Linear Algebra II

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We cover the following topics in this note.

Part I

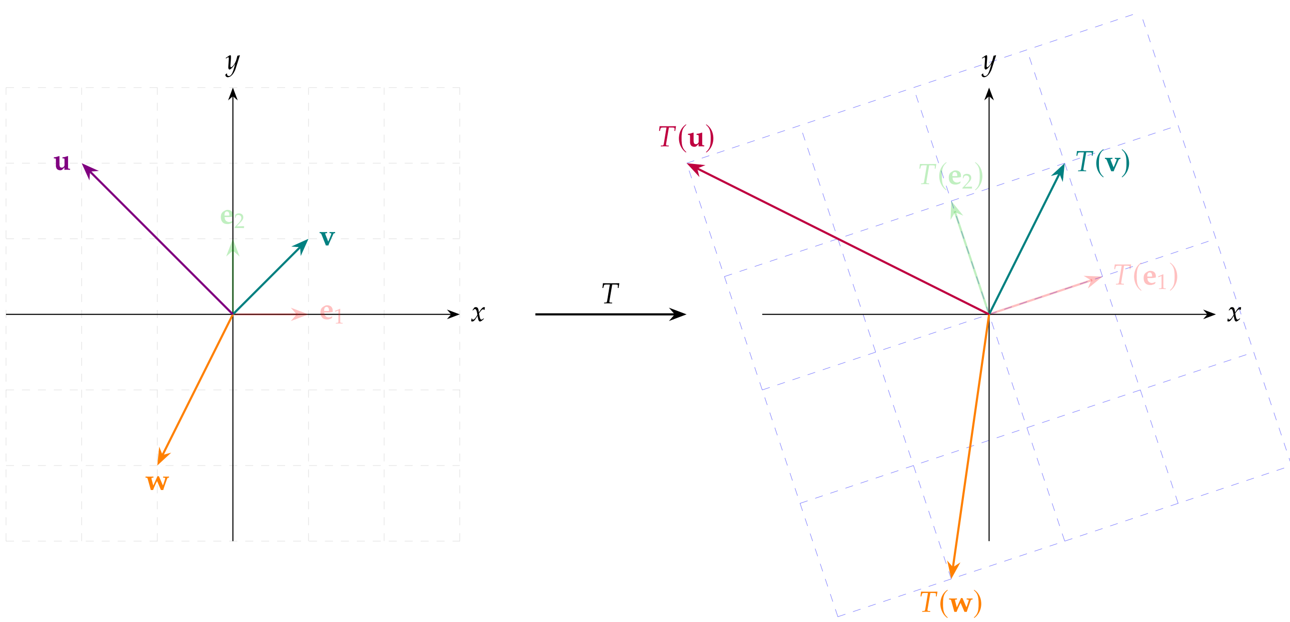
- Coordinate
- Linear Transformation
- Vector Space Isomorphism

Part 2

- Classification of Vector Space (up to Isomorphism)
- Matrix Representation of a Linear Transformation

Part 3

- TBA



1 Part I

Uniqueness of Representation with respect to a Basis

Proposition. *Let V be a vector space over a field $\mathbb{F}$ and let* $\dim V = n < \infty$. *Let*

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\} \subseteq V$$

be a basis of V. Then for every vector $\mathbf{v} \in V$ there exists a unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$ such that

$$\mathbf{v} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2 + \cdots + \alpha_n \mathbf{b}_n = \sum_{i=1}^{n} \alpha_i \, \mathbf{b}_i.$$

Proof. Suppose, for contradiction, that there exist two distinct representations of some vector $\mathbf{v} \in V$ in terms of the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$:

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \, \mathbf{b}_i \quad \text{and} \quad \mathbf{v} = \sum_{j=1}^{n} \beta_j \, \mathbf{b}_j,$$

where $\alpha_i, \beta_j \in \mathbb{F}$ for all i, j. Then

$$\sum_{i=1}^{n} \alpha_i \, \mathbf{b}_i - \sum_{j=1}^{n} \beta_j \, \mathbf{b}_j = \mathbf{0} \implies \sum_{i=1}^{n} (\alpha_i - \beta_i) \, \mathbf{b}_i = \mathbf{0}.$$

Since a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is linearly independent, we have

$$\alpha_i - \beta_i = 0, \quad \text{i.e.,} \quad \alpha_i = \beta_i$$

for all $i = 1, 2, \ldots, n$. Therefore, the representation of any $\mathbf{v} \in V$ as a finite linear combination of elements of the basis $\mathcal{B}$ is unique. □

