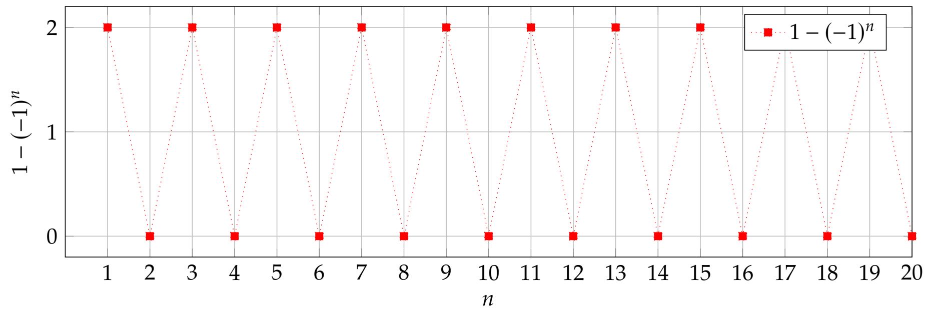
Assume that $n \geq N_\varepsilon$ then

$$|a_n - 0| = \left| \frac{1}{n} \right| = \frac{1}{n} \leq \frac{1}{N_\varepsilon} < \varepsilon.$$

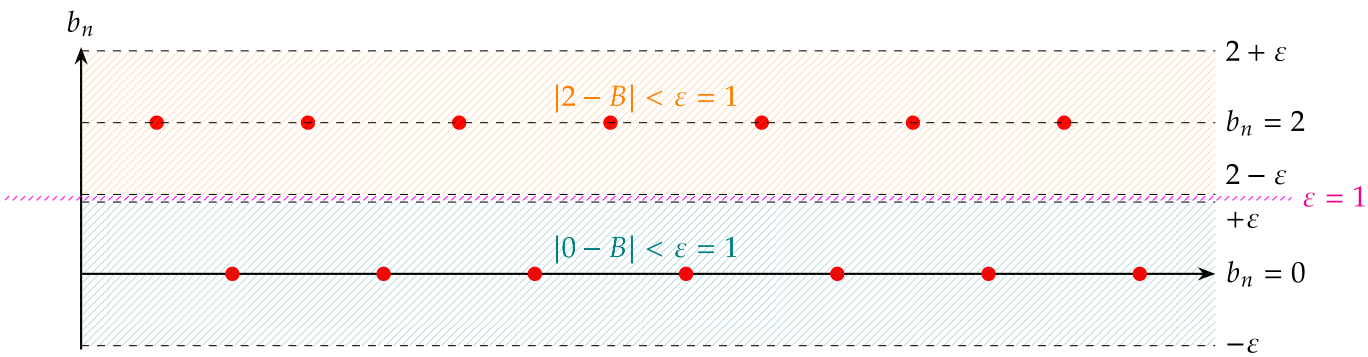
Hence $\lim_{n\to\infty} \frac{1}{n} = 0.$ □

[1] The limit of a sequence is unique. See **Theorem 4**

Example. Consider the sequence defined by $b_n = 1 - (-1)^n$ for all $n \in \mathbb{N}$. Prove that b_n does not converge.



Proof.



$\square$

Absolute Value in Reals

Definition. Let $x \in \mathbb{R}$. A **absolute value** $|x|$ of x is defined by

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Remark. For $x \in \mathbb{R}$,

$$|x| = \begin{cases} x & : x > 0 \\ 0 & : x = 0 \\ -x & : x < 0 \end{cases}$$

Proposition 1. *Let $x, y \in \mathbb{R}$.*

(1) $|x| = +\sqrt{x^2}$

(2) $|x| \geq 0$

(3) $|x| = 0 \Leftrightarrow x = 0$

(4) $|x| = |-x|$

(5) $|xy| = |x||y|$

(6) *(Fundamental Theorem of Absolute Values) For $c \geq 0$, we have*

$$|x| \leq c \iff -c \leq x \leq c$$

(7) $-|x| \leq x \leq |x|$

Proof. (1) If $(x \geq 0)$ then $|x| = x = \sqrt{x^2}$. Similarly if $x < 0$ then $|x| = -x = \sqrt{x^2}$.

(2) $|x| = \begin{cases} x \geq 0 & : x \geq 0 \\ -x > 0 & : x < 0 \end{cases} \geq 0.$

(3) $(\Leftarrow)$ If $x = 0$ then $|x| = x = 0$.

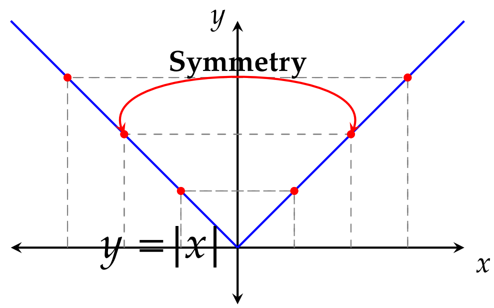
$(\Rightarrow)$ Let $|x| = 0$. Suppose that $x \neq 0$.

