Advanced Calculus III

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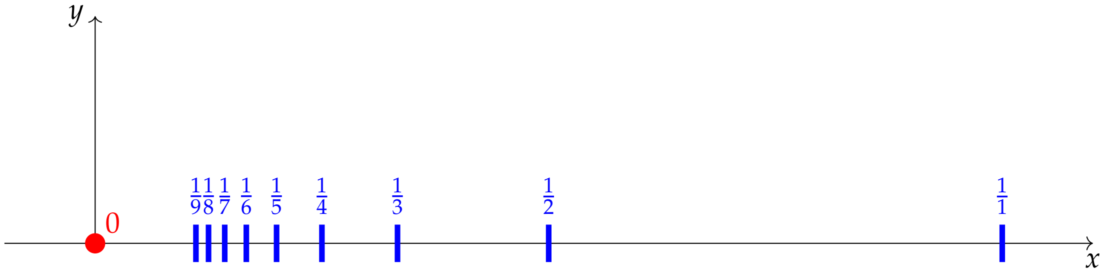
We cover the following topics in this note.

- Limit of a Function
- Continuity of a Function
- Monotone Convergent Theorem (MCT)
- Nested Interval Property (NIP)
- TBA

What is 0 for the set $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$?



Note (Open ε-ball)**.** The open ε-ball of x in S is $B_\varepsilon(x) := \{y \in S : d(x,y) < \varepsilon\}$.

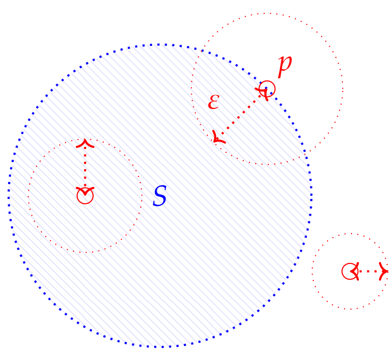
Limit Point (Metric Space)

Definition. Let (X,d) be a metric space. Let $S \subseteq X$. A point $p \in X$ is a **limit point** of S if and only if

$$\forall \varepsilon > 0,\ B_\varepsilon(p) \cap (S \setminus \{p\}) \neq \varnothing.$$

That is,

$$\forall \varepsilon > 0,\ \{x \in S : 0 < d(x,p) < \varepsilon\} \neq \varnothing.$$



Remark. Note that a limit point p may NOT belong to S.

Note (Limit Point (Topology)). Let (X, τ) be a topological space. For a subset $S \subseteq X$. A point $p \in X$ is a limit point of S if and only if

$$\forall U \in \tau \text{ with } p \in U,\ U \cap (S \setminus \{p\}) \neq \varnothing.$$

Example. Let $S = (a, b) \subseteq \mathbb{R}$:

(i) Consider p with $p < a$:

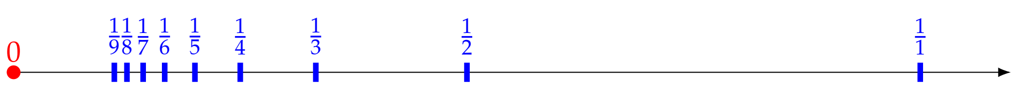
Let $\varepsilon := \frac{a-p}{2} > 0$. Then $B_\varepsilon(p) \cap (S \setminus \{p\}) = \varnothing$. Thus, $p < a$ is NOT a limit point.

(ii) Consider $p = a$:

Let $\varepsilon > 0$. Then $B_\varepsilon(p) \cap (S \setminus \{p\}) \neq \varnothing$. Thus, $p = a$ is a limit point of $S = (a, b)$.

By (i) and (ii), the set of all limit points of (a, b) is $[a, b]$.

Example. Let $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$:



- Consider $p = \frac{1}{n} \in S$. No point of S is a limit point.
- Consider $p = 0$.

Let $\varepsilon > 0$. By Archimedian property, $\exists n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$, and so $1/n \in B_\varepsilon(0) \cap S$. Thus, $p = 0$ is a limit point of $S = \{1/n : n \in \mathbb{N}\}$.

Example. Let $S = \mathbb{Q}$.

- Consider $p \in \mathbb{R}$. Let $\varepsilon > 0$. By density of rationals,

$$\exists r \in \mathbb{Q} \text{ such that } p < r < p + \varepsilon.$$

Then $r \in B_\varepsilon(p) \cap S$ with $r \neq p$, i.e., r is a limit points. Thus, all reals are limit points of $\mathbb{Q}$.