Coordinate in a Finite-Dimensional Vector Space

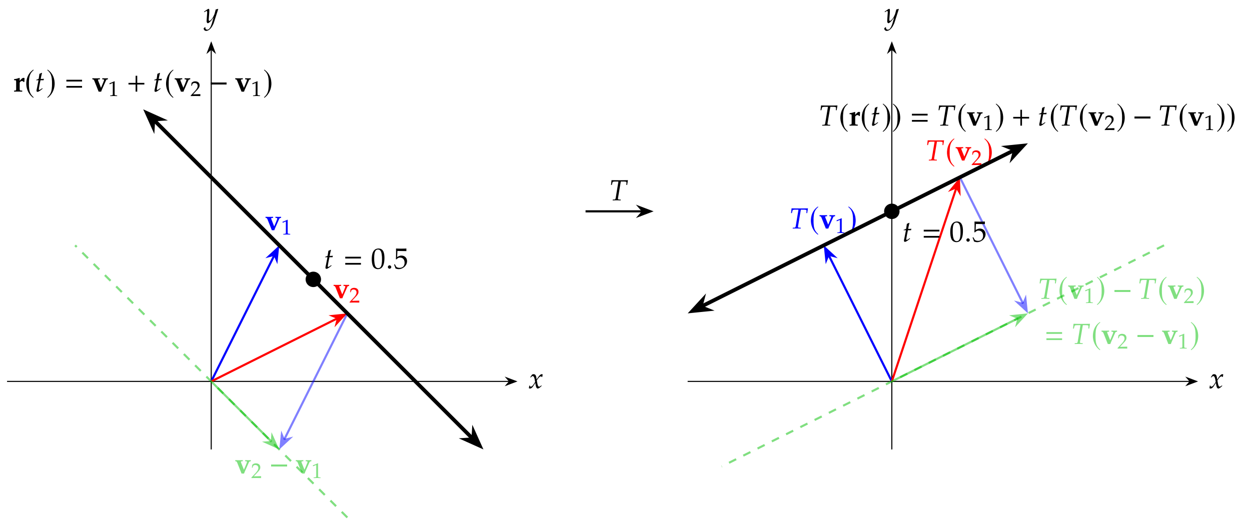
Definition. Let V be a vector space over a field $\mathbb{F}$ with $\dim V = n < \infty$, and let

$$\mathcal{B} = \{\mathbf{b}_i\}_{i=1}^{n} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$$

be a basis of V. The **coordinate of $\mathbf{v} \in V$ with respect to $\mathcal{B}$**, denoted by $[\mathbf{v}]_{\mathcal{B}}$, is the n-tuple

$$[\mathbf{v}]_{\mathcal{B}} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \quad \text{where } \mathbf{v} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2 + \cdots + \alpha_n \mathbf{b}_n.$$

Observation.



⋆ Linear Transformation ⋆

Definition. Let V and W be vector spaces over a field $\mathbb{F}$. A function

$$T : V \longrightarrow W$$

is called a **linear transformation** if for all vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$ and for all scalars $\alpha, \beta \in \mathbb{F}$, the following condition holds:

$$T(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) = \alpha T(\mathbf{v}_1) + \beta T(\mathbf{v}_2).$$

Remark. Equivalently, a function $T : V \longrightarrow W$ is linear if it satisfies

(i) (*Additivity*) For all $\mathbf{v}_1, \mathbf{v}_2 \in V$,

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2);$$

(ii) (*Homogeneity*) For all $\alpha \in \mathbb{F}$ and $\mathbf{v} \in V$,

$$T(\alpha \mathbf{v}) = \alpha T(\mathbf{v}).$$

Remark. This definition ensures T preserves the vector space structure of V in its image in W.

Vector Space Isomorphism

Definition. Let V and W be vector spaces over a field $\mathbb{F}$. A mapping

$$T : V \to W$$

is called a **vector space isomorphism** if it satisfies the following conditions:

(i) (*Linearity*) For any vectors $\mathbf{v}_2, \mathbf{v}_2 \in V$ and any scalars $\alpha, \beta \in \mathbb{F}$,

$$T(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = \alpha T(\mathbf{v}_1) + \beta T(\mathbf{v}_2).$$

(ii) (*Bijectivity*)

- (*Injectivity*) $\forall \mathbf{v}_1, \mathbf{v}_2 \in V, T(\mathbf{v}_1) = T(\mathbf{v}_2) \implies \mathbf{v}_1 = \mathbf{v}_2$;
- (*Surjectivity*) $\forall \mathbf{w} \in W, \exists \mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$.

