Abstract Algebra II

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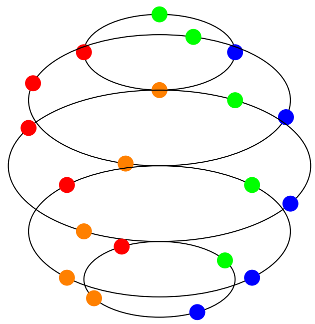
May 13, 2025

We cover the following topics in this note.

- Group Action
- Cayley Theorem
- Normal Subgroups
- Normality of the Kernel



Group Action

Definition. Let $(G, *)$ be a group and let $X \neq \emptyset$. A **(left) group action** of G on X is a function

$$\cdot : G \times X \to X, \quad (g, x) \mapsto g \cdot x$$

satisfying the followings: for all $g, h \in G$ and all $x \in X$,

(i) (Identity) $e \cdot x = x$, where $e \in G$ is the identity element of G;

(ii) (Compatibility) $(g * h) \cdot x = g \cdot (h \cdot x)$.

The pair $(X, \cdot)$ (or simply X) is then called a G-set.

Note (Notation). If a group G acts on a set X, one commonly writes: $G \curvearrowright X$.

Remark. A right group action of G on X is a function $\cdot : X \times G \to X, \quad (x, g) \mapsto x \cdot g$ satisfying:

(i) $x \cdot e = x$ for all $x \in X$;

(ii) $(x \cdot g) \cdot h = x \cdot (gh)$ for all $g, h \in G, x \in X$.

Example (Scalar Multiplication on a Vector Space)**.** Let $\mathbb{F}$ be a field, and let $X = \mathbb{F}^n$ be the n-dimensional vector space over $\mathbb{F}$. Consider the multiplicative group of nonzero scalars in $\mathbb{F}$:

$$G = (\mathbb{F}^\times, \times), \quad \text{where } \mathbb{F}^\times = \mathbb{F} \setminus \{0\}.$$

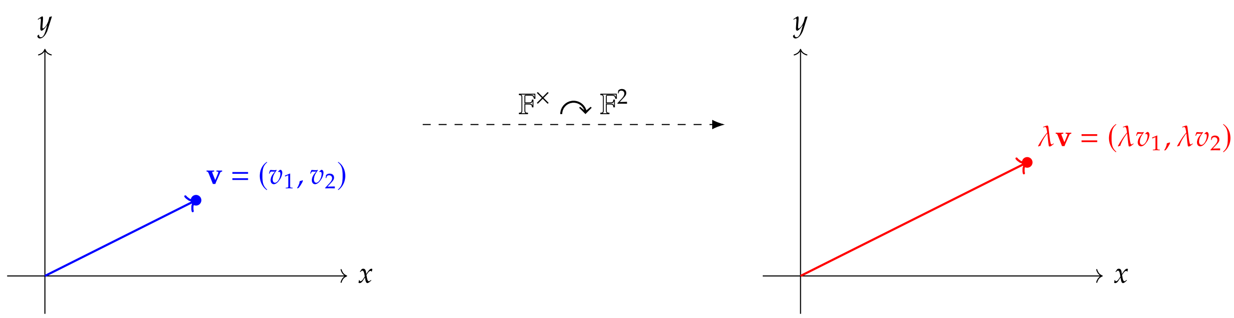
We define an action $G \curvearrowright X$ by scalar multiplication:

$$\begin{array}{rccc} \cdot & : & \mathbb{F}^\times \times \mathbb{F}^n & \longrightarrow & \mathbb{F}^n \\ & & (\lambda, \mathbf{v}) & \longmapsto & \lambda \cdot \mathbf{v} \end{array},$$

where the product $\lambda \cdot \mathbf{v}$ is defined componentwise. Then

(i) $1 \cdot \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{F}^n$.

(ii) $(\lambda\mu) \cdot \mathbf{v} = \lambda \cdot (\mu \cdot \mathbf{v})$ for all $\lambda, \mu \in \mathbb{F}^\times$, $\mathbf{v} \in \mathbb{F}^n$.



Example (Conjugation Action on the Group Itself)**.** Let G be any group, and consider $X = G$. Define an action of G on itself by conjugation:

$$G \curvearrowright G, \quad (g, x) \mapsto g \cdot x := g * x * g^{-1}.$$

Then

(i) $e \cdot x = e * x * e^{-1} = x$ for all $x \in G$.

