Advanced Calculus I

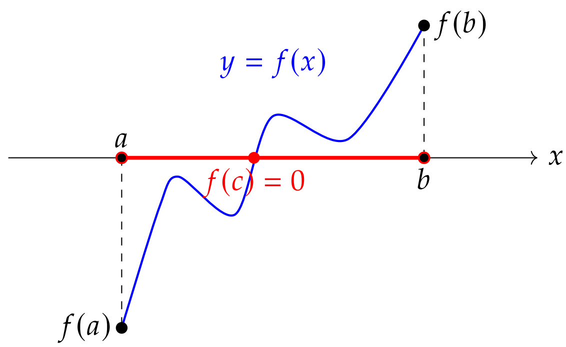
Ji, Yong-hyeon

March 31, 2025

We cover the following topics in this note.

- Boundedness, Supremum and Infimum
- Least Upper Bound Property (Completeness Axiom)
- Well-Ordering Principle and Mathematical Induction
- Archimedean Property

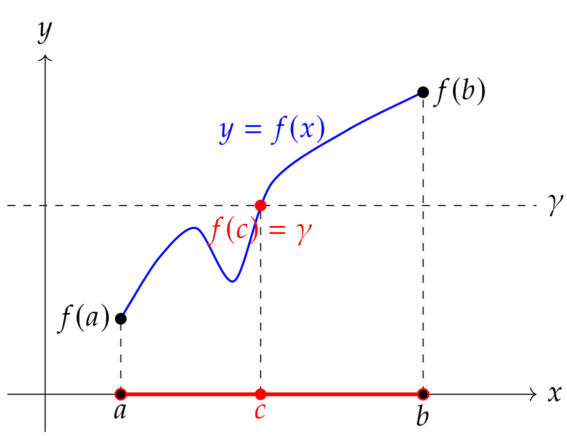
Observation. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Suppose that $f(a)$ and $f(b)$ have opposite signs, i.e., $f(a) \cdot f(b) < 0$. Then, there exists a point $c \in (a, b)$ such that $f(c) = 0$.



Intermediate Value Theorem

Theorem. *Let $[a, b] \subseteq \mathbb{R}$ be a real interval, and let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function on $[a, b]$. Let $f(a) < f(b)$. If $\gamma \in \mathbb{R}$ satisfies $f(a) < \gamma < f(b)$, then*

$$\exists c \in (a, b) \text{ such that } f(c) = \gamma.$$



Contents

1 Numbers

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$	**N**atural Numbers
$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \dots\}$	Integers (**Z**ahlen[1])
$\mathbb{Q} = \left\{\frac{q}{p} : p, q \in \mathbb{Z}, p \neq 0\right\}$	Rationals (**Q**uotient[2])
$\mathbb{R} = \left\{\text{Limit of sequences of rational numbers}\right\}$	**R**eal Numbers
$\mathbb{C} = \left\{p + q\sqrt{-1} : p, q \in \mathbb{R}\right\}$	**C**omplex numbers

Remark. The set $\mathbb{Z}_{\geq 0} := \{0\} \cup \mathbb{N} = \{0, 1, 2, \dots\}$ is called *non-negative integers*.

Remark. Let $n_0 \in \mathbb{Z}$ is given. Then

$$\mathbb{Z}_{\geq n_0} := \{n \in \mathbb{Z} : n \geq n_0\}.$$

[1]The integer set is denoted by $\mathbb{Z}$ because it comes from the German word "Zahlen", meaning "numbers".

[2]The rational set is denoted by $\mathbb{Q}$ because it stands for "Quotient", representing numbers that can be expressed as the quotient of two integers.

2 Least Upper Bound Property of $\mathbb{R}$

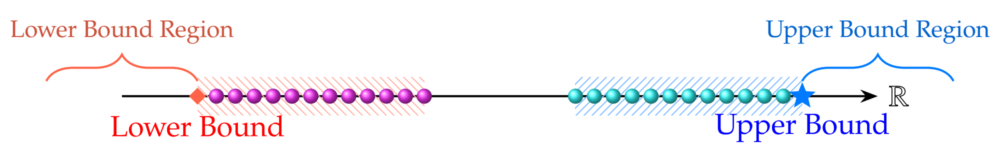
Boundedness

Definition. Let S be a non-empty subset of $\mathbb{R}$.