The bijectivity of T guarantees the existence of an inverse mapping $T^{-1} : W \to V$, which satisfies

$$(\forall \mathbf{v} \in V,\ T^{-1}(T(\mathbf{v})) = \mathbf{v}), \quad \text{and} \quad (\forall \mathbf{w} \in W,\ T(T^{-1}(\mathbf{w})) = \mathbf{w}).$$

Remark. The inverse mapping $T^{-1} : W \to V$ is also a linear transformation.

Proof. Let $\mathbf{w}_1, \mathbf{w}_2 \in W$ and let $\alpha, \beta \in \mathbb{F}$. Since T is bijective, for each $\mathbf{w} \in W$, there exists a unique $\mathbf{v} \in V$ such that $\mathbf{w} = T(\mathbf{v})$. Define

$$\mathbf{v}_1 = T^{-1}(\mathbf{w}_1) \in V \quad \text{and} \quad \mathbf{v}_2 = T^{-1}(\mathbf{w}_2) \in V.$$

Since T is linear, we have

$$T(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = \alpha T(\mathbf{v}_1) + \beta T(\mathbf{v}_2) = \alpha\mathbf{w}_1 + \beta\mathbf{w}_2.$$

Thus,

$$\begin{aligned} T^{-1}(\alpha\mathbf{w}_1 + \beta\mathbf{w}_2) &= T^{-1}(T(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2)) \\ &= \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 \\ &= \alpha T^{-1}(\mathbf{w}_1) + \beta T^{-1}(\mathbf{w}_2). \end{aligned}$$

$\square$

Remark. When a vector space isomorphism $T : V \to W$ exists, the vector spaces V and W are said to be **isomorphic**, denoted by $V \simeq W$.

Lemma. *Let V and W be vector spaces over a field $\mathbb{F}$ with $\dim V < \infty$ and $\dim W < \infty$. The following are equivalent:*

(1) $\dim V = \dim W$

(2) There exists a vector space isomorphism T from V to W

Proof. ((2) $\Rightarrow$ (1)) Assume that there exists a vector space isomorphism $T : V \to W$. Let $\mathcal{B}_V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis of V. Consider the set

$$\operatorname{Img}(\mathcal{B}_V) = T[\mathcal{B}_V] = \big\{T(\mathbf{v}) : \mathbf{v} \in \mathcal{B}_V\big\} = \big\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\big\} \subseteq W.$$

We claim that $T[\mathcal{B}_V]$ is a basis of W:

- (*Linear Independence*) Suppose that for some finite scalars $\{\alpha_i\}_{i=1}^n \subseteq \mathbb{F}$ we have

  $$\alpha_1 T(\mathbf{v}_1) + \alpha_2 T(\mathbf{v}_2) + \cdots + \alpha_n T(\mathbf{v}_n) = \mathbf{0}_W.$$

  By the linearity of T, we obtain $T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n) = \mathbf{0}_W$. Note that $T(\mathbf{0}_V) = T(0 \cdot \mathbf{v}) = 0 \cdot T(\mathbf{v}) = \mathbf{0}_W$ for any $\mathbf{v} \in V$. Since T is injective, it follows that

  $$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}_V.$$

  As $\mathcal{B}_V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis (and hence linearly independent), $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Thus, $T[\mathcal{B}_V]$ is linearly independent.
- (*Spanning Property*) Let $\mathbf{w} \in W$. Since T is surjective, there exists $\mathbf{v} \in V$ such that

  $$T(\mathbf{v}) = \mathbf{w}.$$

  By Uniqueness of Representation w.r.t. a Basis, we know that there exists a unique scalars $\{\alpha\}_{i=1}^n \subseteq \mathbb{F}$ such that

  $$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots \alpha_n\mathbf{v}_n.$$

  Then

  $$\mathbf{w} = T(\mathbf{v}) = T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots \alpha_n\mathbf{v}_n) \overset{\text{linearity}}{=} \alpha_1 T(\mathbf{v}_1) + \alpha_2 T(\mathbf{v}_2) + \cdots + \alpha_n T(\mathbf{v}_n) \in \operatorname{span} T[\mathcal{B}_V].$$

  That is, $\mathbf{w} \in W$ is a linear combination of elements of $T[\mathcal{B}_V]$. Therefore, $\operatorname{span} T[\mathcal{B}_V] = W$.

