

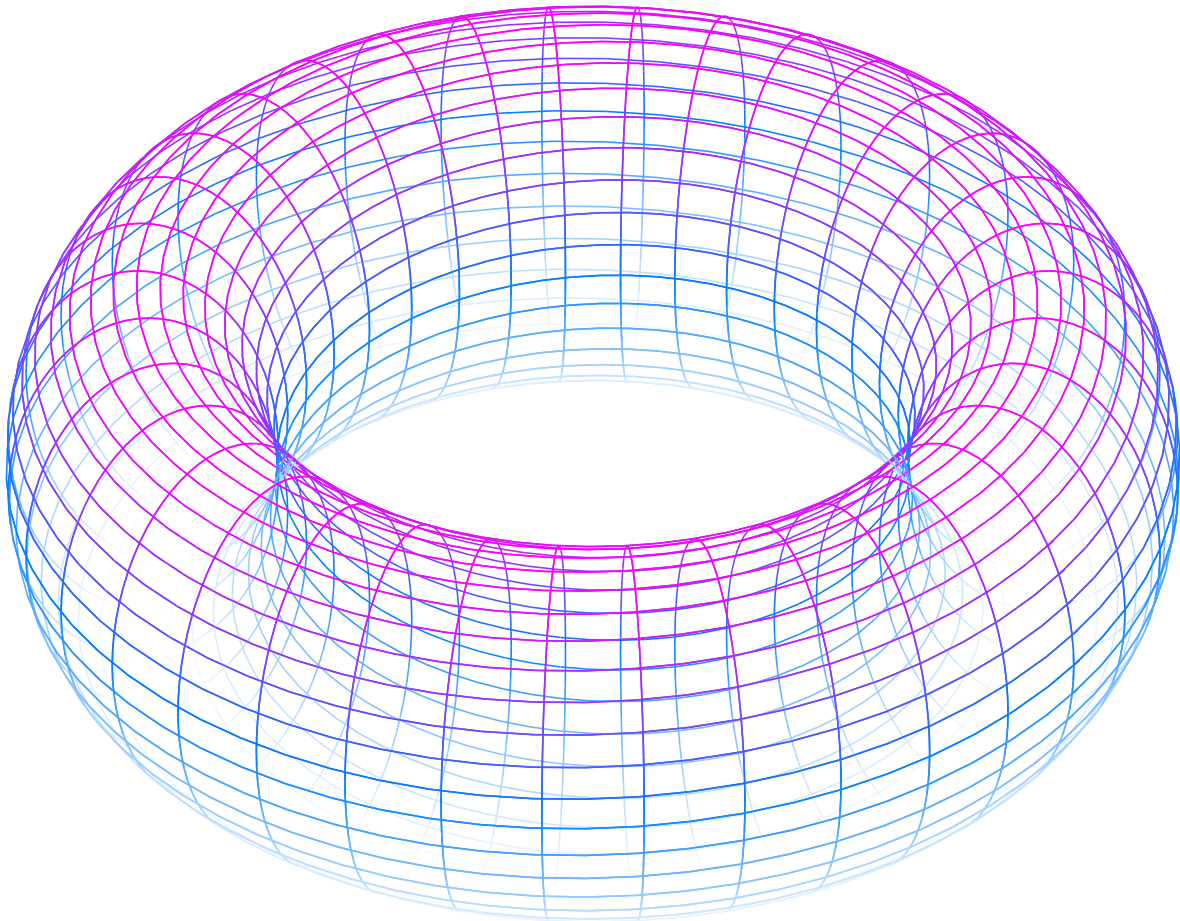
# Torus and Algebra

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We cover the following topics in this note.

