# Homework1: Matrix Representations of Linear Transformations

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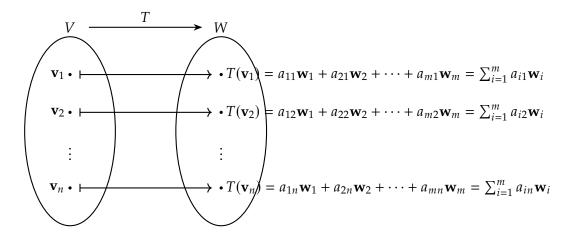
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Let V and W be vector spaces over a field  $\mathbb{F}$  with bases

$$\mathcal{B}_V = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$$
 and  $\mathcal{B}_W = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$ 

and let  $T: V \to W$  be a linear transformation whose matrix with respect to  $\mathcal{B}_V$  and  $\mathcal{B}_W$  is

$$[T]_{\mathcal{B}_{V}}^{\mathcal{B}_{W}} = \begin{bmatrix} \vdots & & \vdots \\ T(\mathbf{v}_{1}) & \cdots & T(\mathbf{v}_{n}) \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$



**Problem 1.** Let  $T: \mathbb{R}^2 \to \mathbb{R}^2$  be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

- 1. Compute  $T\left(\begin{bmatrix} 5 & 6 \end{bmatrix}^T\right)$ .
- 2. Verify that *A* represents *T* by checking  $T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 & 0 \end{bmatrix}^T\right)$  and  $T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 & 1 \end{bmatrix}^T\right)$ .

**Problem 2.** Let  $T: \mathbb{R}^3 \to \mathbb{R}^2$  be defined by

$$T(x, y, z) = (x + 2y - z, 3x - y + 4z).$$

Find the matrix  $[T]_{\mathcal{E}_3}^{\mathcal{E}_2}$  with respect to the standard bases  $\mathcal{E}_3$  of  $\mathbb{R}^3$  and  $\mathcal{E}_2$  of  $\mathbb{R}^2$ .

**Problem 3.** Let  $V = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$  with basis

$$\mathcal{B}_V = \{1 + x, x - 1, x^2 + 2\},\$$

and  $W = \mathbb{R}^2$  with basis

$$\mathcal{B}_W = \{ \mathbf{e}_1 = (1,0), \mathbf{e}_2 = (1,1) \}.$$

Suppose the matrix of  $T: V \to W$  with respect to these bases is

$$[T]_{\mathcal{B}_V}^{\mathcal{B}_W} = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \end{bmatrix}.$$

Compute  $T(-x^2 + 3x + 2)$  in standard coordinates of  $\mathbb{R}^2$ .

**Problem 4.** Let  $V = \mathbb{R}^{2\times 2}$  with basis

$$\mathcal{B}_V = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},\,$$

and  $W = \mathbb{R}^2$  with its standard basis. Define

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b, c-d).$$

Find the matrix  $[T]_{\mathcal{B}_V}^{\mathcal{E}_2}$ .

**Problem 5.** Let  $V = \{ax^3 + bx^2 + cx + d : a, b, c, d \in \mathbb{R}\}$  with standard basis  $\mathcal{B}_V = \{1, x, x^2, x^3\}$  and  $W = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$  with basis  $\mathcal{B}_W = \{1 + x, x^2, 1\}$ . Define

$$T(p(x)) = p'(x) + p(1).$$

Find the matrix  $[T]_{\mathcal{E}_4}^{\mathcal{B}_W}$ .

## **Problem 6.** Choose any nonzero matrix

$$A \in \mathbb{F}^{m \times n}$$

and choose bases

- $\mathcal{B}_V$  for an *n*-dimensional vector space V over a field  $\mathbb F$  and
- $\mathcal{B}_W$  for an m-dimensional vector space W over a field  $\mathbb{F}$ .

Define the linear map  $T: V \to W$  by

$$[T]_{\mathcal{B}_V}^{\mathcal{B}_W} = A.$$

- 1. Pick a vector  $\mathbf{v} \in V$  (in coordinates relative to  $\mathcal{B}_V$ ) and compute  $T(\mathbf{v})$  in the coordinates of W.
- 2. Determine  $\ker T$ ,  $\operatorname{im} T$ , and  $\operatorname{rank} T$ .