Since $\big|\mathcal{B}_V\big| = \big|T[\mathcal{B}_V]\big| = n$, thus, we have

$$\dim V = \dim W.$$

((1) $\Rightarrow$ (2)) Conversely, assume that $\dim V = \dim W =: n$. Consider bases

$$\mathcal{B}_V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \quad \text{and} \quad \mathcal{B}_W = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$$

for V and W, respectively. By Uniqueness of Representation w.r.t. a Basis, for each vector $\mathbf{v} \in V$, $\exists!$ finite scalars $\{\alpha_i\}_{i=1}^n \subseteq \mathbb{F}$ such that $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$. Define a mapping

$$T : V \to W, \quad \mathbf{v} \mapsto T(\mathbf{v}) = T\left(\sum_{i=1}^n \alpha_i \mathbf{v}_i\right) := \sum_{j=1}^n \alpha_j \mathbf{w}_j.$$

for each $\mathbf{v} \in V$. We NTS that T be a one-to-one and onto linear transformation:

(i) (*Linearity*) Let $\mathbf{v}, \mathbf{v}' \in V$ with $\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{v}_i$ and $\mathbf{v}' = \sum_{j=1}^n \beta_j \mathbf{v}_j$. For any $\lambda, \mu \in \mathbb{F}$, we have

$$\begin{aligned}
\lambda\mathbf{v} + \mu\mathbf{v}' = \lambda \sum_{i=1}^n \alpha_i \mathbf{v}_i + \mu \sum_{j=1}^n \beta_j \mathbf{v}_j &= \lambda(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n) + \mu(\beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_n\mathbf{v}_n) \\
&= (\lambda\alpha_1 + \mu\beta_1)\mathbf{v}_1 + (\lambda\alpha_2 + \mu\beta_2)\mathbf{v}_2 + \cdots + (\lambda\alpha_n + \mu\beta_n)\mathbf{v}_n \\
&= \sum_{k=1}^n (\lambda\alpha_k + \mu\beta_k)\mathbf{v}_k.
\end{aligned}$$

By definition of T, we have

$$T(\lambda\mathbf{v} + \mu\mathbf{v}') = \sum_{k=1}^n (\lambda\alpha_k + \mu\beta_k)\mathbf{w}_k = \lambda \sum_{i=1}^n \alpha_i \mathbf{w}_i + \mu \sum_{j=1}^n \beta_j \mathbf{w}_j = \lambda T(\mathbf{v}) + \mu T(\mathbf{v}').$$

(ii) (*Injectivity*) Let $\mathbf{v}, \mathbf{v}' \in V$ with $\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{v}_i$ and $\mathbf{v}' = \sum_{j=1}^n \beta_j \mathbf{v}_j$. Suppose $T(\mathbf{v}) = T(\mathbf{v}')$. Then

$$T(\mathbf{v}) - T(\mathbf{v}') = \sum_{k=1}^n (\alpha_k - \beta_k)\mathbf{w}_k = \mathbf{0}_W.$$

Since $\mathcal{B}_W = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is a basis of W, the linear independence of $\mathcal{B}_W$ implies that $\alpha_k = \beta_k$ for all $k = 1, 2, \ldots, n$. Thus $\mathbf{v} = \mathbf{v}'$, and so T is injective.

(iii) (*Surjectivity*) Let $\mathbf{w} \in W$. Since $\mathcal{B}_W = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is a basis of W, there exists a unique finite scalars $\{\alpha_i\}_{i=1}^n \subseteq \mathbb{F}$ such that $\mathbf{w} = \alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2 + \cdots + \alpha_n\mathbf{w}_n$. Define a vector

$$\mathbf{v} := \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \sum_{i=1}^n \alpha_i \mathbf{v}_i \in V.$$

Then $T(\mathbf{v}) = \sum_{i=1}^n \alpha_i \mathbf{w}_i = \mathbf{w}$. Thus, T is surjective.

$\square$

2 Part II

Classification of Vector Spaces up to Isomorphism

Theorem. *Let*

$$\mathcal{V}_{\mathbb{F}} := \left\{ V : V \text{ is a vector space over a field } \mathbb{F} \right\}.$$

Define a relation $\sim$ on $\mathcal{V}_{\mathbb{F}}$ by

$$\forall V, W \in \mathcal{V}_{\mathbb{F}}, \quad V \sim W \iff \exists T \in W^V \quad \text{such that} \quad T \text{ is a vector space isomorphism}.$$

Then

(1) $\sim$ is an equivalence relation on $\mathcal{V}_{\mathbb{F}}$;

(2) $\forall V, W \in \mathcal{V}_{\mathbb{F}}$, $V \simeq W \iff \dim V = \dim W$.