- Unit Circle
  - Torus
  - Elliptic Curve
  - TBA
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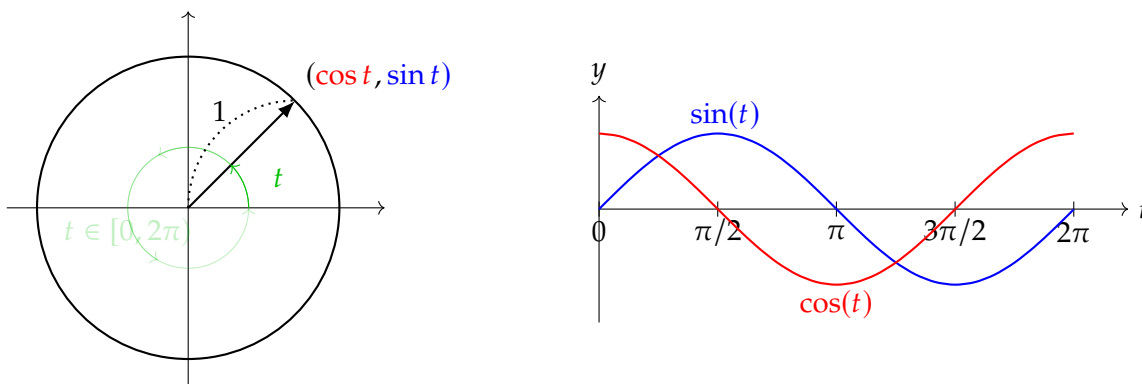


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## 1 Unit Circle

The set  $\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$  is called the **unit circle**.



The standard parametrization of  $\mathbb{S}^1$  is given by

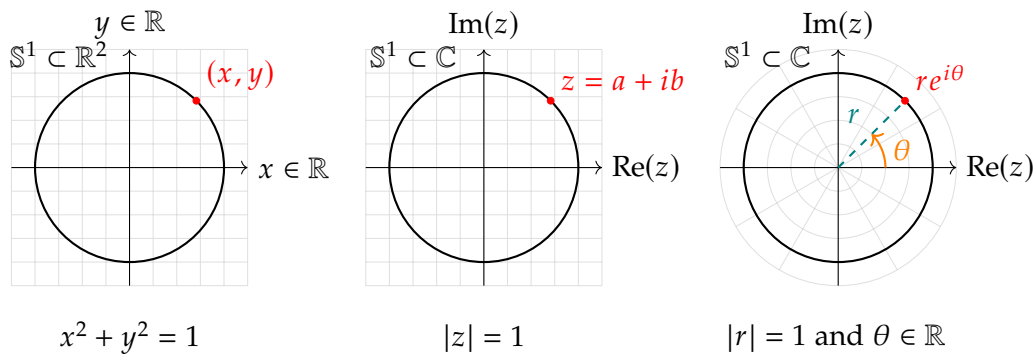
$$t \mapsto (\cos t, \sin t), \quad t \in [0, 2\pi),$$

which implies the *trigonometric identity*  $\cos^2 t + \sin^2 t = 1$ . The mapping

$$\begin{aligned} \varphi : [0, 2\pi) &\longrightarrow \mathbb{S}^1 \\ t &\longmapsto (\cos t, \sin t) \end{aligned}$$

provides a bijection between the half-open interval  $[0, 2\pi)$  and the unit circle  $\mathbb{S}^1$ .

Geometrically, it represents the set of points at a fixed distance 1 from the origin in  $\mathbb{R}^2$ , while algebraically it can be seen as a group under complex multiplication.



The unit circle can be described in several equivalent ways. In  $\mathbb{R}^2$ , it is given by:

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

In the complex plane, we write:

$$\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\} = \{re^{i\theta} : |r| = 1 \text{ and } \theta \in \mathbb{R}\}.$$

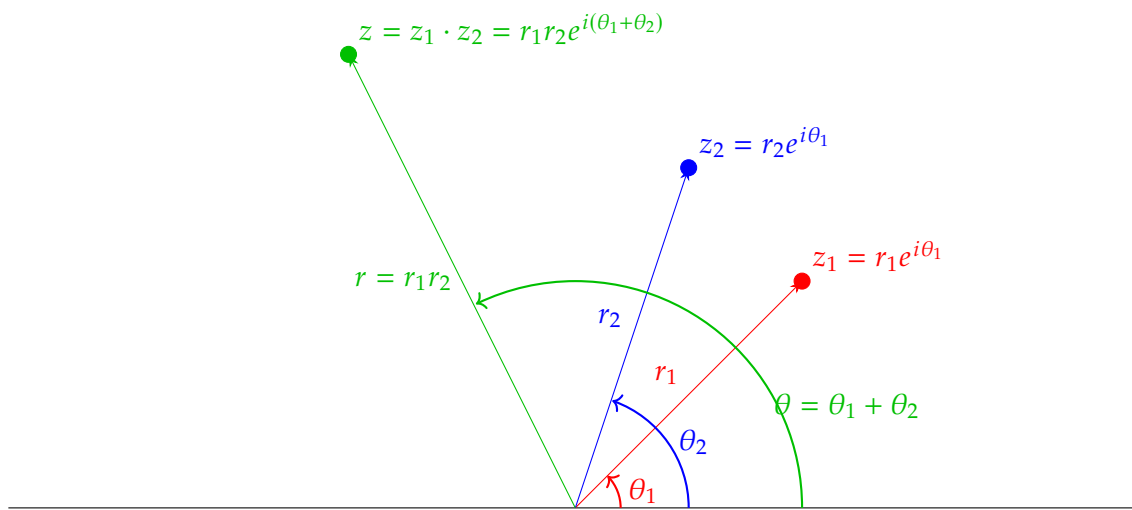
We show that **multiplication of complex number is equivalent to addition of angles**: let

