Topology I

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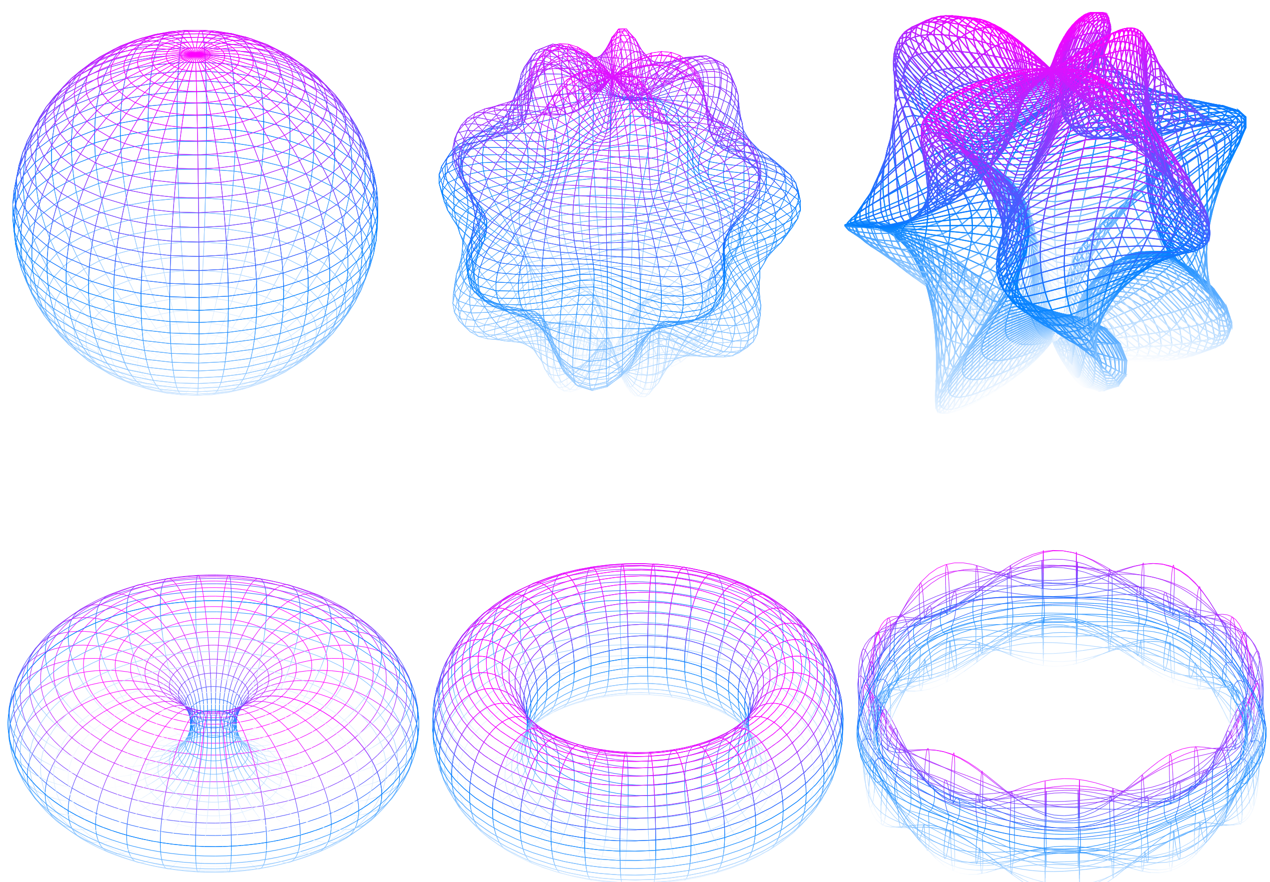
We cover the following topics in this note.

- Topology and Topological Space
- Open Set
- Continuous Mapping
- Distance Function and Metric Space
- Convergence of Sequences; Continuity of Functions
- TBA



Topology; Topological Space

Definition. Let S be a non-empty set. A **topology**[a] on S is a subset $\mathscr{T} \subseteq 2^S$, where 2^S denotes the power set of S, that satisfies the following axioms:

(O1)[b] The empty set and the entire set S belong to $\mathscr{T}$: $\boxed{S \in \mathscr{T} \text{ and } \varnothing \in \mathscr{T}}$.