The isomorphism classes of vector spaces over $\mathbb{F}$ are completely determined by their dimensions.

Proof.

(1) We NTS that the relation $\sim$ is reflexive, symmetric, and transitive:

   (i) (*Reflexivity*) For each $V \in \mathcal{V}_{\mathbb{F}}$, the identity map $\mathrm{id}_V : V \longrightarrow V$ is a linear isomorphism, so $V \sim V$.

   (ii) (*Symmetry*) If $V \sim W$ via an isomorphism $T : V \longrightarrow W$, then its inverse $T^{-1} : W \longrightarrow V$ is also linear, implying $W \sim V$.

   (iii) (*Transitivity*) If $V \sim W$ via $T : V \longrightarrow W$ and $W \sim U$ via $S : W \longrightarrow U$, then the composition $S \circ T : V \longrightarrow U$ is a linear isomorphism, so $V \sim U$.

(2) It is proved by previous lemma.

□

Coordinate Isomorphism

Corollary. *Let V be a vector space over a field $\mathbb{F}$ with* $\dim V = n \in \mathbb{N}$, *and let*

$$\mathbb{F}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in F,\ 1 \le i \le n\}$$

is the space of n-tuples over $\mathbb{F}$ equipped with the usual operations of vector addition and scalar multiplication. . Then there exists a vector space isomorphism

$$\Phi : V \longrightarrow \mathbb{F}^n, \quad \text{i.e.,} \quad V \simeq \mathbb{F}^n.$$

Example. Consider the vector space

$$\mathrm{Mat}_{n\times m}(\mathbb{R}) = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} : a_{ij} \in \mathbb{R},\ 1 \le i \le n,\ 1 \le j \le m \right\}$$

which consists of all $n \times m$ matrices with entries in $\mathbb{R}$ (and where the vector space structure is defined over the field $\mathbb{R}$). Also, let

$$\mathbb{R}^{nm} = \{(x_1, x_2, \ldots, x_{nm}) : x_k \in \mathbb{R},\ 1 \le k \le nm\}$$

the vector space of nm-tuples of real numbers, with the usual coordinate-wise addition and scalar multiplication (again, over the field $\mathbb{R}$). Then there exists a vector space isomorphism

$$\Phi : \mathrm{Mat}_{n\times m}(\mathbb{R}) \longrightarrow \mathbb{R}^{nm},$$

i.e., $\mathrm{Mat}_{n\times m}(\mathbb{R}) \simeq \mathbb{R}^{nm}$.

Note. We also denote the set of all $n \times m$ matrices with real entries, namely $\mathrm{Mat}_{n\times m}(\mathbb{R})$ by $\mathbb{R}^{n\times m}$.

Observation. Let V and W be vector spaces over a field $\mathbb{F}$, and let $T : V \to W$ be a linear transformation. Suppose that

$$\mathcal{B}_V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \quad \text{and} \quad \mathcal{B}_W = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$$

are bases for V and W, respectively. Then for each $1 \leq j \leq n$, there exist unique scalars $\{a_{ij}\}_{i=1}^{m} \subseteq \mathbb{F}$ such that $T(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + a_{2j}\mathbf{w}_2 + \cdots + a_{mj}\mathbf{w}_m$:

$$V \xrightarrow{T} W$$

$$\mathbf{v}_1 \mapsto T(\mathbf{v}_1) = a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{m1}\mathbf{w}_m = \textstyle\sum_{i=1}^{m} a_{i1}\mathbf{w}_i$$

$$\mathbf{v}_2 \mapsto T(\mathbf{v}_2) = a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{m2}\mathbf{w}_m = \textstyle\sum_{i=1}^{m} a_{i2}\mathbf{w}_i$$

$$\vdots \qquad \vdots$$

$$\mathbf{v}_n \mapsto T(\mathbf{v}_n) = a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \cdots + a_{mn}\mathbf{w}_m = \textstyle\sum_{i=1}^{m} a_{in}\mathbf{w}_i$$

In other words, the action of T on the basis of V is completely determined by the matrix

$$[T]_{\mathcal{B}_V}^{\mathcal{B}_W} := \begin{bmatrix} \vdots & \vdots & & \vdots \\ T(\mathbf{v}_1) & T(\mathbf{v}_2) & \cdots & T(\mathbf{v}_n) \\ \vdots & \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{bmatrix} \in \mathrm{Mat}_{m \times n}(\mathbb{F}).$$