$$z_1 = r_1 e^{i\theta_1} = r_1 (\cos \theta_1 + i \sin \theta_1) \in \mathbb{C} \text{ and}$$

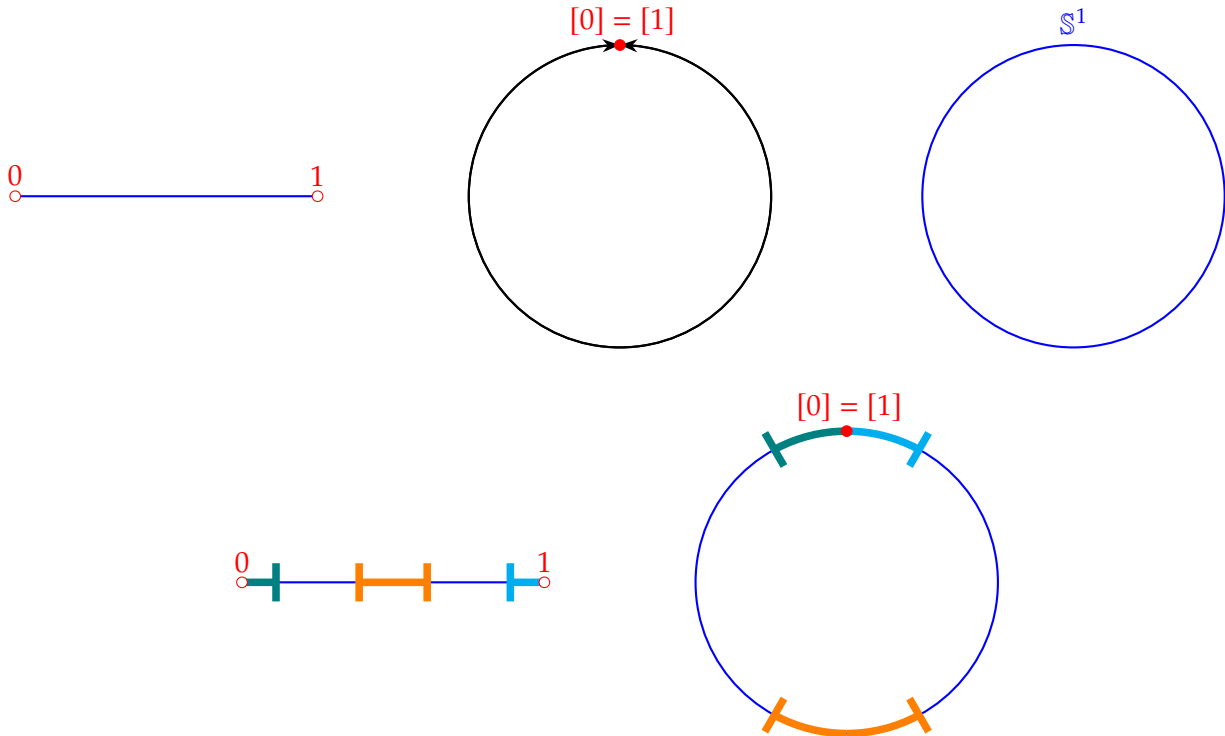
$$z_2 = r_2 e^{i\theta_2} = r_2 (\cos \theta_2 + i \sin \theta_2) \in \mathbb{C}.$$

Then

$$\begin{aligned} z_1 \cdot z_2 &= r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 (\cos \theta_1 + i \sin \theta_1) (\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i (\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2)] \\ &= r_1 r_2 [\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)] \\ &= r (\cos \theta + i \sin \theta) \text{ with } \begin{cases} r = r_1 r_2 \\ \theta = \theta_1 + \theta_2. \end{cases} \end{aligned}$$