(1) A set S is said to be **bounded above** if $\boxed{\exists \beta \in \mathbb{R} \text{ such that for all } x \in S,\ x \leq \beta}$.
A real number $\beta \in \mathbb{R}$ is called an **upper bound** of S.

(2) A set S is said to be **bounded below** if $\boxed{\exists \alpha \in \mathbb{R} \text{ such that for all } x \in S,\ \alpha \leq x}$.
A real number $\alpha \in \mathbb{R}$ is called an **lower bound** of S.

(3) A set S is **bounded** if it is bounded above and below.



Remark (Caution!). It is not guaranteed that $\beta \in S$ and $\alpha \in S$.

Remark. Let $\varnothing \neq S \subseteq \mathbb{R}$, and let $\alpha, \beta \in \mathbb{R}$.

$$S \text{ is bounded above (by } \beta) \iff S \text{ has an upper bound } \beta$$
$$\beta \text{ is an upper bound of } S \iff \forall x \in S,\ x \leq \beta$$

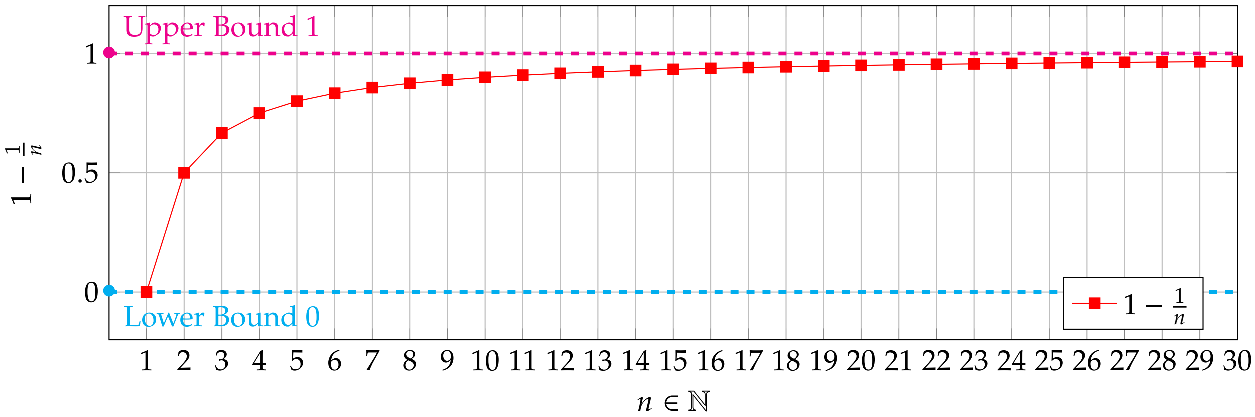
$$S \text{ is bounded below (by } \alpha) \iff S \text{ has an lower bound } \alpha$$
$$\alpha \text{ is an lower bound of } S \iff \forall x \in S,\ \alpha \leq x$$

Remark.

1. The empty $S = \varnothing$ is bounded.
   (i) ($\varnothing$ is bounded above) We need to find a real number $\beta \in \mathbb{R}$ s.t. for all $x \in \varnothing,\ x \leq \beta$. Since "$\forall x \in \varnothing, x \leq \beta$" is vacuously true, we can choose any real number as β.
   (ii) ($\varnothing$ is bounded below) Similarly, we can choose any $\alpha \in \mathbb{R}$ s.t. for all $x \in \varnothing$, $\alpha \leq x$.
2. An upper bound and a lower bound may not be unique. A set $S(\neq \varnothing) \subseteq \mathbb{R}$ may have multiple upper bounds and multiple lower bounds.

Exercise. Show that $A = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$ has an upper bound and a lower bound.

Sol.



□

Exercise. Show that $\mathbb{N}$ has a lower bound but does not have an upper bound.

Sol.

