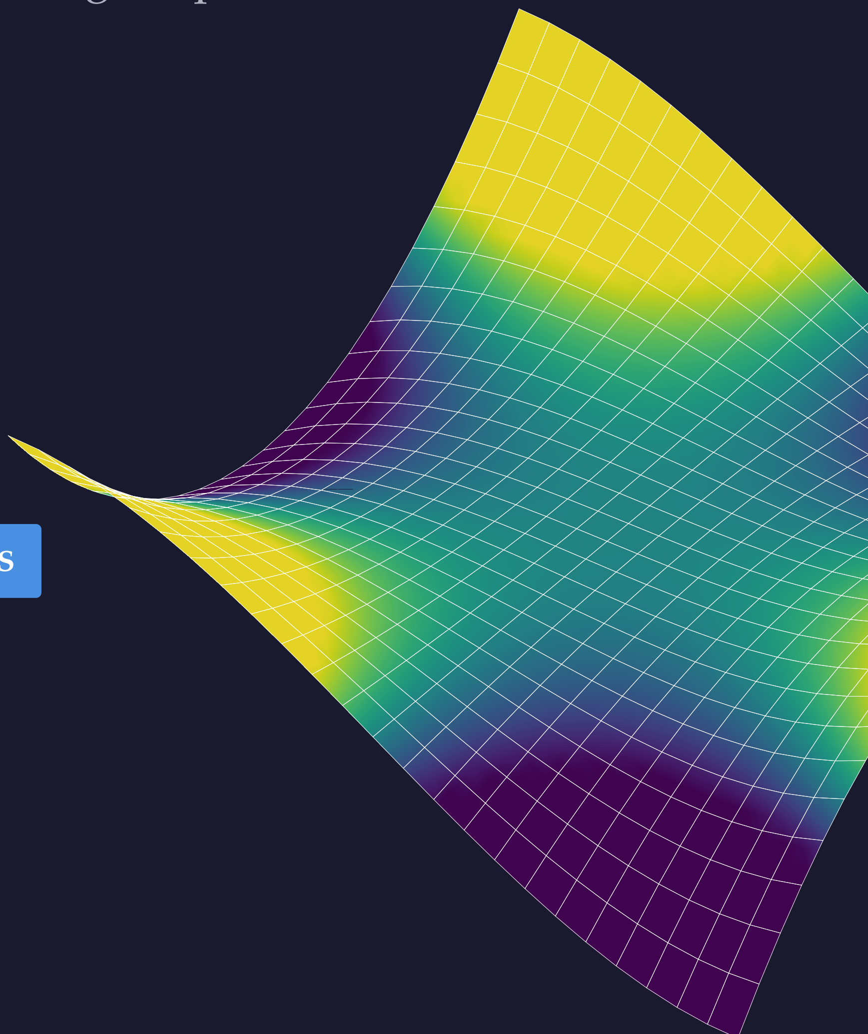
Riemann Surfaces and Algebraic Curves

A framework for understanding Elliptic Curves

Ji, Yonghyeon



PART I — MULTIVARIABLE CALCULUS

A Comprehensive Guide to Modern Algebraic Geometry

Riemann Surfaces and Algebraic Curves

A Framework for Understanding Elliptic Curves

Part I — Multivariable Calculus

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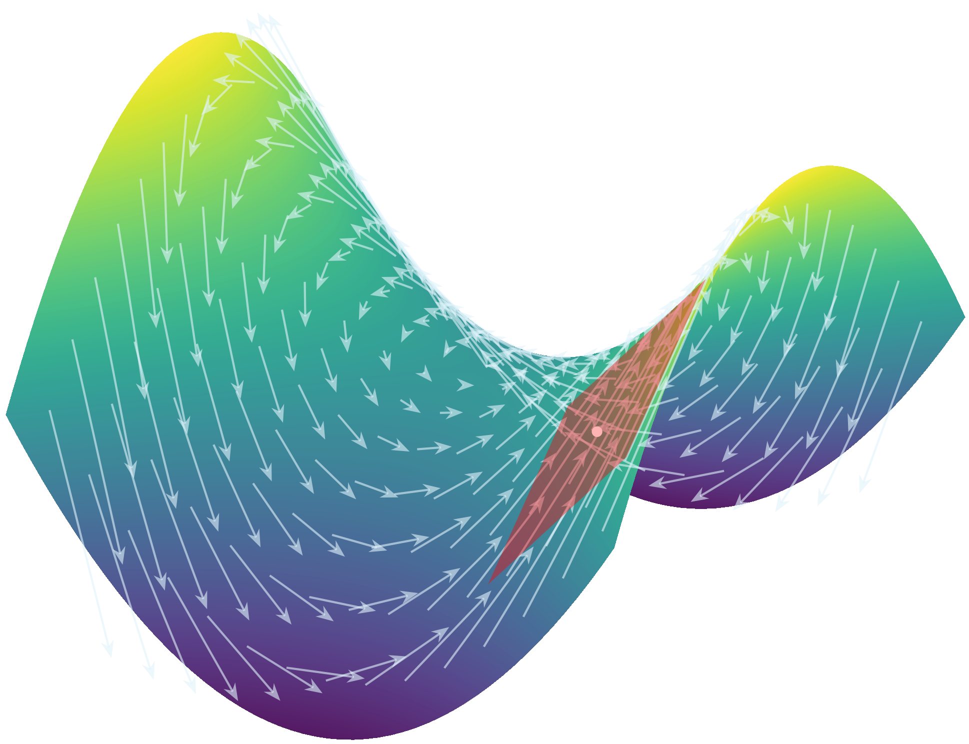

