

# Complex Analysis

Ji, Yong-hyeon

December 3, 2025

## Contents

<b>1</b>	<b>Complex Numbers</b>	<b>3</b>
1.1	Complex Numbers . . . . .	3
1.2	Vectors and Moduli . . . . .	3
1.3	Complex Conjugation . . . . .	3
1.4	Polar and Exponential Form . . . . .	4
1.5	Products, Powers, and Arguments . . . . .	4
1.6	Roots of Complex Numbers . . . . .	4
1.7	Regions in the Complex Plane . . . . .	5
1.8	Exercises . . . . .	6
<b>2</b>	<b>Analytic Functions</b>	<b>14</b>
2.1	Functions of a Complex Variable . . . . .	14
2.2	Mappings . . . . .	14
2.3	Limits and Related Theorems . . . . .	14
2.3.1	Limits involving $\infty$ . . . . .	15
2.4	Continuity . . . . .	15
2.5	Derivatives . . . . .	15
2.6	Cauchy–Riemann Equations . . . . .	16
2.7	Analytic Functions . . . . .	16
2.8	Harmonic Functions . . . . .	17
<b>3</b>	<b>Elementary Functions</b>	<b>27</b>
3.1	The Exponential Function . . . . .	27
3.2	The Logarithmic Function . . . . .	29
3.3	Branches and Derivatives of Logarithms . . . . .	30
3.4	Complex Exponents . . . . .	31
3.5	Trigonometric Functions . . . . .	32
3.6	Hyperbolic Functions . . . . .	33
3.7	Inverse Trigonometric and Hyperbolic Functions . . . . .	34
3.8	Exercises . . . . .	36

<b>4</b>	<b>Integrals: Part I</b>	<b>44</b>
4.1	Definite Integrals of Complex-Valued Functions . . . . .	44
4.2	Contours . . . . .	45
4.3	Contour Integrals . . . . .	46
4.4	Upper Bounds for Moduli of Contour Integrals . . . . .	48
4.5	Antiderivatives and Path Independence . . . . .	49
4.6	Cauchy–Goursat Theorem . . . . .	50
4.7	Integrals on Simply and Multiply Connected Domains . . . . .	51
4.8	Cauchy Integral Formula . . . . .	54
4.8.1	Cauchy formulas for derivatives . . . . .	54
4.9	Consequences of the Cauchy Integral Formula . . . . .	56
4.9.1	Analyticity of derivatives . . . . .	56
4.10	Morera’s Theorem . . . . .	56
4.10.1	Cauchy’s Inequalities . . . . .	56
4.11	Liouville’s Theorem and the Fundamental Theorem of Algebra . . . . .	57
4.12	Maximum Modulus Principle . . . . .	57
4.13	Exercises . . . . .	58
<b>5</b>	<b>Series</b>	<b>66</b>
5.1	Convergence of Sequences . . . . .	66
5.2	Convergence of Series . . . . .	67
5.3	Power Series and Taylor Series . . . . .	67
5.4	Laurent Series . . . . .	68
5.5	Absolute and Uniform Convergence of Power Series . . . . .	69
5.6	Consequences for Sums of Power/Laurent Series . . . . .	69

# 1 Complex Numbers

## 1.1 Complex Numbers

**Definition 1.1** (Complex numbers and parts). A complex number is an ordered pair  $(x, y) \in \mathbb{R}^2$ , denoted  $z = (x, y)$  or  $z = x + iy$ , with **real part**  $\operatorname{Re} z = x$  and **imaginary part**  $\operatorname{Im} z = y$ .

*Remark.* Two complex numbers are equal iff they have the same real and imaginary parts.

**Definition 1.2** (Algebra). For  $z_1 = (x_1, y_1)$  and  $z_2 = (x_2, y_2)$ ,

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2), \quad z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2).$$

Let  $i = (0, 1)$ . then  $i^2 = -1$  and every  $z$  can be written  $x + iy$ .

*Remark* (Basic properties). The complex numbers satisfy the usual commutative, associative, and distributive laws;  $0 = (0, 0)$  and  $1 = (1, 0)$  are additive/multiplicative identities. For  $z \neq 0$ , the multiplicative inverse is

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}.$$

If  $z_2 \neq 0$ ,

$$\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}.$$

**Definition 1.3** (Binomial formula). For  $n \in \mathbb{N}$  and  $z_1, z_2 \in \mathbb{C}$ ,

$$(z_1 + z_2)^n = \sum_{k=0}^n \binom{n}{k} z_1^k z_2^{n-k}.$$

## 1.2 Vectors and Moduli

**Definition 1.4** (Modulus and distance). For  $z = x + iy$ , the **modulus** is  $|z| = \sqrt{x^2 + y^2}$ . The distance between  $z_1 = (x_1, y_1)$  and  $z_2 = (x_2, y_2)$  is  $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ .

*Remark.* The circle of center  $z_0$  and radius  $R > 0$  is  $\{z : |z - z_0| = R\}$ .

## 1.3 Complex Conjugation

**Definition 1.5** (Conjugate). For  $z = x + iy$ , the **conjugate** is  $\bar{z} = x - iy$ .

**Theorem 1.6** (Conjugation identities). For any  $z, z_1, z_2 \in \mathbb{C}$  (with  $z_2 \neq 0$ ),

$$\overline{\bar{z}} = z, \quad |z| = |\bar{z}|, \quad \overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \quad \frac{\bar{z}_1}{z_2} = \frac{\bar{\bar{z}}_1}{\bar{z}_2},$$

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i}, \quad |z|^2 = z \bar{z}, \quad |z_1 z_2| = |z_1| |z_2|.$$

**Theorem 1.7** (Triangle inequality). For all  $z_1, z_2 \in \mathbb{C}$ ,

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Consequently,  $|z_1 + z_2| \geq ||z_1| - |z_2||$  and for any  $n \in \mathbb{N}$ ,

$$\left| \sum_{k=1}^n z_k \right| \leq \sum_{k=1}^n |z_k|.$$

## 1.4 Polar and Exponential Form

**Definition 1.8** (Polar form, argument). For  $z \neq 0$ , write  $z = r(\cos \theta + i \sin \theta) = re^{i\theta}$  with  $r = |z|$  and any **argument**  $\theta \in \arg z = \{\text{Arg} z + 2\pi k : k \in \mathbb{Z}\}$ , where  $\text{Arg} z \in (-\pi, \pi]$  is the principal value. (For  $z = 0$ ,  $\theta$  is undefined.)

**Definition 1.9** (Euler's formula).  $e^{i\theta} = \cos \theta + i \sin \theta$  ( $\theta \in \mathbb{R}$ ).

*Remark* (Parametrizing circles). The circle  $|z| = R$  has parametrization  $z = Re^{i\theta}$ ,  $0 \leq \theta \leq 2\pi$ . The circle  $|z - z_0| = R$  has  $z = z_0 + Re^{i\theta}$ .

## 1.5 Products, Powers, and Arguments

**Proposition 1.10** (Product/quotient in polar form). If  $z_j = r_j e^{i\theta_j}$  ( $j = 1, 2$ ) with  $r_j > 0$ , then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}, \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}, \quad z^{-1} = \frac{1}{r} e^{-i\theta}.$$

For  $n \in \mathbb{Z}$ ,  $z^n = r^n e^{in\theta}$ .

**Corollary 1.11** (de Moivre). For  $n \in \mathbb{Z}$ ,  $(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$ .

**Theorem 1.12** (Arguments). If  $z_j = r_j e^{i\theta_j}$  ( $j = 1, 2$ ), then  $\arg(z_1 z_2) = \arg z_1 + \arg z_2$  and  $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \pmod{2\pi}$ . Using principal values requires care at the branch cut.

## 1.6 Roots of Complex Numbers

**Theorem 1.13** (All  $n$ th roots). Let  $z_0 = r_0 e^{i\theta_0} \neq 0$  and  $n \in \mathbb{N}$ . The solutions of  $z^n = z_0$  are

$$z_k = \sqrt[n]{r_0} \exp\left(i \frac{\theta_0 + 2\pi k}{n}\right), \quad k = 0, 1, \dots, n-1.$$

These  $n$  distinct roots lie on the circle  $|z| = \sqrt[n]{r_0}$  at equal angular spacing  $2\pi/n$ . The root with  $k = 0$  (when  $\theta_0 = \text{Arg} z_0$ ) is the **principal root**.

*Remark* (Roots of unity). For  $z_0 = 1$ , the  $n$ th roots are  $e^{2\pi i k/n}$ ,  $k = 0, \dots, n-1$ .

**Example 1.14** (Cube roots of  $-8i$ ). Since  $-8i = 8 e^{-i\pi/2}$ , the cube roots are  $2 e^{i(-\pi/6 + 2\pi k/3)}$ ,  $k = 0, 1, 2$ .

## 1.7 Regions in the Complex Plane

**Definition 1.15** (Neighborhoods). An  $\varepsilon$ -neighborhood of  $z_0$  is  $\{z : |z - z_0| < \varepsilon\}$ . The **deleted** (punctured) neighborhood is  $\{z : 0 < |z - z_0| < \varepsilon\}$ .

**Definition 1.16** (Interior, exterior, boundary). A point  $z_0$  is an interior point of  $S$  if some neighborhood of  $z_0$  lies in  $S$ ; an exterior point if some neighborhood lies in  $S^c$ ; otherwise  $z_0$  is on the boundary  $\partial S$ .

**Definition 1.17** (Open/closed, closure). A set is **open** if it contains none of its boundary points; **closed** if it contains all of them. The **closure**  $\bar{S}$  is  $S \cup \partial S$ .

**Definition 1.18** (Connected, domain, region). An open set  $S$  is **connected** if any two points can be joined by a polygonal line lying in  $S$ . A nonempty connected open set is a **domain**. A **region** is a domain together with some (possibly all or none) of its boundary points.

**Definition 1.19** (Boundedness).  $S$  is **bounded** if  $S \subset \{z : |z| < R\}$  for some  $R > 0$ .

**Definition 1.20** (Accumulation points). A point  $z_0$  is an accumulation point of  $S$  if every deleted neighborhood of  $z_0$  contains a point of  $S$ . A set is closed iff it contains all its accumulation points.

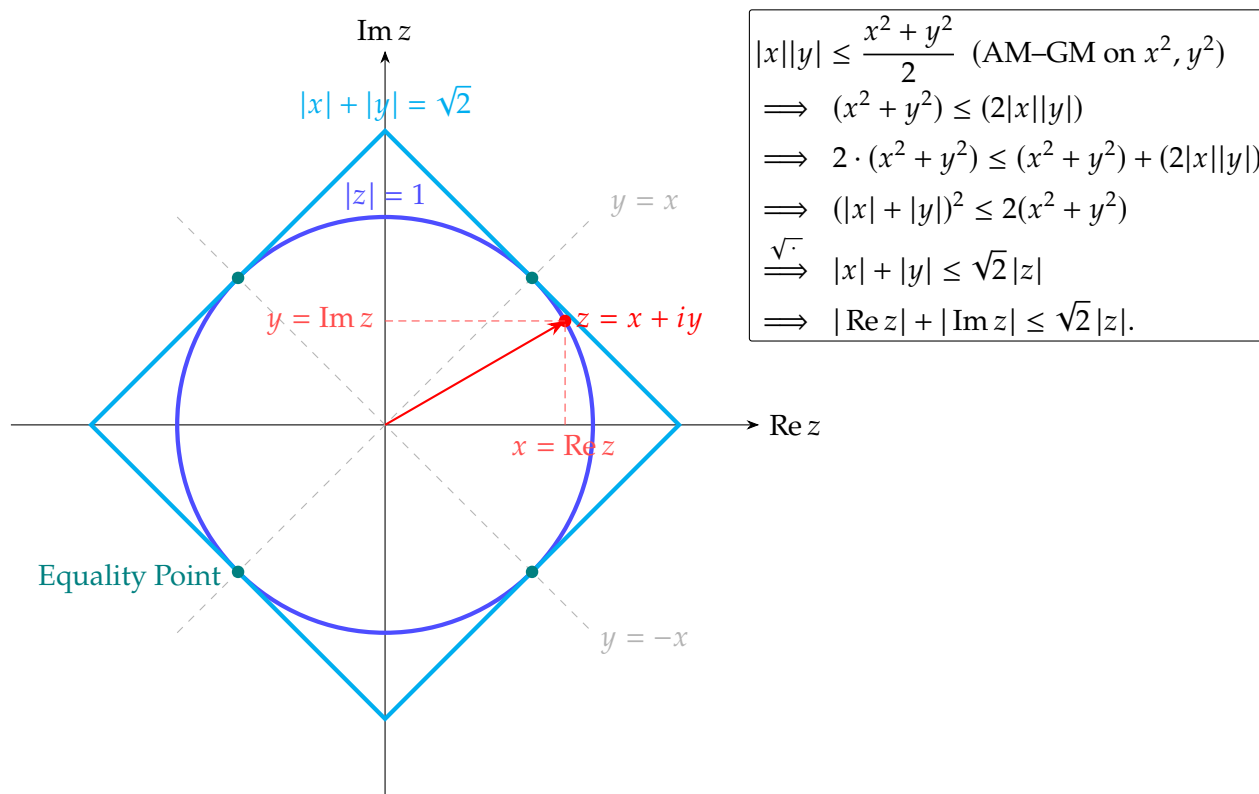
## 1.8 Exercises

1. Verify that  $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$ .

**Sol.** Let  $z = x + iy$ , so that  $x = \operatorname{Re} z$ ,  $y = \operatorname{Im} z$ , and  $|z| = \sqrt{x^2 + y^2}$ . Then

$$\begin{aligned}
 \sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z| &\iff \sqrt{2}\sqrt{x^2 + y^2} \geq |x| + |y| \\
 &\iff 2(x^2 + y^2) \geq (|x| + |y|)^2 \\
 &\iff 2(x^2 + y^2) \geq x^2 + y^2 + 2|x||y| \quad (\because |x|^2 = x^2, |y|^2 = y^2) \\
 &\iff x^2 + y^2 \geq 2|x||y| \quad \text{by subtracting } x^2 + y^2 \text{ from both sides} \\
 &\iff x^2 + y^2 \geq 2\sqrt{x^2 y^2} \\
 &\iff \frac{x^2 + y^2}{2} \geq \sqrt{x^2 y^2} \\
 &\iff \frac{a + b}{2} \geq \sqrt{ab} \quad \text{by setting } a := x^2 \text{ and } b := y^2; \quad (\text{AM-GM inequality})
 \end{aligned}$$

Hence it holds.



□

2. By factoring  $z^4 - 4z^2 + 3$  into two quadratic factors show that if  $z$  lies on the circle  $|z| = 2$ , then

$$\left| \frac{1}{z^4 - 4z^2 + 3} \right| \leq \frac{1}{3}.$$

**Sol.** Since  $z^4 - 4z^2 + 3 = (z^2 - 1)(z^2 - 3)$ , we have

$$|z^4 - 4z^2 + 3| = |z^2 - 1| |z^2 - 3|.$$

For  $|z| = 2$  one has  $|z^2| = |z|^2 = 4$ . By the triangle inequality,

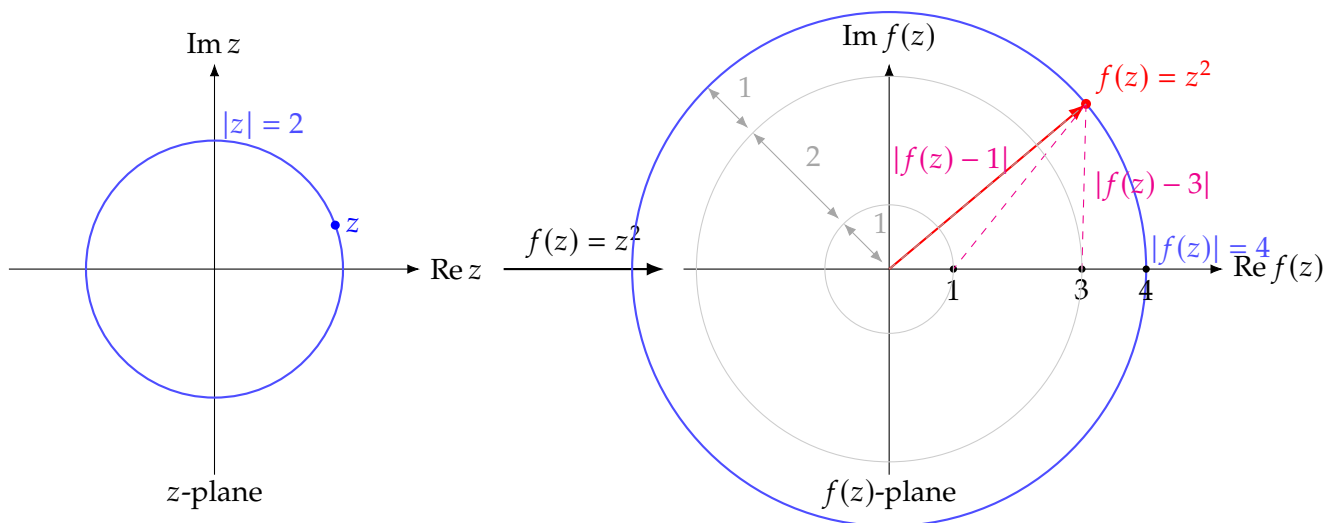
$$|z^2 - 1| \geq ||z^2| - |1|| = |4 - 1| = 3, \quad |z^2 - 3| \geq ||z^2| - |3|| = |4 - 3| = 1.$$

Hence

$$|z^4 - 4z^2 + 3| \geq 3 \cdot 1 = 3,$$

and therefore

$$\left| \frac{1}{z^4 - 4z^2 + 3} \right| = \frac{1}{|z^4 - 4z^2 + 3|} \leq \frac{1}{3}.$$



For equality in the reverse triangle inequalities we must have  $z^2$  and the positive reals 1, 3 on the same ray from the origin, i.e.  $z^2 = 4$ . Together with  $|z| = 2$  this forces  $z = \pm 2$ , and indeed

$$|(\pm 2)^4 - 4(\pm 2)^2 + 3| = |16 - 16 + 3| = 3,$$

so the bound is sharp precisely at  $z = \pm 2$ . □

3. Prove the finite geometric sum

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \quad (z \neq 1)$$

and deduce Lagrange's trigonometric identity

$$1 + \cos \theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin((2n+1)\theta/2)}{2\sin(\theta/2)} \quad (0 < \theta < 2\pi).$$



4. Prove that the usual formula solves the quadratic equation

$$az^2 + bz + c = 0 \quad (a \neq 0)$$

when the coefficient  $a, b$ , and  $c$  are complex numbers. Specifically, by completing the square on the left-hand side, derive the **quadratic formula**

$$z = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

where both square roots are to be considered when  $b^2 - 4ac \neq 0$ . Use this result to find the roots of the equation

$$z^2 + 2z + (1 - i) = 0.$$

**Sol.** Since

$$az^2 + bz + c = a\left(z^2 + \frac{b}{a}z\right) + c = a\left(z + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c = a\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c,$$

we have

$$a\left(z + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c \iff \left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking square roots of both sides yields

$$z + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}, \quad \text{whence} \quad z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Consider  $z^2 + 2z + (1 - i)$  with  $a = 1$ ,  $b = 2$ , and  $c = 1 - i$ . The discriminant is

$$\Delta = b^2 - 4ac = 4 - 4(1 - i) = 4i.$$

Since

$$\sqrt{i} = \frac{1+i}{\sqrt{2}} \quad \left( \text{indeed, } \left( \frac{1+i}{\sqrt{2}} \right)^2 = \frac{1+2i-1}{2} = i \right),$$

