Abstract Algebra II

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We cover the following topics in this note.

- Group Action
- Cayley Theorem
- Normal Subgroups
- TBA

Contents

Group Action

Definition. Let $(G, *)$ be a group and let $X \neq \varnothing$. A **(left) group action** of G on X is a function

$$\cdot : G \times X \to X, \quad (g, x) \mapsto g \cdot x$$

satisfying the followings: for all $g, h \in G$ and all $x \in X$,

(i) (Identity) $e \cdot x = x$, where $e \in G$ is the identity element of G;

(ii) (Compatibility) $(g * h) \cdot x = g \cdot (h \cdot x)$.

The pair $(X, \cdot)$ (or simply X) is then called a G-set.

Note (Notation)**.** If a group G acts on a set X, one commonly writes: $G \curvearrowright X$.

Remark. A right group action of G on X is a function $\cdot : X \times G \to X, \quad (x, g) \mapsto x \cdot g$ satisfying:

(i) $x \cdot e = x$ for all $x \in X$;

(ii) $(x \cdot g) \cdot h = x \cdot (gh)$ for all $g, h \in G$, $x \in X$.

Example (Scalar Multiplication on a Vector Space)**.** Let $\mathbb{F}$ be a field, and let $X = \mathbb{F}^n$ be the n-dimensional vector space over $\mathbb{F}$. Consider the multiplicative group of nonzero scalars in $\mathbb{F}$:

$$G = (\mathbb{F}^\times, \times), \quad \text{where } \mathbb{F}^\times = \mathbb{F} \setminus \{0\}.$$

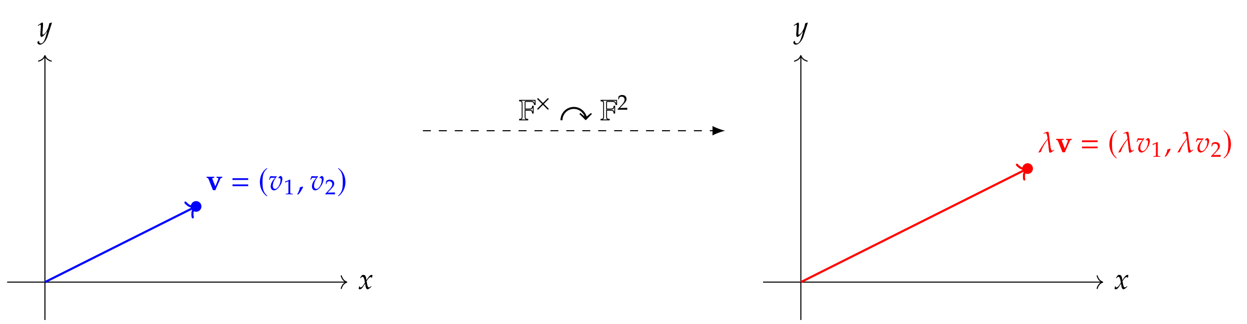
We define an action $G \curvearrowright X$ by scalar multiplication:

$$\begin{array}{cccc} \cdot & : & \mathbb{F}^\times \times \mathbb{F}^n & \longrightarrow & \mathbb{F}^n \\ & & (\lambda, \mathbf{v}) & \longmapsto & \lambda \cdot \mathbf{v} \end{array},$$

where the product $\lambda \cdot \mathbf{v}$ is defined componentwise. Then

(i) $1 \cdot \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{F}^n$.

(ii) $(\lambda\mu) \cdot \mathbf{v} = \lambda \cdot (\mu \cdot \mathbf{v})$ for all $\lambda, \mu \in \mathbb{F}^\times$, $\mathbf{v} \in \mathbb{F}^n$.



Example (Conjugation Action on the Group Itself)**.** Let G be any group, and consider $X = G$. Define an action of G on itself by conjugation:

$$G \curvearrowright G, \quad (g, x) \mapsto g \cdot x := g * x * g^{-1}.$$

Then

(i) $e \cdot x = e * x * e^{-1} = x$ for all $x \in G$.

(ii) Note that

$$\begin{aligned}(g * h) \cdot x &= (g * h) * x * (g * h)^{-1} \\ &= (g * h) * x * (h^{-1} * g^{-1}) \\ &= g * (h * x * h^{-1}) * g^{-1} \\ &= g * (h \cdot x) * g^{-1} \\ &= g \cdot (h \cdot x).\end{aligned}$$

Thus, this is a left group action.

Example (Trivial G-Set)**.** Let G be any group and define the set $X = \{x\}$, a singleton. Define the action

$$G \curvearrowright X, \quad (g, x) \mapsto g \cdot x := x \quad \text{for all } g \in G.$$

This is the **trivial action**, where every group element acts as the identity on X:

(i) $e \cdot x = x$.

