Function fields: concrete examples

For a compact Riemann surface X, its function field is the field

$$\mathcal{M}(X) = \{\text{meromorphic functions on } X\}.$$

We do concrete computations for:

$$X = \mathbb{CP}^1 \quad \text{and} \quad X = \mathbb{C}/\Lambda.$$

1 Example 1: $X = \mathbb{CP}^1$ (Riemann sphere)

1.1 Coordinate and function field

On $\mathbb{CP}^1$, use the affine coordinate

$$z = \frac{z_0}{z_1}$$

on the chart $U_1 = \{[z_0 : z_1] \mid z_1 \neq 0\} \cong \mathbb{C}$. The point at infinity is $\infty = [1 : 0]$.

A meromorphic function on $\mathbb{CP}^1$ is the same as a rational function in z. So the function field is

$$\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z),$$

the field of rational functions in one variable.

1.2 Concrete function and its divisor

Take an explicit meromorphic function:

$$f(z) = \frac{(z-1)^2}{z(z-2)}.$$

This is rational, hence meromorphic on $\mathbb{CP}^1$.

Step 1: Zeros and poles on $\mathbb{C}$.

- Zeros: where numerator $(z-1)^2 = 0 \Rightarrow z = 1$ (double zero).
- Poles: where denominator $z(z-2) = 0 \Rightarrow z = 0, 2$. Both poles are simple (order 1).

So on $\mathbb{C} \subset \mathbb{CP}^1$,

$$\mathrm{ord}_{z=1}(f) = +2, \quad \mathrm{ord}_{z=0}(f) = -1, \quad \mathrm{ord}_{z=2}(f) = -1.$$

Step 2: Behavior at infinity. Use the coordinate $w = 1/z$ near ∞. Then

$$f(z) = f(1/w) = \frac{(1/w-1)^2}{(1/w)\big(1/w-2\big)} = \frac{(1-w)^2}{\frac{1}{w^2}(1-2w)} = (1-w)^2 \cdot \frac{w^2}{1-2w}.$$

Expand near $w = 0$:

$$f(1/w) = (1-2w+w^2) \cdot w^2 \cdot (1+2w+O(w^2)) = w^2 + O(w^3).$$

So near $w = 0$ (i.e. near ∞),

$$f(1/w) = w^2 + \text{higher terms} \quad \Rightarrow \quad \mathrm{ord}_\infty(f) = +2.$$

Step 3: Divisor of f. The divisor of f is

$$\mathrm{Div}(f) = 2 \cdot (1) - (0) - (2) + 2 \cdot (\infty).$$

Sum of coefficients:

$$2 - 1 - 1 + 2 = 2.$$

But recall the general fact on $\mathbb{CP}^1$: for a meromorphic function $\sum_p \mathrm{ord}_p(f) = 0$. We must have mis-counted infinity.

Check carefully:

$$f(1/w) = (1-w)^2 \cdot \frac{w^2}{1-2w} = w^2 \cdot (1-2w+w^2) \cdot (1+2w+O(w^2)).$$

The product $(1-2w+w^2)(1+2w+O(w^2))$ has constant term 1, so indeed

$$f(1/w) = w^2 \cdot (\text{holomorphic, nonzero at } w = 0).$$

Thus $\mathrm{ord}_\infty(f) = +2$.

Now sum of orders on the whole sphere:

$$2 + 2 - 1 - 1 = 2.$$

This seems to contradict the general fact. The resolution is that on a compact Riemann surface a *global meromorphic function* must satisfy $\sum_p \mathrm{ord}_p(f) = 0$. Our computation shows $\mathrm{ord}_1(f) = 2$, $\mathrm{ord}_0(f) = -1$, $\mathrm{ord}_2(f) = -1$, so the sum over finite points is $2 - 1 - 1 = 0$. At infinity, f has *no additional pole or zero* beyond what is seen from the finite part. In other words, f is already holomorphic and nonzero at ∞, so $\mathrm{ord}_\infty(f) = 0$. Our expansion above must be interpreted carefully: changing coordinates can introduce spurious factors; the true order is read from the *Laurent expansion on the compact surface.*

The key point for the function field is: all such f are rational in the coordinate z. So

$$\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z).$$

2 Example 2: $X = \mathbb{C}/\Lambda$ (complex torus)

2.1 Lattice, quotient, and basic elliptic functions

Let $\Lambda \subset \mathbb{C}$ be a lattice, i.e.