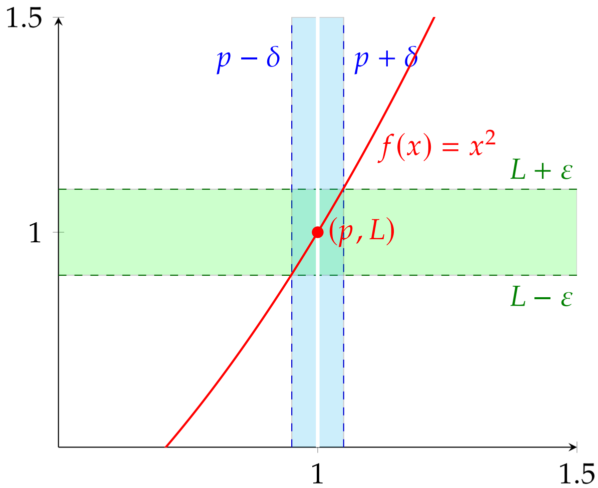
⋆ Limit of a Function ($\varepsilon - \delta$) ⋆

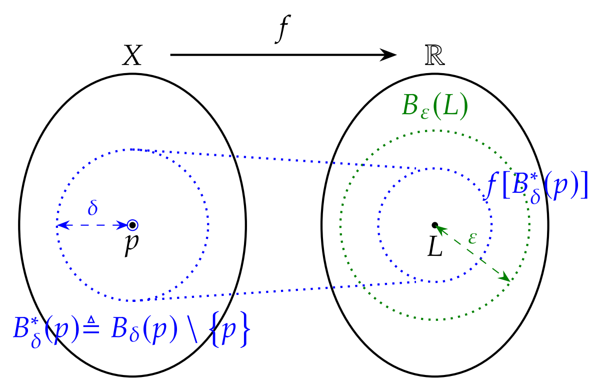
Definition. Let $f : X \longrightarrow \mathbb{R}$ be a function defined on a subset $X(\subseteq \mathbb{R})$ of a metric space, and let $p \in X$ be a limit point of X. We say that $L \in \mathbb{R}$ is the **limit of the function** f **as** x **approaches** p if

$$\boxed{\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x \in X,\ 0 < |x - p| < \delta \implies |f(x) - L| < \varepsilon}.$$

We write

$$\boxed{\lim_{x \to p} f(x) = L}.$$





Remark.

$$\lim_{x \to p} f(x) \neq L \iff \exists \varepsilon > 0 : [\forall \delta > 0 : \exists x \in X : 0 < |x - p| < \delta \text{ but } |f(x) - L| > 0].$$

Continuity of a Function

Definition. Let $f : X \to \mathbb{R}$ be a function defined on a subset $X \subseteq \mathbb{R}$ of a metric space, and let $p \in X$. The function f is **continuous at** p if and only if

$$\lim_{x \to p} f(x) = f(p).$$

That is,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } |x - p| < \delta \implies |f(x) - f(p)| < \varepsilon.$$

Remark (Continuity of a Set). The function f is continuous on subset $S \subseteq X$ if it is continuous at every point $p \in S$.

Remark (Continuity in a Topological Space). Let (X, τ_X) and (Y, τ_Y) are topological spaces. $f : X \to Y$ is **continuous** if and only if

$$U_Y \in \tau_Y \implies f^{-1}[U_Y] \in \tau_X,$$

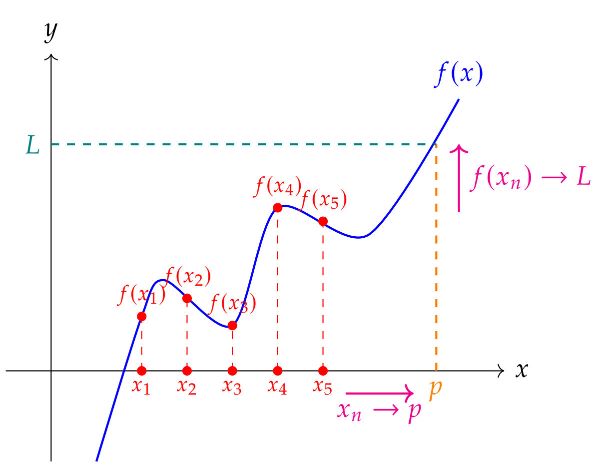
where $f^{-1}[U_Y] = \left\{x \in X : f(x) \in U_Y\right\}$ is the preimage of U_Y under f.