(O2)[c] The union of any collection of elements in $\mathscr{T}$ is also an element of $\mathscr{T}$:

$$\boxed{\{U_i\}_{i\in I} \subseteq \mathscr{T} \implies \bigcup_{i\in I} U_i \in \mathscr{T}}.$$

(O3)[d] The intersection of any finite number of elements in $\mathscr{T}$ is also an element of $\mathscr{T}$:

$$\boxed{\{U_i\}_{i=1}^n \subseteq \mathscr{T} \implies \bigcap_{i=1}^n U_i \in \mathscr{T}}.$$

The pair $(S, \mathscr{T})$ is called a **topological space**.

[a] The word "topology" comes from the Greek roots "topos" meaning "place" and "logos" meaning "study".
[b] Empty set and Whole space
[c] Closure under *arbitrary* unions
[d] Closure under *finite* intersections

Remark. By mathematical induction, we have

$$\text{O3} \iff [\{U_1, U_2\} \subseteq \mathscr{T} \Rightarrow U_1 \cap U_2 \in \mathscr{T}].$$

Open Set (Topology)

Definition. Let $(S, \mathscr{T})$ be a topological space. $U \subseteq S$ is an **open set**, or **open** (in S) iff $U \in \mathscr{T}$.

Remark. A subset $\mathscr{T}$ of power set 2^S is a topology on S if and only if

(i) $\varnothing$ and S are open;

(ii) Let $U_1, U_2, \cdots \in \mathscr{T}$, i.e., $\{U_i\}_{i\in I} \subseteq \mathscr{T}$. Then $\bigcup_{i\in I} U_i$ is open.

(iii) Let $U_1, U_2, \ldots, U_n \in \mathscr{T}$, i.e., $\{U_i\}_{i=1}^n \subseteq \mathscr{T}$. Then $\bigcap_{i=1}^n U_i$ is open.

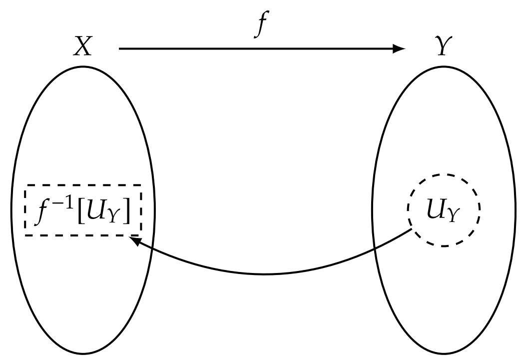
Continuous Mapping by Open Sets

Definition. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are topological spaces. Let $f : X \to Y$ be a mapping from X to Y.

(1) (Continuous Everywhere) The mapping f is **continuous on X** if and only if

$$U_Y \in \mathscr{T}_Y \implies f^{-1}[U_Y] \in \mathscr{T}_X,$$

where $f^{-1}[U_Y] = \{x \in X : f(x) \in U_Y\}$ is the preimage of U_Y under f.



Note (Preparation for **Example 1**). Let $S \neq \varnothing$ be a set, and let $\{A_\alpha\}_{\alpha \in \Lambda} \subseteq S$. Then

$$\begin{aligned} S \setminus \bigcup_{\alpha \in \Lambda} A_\alpha = S \setminus \{x \in S : \exists \alpha \in \Lambda \text{ s.t. } x \in A_\alpha\} &= \left\{x \in S : \neg[\exists \alpha \in \Lambda \text{ s.t. } x \in A_\alpha]\right\} \\ &= \{x \in S : \forall \alpha \in \Lambda,\ x \notin A_\alpha\} \\ &= \left\{x \in S : \forall \alpha \in \Lambda,\ x \in S \setminus A_\alpha\right\} \\ &= \bigcap_{\alpha \in \Lambda} (S \setminus A_\alpha). \end{aligned}$$

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Note (Preparation for **Example 1**).

(1) A Subset of a Finite Set is Finite.

(2) The Intersection of Finite Sets is Finite.