$\square$

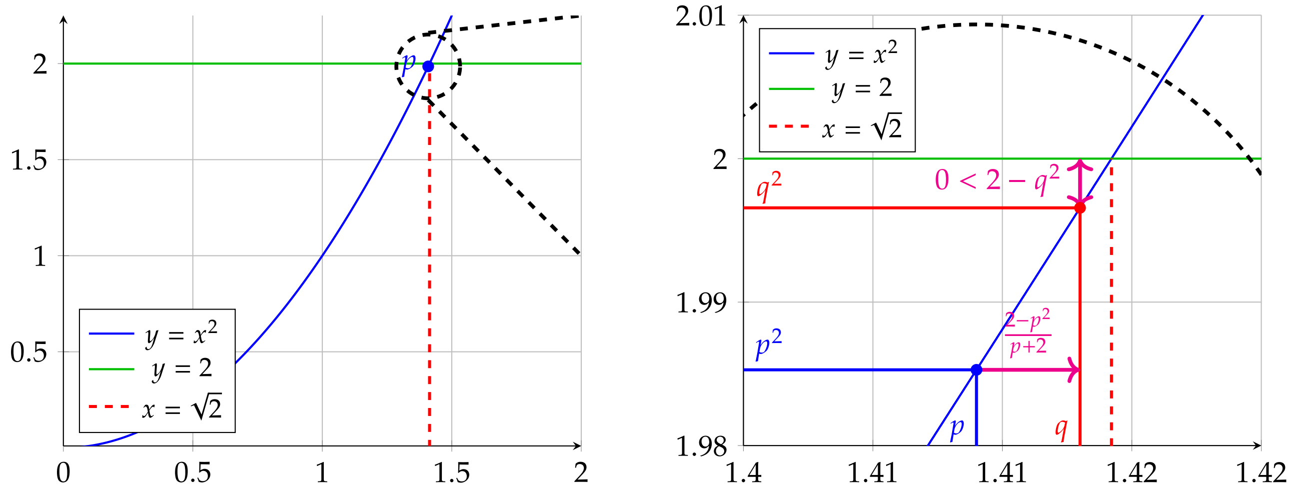
Exercise (⋆). Consider a set

$$A := \left\{ r \in \mathbb{Q} : r > 0,\ r^2 < 2 \right\}$$

of positive rational numbers whose squares are less than 2. Then A has a lower bound 0. Prove that A does not have the maximum element.

Sol. Note that

$$A := \left\{ r \in \mathbb{Q} : r > 0,\ r^2 < 2 \right\} = \{\cdots, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.4141213, 1.4142135, \cdots\}.$$



□

Note (Existence of $\sqrt{2}$)**.** There exists $x \in \mathbb{R}$ such that $x^2 = 2$. We write $x = \sqrt{2} > 0$.

Proof.

$\square$

Supremum and Infimum

Definition. Let $\varnothing \neq S \subseteq \mathbb{R}$.

(1) The number $\beta \in \mathbb{R}$ is the **supremum** (or the **least upper bound**) of S if and only if

  (i) β is an upper bound of S, i.e., $\forall x \in S,\ x \leq \beta$;

  (ii) u is any upper bound of $S \implies \beta \leq u$.

  We write $\boxed{\beta = \sup S} \in \mathbb{R}$.

(2) The number $\alpha \in \mathbb{R}$ is the **infimum** (or the **greatest lower bound**) of S if and only if

  (i) α is an lower bound of S, i.e., $\forall x \in S,\ \alpha \leq x$;

  (ii) if ℓ is any lower bound of S then $\ell \leq \alpha$.

  We write $\boxed{\alpha = \inf S} \in \mathbb{R}$.

Remark (Caution!). It is not guaranteed that $\sup S \in S$ and that $\inf S \in S$.

Remark. Let $\varnothing \neq S \subseteq \mathbb{R}$.