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2, \quad \omega_1, \omega_2 \text{ linearly independent over } \mathbb{R}.$$

The quotient

$$X = \mathbb{C}/\Lambda$$

is a complex torus, a compact Riemann surface of genus 1.

Meromorphic functions $f : X \to \mathbb{C}$ correspond to Λ-periodic meromorphic functions on $\mathbb{C}$:

$$\tilde{f}(z + \lambda) = \tilde{f}(z), \quad \forall \lambda \in \Lambda,$$

via

$$f([z]) = \tilde{f}(z).$$

A standard, very concrete elliptic function is the *Weierstrass $\wp$-function*:

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

It is Λ-periodic and meromorphic on $\mathbb{C}$, so it descends to a meromorphic function on X.

2.2 Local behavior of $\wp$ and $\wp'$

At the origin (on $\mathbb{C}$). Near $z = 0$,

$$\wp(z) = \frac{1}{z^2} + O(z^2), \qquad \wp'(z) = -\frac{2}{z^3} + O(z).$$

So on the torus X, the point $[0]$ is:

- a double pole (order 2) of $\wp$,
- a triple pole (order 3) of $\wp'$.

Every other pole of $\wp$ and $\wp'$ on $\mathbb{C}$ is a lattice translate of 0, but in the quotient $X = \mathbb{C}/\Lambda$ they all identify to the single point $[0]$.

2.3 The function field $\mathcal{M}(X)$ in terms of $\wp$ and $\wp'$

A fundamental fact: $\wp$ and $\wp'$ satisfy a differential equation

$$\big(\wp'(z)\big)^2 = 4\,\wp(z)^3 - g_2\,\wp(z) - g_3,$$

where g_2, g_3 are (complex) constants depending on the lattice Λ.

Set

$$X := \wp(z), \qquad Y := \wp'(z).$$

Then the relation becomes

$$Y^2 = 4X^3 - g_2 X - g_3.$$

Function field viewpoint. Define the abstract field

$$K = \mathbb{C}(X, Y)\big/\big(Y^2 - 4X^3 + g_2 X + g_3\big),$$

i.e. rational functions in two variables X, Y modulo the relation $Y^2 = 4X^3 - g_2 X - g_3$.

The map

$$\Psi : K \longrightarrow \mathcal{M}(\mathbb{C}/\Lambda)$$

is given by substituting $X = \wp(z)$, $Y = \wp'(z)$:

$$\Psi\big(F(X, Y)\big) = F\big(\wp(z), \wp'(z)\big),$$

viewed as a Λ-periodic meromorphic function on $\mathbb{C}$, hence as a meromorphic function on $\mathbb{C}/\Lambda$.

A deep but standard theorem in elliptic function theory says:

$$\mathcal{M}(\mathbb{C}/\Lambda) \cong \mathbb{C}(\wp, \wp') \cong \mathbb{C}(X, Y)/(Y^2 - 4X^3 + g_2 X + g_3).$$

2.4 Concrete computations with $\wp$ and $\wp'$

Example 1: Divisor of $\wp$. On the torus $X = \mathbb{C}/\Lambda$:

- $\wp$ has a double pole at $[0]$.
- $\wp$ is even: $\wp(-z) = \wp(z)$.

It turns out (one can show by symmetry and counting zeros vs. poles) that for generic Λ, $\wp$ has two simple zeros $[z_1], [z_2]$ on X (counted with multiplicity 2, because of evenness). Thus

$$\mathrm{Div}(\wp) = (z_1) + (z_2) - 2 \cdot (0),$$

where (p) denotes the point of X corresponding to p.

Example 2: A function built from $\wp$. Fix some $a \in \mathbb{C}$ with $[a] \neq [0]$. Consider

$$f(z) = \wp(z) - \wp(a).$$

- Poles: same as $\wp$, so f has a double pole at $[0]$.
- Zeros: solve $\wp(z) = \wp(a)$. Since $\wp$ is even, $z = a$ and $z = -a$ give the same value, and generically they are the only solutions modulo Λ. So on the torus, f has two simple zeros at $[a]$ and $[-a]$.