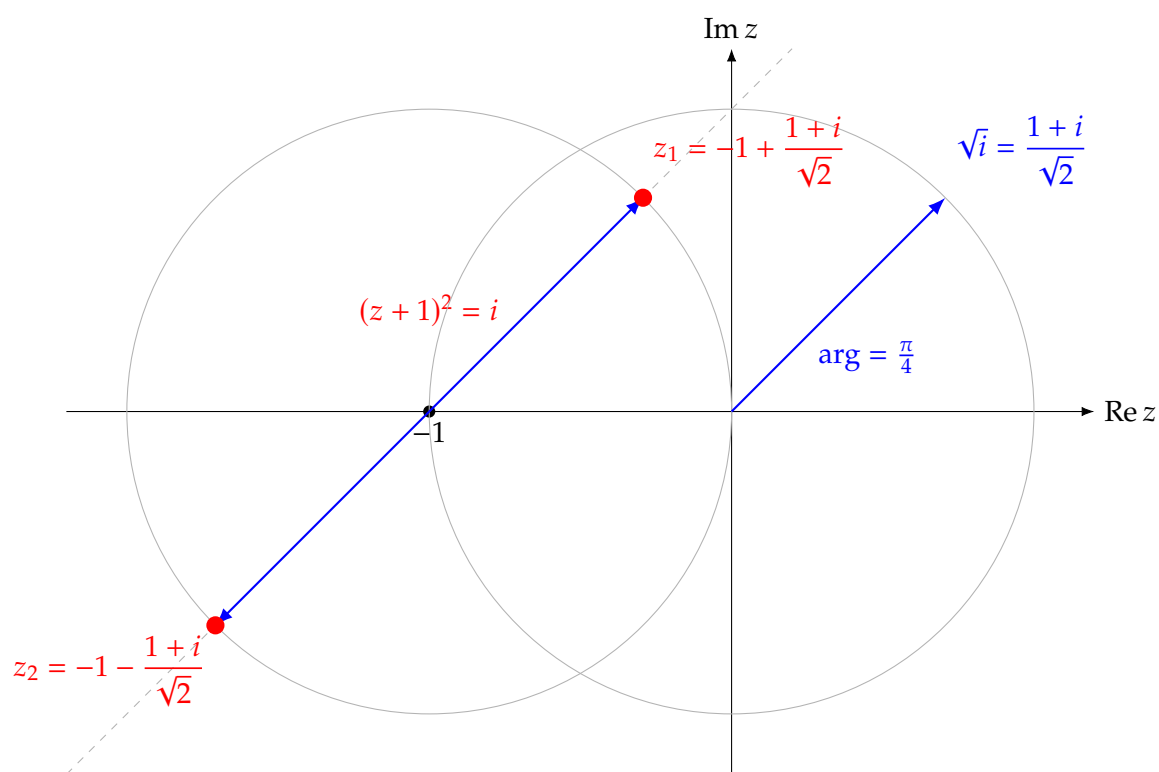
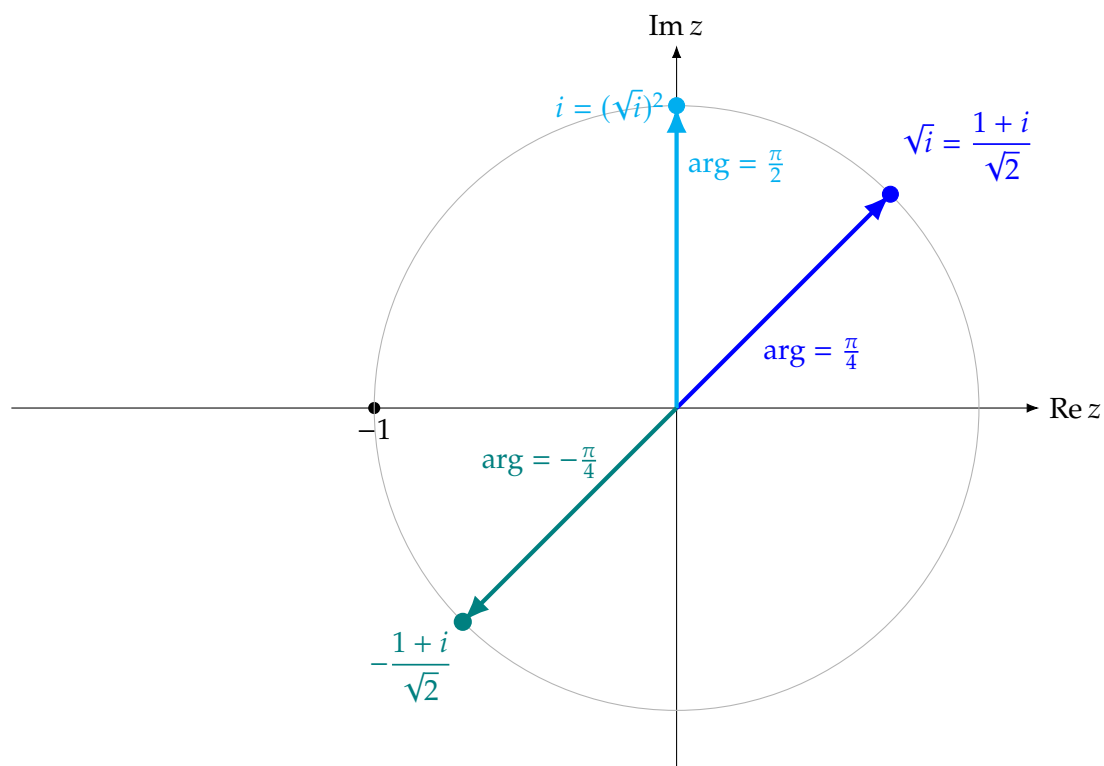
we may take  $\sqrt{\Delta} = \sqrt{4i} = 2\sqrt{i} = \sqrt{2}(1+i)$ . Therefore

$$z = \frac{-2 \pm \sqrt{4i}}{2} = -1 \pm \sqrt{i} = -1 \pm \frac{1+i}{\sqrt{2}}.$$

Thus the roots are

$$z_1 = -1 + \frac{1+i}{\sqrt{2}}, \quad z_2 = -1 - \frac{1+i}{\sqrt{2}}.$$

Note that  $z_1, z_2$  are roots of  $(z+1)^2 = i$ .



□

5. Determine the accumulation points of each sequence:

$$z_n = i^n, \quad z_n = \frac{i^n}{n}, \quad z_n = (-1)^n(1+i)\frac{n-1}{n}.$$

**Sol.** content...

□

6. Prove that a finite set of points  $z_1, z_2, \dots, z_n$  cannot have any accumulation points.

**Sol.** Recall that  $w \in \mathbb{C}$  is an accumulation point of  $F$  iff for every  $\varepsilon > 0$  the punctured ball  $B(w, \varepsilon) \setminus \{w\}$  intersects  $F$  (equivalently,  $B(w, \varepsilon)$  contains a point of  $F$  distinct from  $w$ ).

Fix  $w \in \mathbb{C}$ . Consider the finite set of distances

$$D := \{|w - z_k| : 1 \leq k \leq n\} \subset [0, \infty).$$

Let  $d := \min D$ . There are two cases.

**Case 1:**  $w \notin F$ . Then  $d > 0$ . For  $\varepsilon := \frac{d}{2}$  we have  $B(w, \varepsilon) \cap F = \emptyset$ , hence  $w$  is not an accumulation point.

**Case 2:**  $w = z_j$  for some  $j$ . If  $n = 1$ , then  $F = \{w\}$  and for any  $\varepsilon > 0$  small enough,  $B(w, \varepsilon) \cap (F \setminus \{w\}) = \emptyset$ , so  $w$  is not an accumulation point. If  $n \geq 2$ , put

$$d' := \min_{k \neq j} |z_j - z_k| > 0$$

(since the minimum of finitely many positive numbers is positive). For  $\varepsilon := \frac{d'}{2}$  we have  $B(w, \varepsilon) \cap (F \setminus \{w\}) = \emptyset$ , so again  $w$  is not an accumulation point.

Since **no**  $w \in \mathbb{C}$  can be an accumulation point of  $F$ , the set  $F$  has no accumulation points.

□

**Definition 1.21.** Let  $(z_n)_{n \geq 1}$  be a sequence in  $\mathbb{C}$ . A point  $w \in \mathbb{C}$  is an **accumulation point** (or **subsequential limit**) of  $(z_n)$  if there exists a strictly increasing map  $k \mapsto n_k$  such that  $\lim_{k \rightarrow \infty} z_{n_k} = w$ .

(1)  $z_n = i^n$ .

**Claim.** The set of accumulation points is  $\{1, i, -1, -i\}$ .

**Proof.** Since  $i^n$  is 4-periodic, the image set is  $S := \{1, i, -1, -i\}$ , and each element of  $S$  occurs infinitely many times. Hence for each  $s \in S$  there exists the constant subsequence  $z_{n_k} \equiv s$ , so  $s$  is an accumulation point. Conversely, any subsequence takes all its values in the finite set  $S$ , thus has a further constant subsequence by the pigeonhole principle; hence every accumulation point lies in  $S$ . Therefore the accumulation set equals  $S$ .

(2)  $z_n = \frac{i^n}{n}$ .

**Claim.** The only accumulation point is 0.

**Proof.** Since  $|i^n| = 1$  for all  $n$ , we have

$$|z_n| = \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0.$$

Thus  $z_n \rightarrow 0$ , and a convergent sequence has the singleton set  $\{0\}$  as its accumulation set.

(3)  $z_n = (-1)^n(1+i) \frac{n-1}{n}$ .

**Claim.** The accumulation points are  $\{1+i, -(1+i)\}$ .

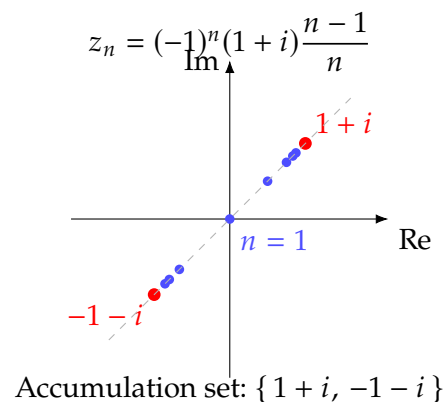
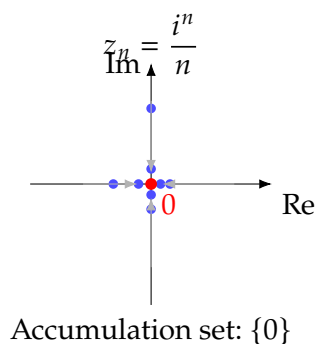
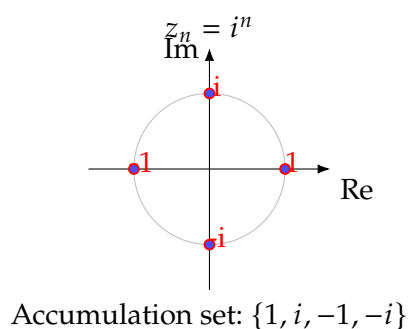
**Proof.** Decompose into even/odd subsequences. For  $n = 2m$ ,

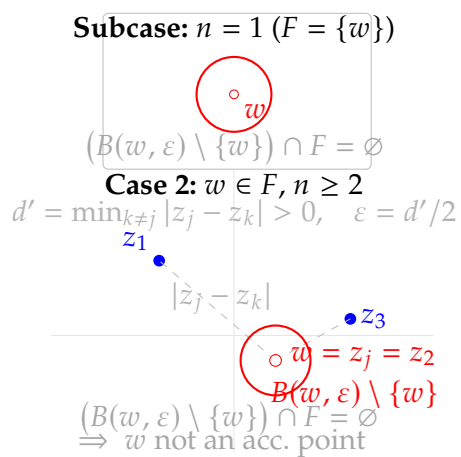
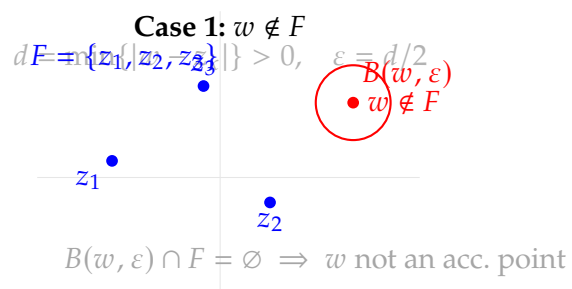
$$z_{2m} = (1+i) \frac{2m-1}{2m} \xrightarrow{m \rightarrow \infty} 1+i.$$

For  $n = 2m+1$ ,

$$z_{2m+1} = -(1+i) \frac{2m}{2m+1} \xrightarrow{m \rightarrow \infty} -(1+i).$$

Hence  $1+i$  and  $-(1+i)$  are accumulation points. If  $w$  is an accumulation point, then there exists  $n_k \rightarrow \infty$  with  $z_{n_k} \rightarrow w$ . Since  $\frac{n_k-1}{n_k} \rightarrow 1$  and  $(-1)^{n_k} \in \{\pm 1\}$ , every limit  $w$  must belong to  $\{\pm(1+i)\}$ . Thus the accumulation set is exactly  $\{1+i, -(1+i)\}$ .





## 2 Analytic Functions

### 2.1 Functions of a Complex Variable

**Definition 2.1** (Function and domain). Let  $S \subset \mathbb{C}$ . A **function**  $f$  on  $S$  assigns to each  $z \in S$  a complex number  $w = f(z)$ . The set  $S$  is the **domain** (domain of definition) of  $f$ . As real functions, we write  $f(z) = u(x, y) + iv(x, y)$  for  $z = x + iy$ ; in polar form,  $f(z) = u(r, \theta) + iv(r, \theta)$ .

**Definition 2.2** (Polynomials and rational functions). If  $n \in \mathbb{Z}_{\geq 0}$  and  $a_0, \dots, a_n \in \mathbb{C}$  with  $a_n \neq 0$ , the polynomial

$$P(z) = a_0 + a_1z + \dots + a_nz^n$$

has degree  $n$ . A **rational function** is  $P(z)/Q(z)$ , defined where  $Q(z) \neq 0$ .

**Example 2.3** (Single-valued choice of a multiple-valued expression). For  $z \neq 0$  with  $z = re^{i\theta}$  ( $-\pi < \theta \leq \pi$ ), the square root has two values  $z^{1/2} = \pm\sqrt{r}e^{i\theta/2}$ . Selecting the “+” value defines a single-valued branch on  $\mathbb{C}^\times$ ; setting  $f(0) = 0$  extends it to  $z = 0$  (not analytic there).

### 2.2 Mappings

**Definition 2.4** (Mapping, image, range, inverse image). Viewing  $f$  as a mapping  $f : S \rightarrow \mathbb{C}$ , the **image** of  $z$  is  $w = f(z)$ ; the image of  $T \subset S$  is  $f(T)$ ; the **range** is  $f(S)$ . The **inverse image** of  $w_0$  is  $\{z \in S : f(z) = w_0\}$ .

**Observation** (Basic geometric actions). [leftmargin=1.5em]

- $w = z + 1$  translates one unit to the right.
- $w = iz = re^{i(\theta+\pi/2)}$  rotates by  $\pi/2$  counterclockwise.
- $w = \bar{z} = x - iy$  reflects across the real axis.

**Example 2.5** ( $w = z^2$  as a mapping). With  $z = x + iy$ , we have  $w = u + iv$  where  $u = x^2 - y^2$ ,  $v = 2xy$ . The first quadrant region  $\{x \geq 0, y \geq 0, xy \leq 1\}$  maps onto the horizontal strip  $\{0 \leq v \leq 2\}$ .

#### Mapping by the exponential

If  $w = e^z = e^{x+iy} = e^x(\cos y + i \sin y) = \rho e^{i\theta}$ , then  $\rho = e^x$  and  $\theta = y$ . Thus vertical lines  $\{x = \text{const}\}$  map to circles  $\{|w| = \text{const}\}$  and horizontal lines  $\{y = \text{const}\}$  map to rays  $\{\arg w = \text{const}\}$ .

### 2.3 Limits and Related Theorems

**Definition 2.6** (Limit). Let  $f$  be defined on a deleted neighborhood of  $z_0$ . We say  $\lim_{z \rightarrow z_0} f(z) = w_0$  if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|f(z) - w_0| < \varepsilon$  whenever  $0 < |z - z_0| < \delta$ .

**Theorem 2.7** (Uniqueness of limits). *If the limit  $\lim_{z \rightarrow z_0} f(z)$  exists, it is unique.*

**Example 2.8.** For  $f(z) = \frac{i}{2}z$  on  $|z| < 1$ ,  $\lim_{z \rightarrow 1} f(z) = \frac{i}{2}$ . For  $f(z) = \bar{z}/z$ ,  $\lim_{z \rightarrow 0} f(z)$  does **not** exist: approaching along the real axis gives 1, along the imaginary axis gives  $-1$ .

**Theorem 2.9** (Limit laws). *If  $\lim_{z \rightarrow z_0} f(z) = f_0$  and  $\lim_{z \rightarrow z_0} g(z) = g_0$ , then*

$$\lim_{z \rightarrow z_0} (f + g) = f_0 + g_0, \quad \lim_{z \rightarrow z_0} f g = f_0 g_0, \quad \lim_{z \rightarrow z_0} \frac{f}{g} = \frac{f_0}{g_0} \quad (g_0 \neq 0).$$

*In particular, polynomials are continuous:  $\lim_{z \rightarrow z_0} P(z) = P(z_0)$ .*

### 2.3.1 Limits involving $\infty$

Neighborhoods of  $\infty$  are exteriors of large disks. One has

$$\lim_{z \rightarrow z_0} f(z) = \infty \iff \lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0, \quad \lim_{z \rightarrow \infty} f(z) = w_0 \iff \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0,$$

$$\text{and } \lim_{z \rightarrow \infty} f(z) = \infty \iff \lim_{z \rightarrow 0} \frac{1}{f(1/z)} = 0.$$

## 2.4 Continuity

**Definition 2.10** (Continuity).  $f$  is continuous at  $z_0$  if  $\lim_{z \rightarrow z_0} f(z) = f(z_0)$ . It is continuous on a region  $R$  if continuous at each  $z_0 \in R$ .

**Theorem 2.11** (Basic properties). *Composition of continuous functions is continuous. If  $f$  is continuous and  $f(z_0) \neq 0$ , then  $f$  is nonzero on some neighborhood of  $z_0$ . If  $f = u + iv$ , then  $f$  is continuous at  $z_0$  iff  $u$  and  $v$  are continuous there. If  $R$  is closed and bounded and  $f$  continuous on  $R$ , then  $|f|$  attains a maximum on  $R$  (boundedness).*

## 2.5 Derivatives

**Definition 2.12** (Complex derivative). If  $f$  is defined on a neighborhood of  $z_0$ , the derivative at  $z_0$  is

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z}, \quad \Delta w = f(z_0 + \Delta z) - f(z_0),$$

when the limit exists.

**Theorem 2.13** (Consequences). *If  $f'(z_0)$  exists then  $f$  is continuous at  $z_0$ . Moreover,*

$$\frac{d}{dz} c = 0, \quad \frac{d}{dz} z = 1, \quad \frac{d}{dz} [cf] = cf', \quad \frac{d}{dz} z^n = nz^{n-1} \quad (n \in \mathbb{Z}, z \neq 0 \text{ if } n < 0),$$

*and the sum/product/quotient rules and chain rule hold exactly as in calculus.*

**Example 2.14.**  $f(z) = z^2 \Rightarrow f'(z) = 2z$ . The function  $f(z) = \bar{z}$  has no complex derivative anywhere. The function  $f(z) = |z|^2$  has derivative only at  $z = 0$  (value 0).

## 2.6 Cauchy–Riemann Equations

Let  $f = u + iv$  with  $u, v$  real-valued.

**Theorem 2.15** (Cauchy–Riemann (CR) equations). *If  $f'(z_0)$  exists then the first partials of  $u, v$  exist at  $(x_0, y_0)$  and satisfy*

$$u_x(x_0, y_0) = v_y(x_0, y_0), \quad u_y(x_0, y_0) = -v_x(x_0, y_0),$$

*and  $f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$ .*

**Theorem 2.16** (Sufficient conditions). *If  $u_x, u_y, v_x, v_y$  exist in a neighborhood of  $z_0$ , are continuous at  $z_0$ , and satisfy the CR equations at  $z_0$ , then  $f'(z_0)$  exists and equals  $u_x + i v_x$ .*

**Example 2.17.**  $f(z) = z^2 = x^2 - y^2 + i 2xy$  satisfies CR everywhere and  $f'(z) = 2z$ . For  $f(z) = |z|^2 = x^2 + y^2$ , the CR equations force  $(x, y) = (0, 0)$ ; hence  $f'$  exists only at 0. For  $f(z) = e^z = e^x(\cos y + i \sin y)$  we have  $f'(z) = e^z$  for all  $z$ .