## 1.1 Quotient Space



Let

$$\pi : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}, \quad x \mapsto x + \mathbb{Z},$$

be the canonical projection onto the quotient group, where the equivalence relation is given by

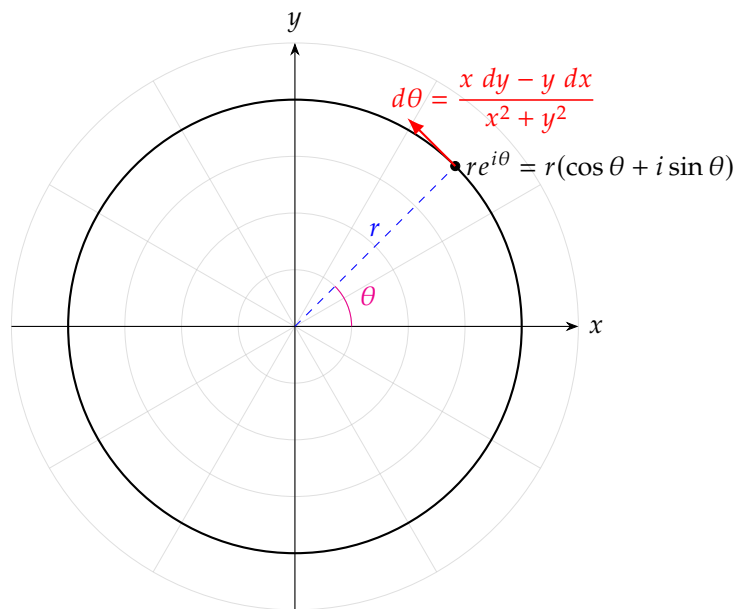
$$x \sim y \iff x - y \in \mathbb{Z}.$$

Denote by

$$[x] = \{ y \in \mathbb{R} \mid y \sim x \} = x + \mathbb{Z}$$

the equivalence class of  $x$ . Then  $\mathbb{R}/\mathbb{Z} = \{[x] : x \in \mathbb{R}\}$ .

## 1.2 Total differential of $d\theta$



$d\theta$  is globally defined, whereas  $\theta$  is local (mod  $2\pi$ ).

Recall that if we express a point in the plane in polar coordinates, then

$$x = r \cos \theta, \quad y = r \sin \theta.$$

One observe that the polar angle  $\theta$  may be expressed as

$$\tan \theta = \frac{y}{x} \implies \theta = \arctan\left(\frac{y}{x}\right), \quad x \neq 0.$$

Let  $\theta(x, y) = \arctan\left(\frac{y}{x}\right)$  with  $x \neq 0$ . We compute the total differential:

$$d\theta = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy.$$

Since

$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x} \left[ \arctan\left(\frac{y}{x}\right) \right] = \frac{1}{1 + (y/x)^2} \cdot \frac{\partial}{\partial x} \left[ \frac{y}{x} \right] = \frac{1}{(x^2 + y^2)/x^2} \cdot \left( -\frac{y}{x^2} \right) = \frac{x^2}{x^2 + y^2} \cdot \left( \frac{-y}{x^2} \right) = \frac{-y}{x^2 + y^2}$$

and

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + (y/x)^2} \cdot \frac{\partial}{\partial y} \left[ \frac{y}{x} \right] = \frac{x^2}{x^2 + y^2} \cdot \left( \frac{1}{x} \right) = \frac{x}{x^2 + y^2},$$

we have the total differential of  $\theta(x, y)$  is

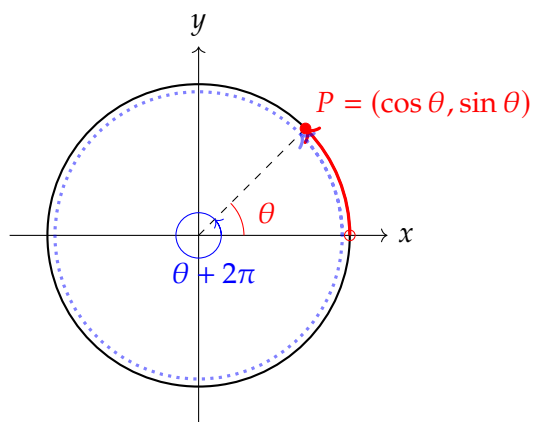
$$d\theta = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy.$$

This can be neatly written as

$$d\theta = \frac{x dy - y dx}{x^2 + y^2}.$$

### 1.3 Local Coordinate Function $\theta : U \rightarrow \mathbb{R}$

Consider the unit circle defined by  $\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ .



