Three Tests for an Exact Differential Form

A Penetrating Example

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We will investigate the properties of the 1-form ω on the simply connected domain $\mathbb{R}^2$:

$$\omega = \underbrace{(2xy^3 - \sin(x))}_{P(x,y)} \underset{\cdot}{\mathrm{x}} + \underbrace{(3x^2y^2)}_{Q(x,y)} \underset{\cdot}{\mathrm{y}}$$

We will apply three distinct tests to determine if ω is exact (i.e., if it is the differential of some potential function $f(x, y)$).

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1 Test 1: Equality of Mixed Partials (The Local Test)

Purpose

This is the fastest diagnostic check. Its purpose is to answer the **local** question: "Does this field have zero curl at every point?" It's a rapid disqualifier—if this test fails, the form is not exact, and we can stop. On a simply connected domain like $\mathbb{R}^2$, this test is also sufficient to prove exactness.

Application

We must check if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

$$P(x, y) = 2xy^3 - \sin(x) \implies \frac{\partial P}{\partial y} = 2x \cdot (3y^2) = \mathbf{6xy^2}$$

$$Q(x, y) = 3x^2y^2 \implies \frac{\partial Q}{\partial x} = 3y^2 \cdot (2x) = \mathbf{6xy^2}$$

Since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, the condition is met. The form ω is **closed**. Because the domain is simply connected, we conclude it is also **exact**.

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2 Test 2: Path Independence (The Global Test)

Purpose

This test verifies the fundamental physical meaning of a conservative field. Its purpose is to answer the **global** question: "Is the work done between two points independent of the path taken?" This confirms the field is globally coherent.

Application

Let's calculate the line integral of ω from point $A = (0,0)$ to point $B = (1,2)$ along two different paths. If the form is exact, the results must be identical.

Path γ_1: The "City Block" Path

Move from $(0,0) \to (1,0)$ and then from $(1,0) \to (1,2)$.

- **Segment 1:** $(0,0) \to (1,0)$**.** Here, $y = 0$ and $\underset{\cdot}{\mathrm{y}} = 0$.

$$\int_{\text{seg1}} \omega = \int_0^1 (2x(0)^3 - \sin(x))\,\underset{\cdot}{\mathrm{x}} = \int_0^1 -\sin(x)\,\underset{\cdot}{\mathrm{x}} = [\cos(x)]_0^1 = \cos(1) - 1$$

- **Segment 2:** $(1,0) \to (1,2)$**.** Here, $x = 1$ and $\underset{\cdot}{\mathrm{x}} = 0$.

$$\int_{\text{seg2}} \omega = \int_0^2 (3(1)^2 y^2)\,\underset{\cdot}{\mathrm{y}} = \int_0^2 3y^2\,\underset{\cdot}{\mathrm{y}} = [y^3]_0^2 = 8$$

The total integral for path γ_1 is $(\cos(1) - 1) + 8 = \mathbf{7} + \cos(\mathbf{1})$.

Path γ_2: The Direct Straight Line

Parameterize the line as $\mathbf{r}(t) = \langle t, 2t \rangle$ for $t \in [0,1]$. This gives $x = t, \underset{\cdot}{\mathrm{x}} = \underset{\cdot}{\mathrm{t}}$ and $y = 2t, \underset{\cdot}{\mathrm{y}} = 2\,\underset{\cdot}{\mathrm{t}}$.

$$\begin{aligned}
\int_{\gamma_2} \omega &= \int_0^1 \left[(2(t)(2t)^3 - \sin(t))\,\underset{\cdot}{\mathrm{t}} + (3(t)^2(2t)^2)(2\,\underset{\cdot}{\mathrm{t}})\right] \\
&= \int_0^1 \left((16t^4 - \sin(t)) + 24t^4\right)\,\underset{\cdot}{\mathrm{t}} \\
&= \int_0^1 (40t^4 - \sin(t))\,\underset{\cdot}{\mathrm{t}} \\
&= [8t^5 + \cos(t)]_0^1 \\
&= (8 + \cos(1)) - (0 + \cos(0)) = \mathbf{7} + \cos(\mathbf{1})
\end{aligned}$$

The results are identical, demonstrating path independence and confirming the form is **exact**.

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3 Test 3: Potential Recovery (The Constructive Test)

Purpose

This is the ultimate proof by construction. Its purpose is to answer the question: "If a potential function exists, what is it?" This is the most practical test, as it yields the potential function f needed for applications, such as using the Fundamental Theorem of Line Integrals.

Application

We construct $f(x, y)$ such that $\underset{.}{\mathrm{f}} = \omega$.

1. **Integrate $P(x, y)$ with respect to x:**

$$\begin{aligned} f(x, y) &= \int P(x, y) \, \underset{.}{\mathrm{x}} + h(y) \\ &= \int (2xy^3 - \sin(x)) \, \underset{.}{\mathrm{x}} + h(y) \\ &= x^2y^3 + \cos(x) + h(y) \end{aligned}$$

   Here, $h(y)$ is an unknown function of y that acts as the constant of integration.

2. **Differentiate with respect to y and set equal to $Q(x, y)$:**

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y^3 + \cos(x) + h(y)) = 3x^2y^2 + h'(y)$$

   We know this must equal $Q(x, y) = 3x^2y^2$.

$$3x^2y^2 + h'(y) = 3x^2y^2$$

3. **Solve for $h(y)$:** The equation simplifies to $h'(y) = 0$. Integrating gives $h(y) = C$, a constant. We can choose $C = 0$ for simplicity.

We have successfully constructed the potential function:

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{x^2y^3} + \cos(\mathbf{x})$$

Finding an explicit potential function is the definitive proof that ω is **exact**.