### CR equations in polar coordinates

If  $f = u(r, \theta) + iv(r, \theta)$  near  $z_0 = r_0 e^{i\theta_0} \neq 0$ , the polar CR system is

$$u_r = \frac{1}{r} v_\theta, \quad v_r = -\frac{1}{r} u_\theta,$$

and  $f'(z_0) = e^{-i\theta_0} (u_r(r_0, \theta_0) + i v_r(r_0, \theta_0))$ .

## 2.7 Analytic Functions

**Definition 2.18** (Analytic/entire/singularity).  $f$  is **analytic** at  $z_0$  if it has a derivative at every point of some neighborhood of  $z_0$ . If analytic at every point of  $\mathbb{C}$ ,  $f$  is **entire**. If  $f$  fails to be analytic at  $z_0$  but is analytic arbitrarily close to  $z_0$ , then  $z_0$  is a **singular point** (singularity) of  $f$ .

**Theorem 2.19** (Algebra and composition). *Sums and products of analytic functions are analytic; a quotient  $f/g$  is analytic where  $g \neq 0$ . If  $f$  is analytic in  $D$  and  $g$  is analytic on  $f(D)$ , then  $g \circ f$  is analytic in  $D$  with  $(g \circ f)' = (g' \circ f) f'$ .*

**Theorem 2.20** (Zero derivative). *If  $f'(z) = 0$  for all  $z$  in a domain  $D$ , then  $f$  is constant on  $D$ .*

**Example 2.21.**

$$f(z) = \frac{z^3 + 4}{(z^2 - 3)(z^2 + 1)}$$

is analytic on  $\mathbb{C} \setminus \{\pm\sqrt{3}, \pm i\}$ . Also  $f(z) = \cosh x \cos y + i \sinh x \sin y$  is entire since CR holds everywhere.

**Theorem 2.22** (Conjugate tests). *If  $f$  and  $\bar{f}$  are both analytic in  $D$ , then  $f$  is constant in  $D$ . If  $f$  is analytic in  $D$  and  $|f|$  is constant, then  $f$  is constant.*



## 2.8 Harmonic Functions

**Definition 2.23** (Harmonicity). A real function  $h(x, y)$  is **harmonic** on a domain if it has continuous second partials and satisfies Laplace's equation

$$\Delta h = h_{xx} + h_{yy} = 0.$$

**Theorem 2.24** (Harmonic components). If  $f = u + iv$  is analytic in  $D$ , then  $u$  and  $v$  are harmonic in  $D$ . Conversely, if  $u$  and  $v$  are harmonic and satisfy the CR equations in  $D$ , then  $f = u + iv$  is analytic in  $D$ ;  $v$  is then a **harmonic conjugate** of  $u$ .

**Example 2.25.**  $f(z) = \frac{i}{z^2}$  is analytic on  $\mathbb{C} \setminus \{0\}$ ; writing it as

$$\frac{i}{z^2} = \frac{2xy + i(x^2 - y^2)}{(x^2 + y^2)^2} = u + iv,$$

both  $u$  and  $v$  are harmonic away from the origin. For  $u(x, y) = y^3 - 3x^2y$ , a harmonic conjugate is  $v(x, y) = -3xy^2 + x^3 + C$ .

### Uniqueness and reflection

**Lemma 2.26** (Identity lemma). If  $f$  is analytic in  $D$  and vanishes on a set with a limit point in  $D$  (e.g. a subdomain or line segment), then  $f \equiv 0$  in  $D$ .

**Theorem 2.27** (Uniqueness from values). An analytic function in  $D$  is uniquely determined in  $D$  by its values on any subdomain or line segment contained in  $D$ .

**Theorem 2.28** (Reflection principle (real axis)). Let  $D$  contain a symmetric neighborhood of a real segment. Then  $f(\bar{z}) = \overline{f(z)}$  in  $D$  iff  $f(x) \in \mathbb{R}$  for all  $x$  on that segment.

## Exercises

1. Show that the following limit does not exist

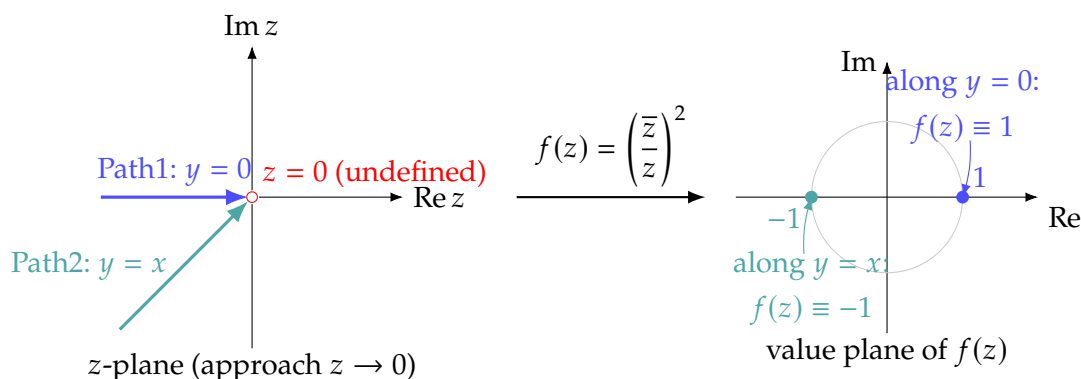
$$\lim_{z \rightarrow 0} \left( \frac{\bar{z}}{z} \right)^2$$

Do this by letting nonzero points  $z = (x, 0)$  and  $z = (x, x)$  approach the origin. (Note that it is not sufficient to simply consider points  $z = (x, 0)$  and  $z = (0, y)$ .)

**Sol.** Let  $z = x + iy \in \mathbb{C}$  with  $x, y \in \mathbb{R}$ . Then

$$\left( \frac{\bar{z}}{z} \right)^2 = \left( \frac{x - iy}{x + iy} \right)^2.$$

If  $z = re^{i\theta}$  with  $r > 0$ , then  $\bar{z}/z = e^{-2i\theta}$ , so  $|(\bar{z}/z)^2| = |e^{-4i\theta}| = 1$ .



**(1) Path 1: approach along the real axis  $y = 0$**

Let  $z = x + 0i = x$  with  $x \in \mathbb{R} \setminus \{0\}$  and  $x \rightarrow 0$ . Then  $\left( \frac{\bar{z}}{z} \right)^2 = \left( \frac{x}{x} \right)^2 = 1$ .

**(2) Path 2: approach along the diagonal  $y = x$**

Let  $z = x + ix = (1 + i)x$  with  $x \in \mathbb{R} \setminus \{0\}$  and  $x \rightarrow 0$ . Then

$$\frac{\bar{z}}{z} = \frac{\overline{(1+i)x}}{(1+i)x} = \frac{(1-i)x}{(1+i)x} = \frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{1-2i+i^2}{1-i^2} = \frac{1-2i-1}{1-(-1)} = \frac{-2i}{2} = -i.$$

Hence

$$\left( \frac{\bar{z}}{z} \right)^2 = (-i)^2 = -1.$$

**(3) Conclusion**

Since the limits along these two paths are different (namely 1 and  $-1$ ), the limit cannot exist.  $\square$

2. Let

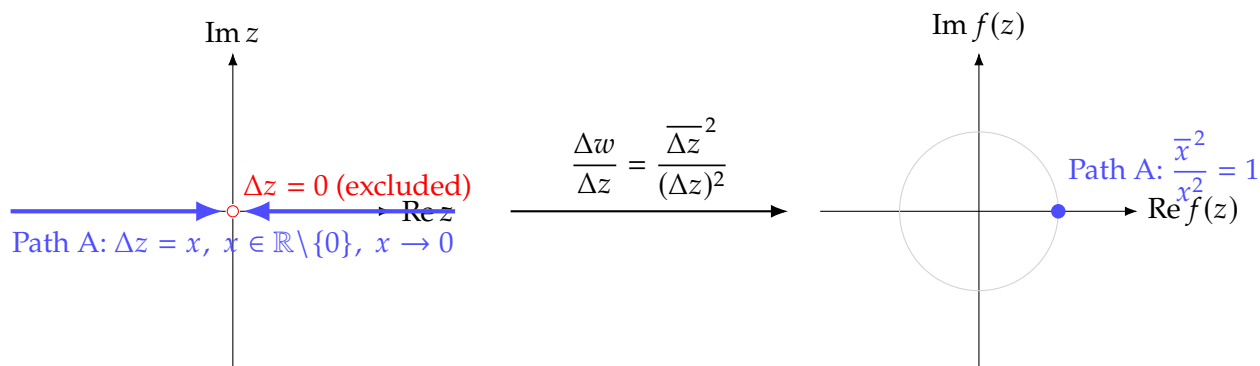
$$f(z) = \begin{cases} \bar{z}^2/z, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Show that if  $z = 0$ , then  $\Delta w / \Delta z = 1$  at each nonzero point on the real and imaginary axes in the  $\Delta z$ , or  $\Delta x \Delta y$ , plane. Then show that  $\Delta w / \Delta z = -1$  at each nonzero point  $(\Delta x, \Delta y)$  on the line  $\Delta y = \Delta x$  in that plane. Conclude from these observations that  $f'(0)$  does not exist. Note that to obtain this result, it is not sufficient to consider only horizontal and vertical approaches to the origin in the  $\Delta z$  plane.

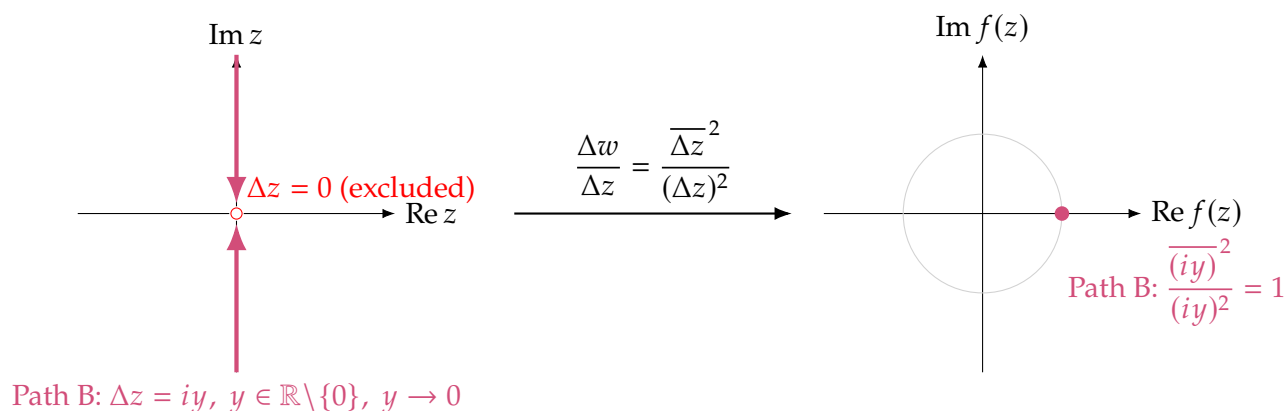
*Proof.* Let  $\frac{\Delta w}{\Delta z} = \frac{f(\Delta z) - f(0)}{\Delta z}$  ( $\Delta z \neq 0$ ). Since  $f(0) = 0$ , for  $\Delta z \neq 0$ ,  $\frac{\Delta w}{\Delta z} = \frac{f(\Delta z)}{\Delta z} = \frac{\overline{\Delta z}^2}{(\Delta z)^2}$ .

(1) Real and imaginary axes.

- Real axis:  $\Delta z = x$  with  $x \in \mathbb{R} \setminus \{0\}$ ,  $\frac{\Delta w}{\Delta z} = \frac{\bar{x}^2}{x^2} = \frac{x^2}{x^2} = 1$ .

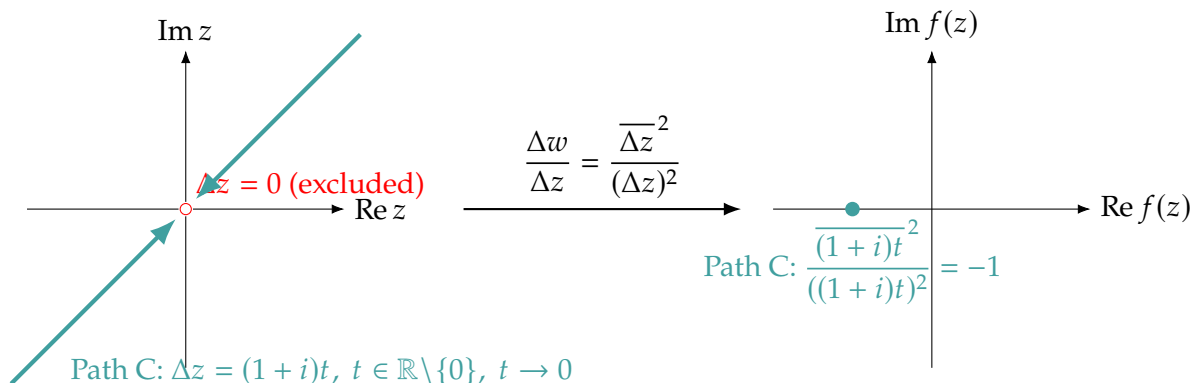


- Imaginary axis:  $\Delta z = iy$  with  $y \in \mathbb{R} \setminus \{0\}$ ,  $\frac{\Delta w}{\Delta z} = \frac{\overline{iy}^2}{(iy)^2} = \frac{(-iy)^2}{(iy)^2} = \frac{-y^2}{-y^2} = 1$ .



(2) **Line**  $\Delta y = \Delta x$ . Let  $\Delta z = (1 + i)x$  with  $x \in \mathbb{R} \setminus \{0\}$ . Then

$$\frac{\Delta w}{\Delta z} = \frac{\overline{(1+i)x}^2}{((1+i)x)^2} = \frac{((1-i)x)^2}{((1+i)x)^2} = \frac{(1-i)^2}{(1+i)^2} = \frac{-2i}{2i} = -1.$$



(3) **Conclusion.** Since the difference quotient equals 1 along the axes but  $-1$  along the line  $\Delta y = \Delta x$ , the limit

$$\lim_{\Delta z \rightarrow 0} \frac{f(\Delta z) - f(0)}{\Delta z}$$

depends on the path and therefore does not exist. Consequently,  $f'(0)$  does not exist.  $\square$

3. Let

$$f(z) = \bar{z}, \quad f(z) = 2x + ixy^2, \quad f(z) = e^{\bar{z}}$$

Then show that  $f'(z)$  does not exist at any point.

**Sol.** Let  $z = x + iy$  and  $f(z) = u + iv$  with  $x, y, u, v \in \mathbb{R}$ . The Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

are necessary and sufficient for complex differentiability.

(1)  $f_1(z) = \bar{z} = \overline{x + iy} = x - iy.$

Here  $u(x, y) = x$ ,  $v(x, y) = -y$ . Thus

$$u_x = 1, \quad u_y = 0, \quad v_x = 0, \quad v_y = -1.$$

The CR require  $u_x = v_y$ , i.e.  $1 = -1$ , which is impossible. Hence  $f'_1$  does not exist anywhere.

(2)  $f_2(z) = 2x + ixy^2.$

Here  $u(x, y) = 2x$ ,  $v(x, y) = xy^2$ . Thus

$$u_x = 2, \quad u_y = 0, \quad v_x = y^2, \quad v_y = 2xy.$$

The CR demand

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \implies \begin{cases} 2 = 2xy \\ 0 = y^2 \end{cases} \implies \begin{cases} xy = 1 \\ y = 0 \end{cases}.$$

These cannot hold simultaneously for any  $x$ . Hence CR fail at every point, so  $f'_2$  exists nowhere.

(3)  $f_3(z) = e^{\bar{z}}.$

Let  $\bar{z} = x - iy$ . Then  $f_3(x, y) = e^{x-iy} = e^x(\cos y - i \sin y)$ , so

$$u(x, y) = e^x \cos y, \quad v(x, y) = -e^x \sin y.$$

Compute

$$u_x = e^x \cos y, \quad u_y = -e^x \sin y, \quad v_x = -e^x \sin y, \quad v_y = -e^x \cos y.$$

The CR give

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \implies \begin{cases} e^x \cos y = -e^x \cos y \\ -e^x \sin y = +e^x \sin y \end{cases} \implies \begin{cases} \cos y = 0 \\ \sin y = 0 \end{cases}.$$

These cannot hold simultaneously for any  $y$ . Hence CR fail everywhere and  $f'_3$  exists nowhere.  $\square$

4. Let  $f(z) = u(x, y) + iv(x, y)$  be given by

$$f(z) = \begin{cases} \bar{z}^2/z & : z \neq 0 \\ 0 & : z = 0. \end{cases}$$

Verify that the Cauchy–Riemann equations  $u_x = v_y$  and  $u_y = -v_x$  are satisfied at the origin  $z = (0, 0)$ .

*Proof.* Write  $z = x + iy$  and define

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Let  $f = u + iv$ . We compute the first-order partials of  $u, v$  at  $(0, 0)$  by restricting to the coordinate axes.

**Along the  $x$ -axis ( $y = 0$ ):** For  $x \neq 0$ ,

$$f(x, 0) = \frac{\bar{x}^2}{x} = x,$$

hence  $u(x, 0) = x$  and  $v(x, 0) = 0$ . Therefore

$$u_x(0, 0) = \lim_{h \rightarrow 0} \frac{u(h, 0) - u(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{h - 0}{h} = 1, \quad v_x(0, 0) = \lim_{h \rightarrow 0} \frac{v(h, 0) - v(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0.$$

**Along the  $y$ -axis ( $x = 0$ ):** For  $y \neq 0$ ,

$$f(0, y) = \frac{\overline{iy}^2}{iy} = \frac{(-iy)^2}{iy} = \frac{-y^2}{iy} = iy,$$

so  $u(0, y) = 0$  and  $v(0, y) = y$ . Hence

$$u_y(0, 0) = \lim_{k \rightarrow 0} \frac{u(0, k) - u(0, 0)}{k} = \lim_{k \rightarrow 0} \frac{0 - 0}{k} = 0, \quad v_y(0, 0) = \lim_{k \rightarrow 0} \frac{v(0, k) - v(0, 0)}{k} = \lim_{k \rightarrow 0} \frac{k - 0}{k} = 1.$$

Thus at  $(0, 0)$  we have

$$u_x(0, 0) = 1, \quad v_y(0, 0) = 1, \quad u_y(0, 0) = 0, \quad v_x(0, 0) = 0,$$

and consequently the Cauchy–Riemann equations  $u_x = v_y$  and  $u_y = -v_x$  hold at the origin.

*Remark.* Although the Cauchy–Riemann equations hold at  $(0,0)$ , the complex derivative  $f'(0)$  does not exist (since  $\frac{f(\Delta z) - f(0)}{\Delta z} = \frac{\overline{\Delta z}^2}{(\Delta z)^2}$  takes different values along different approach directions).

□

5. Let

$$f(z) = \sin x \cosh y + i \cos x \sinh y \quad \text{and} \quad f(z) = e^{-y}(\sin x - i \cos x).$$

Then show that all  $f$  are entire.