A natural idea is to assign to every point  $P = (x, y)$  its angle  $\theta$  so that

$$P = (x, y) = (\cos \theta, \sin \theta)$$

Both  $\theta$  and  $\theta + 2\pi$  give the same point on the circle, because

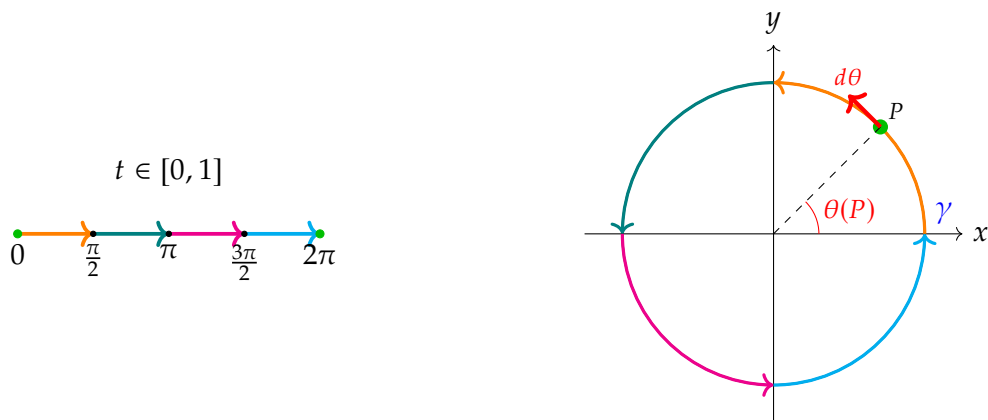
$$(\cos(\theta + 2\pi), \sin(\theta + 2\pi)) = (\cos \theta, \sin \theta)$$

In  $U := \mathbb{S}^1 \setminus \{(1, 0)\}$ , we can define an angular coordinate function

$$\begin{aligned} \theta &: U(\subseteq \mathbb{S}^1) \longrightarrow \mathbb{R} \\ P &\longmapsto \theta(P) \end{aligned}.$$

Here,  $\theta$  is only locally well defined.

## 1.4 Line Integral $\oint_{\gamma} d\theta$



**Parameterize the Unit Circle** We represent the unit circle as

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

A standard parameterization is given by the function

$$\gamma : [0, 2\pi] \rightarrow \mathbb{S}^1, \quad t \mapsto \gamma(t) = (\cos t, \sin t).$$

**Differential Form  $d\theta$**  Since every point on the circle satisfies  $x^2 + y^2 = 1$ , we have

$$d\theta = \frac{-y dx + x dy}{x^2 + y^2} = -y dx + x dy.$$

Since  $\gamma(t) = (\cos t, \sin t)$ ,

(i) The  $x$ -coordinate is  $x(t) = \cos t$ . Then  $dx = \frac{dx}{dt} dt = -\sin t dt$ .

(ii) The  $y$ -coordinate is  $y(t) = \sin t$ . Then  $dy = \frac{dy}{dt} dt = \cos t dt$ .

By (i) and (ii), we obtain

$$\begin{aligned} d\theta &= -y dx + x dy \\ &= -\sin t (dx) + \cos t (dy) \\ &= -\sin t (-\sin t dt) + \cos t (\cos t dt) \\ &= (\sin^2 t + \cos^2 t) dt \\ &= dt. \end{aligned}$$



### Perform the Line Integral

$$\oint_{\gamma} d\theta = \int_0^{2\pi} dt = \int_0^{2\pi} 1 dt = \left[ t \right]_{t=0}^{t=2\pi} = 2\pi - 0 = 2\pi.$$

**Interpretation** The value  $2\pi$  represents the total angular change as one goes around the circle *once*. In a general situation, if a closed curve  $\gamma$  on  $\mathbb{S}^1$  winds around the circle  $k$  times, the integral would yield

$$\oint_{\gamma} d\theta = 2\pi k, \quad \text{where } k \in \mathbb{Z}.$$

Here,  $k \in \mathbb{Z}$  is called the **winding number**.