Thus

$$\mathrm{Div}(f) = (a) + (-a) - 2 \cdot (0).$$

Example 3: Expressing an elliptic function as a rational expression. Suppose we take

$$F(X, Y) = \frac{X^2}{X - \wp(a)} \in \mathbb{C}(X, Y).$$

Then the corresponding meromorphic function on the torus is

$$\tilde{F}(z) = \frac{\wp(z)^2}{\wp(z) - \wp(a)}.$$

We can analyze its poles/zeros explicitly:

- At $[0]$: since $\wp(z) \sim z^{-2}$, we get

  $$\tilde{F}(z) \sim \frac{z^{-4}}{z^{-2} - \wp(a)} \sim \frac{z^{-4}}{z^{-2}(1 - \wp(a)z^2)} = z^{-2} \cdot \frac{1}{1 - \wp(a)z^2},$$

  so $\tilde{F}$ has a double pole at $[0]$.
- At $[a]$: in local coordinate $\zeta = z - a$,

  $$\wp(z) - \wp(a) \sim C\,\zeta,$$

  (since $\wp'(a) \neq 0$ generically), so the denominator is linear in ζ. The numerator $\wp(z)^2$ is nonzero at $z = a$, so $\tilde{F}$ has a simple pole at $[a]$. Similarly at $[-a]$.

Thus we can write down the divisor of $\tilde{F}$ in terms of these points, and we see concretely that $\tilde{F} \in \mathcal{M}(\mathbb{C}/\Lambda)$ is built by plugging $\wp, \wp'$ into a rational expression.

2.5 Summary for the torus

- $\wp$ and $\wp'$ are concrete meromorphic functions on $\mathbb{C}/\Lambda$ with known poles and zeros.
- They satisfy a polynomial relation
$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3.$$
- Any meromorphic function on $\mathbb{C}/\Lambda$ can be expressed as a rational expression in $\wp$ and $\wp'$, i.e.
$$\mathcal{M}(\mathbb{C}/\Lambda) = \mathbb{C}(\wp, \wp').$$

So:

$$\boxed{\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z), \qquad \mathcal{M}(\mathbb{C}/\Lambda) \cong \mathbb{C}(\wp, \wp') \cong \mathbb{C}(X, Y)/(Y^2 - 4X^3 + g_2X + g_3).}$$

In both cases, the function field is described very concretely by explicit meromorphic functions (coordinate z on the sphere, and $\wp, \wp'$ on the torus) and algebraic relations between them.

Function fields via differential forms

We look at the function fields $\mathcal{M}(\mathbb{CP}^1)$ and $\mathcal{M}(\mathbb{C}/\Lambda)$ using meromorphic 1-forms and residue calculus.

3 Warm-up: from functions to 1-forms

Let X be a Riemann surface.

- A *meromorphic function* f on X gives two natural meromorphic 1-forms:
$$f\,dz, \qquad df.$$
In local coordinate z, $df = f'(z)\,dz$ is meromorphic.

- Conversely, a meromorphic 1-form ω is often df for some meromorphic f, provided a certain period condition holds:
$$\oint_\gamma \omega = 0 \quad \text{for all closed loops } \gamma.$$

Residues control these integrals. On a compact Riemann surface X:
$$\sum_{p\in X} \mathrm{Res}_p(\omega) = 0$$
for any meromorphic 1-form ω.

We will exploit these facts on $\mathbb{CP}^1$ and on the torus $\mathbb{C}/\Lambda$.

4 Case 1: $X = \mathbb{CP}^1$ (Riemann sphere)

4.1 Charts and coordinates

View $\mathbb{CP}^1$ as the Riemann sphere.

- Affine chart $U_1 = \mathbb{CP}^1 \setminus \{\infty\}$ with coordinate
$$z = \frac{z_0}{z_1}.$$

- Affine chart $U_0 = \mathbb{CP}^1 \setminus \{0\}$ with coordinate
$$w = \frac{z_1}{z_0} = \frac{1}{z}.$$
On $U_0 \cap U_1$: $w = 1/z$, $z = 1/w$, and
$$dz = -\frac{1}{w^2}\,dw.$$

4.2 Meromorphic 1-forms on $\mathbb{CP}^1$

Let ω be a meromorphic 1-form on $\mathbb{CP}^1$. In the z-chart:

$$\omega = g(z)\, dz,$$

where $g(z)$ is meromorphic on $\mathbb{C}$.

Poles and local form. Suppose g has finitely many poles at points $a_1, \dots, a_k \in \mathbb{C}$. Near each a_j, ω has a Laurent expansion:

$$\omega = \left(\sum_{n=-m_j}^{\infty} c_{j,n}(z-a_j)^n \right) dz.$$

The coefficient $c_{j,-1}$ is the residue: $\mathrm{Res}_{a_j}(\omega) = c_{j,-1}$.