(ii) $(g * h) \cdot x = x = g \cdot (h \cdot x)$.

Example (Action on Coset Space G/H)**.** Let $(G, *)$ be a group, and let $H \leq G$. Let $X = G/H$ be the set of left cosets of H in G, i.e.,

$$X = G/H = \{gH \mid g \in G\}.$$

Define an action

$$G \curvearrowright G/H, \quad (g, aH) \mapsto (ga)H.$$

This is well-defined because if $a_1H = a_2H$, then $a_1^{-1}a_2 \in H$, so: $ga_1H = ga_2H$.. Since

(i) $e \cdot aH = aH$;

(ii) $(gh) \cdot aH = g \cdot (h \cdot aH)$,

this is a **transitive action**.

Group Elements Act as Permutations

Proposition. *Let G be a group action on a set X via a left action $G \curvearrowright X$, given by $(g, x) \mapsto g \cdot x$. Then for each $g \in G$, the map*

$$\sigma_g : X \to X, \quad x \mapsto g \cdot x$$

is one-to-one and onto. That is, $\sigma_g \in \mathrm{Sym}(X)$, the group of all permutations of X.

Proof. TBA □

Group Actions Induce Permutation Representations

Theorem. *Let G be a group action on a set X via a left group action $G \curvearrowright X$, $\quad (g, x) \mapsto g \cdot x$. For each $g \in G$, define the bijection $\sigma_g : X \to X$ by $\sigma_g(x) := g \cdot x$. Then the map*

$$\phi : G \to \mathrm{Sym}(X), \quad g \mapsto \sigma_g,$$

*is a **group homomorphism** from G to the symmetric group $\mathrm{Sym}(X)$. In other words, for all $g, h \in G$,*

$$\phi(g * h) = \sigma_{g*h} = \sigma_g \circ \sigma_h = \phi(g) \circ \phi(h).$$

Remark. A group action $G \curvearrowright X$ is equivalent to a group homomorphism $G \to \mathrm{Sym}(X)$, i.e., a **permutation representation** of G.

Proof. TBA □

Cayley Theorem

Theorem. *Let G be a group. Consider the action of G on itself by left multiplication. For each $g \in G$, define*

$$\sigma_g \ : \ G \longrightarrow G, \qquad x \ \mapsto \ g \cdot x.$$

Then the map

$$\phi \ : \ G \longrightarrow \mathrm{Sym}(G), \qquad g \ \mapsto \ \sigma_g$$

is an injective group homomorphism (group monomorphism). In particular,

$$\phi(G) \simeq G \quad \text{and} \quad \varphi(G) \leq \mathrm{Sym}(G).$$

Proof. □

Normal Subgroups

Existence of the Quotient Group

Proposition. *Let* $(G, *)$ *be a group and let* $H \leq G$ *be a subgroup. Define a binary operation* $\boxast$ *on the set of left cosets* G/H *by*

$$(g * H) \boxast (g' * H) \;=\; (g * g') * H$$

where $g, g' \in G$*. Then this operation is well-defined (and makes* G/H *into a group) if and only if*

$$g * h * g^{-1} \;\in\; H.$$

for all $g \in G$, $h \in H$.

Proof. $(\Rightarrow)$ Let $g \in G$ and $h \in H$. Then

$$h * g^{-1} \in H \implies g * H = g(h)$$

$(\Leftarrow)$

□

Normal Subgroup

Definition.

References

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[2] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 24. 추상대수학 (e) 정규부분군의 정의 def of normal subgroups" YouTube Video, 23:00. Published October 25, 2019. URL: `https://www.youtube.com/watch?v=3UJILZr4CNo`.

Appendices

Orbit and Stabilizer

Definition. Let G be a group acting on a set X via a (left) group action:

$$G \curvearrowright X, \quad (g, x) \mapsto g \cdot x.$$

Let $x \in X$.

(1) The **orbit** of x under the action of G is defined by

$$\mathrm{Orb}_G(x) := G \cdot x = \{g \cdot x \mid g \in G\} \subseteq X.$$

This is the set of all elements of X to which x can be moved by the action of $g \in G$.

(2) The **stabilizer subgroup** of an $x \in X$, also called the **isotropy subgroup** or **fixer**, is defined by

$$\mathrm{Stab}_G(x) := \{g \in G \mid g \cdot x = x\}.$$

This is a subgroup of G consisting of all elements that fix x under the action.

Remark.

- The orbits form a *partition* of X; that is, X is the disjoint union of its orbits under the action of G.