Note. $[p \Rightarrow (q \Rightarrow r)] \equiv [p \Rightarrow (\neg q \vee r)] \equiv [\neg p \vee (\neg q \vee r)] \equiv [\neg(p \wedge q) \vee r] \equiv [(p \wedge q) \Rightarrow r]$.

Limit of Function by Convergent Sequences

Theorem. *Let $f : X \to \mathbb{R}$ be a function defined on a subset $\varnothing \neq X \subseteq \mathbb{R}$ of a metric space, and let p is a limit point of X. Then*

$$\lim_{x \to p} f(x) = L \iff \left[\forall \{x_n\} \subseteq X \setminus \{p\},\ \left(\lim_{n \to \infty} x_n = p \implies \lim_{n \to \infty} f(x_n) = L\right)\right].$$



Proof. ($\Rightarrow$) Suppose that $\lim_{x\to p} f(x) = L$. Let $\{x_n\} \subseteq X \setminus \{p\}$ be a sequence, and let $\lim_{n\to\infty} x_n = p$. We NTS that

$$\lim_{n\to\infty} f(x_n) = L, \quad \text{i.e.,} \quad \forall \varepsilon > 0 : \exists N \in \mathbb{N} : n \geq N \Rightarrow |f(x_n) - L| < \varepsilon.$$

Let $\varepsilon > 0$. Since $\lim_{x\to p} f(x) = L$, we know

$$\exists \delta > 0 \text{ such that } 0 < |x - p| < \delta \implies |f(x) - L| < \varepsilon. \tag{*}$$

Since $\lim_{n\to\infty} x_n = p$, we obtain

$$\exists N \in \mathbb{N} \text{ such that } n \geq N \implies |x_n - p| < \delta.$$

Thus, if $n \geq N$ then,

$$\begin{aligned} |x_n - p| < \delta &\implies 0 < |x_n - p| < \delta \quad \because x_n \neq p \\ &\implies |f(x_n) - L| < \varepsilon \quad \text{by (*)} \end{aligned}$$

Thus, $\lim_{n\to\infty} f(x_n) = L$.

($\Leftarrow$) Let the RHS holds. Assume, for the contradiction, that $\lim_{x\to p} f(x) \neq L$, i.e.,

$$\exists \varepsilon > 0 : \forall \delta > 0 : \exists x_\delta \in X : 0 < |x_\delta - p| < \delta \text{ but } |f(x_\delta) - L| \geq \varepsilon.$$

Take $\delta = 1/n$ for $n \in \mathbb{N}$. Then

$$\exists x_n \in X \text{ such that } 0 < |x_n - p| < \frac{1}{n} \text{ but } |f(x_n) - L| \geq \varepsilon.$$

~~(Axiom of Countable Choice)~~ This means that

$$\forall n \in \mathbb{N} : \exists \{x_n\} \subseteq X \setminus \{p\} \text{ such that } 0 < |x_n - p| < \frac{1}{n} \text{ but } |f(x_n) - L| \geq \varepsilon.$$

By Squeeze Theorem, we have $\lim_{n\to\infty} x_n = p$ since $0 < |x_n - p| < 1/n$. Since the RHS holds, we obtain $\lim_{n\to\infty} f(x_n) = L$. Then, for some $\varepsilon > 0$,

$$\exists N \in \mathbb{N} \text{ such that } n \geq N \implies |f(x_n) - L| < \varepsilon \text{ ↯}.$$

Hence it is proved. ∎

Continuity of Function by Convergent Sequences

Corollary. *Let $f : X \to \mathbb{R}$ be a function defined on a subset $\varnothing \neq X \subseteq \mathbb{R}$ of a metric space, and let p is a limit point of X. Then*

$$\lim_{x\to p} f(x) = f(p) \iff \left[\forall \{x_n\} \subseteq X, \left(\lim_{n\to\infty} x_n = p \implies \lim_{n\to\infty} f(x_n) = f(p)\right)\right].$$