Example 1 (Cofinite Topology)**.** Let $S \neq \varnothing$ be a set. Define the cofinite topology $\mathscr{T}_C \subseteq 2^S$ by

$$\begin{aligned}\mathscr{T}_C := &\left\{U \subseteq S : S \setminus U \text{ is finite}\right\} \cup \{\varnothing\} \\ = &\left\{U \subseteq S : U = \varnothing \text{ or } S \setminus U \text{ is finite}\right\}.\end{aligned}$$

In other words, U is open in the cofinite topology if U is the empty, or if the complement $S \setminus U$ is a finite set. We claim that $\mathscr{T}_C$ be a topology on S:

(O1) By definition, $\varnothing \in \mathscr{T}_C$. For $U = S$, the complement $S \setminus S = \varnothing$, which is finite, so $S \in \mathscr{T}_C$. Hence, both $\varnothing$ and S are elements of $\mathscr{T}_C$.

(O2) Let $\{U_i\}_{i \in I} \subseteq \mathscr{T}_C$.

(Case 1) If $U_i = \varnothing$ for all $i \in I$, then $\bigcup_{i \in I} U_i = \varnothing \in \mathscr{T}_C$.

(Case 2) Suppose that there exists $i_0 \in I$ such that $U_{i_0} \neq \varnothing$. Then

$$S \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} \left(S \setminus U_i\right) \subseteq (S \setminus U_{i_0}).$$

Since $S \setminus U_{i_0}$ is finite, $S \setminus \bigcup_{i \in I} U_i$ if finite, so $\bigcup_{i \in I} U_i \in \mathscr{T}_C$.

(O3) Let $U_1 \in \mathscr{T}_C$ and $U_2 \in \mathscr{T}_C$.

(Case 1) If $U_1 = \varnothing$ or $U_2 = \varnothing$, then $U_1 \cap U_2 = \varnothing \in \mathscr{T}_C$.

(Case 2) Suppose that $U_1 \neq \varnothing$ and $U_2 \neq \varnothing$. Then $S \setminus U_1$ and $S \setminus U_2$ are finite. By the De Morgan law, we have

$$S \setminus (U_1 \cap U_2) = (S \setminus U_1) \cup (S \setminus U_2),$$

which is a finite set. Thus, $U_1 \cap U_2 \in \mathscr{T}_C$.

Example 2 (Discrete Topology). Let $S \neq \varnothing$ be a set, and let $\mathscr{T} = 2^S$ be the power set of S. Then $\mathscr{T}$ is called the **discrete topology** on S and $(S, \mathscr{T}) = (S, 2^S)$ the **discrete (topological) space** on S.

Example 3 (Indiscrete Topology). Let $S \neq \varnothing$ be a set, and let $\mathscr{T} = \{S, \varnothing\}$. Then $\mathscr{T}$ is called the **indiscrete topology** on S and $(S, \mathscr{T}) = (S, \{S, \varnothing\})$ the **indiscrete (topological) space** on S.

Note.

(1) Discrete Topology is Finest Topology.

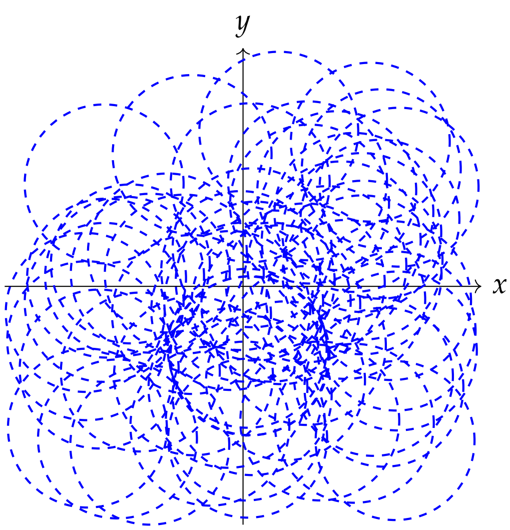
(2) Indiscrete Topology is Coarsest Topology.

Coarser Topology and Finer Topology

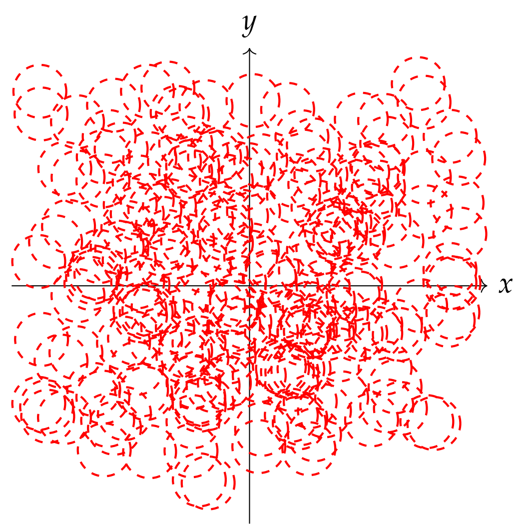
Definition. Let $S \neq \varnothing$ be a set. Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be topologies on S.

