Complex Analysis

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1 Complex Numbers

1.1 Complex Numbers

Definition 1.1 (Complex numbers and parts)**.** A complex number is an ordered pair $(x, y) \in \mathbb{R}^2$, denoted $z = (x, y)$ or $z = x + iy$, with **real part** $\operatorname{Re} z = x$ and **imaginary part** $\operatorname{Im} z = y$.

Remark. Two complex numbers are equal iff they have the same real and imaginary parts.

Definition 1.2 (Algebra)**.** For $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$,

$$z_1 + z_2 = (x_1 + x_2,\ y_1 + y_2), \qquad z_1 z_2 = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + y_1 x_2).$$

Let $i = (0, 1)$. then $i^2 = -1$ and every z can be written $x + iy$.

Remark (Basic properties)*.* The complex numbers satisfy the usual commutative, associative, and distributive laws; $0 = (0, 0)$ and $1 = (1, 0)$ are additive/multiplicative identities. For $z \neq 0$, the multiplicative inverse is

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}.$$

If $z_2 \neq 0$,

$$\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}.$$

Definition 1.3 (Binomial formula)**.** For $n \in \mathbb{N}$ and $z_1, z_2 \in \mathbb{C}$,

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}.$$

1.2 Vectors and Moduli

Definition 1.4 (Modulus and distance)**.** For $z = x + iy$, the **modulus** is $|z| = \sqrt{x^2 + y^2}$. The distance between $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Remark. The circle of center z_0 and radius $R > 0$ is $\{z : |z - z_0| = R\}$.

1.3 Complex Conjugation

Definition 1.5 (Conjugate)**.** For $z = x + iy$, the **conjugate** is $\bar{z} = x - iy$.

Theorem 1.6 (Conjugation identities)**.** *For any $z, z_1, z_2 \in \mathbb{C}$ (with $z_2 \neq 0$),*

$$\bar{\bar{z}} = z, \quad |z| = |\bar{z}|, \quad \overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2},$$

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i}, \quad |z|^2 = z\bar{z}, \quad |z_1 z_2| = |z_1||z_2|.$$

Theorem 1.7 (Triangle inequality)**.** *For all $z_1, z_2 \in \mathbb{C}$,*

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Consequently, $|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$ and for any $n \in \mathbb{N}$,

$$\Big|\sum_{k=1}^{n} z_k\Big| \leq \sum_{k=1}^{n} |z_k|.$$

1.4 Polar and Exponential Form

Definition 1.8 (Polar form, argument)**.** For $z \neq 0$, write $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ with $r = |z|$ and any **argument** $\theta \in \arg z = \{\mathrm{Arg}z + 2\pi k : k \in \mathbb{Z}\}$, where $\mathrm{Arg}z \in (-\pi, \pi]$ is the principal value. (For $z = 0$, θ is undefined.)

Definition 1.9 (Euler's formula)**.** $e^{i\theta} = \cos\theta + i\sin\theta \quad (\theta \in \mathbb{R})$.

Remark (Parametrizing circles)*.* The circle $|z| = R$ has parametrization $z = Re^{i\theta}$, $0 \leq \theta \leq 2\pi$. The circle $|z - z_0| = R$ has $z = z_0 + Re^{i\theta}$.

1.5 Products, Powers, and Arguments

Proposition 1.10 (Product/quotient in polar form)**.** *If $z_j = r_je^{i\theta_j}$ $(j = 1, 2)$ with $r_j > 0$, then*

$$z_1z_2 = r_1r_2\, e^{i(\theta_1+\theta_2)}, \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}, \qquad z^{-1} = \frac{1}{r}\, e^{-i\theta}.$$

For $n \in \mathbb{Z}$, $z^n = r^n e^{in\theta}$.

Corollary 1.11 (de Moivre)**.** *For $n \in \mathbb{Z}$, $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.*

Theorem 1.12 (Arguments)**.** *If $z_j = r_je^{i\theta_j}$ $(j = 1, 2)$, then $\arg(z_1z_2) = \arg z_1 + \arg z_2$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$ (mod 2π). Using principal values requires care at the branch cut.*

1.6 Roots of Complex Numbers

Theorem 1.13 (All nth roots)**.** *Let $z_0 = r_0e^{i\theta_0} \neq 0$ and $n \in \mathbb{N}$. The solutions of $z^n = z_0$ are*

$$z_k = \sqrt[n]{r_0}\, \exp\left(i\, \frac{\theta_0 + 2\pi k}{n}\right), \qquad k = 0, 1, \ldots, n-1.$$

These n distinct roots lie on the circle $|z| = \sqrt[n]{r_0}$ at equal angular spacing $2\pi/n$. The root with $k = 0$ (when $\theta_0 = \mathrm{Arg}z_0$) is the ***principal root****.*

Remark (Roots of unity)*.* For $z_0 = 1$, the nth roots are $e^{2\pi ik/n}$, $k = 0, \ldots, n-1$.

Example 1.14 (Cube roots of $-8i$)**.** Since $-8i = 8\, e^{-i\pi/2}$, the cube roots are $2\, e^{i(-\pi/6+2\pi k/3)}$, $k = 0, 1, 2$.

1.7 Regions in the Complex Plane

Definition 1.15 (Neighborhoods)**.** An ε-neighborhood of z_0 is $\{z : |z - z_0| < \varepsilon\}$. The **deleted** (punctured) neighborhood is $\{z : 0 < |z - z_0| < \varepsilon\}$.

Definition 1.16 (Interior, exterior, boundary)**.** A point z_0 is an interior point of S if some neighborhood of z_0 lies in S; an exterior point if some neighborhood lies in S^c; otherwise z_0 is on the boundary ∂S.

Definition 1.17 (Open/closed, closure)**.** A set is **open** if it contains none of its boundary points; **closed** if it contains all of them. The **closure** $\overline{S}$ is $S \cup \partial S$.