(i) $x > 0 \implies |x| = x > 0$ ↯

(ii) $x < 0 \implies |x| = -x > 0$ ↯

Thus x must be zero.

(4) $|-x| = \begin{cases} -x & : -x \geq 0 \text{ (i.e., } x \leq 0) \\ -(-x) = x & : -x < 0 \text{ (i.e., } x > 0) \end{cases} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} = |x|.$



(5) $|xy| = \begin{cases} xy = |x||y| & : x \geq 0, y \geq 0 \\ -xy = x(-y) = |x||y| & : x \geq 0, y < 0 \\ -xy = (-x)y = |x||y| & : x < 0, y \geq 0 \\ xy = (-x)(-y) = |x||y| & : x < 0, y < 0 \end{cases}$

(6) $(\Rightarrow)$ Let $|x| \leq c$.

(i) $x \geq 0 \implies x = |x| \leq c$, i.e., $-c \leq 0 \leq x \leq c$.

(ii) $x < 0 \implies -x = |x| \leq c$, i.e., $-c \leq x < 0 \leq c$.

Thus, $-c \leq x \leq c$.

$(\Leftarrow)$ Let $-c \leq x \leq c$.

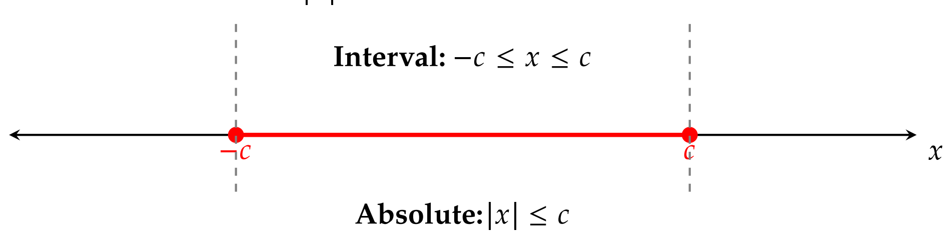
(i) $x \geq 0 \implies |x| = x \leq c$.

(ii) $x < 0 \implies |x| = -x \leq c$.

Thus, $|x| \leq c$.

Key equivalence:

$|x| \leq c \iff -c \leq x \leq c$



(7) Let $c = |x|$, where $c \geq 0$. By (6), thus, the result follows.

$\square$

Triangle Inequality

Proposition 2. *Let* $x, y \in \mathbb{R}$.

(1) $|x+y| \le |x| + |y|$

(2) $\big||x| - |y|\big| \le |x-y|$.

Proof. (1) By (7) of **Proposition 1**, we have

$$-|x| \le x \le |x|, \quad -|y| \le y \le |y|.$$

Then

$$\begin{array}{rcccccc} & -|x| & \le & x & \le & |x| \\ + & -|y| & \le & y & \le & |y| \\ \hline & -(|x|+|y|) & \le & x+y & \le & |x|+|y| \end{array}$$

Thus, we have $|x+y| \le |x| + |y|$.

(2)

(i) Note that

$$\begin{aligned} |x| &= |x-y+y| \\ &\le |x-y| + |y| \quad \text{by (1) of } \textbf{Proposition 2} \end{aligned}$$

Thus $|x| - |y| \le |x-y|$.

(ii) Note that

$$\begin{aligned} |y| &= |x-(x-y)| \\ &\le |x| + |-(x-y)| \quad \text{by (1) of } \textbf{Proposition 2} \\ &= |x| + |x-y| \quad \text{by (4) of } \textbf{Proposition 1} \end{aligned}$$

Therefore $-|x-y| \le |x| - |y|$.

By (i) and (ii), we know

$$-|x-y| \le |x| - |y| \le |x-y|, \quad \text{i.e.,} \quad \big||x| - |y|\big| \le |x-y|.$$

□

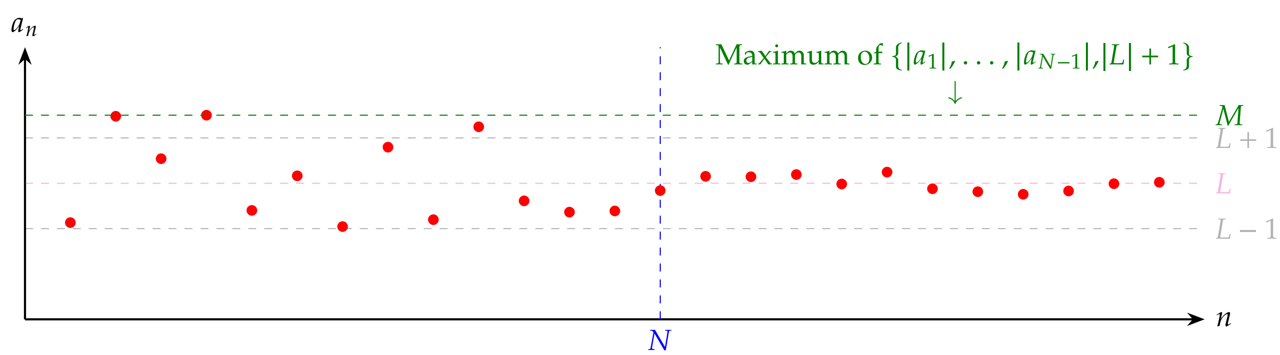
Boundedness of Sequence

Definition. Let $\{a_n\}_{n=1}^{\infty}\,(\subseteq \mathbb{R})$ is a sequence. $\{a_n\}$ is said to be **bounded** if