(1) $\mathscr{T}_1$ is said to be **coarser** than $\mathscr{T}_2$ if $\mathscr{T}_1 \subseteq \mathscr{T}_2$.

(2) $\mathscr{T}_1$ is said to be **finer** than $\mathscr{T}_2$ if $\mathscr{T}_2 \subseteq \mathscr{T}_1$.



Coarser Topology



Finer Topology

Distance Function

Definition. Let S be a set. The real-valued function of two variable

$$d : S \times S \to \mathbb{R}$$

is called a **distance function** (or **metric**) if it satisfies the following properties:

(i)[a] $\forall x, y \in S,\ d(x,y) \geq 0 \quad \text{and} \quad d(x,y) = 0 \Leftrightarrow x = y.$

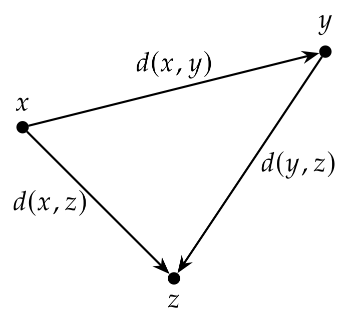
(ii)[b] $\forall x, y \in S,\ d(x,y) = d(y,x).$

(iii)[c] $\forall x, y, z \in S,\ d(x,z) \leq d(x,y) + d(y,z).$

The pair (S, d) is called a **metric space**.

[a] Non-negativity and Zero only for identical points
[b] Symmetry
[c] Triangle inequality

Remark.



Example 4.

- Let $S = \mathbb{R}$, the set of real numbers. Define the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$d(x,y) = \left| x - y \right|$$

  for $x, y \in \mathbb{R}$.

- Let $S = \mathbb{R}^n$, the n-dimensional Euclidean space. Define the function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$d(\mathbf{x}, \mathbf{y}) = \left\| \mathbf{x} - \mathbf{y} \right\| = \sqrt{\sum_{i=0}^{n-1} \left| x_i - y_i \right|^2},$$

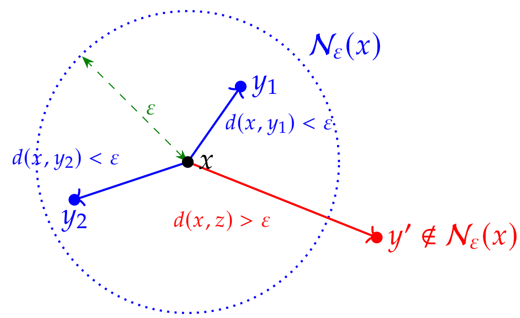
  where $\mathbf{x} = (x_0, x_1, \ldots x_{n-1})$ and $\mathbf{y} = (y_0, \ldots, y_{n-1})$ are vectors in $\mathbb{R}^n$.

Neighborhood (Metric Space)

Definition. Let (S, d) be a metric space, where S is a set and $d : S \times S \to \mathbb{R}$ is a metric. For $x \in S$ and $\varepsilon > 0$, the **ε-neighborhood of x**, denoted by $\mathcal{N}_\varepsilon(x)$, is defined as

$$\mathcal{N}_\varepsilon(x) := \{y \in S : d(x, y) < \varepsilon\}.$$

Remark.