Definition 1.18 (Connected, domain, region)**.** An open set S is **connected** if any two points can be joined by a polygonal line lying in S. A nonempty connected open set is a **domain**. A **region** is a domain together with some (possibly all or none) of its boundary points.

Definition 1.19 (Boundedness)**.** S is **bounded** if $S \subset \{z : |z| < R\}$ for some $R > 0$.

Definition 1.20 (Accumulation points)**.** A point z_0 is an accumulation point of S if every deleted neighborhood of z_0 contains a point of S. A set is closed iff it contains all its accumulation points.

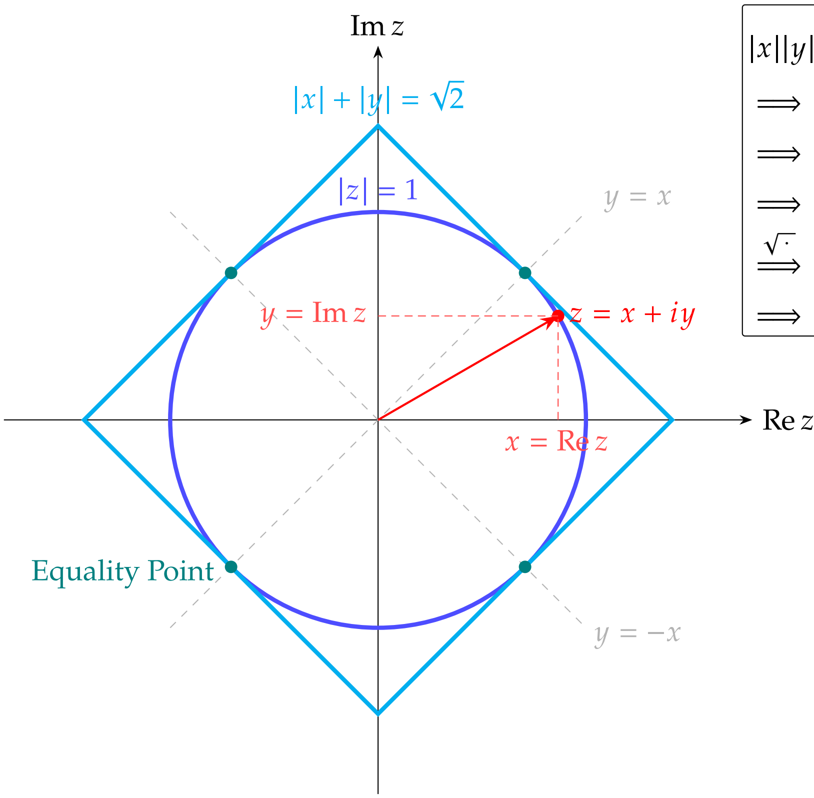
1.8 Exercises

1. Verify that $\sqrt{2}\,|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$.

Sol. Let $z = x + iy$, so that $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, and $|z| = \sqrt{x^2 + y^2}$. Then

$$
\begin{aligned}
\sqrt{2}\,|z| \;\geq\; |\operatorname{Re} z| + |\operatorname{Im} z| &\iff \sqrt{2}\sqrt{x^2+y^2} \;\geq\; |x| + |y| \\
&\iff 2(x^2+y^2) \;\geq\; (|x|+|y|)^2 \\
&\iff 2(x^2+y^2) \;\geq\; x^2+y^2+2|x||y| \quad (\because\ |x|^2 = x^2,\ |y|^2 = y^2) \\
&\iff x^2+y^2 \;\geq\; 2|x||y| \quad \text{by subtracting } x^2+y^2 \text{ from both sides} \\
&\iff x^2+y^2 \;\geq\; 2\sqrt{x^2y^2} \\
&\iff \frac{x^2+y^2}{2} \;\geq\; \sqrt{x^2y^2} \\
&\iff \frac{a+b}{2} \;\geq\; \sqrt{ab} \quad \text{by setting } a := x^2 \text{ and } b := y^2; \quad \text{(AM-GM inequality)}
\end{aligned}
$$

Hence it holds.



$$
\begin{aligned}
&|x||y| \leq \frac{x^2+y^2}{2} \quad \text{(AM–GM on } x^2, y^2) \\
&\implies (x^2+y^2) \leq (2|x||y|) \\
&\implies 2\cdot(x^2+y^2) \leq (x^2+y^2) + (2|x||y|) \\
&\implies (|x|+|y|)^2 \leq 2(x^2+y^2) \\
&\overset{\sqrt{\cdot}}{\implies} |x|+|y| \leq \sqrt{2}\,|z| \\
&\implies |\operatorname{Re} z| + |\operatorname{Im} z| \leq \sqrt{2}\,|z|.
\end{aligned}
$$

$\square$

2. By factoring $z^4 - 4z + 3$ into two quadratic factors show that if z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \le \frac{1}{3}.$$