**Key Point** Even though the local function  $\theta$  is defined only up to an additive constant of  $2\pi$ , the line integral of its differential  $d\theta$  gives a well-defined, global number measuring the rotation.

## 1.5 Winding Number

### Winding Number via the Angular 1-form

**Definition.** Let

$$\gamma : [0, 1] \rightarrow \mathbb{R}^2 \setminus \{0\}$$

be a piecewise  $C^1$  map with  $\gamma(0) = \gamma(1)$ ; that is,  $\gamma$  is a closed, piecewise smooth curve in  $\mathbb{R}^2 \setminus \{0\}$ . Define the angular 1-form  $\omega$  by

$$\omega := \frac{-y dx + x dy}{x^2 + y^2}.$$

Then the **winding number of  $\gamma$  about the origin** is defined by

$$\text{wind}(\gamma, 0) := \frac{1}{2\pi} \oint_{\gamma} d\omega.$$

It is a standard result that  $\oint_{\gamma} \omega \in 2\pi\mathbb{Z}$ , so that  $\text{wind}(\gamma, 0) \in \mathbb{Z}$ .

We can indeed define the winding number not just about the origin 0 but relative to any point  $p \in \mathbb{R}^2$  (provided that  $p$  is not on the image of the curve). In such a case, one writes the winding number as  $\text{wind}(\gamma, p)$  rather than  $\text{wind}(\gamma, 0)$ . The construction is analogous; one “centers” the angular coordinate at the point  $p$  instead of at 0.

Let  $p \in \mathbb{R}^2$  be a fixed point and let

$$\gamma : [0, 1] \rightarrow \mathbb{R}^2 \setminus \{p\}$$

be a piecewise  $C^1$  closed curve, that is,  $\gamma(0) = \gamma(1)$  and  $\gamma(t) \neq p$  for all  $t \in [0, 1]$ . Define the map

$$\tilde{\gamma}(t) = \frac{\gamma(t) - p}{\|\gamma(t) - p\|},$$

which is a well-defined map from  $[0, 1]$  to the unit circle

$$S^1 = \{z \in \mathbb{R}^2 : \|z\| = 1\}.$$

Since  $\tilde{\gamma}(0) = \tilde{\gamma}(1)$ , the map  $\tilde{\gamma}$  is a loop in  $S^1$ . The *winding number* of  $\gamma$  about  $p$  is defined as the degree of  $\tilde{\gamma}$ :

$$\text{wind}(\gamma, p) := \deg(\tilde{\gamma}) \in \mathbb{Z}.$$

Equivalently, if we let  $\omega$  denote the standard angular 1-form on  $\mathbb{R}^2 \setminus \{0\}$ ,

$$\omega = \frac{-y dx + x dy}{x^2 + y^2},$$

then by substituting  $x' = x - p_1$  and  $y' = y - p_2$  (where  $p = (p_1, p_2)$ ), one can define an angular coordinate about  $p$  and obtain

$$\text{wind}(\gamma, p) = \frac{1}{2\pi} \int_{\gamma} \omega_p,$$

where

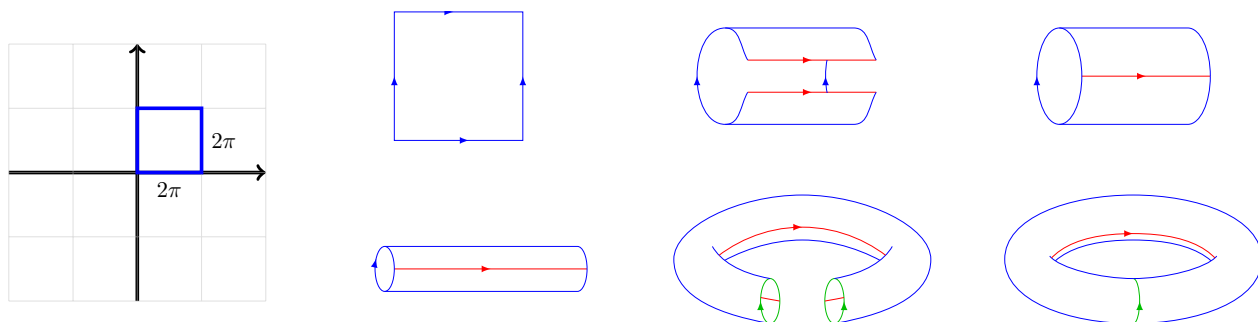
$$\omega_p = \frac{-(y - p_2) dx + (x - p_1) dy}{(x - p_1)^2 + (y - p_2)^2}.$$

It is a standard result that

$$\int_{\gamma} \omega_p \in 2\pi\mathbb{Z},$$

so that  $\text{wind}(\gamma, p) \in \mathbb{Z}$ .

