The Exact Mathematical Definition of a Function

A **function** f from a set A to a set B, written as

$$f : A \to B,$$

is defined as a relation that assigns to each element $a \in A$ *exactly one* element $b \in B$. More formally, f is a subset of the Cartesian product $A \times B$ with the property that for every $a \in A$, there exists a unique $b \in B$ such that

$$(a, b) \in f.$$

This uniqueness is essential: it means that an element in the domain cannot be assigned more than one output in the codomain.

Motivation from Finite and Infinite Perspectives

1. Finite Table Representation:

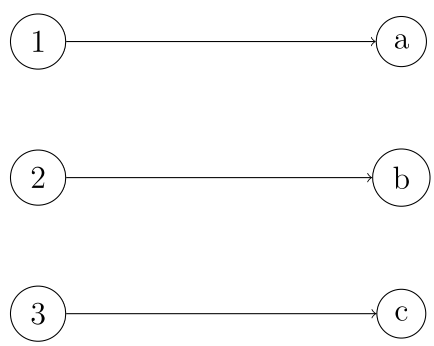
When the domain A is finite, a function can be completely described by a table that lists each input and its corresponding output. For example, consider the function

$$f : \{1, 2, 3\} \to \{a, b, c\}$$

given by:

$a \in A$	$f(a) \in B$
1	a
2	b
3	c

The following TikZ diagram illustrates this mapping:



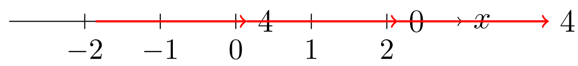
This table and diagram clearly show that each input from the set $\{1, 2, 3\}$ is paired with a unique output.

2. Infinite Function Expression:

When the domain A is infinite, such as $A = \mathbb{R}$, it is impractical to list all the pairs. Instead, a function is described by a rule or formula. For example, consider the function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2.$$

This single formula compactly describes an infinite number of input-output pairs. The following TikZ diagram gives a conceptual visualization of this idea:



Conceptual view of $f(x) = x^2$ mapping each x to a unique x^2

Why the Definition Must be This Way:

- *Uniqueness:* The requirement that each $a \in A$ is assigned exactly one $b \in B$ ensures that the function behaves predictably—knowing the input fully determines the output.
- *Consistency Across Finite and Infinite Cases:* Whether a function is given by a finite table or by an infinite rule, the concept remains the same: a well-defined assignment of outputs to inputs.
- *Foundational Role:* This strict definition underpins much of mathematics. It ensures that functions can be composed, inverted (when appropriate), and analyzed in a rigorous manner.

Conclusion: The exact definition of a function—requiring that every input is paired with a unique output—is motivated by the need for consistency, predictability, and clarity. Finite tables and infinite formulas are simply two perspectives on the same fundamental concept.