Sol. Since $z^4 - 4z^2 + 3 \;=\; (z^2-1)(z^2-3)$, we have

$$\left|z^4 - 4z^2 + 3\right| = |z^2 - 1|\,|z^2 - 3|.$$

For $|z| = 2$ one has $|z^2| = |z|^2 = 4$. By the triangle inequality,

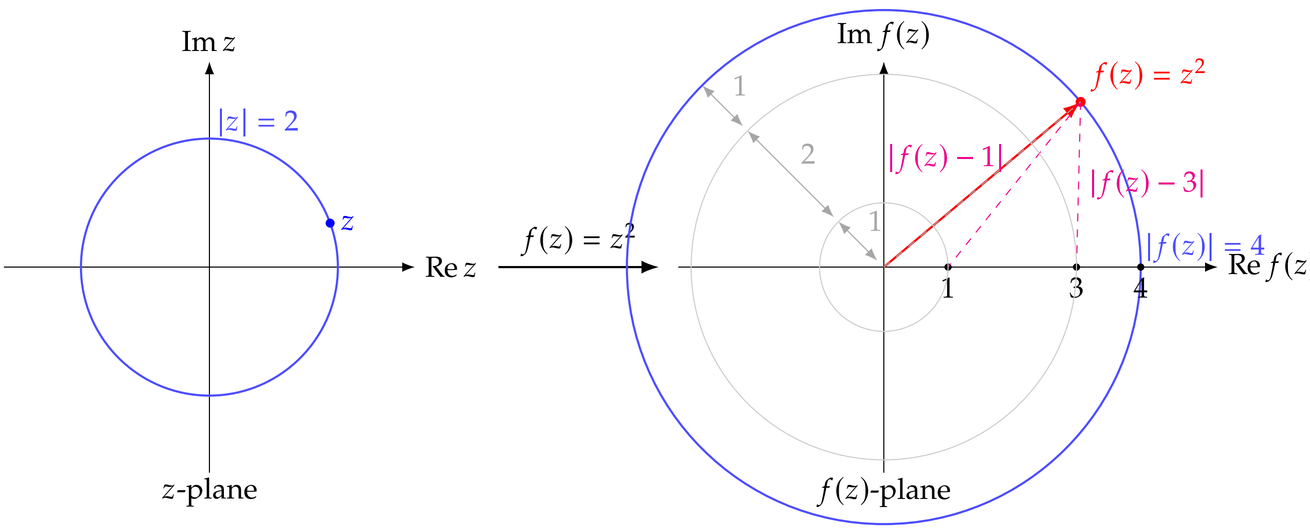
$$|z^2 - 1| \;\ge\; \big||z^2| - |1|\big| = |4-1| = 3, \qquad |z^2 - 3| \;\ge\; \big||z^2| - |3|\big| = |4-3| = 1.$$

Hence

$$|z^4 - 4z^2 + 3| \;\ge\; 3 \cdot 1 \;=\; 3,$$

and therefore

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \;=\; \frac{1}{|z^4 - 4z^2 + 3|} \;\le\; \frac{1}{3}.$$



For equality in the reverse triangle inequalities we must have z^2 and the positive reals $1, 3$ on the same ray from the origin, i.e. $z^2 = 4$. Together with $|z| = 2$ this forces $z = \pm 2$, and indeed

$$|(\pm 2)^4 - 4(\pm 2)^2 + 3| = |16 - 16 + 3| = 3,$$

so the bound is sharp precisely at $z = \pm 2$. $\square$

3. Prove the finite geometric sum

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \quad (z \neq 1)$$

and deduce Lagrange's trigonometric identity

$$1 + \cos\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\big((2n+1)\theta/2\big)}{2\sin\big(\theta/2\big)} \quad (0 < \theta < 2\pi).$$

4. Prove that the usual formula solves the quadratic equation

$$az^2 + bz + c = 0 \quad (a \neq 0)$$

when the coefficient a,b, and c are complex numbers. Specifically, by completing the square on the left-hand side, derive the **quadratic formula**

$$z = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

where both square roots are to be considered when $b^2 - 4ac \neq 0$. Use this result to find the roots of the equation

$$z^2 + 2z + (1 - i) = 0.$$

Sol. Since

$$az^2 + bz + c = a\left(z^2 + \frac{b}{a}z\right) + c = a\left(z + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c = a\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c,$$

we have

$$a\left(z + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c \quad \Longleftrightarrow \quad \left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking square roots of both sides yields

$$z + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}, \quad \text{whence} \quad z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Consider $z^2 + 2z + (1 - i)$ with $a = 1$, $b = 2$, and $c = 1 - i$. The discriminant is

$$\Delta = b^2 - 4ac = 4 - 4(1 - i) = 4i.$$

Since

$$\sqrt{i} = \frac{1 + i}{\sqrt{2}} \quad \left(\text{indeed, } \left(\frac{1 + i}{\sqrt{2}}\right)^2 = \frac{1 + 2i - 1}{2} = i\right),$$

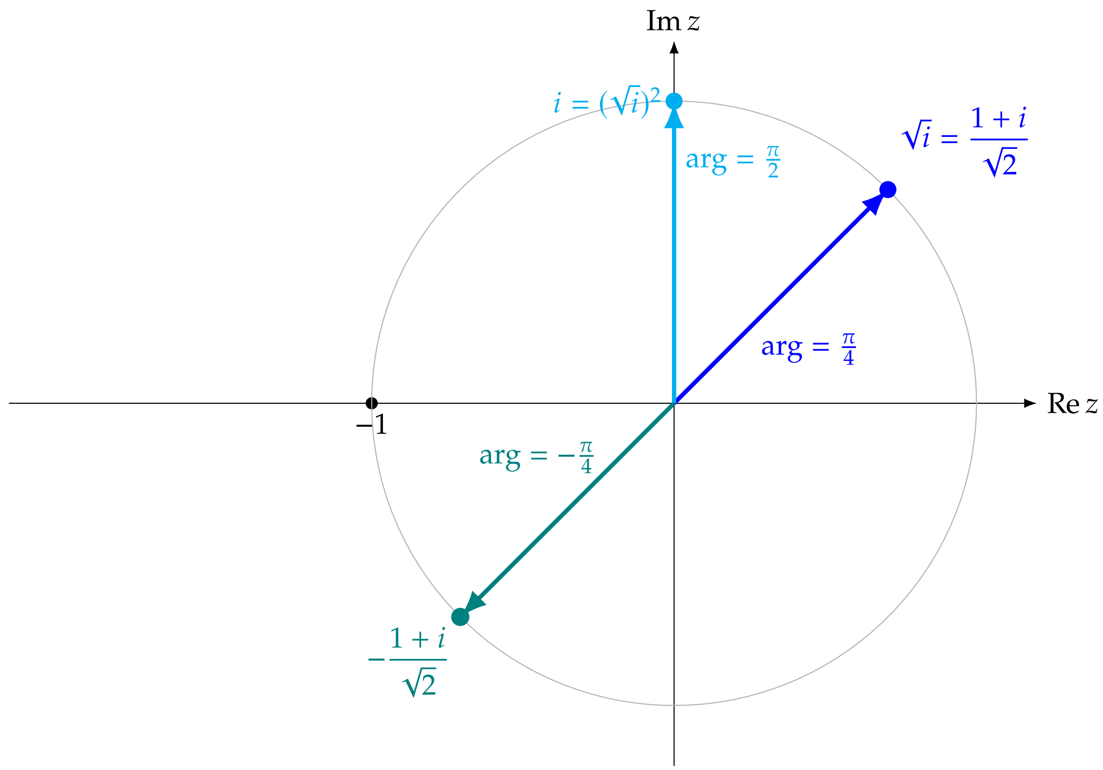
we may take $\sqrt{\Delta} = \sqrt{4i} = 2\sqrt{i} = \sqrt{2}\,(1 + i)$. Therefore