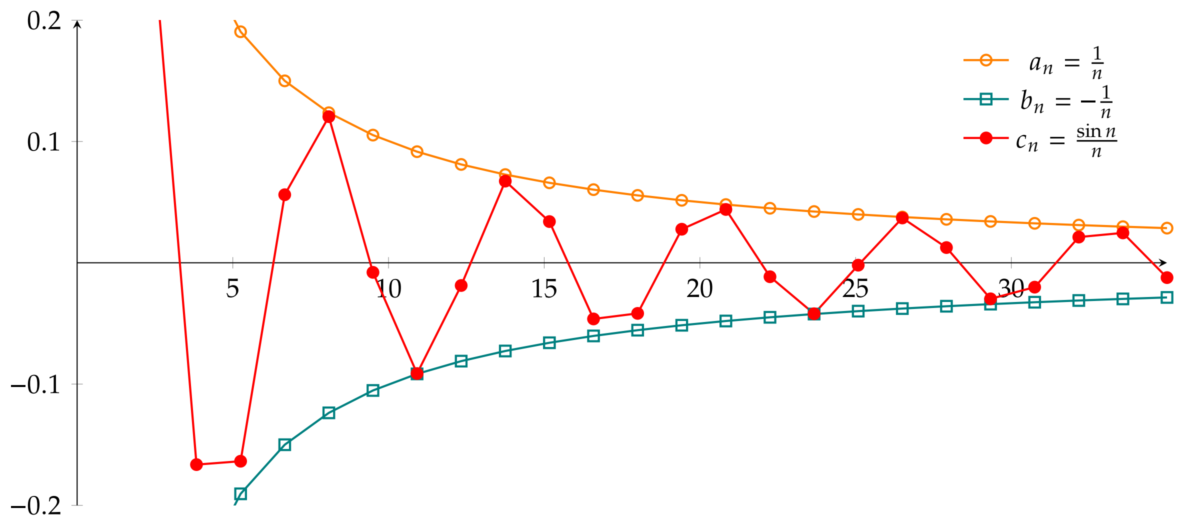
Squeeze Theorem; Sandwich Theorem

Theorem. *Let*

(i) $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$;

(ii) $\exists n_0 \in \mathbb{N}$ *such that* $a_n \leq c_n \leq b_n$ *for all* $n \geq n_0$.

Then $\lim_{n\to\infty} c_n = L$.



Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = L$, we have

$$\exists n_1 \in \mathbb{N} \text{ such that } n \geq n_1 \implies L - \varepsilon < a_n < L + \varepsilon,$$
$$\exists n_2 \in \mathbb{N} \text{ such that } n \geq n_2 \implies L - \varepsilon < b_n < L + \varepsilon.$$

Let $N := \max\{n_0, n_1, n_2\}$. If $n \geq N$ then

$$L - \varepsilon < a_n \leq c_n \leq b_n < L_+\varepsilon,$$

and so $|c_n - L| < \varepsilon$. □

Note. Recall that

"A convergent sequence is bounded."

Formally,

$$\exists A \in \mathbb{R} \text{ s.t. } A = \lim_{n\to\infty} a_n \implies \exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, |a_n| \le M.$$

However, the converse is not necessarily true:

$$\exists A \in \mathbb{R} \text{ s.t. } A = \lim_{n\to\infty} a_n \not\Leftarrow \exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, |a_n| \le M.$$

To illustrate, consider the sequence $\{a_n\} = 1 - (-1)^n$ that is bounded, yet it does not converge.

Monotone Sequence

Definition. A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be **monotone** if it is either **monotonically increasing** or **monotonically decreasing**.

(1) A sequence $\{a_n\}_{n=1}^{\infty}$ is **monotonically increasing** if $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$.
Alternatively, it is **strictly increasing** if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.

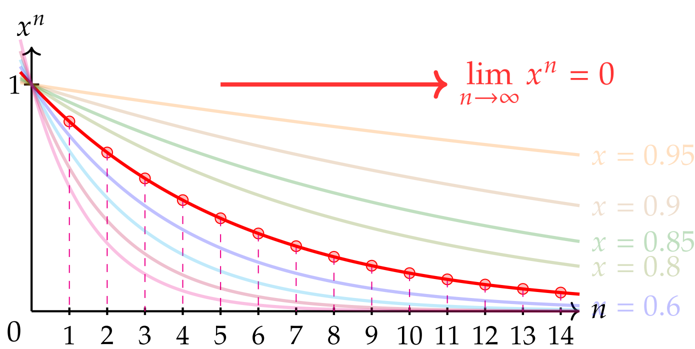
(2) A sequence $\{a_n\}_{n=1}^{\infty}$ is **monotonically decreasing** if $a_{n+1} \le a_n$ for all $n \in \mathbb{N}$.
Alternatively, it is **strictly decreasing** if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$.