Example. Consider the linear transformation

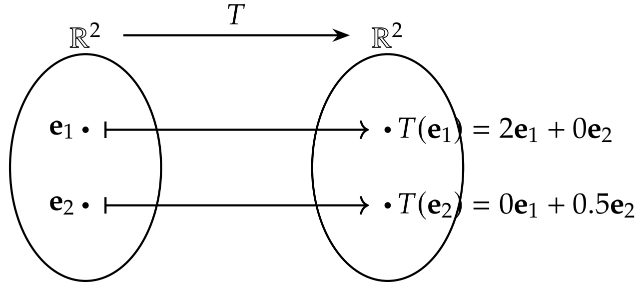
$$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T(x, y) = (2x, 0.5y).$$

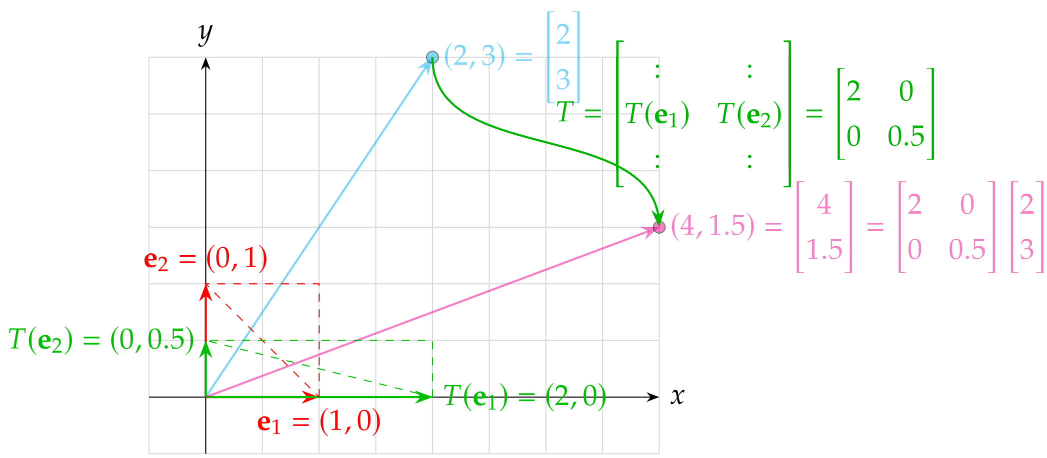
Its effect on the standard basis vectors is

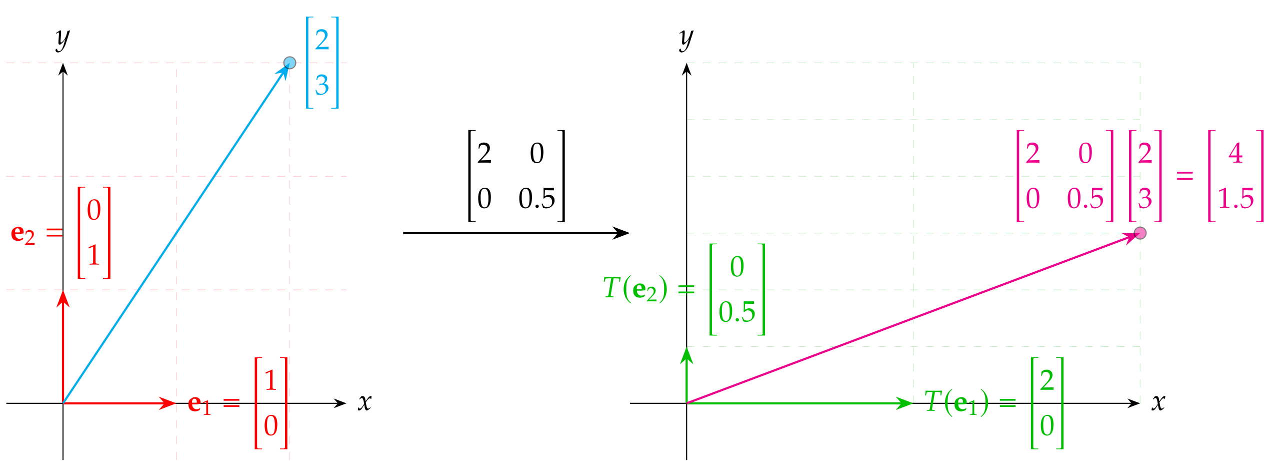
$$T(\mathbf{e}_1) = T(1, 0) = (2, 0) \quad \text{and} \quad T(\mathbf{e}_2) = T(0, 1) = (0, 0.5).$$