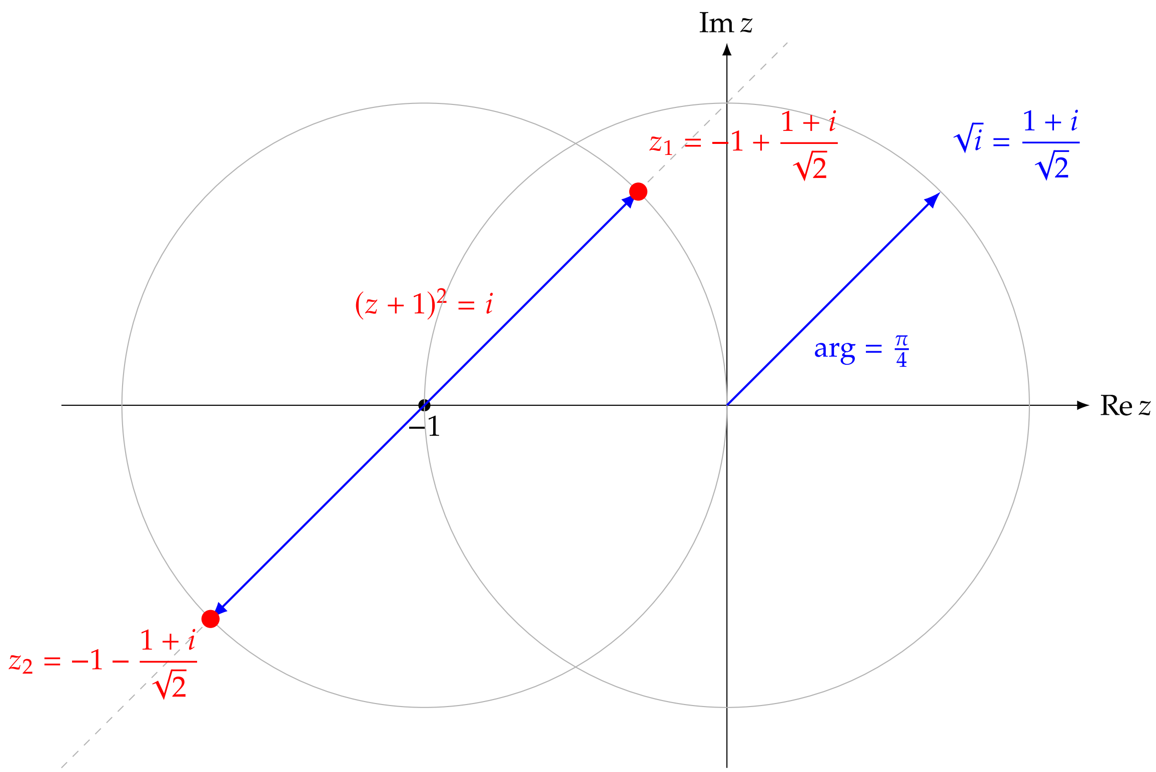
$$z = \frac{-2 \pm \sqrt{4i}}{2} = -1 \pm \sqrt{i} = -1 \pm \frac{1 + i}{\sqrt{2}}.$$

Thus the roots are

$$z_1 = -1 + \frac{1 + i}{\sqrt{2}}, \qquad z_2 = -1 - \frac{1 + i}{\sqrt{2}}.$$

Note that z_1, z_2 are roots of $(z + 1)^2 = i$.


Im z
$i = (\sqrt{i})^2$
$\sqrt{i} = \frac{1+i}{\sqrt{2}}$
$\arg = \frac{\pi}{2}$
$\arg = \frac{\pi}{4}$
Re z
-1
$\arg = -\frac{\pi}{4}$
$-\frac{1+i}{\sqrt{2}}$


Im z
$z_1 = -1 + \frac{1+i}{\sqrt{2}}$
$\sqrt{i} = \frac{1+i}{\sqrt{2}}$
$(z+1)^2 = i$
$\arg = \frac{\pi}{4}$
Re z
-1
$z_2 = -1 - \frac{1+i}{\sqrt{2}}$

$\square$

5. Determine the accumulation points of each sequence:

$$z_n = i^n, \quad , z_n = \frac{i^n}{n}, \quad z_n = (-1)^n(1+i)\frac{n-1}{n}.$$

Sol. content... □

6. Prove that a finite set of points $z_1, z_2, \cdots, z_n$ cannot have any accumulation points.

Sol. Recall that $w \in \mathbb{C}$ is an accumulation point of F iff for every $\varepsilon > 0$ the punctured ball $B(w, \varepsilon) \setminus \{w\}$ intersects F (equivalently, $B(w, \varepsilon)$ contains a point of F distinct from w).