(1) $$\beta = \sup S \iff \begin{cases} \text{(i) } \forall x \in S, x \leq \beta; \\ \text{(ii) } \forall u \in \mathbb{R},\ [(\forall x \in S,\ x \leq u) \implies \beta \leq u]. \end{cases}$$

(2) $$\alpha = \inf S \iff \begin{cases} \text{(i) } \forall x \in S, \alpha \leq x; \\ \text{(ii) } \forall \ell \in \mathbb{R},\ [(\forall x \in S,\ \ell \leq x) \implies \ell \leq \alpha]. \end{cases}$$

Remark.

$$\begin{aligned} [u \text{ is any upper bound of } S \implies \beta \leq u] &\iff [u < \beta \implies u \text{ is NOT an upper bound of } S] \\ &\iff \beta \leq u \text{ for all upper bound } u \text{ of } S. \end{aligned}$$

$$\begin{aligned} [\ell \text{ is any lower bound of } S \implies \ell \leq \alpha] &\iff [\alpha < \ell \implies \ell \text{ is NOT an lower bound of } S] \\ &\iff \ell \leq \alpha \text{ for all lower bound } \ell \text{ of } S. \end{aligned}$$

Remark (Uniqueness of Supremum and Infimum).

(*Proof by Trichotomy*) Let $\varnothing \neq S \subseteq \mathbb{R}$ and S is bounded above. Suppose that $\sup S = a$ and that $\sup S = b$ also. By trichotomy, exactly one of the following holds:

$$a = b, \ a < b, \ \text{or} \ b < a.$$

However, $a < b$ and $b < a$ are impossible, as a and b are upper bounds, respectively. Hence $a = b$. Similarly, infimum also is unique. $\square$

(*Proof by Anti-symmetry*[3] *of* $\leq$) Let $\varnothing \neq T \subseteq \mathbb{R}$ and T is bounded above. Suppose that $\sup T = a$ and that $\sup T = b$ also. Then

(a) a is an upper bound of T in $\mathbb{R}$; (b) a is a supremum of T in $\mathbb{R}$;

(c) b is an upper bound of T in $\mathbb{R}$; (d) b is a supremum of T in $\mathbb{R}$.

Then

$$\text{(a) and (d)} \implies b \leq a, \qquad \text{(b) and (c)} \implies a \leq b.$$

By the anti-symmetry of $\leq$, we obtain $a = b$. Similarly, infimum also is unique. $\square$

Unbounded Sets

Definition. Let $\varnothing \neq S \subseteq \mathbb{R}$.

(1) If S is unbounded above, then we write $\boxed{\sup S = \infty}$.

(2) If S is unbounded below, then we write $\boxed{\inf S = -\infty}$.

(3) $\sup \varnothing := -\infty$ and $\inf \varnothing := \infty$.

Example. $\sup \mathbb{N} = \infty$ and $\inf \mathbb{Z} = -\infty$.

Remark. Suppose that $\varnothing \neq S \subseteq \mathbb{R}$ is unbounded above. Then

$$\neg[\exists \beta \in \mathbb{R} \text{ s.t. } \forall x \in S, \ x \leq \beta], \quad \text{i.e.,} \quad [\forall \beta \in \mathbb{R}, \ \exists x \in S \text{ s.t. } \beta < x].$$

Suppose that $\varnothing \neq T \subseteq \mathbb{R}$ is unbounded below. Then

$$\neg[\exists \alpha \in \mathbb{R} \text{ s.t. } \forall x \in T, \ \alpha \leq x], \quad \text{i.e.,} \quad [\forall \alpha \in \mathbb{R}, \ \exists x \in T \text{ s.t. } x < \alpha].$$

[3] A relation $\mathcal{R}$ on a set S is anti-symmetric if, for $a, b \in \mathcal{R}$, $a \,\mathcal{R}\, b \wedge b \,\mathcal{R}\, a \implies a = b$.

Approximation Property for Supremum and Infinum I

Proposition 1.

(1) Let $\varnothing \neq S \subseteq \mathbb{R}$ *which is bounded above, and let* λ *be an upper bound of* S *in* $\mathbb{R}$*.*

$$\lambda = \sup S \iff \forall \varepsilon > 0,\ \exists x_\varepsilon \in S \text{ s.t. } \lambda - \varepsilon < x_\varepsilon \leq \lambda.$$

(2) Let $\varnothing \neq T \subseteq \mathbb{R}$ *which is bounded below, and let* γ *be a lower bound of* T *in* $\mathbb{R}$*.*

$$\gamma = \inf T \iff \forall \varepsilon > 0,\ \exists x_\varepsilon \in T \text{ s.t. } \gamma \leq x_\varepsilon < \gamma + \varepsilon.$$

Proof.