(ii) Note that

$$\begin{aligned} (g * h) \cdot x &= (g * h) * x * (g * h)^{-1} \\ &= (g * h) * x * (h^{-1} * g^{-1}) \\ &= g * (h * x * h^{-1}) * g^{-1} \\ &= g * (h \cdot x) * g^{-1} \\ &= g \cdot (h \cdot x). \end{aligned}$$

Thus, this is a left group action.

Example (Trivial G-Set)**.** Let G be any group and define the set $X = \{x\}$, a singleton. Define the action

$$G \curvearrowright X, \quad (g, x) \mapsto g \cdot x := x \quad \text{for all } g \in G.$$

This is the **trivial action**, where every group element acts as the identity on X:

(i) $e \cdot x = x$.

(ii) $(g * h) \cdot x = x = g \cdot (h \cdot x)$.

Example (Action on Coset Space G/H)**.** Let $(G, *)$ be a group, and let $H \leq G$. Let $X = G/H$ be the set of left cosets of H in G, i.e.,

$$X = G/H = \{gH \mid g \in G\}.$$

Define an action

$$G \curvearrowright G/H, \quad (g, aH) \mapsto (ga)H.$$

This is well-defined because if $a_1H = a_2H$, then $a_1^{-1}a_2 \in H$, so: $ga_1H = ga_2H$.. Since

(i) $e \cdot aH = aH$;

(ii) $(gh) \cdot aH = g \cdot (h \cdot aH)$.

Group Elements Act as Permutations

Proposition. *Let G be a group action on a set X via a left action $G \curvearrowright X$, given by $(g, x) \mapsto g \cdot x$. Then for each $g \in G$, the map*

$$\sigma_g : X \to X, \quad x \mapsto g \cdot x$$

is one-to-one and onto. That is, $\sigma_g \in \mathrm{Sym}(X)$, the group of all permutations of X.

Proof. TBA □

Group Actions Induce Permutation Representations

Theorem. *Let G be a group action on a set X via a left group action $G \curvearrowright X, \quad (g, x) \mapsto g \cdot x$. For each $g \in G$, define the bijection $\sigma_g : X \to X$ by $\sigma_g(x) := g \cdot x$. Then the map*

$$\phi : G \to \mathrm{Sym}(X), \quad g \mapsto \sigma_g,$$

*is a **group homomorphism** from G to the symmetric group $\mathrm{Sym}(X)$. In other words, for all $g, h \in G$,*

$$\phi(g * h) = \sigma_{g*h} = \sigma_g \circ \sigma_h = \phi(g) \circ \phi(h).$$

Remark. A group action $G \curvearrowright X$ is equivalent to a group homomorphism $G \to \mathrm{Sym}(X)$, i.e., a **permutation representation** of G.

Proof. TBA □

Cayley Theorem

Theorem. *Let G be a group. Consider the action of G on itself by left multiplication. For each $g \in G$, define*

$$\sigma_g \; : \; G \longrightarrow G, \qquad x \mapsto g \cdot x.$$

Then the map

$$\phi \; : \; G \longrightarrow \mathrm{Sym}(G), \qquad g \mapsto \sigma_g$$

is an injective group homomorphism (group monomorphism). In particular,

$$\phi(G) \simeq G \quad \text{and} \quad \phi(G) \leq \mathrm{Sym}(G).$$

Proof. TBA □

Normal Subgroups

Observation. Consider $4\mathbb{Z} \leq \mathbb{Z}$. Then

$$\mathbb{Z}/4\mathbb{Z} = \{0 + 4\mathbb{Z}, 1 + 4\mathbb{Z}, 2 + 4\mathbb{Z}, 3 + 4\mathbb{Z}\} = \{[0], [1], [2], [3]\}.$$

- $[0] + [1] = (0 + 4\mathbb{Z}) + (1 + 4\mathbb{Z}) \overset{\text{def.}}{=} (0 + 1) + 4\mathbb{Z} = 1 + 4\mathbb{Z} = [1]$.
- $[1] + [2] = (1 + 4\mathbb{Z}) + (2 + 4\mathbb{Z}) \overset{\text{def.}}{=} (1 + 2) + 4\mathbb{Z} = 3 + 4\mathbb{Z} = [3]$.
- $[1] + [3] = (2 + 4\mathbb{Z}) + (3 + 4\mathbb{Z}) \overset{\text{def.}}{=} (1 + 3) + 4\mathbb{Z} = 4 + 4\mathbb{Z} = 0 + 4\mathbb{Z} = [0]$.