**Sol.** Let

$$f_1(z) = \sin x \cosh y + i \cos x \sinh y \quad \text{and}$$

$$f_2(z) = e^{-y}(\sin x - i \cos x).$$

(a) Let  $z = x + iy$  with  $x, y \in \mathbb{R}$ . Note that

$$\begin{aligned} e^{iz} &= \cos z + i \sin z & \cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ &\longleftrightarrow & & \\ e^{-iz} &= \cos z - i \sin z & \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}$$

By definition,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Substitute  $z = x + iy$ :

$$iz = i(x + iy) = ix - y, \quad -iz = -ix + y,$$

so

$$e^{iz} = e^{ix-y} = e^{-y}e^{ix}, \quad e^{-iz} = e^{-ix+y} = e^ye^{-ix}.$$

Using Euler's formula  $e^{ix} = \cos x + i \sin x$  and  $e^{-ix} = \cos x - i \sin x$ , we get

$$e^{iz} = e^{-y}(\cos x + i \sin x), \quad e^{-iz} = e^y(\cos x - i \sin x).$$

Then

$$\begin{aligned} e^{iz} - e^{-iz} &= e^{-y}(\cos x + i \sin x) - e^y(\cos x - i \sin x) \\ &= \cos x(e^{-y} - e^y) + i \sin x(e^{-y} + e^y). \end{aligned}$$

Recall the hyperbolic functions

$$\cosh y = \frac{e^y + e^{-y}}{2}, \quad \sinh y = \frac{e^y - e^{-y}}{2},$$

so that

$$e^{-y} + e^y = 2 \cosh y, \quad e^{-y} - e^y = -2 \sinh y.$$

Hence

$$\begin{aligned} e^{iz} - e^{-iz} &= \cos x(-2 \sinh y) + i \sin x(2 \cosh y) \\ &= 2(i \sin x \cosh y - \cos x \sinh y). \end{aligned}$$

Therefore

$$\begin{aligned} \sin z &= \frac{e^{iz} - e^{-iz}}{2i} = \frac{2(i \sin x \cosh y - \cos x \sinh y)}{2i} \\ &= \frac{i \sin x \cosh y}{i} - \frac{\cos x \sinh y}{i} \\ &= \sin x \cosh y + i \cos x \sinh y, \end{aligned}$$

since  $\frac{1}{i} = -i$ .

Thus

$$\boxed{\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y.}$$

Using the standard identity for the complex sine,

$$\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$$

we see immediately that  $f_1(z) = \sin z$ . Since  $\sin z$  is an entire function (power series with infinite radius of convergence),  $f_1$  is entire.

**(b)** Note that

$$e^{iz} = e^{i(x+iy)} = e^{ix-y} = e^{-y}(\cos x + i \sin x).$$

Multiplying by  $-i$  gives

$$-i e^{iz} = e^{-y}(\sin x - i \cos x) = f_2(z).$$

Thus  $f_2(z) = -i e^{iz}$ . Since the exponential is entire and multiplication by a constant preserves holomorphy,  $f_2$  is entire.

Therefore both functions are entire. □



6. Show that the function

$$f(z) = \ln r + i\theta \quad (r > 0, 0 < \theta < 2\pi)$$

is analytic in the indicated domain of definition, with derivative  $f'(z) = 1/z$ . Then show that the composite function  $g(z) = f(z^2 + 1)$  is analytic in the quadrant  $x > 0, y > 0$  with derivative

$$g'(z) = \frac{2z}{z^2 + 1}.$$

(Suggestion: Observe that  $\operatorname{Im}(z^2 + 1) > 0$  when  $x > 0, y > 0$ )

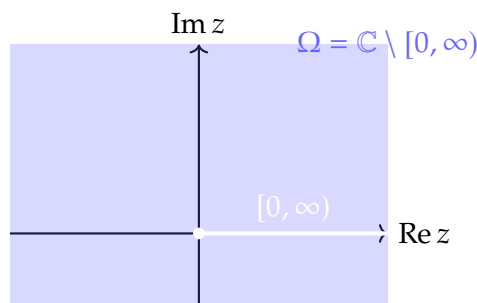
**Sol.** Let  $z = x + iy = re^{i\theta}$  with  $r = \sqrt{x^2 + y^2} > 0$  and  $0 < \theta < 2\pi$ . Define

$$f(z) = \ln r + i\theta.$$

Then  $f$  is analytic on the slit plane

$$\Omega := \{z \in \mathbb{C} : r > 0, 0 < \theta < 2\pi\} = \mathbb{C} \setminus [0, \infty),$$

$$\text{and } f'(z) = \frac{1}{z} \quad (z \in \Omega).$$



Write  $f = u + iv$  with

$$u(x, y) = \ln r = \frac{1}{2} \ln(x^2 + y^2), \quad v(x, y) = \theta = \operatorname{Arg}(z) \in (0, 2\pi).$$

On  $\Omega$  the functions  $u, v$  are  $C^1$  and their partials are:

$$u_x = \frac{x}{x^2 + y^2}, \quad u_y = \frac{y}{x^2 + y^2}, \quad v_x = -\frac{y}{x^2 + y^2}, \quad v_y = \frac{x}{x^2 + y^2}.$$

Hence the Cauchy–Riemann equations hold on  $\Omega$ :

$$u_x = v_y = \frac{x}{x^2 + y^2}, \quad u_y = -v_x = \frac{y}{x^2 + y^2}.$$

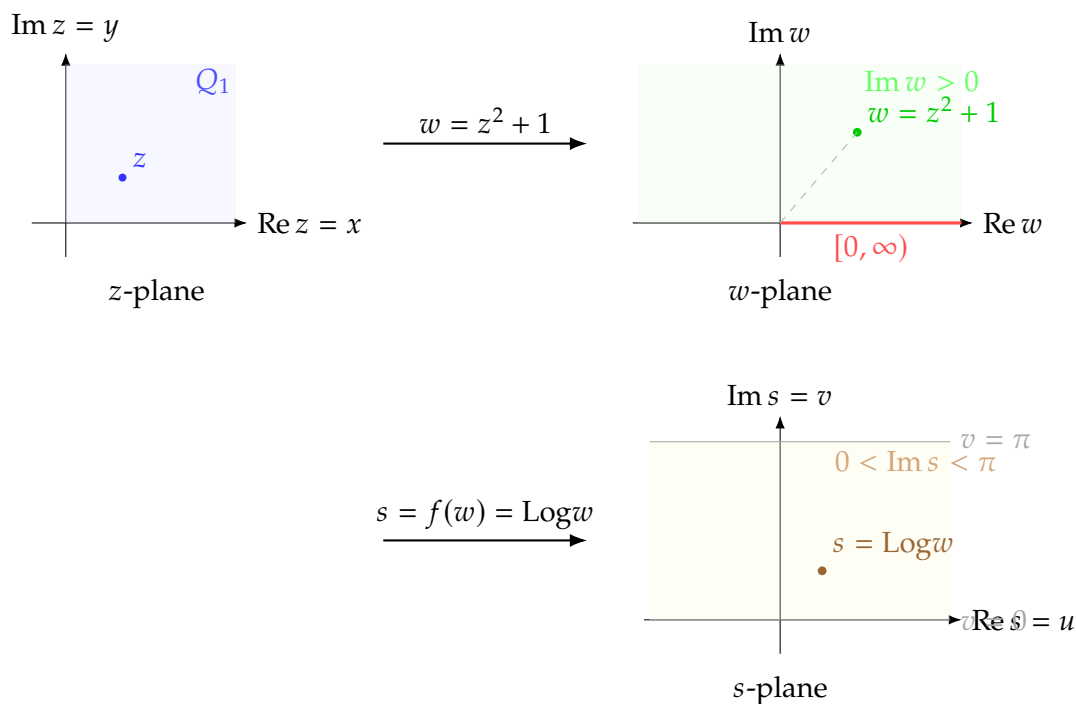
Since these partials are continuous on  $\Omega$ ,  $f$  is analytic there. Its complex derivative is

$$f'(z) = u_x + iv_x = \frac{x}{x^2 + y^2} + i\left(-\frac{y}{x^2 + y^2}\right) = \frac{x - iy}{x^2 + y^2} = \frac{1}{x + iy} = \frac{1}{z}.$$

For  $g(z) = f(z^2 + 1)$ , compute

$$\begin{aligned} z^2 + 1 &= (x + iy)^2 + 1 \\ &= (x^2 - y^2 + 2ixy) + 1 \\ &= (x^2 - y^2 + 1) + i(2xy). \end{aligned}$$

If  $x > 0$  and  $y > 0$ , then  $\operatorname{Im}(z^2 + 1) = 2xy > 0$ , so  $z^2 + 1$  lies in the open upper half-plane  $\mathbb{H}$ , in particular in  $\Omega$  (its argument lies in  $(0, \pi) \subset (0, 2\pi)$ ).



Thus  $g$  is the composition of analytic functions on the first quadrant  $Q_1$ , hence analytic on  $Q_1$ . By the chain rule,

$$g'(z) = f'(z^2 + 1) \cdot (2z) = \frac{2z}{z^2 + 1} \quad (z \in Q_1).$$

□

### 3 Elementary Functions

#### 3.1 The Exponential Function

##### Exponential Function

**Definition 3.1.** For  $z = x + iy \in \mathbb{C}$ , define

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y),$$

where  $y$  is in radians. We also write  $\exp z$  for  $e^z$ .

**Theorem 3.2.** For  $z_1, z_2 \in \mathbb{C}$ ,

$$e^{z_1+z_2} = e^{z_1}e^{z_2}, \quad \frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}.$$

Moreover,  $e^z$  is entire, satisfies

$$\frac{d}{dz}e^z = e^z$$

for all  $z$ , and  $e^z \neq 0$  for all  $z \in \mathbb{C}$ .

**Observation.** Writing  $e^z = \rho e^{i\theta}$  gives  $\rho = e^x$  and  $\theta = y$ , hence

$$|e^z| = e^x, \quad \arg(e^z) = y + 2\pi n \ (n \in \mathbb{Z}).$$

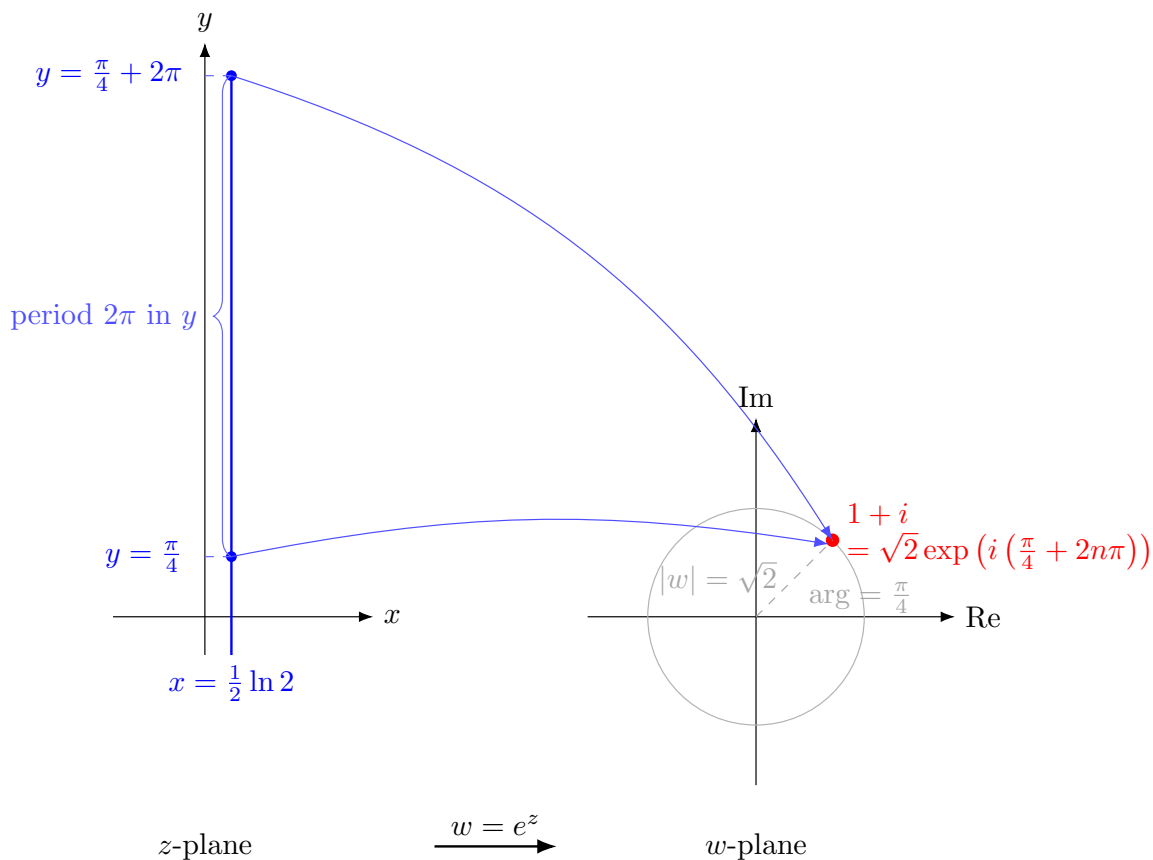
Thus  $e^{z+2\pi i} = e^z$ , so  $e^z$  is periodic with pure imaginary period  $2\pi i$ .

##### Euler's Identity

**Corollary 3.3.** Euler's identity is given by

$$e^{i\pi} = -1 \quad \text{equivalently,} \quad e^{i\pi} + 1 = 0.$$

**Example 3.4.** Solve  $e^z = 1 + i$  for  $z = x + iy$ .



**Sol.** Since  $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$  and  $1 + i = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$ , we have

$$e^x = \sqrt{2} \quad \text{and} \quad y = \frac{\pi}{4} + 2n\pi \quad (n = 0, \pm 1, \pm 2, \dots).$$

Thus,

$$x = \frac{1}{2} \ln 2 \quad \text{and} \quad y = \left( 2n + \frac{1}{4} \right) \pi \quad (n = 0, \pm 1, \pm 2, \dots)$$

and so

$$z = \frac{1}{2} \ln 2 + i \left( 2n + \frac{1}{4} \right) \pi \quad (n = 0, \pm 1, \pm 2, \dots).$$

□

### 3.2 The Logarithmic Function

**Observation.** To solve

$$z = e^w$$

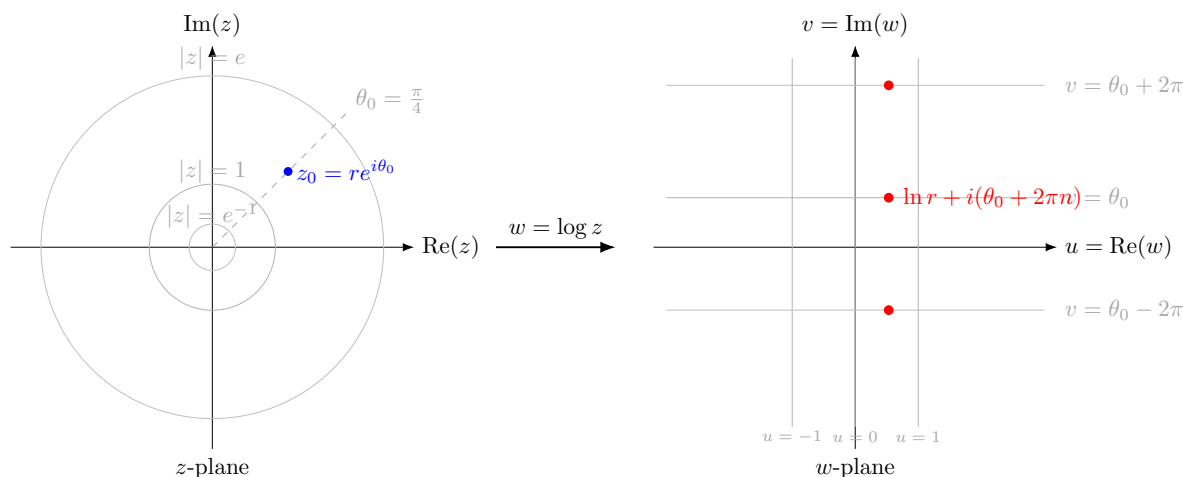
for  $w$  when  $z \neq 0$ , write  $z = re^{i\theta}$ ,  $w = u + iv$ . Then  $e^u = r$  and  $v = \theta + 2\pi n$ , hence

$$\log z = \ln r + i(\theta + 2\pi n), \quad n \in \mathbb{Z},$$

a multiple-valued function with

$$e^{\log z} = z$$

for  $z \neq 0$ .



**Example 3.5.** If  $z = -1 - \sqrt{3}i$ , then  $r = 2$  and  $\theta = -\frac{2\pi}{3}$ , so

$$\log z = \ln 2 + i\left(-\frac{2\pi}{3} + 2\pi n\right), \quad n \in \mathbb{Z}.$$

## Argument and Principal Value

**Definition 3.6.** For  $z \neq 0$ , the set of all arguments is  $\arg z = \{\theta + 2\pi n : n \in \mathbb{Z}\}$  when  $z = re^{i\theta}$ . The principal value  $\text{Arg}z$  is the unique  $\theta$  with  $-\pi < \theta \leq \pi$ .

**Observation.** In general,

$$\log(e^z) = z + 2\pi i n, \quad n \in \mathbb{Z}.$$

**Definition 3.7** (Principal Value of the Logarithm). The principal value is

$$\text{Log}z = \ln r + i\theta \quad (z = re^{i\theta}, r > 0, -\pi < \theta < \pi).$$

Then  $\log z = \text{Log}z + 2\pi i n$  for  $n \in \mathbb{Z}$ .

**Example 3.8.**  $\log 1 = 2\pi i n$  with  $\text{Log}1 = 0$ ; and  $\log(-1) = (2n + 1)\pi i$  with  $\text{Log}(-1) = \pi i$ . The function  $\text{Log}z$  is not continuous along the negative real axis.

### 3.3 Branches and Derivatives of Logarithms

**Observation.** Let  $\alpha \in \mathbb{R}$ . Restrict  $\theta$  in

$$\log z = \ln r + i\theta \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

to obtain a single-valued continuous branch on that domain; it is in fact analytic there.

**Theorem 3.9.** For a branch as above,

$$\frac{d}{dz} \log z = \frac{1}{z} \quad (|z| > 0, \alpha < \arg z < \alpha + 2\pi).$$

In particular, on the principal branch,

$$\frac{d}{dz} \text{Log}z = \frac{1}{z} \quad (|z| > 0, -\pi < \text{Arg}z < \pi).$$

**Definition 3.10** (Branch, Principal Branch, Branch Cut/Point). A **branch** of a multiple-valued  $f$  is any single-valued analytic function  $F$  whose values are among those of  $f$ . The **principal branch** of  $\log$  is  $\text{Log}z$  on  $r > 0, -\pi < \theta < \pi$ . A **branch cut** is a curve removed to render a single-valued branch; points on it are singular for that branch. The origin is a branch point for  $\log$ .

**Example 3.11.**  $\text{Log}(i^3) = \text{Log}(-i) = \ln 1 - i\frac{\pi}{2} = -i\frac{\pi}{2}$ , while  $3\text{Log}i = 3 \cdot i\frac{\pi}{2} = \frac{3\pi i}{2}$ . Hence  $\text{Log}(i^3) \neq 3\text{Log}i$ .

**Theorem 3.12.** For nonzero  $z_1, z_2$ ,

$$\log(z_1 z_2) = \log z_1 + \log z_2, \quad \arg(z_1 z_2) = \arg z_1 + \arg z_2,$$

and thus  $\ln |z_1 z_2| + i \arg(z_1 z_2) = (\ln |z_1| + i \arg z_1) + (\ln |z_2| + i \arg z_2)$ .

**Example 3.13.** Let  $z_1 = z_2 = -1$ . Then  $\log 1 = 0$ , while  $\log(-1) = (2n + 1)\pi i$ . Equality can require compatible choices of values. Using principal values everywhere may fail:  $\text{Log}(z_1 z_2) = 0$  but  $\text{Log} z_1 + \text{Log} z_2 = 2\pi i$ .

**Theorem 3.14.** For nonzero  $z_1, z_2$ ,

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2.$$

**Observation** (Roots via Logarithm). For  $z \neq 0$  and  $n \in \mathbb{N}$ ,

$$z^{1/n} = \exp\left(\frac{1}{n} \log z\right),$$

which gives exactly the  $n$  distinct  $n$ th roots when  $k = 0, 1, \dots, n - 1$  are taken in the angles.

### 3.4 Complex Exponents

**Definition 3.15** (Complex Power). For  $z \neq 0$  and  $c \in \mathbb{C}$ ,

$$z^c = e^{c \log z},$$

a multiple-valued function in general.

**Example 3.16.**

$$i^{-2i} = e^{-2i \log i}, \quad \log i = \ln 1 + i\left(\frac{\pi}{2} + 2\pi n\right) = \left(2n + \frac{1}{2}\right)\pi i.$$

Hence  $i^{-2i} = \exp((4n + 1)\pi)$ , which are real numbers.