□

Remark. See **Approximation Property for Supremum and Infinum II**.

Least Upper Bound Property (Completeness Axiom) of Real Number

Axiom. Every non-empty subset of $\mathbb{R}$ that is bounded above has the supremum in $\mathbb{R}$.

Example. $\mathbb{Q}$ does NOT hold completeness axiom. We already showed that $\{x \in \mathbb{Q} : x > 0,\ x^2 < 2\}$ has NO supremum in $\mathbb{Q}$.

Infimum Property

Axiom. Every non-empty subset of $\mathbb{R}$ that is bounded below has the infimum in $\mathbb{R}$.

3 Well-Ordering Principle and Mathematical Induction

Well-Ordering Principle (Principle of the Least Element)

Axiom. Every non-empty subset S of $\mathbb{N}$ has a least element, i.e.,

$$\varnothing \neq S \subseteq \mathbb{N} \implies \exists n \in S \text{ s.t. } \forall k \in S,\ n \leq k.$$

In other words, $[\varnothing \neq S \subseteq \mathbb{N} \Rightarrow \exists n \in S \text{ s.t. } n = \min(S)]$.

Remark (general version). $\varnothing \neq S \subseteq \mathbb{Z}_{\geq n_0} \implies \exists n \in S \text{ s.t. } n = \min S \geq n_0$.

Principle of Mathematical Induction

Axiom. Suppose that $S \subseteq \mathbb{N}$ satisfies the following two conditions:

1. (Basic Step) $1 \in S$, and
2. (Inductive Step) $n \in S \implies n+1 \in S$.

Then $S = \mathbb{N}$.

Remark (general version). Let $n_0 \in \mathbb{Z}$ be given, and let $S \subseteq \mathbb{Z}_{\geq n_0}$. Suppose that S satisfies the following two conditions:

1. (Basic Step) $n_0 \in S$, and
2. (Inductive Step) $\forall n \in \mathbb{Z}_{\geq n_0} : [n \in S \implies n+1 \in S]$.

Then $\forall n \in \mathbb{Z}_{\geq n_0} : n \in S$, i.e., $S = \mathbb{Z}_{\geq n_0}$.

Remark. To show that a mathematical statement $P(n)$ (property for n) holds for $n \in \mathbb{N}$, simply verify that the set

$$S := \{n \in \mathbb{N} : P(n) \text{ holds}\}$$

satisfies the following conditions:

(Step 1) Show that $P(1)$ holds.

(Step 2) Show that $P(n+1)$ holds with the assumption $P(n)$ holds.

Equivalence of Well-Ordering Principle and Induction

Theorem.

The Well-Ordering Principle and Principle of Mathematical Induction are equivalent.

Proof.

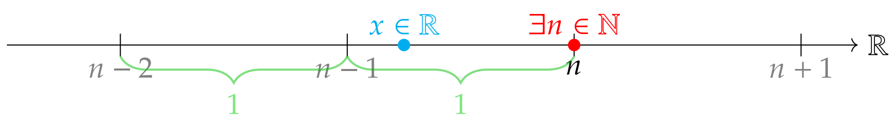
□

4 Archimedean Principle

Archimedean Property (The Unboundedness of Natural Numbers)

Theorem. *Let $x \in \mathbb{R}$. Then*

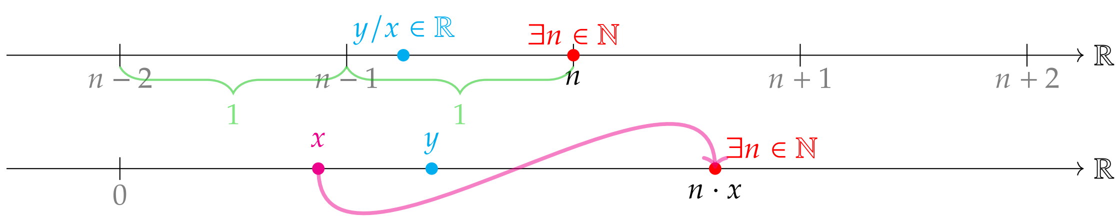
$$\exists n \in \mathbb{N} \text{ such that } x < n.$$



Proof.