Fix $w \in \mathbb{C}$. Consider the finite set of distances

$$D := \{|w - z_k| : 1 \le k \le n\} \subset [0, \infty).$$

Let $d := \min D$. There are two cases.

Case 1: $w \notin F$**.** Then $d > 0$. For $\varepsilon := \frac{d}{2}$ we have $B(w, \varepsilon) \cap F = \varnothing$, hence w is not an accumulation point.

Case 2: $w = z_j$ **for some** j**.** If $n = 1$, then $F = \{w\}$ and for any $\varepsilon > 0$ small enough, $B(w, \varepsilon) \cap (F \setminus \{w\}) = \varnothing$, so w is not an accumulation point. If $n \ge 2$, put

$$d' := \min_{k \neq j} |z_j - z_k| \ > \ 0$$

(since the minimum of finitely many positive numbers is positive). For $\varepsilon := \frac{d'}{2}$ we have $B(w, \varepsilon) \cap (F \setminus \{w\}) = \varnothing$, so again w is not an accumulation point.

Since **no** $w \in \mathbb{C}$ can be an accumulation point of F, the set F has no accumulation points.

□

Definition 1.21. Let $(z_n)_{n \ge 1}$ be a sequence in $\mathbb{C}$. A point $w \in \mathbb{C}$ is an **accumulation point** (or **subsequential limit**) of (z_n) if there exists a strictly increasing map $k \mapsto n_k$ such that $\lim_{k \to \infty} z_{n_k} = w$.

(1) $z_n = i^n$.

Claim. The set of accumulation points is $\{1, i, -1, -i\}$.

Proof. Since i^n is 4-periodic, the image set is $S := \{1, i, -1, -i\}$, and each element of S occurs infinitely many times. Hence for each $s \in S$ there exists the constant subsequence $z_{n_k} \equiv s$, so s is an accumulation point. Conversely, any subsequence takes all its values in the finite set S, thus has a further constant subsequence by the pigeonhole principle; hence every accumulation point lies in S. Therefore the accumulation set equals S.

(2) $z_n = \dfrac{i^n}{n}$.

Claim. The only accumulation point is 0.

Proof. Since $|i^n| = 1$ for all n, we have

$$|z_n| = \frac{1}{n} \xrightarrow[n\to\infty]{} 0.$$

Thus $z_n \to 0$, and a convergent sequence has the singleton set $\{0\}$ as its accumulation set.

(3) $z_n = (-1)^n(1+i)\,\dfrac{n-1}{n}$.

Claim. The accumulation points are $\{\,1+i,\ -(1+i)\,\}$.

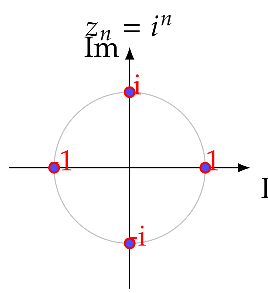
Proof. Decompose into even/odd subsequences. For $n = 2m$,

$$z_{2m} = (1+i)\,\frac{2m-1}{2m} \xrightarrow[m\to\infty]{} 1+i.$$

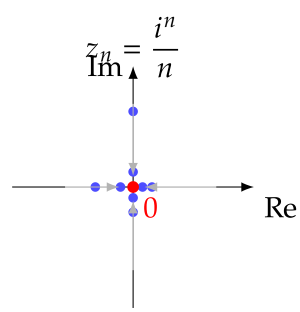
For $n = 2m+1$,

$$z_{2m+1} = -(1+i)\,\frac{2m}{2m+1} \xrightarrow[m\to\infty]{} -(1+i).$$

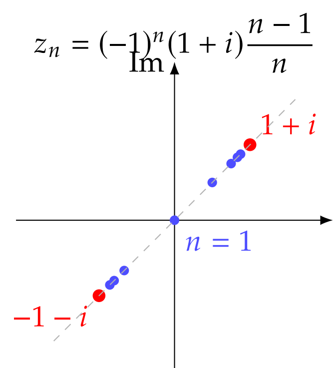
Hence $1+i$ and $-(1+i)$ are accumulation points. If w is an accumulation point, then there exists $n_k \to \infty$ with $z_{n_k} \to w$. Since $\frac{n_k-1}{n_k} \to 1$ and $(-1)^{n_k} \in \{\pm 1\}$, every limit w must belong to $\{\pm(1+i)\}$. Thus the accumulation set is exactly $\{\,1+i,\ -(1+i)\,\}$.