Remark. A sequence $\{a_n\}$ is monotone if $\begin{cases} a_n \le a_{n+1} & \text{(monotonically increasing)} \\ a_{n+1} \le a_n & \text{(monotonically decreasing)} \end{cases}$.

Example.

- $\{n\}_{n=1}^{\infty}$ is monotonically increasing.
- $\{1/n\}_{n=1}^{\infty}$ is monotonically decreasing.

Example. Let $0 < x < 1$.



Monotone Convergence Theorem (MCT)

Theorem. *A monotone sequence of real numbers $\{a_n\}$ is convergent if and only if it is bounded.*

(1) Let $\{a_n\}$ be an monotonically increasing sequence of real numbers that is bounded above. Then

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

(2) Let $\{b_n\}$ be an monotonically decreasing sequence of real numbers that is bounded below. Then

$$\lim_{n\to\infty} b_n = \inf\{b_n : n \in \mathbb{N}\}.$$

Proof.

(1) Suppose that a sequence $\{a_n\}$ is monotonically increasing and bounded above. Consider the set $\{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$, which is non-empty and bounded above by assumption. By **Least Upper Bound Property**[1],

$$\exists \alpha \in \mathbb{R} \text{ such that } \alpha = \sup\{a_n : n \in \mathbb{N}\}.$$

We claim that:

$$\lim_{n\to\infty} a_n = \alpha = \sup\{a_n : n \in \mathbb{N}\}.$$

Let $\varepsilon > 0$. Since α is the supremum (*least* upper bound) of $\{a_n : n \in \mathbb{N}\}$, it follows that $\alpha - \varepsilon$ is not an upper bound of $\{a_n : n \in \mathbb{N}\}$. Thus, $\neg[\forall N \in \mathbb{N},\ a_N \leq \alpha - \varepsilon]$, i.e.,

$$\exists N \in \mathbb{N} \text{ such that } \alpha - \varepsilon < a_N.$$

Since $\{a_n\}$ is monotonically increasing,

$$\alpha - \varepsilon < a_N \leq a_n$$

for all $n \geq N$. Therefore,

$$\alpha - \varepsilon \underset{\varepsilon>0}{\overset{\alpha=\sup\{a_n\}}{<}} a_N \underset{n\geq N}{\overset{\{a_n\}\text{ is monotonically increasing}}{\leq}} a_n \overset{\alpha\text{ is an upper bound}}{\leq} \alpha \overset{\varepsilon>0}{<} \alpha + \varepsilon.$$

This implies that $|a_n - \alpha| < \varepsilon$ for all $n \geq N$.

[1]Every non-empty subset of $\mathbb{R}$ that is bounded above has the supremum in $\mathbb{R}$.

(2) Suppose that a sequence $\{b_n\}$ is monotonically decreasing and bounded below. Consider the set $\{b_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$, which is non-empty and bounded below by assumption. By **Greatest Lower Bound Property**[2],

$$\exists \beta \in \mathbb{R} \text{ such that } \beta = \inf \{b_n : n \in \mathbb{N}\}.$$

We claim that:

$$\lim_{n\to\infty} b_n = \beta = \inf \{b_n : n \in \mathbb{N}\}.$$

Let $\varepsilon > 0$. Since β is the infimum (*greatest* lower bound) of $\{b_n : n \in \mathbb{N}\}$, it follows that $\beta + \varepsilon$ is not a lower bound of $\{b_n : n \in \mathbb{N}\}$. Thus, $\neg[\forall N \in \mathbb{N},\ \beta + \varepsilon \leq b_N]$, i.e.,

$$\exists N \in \mathbb{N} \text{ such that } b_N < \beta + \varepsilon.$$

Since $\{b_n\}$ is monotonically decreasing,

$$b_n \leq b_N < \beta + \varepsilon$$

for all $n \geq N$. Therefore,

$$\beta - \varepsilon \overset{\varepsilon>0}{<} \beta \overset{\beta \text{ is a lower bound}}{\leq} b_n \underset{n\geq N}{\overset{\{b_n\} \text{ is monotonically decreasing}}{\leq}} b_N \underset{\varepsilon>0}{\overset{\beta=\inf\{b_n\}}{<}} \beta + \varepsilon$$

This implies that $|b_n - \beta| < \varepsilon$ for all $n \geq N$.

