# Linear Algebra I

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We cover the following topics in this note.

- Linear Combination, Spanning Set
- Linearly Independent and Dependent
- (Hamel) Basis
- Partial Order; POSET
- Total Order (Linear Order); TOSET
- Maximal, Minimal, Hasse Diagram
- Chain, Zorn's Lemma
- Hamel Basis Theorem (Existence of Basis)
- Invariance of Basis Cardinality; Dimension of Vector Space

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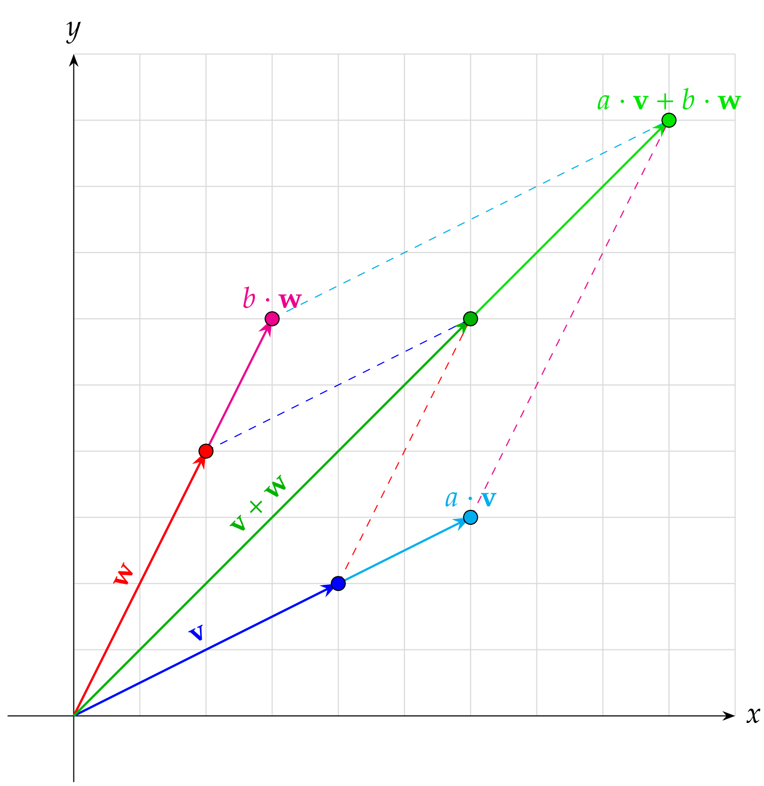


**Vector Space**

**Definition.** Let $F$ be a field. A **vector space** over $F$ (or a $F$-vector space) is a structure $(V, +, \cdot)$ satisfying the following axioms:

(i) $(V, +)$ is an abelian group with additive identity $\mathbf{0} \in V$.

(ii) Define *scalar multiplication* as the function $\cdot : F \times V \to V, \quad (a, \mathbf{v}) \mapsto a \cdot \mathbf{v}$.

(iii) (Compatibility) For all $a, b \in F$ and $\mathbf{v}, \mathbf{w} \in V$,

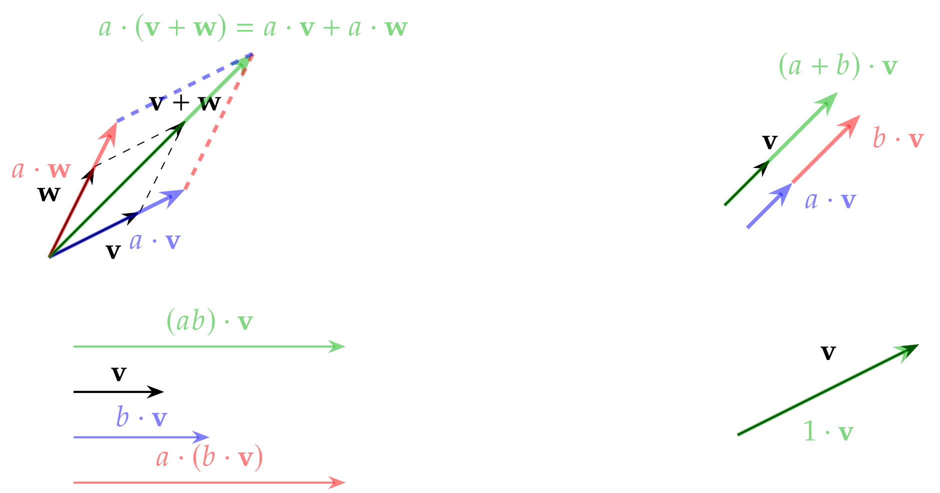
(a) $a \cdot (\mathbf{v} + \mathbf{w}) = a \cdot \mathbf{v} + a \cdot \mathbf{w}$. (Distributivity over vector addition)