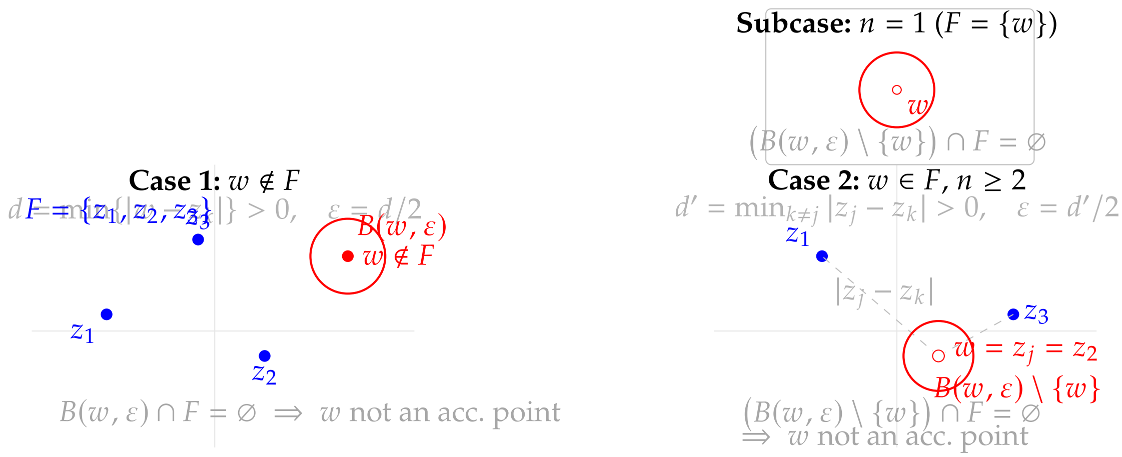
Accumulation set: $\{1, i, -1, -i\}$



Accumulation set: $\{0\}$



Accumulation set: $\{\,1+i,\ -1-i\,\}$


Subcase: $n = 1$ ($F = \{w\}$)
w
$(B(w,\varepsilon) \setminus \{w\}) \cap F = \varnothing$
Case 1: $w \notin F$
$F = \{z_1, z_2, z_3\}$
$d = \min\{|z_k - w|\} > 0, \quad \varepsilon = d/2$
$B(w,\varepsilon)$
$w \notin F$
z_1
z_2
z_3
$B(w,\varepsilon) \cap F = \varnothing \implies w$ not an acc. point
Case 2: $w \in F$, $n \geq 2$
$d' = \min_{k \neq j} |z_j - z_k| > 0, \quad \varepsilon = d'/2$
z_1
$|z_j - z_k|$
z_3
$w = z_j = z_2$
$B(w,\varepsilon) \setminus \{w\}$
$(B(w,\varepsilon) \setminus \{w\}) \cap F = \varnothing$
$\implies w$ not an acc. point

2 Analytic Functions

2.1 Functions of a Complex Variable

Definition 2.1 (Function and domain)**.** Let $S \subset \mathbb{C}$. A **function** f on S assigns to each $z \in S$ a complex number $w = f(z)$. The set S is the **domain** (domain of definition) of f. As real functions, we write $f(z) = u(x, y) + iv(x, y)$ for $z = x + iy$; in polar form, $f(z) = u(r, \theta) + iv(r, \theta)$.

Definition 2.2 (Polynomials and rational functions)**.** If $n \in \mathbb{Z}_{\geq 0}$ and $a_0, \ldots, a_n \in \mathbb{C}$ with $a_n \neq 0$, the polynomial

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$

has degree n. A **rational function** is $P(z)/Q(z)$, defined where $Q(z) \neq 0$.

Example 2.3 (Single-valued choice of a multiple-valued expression)**.** For $z \neq 0$ with $z = re^{i\theta}$ ($-\pi < \theta \leq \pi$), the square root has two values $z^{1/2} = \pm\sqrt{r}\, e^{i\theta/2}$. Selecting the "+" value defines a single-valued branch on $\mathbb{C}^{\times}$; setting $f(0) = 0$ extends it to $z = 0$ (not analytic there).

2.2 Mappings

Definition 2.4 (Mapping, image, range, inverse image)**.** Viewing f as a mapping $f : S \to \mathbb{C}$, the **image** of z is $w = f(z)$; the image of $T \subset S$ is $f(T)$; the **range** is $f(S)$. The **inverse image** of w_0 is $\{z \in S : f(z) = w_0\}$.

Observation (Basic geometric actions)**.** [leftmargin=1.5em]

- $w = z + 1$ translates one unit to the right.
- $w = iz = re^{i(\theta+\pi/2)}$ rotates by $\pi/2$ counterclockwise.
- $w = \bar{z} = x - iy$ reflects across the real axis.

Example 2.5 ($w = z^2$ as a mapping)**.** With $z = x + iy$, we have $w = u + iv$ where $u = x^2 - y^2$, $v = 2xy$. The first quadrant region $\{x \geq 0,\ y \geq 0,\ xy \leq 1\}$ maps onto the horizontal strip $\{0 \leq v \leq 2\}$.

Mapping by the exponential

If $w = e^z = e^{x+iy} = e^x(\cos y + i \sin y) = \rho e^{i\theta}$, then $\rho = e^x$ and $\theta = y$. Thus vertical lines $\{x = \text{const}\}$ map to circles $\{|w| = \text{const}\}$ and horizontal lines $\{y = \text{const}\}$ map to rays $\{\arg w = \text{const}\}$.

2.3 Limits and Related Theorems

Definition 2.6 (Limit)**.** Let f be defined on a deleted neighborhood of z_0. We say $\lim\limits_{z \to z_0} f(z) = w_0$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

Theorem 2.7 (Uniqueness of limits)**.** *If the limit* $\lim_{z\to z_0} f(z)$ *exists, it is unique.*

Example 2.8. For $f(z) = \frac{i}{2}z$ on $|z| < 1$, $\lim_{z\to 1} f(z) = \frac{i}{2}$. For $f(z) = \bar{z}/z$, $\lim_{z\to 0} f(z)$ does **not** exist: approaching along the real axis gives 1, along the imaginary axis gives -1.

Theorem 2.9 (Limit laws)**.** *If* $\lim_{z\to z_0} f(z) = f_0$ *and* $\lim_{z\to z_0} g(z) = g_0$*, then*

$$\lim_{z\to z_0} (f+g) = f_0 + g_0, \quad \lim_{z\to z_0} f\,g = f_0 g_0, \quad \lim_{z\to z_0} \frac{f}{g} = \frac{f_0}{g_0}\ (g_0 \neq 0).$$

In particular, polynomials are continuous: $\lim_{z\to z_0} P(z) = P(z_0)$.

2.3.1 Limits involving ∞

Neighborhoods of ∞ are exteriors of large disks. One has

$$\lim_{z\to z_0} f(z) = \infty \iff \lim_{z\to z_0} \frac{1}{f(z)} = 0, \qquad \lim_{z\to\infty} f(z) = w_0 \iff \lim_{z\to 0} f\left(\tfrac{1}{z}\right) = w_0,$$

and $\lim_{z\to\infty} f(z) = \infty \iff \lim_{z\to 0} \frac{1}{f(1/z)} = 0$.