Then, we have

$$\begin{aligned} T(x, y) &= (2x,\ 0.5y) \\ &= \begin{bmatrix} \vdots & \vdots \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$







⋆ Matrix Representation of a Linear Transformation ⋆

Definition. Let V and W be vector spaces over a field $\mathbb{F}$, and let $T : V \to W$ be a linear transformation. Suppose that

$$\mathcal{B}_V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \quad \text{and} \quad \mathcal{B}_W = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$$

are bases for V and W, respectively. The **matrix representation of T with respect to the bases $\mathcal{B}_V$ and $\mathcal{B}_W$** is the unique matrix

$$[T]_{\mathcal{B}_V}^{\mathcal{B}_W} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{bmatrix} \in \mathrm{Mat}_{m \times n}(\mathbb{F})$$

whose $a_{ij} \in \mathbb{F}$ are defined by $T(\mathbf{v}_j) = \sum_{i=1}^{m} a_{ij}\mathbf{w}_i$ for each $j = 1, 2, \ldots, n$. In other words, if

$$[T(\mathbf{v}_j)]_{\mathcal{B}_W} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj}, \end{bmatrix},$$

then the j-th column of $[T]_{\mathcal{B}_V}^{\mathcal{B}_W}$ is given by the coordinate vector $[T(\mathbf{v}_j)]_{\mathcal{B}_W}$ of $T(\mathbf{v}_j)$ w.r.t. $\mathcal{B}_W$.

Remark. For each $\mathbf{v} \in V$, we have $[T(\mathbf{v})]_{\mathcal{B}_W} = [T]_{\mathcal{B}_V}^{\mathcal{B}_W}[\mathbf{v}]_{\mathcal{B}_V}$.

Note (Standard Basis for $\mathbb{F}^n$)**.** Consider the vector space of n-tuples over a field $\mathbb{F}$, that is,

$$\mathbb{F}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{F} \text{ for } i = 1, \ldots, n\}.$$

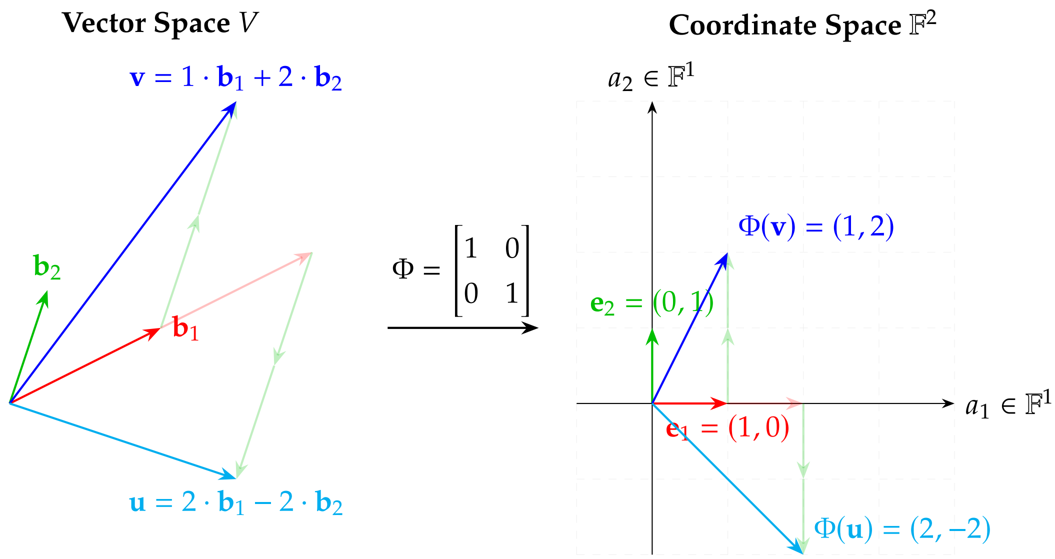
The *standard basis* for $\mathbb{F}^n$ is the set $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, where each $\mathbf{e}_i$ is defined by

$$\mathbf{e}_i = (0, \ldots, 0, \underbrace{1}_{i\text{-th position}}, 0, \ldots, 0),$$

Equivalently, in terms of the Kronecker delta, $\mathbf{e}_i = (\delta_{1i}, \delta_{2i}, \ldots, \delta_{ni})$, with $\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$

Every vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{F}^n$ can be uniquely expressed as $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$.