(b) $(a + b) \cdot \mathbf{v} = a \cdot \mathbf{v} + b \cdot \mathbf{v}$. (Distributivity over field addition)

(c) $a \cdot (b \cdot \mathbf{v}) = (ab) \cdot \mathbf{v}$. (Associativity of scalar multiplication)

(d) $1_F \cdot \mathbf{v} = \mathbf{v}$. (Identity of scalar multiplication)

(e) $0_F \cdot \mathbf{v} = \mathbf{0}$.




**Remark.** Consider a vector space $V$ over a field $F$. Let $\mathbf{v} \in V$. Since $0_F = 0_F + 0_F$ (over $F$), we have

$$0_F \cdot \mathbf{v} = (0_F + 0_F) \cdot \mathbf{v} \overset{\text{(iii)-(b)}}{=} 0_F \cdot \mathbf{v} + 0_F \cdot \mathbf{v}.$$

Then

$$\begin{aligned} 0_F \cdot \mathbf{v} + (-0_F \cdot \mathbf{v}) &= 0_F \cdot \mathbf{v} + 0_F \cdot \mathbf{v} + (-0_F \cdot \mathbf{v}), \\ \mathbf{0} &= 0_F \cdot \mathbf{v} + \mathbf{0}, \\ \mathbf{0} &= 0_F \cdot \mathbf{v}. \end{aligned}$$

**Linear Combination and Spanning Set**

**Definition.** Let $V$ be a vector space over a field $F$, and let $S$ be a subset of $V$

(1) A vector $\mathbf{v} \in V$ is called a **linear combination** of elements of $S$ if there exists finite number of vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \in S$ and scalars $a_1, a_2, \ldots, a_n \in F$ such that

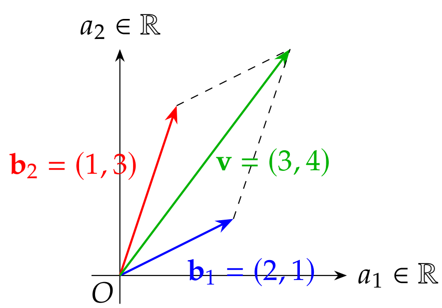
$$\mathbf{v} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n = \sum_{i=1}^{n} a_i\mathbf{b}_i.$$

(2) The **subspace spanned by** $S$ (or **spanning set** $S$), denoted by $\mathrm{span}(S)$, is the set of all finite linear combinations of elements of $S$:

$$\begin{aligned}\mathrm{span}(S) &= \left\{a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n \mid a_i \in F, \mathbf{b}_i \in S \text{ for all } i = 1, 2, \ldots, n\right\}\\ &= \left\{\sum_{i=1}^{n} a_i\mathbf{b}_i \;\middle|\; a_i \in F, \mathbf{b}_i \in S \text{ for all } i = 1, 2, \ldots, n\right\}\end{aligned}$$

**Example.** Consider the vector space $\mathbb{R}^2$ and the subset

$$S = \{\mathbf{b}_1, \mathbf{b}_2\} \quad \text{with} \quad \mathbf{b}_1 = (2, 1) \text{ and } \mathbf{b}_2 = (1, 3).$$