$\square$

[2]Every non-empty subset of $\mathbb{R}$ that is bounded below has the infimum in $\mathbb{R}$.

Divergence of Sequence

Definition. Let $\{a_n\}$ be a sequence of real numbers.

(1) We say that the sequence $\{a_n\}$ **diverges to infinity** (or **tends to infinity**) if

$$\forall M \in \mathbb{R},\ \exists N \in \mathbb{N} \text{ such that } n \geq N \implies M < a_n,$$

and write $\lim_{n\to\infty} a_n = +\infty$.

(2) We say that the sequence $\{a_n\}$ **diverges to minus infinity** (or **tends to infinity**) if

$$\forall M \in \mathbb{R},\ \exists N \in \mathbb{N} \text{ such that } n \geq N \implies a_n < M,$$

and write $\lim_{n\to\infty} a_n = -\infty$.

(3) We say that $\{a_n\}$ is properly divergent in case we have either $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} = -\infty$.

Note. Recall that

[Monotonicity] A sequence $\{a_n\}$ is monotonically increasing if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$;

[Not Bounded Above] The sequence $\{a_n\}$ is not bounded above if

$$\neg[\exists M \in \mathbb{R},\ \forall n \in \mathbb{N},\ a_n \leq M] \equiv [\forall M \in \mathbb{R},\ \exists n \in \mathbb{N} \text{ such that } a_n > M].$$

We claim that a sequence $\{a_n\}$ that is monotonically increasing and not bounded above diverges to infinity:

Proof. Let $M \in \mathbb{R}$. Since $\{a_n\}$ is not bounded above,

$$\exists n_0 \in \mathbb{N} \text{ such that } a_{n_0} > M.$$

Since $\{a_n\}$ is monotonic increasing, it fllows that

$$a_{n_0} \leq a_n,\ \forall n \geq n_0.$$

Thus

$$n \geq n_0 \overset{\text{monotonically increasing}}{\Longrightarrow} a_{n_0} \leq a_n \overset{\text{Not Bounded Above}}{\Longrightarrow} M < a_{n_0} < a_n.$$

Hence it is proved. □

Comparison Theorem

Lemma. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Then*

$$[\forall n \in \mathbb{N},\ a_n \le b_n] \implies \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n.$$

Proof. Let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Suppose that $a > b$. Let $\varepsilon = a - b > 0$. Then

$$\begin{aligned} \exists N_1 \in \mathbb{N} \text{ such that } n \ge N_1 &\implies |a_n - a| < \varepsilon, \\ \exists N_2 \in \mathbb{N} \text{ such that } n \ge N_2 &\implies |b_n - b| < \varepsilon. \end{aligned}$$

Let $N := \max\{N_1, N_2\}$. Then $b_N < b + \varepsilon < a + \varepsilon < a_N$ $\lightning$. Hence $a \le b$, i.e., $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$. □

Note. Let $I_n = \left(0, \frac{1}{n}\right) \subseteq \mathbb{R}$ for all $n \in \mathbb{N}$.

Suppose that $x \in \bigcap_{n=1}^{\infty} I_n$ then $x \in I_n$ for all $n \ge 1$. That is,

$$0 < x < \frac{1}{n} \quad \text{for all} \quad n \ge 1.$$

By Archimedian property, $\exists n_0 \in \mathbb{N}$ s.t. $n_0 x > 1$ $\lightning$. Hence $\bigcap_{n=1}^{\infty} I_n = \varnothing$.

Note. Let $I_n = [n, \infty) \subseteq \mathbb{R}$ for all $n \in \mathbb{N}$.