Example (Coordinate Isomorphism)**.**



Every $\mathbf{v} \in V$ can be uniquely expressed as $\mathbf{v} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2$ and $\Phi(\mathbf{v}) = (a_1, a_2) \in \mathbb{F}^2$.

Let V be an n-dimensional vector space over a field $\mathbb{F}$. Suppose that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis of V and that $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a standard basis of $\mathbb{F}^n$. Define the mapping

$$\begin{array}{rccl} \Phi & : & V & \longrightarrow & \mathbb{F}^n \\ & & \mathbf{v} & \longmapsto & \Phi(\mathbf{v}) = \sum_{i=1}^n \alpha_i \mathbf{e}_i \end{array},$$

where $\mathbf{v} \in V$ is uniquely expressed as $\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{b}_i$ with unique scalars $\{\alpha_i\}_{i=1}^n \subseteq \mathbb{F}$. Then

$$\begin{aligned} \Phi(\mathbf{b}_1) &= \Phi(1 \cdot \mathbf{b}_1) = \mathbf{e}_1 = 1\mathbf{e}_1 + 0\mathbf{e}_2 + \cdots + 0\mathbf{e}_n, \\ \Phi(\mathbf{b}_2) &= \Phi(1 \cdot \mathbf{b}_2) = \mathbf{e}_2 = 0\mathbf{e}_1 + 1\mathbf{e}_2 + \cdots + 0\mathbf{e}_n, \\ &\vdots \\ \Phi(\mathbf{b}_n) &= \Phi(1 \cdot \mathbf{b}_n) = \mathbf{e}_n = 0\mathbf{e}_1 + 0\mathbf{e}_2 + \cdots + 1\mathbf{e}_n. \end{aligned}$$

Thus, the matrix representation of Φ w.r.t. the bases $\mathcal{B}$ and $\mathcal{E}$ is the unique matrix

$$[\Phi]_{\mathcal{B}}^{\mathcal{E}} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ \Phi(\mathbf{b}_1) & \Phi(\mathbf{b}_2) & \cdots & \Phi(\mathbf{b}_n) \\ \vdots & \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} =: I_{n \times n} \text{ (or just } I_n\text{)}.$$

Hence each vector $\mathbf{v} \in V$ is uniquely represented by its coordinate vector w.r.t. a fixed basis, thereby establishing an isomorphism.

Example (Transpose Map)**.** Consider the vector space of 2×2 matrices over $\mathbb{F}$,

$$\mathrm{Mat}_2(\mathbb{F}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;:\; a, b, c, d \in \mathbb{F} \right\}.$$

Define the mapping

$$\Phi : \mathrm{Mat}_2(\mathbb{F}) \longrightarrow \mathrm{Mat}_2(\mathbb{F}), \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Here Φ is linear: for any $A, B \in \mathrm{Mat}_2(\mathbb{F})$,

$$\Phi(A + B) = (A + B)^T = A^T + B^T \quad \text{and} \quad \Phi(cA) = (cA)^T = cA^T.$$

To express the matrix representation of Φ w.r.t. a fixed basis, choose the standard basis for $\mathrm{Mat}_2(\mathbb{F})$:

$$\mathcal{E} = \left\{ E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Then

$$\begin{aligned}
\Phi(E_{11}) &= (E_{11})^T = E_{11} = 1E_{11} + 0E_{12} + 0E_{21} + 0E_{22}, \\
\Phi(E_{12}) &= (E_{12})^T = E_{21} = 0E_{11} + 0E_{12} + 1E_{21} + 0E_{22}, \\
\Phi(E_{21}) &= (E_{21})^T = E_{12} = 0E_{11} + 1E_{12} + 0E_{21} + 0E_{22}, \\
\Phi(E_{22}) &= (E_{11})^T = E_{22} = 0E_{11} + 0E_{12} + 0E_{21} + 1E_{22}.
\end{aligned}$$

Thus,

$$[T]_{\mathcal{E}}^{\mathcal{E}} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \Phi(E_{11}) & \Phi(E_{12}) & \Phi(E_{21}) & \Phi(E_{22}) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Remark. The matrix representation of a linear transformation $T : V \longrightarrow W$ is not canonical; it depends explicitly on the choices of bases for the domain V and the codomain W.

3 Part III

TBA