- A vector $\mathbf{v} = (3, 4) \in \mathbb{R}^2$ is a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$ since

$$\mathbf{v} = (3, 4) = (2 \cdot 1 + 1, 1 + 3 \cdot 1) = 1 \cdot (2, 1) + 1 \cdot (1, 3), \quad \text{i.e.,} \quad \mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

- Since $\mathbf{b}_1$ and $\mathbf{b}_2$ are not colinear (they are lineary independent), every vector in $\mathbb{R}^2$ can be expressed in the form $(2a_1 + a_2, a_1 + 3a_2)$ for some $a_1, a_2 \in \mathbb{R}$. Hence

$$\mathrm{span}(S) = \mathbb{R}^2.$$

**Linearly Independent and Dependent**

**Definition.** Let $V$ be a vector space over a field $F$ and let $S \subseteq V$.

(1) The set $S$ said to be **linearly independent** if, for any finite collection of distinct vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \in S$ and any scalars $a_1, a_2, \ldots, a_n \in F$,

$$a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n = \mathbf{0} \implies a_1 = a_2 = \cdots = a_n = 0.$$

(2) The set $S$ is said to be **linearly dependent** (i.e., not linearly independent) if there exists finitely many distinct vectors $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n \in S$ and scalars $a_1, a_2, \ldots, a_n \in F$, not all zeros, such that

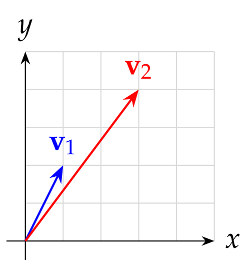
$$a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n = \mathbf{0}.$$

**Remark.** In (2), suppose that $a_1 \neq 0$, Then

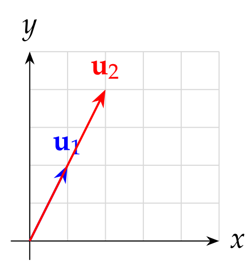
$$a_1\mathbf{b}_1 = -a_2\mathbf{b}_2 - \cdots - a_n\mathbf{b}_n \iff \mathbf{b}_1 = -a_1^{-1}(a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n).$$

That is, a set $S$ is linearly dependent if at least one vector in $S$ can be expressed as a linear combination of the others.

**Example.**




Linearly Independent Vectors




Linearly Dependent Vectors (Collinear)

- The vectors $\mathbf{v}_1 = (1, 2)$ and $\mathbf{v}_2 = (3, 4)$ are linearly independent because the only solution to

$$a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{0}$$

  is $a = 0$ and $b = 0$.

- The vectors $\mathbf{u}_1 = (1, 2)$ and $\mathbf{u}_2 = (2, 4)$ are linearly dependent because $\mathbf{u}_2$ is a multiple of $\mathbf{u}_1$; nontrivial solutions exist for

$$a\mathbf{u}_1 + b\mathbf{u}_2 = \mathbf{0}.$$

**Remark.** In any vector space $V$, we can always find a subset of $S$ such that

$$\operatorname{span}(S) = V.$$

For instance, taking $S = V$ gives $\operatorname{span}(S) = V$. Since $S = V$, each vector $\mathbf{v} \in V$ can be expressed as a trivial linear combination $\mathbf{v} = 1 \cdot \mathbf{v}$. Thus, there exists a subset $S \subseteq V$ such that $\operatorname{span}(S) = V$.

**Remark.**

- A singleton set $\mathcal{B} = \{\mathbf{b}\}$ is linearly independent since $k\mathbf{b} = 0 \implies k = 0$ for any $k \in F$.
- The empty set $\varnothing$ is linearly independent; this holds vacuously.