**Observation.** Since  $1/e^z = e^{-z}$ , we have  $z^{-c} = \exp(-c \log z)$  and in particular  $1/i^{2i} = i^{-2i} = \exp((4n + 1)\pi)$ .

**Observation.** Fix a branch  $\log z = \ln r + i\theta$  on  $\alpha < \theta < \alpha + 2\pi$ . Then  $z^c = \exp(c \log z)$  is single-valued and analytic there, and

$$\frac{d}{dz} z^c = c z^{c-1} \quad (|z| > 0, \alpha < \arg z < \alpha + 2\pi).$$

The principal value is P.V.  $z^c = \exp(c \text{Log} z)$ .

**Example 3.17.**

$$\text{P.V. } (-i)^i = \exp(i \text{Log}(-i)) = \exp\left(i \left[\ln 1 - i\frac{\pi}{2}\right]\right) = e^{\pi/2}.$$

For  $z^{2/3}$  on the principal branch  $(-\pi < \text{Arg} z < \pi)$ ,

$$\text{P.V. } z^{2/3} = r^{2/3} \left( \cos \frac{2\varphi}{3} + i \sin \frac{2\varphi}{3} \right) \quad (z = r e^{i\varphi}).$$

**Example 3.18.** Let  $z_1 = 1 + i$ ,  $z_2 = 1 - i$ ,  $z_3 = -1 - i$ . Then

$$(z_1 z_2)^i = e^{i \ln 2}, \quad z_1^i = e^{-\pi/4} e^{i(\ln 2)/2}, \quad z_2^i = e^{\pi/4} e^{i(\ln 2)/2},$$

so  $(z_1 z_2)^i = z_1^i z_2^i$ . But

$$(z_2 z_3)^i = e^{-\pi} e^{i \ln 2}, \quad z_2^i = e^{3\pi/4} e^{i(\ln 2)/2},$$

whence  $(z_2 z_3)^i = z_2^i z_3^i e^{-2i}$ , showing branch subtleties.

**Definition 3.19** (Exponential with Base  $c \neq 0$ ). For fixed  $c \in \mathbb{C} \setminus \{0\}$  and a chosen value of  $\log c$ , define

$$c^z = e^{z \log c}.$$

Then  $c^z$  is entire and  $\frac{d}{dz} c^z = c^z \log c$ .

### 3.5 Trigonometric Functions

**Definition 3.20.** For  $z \in \mathbb{C}$ ,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

**Theorem 3.21.** The functions  $\sin z$  and  $\cos z$  are entire and satisfy

$$\frac{d}{dz} \sin z = \cos z, \quad \frac{d}{dz} \cos z = -\sin z,$$

and remain odd/even respectively:  $\sin(-z) = -\sin z$ ,  $\cos(-z) = \cos z$ . Moreover  $e^{iz} = \cos z + i \sin z$ .

**Theorem 3.22** (Formulas). For  $z, z_1, z_2 \in \mathbb{C}$ ,

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2, \quad \cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2,$$

$$\sin 2z = 2 \sin z \cos z, \quad \cos 2z = \cos^2 z - \sin^2 z,$$

$$\sin\left(z + \frac{\pi}{2}\right) = \cos z, \quad \cos\left(z - \frac{\pi}{2}\right) = -\sin z,$$

$$\sin^2 z + \cos^2 z = 1, \quad \sin(z + \pi) = -\sin z, \quad \cos(z + \pi) = -\cos z,$$

$$\sin(z + 2\pi) = \sin z, \quad \cos(z + 2\pi) = \cos z.$$

**Observation.** For real  $y$ ,

$$\cos(iy) = \cosh y, \quad \sin(iy) = i \sinh y.$$

Writing  $z = x + iy$ ,

$$\sin z = \sin x \cosh y + i \cos x \sinh y, \quad \cos z = \cos x \cosh y - i \sin x \sinh y.$$



*Remark.*  $\sin z$  and  $\cos z$  are unbounded on  $\mathbb{C}$ .

**Observation (Zeros).**  $\sin z = 0$  iff  $z = n\pi$ ;  $\cos z = 0$  iff  $z = \frac{\pi}{2} + n\pi$  for  $n \in \mathbb{Z}$ .

**Definition 3.23.** Define

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}.$$

**Theorem 3.24.**

$$\frac{d}{dz} \tan z = \sec^2 z, \quad \frac{d}{dz} \sec z = \sec z \tan z, \quad \frac{d}{dz} \cot z = -\csc^2 z, \quad \frac{d}{dz} \csc z = -\csc z \cot z.$$

**Observation.**  $\tan z$  and  $\sec z$  are analytic off  $z = \frac{\pi}{2} + n\pi$ ;  $\cot z$  and  $\csc z$  are analytic off  $z = n\pi$ .

### 3.6 Hyperbolic Functions

**Definition 3.25.**

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}.$$

**Theorem 3.26.**  $\sinh z$  and  $\cosh z$  are entire and  $\frac{d}{dz} \sinh z = \cosh z$ ,  $\frac{d}{dz} \cosh z = \sinh z$ .

**Theorem 3.27.** For  $z = x + iy$  and  $z_1, z_2 \in \mathbb{C}$ ,

$$\begin{aligned} -i \sinh(iz) &= \sin z, & \cosh(iz) &= \cos z, & -i \sin(iz) &= \sinh z, & \cos(iz) &= \cosh z, \\ \sinh(-z) &= -\sinh z, & \cosh(-z) &= \cosh z, & \cosh^2 z - \sinh^2 z &= 1, \\ \sinh(z_1 + z_2) &= \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2, \\ \cosh(z_1 + z_2) &= \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2, \\ \sinh z &= \sinh x \cos y + i \cosh x \sin y, \\ \cosh z &= \cosh x \cos y + i \sinh x \sin y, \\ |\sinh z|^2 &= \sinh^2 x + \sin^2 y, & |\cosh z|^2 &= \cosh^2 x + \cos^2 y. \end{aligned}$$

*Remark.*  $\sinh z$  and  $\cosh z$  are periodic with period  $2\pi i$ .

**Observation (Zeros).**  $\sinh z = 0$  iff  $z = n\pi i$ ;  $\cosh z = 0$  iff  $z = (\frac{\pi}{2} + n\pi)i$  ( $n \in \mathbb{Z}$ ).

**Definition 3.28.** Define  $\tanh z = \frac{\sinh z}{\cosh z}$  (analytic where  $\cosh z \neq 0$ ). Set  $\coth z = 1/\tanh z$ ,  $\operatorname{sech} z = 1/\cosh z$ ,  $\operatorname{csch} z = 1/\sinh z$ .

**Theorem 3.29.**

$$\frac{d}{dz} \tanh z = \operatorname{sech}^2 z, \quad \frac{d}{dz} \operatorname{sech} z = -\operatorname{sech} z \tanh z, \quad \frac{d}{dz} \coth z = -\operatorname{csch}^2 z, \quad \frac{d}{dz} \operatorname{csch} z = -\operatorname{csch} z \coth z.$$

### 3.7 Inverse Trigonometric and Hyperbolic Functions

**Observation.** To define  $\sin^{-1} z$ , write

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}.$$

Then

$$\begin{aligned} e^{iw} - e^{-iw} &= 2iz \\ (e^{iw})^2 - 2iz - 1 &= 0 \end{aligned}$$

Solving the quadratic in  $e^{iw}$  yields

$$e^{iw} = iz + (1 - z^2)^{1/2},$$

where  $(1 - z^2)^{1/2}$  is double-valued.

**Definition 3.30.** Multiple-valued inverses:

$$\sin^{-1} z = -i \log[iz + (1 - z^2)^{1/2}],$$

$$\cos^{-1} z = -i \log[z + i(1 - z^2)^{1/2}],$$

$$\tan^{-1} z = \frac{i}{2} \log\left(\frac{i+z}{i-z}\right).$$

With specific branches of  $\sqrt{\cdot}$  and  $\log$ , these become single-valued and analytic on suitable domains.

**Theorem 3.31** (Derivatives).

$$\frac{d}{dz} \sin^{-1} z = \frac{1}{(1 - z^2)^{1/2}}, \quad \frac{d}{dz} \cos^{-1} z = -\frac{1}{(1 - z^2)^{1/2}}, \quad \frac{d}{dz} \tan^{-1} z = \frac{1}{1 + z^2}.$$

**Example 3.32.**

$$\sin^{-1}(-i) = -i \log(1 \pm \sqrt{2}).$$

Since  $\log(1 + \sqrt{2}) = \ln(1 + \sqrt{2}) + 2\pi i n$  and  $\log(1 - \sqrt{2}) = \ln(\sqrt{2} - 1) + (2n+1)\pi i$  with  $\ln(\sqrt{2} - 1) = -\ln(1 + \sqrt{2})$ , the values of  $\sin^{-1}(-i)$  are

$$n\pi + i(-1)^{n+1} \ln(1 + \sqrt{2}), \quad n \in \mathbb{Z}.$$

**Observation** (Inverse Hyperbolic Functions).

$$\sinh^{-1} z = \log[z + (z^2 + 1)^{1/2}], \quad \cosh^{-1} z = \log[z + (z^2 - 1)^{1/2}], \quad \tanh^{-1} z = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right).$$

### 3.8 Exercises

1. Show that  $f(z) = \exp(\bar{z})$  is not analytic anywhere.

(Hint: use the Cauchy–Riemann equations.)

**Sol.** (Proof via Cauchy–Riemann equations) Write  $z = x + iy$ . Then

$$f(z) = e^{\bar{z}} = e^{x-iy} = e^x (\cos y - i \sin y),$$

so

$$u(x, y) = e^x \cos y, \quad v(x, y) = -e^x \sin y.$$

Then

$$u_x = e^x \cos y, \quad u_y = -e^x \sin y, \quad v_x = -e^x \sin y, \quad v_y = -e^x \cos y.$$

If  $f$  is complex differentiable at  $(x, y)$ , the Cauchy–Riemann equations would hold:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

That is,

$$\begin{aligned} u_x = v_y &\implies e^x \cos y = -e^x \cos y &&\implies \cos y = 0, \\ u_y = -v_x &\implies -e^x \sin y = e^x \sin y &&\implies \sin y = 0. \end{aligned}$$

There is no  $y \in \mathbb{R}$  with  $\cos y = 0$  and  $\sin y = 0$  simultaneously. Hence the Cauchy–Riemann equations fail at every point, so  $f$  is nowhere analytic.

(Proof via Wirtinger derivatives) Using  $\partial/\partial z = \frac{1}{2}(\partial_x - i \partial_y)$  and  $\partial/\partial \bar{z} = \frac{1}{2}(\partial_x + i \partial_y)$ , one checks directly that

$$\frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial \bar{z}} = e^{\bar{z}} \neq 0 \quad \text{for all } z.$$

A function is holomorphic iff  $\partial f/\partial \bar{z} \equiv 0$  on its domain. Since this is not the case,  $f$  is nowhere holomorphic.  $\square$

2. Let  $f(z) = u(x, y) + iv(x, y)$  be analytic in a domain  $D$ . Show that

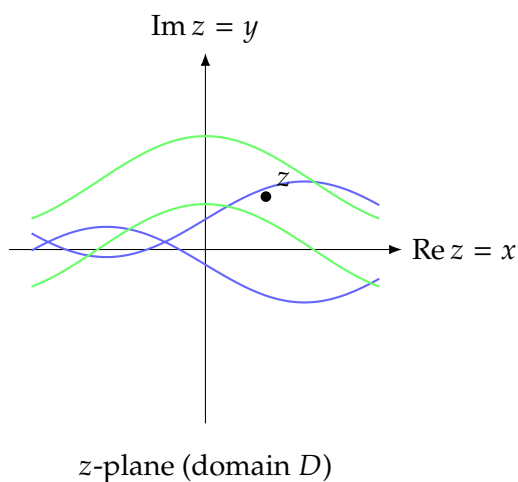
$$U(x, y) = e^{u(x, y)} \cos v(x, y), \quad V(x, y) = e^{u(x, y)} \sin v(x, y)$$

are harmonic in  $D$ , and that  $V$  is a harmonic conjugate of  $U$ .

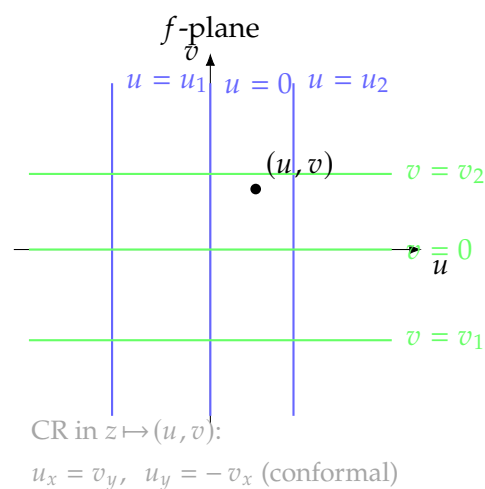
**Sol.** Since  $f$  is analytic on  $D$ , the composition

$$F(z) := e^{f(z)} = e^{u(x, y)} (\cos v(x, y) + i \sin v(x, y)) = U(x, y) + iV(x, y)$$

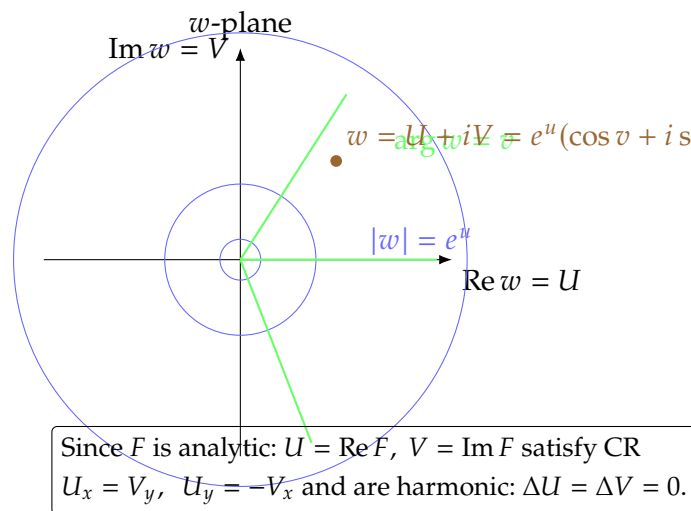
is analytic on  $D$  (composition of analytic maps). It follows that  $U = \operatorname{Re} F$  and  $V = \operatorname{Im} F$  are harmonic and satisfy the Cauchy–Riemann equations.



$$f(z) = u + iv \longrightarrow$$



$$F(z) = e^{f(z)} \longrightarrow$$



Note that

$$\begin{aligned} U_x &= e^{u(x,y)} (u_x \cos v - v_x \sin v), & U_y &= e^{u(x,y)} (u_y \cos v - v_y \sin v), \\ V_x &= e^{u(x,y)} (u_x \sin v + v_x \cos v), & V_y &= e^{u(x,y)} (u_y \sin v + v_y \cos v). \end{aligned}$$

Because  $f$  is analytic,  $u, v$  satisfy the CR equations  $u_x = v_y$  and  $u_y = -v_x$ . Substituting,

$$\begin{aligned} U_x &= e^{u(x,y)} (v_y \cos v - v_x \sin v) = V_y, \\ U_y &= e^{u(x,y)} (-v_x \cos v - v_y \sin v) = -V_x. \end{aligned}$$

Thus  $U_x = V_y$  and  $U_y = -V_x$ , i.e.  $V$  is a harmonic conjugate of  $U$ .

To show harmonicity, differentiate the CR relations and use equality of mixed partials:

$$U_{xx} = (V_y)_x = V_{yx}, \quad U_{yy} = (-V_x)_y = -V_{xy}.$$

Hence  $\Delta U := U_{xx} + U_{yy} = V_{yx} - V_{xy} = 0$ . Similarly,

$$V_{xx} = (-U_y)_x = -U_{yx}, \quad V_{yy} = (U_x)_y = U_{xy},$$

so  $\Delta V := V_{xx} + V_{yy} = -U_{yx} + U_{xy} = 0$ . Therefore  $U$  and  $V$  are harmonic on  $D$ , and  $V$  is a harmonic conjugate of  $U$ .  $\square$

3. Show that  $f(z) = \text{Log}(z - i)$  is analytic except on portion  $x \leq 0$  of the line  $y = 1$  and that the function

$$f(z) = \frac{\text{Log}(z + 4)}{z^2 + i}$$

is analytic everywhere except at the points  $\pm(1 - i)/\sqrt{2}$  and on the portion  $x \leq -4$  of the real axis.

**Sol.** Consider  $\text{Log } z = \ln|z| + i\text{Arg } z$ , the principal branch of the complex logarithm, with  $\text{Arg } z \in (-\pi, \pi)$ , so that  $\text{Log}$  is analytic on

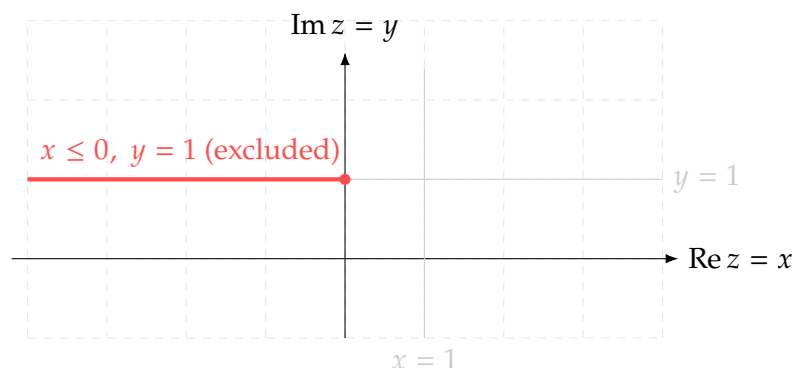
$$\begin{aligned}\mathbb{C} \setminus (-\infty, 0] &= \mathbb{C} \setminus \{z \in \mathbb{C} : \text{Re } z \leq 0 \wedge \text{Im } z = 0\} \\ &= \{z \in \mathbb{C} : \text{Re } z > 0 \vee \text{Im } z \neq 0\}.\end{aligned}$$

Then

- (1) Since  $\text{Log}$  is analytic on  $\mathbb{C} \setminus (-\infty, 0]$  and the map  $z \mapsto z - i$  is entire, the composition  $z \mapsto \text{Log}(z - i)$  is analytic precisely where  $z - i \notin (-\infty, 0]$ . Equivalently,

$$\begin{aligned}z - i \in (-\infty, 0] &\iff \text{Re}(z - i) \leq 0 \text{ and } \text{Im}(z - i) = 0 \\ &\iff \text{Re}(x + i(y - 1)) = x \leq 0 \text{ and } \text{Im}(x + i(y - 1)) = y - 1 = 0.\end{aligned}$$

That is,  $f(z) := \text{Log}(z - i)$  is analytic on  $\mathbb{C} \setminus \{x + iy : x \leq 0, y = 1\}$ .



$$f(z) = \text{Log}(z - i)$$

Analytic domain:  $\mathbb{C} \setminus \{(x, y) \mid y = 1, x \leq 0\}$ .

- (2) The numerator  $z \mapsto \text{Log}(z + 4)$  is analytic wherever  $z + 4 \notin (-\infty, 0]$ , i.e.,  $z \notin (-\infty, -4]$ . In other words,  $\text{Log}(z + 4)$  is analytic on

$$\mathbb{C} \setminus (-\infty, -4] = \mathbb{C} \setminus \{z \in \mathbb{C} : \text{Re } z \leq -4 \wedge \text{Im } z = 0\} = \{z \in \mathbb{C} : \text{Re } z > -4 \vee \text{Im } z \neq 0\}$$

$\mathbb{C} \setminus (-\infty, -4]$  for  $z \notin (-\infty, -4]$ , which is the portion  $x \leq -4$  of the real axis. The denominator

$z^2 + i$  vanishes exactly at the zeros of  $z^2 = -i$ , namely

$$z = \pm(-i)^{1/2} = \pm e^{-i\pi/4} = \pm \frac{1-i}{\sqrt{2}}.$$

Therefore  $g$  is analytic on the domain where the numerator is analytic and the denominator is nonzero, i.e.