Suppose that $x \in \bigcap_{n=1}^{\infty} I_n$ then $x \in I_n$ for all $n \ge 1$. That is,

$$n \le x \quad \text{for all} \quad n \ge 1.$$

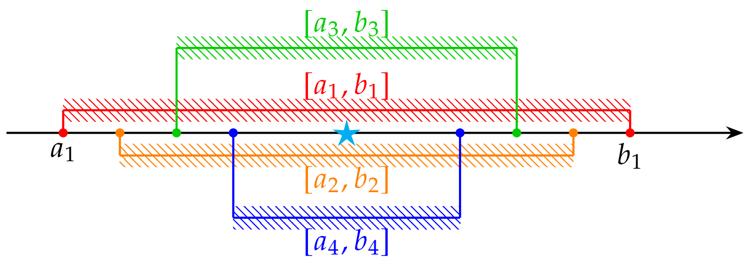
By Archimedian property, $\exists n_0 \in \mathbb{N}$ s.t. $x > n$ $\lightning$. Hence $\bigcap_{n=1}^{\infty} I_n = \varnothing$.

Nested Interval Property (NIP)

Theorem. *Let $a_n \le b_n$ for all $n \in \mathbb{N}$, and let $\{[a_n, b_n]\}_{i=1}^{\infty} \subseteq \mathbb{R}$ be a sequence of bounded and closed intervals satisfying $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for all $n \in \mathbb{N}$. Then*

$$\bigcap_{n=1}^{\infty} [a_n, b_n] := \left\{x \in \mathbb{R} : x \in [a_n, b_n] \text{ for all } n \in \mathbb{N}\right\} \neq \varnothing.$$

If $\lim_{n\to\infty} (b_n - a_n) = 0$, then $\left|\bigcap_{n=1}^{\infty} [a_n, b_n]\right| = 1$.



Proof. Since $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for all $n \in \mathbb{N}$, we know the sequence $\{a_n\}$ is monotonically increasing, and the sequence $\{b_n\}$ is monotonically decreasing. In other words,

$$a_1 \le a_2 \le \cdots \le a_n \le \cdots \le b_n \le \cdots b_2 \le b_1.$$

By Monotone Convergence Theorem, we obtain

$$\lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} a_n \quad \text{and} \quad \lim_{n\to\infty} b_n = \inf_{n\in\mathbb{N}} b_n.$$

Thus,

$$[\forall n \in \mathbb{N},\ a_n \le b_n] \implies \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n \implies \sup_{n\in\mathbb{N}} a_n \le \inf_{n\in\mathbb{N}} b_n.$$

Then

$$\begin{aligned} x \in \bigcap_{n=1}^{\infty} [a_n, b_n] \iff \forall n \in \mathbb{N},\ a_n \le x \le b_n &\iff \sup_{n\in\mathbb{N}} a_n \le x \le \inf_{n\in\mathbb{N}} b_n \\ &\iff x \in \left[\sup_{n\in\mathbb{N}} a_n, \inf_{n\in\mathbb{N}} b_n\right]. \end{aligned}$$

By Set Equality, we have

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = \left[\sup_{n\in\mathbb{N}} a_n, \inf_{n\in\mathbb{N}} b_n\right],$$

and so $\left[\sup_{n\in\mathbb{N}} a_n, \inf_{n\in\mathbb{N}} b_n\right] \neq \varnothing$ by Least Upper Bound Property. □

Limit Superior

Definition. TBA

Limit Inferior

Definition. TBA

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[1] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 10. 해석학 개론 (e) 엡실론-델타와 수열의 수렴성” YouTube Video, 25:57. Published September 29, 2019. URL: https://youtu.be/2Ml3G_Duffk?si=qo-CVgW3Ukd4ADRL.

[2] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 11. 해석학 개론 (f) MCT and NIP” YouTube Video, 20:17. Published October 1, 2019. URL: https://youtu.be/YdnBQaY5eDk?si=BNe0Ue4iq2P9Fxsd.

[3] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 12. 해석학 개론 (g) Limsup, Liminf” YouTube Video, 34:31. Published October 2, 2019. URL: https://youtu.be/4Q1cm3VQPUE?si=phAhKwnOxdnRAiRR.

A Equivalent Statements of the Least Upper Bound Property

$$\begin{aligned}\text{Least Upper Bound Property} &\iff \text{Monotone Convergence Theorem}\\ &\iff \text{Nested Interval Property}\end{aligned}$$

Theorem. *Monotone Convergence Theorem* $\iff$ *Nested Interval Property*

Proof. ($\Rightarrow$) See **Nested Interval Property**.

($\Leftarrow$) TBA

□

Theorem. *Least Upper Bound Property* $\iff$ *Monotone Convergence Theorem*

Proof. ($\Rightarrow$) See **Monotone Convergence Theorem**.

($\Leftarrow$) TBA

□