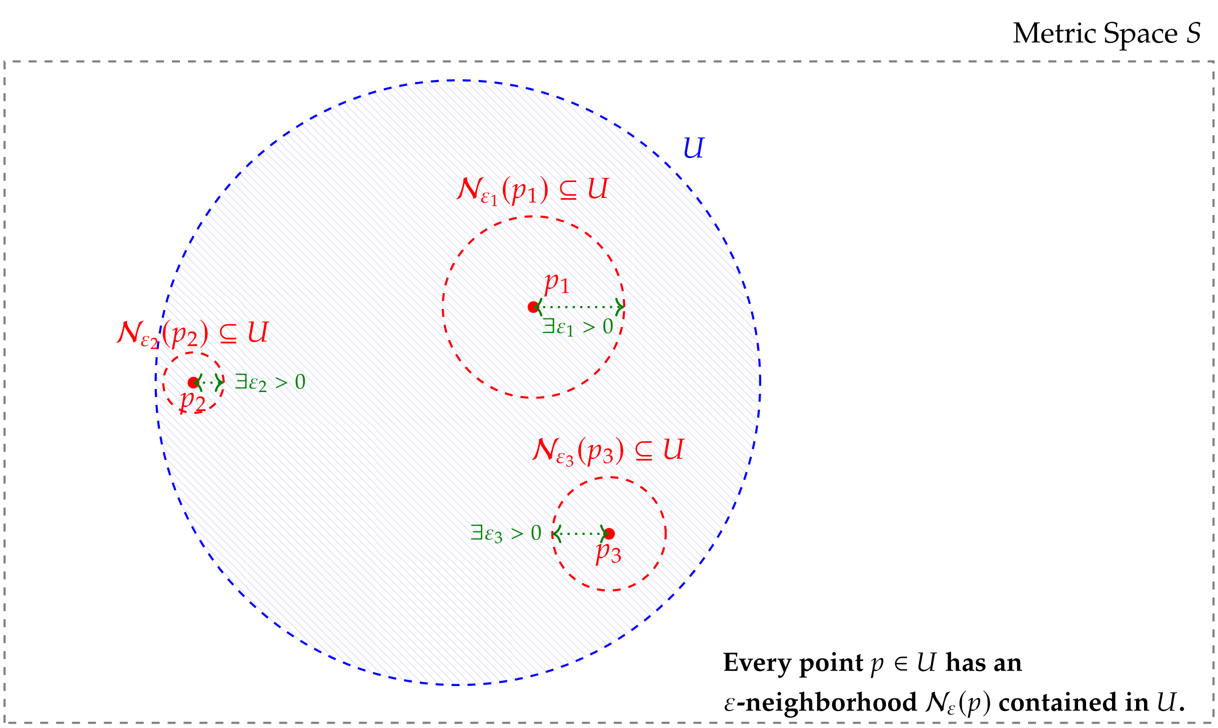


Open Set (Metric Space)

Definition. Let (S, d) be a metric space, where S is a set and $d : S \times S \to \mathbb{R}$ is a metric. Then

$$U \subseteq S \text{ is } \textbf{open} \text{ in } S \overset{\text{def}}{\iff} \forall p \in U,\ \exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(p) \subseteq U.$$

Remark.



Exercise (Metric Topology). Let (S, d) be a metric space, where S is a set and $d : S \times S \to \mathbb{R}$ is a metric. Consider the set τ of all open sets of S:

$$\begin{aligned}\tau :&= \left\{U \subseteq S : U \text{ is open in } S\right\} \\ &= \left\{U \subseteq S : \forall p \in U,\ \exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(p) \subseteq U\right\}.\end{aligned}$$

We claim that τ is the topology induced by the metric d on the space S:

(O1) $S \in \tau$ and $\varnothing \in \tau$:

($\varnothing \in \tau$) The condition

$$\text{“}\forall p \in U,\ \exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(p) \subseteq U\text{”}$$

is vacuously true for $U = \varnothing$. Therefore $\varnothing \in \tau$.

($S \in \tau$) For $p \in S$, the ε-neighborhhod of p is defined as

$$\mathcal{N}_\varepsilon(p) = \left\{q \in S : d(p, q) < \varepsilon\right\} \subseteq S.$$

Since S is the entire space, $\mathcal{N}_\varepsilon(p) \subseteq S$ for any $\varepsilon > 0$.

(O2) τ is closed under arbitrary unions:

Let $\{U_i\}_{i\in I}$ be an arbitrary collection of sets in τ. Let $p \in \bigcup_{i\in I} U_i$. Then

$$\exists i_0 \in I \quad \text{such that} \quad p \in U_{i_0}.$$

Since $U_{i_0} \in \tau$, there exists $\epsilon > 0$ such that $\mathcal{N}_\epsilon(p) \subseteq U_{i_0}$. Then

$$\mathcal{N}_\epsilon(p) \subseteq U_{i_0} \subseteq \bigcup_{i\in I} U_i.$$

Thus, $\bigcup_{i\in I} U_i \in \tau$.