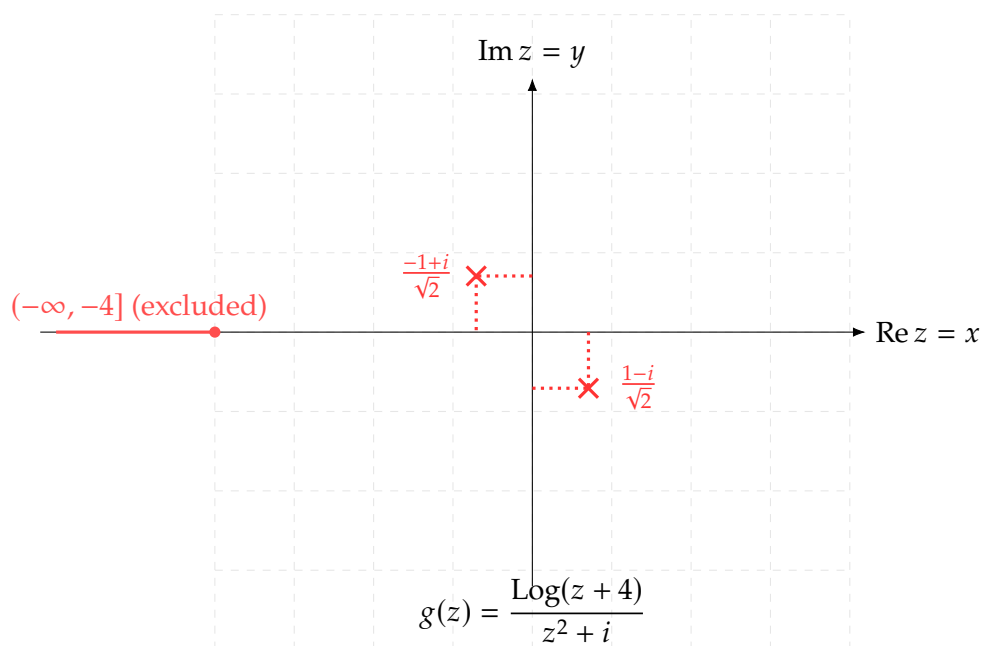
$$\mathbb{C} \setminus \left( (-\infty, -4] \cup \left\{ \pm \frac{1-i}{\sqrt{2}} \right\} \right),$$

which is exactly the stated set.

$g(z) := \frac{\text{Log}(z+4)}{z^2+i}$  is analytic on

$$\mathbb{C} \setminus \left( \{x+iy : y=0, x \leq -4\} \cup \left\{ \pm \frac{1-i}{\sqrt{2}} \right\} \right),$$

i.e. everywhere except at the branch cut  $x \leq -4$  on the real axis and at the two points  $\pm(1-i)/\sqrt{2}$ .



□



4. Show that the function  $\ln(x^2 + y^2)$  is harmonic in every domain that does not contain the origin.

**Sol.** For  $(x, y) \neq (0, 0)$ , we can differentiate:

$$u_x = \frac{\partial}{\partial x} \ln(x^2 + y^2) = \frac{2x}{x^2 + y^2},$$

$$u_y = \frac{\partial}{\partial y} \ln(x^2 + y^2) = \frac{2y}{x^2 + y^2}.$$

And then

$$u_{xx} = \frac{2(x^2 + y^2) - 2x \cdot 2x}{(x^2 + y^2)^2} = \frac{2(x^2 + y^2) - 4x^2}{(x^2 + y^2)^2} = \frac{-2x^2 + 2y^2}{(x^2 + y^2)^2}$$

$$u_{yy} = \frac{2(x^2 + y^2) - 2y \cdot 2y}{(x^2 + y^2)^2} = \frac{2(x^2 + y^2) - 4y^2}{(x^2 + y^2)^2} = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}.$$

Now compute the Laplacian:

$$u_{xx} + u_{yy} = \frac{-2x^2 + 2y^2}{(x^2 + y^2)^2} + \frac{2x^2 - 2y^2}{(x^2 + y^2)^2} = \frac{(-2x^2 + 2y^2) + (2x^2 - 2y^2)}{(x^2 + y^2)^2} = \frac{0}{(x^2 + y^2)^2} = 0$$

for all  $(x, y) \neq (0, 0)$ .

**(Proof via Wirtinger-operator)** Let  $z = x + iy$  and

$$u(x, y) = \ln(x^2 + y^2) = \ln(|z|^2) = \ln(z\bar{z}).$$

Recall the Wirtinger operators  $\partial := \frac{1}{2}(\partial_x - i\partial_y)$  and  $\bar{\partial} := \frac{1}{2}(\partial_x + i\partial_y)$ , so that the Laplacian satisfies

$$\Delta = \partial_{xx} + \partial_{yy} = 4\partial\bar{\partial} = 4\bar{\partial}\partial.$$

On  $\mathbb{C} \setminus \{0\}$  the chain rule gives

$$\partial u = \partial(\ln(z\bar{z})) = \frac{1}{z\bar{z}} \partial(z\bar{z}) = \frac{1}{z\bar{z}} \bar{z} = \frac{1}{z},$$

$$\bar{\partial} u = \bar{\partial}(\ln(z\bar{z})) = \frac{1}{z\bar{z}} \bar{\partial}(z\bar{z}) = \frac{1}{z\bar{z}} z = \frac{1}{\bar{z}}.$$

Therefore,

$$\Delta u = 4\partial\bar{\partial}u = 4\partial\left(\frac{1}{\bar{z}}\right) = 0 \quad \text{on } \mathbb{C} \setminus \{0\}.$$

□

5. Show that neither  $\sin \bar{z}$  nor  $\cos \bar{z}$  is an analytic function of  $z$  anywhere.

(Hint: use the Cauchy–Riemann equation.)

**Sol.** (Proof via CR-equations) Write  $z = x + iy$ .

(1) Let

$$f(z) = \sin(\bar{z}) = \sin(x - iy) = \sin(x) \cos(iy) - \cos x \sin(iy) = \sin x \cosh y - i \cos x \sinh y$$

then

$$u(x, y) = \sin x \cosh y, \quad v(x, y) = -\cos x \sinh y.$$

Compute the partials:

$$u_x = \cos x \cosh y, \quad u_y = \sin x \sinh y, \quad v_x = \sin x \sinh y, \quad v_y = -\cos x \cosh y.$$

The Cauchy–Riemann (CR) equations  $u_x = v_y$  and  $u_y = -v_x$  become

$$\cos x \cosh y = -\cos x \cosh y \implies \cos x \cosh y = 0,$$

$$\sin x \sinh y = -\sin x \sinh y \implies \sin x \sinh y = 0.$$

Since  $\cosh y \neq 0$  for all  $y$ , the first forces  $\cos x = 0$ . Then the second gives either  $\sin x = 0$  (impossible simultaneously with  $\cos x = 0$ ) or  $\sinh y = 0$ , i.e.  $y = 0$ . Hence the CR equations can hold only at isolated points with  $y = 0$  and  $\cos x = 0$  (i.e.  $x = \frac{\pi}{2} + k\pi$ ). They **cannot** hold on any open set. Therefore  $f$  is not analytic anywhere.

(2)  $g(z) = \cos(\bar{z}) = \cos(x - iy) = \cos x \cosh y + i \sin x \sinh y$ . Thus

$$u(x, y) = \cos x \cosh y, \quad v(x, y) = \sin x \sinh y.$$

Compute the partials:

$$u_x = -\sin x \cosh y, \quad u_y = \cos x \sinh y, \quad v_x = \cos x \sinh y, \quad v_y = \sin x \cosh y.$$

The CR equations give

$$u_x = v_y \implies -\sin x \cosh y = \sin x \cosh y \implies \sin x \cosh y = 0,$$

$$u_y = -v_x \implies \cos x \sinh y = -\cos x \sinh y \implies \cos x \sinh y = 0.$$

Again, since  $\cosh y \neq 0$ , the first forces  $\sin x = 0$ ; then the second forces either  $\cos x = 0$  (incompatible) or  $\sinh y = 0$ , i.e.  $y = 0$ . Thus CR can hold only at isolated points with  $y = 0$  and  $\sin x = 0$  (i.e.  $x = k\pi$ ), and not on any open set. Therefore  $g$  is not analytic anywhere.

(Proof via Wirtinger-operators) Recall the Wirtinger operators

$$\partial = \frac{1}{2}(\partial_x - i \partial_y), \quad \bar{\partial} = \frac{1}{2}(\partial_x + i \partial_y),$$

and the criterion: a  $C^1$  function  $F$  is holomorphic on an open set iff  $\bar{\partial}F \equiv 0$  there.

Let  $h(w) = \sin w$ . Then  $f_1(z) := \sin(\bar{z}) = h(\bar{z})$  satisfies

$$\partial f_1(z) = 0, \quad \bar{\partial} f_1(z) = h'(\bar{z}) = \cos(\bar{z}).$$

Similarly, with  $h(w) = \cos w$ ,  $f_2(z) := \cos(\bar{z})$  satisfies

$$\partial f_2(z) = 0, \quad \bar{\partial} f_2(z) = h'(\bar{z}) = -\sin(\bar{z}).$$

In each case,  $\bar{\partial}f_j$  is **not** identically zero on any open set (its zero set is discrete). Hence neither  $f_1$  nor  $f_2$  is holomorphic on any nonempty open set; i.e. they are nowhere analytic.

**Remark.** At isolated points where  $\cos(\bar{z}) = 0$  (respectively  $\sin(\bar{z}) = 0$ ), the complex difference quotient may happen to have limit 0; however, analyticity requires  $\bar{\partial}f \equiv 0$  on a neighborhood, which fails here.  $\square$

6. Show that  $\cosh^2 z - \sinh^2 z = 1$  and  $\sinh z + \cosh z = e^z$ .

**Sol.** Recall the exponential definitions (valid for all  $z \in \mathbb{C}$ ):

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

(1)  $\cosh^2 z - \sinh^2 z = 1$ .

$$\cosh^2 z - \sinh^2 z = \left( \frac{e^z + e^{-z}}{2} \right)^2 - \left( \frac{e^z - e^{-z}}{2} \right)^2 = \frac{(e^z + e^{-z})^2 - (e^z - e^{-z})^2}{4}.$$

Expanding,

$$(e^z + e^{-z})^2 - (e^z - e^{-z})^2 = e^{2z} + 2 + e^{-2z} - (e^{2z} - 2 + e^{-2z}) = 4,$$

so  $\cosh^2 z - \sinh^2 z = \frac{4}{4} = 1$ .

(2)  $\sinh z + \cosh z = e^z$ .

$$\sinh z + \cosh z = \frac{e^z - e^{-z}}{2} + \frac{e^z + e^{-z}}{2} = e^z.$$

$\square$

## 4 Integrals: Part I

### 4.1 Definite Integrals of Complex-Valued Functions

#### Derivative

**Definition 4.1.** If  $w(t) = u(t) + iv(t)$  with real-valued  $u, v$ , the derivative is

$$\frac{d}{dt}w(t) = w'(t) = u'(t) + iv'(t),$$

whenever  $u'$  and  $v'$  exist. If  $z_0 = x_0 + iy_0$  is constant, then

$$\frac{d}{dt}[z_0 w(t)] = z_0 w'(t), \quad \frac{d}{dt}e^{z_0 t} = z_0 e^{z_0 t}.$$

**Observation** (No mean value theorem for derivatives). If  $w(t)$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , there need **not** exist  $c \in (a, b)$  with

$$w'(c) = \frac{w(b) - w(a)}{b - a}.$$

For  $w(t) = e^{it}$  on  $[0, 2\pi]$ , we have  $|w'(t)| = 1$  but  $[w(2\pi) - w(0)]/(2\pi) = 0$ .

**Definition 4.2** (Definite integral). For  $w(t) = u(t) + iv(t)$ ,

$$\int_a^b w(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt,$$

with analogous definitions for improper integrals.

**Example 4.3.**

$$\int_0^1 (1 + it)^2 dt = \int_0^1 (1 - t^2) dt + i \int_0^1 2t dt = \frac{2}{3} + i.$$

**Theorem 4.4** (Additivity). For  $a \leq c \leq b$ ,

$$\int_a^b w(t) dt = \int_a^c w(t) dt + \int_c^b w(t) dt.$$

### Fundamental Theorem of Calculus

**Theorem 4.5.** If  $W'(t) = w(t)$  and  $W, w$  are continuous on  $[a, b]$ , then

$$\int_a^b w(t) dt = W(b) - W(a).$$

**Example 4.6.** Since  $\frac{d}{dt}(e^{it}/i) = e^{it}$ ,

$$\int_0^{\pi/4} e^{it} dt = \left[ \frac{e^{it}}{i} \right]_0^{\pi/4} = \frac{1}{\sqrt{2}} + i \left( 1 - \frac{1}{\sqrt{2}} \right).$$

*Remark* (No mean value theorem for integrals). There need not be  $c \in (a, b)$  with

$$w(c) = \frac{1}{b-a} \int_a^b w(t) dt$$

when  $w$  is complex-valued.

## 4.2 Contours

### Arc

**Definition 4.7.** An **arc**  $C$  is a set  $z(t) = x(t) + iy(t)$ ,  $a \leq t \leq b$ , where  $x, y$  are continuous.

**Definition 4.8** (Simple arc / Jordan curve).  $C$  is **simple** if  $z(t_1) \neq z(t_2)$  for  $t_1 \neq t_2$ . If  $C$  is simple with  $z(a) = z(b)$ , it is a **simple closed curve** (Jordan curve). Positive orientation is counterclockwise.

**Example 4.9.** The polygonal line from 0 to  $1 + i$  to  $2 + i$  is a simple arc;  $z = e^{i\theta}$ ,  $0 \leq \theta \leq 2\pi$ , is a positively oriented unit circle;  $z = z_0 + Re^{i\theta}$  is a circle centered at  $z_0$  of radius  $R$ . Traversing  $z = e^{-i\theta}$  reverses orientation;  $z = e^{i2\theta}$  traverses the unit circle twice.

**Observation** (Arc length). If  $z'(t) = x'(t) + iy'(t)$  is continuous on  $[a, b]$ , then

$$L(C) = \int_a^b |z'(t)| dt, \quad |z'(t)| = \left( [x'(t)]^2 + [y'(t)]^2 \right)^{1/2}.$$

The unit tangent is  $T = z'(t)/|z'(t)|$  where  $z'(t) \neq 0$ ; such an arc is **smooth**.

### Smooth arc and contour

**Definition 4.10.** An arc is **smooth** if  $z'(t)$  is continuous on  $[a, b]$  and nonzero on  $(a, b)$ . A **contour** (piecewise smooth arc) is a finite concatenation of smooth arcs. A contour with identical initial and final points is a **simple closed contour**.

**Jordan Curve Theorem**

**Theorem 4.11.** *A simple closed curve  $C$  is the boundary of exactly two domains: a bounded interior and an unbounded exterior.*

**4.3 Contour Integrals****Contour integral**

**Definition 4.12.** If  $C$  is given by  $z = z(t)$ ,  $a \leq t \leq b$ , and  $f$  is (piecewise) continuous on  $C$ , define

$$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt.$$

This is invariant under reparametrization of  $C$ .

**Linearity**

**Proposition 4.13.** *For a contour  $C$  and constant  $z_0$ ,*

$$\int_C z_0 f(z) dz = z_0 \int_C f(z) dz, \quad \int_C [f(z) + g(z)] dz = \int_C f + \int_C g.$$

**Orientation reversal**

**Proposition 4.14.** *If  $-C$  is  $C$  with reversed direction, then*

$$\int_{-C} f(z) dz = - \int_C f(z) dz.$$

**Additivity over legs**

**Proposition 4.15.** *If  $C = C_1 + C_2$  (concatenation), then*

$$\int_C f(z) dz = \int_{C_1} f + \int_{C_2} f.$$

**Example 4.16** (Half-circle integral). Let  $C : z = 2e^{i\theta}$ ,  $-\pi/2 \leq \theta \leq \pi/2$  (right half of  $|z| = 2$ ). Then

$$\int_C z dz = \int_{-\pi/2}^{\pi/2} 2e^{i\theta} (2ie^{i\theta}) d\theta = 4i \int_{-\pi/2}^{\pi/2} e^{2i\theta} d\theta = 4\pi i.$$

**Example 4.17** (Polygonal and diagonal paths). Let  $f(z) = y - x - i3x^2$  with  $z = x + iy$ . With  $C_1 :$

$O \rightarrow A \rightarrow B$  (up then right) and  $C_2 : O \rightarrow B$  along  $y = x$ , one finds

$$\int_{C_1} f(z) dz = \frac{1-i}{2}, \quad \int_{C_2} f(z) dz = 1-i, \quad \int_{C_1-C_2} f(z) dz = \frac{-1+i}{2}.$$

**Example 4.18** (Path-independence for  $f(z) = z$ ). For any smooth arc  $C$  from  $z_1$  to  $z_2$ ,

$$\int_C z dz = \frac{z_2^2 - z_1^2}{2}.$$

Hence the value depends only on endpoints, so the same holds for any piecewise smooth contour by telescoping the legs.

**Example 4.19** (Square-root branch on a semicircle). Let  $C : z = 3e^{i\theta}$ ,  $0 \leq \theta \leq \pi$  and take the branch  $z^{1/2} = \exp(\frac{1}{2} \log z)$  on  $|z| > 0$ ,  $0 < \arg z < 2\pi$ . Then  $z^{1/2}$  is piecewise continuous on  $C$  and

$$\int_C z^{1/2} dz = 3\sqrt{3} i \int_0^\pi e^{i(3\theta/2)} d\theta = -2\sqrt{3}(1+i).$$

**Example 4.20** (Power integral on a circle). On the principal branch  $z^{a-1} = \exp[(a-1)\text{Log}z]$  with  $|z| > 0$ ,  $-\pi < \text{Arg}z < \pi$ , for  $C : z = Re^{i\theta}$ ,  $-\pi < \theta < \pi$ ,

$$\int_C z^{a-1} dz = iR^a \int_{-\pi}^\pi e^{ia\theta} d\theta = \frac{i2R^a}{a} \sin(a\pi).$$

If  $a = n \in \mathbb{Z} \setminus \{0\}$ , this vanishes; for  $a = 0$  it yields

$$\int_C \frac{1}{z} dz = \int_{-\pi}^\pi \frac{iRe^{i\theta}}{Re^{i\theta}} d\theta = 2\pi i.$$

#### 4.4 Upper Bounds for Moduli of Contour Integrals

**Lemma 4.21.** If  $w(t)$  is piecewise continuous on  $[a, b]$ , then

$$\left| \int_a^b w(t) dt \right| \leq \int_a^b |w(t)| dt.$$

#### ML-inequality

**Theorem 4.22.** Let  $C$  be a contour of length  $L = b - a$ , and suppose  $f$  is piecewise continuous on  $C$  with  $|f(z)| \leq M$  on  $C$ . Then

$$\left| \int_C f(z) dz \right| \leq ML (= M(b - a)).$$

**Example 4.23.** On the quarter-circle  $C : |z| = 2$  from  $2$  to  $2i$ ,

$$\left| \int_C \frac{z+4}{z^3-1} dz \right| \leq \frac{6}{7} \cdot \frac{\pi}{2} \cdot 2 = \frac{6\pi}{7},$$

since  $|z+4| \leq 6$ ,  $|z^3-1| \geq 7$ , and  $L = \pi$ .

**Example 4.24** (Large semicircle vanishing). Let  $C_R : z = Re^{i\theta}$ ,  $0 \leq \theta \leq \pi$ , and take  $z^{1/2} = \exp(\frac{1}{2} \log z)$  on  $|z| > 0$ ,  $-\pi/2 < \theta < 3\pi/2$ . Then

$$\left| \int_{C_R} \frac{z^{1/2}}{z^2+1} dz \right| \leq \max_{C_R} \frac{\sqrt{R}}{R^2-1} \cdot (\pi R) = \frac{\pi R \sqrt{R}}{R^2-1} \xrightarrow{R \rightarrow \infty} 0.$$

Hence  $\lim_{R \rightarrow \infty} \int_{C_R} \frac{z^{1/2}}{z^2+1} dz = 0$ .