## 2 Torus



**Note.** The torus rotation parameter is set to  $\theta_1$  and  $\theta_2$ , each  $\theta$  expresses

$$d\theta = \frac{-y dx + x dy}{x^2 + y^2}$$

globally (where  $\theta$  itself is only a local parameter), and  $\theta$  is not a gradient vector field; hence, the integral value over a circle is expressed by a nonzero integer multiple (of  $2\pi$ ).

These two (normalized) 1-forms constitute a basis of  $\mathbb{R}^2$  when one identifies the collection of closed differential forms modulo exact forms.

### 2.1 Quotient and Its Lattice Structure

Consider

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\} \quad \text{and} \quad \mathbb{Z}^2 = \{(m, n) : m, n \in \mathbb{Z}\}.$$

The quotient space  $\mathbb{T}^2 := \mathbb{R}^2 / \mathbb{Z}^2$  is defined by the equivalence relation on  $\mathbb{R}^2$

$$(x, y) \sim (x', y') \iff (x - x', y - y') \in \mathbb{Z}^2.$$

Consider

$$\mathbb{T}^2 \cong \mathbb{R}^2 / \mathbb{Z}^2,$$

where the subgroup  $\mathbb{Z}^2$  is regarded as a free (abelian)  $\mathbb{Z}$ -module of rank 2. That is, there exist vectors

$$v_1, v_2 \in \mathbb{R}^2$$

such that every element of  $\mathbb{Z}^2$  is uniquely expressible as

$$m v_1 + n v_2, \quad m, n \in \mathbb{Z}.$$

Consequently, the torus possesses two independent cycles which, topologically, are isomorphic to  $\mathbb{S}^1$ . In particular, one may write

$$\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1.$$

**[ $\mathbb{Z}^2$  as Free  $\mathbb{Z}$ -Module of Rank 2]** Consider the abelian group

$$\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z} := \{(a, b) : a, b \in \mathbb{Z}\}$$

with addition defined coordinate-wise:  $(a, b) + (c, d) = (a + c, b + d)$ . Since every element of  $\mathbb{Z}^2$  can be uniquely expressed in the form

$$(a, b) = a \cdot (1, 0) + b \cdot (0, 1),$$

$\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$  is a free  $\mathbb{Z}$ -module with basis  $\{(1, 0), (0, 1)\}$  (rank 2).

## 2.2 Local Angular Coordinates and Their Differential Forms

On each  $S^1$  factor of  $T^2$  one can introduce a local angular coordinate. More precisely, let

$$\theta_j : U_j \rightarrow \mathbb{R}, \quad j = 1, 2,$$

be smooth functions defined on open subsets  $U_j \subset S^1$  such that for a point  $P$  in the corresponding open set we have the (local) identification

$$P = (\cos \theta_j(P), \sin \theta_j(P)).$$

Due to the periodicity of the trigonometric functions,

$$e^{i\theta} = e^{i(\theta+2\pi k)} \quad \text{for all } k \in \mathbb{Z},$$

the function  $\theta_j$  is defined only locally; its values are determined modulo  $2\pi$ . Thus, while  $\theta_j$  serves as a local coordinate (or parameter) on  $S^1$ , it cannot be defined globally as a real-valued function on  $S^1$ .

However, the exterior derivative of  $\theta_j$ ,

$$d\theta_j,$$

is independent of the additive ambiguity. To be explicit, if  $\tilde{\theta}_j$  is any other local angular coordinate with

$$\tilde{\theta}_j = \theta_j + 2\pi k, \quad k \in \mathbb{Z},$$

then

$$d\tilde{\theta}_j = d(\theta_j + 2\pi k) = d\theta_j,$$

since the exterior derivative of a constant is zero. This shows that the 1-form  $d\theta_j$  is uniquely defined on each  $S^1$  factor, and thereby on the torus  $T^2$ .