WINTER 2026

The FTC hierarchy

Name	Formula
FTC I (Accumulation)	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\,\mathrm{d}t\right) = f(x).$
FTC II (Evaluation)	$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a).$
Fundamental Theorem of Line Integrals	$\int_C \nabla\phi \cdot \mathrm{d}\mathbf{r} = \phi(B) - \phi(A).$
Green's Theorem	$\oint_{\partial R} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_R \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\,\mathrm{d}A.$
Stokes' Theorem (3D)	$\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,\mathrm{d}S.$
Divergence Theorem	$\iint_{\partial V} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S = \iiint_V (\nabla \cdot \mathbf{F})\,\mathrm{d}V.$
Generalized Stokes	$\int_{\partial\Omega} \omega = \int_\Omega \mathrm{d}\omega.$

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Changelog

v1.0	2025-12-29	Initial release.

Contents

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1 Fundamental Theorem of Calculus

Fundamental Theorem for Gradient Fields

If $\mathbf{F} = \nabla f$ is a conservative vector field and C is a smooth curve from A to B, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Green's Theorem

For a positively oriented, simple closed curve C bounding a region R in the plane,

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Divergence Theorem

Let $\mathbf{F}$ be a vector field defined on a region E with closed boundary surface S (outward-oriented). Then

$$\iiint_E \nabla \cdot \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot \mathbf{n}\,dS.$$

Stokes' Theorem

Let S be an oriented surface with boundary curve C, and let $\mathbf{F}$ be a vector field. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Triple Integral

To integrate a scalar function $f(x, y, z)$ over a region E in $\mathbb{R}^3$,

$$\iiint_E f(x, y, z)\,dV.$$

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1.1 Gradient Vector Fields

1. Let $\mathbf{F} = \langle 2x, 2y \rangle$. Show that $\mathbf{F}$ is conservative and compute

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where C is any path from $(0,0)$ to $(1,1)$.

2. Determine whether the vector field $\mathbf{F} = \langle y, x \rangle$ is conservative. If so, find a potential function.

3. Let $f(x,y,z) = xyz$. Compute ∇f and evaluate the line integral of ∇f over the path from $(1,0,0)$ to $(1,2,3)$.

4. Let $\mathbf{F} = \nabla f$ for $f(x,y) = x^2 + y^2$. Compute the line integral over a circular path from $(1,0)$ to $(0,1)$ and explain the result.

1.2 Green's Theorem

1. Use Green's Theorem to evaluate

$$\oint_C x\, dy - y\, dx$$

where C is the unit circle oriented counterclockwise.

2. Let $\mathbf{F} = \langle y^2, 2xy \rangle$. Use Green's Theorem to evaluate the line integral around the boundary of the square $[0,1] \times [0,1]$.

3. Evaluate

$$\oint_C (x+y)\mathrm{d}x + (x-y)\mathrm{d}y$$

where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$ oriented counterclockwise.

4. Determine if

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$$

for $\mathbf{F} = \langle y, -x \rangle$ around a circle of radius r centered at the origin.

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1.3 Divergence Theorem

1. Let $\mathbf{F} = \langle x, y, z \rangle$. Use the Divergence Theorem to compute the flux across the surface of the unit sphere.
2. Let $\mathbf{F} = \langle x^2, y^2, z^2 \rangle$. Compute both the divergence and the surface integral over the unit cube $[0,1]^3$.
3. Use the Divergence Theorem to find the outward flux of $\mathbf{F} = \langle yz, xz, xy \rangle$ through the unit cube.
4. Let $\mathbf{F} = \langle x, -y, z \rangle$. Verify the Divergence Theorem on the upper hemisphere of radius 1 centered at the origin.

1.4 Stokes' Theorem

1. Let $\mathbf{F} = \langle -y, x, 0 \rangle$. Use Stokes' Theorem to compute the circulation around the boundary of the disk $x^2 + y^2 \leq 1$ in the xy-plane.
2. Let $\mathbf{F} = \langle z, 0, x \rangle$. Use Stokes' Theorem on the triangular surface with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$.
3. Compute both sides of Stokes' Theorem for $\mathbf{F} = \langle y, z, x \rangle$ on the surface $z = 0$ bounded by the unit circle.
4. Use Stokes' Theorem to show that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

if $\mathbf{F}$ is the gradient of some scalar field f.

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2 Differential Forms

TBA

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3 Winding Numbers and Complexification

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