## 4.5 Antiderivatives and Path Independence

**Theorem 4.25.** Let  $f$  be continuous on a domain  $D \subset \mathbb{C}$ . The following are equivalent:

- (1)  $f$  has an antiderivative  $F$  on  $D$ ;
- (2) For any  $z_1, z_2 \in D$  and any contour  $C$  in  $D$  from  $z_1$  to  $z_2$ ,

$$\int_C f(z) dz = F(z_2) - F(z_1);$$

- (3)  $\int_C f(z) dz = 0$  for every closed contour  $C$  in  $D$ .

**Example 4.26.**  $f(z) = z^2$  has antiderivative  $F(z) = z^3/3$  on  $\mathbb{C}$ ; thus for any contour  $0 \rightarrow 1+i$ ,

$$\int_0^{1+i} z^2 dz = \left[ \frac{z^3}{3} \right]_0^{1+i} = \frac{2}{3}(-1+i).$$

**Example 4.27.**  $f(z) = z^{-2}$  is continuous on  $\mathbb{C} \setminus \{0\}$  with antiderivative  $F(z) = -1/z$  on  $|z| > 0$ . Therefore for the circle  $z = 2e^{i\theta}$ ,

$$\int_{|z|=2} \frac{1}{z^2} dz = 0.$$

**Example 4.28** (Using branches of  $\log$  for  $1/z$ ). On the right semicircle  $C_1 : z = 2e^{i\theta}, -\pi/2 \leq \theta \leq \pi/2$ , the principal branch

$$\text{Log } z = \ln r + i\varphi \quad (r > 0, -\pi < \varphi < \pi)$$

is an antiderivative of  $1/z$ , hence

$$\int_{C_1} \frac{1}{z} dz = \text{Log}(2i) - \text{Log}(-2i) = \pi i.$$

On the left semicircle  $C_2 : \pi/2 \leq \theta \leq 3\pi/2$  using the branch

$$\log z = \ln r + i\theta \quad (r > 0, 0 < \theta < 2\pi),$$

we likewise obtain

$$\int_{C_2} \frac{1}{z} dz = \log(-2i) - \log(2i) = \pi i.$$

Therefore  $\oint_{|z|=2} \frac{1}{z} dz = 2\pi i$ .

## 4.6 Cauchy–Goursat Theorem

We begin with the real-variable result which motivates Cauchy’s theorem.

### Green’s Theorem

**Theorem 4.29.** Let  $C(= \partial R)$  be a positively oriented simple closed contour in the plane, and let  $R$  be the region it encloses. Suppose  $P(x, y), Q(x, y)$  are continuous on  $C \cup R$  and have continuous first partial derivatives  $P_x, P_y, Q_x, Q_y$  there. Then

$$\int_{C=\partial R} P(x, y) dx + Q(x, y) dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

This gives rise to one of the central theorems of complex integration.

### Cauchy’s Theorem (elementary form)

**Theorem 4.30.** Let  $f$  be analytic and  $f'$  continuous in a simply connected domain  $D \subset \mathbb{C}$ . If  $C$  is a positively oriented simple closed contour in  $D$ , then

$$\int_C f(z) dz = 0.$$

*Remark.* Let  $f = u + iv$  and  $z = z(t) = x(t) + iy(t)$  ( $dz = dx + idy$ ). Then

$$\begin{aligned} \int f(z) dz &= \int (u dx - v dy + i(v dx + u dy)) \\ &= \int (u dx - v dy) + i \int (v dx + u dy) \\ &= \iint_D \end{aligned}$$

**Example 4.31.** Let  $C$  be any simple closed contour. The function  $f(z) = e^{z^3}$  is entire. Hence

$$\int_C e^{z^3} dz = 0.$$

To remove the hypothesis “ $f'$  is continuous”, we use a covering lemma.

**Lemma 4.32.** *Let  $f$  be analytic throughout a closed region  $R$  consisting of the interior of a positively oriented simple closed contour  $C$  together with the points of  $C$  itself. For any  $\varepsilon > 0$ , the region  $R$  can be covered by finitely many (possibly partial) squares indexed by  $j = 1, \dots, n$  such that in each square there is a point  $z_j$  with*

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \varepsilon$$

*for all  $z$  in that square distinct from  $z_j$ .*

### Cauchy–Goursat Theorem

**Theorem 4.33.** *If  $f$  is analytic at all points on and inside a positively oriented simple closed contour  $C$ , then*

$$\int_C f(z) dz = 0.$$

## 4.7 Integrals on Simply and Multiply Connected Domains

### Simply connected domain

**Definition 4.34.** A domain  $D \subset \mathbb{C}$  is **simply connected** if every simple closed contour contained in  $D$  encloses only points of  $D$  (equivalently: any closed contour in  $D$  can be continuously deformed to a point while remaining in  $D$ ).

**Example 4.35.** The interior of a simple closed curve is simply connected. The annulus

$$\{z : r < |z| < R\}$$

is not simply connected because closed contours can wind around the missing center point.

**Definition 4.36** (Multiply connected domain). A domain that is not simply connected is called **multiply connected**.

### Cauchy on simply connected domains

**Theorem 4.37.** *If  $f$  is analytic throughout a simply connected domain  $D$ , then*

$$\int_C f(z) dz = 0$$

*for every closed contour  $C$  contained in  $D$ .*

**Example 4.38.** Let  $C$  be any closed contour in the disk  $\{z : |z| < 2\}$ . Consider

$$f(z) = \frac{ze^z}{(z^2 + 9)^5}.$$

The poles at  $z = \pm 3i$  lie outside  $|z| < 2$ . On  $|z| < 2$  the function  $f$  is analytic. Hence

$$\int_C \frac{ze^z}{(z^2 + 9)^5} dz = 0.$$

The following result ties together antiderivatives, path-independence, and zero integral over closed contours.

**Theorem 4.39** (Equivalence). *Let  $f$  be continuous on a domain  $D$ . The following are equivalent:*

1.  $f$  has an antiderivative  $F$  on  $D$ ;
2. For any  $z_1, z_2 \in D$ , and any contour  $C$  in  $D$  from  $z_1$  to  $z_2$ ,

$$\int_C f(z) dz = F(z_2) - F(z_1);$$

3. For every closed contour  $C$  in  $D$ ,

$$\int_C f(z) dz = 0.$$

**Corollary 4.40.** *If  $f$  is analytic throughout a simply connected domain  $D$ , then  $f$  has an antiderivative on  $D$ .*

*Remark.* Since the entire plane  $\mathbb{C}$  is simply connected, every entire function possesses an entire antiderivative.

## Multiply connected case

## Cauchy for multiply connected regions

**Theorem 4.41.** Suppose

1.  $C$  is a positively oriented simple closed contour;
2.  $C_1, \dots, C_n$  are negatively oriented (clockwise) simple closed contours interior to  $C$ , pairwise disjoint, and their interiors do not intersect;
3.  $f$  is analytic on  $C$ , on each  $C_k$ , and on the region consisting of points inside  $C$  and outside all  $C_k$ .

Then

$$\int_C f(z) dz + \sum_{k=1}^n \int_{C_k} f(z) dz = 0.$$

## Deformation of Paths

**Corollary 4.42.** Let  $C_1$  and  $C_2$  be positively oriented simple closed contours with  $C_1$  inside  $C_2$ . If  $f$  is analytic on and between these contours, then

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz.$$

**Example 4.43.** Let  $C$  be any positively oriented simple closed contour around the origin. Then

$$\int_C \frac{1}{z} dz = 2\pi i.$$

Indeed, by deformation we may replace  $C$  by the unit circle.

## 4.8 Cauchy Integral Formula

We now reach one of the most powerful formulas in complex analysis.

### Cauchy Integral Formula

**Theorem 4.44.** Let  $f$  be analytic on and inside a positively oriented simple closed contour  $C$ , and let  $z_0$  be a point interior to  $C$ . Then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

*Remark.*

$$\begin{aligned} \int_C \frac{f(z)}{z - z_0} dz &= \int_C \frac{f(z) - f(z_0) + f(z_0)}{z - z_0} dz \\ &= \int_C \frac{f(z) - f(z_0)}{z - z_0} dz + \int_C \frac{f(z_0)}{z - z_0} dz \end{aligned}$$

*Remark.* This formula shows that the values of  $f$  inside  $C$  are **completely determined** by the values of  $f$  on  $C$ .

**Observation.** Written as

$$\int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0),$$

the formula is very convenient for evaluating contour integrals.

**Example 4.45.** Let  $C$  be the positively oriented circle  $|z| = 2$ . Consider

$$\int_C \frac{z}{(9 - z^2)(z + i)} dz.$$

Write  $f(z) = \frac{z}{9 - z^2}$ , which is analytic on  $|z| \leq 2$ , and  $z_0 = -i$  lies inside  $C$ . Then

$$\int_C \frac{f(z)}{z - (-i)} dz = 2\pi i f(-i) = 2\pi i \cdot \frac{-i}{9 - (-i)^2} = 2\pi i \cdot \frac{-i}{9 + 1} = \frac{\pi}{5}.$$

### 4.8.1 Cauchy formulas for derivatives

#### Cauchy formula for $f'$

**Theorem 4.46.** Under the hypotheses of Theorem 4.44, for  $z$  interior to  $C$ ,

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z)^2} ds.$$

Remark.

$$f(z) - \frac{1}{2\pi i} \int_C \frac{f(s)}{s-z} dz \implies f'(z) = \frac{df}{dz} = \frac{d}{dz} \left( \frac{1}{2\pi i} \int_C \frac{f(s)}{s-z} ds \right)$$

### Cauchy formula for higher derivatives

**Corollary 4.47.** If  $f$  is analytic on and inside  $C$ , then for  $n = 1, 2, \dots$ ,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s-z)^{n+1}} ds.$$

Equivalently,

$$\int_C \frac{f(s)}{(s-z)^{n+1}} ds = \frac{2\pi i}{n!} f^{(n)}(z).$$

**Example 4.48.** Let  $C$  be the positively oriented unit circle  $|z| = 1$ . Evaluate

$$\int_C \frac{e^{2z}}{z^4} dz.$$

Here  $f(z) = e^{2z}$  is analytic everywhere and we want the coefficient corresponding to  $(s-0)^{-4}$ . Take  $z = 0$  and  $n = 3$  in the corollary:

$$\int_C \frac{f(s)}{s^4} ds = \frac{2\pi i}{3!} f^{(3)}(0).$$

But  $f^{(3)}(z) = 2^3 e^{2z} = 8e^{2z}$ , so  $f^{(3)}(0) = 8$ . Hence

$$\int_C \frac{e^{2z}}{z^4} dz = \frac{2\pi i}{6} \cdot 8 = \frac{8\pi i}{3}.$$

**Example 4.49.** Let  $f(z) = 1$ . Then

$$\int_C \frac{1}{z-z_0} dz = 2\pi i, \quad \int_C \frac{1}{(z-z_0)^{n+1}} dz = 0, \quad n = 1, 2, \dots$$

whenever  $z_0$  is inside  $C$ .

## 4.9 Consequences of the Cauchy Integral Formula

### 4.9.1 Analyticity of derivatives

**Theorem 4.50.** *If  $f$  is analytic at a point  $z_0$ , then all derivatives  $f^{(n)}$  exist and are analytic at  $z_0$ .*

**Corollary 4.51.** *If  $f(z) = u(x, y) + iv(x, y)$  is analytic at  $z_0$ , then  $u$  and  $v$  have continuous partial derivatives of all orders in a neighborhood of  $z_0$ .*

## 4.10 Morera's Theorem

**Theorem 4.52** (Morera). *Let  $f$  be continuous on a domain  $D$ . If*

$$\int_C f(z) dz = 0$$

*for every closed contour  $C$  in  $D$ , then  $f$  is analytic throughout  $D$ .*

### 4.10.1 Cauchy's Inequalities

**Theorem 4.53** (Cauchy's inequality). *Suppose  $f$  is analytic on and inside the circle  $C_R = \{z : |z - z_0| = R\}$ , and let*

$$M_R = \max_{|z - z_0| = R} |f(z)|.$$

*Then for  $n = 0, 1, 2, \dots$ ,*

$$|f^{(n)}(z_0)| \leq \frac{n! M_R}{R^n}.$$



### 4.11 Liouville's Theorem and the Fundamental Theorem of Algebra

#### Liouville

**Theorem 4.54.** *If  $f$  is entire and bounded in the whole complex plane, then  $f$  is constant.*

#### Fundamental Theorem of Algebra

**Theorem 4.55.** *Let*

$$P(z) = a_0 + a_1z + \cdots + a_nz^n, \quad a_n \neq 0, \quad n \geq 1,$$

*be a complex polynomial. Then  $P$  has at least one zero in  $\mathbb{C}$ .*

*Remark.* It follows that any polynomial of degree  $n$  can be factored into linear factors:

$$P(z) = c(z - z_1)(z - z_2) \cdots (z - z_n), \quad c, z_k \in \mathbb{C}.$$

### 4.12 Maximum Modulus Principle

**Lemma 4.56.** *Suppose  $f$  is analytic in a disk  $|z - z_0| < \varepsilon$  and  $|f(z)| \leq |f(z_0)|$  for all such  $z$ . Then  $f$  is constant in that disk.*

**Theorem 4.57** (Maximum Modulus Principle). *If  $f$  is analytic and non-constant in a domain  $D$ , then  $|f(z)|$  has no maximum value in  $D$ .*

**Corollary 4.58.** *Let  $f$  be continuous on a closed bounded region  $R$ , analytic and non-constant on the interior of  $R$ . Then  $\max_{z \in R} |f(z)|$  is attained on the boundary of  $R$ , not in the interior.*

*Remark.* If  $f = u + iv$  is analytic, then  $u$  is harmonic. The corollary implies a maximum principle for  $u$  as well.

**Example 4.59.** Let  $R = \{0 \leq x \leq \pi, 0 \leq y \leq 1\}$  and  $f(z) = \sin z$ . Since

$$\sin z = \sin x \cosh y + i \cos x \sinh y,$$

we have

$$|f(z)|^2 = \sin^2 x + \sinh^2 y.$$

On  $R$ ,  $\sin^2 x$  is largest at  $x = \pi/2$  and  $\sinh^2 y$  is largest at  $y = 1$ , so the maximum of  $|f(z)|$  on  $R$  is attained at  $z = \pi/2 + i$  and nowhere in the interior.

## 4.13 Exercises

1. Let  $C_0$  be the positively oriented circle  $|z - z_0| = R$ . Show that

$$\int_{C_0} (z - z_0)^{n-1} dz = \begin{cases} 0, & n = \pm 1, \pm 2, \dots \\ 2\pi i, & n = 0. \end{cases}$$

**Sol.** Parametrize  $C_0$  by  $z(t) = z_0 + Re^{it}$  ( $t \in [0, 2\pi]$ ) then  $dz = iRe^{it} dt$  and

$$\int_{C_0} (z - z_0)^{n-1} dz = \int_0^{2\pi} (Re^{it})^{n-1} iRe^{it} dt = iR^n \int_0^{2\pi} e^{int} dt.$$

(1) If  $n \neq 0$ , then

$$\int_0^{2\pi} e^{int} dt = \left[ \frac{1}{in} e^{int} \right]_0^{2\pi} = \frac{e^{in2\pi} - 1}{in} = \frac{1 - 1}{in} = 0, \quad \text{so the integral is 0.}$$

(2) If  $n = 0$ , then  $e^{int} \equiv 1$  and the integral equals  $iR^0 \int_0^{2\pi} 1 dt = 2\pi i$ .

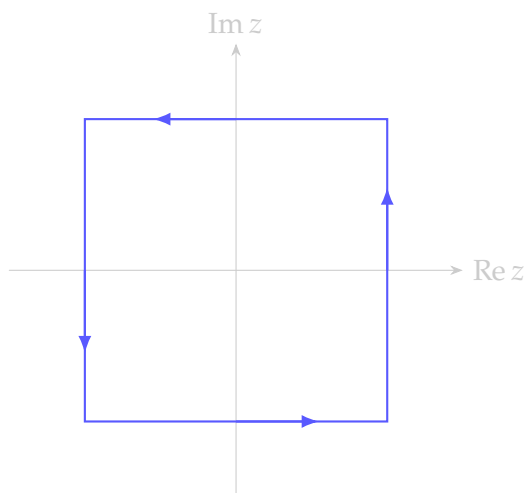
□

2. Let  $C$  be the boundary of the square with sides  $x = \pm 2$ ,  $y = \pm 2$ , oriented positively. Show that

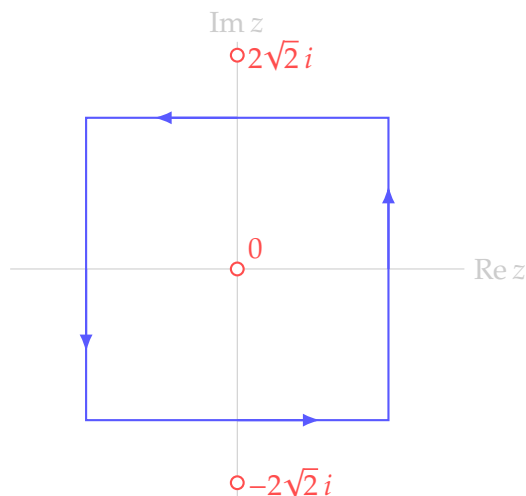
$$\int_C \frac{\cos z}{z(z^2 + 8)} dz = \frac{i\pi}{4}, \quad \int_C \frac{\cosh z}{z^4} dz = 0, \quad \int_C \frac{\tan(z/2)}{(z - x_0)^2} dz = i\pi \sec^2\left(\frac{x_0}{2}\right),$$

where  $-2 < x_0 < 2$ .

**Sol.** Let  $C$  be the positively oriented boundary of the square  $\{x + iy : |x| \leq 2, |y| \leq 2\}$ .



(1)  $\int_C \frac{\cos z}{z(z^2 + 8)} dz$ . The integrand is meromorphic with simple poles at  $z = 0$  and  $z = \pm 2\sqrt{2}i$ .



Only  $z = 0$  is inside  $C$ . Around  $z = 0$ ,

$$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \cdots, \quad \frac{1}{z(z^2 + 8)} = \frac{1}{z^3 + 8z} = \frac{1}{8z} \frac{1}{1 + z^2/8} = \frac{1}{8z} \left( 1 - \frac{z^2}{8} + \frac{z^4}{8^2} - \cdots \right).$$

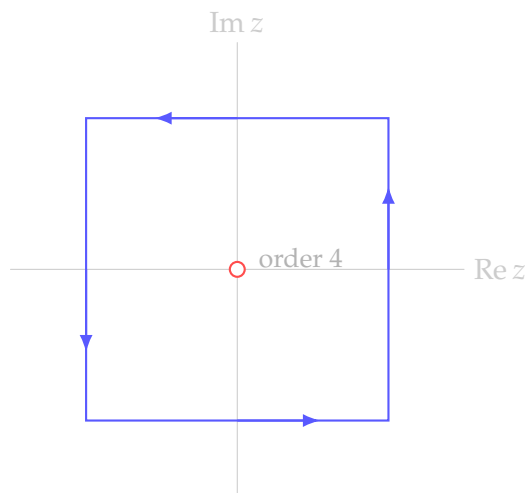
Thus,

$$\operatorname{Res}_{z=0} \frac{\cos z}{z(z^2 + 8)} = \operatorname{Res}_{z=0} \left( \frac{1}{8z} \left( 1 - \frac{z^2}{8} + \frac{z^4}{8^2} - \cdots \right) \left( 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \cdots \right) \right) = \frac{1}{8}.$$

By the residue theorem,

$$\int_C \frac{\cos z}{z(z^2 + 8)} dz = 2\pi i \cdot \frac{1}{8} = \frac{i\pi}{4}.$$

(2)  $\int_C \frac{\cosh z}{z^4} dz$ . Here the only singularity is at  $z = 0$  (order 4).