**⋆ ~~(Hamel)~~ Basis ⋆**

**Definition.** Let $V$ be a vector space over a field $F$. A subset $\mathcal{B} \subseteq V$ is called a **~~(Hamel)~~** **basis** for $V$ if it satisfies the following two conditions:

(i) (*Linearly Independent*) The set $\mathcal{B}$ is linearly independent; that is, for any *finite* collection of distinct elements $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \in \mathcal{B}$ and scalars $a_1, a_2, \ldots, a_n \in F$,

$$a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n = 0 \implies a_1 = a_2 = \cdots = a_n = 0.$$

(ii) (*Spanning Property*) The set $\mathcal{B}$ spans $V$ ($\operatorname{span}(\mathcal{B}) = V$); that is, every vector $\mathbf{v} \in V$, there exist a positive integer $n \in \mathbb{Z}^+$, scalars $a_1, a_2, \ldots, a_n \in F$, and distinct elements $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \in \mathcal{B}$ such that

$$\mathbf{v} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n,$$

**Remark** (Schauder Basis). Let $X$ be a Banach space (or more generally, a complete normed vector space) over the field $F$. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ is called a **Schauder basis** for $X$ it if satisfies the following condition:
For every vector $x \in X$, there exits a unique sequence of scalars $\{a_n\}_{n=1}^{\infty} \subseteq F$ such that

$$x = \sum_{i=1}^{\infty}(a_n \cdot x_n),$$

where the series converges in the norm topology of $X$, i.e., $\lim_{N\to\infty}\left\| x - \sum_{n=1}^{N}(a_n \cdot x_n)\right\| = 0.$

**Remark.** A Hamel basis is unique in the sense that every vector in $V$ has a unique representation as a finite linear combination of the elements of $\mathcal{B}$.

**Partial Order**

**Definition.** Let $S$ be a set. A binary relation $\preceq$ on $S$ (i.e., $\preceq \subseteq S \times S$) is called a **partial order** if it satisfies the following three axioms for all $a, b, c \in X$,

(i) (Reflexivity) $a \preceq a$;

(ii) (Anti-symmetry) $a \preceq b$ and $b \preceq a \implies a = b$;

(iii) (Transitivity) $a \preceq b$ and $b \preceq c \implies a \preceq c$.

**Note.** A **partially ordered set (POSET)** is an $(S, \preceq)$, where $S$ is a set and $\preceq$ is a partial order on $S$.

**Example** (Poset of the Power Set with Set Inclusion)**.** Let $S$ be any set. Consider the power set of $S$:

$$2^S = \{A : A \subseteq S\} \quad \text{with} \quad \text{binary operation} \subseteq \text{on } 2^S.$$

We claim that $(2^S, \subseteq)$ is partially ordered set: for any $A, B, C \in 2^S$,

(i) Reflexivity: $A \subseteq A$;

(ii) Anti-symmetry: $A \subseteq B$ and $B \subseteq A \implies A = B$;

(iii) Transitivity: $A \subseteq B$ and $B \subseteq C \implies A \subseteq C$.

Hence, $(2^S, \subseteq)$ forms a poset.

**Total Order (Linear Order)**

**Definition.** Let $(S, \preceq)$ be a poset; that is, $\preceq$ is a partial order on $S$. We say that $\preceq$ is a **total order** (or **linear order**) on $S$ if it satisfies the *comparability condition*: for each $a, b \in S$, either

$$a \preceq b \quad \text{or} \quad b \preceq a.$$

**Note.** A **totally ordered set (TOSET)** is a poset $(S, \preceq)$ in which the relation $\preceq$ is a total order. In other words, $(S, \preceq)$ is totally ordered if every pair of elements in $S$ is comparable.

**Example.** Consider all binary string of length 3:

$$\{000, 001, 010, 011, 100, 101, 110, 111\}.$$

They are ordered as follows:

$$000 \longrightarrow 001 \longrightarrow 010 \longrightarrow 011 \longrightarrow 100 \longrightarrow 101 \longrightarrow 110 \longrightarrow 111$$

**Maximal and Minimal**

**Definition.** Let $(P, \preceq)$ be a poset.

(1) An element $m \in P$ is said to be **maximal** in $P$ if

$$\forall a \in P, \quad (m \preceq a) \implies (m = a).$$

In other words, there exits no element in $P$ that is strictly greater than $m$.