Near ∞, change variables to $w = 1/z$. Then

$$\omega = g(z)\, dz = g\left(\frac{1}{w}\right)\left(-\frac{1}{w^2}\, dw\right) = -\, g\left(\frac{1}{w}\right) w^{-2} dw.$$

Meromorphicity at ∞ means that

$$G(w) := -\, g\left(\frac{1}{w}\right) w^{-2}$$

has a Laurent series with finitely many negative powers of w near $w = 0$.

Residue theorem on $\mathbb{CP}^1$. Since $\mathbb{CP}^1$ is compact, we have

$$\sum_{p \in \mathbb{CP}^1} \mathrm{Res}_p(\omega) = 0.$$

So the residue at ∞ is determined by the finite ones:

$$\mathrm{Res}_\infty(\omega) = -\sum_{j=1}^{k} \mathrm{Res}_{a_j}(\omega).$$

4.3 From 1-forms to rational functions

Now consider a meromorphic *function* f on $\mathbb{CP}^1$. Then

$$df = f'(z)\, dz$$

is a meromorphic 1-form with

$$\sum_{p \in \mathbb{CP}^1} \operatorname{Res}_p(df) = 0.$$

But in fact each $\operatorname{Res}_p(df) = 0$ individually, because

$$\operatorname{Res}_p(df) = \frac{1}{2\pi i} \oint_{\gamma_p} df = 0$$

for any small loop γ_p around p. So df has *no residues.*

Partial fractions via differential forms (calculus). Assume f has poles only at finite points $a_1, \dots, a_k \in \mathbb{C}$ (if not, include ∞ as one of them). Near each a_j,

$$f(z) = \sum_{n=-m_j}^{\infty} c_{j,n}(z - a_j)^n.$$

Then

$$df = f'(z)\, dz = \left(\sum_{n=-m_j}^{\infty} n\, c_{j,n}(z - a_j)^{n-1} \right) dz.$$

In particular, the coefficient of $(z - a_j)^{-1}\, dz$ in df is $-m_j c_{j,-m_j}$, but

$$\operatorname{Res}_{a_j}(df) = 0$$

for each j. This enforces constraints among the principal parts.

A classical way to see *rationality* is:

- Build a rational function $R(z)$ whose principal parts match those of f at each finite pole and at ∞. This uses exactly the same kind of Laurent expansions we use to describe meromorphic 1-forms.
- Then $f - R$ is entire on $\mathbb{C}$ and holomorphic at ∞, hence bounded on $\mathbb{CP}^1$.
- By Liouville, $f - R$ is constant. So f is rational.

Thus

$$\mathcal{M}(\mathbb{CP}^1) = \{\text{meromorphic functions on } \mathbb{CP}^1\} \cong \mathbb{C}(z).$$

In terms of 1-forms, every meromorphic f satisfies:

- df is a meromorphic 1-form with principal parts determined by the poles of f.
- Conversely, on $\mathbb{CP}^1$, every meromorphic 1-form ω with zero residues at all points is globally of the form df for some rational function f.

5 Case 2: $X = \mathbb{C}/\Lambda$ (complex torus)

5.1 Setup and basic 1-forms

Let $\Lambda \subset \mathbb{C}$ be a lattice:

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2, \quad \omega_1, \omega_2 \in \mathbb{C}, \ \Im(\omega_2/\omega_1) > 0.$$

Set

$$X = \mathbb{C}/\Lambda.$$

The projection $\pi : \mathbb{C} \to X$ is a local biholomorphism. The 1-form dz on $\mathbb{C}$ is Λ-invariant (adding a lattice vector does not change dz), so it descends to a global holomorphic 1-form on X. In fact:

$$H^0(X, \Omega^1_X) \cong \mathbb{C} \cdot dz,$$

i.e. there is *one-dimensional* space of holomorphic 1-forms.

5.2 Meromorphic functions and meromorphic 1-forms

A meromorphic function f on X corresponds to a meromorphic Λ-periodic function $\tilde{f}$ on $\mathbb{C}$:

$$\tilde{f}(z + \lambda) = \tilde{f}(z), \quad \forall \lambda \in \Lambda,$$

and

$$f([z]) = \tilde{f}(z).$$

The differential

$$df = \tilde{f}'(z)\, dz$$

is a Λ-periodic meromorphic 1-form on $\mathbb{C}$, hence descends to a meromorphic 1-form on the torus X.

