1-Form as Scalar Projection onto a Tangent Line

Setup

Consider the smooth curve $C \subseteq \mathbb{R}^2$ defined by

$$C = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\},$$

and let $f(x) = x^2$, so that $f'(x) = 2x$. Let

$$p = (1.5, f(1.5)) = (1.5, 2.25),$$

with tangent vector at p:

$$\vec{v} = \langle 1, f'(1.5) \rangle = \langle 1, 3 \rangle .$$

Hence the tangent space at p is

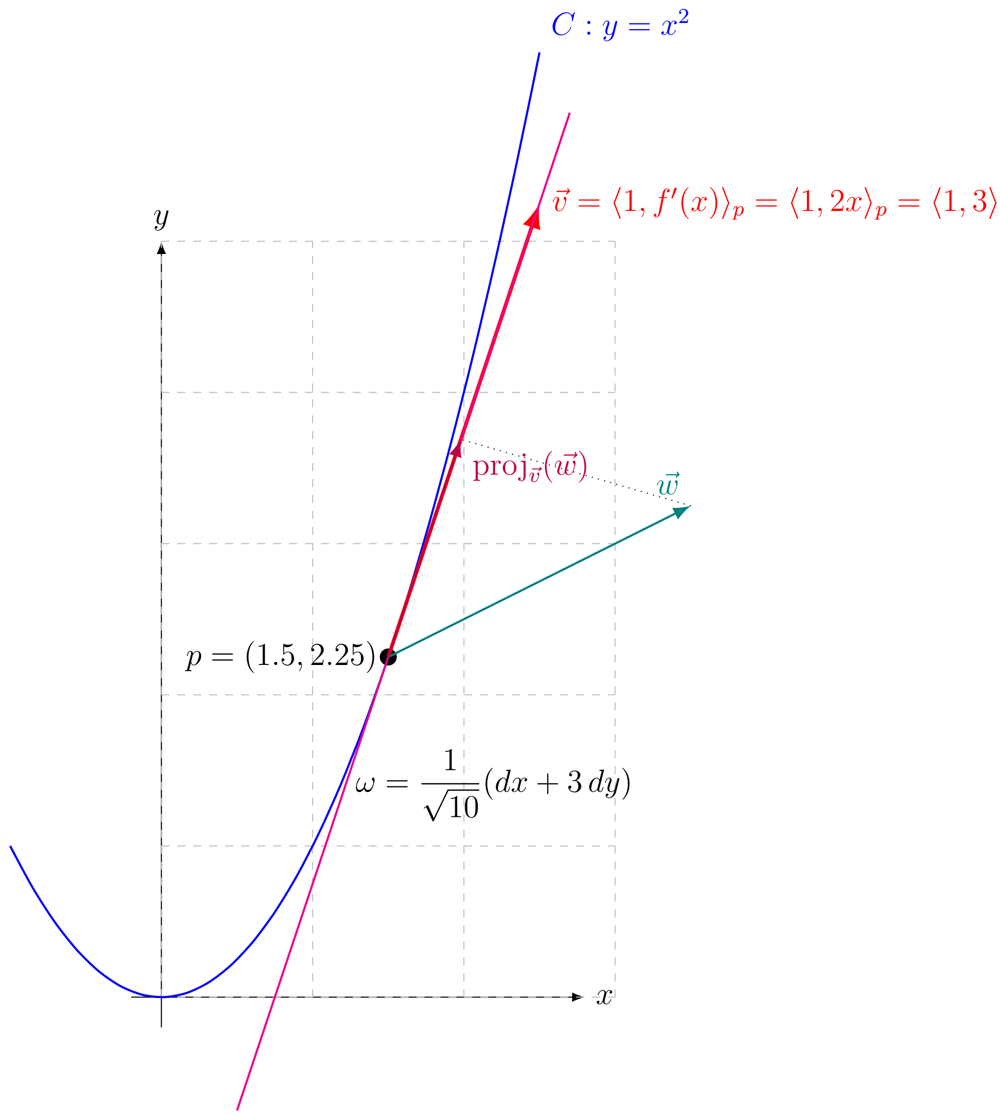
$$T_pC = \operatorname{span} \{ \langle 1, 3 \rangle \} .$$

We define a differential 1-form $\omega \in \Omega^1(\mathbb{R}^2)$ corresponding to the scalar projection onto $\vec{v}$:

$$\omega = \frac{1}{\sqrt{10}}(dx + 3\, dy).$$

This 1-form evaluates, for any $\vec{w} \in T_p\mathbb{R}^2$, the scalar projection of $\vec{w}$ onto the direction of $\vec{v}$.

Visualization of Geometry



Interpretation

- The red vector $\vec{v} = \langle 1, 3 \rangle$ is tangent to the curve $y = x^2$ at the point $p = (1.5, 2.25)$.
- The vector $\vec{w} \in T_p\mathbb{R}^2$ is arbitrary.
- The projection $\text{proj}_{\vec{v}}(\vec{w})$ shows the scalar component of $\vec{w}$ in the direction of $\vec{v}$.
- The 1-form ω returns the component of any input vector along the line in the direction of $\vec{v}$, normalized by $\|\vec{v}\| = \sqrt{10}$.