(2) An element $m \in P$ is said to be **minimal** in $P$ if

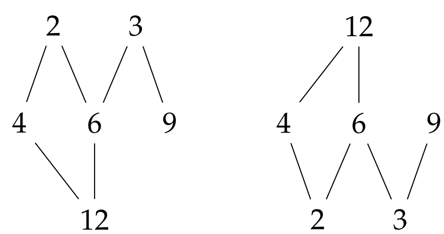
$$\forall a \in P, \quad (a \preceq m) \implies (a = m).$$

That is, there is no element in $P$ that is strictly less than $m$.

**Example.** Consider the set

$$S = \{2, 3, 4, 6, 9, 12\} \subseteq \mathbb{N}$$

with the partial order defined by *divisibility* (i.e., $x \preceq y \iff x \mid y$). See the Hasse diagram:

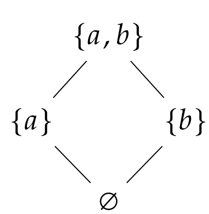



In this example, the minimal elements here are: $\{2, 3\}$.

**Example.** Consider the power set of $\{a, b\}$ with the usual subset relation $\subseteq$. The poset is

$$\{\varnothing, \{a\}, \{b\}, \{a, b\}\},$$

partially ordered by "is a subset of."




- The *minimal element* here is $\varnothing$ (there's nothing strictly smaller).
- The *maximal element* here is $\{a, b\}$ (there's nothing strictly bigger).

## Chain

**Definition.** Let $(P, \preceq)$ be a poset. A subset $C \subseteq P$ is called a **chain** if

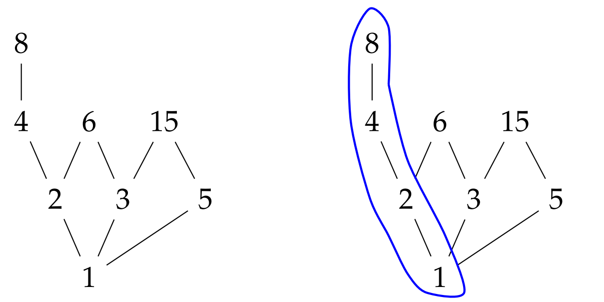
$$\forall a, b \in C, \quad \text{either } a \preceq b \text{ or } b \preceq a.$$

In other words, a chain in a poset is a subset in which every two elements are comparable (i.e.the subset is totally ordered).

**Example.** Consider a poset

$$P = \{1, 2, 3, 4, 5, 6, 8, 15\} \subseteq \mathbb{N}$$

with the partial order defined by divisibility. See the Hasse diagram:



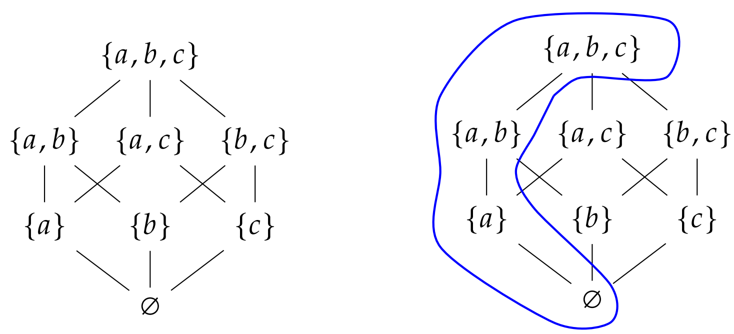

Here, $C = \{1, 2, 4, 8, 16\}$ is a *chain* under divisibility.

**Example.** Let $S = \{a, b, c\}$. Consider all the subsets of $S$ under the subset relation $\subseteq$. The entire power set of $S$ is

$$2^S = \left\{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\right\}.$$

This set $2^S$ (the power set) is partially ordered by $\subseteq$: for any $A, B \in 2^S$,

$$A \preceq B \iff A \subseteq B.$$




Here, $C = \left\{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}\right\}$ is a *chain* in $2^S$.