$$\exists M \in \mathbb{R} \text{ such that } \forall n \in \mathbb{N},\ |a_n| \leq M.$$

Proposition 3. *A convergent sequence is bounded.*

Proof.



□

Note. We have established that if the limit of a sequence a_n exists as n approaches infinity, then there exists a real number M such that $|a_n| \leq M$ for all n:

$$\exists A \in \mathbb{R} \text{ s.t. } A = \lim_{n\to\infty} a_n \implies \exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, |a_n| \leq M.$$

However, the converse is not necessarily true:

$$\exists A \in \mathbb{R} \text{ s.t. } A = \lim_{n\to\infty} a_n \;\not\Longleftarrow\; \exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, |a_n| \leq M.$$

To illustrate, consider the sequence $\{a_n\} = 1 - (-1)^n$. This sequence is bounded, yet it does not converge, serving as a counterexample. Furthermore, we note the following important theorems:

1. **Monotone Convergence Theorem**:
   (i) If a sequence $\{a_n\}$ is bounded above and monotone increasing, then it converges.
   (ii) If a sequence $\{a_n\}$ is bounded below and monotone decreasing, then it converges.
2. **Bolzano-Weierstrass Theorem**: Every bounded sequence of real numbers has a convergent subsequence. That is, if there exists a real number M such that $|a_n| < M$ for all n, then there exists a convergent subsequence $\{a_{n_k}\}$ of $\{a_n\}$.

Limit Theorem (Algebraic Property of Limit of Sequence)

Theorem. *Let* $\lim_{n\to\infty} a_n = \alpha$, $\lim_{n\to\infty} b_n = \beta$, *and* $k \in \mathbb{R}$. *Then*

(1) $\lim_{n\to\infty} ka_n = k\alpha = k \lim_{n\to\infty} a_n$.

(2) $\lim_{n\to\infty} a_n \pm b_n = \alpha \pm \beta = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$.

(3) $\lim_{n\to\infty} a_n b_n = \alpha\beta = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$.

(4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\alpha}{\beta} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$. (Here, $\beta \neq 0$ and $b_n \neq 0$)

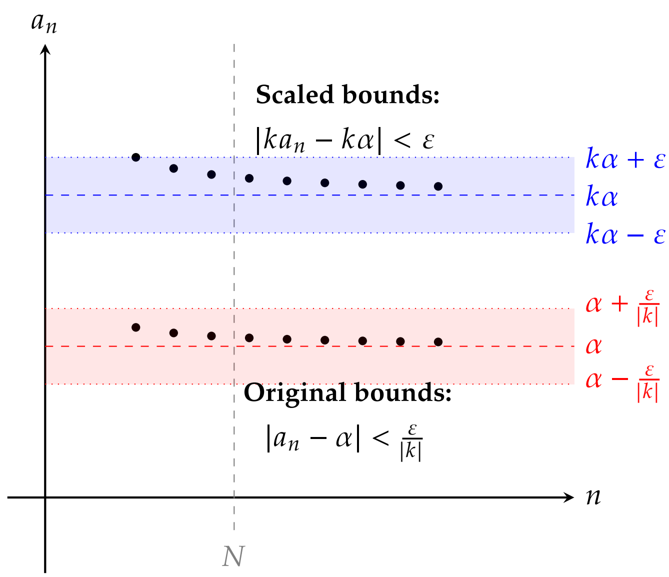
Proof. Let $\varepsilon > 0$.

(1) If $k = 0$, it is trivial. Let $k \neq 0$. Since $\lim_{n\to\infty} a_n = \alpha$, we know

$$\exists N \in \mathbb{N} \text{ such that } \left[n \geq N \implies |a_n - \alpha| < \frac{\varepsilon}{|k|} \right] \tag{*}$$

Thus, if $n \geq N$ then

$$\begin{aligned} |ka_n - k\alpha| &= |k(a_n - \alpha)| \\ &= |k||a_n - \alpha| \quad \because |xy| = |x||y| \\ &< |k| \cdot \frac{\varepsilon}{|k|} \quad \text{by (*)} \\ &= \varepsilon. \end{aligned}$$



(2)

(3)

(4)

□

Uniqueness of Limits

Theorem 4. *The limit of a sequence is unique.*

Proof.

□