2.4 Continuity

Definition 2.10 (Continuity)**.** f is continuous at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$. It is continuous on a region R if continuous at each $z_0 \in R$.

Theorem 2.11 (Basic properties)**.** *Composition of continuous functions is continuous. If f is continuous and $f(z_0) \neq 0$, then f is nonzero on some neighborhood of z_0. If $f = u + iv$, then f is continuous at z_0 iff u and v are continuous there. If R is closed and bounded and f continuous on R, then $|f|$ attains a maximum on R (boundedness).*

2.5 Derivatives

Definition 2.12 (Complex derivative)**.** If f is defined on a neighborhood of z_0, the derivative at z_0 is

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z\to 0} \frac{\Delta w}{\Delta z}, \quad \Delta w = f(z_0 + \Delta z) - f(z_0),$$

when the limit exists.

Theorem 2.13 (Consequences)**.** *If $f'(z_0)$ exists then f is continuous at z_0. Moreover,*

$$\frac{d}{dz}c = 0, \qquad \frac{d}{dz}z = 1, \qquad \frac{d}{dz}[cf] = cf', \qquad \frac{d}{dz}z^n = nz^{n-1}\ (n \in \mathbb{Z},\ z \neq 0 \text{ if } n < 0),$$

and the sum/product/quotient rules and chain rule hold exactly as in calculus.

Example 2.14. $f(z) = z^2 \Rightarrow f'(z) = 2z$. The function $f(z) = \bar{z}$ has no complex derivative anywhere. The function $f(z) = |z|^2$ has derivative only at $z = 0$ (value 0).

2.6 Cauchy–Riemann Equations

Let $f = u + iv$ with u, v real-valued.

Theorem 2.15 (Cauchy–Riemann (CR) equations)**.** *If $f'(z_0)$ exists then the first partials of u, v exist at (x_0, y_0) and satisfy*

$$u_x(x_0, y_0) = v_y(x_0, y_0), \qquad u_y(x_0, y_0) = -v_x(x_0, y_0),$$

and $f'(z_0) = u_x(x_0, y_0) + i\,v_x(x_0, y_0)$.

Theorem 2.16 (Sufficient conditions)**.** *If u_x, u_y, v_x, v_y exist in a neighborhood of z_0, are continuous at z_0, and satisfy the CR equations at z_0, then $f'(z_0)$ exists and equals $u_x + iv_x$.*

Example 2.17. $f(z) = z^2 = x^2 - y^2 + i\,2xy$ satisfies CR everywhere and $f'(z) = 2z$. For $f(z) = |z|^2 = x^2 + y^2$, the CR equations force $(x, y) = (0, 0)$; hence f' exists only at 0. For $f(z) = e^z = e^x(\cos y + i \sin y)$ we have $f'(z) = e^z$ for all z.

CR equations in polar coordinates

If $f = u(r, \theta) + iv(r, \theta)$ near $z_0 = r_0 e^{i\theta_0} \neq 0$, the polar CR system is

$$u_r = \frac{1}{r} v_\theta, \qquad v_r = -\frac{1}{r} u_\theta,$$

and $f'(z_0) = e^{-i\theta_0}\big(u_r(r_0, \theta_0) + i\,v_r(r_0, \theta_0)\big)$.

2.7 Analytic Functions

Definition 2.18 (Analytic/entire/singularity)**.** f is **analytic** at z_0 if it has a derivative at every point of some neighborhood of z_0. If analytic at every point of $\mathbb{C}$, f is **entire**. If f fails to be analytic at z_0 but is analytic arbitrarily close to z_0, then z_0 is a **singular point** (singularity) of f.

Theorem 2.19 (Algebra and composition)**.** *Sums and products of analytic functions are analytic; a quotient f/g is analytic where $g \neq 0$. If f is analytic in D and g is analytic on $f(D)$, then $g \circ f$ is analytic in D with $(g \circ f)' = (g' \circ f)\, f'$.*

Theorem 2.20 (Zero derivative)**.** *If $f'(z) = 0$ for all z in a domain D, then f is constant on D.*

Example 2.21.

$$f(z) = \frac{z^3 + 4}{(z^2 - 3)(z^2 + 1)}$$

is analytic on $\mathbb{C} \setminus \{\pm\sqrt{3}, \pm i\}$. Also $f(z) = \cosh x \cos y + i \sinh x \sin y$ is entire since CR holds everywhere.

Theorem 2.22 (Conjugate tests)**.** *If f and $\bar{f}$ are both analytic in D, then f is constant in D. If f is analytic in D and $|f|$ is constant, then f is constant.*

2.8 Harmonic Functions

Definition 2.23 (Harmonicity). A real function $h(x, y)$ is **harmonic** on a domain if it has continuous second partials and satisfies Laplace's equation

$$\Delta h = h_{xx} + h_{yy} = 0.$$

Theorem 2.24 (Harmonic components). *If $f = u + iv$ is analytic in D, then u and v are harmonic in D. Conversely, if u and v are harmonic and satisfy the CR equations in D, then $f = u + iv$ is analytic in D; v is then a **harmonic conjugate** of u.*

Example 2.25. $f(z) = \dfrac{i}{z^2}$ is analytic on $\mathbb{C} \setminus \{0\}$; writing it as

$$\frac{i}{z^2} = \frac{2xy + i(x^2 - y^2)}{(x^2 + y^2)^2} = u + iv,$$

both u and v are harmonic away from the origin. For $u(x, y) = y^3 - 3x^2 y$, a harmonic conjugate is $v(x, y) = -3xy^2 + x^3 + C$.