**Zorn's Lemma**

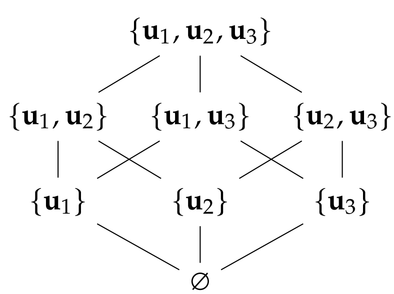
**Axiom.** Let $(P, \preceq)$ be a partially ordered set (poset) with property that every chain $C \subseteq P$ has an upper bound in $P$; that is, for every chain $C \subseteq P$,

$$\exists u \in P \quad \text{such that} \quad \forall c \in C, \quad c \preceq u.$$

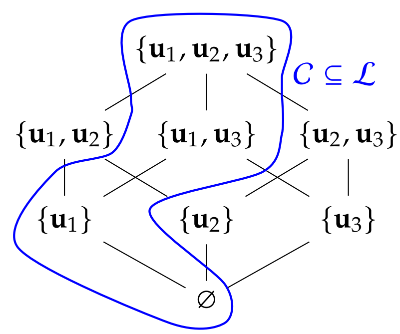
Then $P$ contains at least one maximal element; that is,

$$\exists m \in P \quad \text{such that} \quad \forall a \in P, \quad (m \preceq a) \implies (m = a).$$

**Observation** (Existence of Basis). Let $\mathcal{L} := \{S \subseteq \mathbb{R}^3 : S \text{ is linearly independent}\}$.

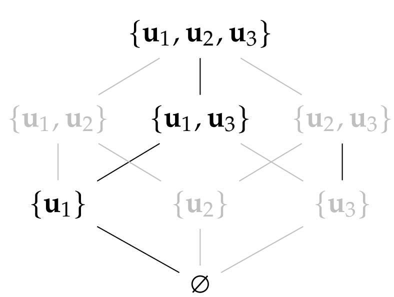



Hasse Diagram for a poset $(\mathcal{L}, \subseteq)$ in $\mathbb{R}^3$

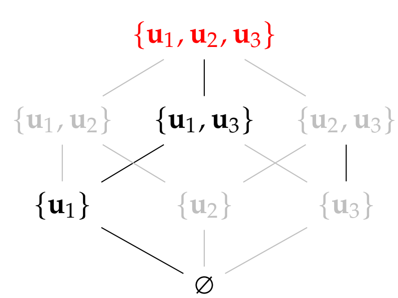



Any chain $C$

$U = \varnothing \cup \{\mathbf{u}_1\} \cup \{\mathbf{u}_1, \mathbf{u}_3\} \cup \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$




Upper Bound $U = \bigcup_{S \in C} S$




Maximal element $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

**⋆ Hamel Basis Theorem ⋆**

**Theorem.** *Every vector space $V$ over a field $F$ has a basis.*

*Proof.*

**Key Idea**: "By considering all linearly independent subsets of $V$ and partially ordering them by inclusion, we use Zorn's Lemma to guarantee a maximal linearly independent set exists."

Step 1 **Definition of Poset.**

Define the set

$$\mathcal{L} := \{S \subseteq V : S \text{ is linearly independent}\}.$$

with the partial order $\preceq$ on $\mathcal{L}$ by set inclusion:

$$\forall S, T \in \mathcal{L}, \quad S \preceq T \iff S \subseteq T.$$

Since $\varnothing \in \mathcal{L}$, we have $\mathcal{L} \neq \varnothing$. Thus, $(\mathcal{L}, \subseteq)$ forms a poset.

Step 2 **Chains and Upper Bounds.**

Let $C \subseteq \mathcal{L}$ be any chain, i.e.,

$$\forall S, T \in C, \quad S \subseteq T \text{ or } T \subseteq S.$$

Now, we need to find an upper bound $U \in \mathcal{L}$ of $C$. Define

$$U := \bigcup_{S \in C} S.$$

Clearly, $U \subseteq V$. We claim that $U$ is linearly independent, i.e., $U \in \mathcal{L}$:

(*Proof of $U \in \mathcal{L}$*) Let $n \in \mathbb{N}$ and suppose

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n = 0 \quad \text{with } a_i \in F, \mathbf{u}_i \in U \text{ for } i = 1, 2, \ldots, n.$$