Existence of the Quotient Group

Proposition. *Let $(G, *)$ be a group and let $H \leq G$ be a subgroup. Define a binary operation $\boxtimes$ on the set of left cosets G/H by*

$$(g * H) \boxtimes (g' * H) \;=\; (g * g') * H$$

where $g, g' \in G$. Then this operation is well-defined if and only if

$$g * h * g^{-1} \;\in\; H.$$

for all $g \in G$, $h \in H$.

Proof. TBA □

Normal Subgroup

Definition. Let $(G, *)$ be a group and let $H \leq G$. We say that H is **normal** in G, written

$$H \trianglelefteq G,$$

if $g * h * g^{-1} \in H$ for any $g \in G$ and $h \in H$.

Remark. The set of (left) cosets G/H be a well-defined group structure via

$$(g * H) \boxast (k * H) = (g * k) * H,$$

making G/H the quotient group of G by H.

Equivalent Definitions of Normal Subgroup

Proposition. *Let $(G, *)$ be a group and let $H \leq G$. The Following Are Equivalent:*

(1)[a] *H is normal in G,* i.e., *$H \trianglelefteq G$;*

(2)[b] *$g * h * g^{-1} \in H$ for all $g \in G$, $h \in H$;*

(3)[c] *$g * H * g^{-1} = H$ for all $g \in G$;*

(4)[d] *$g * H = H * g$ for all $g \in G$.*

[a] Terminology and Notation
[b] (Elementwise Conjugation)
[c] (Conjugation Invariance)
[d] (Coset Equality)

Proof. ((2)⇒(3)) TBA

((3)⇒(4)) TBA

((4)⇒(2)) TBA

□

Normality of Kernel

Theorem. *Let* $\phi : (G, *) \longrightarrow (H, *')$ *be a group homomorphism, and define its kernel by*

$$\ker\phi \;=\; \{ g \in G : \phi(g) = e_H \} .$$

Then $\ker\phi$ *is a normal subgroup of* G*; that is,* $\ker\phi \trianglelefteq G$.

Proof. Since ϕ is a homomorphism, for every $g \in G$ and every $k \in \ker\phi$ we have

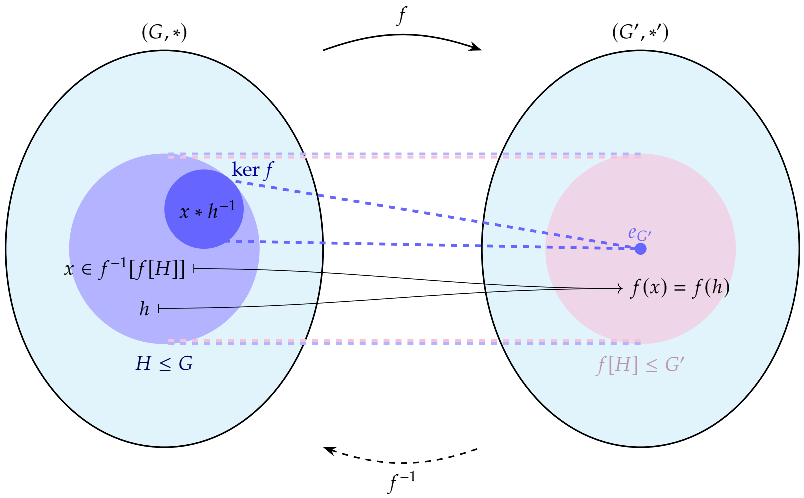
$$\phi\big(g * k * g^{-1}\big) = \phi(g) *' \phi(k) *' \phi(g)^{-1} = \phi(g) *' e_H *' \phi(g)^{-1} = e_H,$$

so $g * k * g^{-1} \in \ker\phi$. Thus,

$$g * (\ker\phi) * g^{-1} \;=\; \ker\phi \quad \forall\, g \in G,$$

i.e. $\ker\phi$ is invariant under conjugation and hence normal in G. □

Illustration.



$f^{-1}[f[H]] = H.$

Preimage of the Image of a Subgroup

Theorem. *Let $f : (G, *) \longrightarrow (G', *')$ be a group homomorphism, and let $H \leq G$ such that*

$$\{ g \in G : f(g) = e_{G'} \} = \ker f \subseteq H.$$

Then

$$f^{-1}[f[H]] = H,$$

with $f[H] = \{f(h) \mid h \in H\}$ and $f^{-1}[f[H]] = \{g \in G \mid f(g) \in f[H]\}$.