□

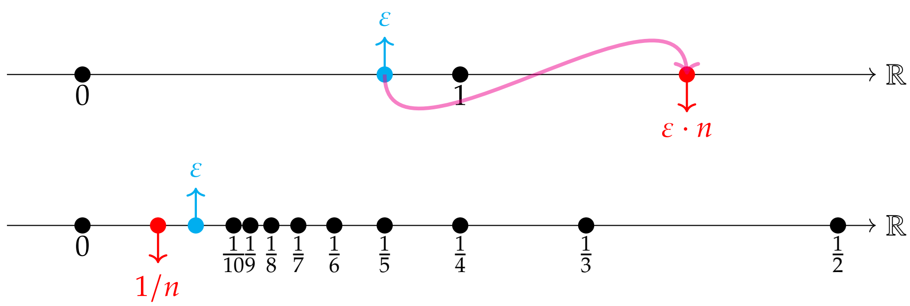
Corollary. *Let $x, y \in \mathbb{R}$ with $x > 0$. Then $\exists n \in \mathbb{N}$ such that $y < n \cdot x$.*



Proof.

□

Corollary. $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ *such that* $\frac{1}{n} < \varepsilon$.



Proof.

□

Note (Archimedean Property in Number Theory). Let $a, b \in \mathbb{N}$. Then $\exists n \in \mathbb{N}$ such that $b < na$.

Proof. Suppose that $\exists a, b \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, na \leq b.$$

Define a set S by

$$S := \{b - na \geq 0 : n \in \mathbb{N}\} \subseteq \mathbb{Z}_{\geq 0}.$$

By the well-ordering principle, $\exists s := \min S$. Since $s = \min S \in S$, we have

$$s = b - ma \text{ for some } m \in \mathbb{N}.$$

Since $m + 1 \in \mathbb{N}$ also, we have $b - (m+1)a \in S$, and so

$$\begin{aligned} b - (m+1)a &= b - ma - a \\ &< b - ma \quad \because a \in \mathbb{N}, \text{i.e., } a > 0 \\ &= \min S \text{ ↯}. \end{aligned}$$

Hence it is proved. □

Approximation Property for Supremum and Infinum II

Proposition 2.

(1) Let $\varnothing \neq S \subseteq \mathbb{R}$ which is bounded above, and let λ be an upper bound of S in $\mathbb{R}$.

$$\begin{aligned}\lambda = \sup S &\iff \forall \varepsilon > 0,\ \exists x_\varepsilon \in S \text{ s.t. } \lambda - \varepsilon < x_\varepsilon \leq \lambda \\ &\iff \forall n \in \mathbb{N},\ \exists x_n \in S \text{ s.t. } \lambda - \frac{1}{n} < x_n \leq \lambda.\end{aligned}$$

(2) Let $\varnothing \neq T \subseteq \mathbb{R}$ which is bounded below, and let γ be a lower bound of T in $\mathbb{R}$.

$$\begin{aligned}\gamma = \inf T \in \mathbb{R} &\iff \forall \varepsilon > 0,\ \exists x_\varepsilon \in T \text{ s.t. } \gamma \leq x_\varepsilon < \gamma + \varepsilon \\ &\iff \forall n \in \mathbb{N},\ \exists x_n \in T \text{ s.t. } \gamma \leq x_n < \gamma + \frac{1}{n}.\end{aligned}$$

Proof.

□

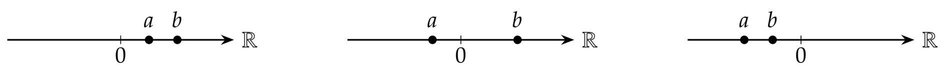
Remark. See **Approximation Property for Supremum and Infinum I**.

Density of the Rationals

Theorem. *Let* $a, b \in \mathbb{R}$.