Since $U = \bigcup_{S \in C} S$,

$$\mathbf{u}_i \in U \iff \exists S_i \in C \text{ such that } \mathbf{u}_i \in S_i.$$

for each $i \in \{1, 2, \ldots, n\}$. Since $C$ is a chain (totally ordered by inclusion), the sets $S_1, S_2, \ldots, S_n$ are comparable. Therefore, there exists at least one set $S^* \in C$ such that

$$(\forall i \in \{1, 2, \ldots, n\}, \ \mathbf{u}_i \in S^*) \quad \text{i.e.,} \quad \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\} \subseteq S^*.$$

Since $S^*$ is an element of $\mathcal{C}$ (and $\mathcal{C} \subseteq \mathcal{L}$, where every element is linearly independent), the linear independence of $S^*$ implies that

$$a_1 = a_2 = \cdots = a_n = 0.$$

Thus, $U$ is linearly independent, i.e., $U \in \mathcal{L}$.

By definition of $U$, we know

$$\forall S \in \mathcal{C},\ S \subseteq U,$$

and so $U \in \mathcal{L}$ be an upper bound of $\mathcal{C}$.

Step 3 **Application of Zorn's Lemma.**

Since every chain $\mathcal{C}$ in $\mathcal{L}$ has an upper bound $U \in \mathcal{L}$, Zorn's Lemma guarantees the existence of a maximal element $\mathcal{B} \in \mathcal{L}$ such that

$$\forall S \in \mathcal{L},\ (\mathcal{B} \subseteq S) \implies (\mathcal{B} = S), \quad \text{i.e.,} \quad \nexists S \in \mathcal{L} \text{ with } \mathcal{B} \subsetneq S.$$

Step 4 **$\mathcal{B}$ is a Basis of $V$.**

We now show that $\mathcal{B}$ spans $V$, i.e., $\operatorname{span}\mathcal{B} = V$. Assume, for contradiction, that

$$\operatorname{span}\mathcal{B} \neq V, \quad \text{i.e.,} \quad \exists \mathbf{v}_0 \in V \setminus \operatorname{span}\mathcal{B}.$$

Consider

$$\mathcal{B}' = \mathcal{B} \cup \{\mathbf{v}_0\}.$$

We NTS that $\mathcal{B}'$ is linearly independent. Suppose that for $n \in \mathbb{N}$, scalars $a_0, a_1, \cdots, a_n \in F$ and distinct vectors $\mathbf{v}_0, \mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n \in \mathcal{B}'$, the followings holds:

$$a_0\mathbf{v}_0 + (a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n) = 0.$$

(Case I) If $a_0 = 0$, then

$$a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n = 0$$

and since $\mathcal{B}$ is linearly independent, $a_i = 0$ for $i = 1, 2, \ldots, n$.

(Case II) If $a_0 \neq 0$, then

$$\mathbf{v}_0 = -\frac{1}{a_0}(a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n) \in \operatorname{span}\mathcal{B},$$

which contradicts the assumption that $\mathbf{v}_0 \notin \operatorname{span}\mathcal{B}$.

Thus, in all cases,

$$a_0 = a_1 = \cdots = a_n = 0.$$

Hence, $\mathcal{B}'$ is linearly independent, i.e., $\mathcal{B}' \in \mathcal{L}$, and $\mathcal{B} \subseteq \mathcal{B}'$, contradicting the maximality of $\mathcal{B}$.

□

**Remark.** This theorem and its proof is a classic demonstration of how abstract set-theoretic principles can yield concrete and essential results in linear algebra.