In standard Cartesian coordinates on  $\mathbb{R}^2 \setminus \{0\}$ , a direct computation yields the coordinate expression

$$d\theta = \frac{-y dx + x dy}{x^2 + y^2}.$$

When restricted to the unit circle (or to any circle via rescaling), this expression computes the infinitesimal change in the angle.

### 2.3 Non-Exactness and the Integral of $d\theta$

If  $d\theta$  were exact—that is, if there existed a global smooth function  $f$  on  $S^1$  such that  $df = d\theta$ —then by the exactness property the integral over any closed curve would vanish:

$$\int_{\gamma} df = f(\gamma(1)) - f(\gamma(0)) = 0,$$

where  $\gamma$  is any closed loop. However, for a loop  $\gamma$  which represents a full rotation around the circle (or a nontrivial element in  $H_1(S^1; \mathbb{Z})$ ), one has

$$\int_{\gamma} d\theta = 2\pi k,$$

with  $k$  being the winding number (typically nonzero). Hence,  $d\theta$  is closed (since  $d(d\theta) = 0$  identically) but not exact. The non-exactness of  $d\theta$  reflects the nontrivial topology of  $S^1$  and, by extension, of  $T^2$ .

### 3 Elliptic Curve

An **elliptic curve**  $E$  over a field  $K$  (like  $\mathbb{C}$ ) is given in Weierstrass form by a cubic polynomials in two variables:

$$E : y^2 = x^3 + ax + b$$

where  $a, b \in K$  satisfy the nonvanishing discriminant condition  $\Delta = -16(4a^3 + 27b^2) \neq 0$ .

## A Calculus

### A.1 Differentiation of Arctangent

Compute  $\frac{d}{dx} \tan^{-1} u$  :

$$\begin{aligned} y &= \tan^{-1}(u), \\ \tan y &= u, \\ \frac{d}{dx} \tan y &= \frac{d}{dx} u, \\ \sec^2 y \frac{dy}{dx} &= \frac{du}{dx}, \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} \frac{du}{dx} \\ &= \frac{1}{1 + \tan^2 y} \frac{du}{dx} = \frac{1}{1 + u^2} \frac{du}{dx}. \end{aligned}$$

Thus,

$$\frac{dy}{dx} = \frac{1}{1 + u^2} \frac{du}{dx}, \quad \text{i.e.,} \quad \frac{d}{dx} \tan^{-1} u = \frac{1}{1 + u^2} \frac{du}{dx}.$$

### A.2 Generalizing Differentials

In single-variable calculus, we learn:

$$y = f(x) \implies \frac{dy}{dx} = f'(x) \implies dy = f'(x) dx.$$

If  $y$  depends on  $x$  alone, then a small change  $dx$  in  $x$  produces a corresponding small change  $dy$  in  $y$ . The amount of change in  $y$  is given by  $f'(x) dx$ .

**Intuitive Idea.** The derivative  $f'(x)$  tells us how fast  $y$  changes when  $x$  changes, and  $dx$  tells you how much  $x$  changed. We multiply the two to get the approximate change in  $y$ .

**Extending to Two Variables.** Now suppose  $x$  depends on *two* quantities, say  $r$  and  $\theta$ . We write:

$$x = x(r, \theta).$$

We want to figure out what happens if *both*  $r$  and  $\theta$  change a little bit. How does  $x$  change?

1. (Think of  $r$  changing while  $\theta$  is frozen.) If we imagine  $\theta$  held fixed, then  $x$  is effectively a one-variable function of  $r$ . So a small change in  $r$  (call it  $dr$ ) would change  $x$  by

$$(\text{rate of change w.r.t. } r) \times dr = \frac{\partial x}{\partial r} dr.$$

The partial derivative of  $x$  with respect to  $r$ ,  $\frac{\partial x}{\partial r}$ , tells us how fast  $x$  changes if only  $r$  changes and  $\theta$  stays fixed.

2. (Think of  $\theta$  changing while  $r$  is frozen.) Similarly, if  $r$  is held fixed, then  $x$  is effectively a one-variable function of  $\theta$ . So a small change in  $\theta$  (call it  $d\theta$ ) would change  $x$  by

$$(\text{rate of change w.r.t. } \theta) \times d\theta = \frac{\partial x}{\partial \theta} d\theta.$$

3. (Add the two contributions together.) If *both*  $r$  and  $\theta$  change at the same time, then the *total* change in  $x$ , which we call  $dx$ , is the sum of the two partial changes:

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta.$$