(O3) τ is closed under finite intersections:

Let $U_1, U_2 \in \tau$, and let $p \in (U_1 \cap U_2)$. Then

$$\begin{aligned}&\exists \varepsilon_1 > 0 \quad \text{such that} \quad \mathcal{N}_{\varepsilon_1}(p) \subseteq U_1, \\ &\exists \varepsilon_2 > 0 \quad \text{such that} \quad \mathcal{N}_{\varepsilon_2}(p) \subseteq U_2.\end{aligned}$$

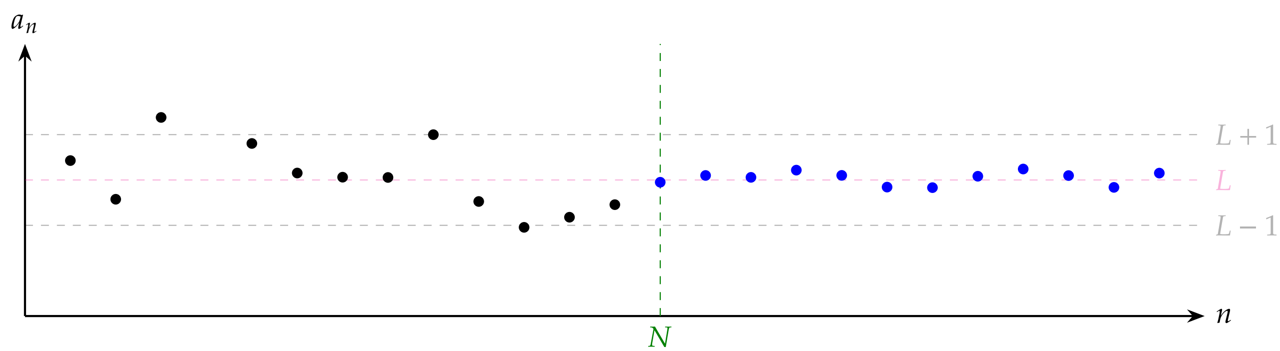
Define $\varepsilon := \min(\varepsilon_1, \varepsilon_2)$. Then

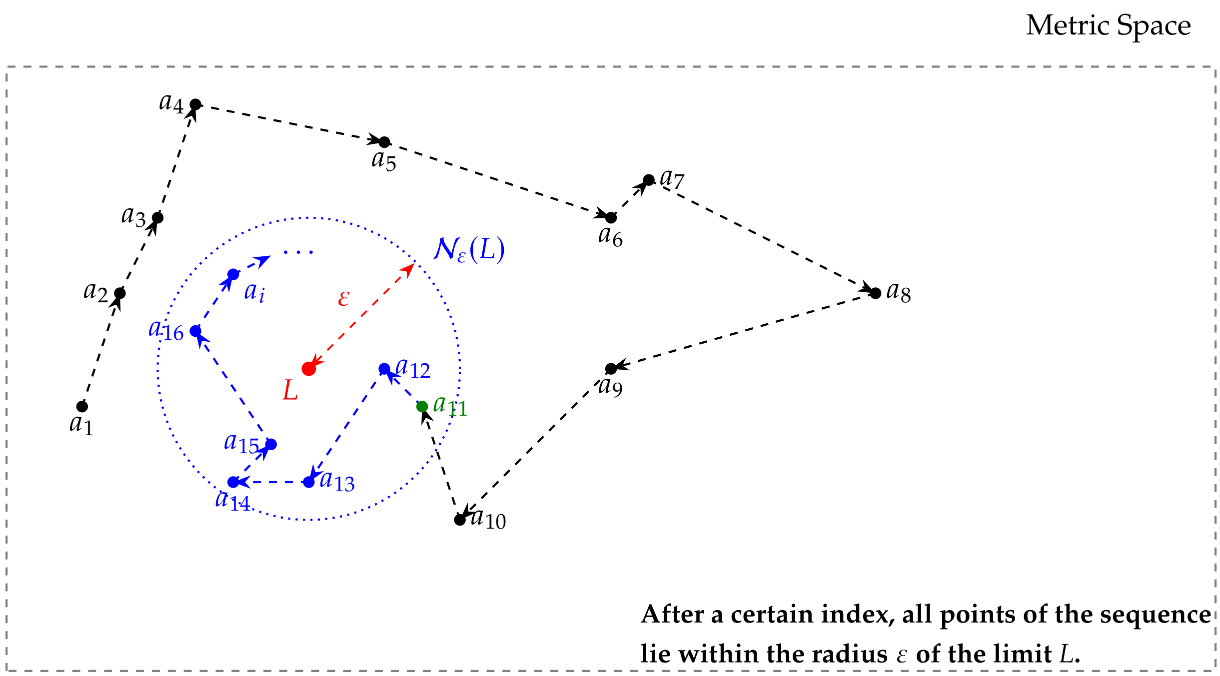
$$\mathcal{N}_\varepsilon(p) \subseteq \mathcal{N}_{\varepsilon_i}(p) \subseteq U_i \quad \text{for} \quad i = 1, 2.$$

Thus $\mathcal{N}_\varepsilon(p) \subset U_1 \cap U_2$, and so $U_1 \cap U_2 \in \tau$.