Using

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots, \quad \frac{\cosh z}{z^4} = \frac{1}{z^4} + \frac{1}{2} \frac{1}{z^2} + \frac{1}{4!} + \cdots,$$

there is no  $1/z$  term; hence  $\text{Res}_{z=0}(\cosh z/z^4) = 0$ , and therefore

$$\int_C \frac{\cosh z}{z^4} dz = 0.$$

(3)  $\int_C \frac{\tan(z/2)}{(z-x_0)^2} dz$  with  $-2 < x_0 < 2$ . We know that

$$\tan w = \frac{\sin w}{\cos w},$$

so the poles of  $\tan w$  occur exactly where  $\cos w = 0$  and  $\sin w \neq 0$ . The zeros of  $\cos w$  are

$$w = \frac{\pi}{2} + k\pi = \frac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z}.$$

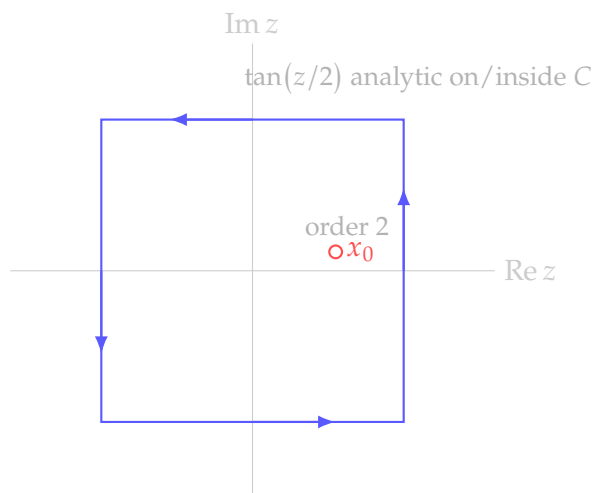
Now consider  $\tan\left(\frac{z}{2}\right)$ . Let  $w = \frac{z}{2}$ . The poles of  $\tan(z/2)$  occur where  $w$  is a pole of  $\tan w$ , i.e. where

$$\frac{z}{2} = \frac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z}.$$

So the poles of  $\tan(z/2)$  are precisely at  $z = (2k+1)\pi$  with  $k \in \mathbb{Z}$ . Since

$$|(2k+1)\pi| \geq \pi > 2,$$

none of these poles lie inside or on  $C$ . Hence  $\tan(z/2)$  is analytic on and inside  $C$ . The only singularity of the integrand  $\tan(z/2)/(z-x_0)^2$  inside  $C$  is at  $z = x_0$ .



Since  $\tan(z/2)$  is analytic at  $z = x_0$ , we may expand it in a Taylor series about  $x_0$ :

$$\begin{aligned}\tan\left(\frac{z}{2}\right) &= \tan\left(\frac{x_0}{2}\right) + \frac{d}{dz} \tan\left(\frac{z}{2}\right) \Big|_{z=x_0} (z - x_0) + \cdots \\ &= \tan\left(\frac{x_0}{2}\right) + \frac{1}{2} \sec^2\left(\frac{x_0}{2}\right) (z - x_0) + \cdots.\end{aligned}$$

Dividing by  $(z - x_0)^2$  gives the Laurent series

$$\frac{\tan(z/2)}{(z - x_0)^2} = \frac{\tan(x_0/2)}{(z - x_0)^2} + \frac{1}{2} \sec^2\left(\frac{x_0}{2}\right) \frac{1}{z - x_0} + \cdots.$$

Thus, we obtain

$$\operatorname{Res}_{z=x_0} \left( \frac{\tan(z/2)}{(z - x_0)^2} \right) = \frac{1}{2} \sec^2\left(\frac{x_0}{2}\right).$$

By the residue theorem,

$$\int_C \frac{\tan(z/2)}{(z - x_0)^2} dz = 2\pi i \cdot \frac{1}{2} \sec^2\left(\frac{x_0}{2}\right) = i\pi \sec^2\left(\frac{x_0}{2}\right).$$

□

3. Let  $C$  be the circle  $|z| = 3$ , positively oriented, and define

$$f(z) = \int_C \frac{2s^2 - s - 2}{s - z} ds, \quad |z| \neq 3.$$

Show that  $f(2) = 8\pi i$ .

**Sol.** Let  $F(s) = 2s^2 - s - 2$ , which is entire. By the Cauchy integral formula, for  $|z| < 3$ ,

$$\int_{|s|=3} \frac{F(s)}{s - z} ds = 2\pi i F(z).$$

Since 2 lies inside the circle  $|s| = 3$ , we have

$$f(2) = 2\pi i F(2) = 2\pi i (2 \cdot 2^2 - 2 - 2) = 2\pi i (8 - 2 - 2) = 2\pi i \cdot 4 = 8\pi i.$$

□

4. Let  $C$  be any positively oriented simple closed contour and

$$f(z) = \int_C \frac{s^3 + 2s}{(s - z)^3} ds.$$

Show that  $f(z) = 6\pi i z$  when  $z$  is inside  $C$ , and  $f(z) = 0$  when  $z$  is outside.

**Sol.** Let  $F(s) = s^3 + 2s$ , an entire function. By the generalized Cauchy integral formula,

$$\int_C \frac{F(s)}{(s - z)^{n+1}} ds = \frac{2\pi i}{n!} F^{(n)}(z),$$

for  $z$  inside  $C$ . Here  $\frac{F(s)}{(s - z)^3}$  corresponds to  $n = 2$ , so

$$f(z) = \int_C \frac{s^3 + 2s}{(s - z)^3} ds = \frac{2\pi i}{2!} F''(z).$$

Compute  $F'(s) = 3s^2 + 2$  and  $F''(s) = 6s$ , hence for  $z$  inside  $C$ ,

$$f(z) = \frac{2\pi i}{2} \cdot 6z = 6\pi i z.$$

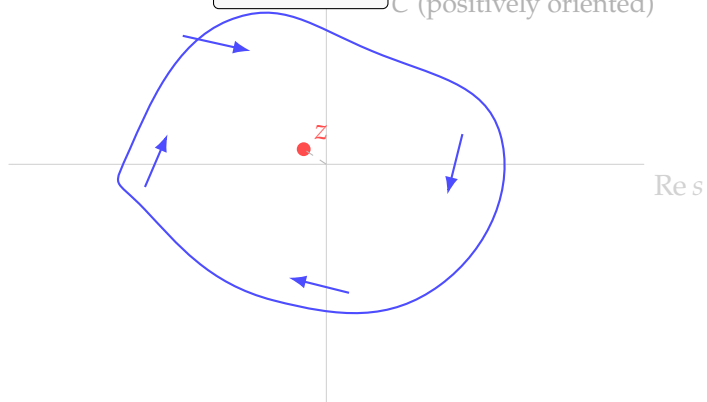
If  $z$  is outside  $C$ , then  $s \mapsto \frac{s^3 + 2s}{(s - z)^3}$  is analytic on and inside  $C$  (the only singularity is at  $s = z$ , which lies outside). By Cauchy's theorem,

$$f(z) = \int_C \frac{s^3 + 2s}{(s - z)^3} ds = 0.$$

$$f(z) = \int_C \frac{s^3 + 2s}{(s-z)^3} ds, \quad F(s) = s^3 + 2s$$

$$f(z) = 6\pi i z$$

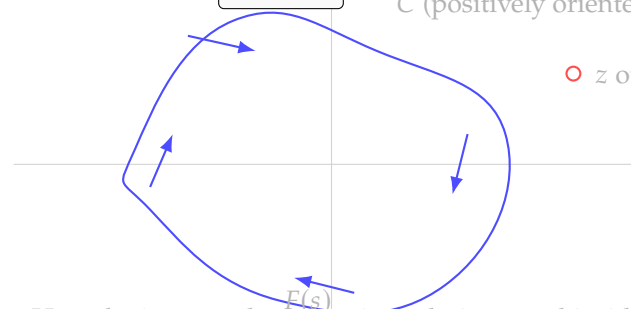
$C$  (positively oriented)



Exterior case:  $z$  outside  $C$

$$f(z) = 0$$

$C$  (positively oriented)



Here the integrand  $\frac{F(s)}{(s-z)^3}$  is analytic on and inside  $C$  (its only singularity is at  $s = z$ , which lies outside).

By Cauchy's theorem,  $f(z) = \int_C \frac{F(s)}{(s-z)^3} ds = 0$ .

□

5. Let  $C$  be the unit circle  $z = e^{i\theta}$ ,  $-\pi \leq \theta \leq \pi$ . Show that for any constant  $a$ ,

$$\int_C \frac{e^{az}}{z} dz = 2\pi i.$$

Then write this integral in term of  $\theta$  to derive the integration formula

$$\int_0^\pi e^{a \cos \theta} \cos(a \sin \theta) d\theta = \pi.$$

**Sol.** Let  $C$  be the unit circle oriented positively. The integrand

$$\frac{e^{az}}{z}$$

has a simple pole at  $z = 0$  with residue  $\text{Res}_{z=0} \frac{e^{az}}{z} = e^{a \cdot 0} = 1$ . By the residue theorem,

$$\int_C \frac{e^{az}}{z} dz = 2\pi i.$$

Now parametrize  $C$  by  $z = e^{i\theta}$ ,  $-\pi \leq \theta \leq \pi$ . Then  $dz = ie^{i\theta} d\theta$  and

$$\int_C \frac{e^{az}}{z} dz = \int_{-\pi}^{\pi} \frac{e^{ae^{i\theta}}}{e^{i\theta}} ie^{i\theta} d\theta = i \int_{-\pi}^{\pi} e^{a(\cos \theta + i \sin \theta)} d\theta = i \int_{-\pi}^{\pi} e^{a \cos \theta} (\cos(a \sin \theta) + i \sin(a \sin \theta)) d\theta.$$

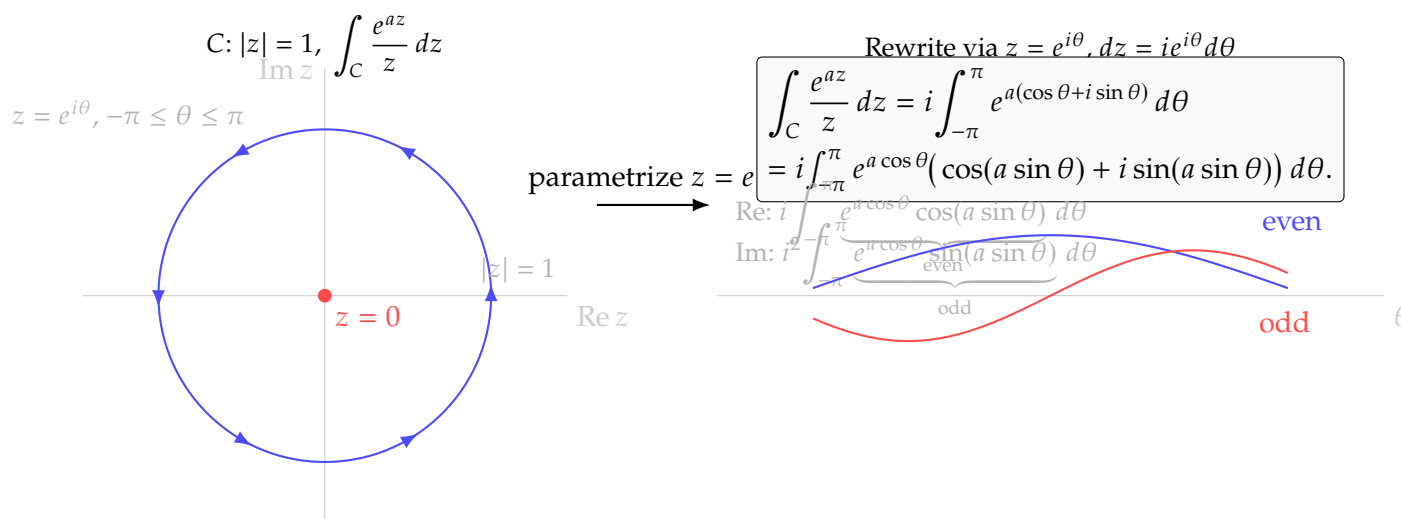
Equating real and imaginary parts with  $2\pi i$  gives

$$\int_{-\pi}^{\pi} e^{a \cos \theta} \sin(a \sin \theta) d\theta = 0, \quad \int_{-\pi}^{\pi} e^{a \cos \theta} \cos(a \sin \theta) d\theta = 2\pi.$$

Since the integrand  $e^{a \cos \theta} \cos(a \sin \theta)$  is even in  $\theta$ , we obtain

$$\int_0^\pi e^{a \cos \theta} \cos(a \sin \theta) d\theta = \pi.$$





□

## 5 Series

### 5.1 Convergence of Sequences

**Definition 5.1** (Limit of a sequence). A sequence  $(z_n)$  of complex numbers converges to  $z \in \mathbb{C}$  if for each  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that

$$|z_n - z| < \varepsilon \quad (n > N).$$

We write  $\lim_{n \rightarrow \infty} z_n = z$ . If no such  $z$  exists, the sequence **diverges**.

*Remark* (Uniqueness). A complex sequence has at most one limit.

**Theorem 5.2** (Componentwise convergence). Let  $z_n = x_n + iy_n$  and  $z = x + iy$ . Then

$$\lim_{n \rightarrow \infty} z_n = z \iff \lim_{n \rightarrow \infty} x_n = x \text{ and } \lim_{n \rightarrow \infty} y_n = y.$$

**Example 5.3.** (a)  $z_n = \frac{1}{n^3} + i \Rightarrow \lim z_n = i$ . (b)  $z_n = -2 + i \frac{(-1)^n}{n^2} \Rightarrow \lim z_n = -2$ .

**Observation** (Polar view). Writing  $z_n = r_n e^{i\theta_n}$  with  $r_n = |z_n|$  and  $\theta_n = \text{Arg} z_n$ , one may have  $r_n \rightarrow r$  while  $(\theta_n)$  fails to converge (e.g. even/odd subsequences approaching  $\pm\pi$ ).

## 5.2 Convergence of Series

### Series and sum

**Definition 5.4.** A series  $\sum_{n=1}^{\infty} z_n$  converges to  $S$  if the partial sums  $S_N = \sum_{n=1}^N z_n$  satisfy  $S_N \rightarrow S$ . Then  $\sum_{n=1}^{\infty} z_n = S$ .

### Componentwise

**Theorem 5.5.** If  $z_n = x_n + iy_n$  and  $S = X + iY$ , then

$$\sum_{n=1}^{\infty} z_n = S \iff \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y.$$

*Remark* (Necessary test and boundedness). If  $\sum z_n$  converges, then  $z_n \rightarrow 0$  (the  $n$ th-term test). In particular, the terms are bounded: there exists  $M$  with  $|z_n| \leq M$  for all  $n$ .

**Definition 5.6** (Absolute convergence).  $\sum z_n$  is **absolutely convergent** if  $\sum |z_n|$  converges. Absolute convergence implies convergence.

*Remark* (Remainders). If  $S = \sum_{n=1}^{\infty} z_n$ , the remainder after  $N$  terms is  $\rho_N = S - S_N$ . Then  $S_N \rightarrow S$  iff  $\rho_N \rightarrow 0$ .

## 5.3 Power Series and Taylor Series

### Power series centered at $z_0$

**Definition 5.7.**

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots,$$

with  $a_n, z_0 \in \mathbb{C}$ .

### Taylor series

**Theorem 5.8.** If  $f$  is analytic on  $|z - z_0| < R_0$ , then for  $|z - z_0| < R_0$ ,

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!}.$$

For  $z_0 = 0$  this is the **Maclaurin series**.

**Example 5.9.** Since  $e^z$  is entire,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z^2 e^{3z} = \sum_{n=0}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n \quad (\text{interpreting } (n-2)! = \infty \text{ for } n < 2 \text{ gives zero terms}).$$

Also

$$\begin{aligned}\sin z &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, & \cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \\ \sinh z &= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, & \cosh z &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.\end{aligned}$$

**Example 5.10** (Geometric series). For  $f(z) = \frac{1}{1-z}$  we have

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1,$$

and similarly  $\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$  for  $|z| < 1$ .

## 5.4 Laurent Series

*Remark.* At a point  $z_0$  where  $f$  is not analytic, Taylor series may fail; on an annulus  $R_1 < |z - z_0| < R_2$  one often has a two-sided power expansion (Laurent series).

**Theorem 5.11** (Laurent). *If  $f$  is analytic on the annulus  $R_1 < |z - z_0| < R_2$  and  $C$  is any positively oriented simple closed contour around  $z_0$  lying in that annulus, then on  $R_1 < |z - z_0| < R_2$ ,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

with

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \quad (n \in \mathbb{Z}).$$

If  $f$  is analytic on  $|z - z_0| < R_2$  then  $b_n = 0$  and Laurent reduces to Taylor.

**Example 5.12.** Since  $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$  for all  $z$ , we get

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}, \quad 0 < |z| < \infty.$$

The coefficient of  $(z^{-1})$  is 1, hence for any positively oriented simple closed contour  $C$  around 0,

$$\int_C e^{1/z} dz = 2\pi i.$$

**Example 5.13** (Partial fractions across annuli). Let

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}.$$

Three Laurent expansions in powers of  $z$  arise:

$$\begin{aligned} |z| < 1: \quad f(z) &= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left( \frac{1}{2^{n+1}} - 1 \right) z^n, \\ 1 < |z| < 2: \quad f(z) &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}, \\ |z| > 2: \quad f(z) &= \sum_{n=1}^{\infty} \frac{1 - 2^{n-1}}{z^n}. \end{aligned}$$

## 5.5 Absolute and Uniform Convergence of Power Series

**Theorem 5.14** (Absolute convergence inside any interior circle). *If a power series  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  converges at some  $z_1 \neq z_0$ , then it converges absolutely for all  $|z - z_0| < |z_1 - z_0|$ .*

**Definition 5.15** (Circle of convergence). The largest open disk centered at  $z_0$  on which the series converges is the **circle of convergence**. Its radius is the **radius of convergence**.

**Theorem 5.16** (Uniform convergence on closed interior disks). *If  $|z_1 - z_0| = R_1$  lies strictly inside the circle  $|z - z_0| = R$ , then the series is uniformly convergent on the closed disk  $|z - z_0| \leq R_1$ .*

## 5.6 Consequences for Sums of Power/Laurent Series

^

**Theorem 5.17** (Continuity and analyticity). *Inside the circle of convergence, the sum  $S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$  is continuous and analytic.*

**Remark** (Exterior series). If  $\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$  converges at  $z_1 \neq z_0$ , then it converges absolutely to a continuous function on  $\{|z - z_0| > |z_1 - z_0|\}$  (the exterior of the circle through  $z_1$ ).

**Remark** (Laurent on annuli). If

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

is valid on  $R_1 < |z - z_0| < R_2$ , then both series converge uniformly on any closed annulus  $R_1 + \varepsilon \leq |z - z_0| \leq R_2 - \varepsilon$  ( $\varepsilon > 0$ ).

**Theorem 5.18** (Termwise integration on interior contours). *Let  $C$  be a contour inside the circle of convergence of  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  and  $f$  continuous on  $C$ . Then*

$$\int_C f(z) \sum_{n=0}^{\infty} a_n(z - z_0)^n dz = \sum_{n=0}^{\infty} a_n \int_C f(z)(z - z_0)^n dz.$$

**Corollary 5.19.** *The sum  $S(z)$  is analytic inside its circle of convergence and may be integrated term by term on interior contours.*

**Example 5.20.** Define

$$f(z) = \begin{cases} \frac{e^z - 1}{z}, & z \neq 0, \\ 1, & z = 0. \end{cases}$$

Since  $e^z - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!}$ , we obtain  $f(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!}$  for all  $z$  with the limit at 0 equal to 1. Thus  $f$  is entire and continuous at 0.

**Theorem 5.21** (Termwise differentiation). *Inside the circle of convergence,*

$$\frac{d}{dz} \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

**Theorem 5.22** (Uniqueness of Taylor/Laurent expansions). *If a power series in  $(z - z_0)$  equals  $f(z)$  on a disk, it is the Taylor series of  $f$ . If a doubly-infinite series  $\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$  equals  $f$  on an annulus, it is the Laurent expansion of  $f$  on that annulus.*

**Corollary 5.23** (Cauchy product). *If*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

*converge on  $|z - z_0| < R$ , then*

$$f(z)g(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k b_{n-k} \right) (z - z_0)^n, \quad |z - z_0| < R.$$