Proof. Suppose that $\ker f \subseteq H \leq G$. We NTS that $f^{-1}[f[H]] = H$:

($\supseteq$) $h \in H \implies f(h) \in f[H] \implies h \in f^{-1}[f[H]]$.

($\subseteq$) Let $x \in f^{-1}[f[H]]$. Then $f(x) \in f[H]$; that is,

$$\exists h \in H \quad \text{such that} \quad f(h) = f(x).$$

Thus,

$$f(x * h^{-1}) = f(x) *' f(h)^{-1} = f(x) *' f(x)^{-1} = e_{G'},$$

so $x * h^{-1} \in \ker f$. Since $\ker f \subseteq H$, we have

$$x = (x * h^{-1}) * h \in H,$$

and hence $f^{-1}[f[H]] \subseteq H$.

□

Theorem. *Let* $\phi : G \longrightarrow G'$ *be a surjective homomorphism of groups. Define two collections:*

$$\mathcal{S} = \{H \subseteq G : \ker\phi \subseteq H \leq G\}, \qquad \mathcal{T} = \{H' \subseteq G' : H' \leq G'\}.$$

Then the map

$$\Phi : \mathcal{S} \longrightarrow \mathcal{T}, \qquad \Phi(H) = \phi(H)$$

is a bijection. Its inverse is

$$\Phi^{-1} : \mathcal{T} \longrightarrow \mathcal{S}, \qquad \Phi^{-1}(H') = \phi^{-1}(H').$$

Moreover,

$$H \trianglelefteq G \iff \phi(H) \trianglelefteq G'.$$

Proof. □

References

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[2] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 24. 추상대수학 (e) 정규부분군의 정의 def of normal subgroups" YouTube Video, 23:00. Published October 25, 2019. URL: `https://www.youtube.com/watch?v=3UJILZr4CNo`.

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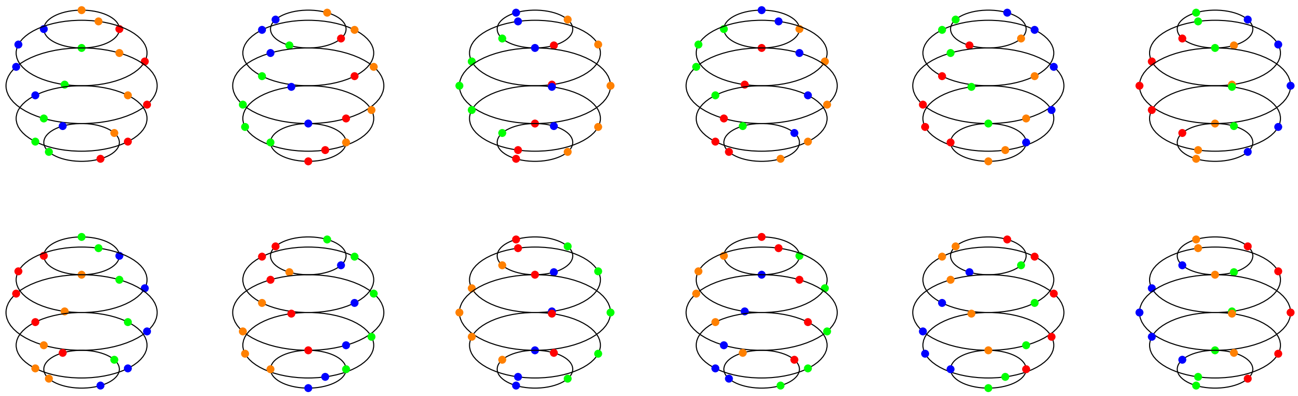
A Appendices

A.1 The Rotation Action of $\mathbb{S}^1$ on $\mathbb{S}^2$



Consider

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\},$$
$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Define the map

$$\Phi\colon \mathbb{S}^1 \times \mathbb{S}^2 \longrightarrow \mathbb{S}^2, \qquad (e^{i\theta}, P) \mapsto \mathsf{Rot}_\theta(P),$$

where, for each $e^{i\theta} \in S^1$, define the rotation

$$\mathsf{Rot}_\theta\colon \mathbb{S}^2 \to \mathbb{S}^2, \qquad \mathsf{Rot}_\theta(x, y, z) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \\ z \end{pmatrix}.$$

Here, $\Phi(e^{i\theta}, P) = \mathsf{Rot}_\theta(P)$. Then

(i) (Identity) The identity in $\mathbb{S}^1$ is $1 = e^{i\cdot 0}$. Since $\cos 0 = 1$, $\sin 0 = 0$, we have

$$\Phi(1, P) = \mathsf{Rot}_0(P) = (x, y, z) = P,$$

for every $P \in \mathbb{S}^2$.