**Definition.** Consider any two sets $S_1$ and $S_2$.

(1) (Equal Cardinalities) We write

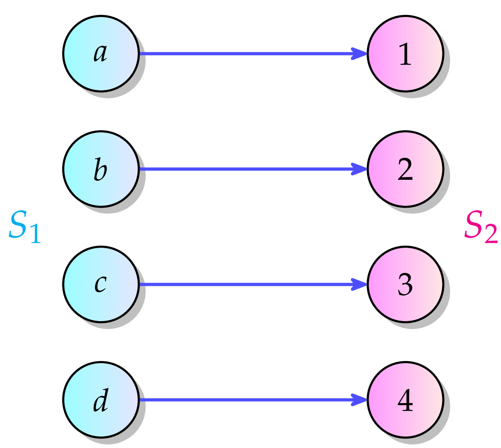
$$|S_1| = |S_2|$$

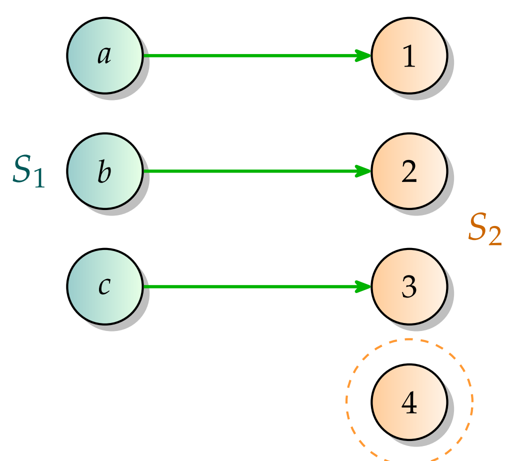
if and only if there exists a bijective (one-to-one and onto) function $f : S_1 \to S_2$.

(2) (Strict Inequality of Cardinalities) We write

$$|S_1| < |S_2|$$

if and only if there exists an injective (one-to-one) function $f : S_1 \to S_2$ but no bijective function from $S_1$ onto $S_2$ exists.



**Steinitz's Exchange Lemma**

**Lemma.** *Let $V$ be a vector space over a field $F$. Suppose that*

*(i)* $\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\} \subseteq V$ *is a linearly independent set, and*

*(ii)* $\mathcal{Y} = \left\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\right\} \subseteq V$ *is a spanning set of* $V$, i.e., $\operatorname{span} \mathcal{Y} = V$.

*Then*

$$\left|\mathcal{X}\right| \leq \left|\mathcal{Y}\right|,$$

*that is, there exists an injective function* $f : \mathcal{X} \rightarrowtail \mathcal{Y}$.

*Proof.* TBA □

**Invariance of Basis Cardinality**

**Theorem.** *Let $V$ be a vector space over a field $F$, and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bases of $V$. Then*

$$|\mathcal{B}_1| = |\mathcal{B}_2|.$$

*Proof.* Suppose, for the contradiction, that

$$|\mathcal{B}_1| < |\mathcal{B}_2|.$$

Since $\mathcal{B}_1$ is a basis, it spans $V$. Also since $\mathcal{B}_2$ is a basis, it is linearly independent. Applying the Steinitz's Exchange Lemma, we obtain

$$|\mathcal{B}_2| \leq |\mathcal{B}_1| \quad \lightning.$$

Thus, it is not possible to have bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of $V$ with different cardinalities. □

**Dimension of Vector Space**

**Definition.** Let $V$ be a vector space over a field $F$. The **dimension** of $V$, denoted by $\dim V$, is defined as the cardinality of any basis $\mathcal{B}$ of $V$:

$$\dim V := |\mathcal{B}|.$$

**Remark.** By the Invariance of Basis Cardinality, this definition does not depend on the choice of the basis.

## References

[1] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 14. 선형대수학 (a) 벡터공간의 정의와 초른의 보조정리" YouTube Video, 27:20. Published October 8, 2019. URL: https://www.youtube.com/watch?v=esLn0FeedyQ.

[2] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 15. 선형대수학 (b) 벡터공간의 기저의 존재성과 차원" YouTube Video, 24:41. Published October 10, 2019. URL: https://www.youtube.com/watch?v=oGC_BU5Erkk&t=875s.

[3] Wikipedia, The Free Encyclopedia. "Steinitz exchange lemma." Wikipedia Article. Accessed February 25, 2025. URL: https://en.wikipedia.org/wiki/Steinitz_exchange_lemma.