Note (Convergence of Sequences). A sequence $\{a_n\}_{n=1}^{\infty} (\subseteq \mathbb{R})$ is **converge** to $L \in \mathbb{R}$ if and only if

$$
\begin{aligned}
&\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \left[n \geq N \implies |a_n - L| < \varepsilon\right] \\
\iff &\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \left[n \geq N \implies d(a_n, L) < \varepsilon\right] \\
\iff &\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \left[n \geq N \implies a_n \in \mathcal{N}_\varepsilon(L)\right]
\end{aligned}
$$





Continuity of Functions

Definition. Let $S \subseteq$ be a non-empty subset of $\mathbb{R}$. Let $f : S \to \mathbb{R}$ be a real-valued function, and let $a \in S$. We say that f is **continuous at** a if and only if

$$\lim_{x \to a} f(x) = f(a).$$

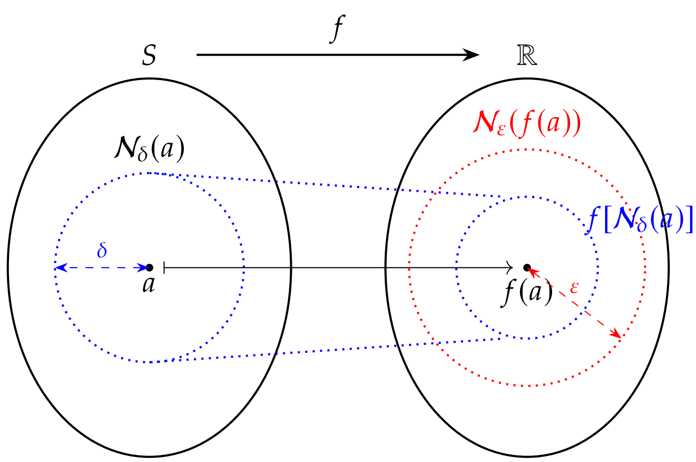
That is,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad |x-a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

If f is continuous on every point of S, then f is called a **continuous function on** S.

Remark.

$$\begin{aligned}
&\forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad |x-a| < \delta \implies |f(x) - f(a)| < \varepsilon \\
\iff &\forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad x \in \mathcal{N}_\delta(a) \implies f(x) \in \mathcal{N}_\varepsilon(f(a)) \\
\iff &\forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad f(x) \in f[\mathcal{N}_\delta(a)] \implies f(x) \in \mathcal{N}_\varepsilon(f(a)) \quad \because f[\mathcal{N}_\delta(a)] = \left\{ f(x) : x \in \mathcal{N}_\delta(a) \right\} \\
\iff &\forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad f[\mathcal{N}_\delta(a)] \subseteq \mathcal{N}_\varepsilon\left(f(a)\right).
\end{aligned}$$



Remark. f is discontinuous at a if and only if

$$\begin{aligned}
&\exists \varepsilon > 0 \quad \text{such that} \quad \forall \delta > 0, |x-a| < \delta \text{ but } |f(x) - f(a)| \geq \varepsilon \\
\iff &\exists \varepsilon > 0 \quad \text{such that} \quad \forall \delta > 0,\ \mathcal{N}_\varepsilon\left(f(a)\right) \subset f\left[\mathcal{N}_\delta(a)\right].
\end{aligned}$$

Note. TBA

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References

[1] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 8. 위상수학 (a) 위상공간의 정의." YouTube Video, 41:25. Published September 27, 2019. URL: https://www.youtube.com/watch?v=q8BtXIFzo2Q.

[2] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 9. 위상수학 (b) 해석학개론과 거리위상" YouTube Video, 33:43. Published September 29, 2019. URL: https://www.youtube.com/watch?v=uJOGw7Yxk7c&t=242s.