(ii) (Compatibility) For any $e^{i\theta}, e^{i\phi} \in \mathbb{S}^1$ and $P \in \mathbb{S}^2$,

$$\Phi\big(e^{i\theta}e^{i\phi}, P\big) = \Phi\big(e^{i(\theta+\phi)}, P\big) = \mathsf{Rot}_{\theta+\phi}(P) = \mathsf{Rot}_\theta\big(\mathsf{Rot}_\phi(P)\big) = \Phi\big(e^{i\theta}, \Phi(e^{i\phi}, P)\big).$$

Hence Φ be a left group action. To be continue $\cdots$.

P	θ	$\mathsf{Rot}_\theta(P)$	Comment
$(1,0,0)$	$\pi/2$	$(0,-1,0)$	"East" $\to$ "South" on equator
$(1,0,0)$	π	$(-1,0,0)$	"East" $\to$ "West"
$(1,0,0)$	$2\pi/3$	$(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$	120° around equator
$(0,1,0)$	$\pi/2$	$(1,0,0)$	"North" $\to$ "East" on equator
$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$	$\pi/4$	$(1,0,0)$	45° NE equator $\to$ East
$(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$	$\pi/2$	$(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	45° above equator (height fixed)
$(1,0,0)$	rotate by $\pi/4$ then $\pi/4$	$(0,-1,0)$	composition = $\mathsf{Rot}_{\pi/2}(1,0,0)$

Table 1: Computations of the rotation Rot_θ on $\mathbb{S}^2$.

θ	$\cos\theta$	$\sin\theta$	$\mathsf{Rot}_\theta(x,y,z)$	Interpretation
0	1	0	(x,y,z)	identity
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$(\frac{\sqrt{3}}{2}x + \frac{1}{2}y,\ -\frac{1}{2}x + \frac{\sqrt{3}}{2}y,\ z)$	30° rotation
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$(\frac{x+y}{\sqrt{2}},\ -\frac{x-y}{\sqrt{2}},\ z)$	45° rotation
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$(\frac{1}{2}x + \frac{\sqrt{3}}{2}y,\ -\frac{\sqrt{3}}{2}x + \frac{1}{2}y,\ z)$	60° rotation
$\frac{\pi}{2}$	0	1	$(y,\ -x,\ z)$	90° rotation
π	-1	0	$(-x,-y,z)$	180° rotation
2π	1	0	(x,y,z)	full 360° rotation

Table 2: Trigonometric values and the induced rotations on $\mathbb{S}^2$ for key angles θ.

P	$\mathsf{Rot}_0(P)$	$\mathsf{Rot}_{\frac{\pi}{4}}(P)$	$\mathsf{Rot}_{\frac{\pi}{2}}(P)$	$\mathsf{Rot}_{\frac{3\pi}{4}}(P)$	$\mathsf{Rot}_{\pi}(P)$	$\mathsf{Rot}_{\frac{5\pi}{4}}(P)$	$\mathsf{Rot}_{\frac{3\pi}{2}}(P)$	$\mathsf{Rot}_{\frac{7\pi}{4}}(P)$
$(1,0,0)$	$(1,0,0)$	$(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0)$	$(0,-1,0)$	$(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0)$	$(-1,0,0)$	$(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0)$	$(0,1,0)$	$(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0)$
$(0,1,0)$	$(0,1,0)$	$(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0)$	$(1,0,0)$	$(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0)$	$(0,-1,0)$	$(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0)$	$(-1,0,0)$	$(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0)$
$(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0)$	$(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0)$	$(0,-1,0)$	$(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0)$	$(-1,0,0)$	$(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0)$	$(0,1,0)$	$(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0)$	$(1,0,0)$
$(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})$	$(\frac{1}{2},-\frac{1}{2},\frac{1}{\sqrt{2}})$	$(0,-\frac{\sqrt{2}}{2},\frac{1}{\sqrt{2}})$	$(-\frac{1}{2},-\frac{1}{2},\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})$	$(-\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}})$	$(0,\frac{\sqrt{2}}{2},\frac{1}{\sqrt{2}})$	$(\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}})$
$(0,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$