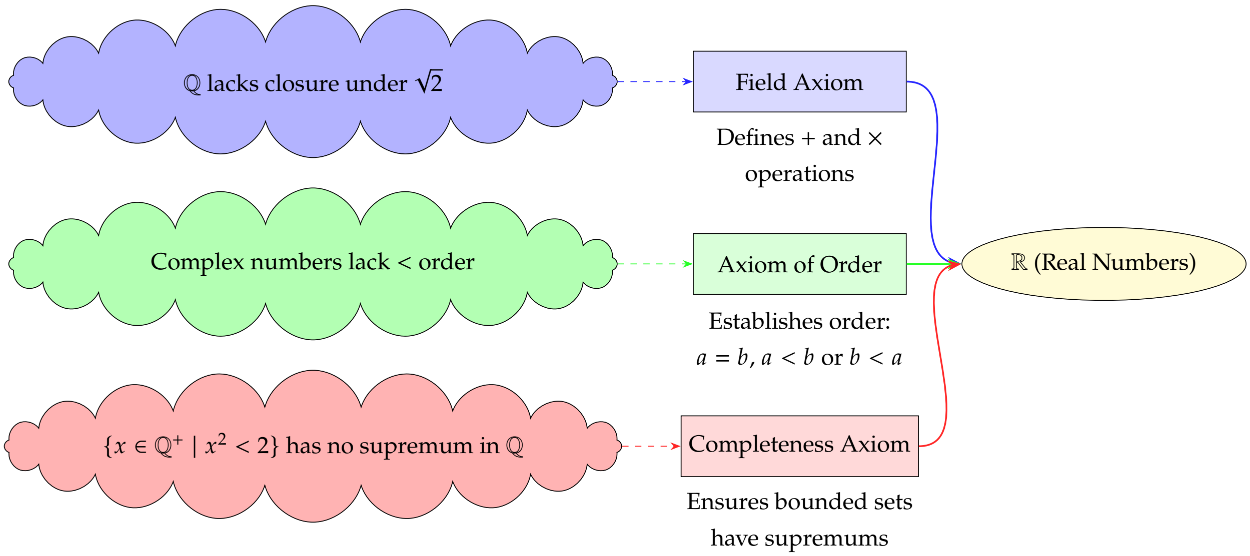
$$a < b \implies \exists q \in \mathbb{Q} \text{ such that } a < q < b.$$

Proof. Let $a, b \in \mathbb{R}$. Suppose that $a < b$.



□

A Axioms of the Real Numbers



The following axioms define the real numbers $\mathbb{R}$ as a complete ordered field.

A.1 Field Axioms

Addition:

1. **Closure under addition**: $\forall a, b \in \mathbb{R},\ a + b \in \mathbb{R}$
2. **Associativity of addition**:$\forall a, b, c \in \mathbb{R},\ (a + b) + c = a + (b + c)$
3. **Commutativity of addition**:$\forall a, b \in \mathbb{R},\ a + b = b + a$
4. **Existence of additive identity**: $\exists 0 \in \mathbb{R}$ such that $\forall a \in \mathbb{R},\ a + 0 = a$
5. **Existence of additive inverses**: $\forall a \in \mathbb{R},\ \exists -a \in \mathbb{R}$ such that $a + (-a) = 0$

Multiplication:

1. **Closure under multiplication**: $\forall a, b \in \mathbb{R},\ a \cdot b \in \mathbb{R}$
2. **Associativity of multiplication**: $\forall a, b, c \in \mathbb{R},\ (a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. **Commutativity of multiplication**: $\forall a, b \in \mathbb{R},\ a \cdot b = b \cdot a$
4. **Existence of multiplicative identity**: $\exists 1 \in \mathbb{R}, 1 \neq 0,$ such that $\forall a \in \mathbb{R},\ a \cdot 1 = a$
5. **Existence of multiplicative inverses**: $\forall a \in \mathbb{R}, a \neq 0,\ \exists a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = 1$

Distributive law:

1. **Distributivity of multiplication over addition**: $\forall a, b, c \in \mathbb{R},\ a \cdot (b + c) = a \cdot b + a \cdot c$

A.2 Axiom of Order

A relation $<$ defined on $\mathbb{R}$ satisfy the followings:

1. **Trichotomy**: For $a, b \in \mathbb{R}$, exactly one of the following holds:

$$a = b, \quad a < b, \quad \text{or} \quad b < a.$$

2. **Transitivity**: For $a, b, c \in \mathbb{R}$,

$$a < b \text{ and } b < c \implies a < c$$

3. **Additive compatibility**: For $a, b, c \in \mathbb{R}$,

$$a < b \implies a + c < b + c$$

4. **Multiplicative compatibility**: For $a, b \in \mathbb{R}$ and $c \in \mathbb{R}^+$,

$$a < b \implies a \cdot c < b \cdot c$$

A.3 Completeness Axiom

The least upper bound property (or **supremum property**):

$$\forall S \subseteq \mathbb{R},\ S \neq \emptyset,\ \text{if } S \text{ is bounded above then } \exists \sup(S) \in \mathbb{R}$$

B Application of Well-Ordering Principle

Theorem. $\sqrt{2}$ *is irrational*, i.e., $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

Proof. Suppose $\sqrt{2} \in \mathbb{Q}$. That is, $\exists p, q \in \mathbb{N}$ s.t. $p\sqrt{2} = q$. Define a set S by

$$S := \left\{ k\sqrt{2} \in \mathbb{N} : k \in \mathbb{N} \right\} \subseteq \mathbb{N}.$$

Since $p\sqrt{2} = q \in \mathbb{N}$, we have $S \neq \varnothing$. By the Well-Ordering Principle,

$$\exists s = \min(S) \in S.$$

Then $s = t\sqrt{2}$ for some $t \in \mathbb{N}$. Define a number

$$r := s\sqrt{2} - s.$$

(Claim 1) $r \in S$:

$$\begin{aligned} r &= s\sqrt{2} - s \\ &= s\sqrt{2} - t\sqrt{2} \\ &= (s-t)\sqrt{2} \\ &\in S && \because s = t\sqrt{2} > t \Rightarrow s - t > 0 \Rightarrow s - t \in \mathbb{N}. \end{aligned}$$

(Claim 2) $r < s = \min(S)$:

$$\begin{aligned} r &= s\sqrt{2} - s \\ &= s(\sqrt{2} - 1) \\ &< s = \min S && \because s \in \mathbb{N} \text{ and } 1 < \sqrt{2} < 2 \Rightarrow 0 < \sqrt{2} - 1 < 1. \end{aligned}$$

It is a contradiction. Hence $\sqrt{2} \notin \mathbb{Q}$. $\square$

C The 2nd Principle of Mathematical Induction

The 2nd Principle of Mathematical Induction

Theorem. *Suppose that $T \subseteq \mathbb{N}$ satisfies the following two conditions:*

1. *(Basic Step) $1 \in T$, and*
2. *(Inductive Step) $1, 2, \ldots n \in T \implies n + 1 \in T$.*

Then $T = \mathbb{N}$.

Proof. We use the first principle of mathematical induction. Define the set T' by

$$T' := \{n \in \mathbb{N} : 1, 2, \ldots, n \in T\} \subseteq \mathbb{N}.$$

For example, if $1, 2, 3 \in T$ then $3 \in T'$; conversely, if $3 \in T'$ then $1, 2, 3 \in T$. Since $n \in T' \Rightarrow n \in T$, we have $T' \subseteq T \subseteq \mathbb{N}$. We claim that T' satisfies the condition of MI:

(i) (Basic Step) Clearly $1 \in T'$.

(ii) (Inductive Step) Suppose that $k \in T'$. This means that $1, 2, \ldots k \in T$. By condition 2,

$$1, 2, \ldots, k, k + 1 \in T, \text{ i.e., } k + 1 \in T'.$$

Therefore by the first principle of mathematical induction, $T' = \mathbb{N}$. That is,

$$\mathbb{N} = T' \subseteq T \subseteq \mathbb{N} \implies T = \mathbb{N}.$$

Hence it is proved. □

Remark. To show that a mathematical statement $P(n)$ (property for n) holds for $n \in \mathbb{N}$, verify that the set

$$S := \{n \in \mathbb{N} : P(n) \text{ holds}\}$$

satisfies the following conditions:

(Step 1) Show that $P(1)$ holds.

(Step 2) Show that $P(n + 1)$ holds assuming $P(k)$ holds for all $k \leq n$.