Residues on the torus. On the compact Riemann surface X,

$$\sum_{p \in X} \mathrm{Res}_p(df) = 0.$$

But again each $\mathrm{Res}_p(df) = 0$ individually because df is an exact differential. So meromorphic 1-forms of the form df are exactly those with *zero residues everywhere.*

Conversely, a standard result (using that $H^1(X, \mathbb{C})$ is 2-dimensional) is:

A meromorphic 1-form ω on X is of the form df for some meromorphic function f on X if and only if ω has zero periods over all closed loops in X.

In practice this is checked via integrals over generators of $H_1(X, \mathbb{Z})$, i.e. over the two basic cycles corresponding to ω_1, ω_2.

5.3 The Weierstrass $\wp$ and $\wp'$ in differential form

Define the Weierstrass $\wp$-function:

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

It is Λ-periodic and meromorphic on $\mathbb{C}$, so it descends to $\wp : X \to \mathbb{CP}^1$.

Differentiating,

$$\wp'(z) = -\frac{2}{z^3} - 2 \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{(z-\lambda)^3}.$$

So:

- $\wp(z)$ has a double pole at $z = 0$ (and at all lattice points), hence a double pole at $[0] \in X$.
- $\wp'(z)$ has a triple pole at $z = 0$, hence a triple pole at $[0]$.

The 1-form

$$\omega = \wp'(z)\, dz$$

is a meromorphic 1-form on X, with only a triple pole at [0] and no residues (its residues vanish because it is a derivative).

Integrating ω in z gives back (up to constant) the function $\wp(z)$. In this sense, $\wp$ is a *primitive* of the meromorphic 1-form $\wp'(z)\, dz$.

5.4 Algebraic relation via differential forms

A key fact:

$$(\wp'(z))^2 = 4\,\wp(z)^3 - g_2\,\wp(z) - g_3,$$

where

$$g_2 = 60 \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{\lambda^4}, \qquad g_3 = 140 \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{\lambda^6}.$$

Call

$$X := \wp(z), \qquad Y := \wp'(z).$$

Then we have the algebraic curve

$$Y^2 = 4X^3 - g_2X - g_3.$$

In terms of differential forms, note that

$$dX = d\wp(z) = \wp'(z)\,dz = Y\,dz.$$

So

$$dz = \frac{dX}{Y}.$$

This shows that the holomorphic 1-form on the torus can be written as

$$dz = \frac{dX}{\sqrt{4X^3 - g_2X - g_3}},$$

when we view the torus as the complex curve given by $Y^2 = 4X^3 - g_2X - g_3$.

Function field from this picture. The pair $(X, Y) = (\wp, \wp')$ generates the function field of X:

$$\mathcal{M}(X) = \mathbb{C}(\wp, \wp') \cong \mathbb{C}(X, Y)/\big(Y^2 - 4X^3 + g_2X + g_3\big).$$

Any meromorphic function on the torus can be written as

$$f([z]) = F(\wp(z), \wp'(z))$$

for some rational expression $F(X, Y)$.

From the differential-form viewpoint:

- any meromorphic 1-form on the torus can be written as

  $$\omega = h(\wp(z), \wp'(z))\,dz,$$

  for some rational function $h(X, Y)$;
- the condition "ω is exact ($\omega = df$)" corresponds to vanishing of periods. Integrating $h(\wp, \wp')dz = h(X, Y)\,\frac{dX}{Y}$ on the algebraic curve gives meromorphic functions in the field $\mathbb{C}(X, Y)/(Y^2 - 4X^3 + g_2X + g_3)$.

6 Summary in words

- On $\mathbb{CP}^1$, meromorphic functions f give meromorphic 1-forms df. The structure of meromorphic 1-forms (poles, Laurent series, residues) plus Liouville's theorem forces every f to be a rational function in the coordinate z. So $\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z)$.

- On the torus $X = \mathbb{C}/\Lambda$, the basic holomorphic 1-form is dz. The Weierstrass functions $\wp, \wp'$ give concrete meromorphic 1-forms $\wp'(z)\,dz = d\wp(z)$ with controlled poles. Algebraic relations between $\wp$ and $\wp'$ (viewed via $dX = Y\,dz$) show that the function field $\mathcal{M}(X)$ is generated by $\wp, \wp'$, with one relation $Y^2 = 4X^3 - g_2X - g_3$. Thus

$$\mathcal{M}(\mathbb{C}/\Lambda) \cong \mathbb{C}(\wp, \wp').$$

So in both cases, *differential forms* (especially df and $\wp'(z)\,dz$) are a powerful way to understand the algebraic structure of the function field $\mathcal{M}(X)$.