Uniqueness and reflection

Lemma 2.26 (Identity lemma). *If f is analytic in D and vanishes on a set with a limit point in D (e.g. a subdomain or line segment), then $f \equiv 0$ in D.*

Theorem 2.27 (Uniqueness from values). *An analytic function in D is uniquely determined in D by its values on any subdomain or line segment contained in D.*

Theorem 2.28 (Reflection principle (real axis)). *Let D contain a symmetric neighborhood of a real segment. Then $f(\bar{z}) = \overline{f(z)}$ in D iff $f(x) \in \mathbb{R}$ for all x on that segment.*

Exercises

1. Show that the following limit does not exitst

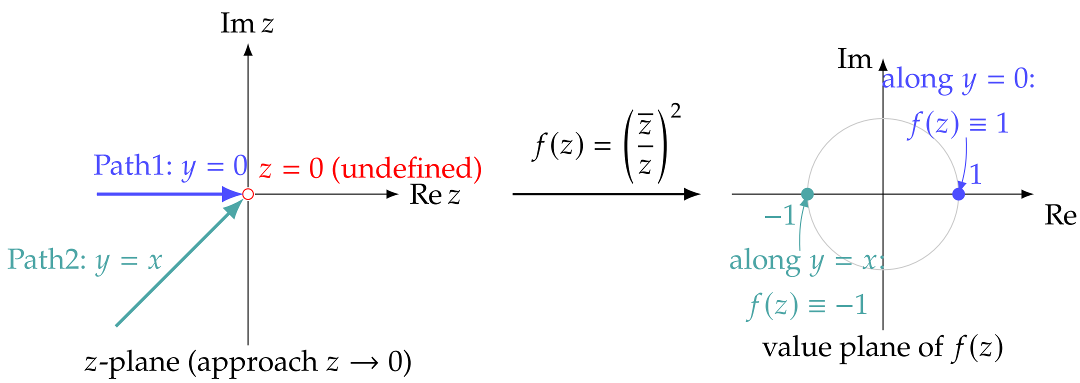
$$\lim_{z\to 0}\left(\frac{\bar{z}}{z}\right)^2$$

Do this by letting nonzero points $z = (x, 0)$ and $z = (x, x)$ approach the origin. (Note that it is not sufficient to simply consider points $z = (x, 0)$ and $z = (0, y)$.)

Sol. Let $z = x + iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$. Then

$$\left(\frac{\bar{z}}{z}\right)^2 = \left(\frac{x-iy}{x+iy}\right)^2.$$

If $z = re^{i\theta}$ with $r > 0$, then $\bar{z}/z = e^{-2i\theta}$, so $|(\bar{z}/z)^2| = |e^{-4i\theta}| = 1$.



(1) Path 1: approach along the real axis $y = 0$

Let $z = x + 0i = x$ with $x \in \mathbb{R} \setminus \{0\}$ and $x \to 0$. Then $\left(\frac{\bar{z}}{z}\right)^2 = \left(\frac{x}{x}\right)^2 = 1.$

(2) Path 2: approach along the diagonal $y = x$

Let $z = x + ix = (1 + i)x$ with $x \in \mathbb{R} \setminus \{0\}$ and $x \to 0$. Then

$$\frac{\bar{z}}{z} = \frac{\overline{(1+i)x}}{(1+i)x} = \frac{(1-i)\,x}{(1+i)\,x} = \frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{1-2i+i^2}{1-i^2} = \frac{1-2i-1}{2} = \frac{-2i}{2} = -i.$$

Hence

$$\left(\frac{\bar{z}}{z}\right)^2 = (-i)^2 = -1.$$

(3) Conclusion

Since the limits along these two paths are different (namely 1 and -1), the limit cannot exist. □

2. Let

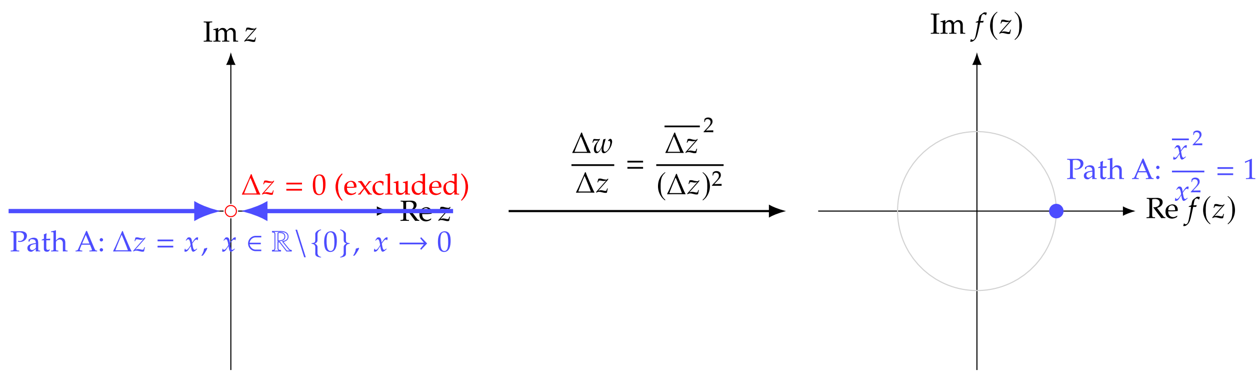
$$f(z) = \begin{cases} \bar{z}^2/z, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Show that if $z = 0$, then $\Delta w/\Delta z = 1$ at eatch nonzero point on the real and imaginary axes in the Δz, or $\Delta x \Delta y$, plane. Then show that $\Delta w/\Delta z = -1$ at each nonzero point $(\Delta x, \Delta y)$ on the line $\Delta y = \Delta x$ in that plane. Conclude from these observations that $f'(0)$ does not exist. Note that to obtain this result, it is not sufficient to consider only horizontal and vertical approaches to the origin in the Δz plane.

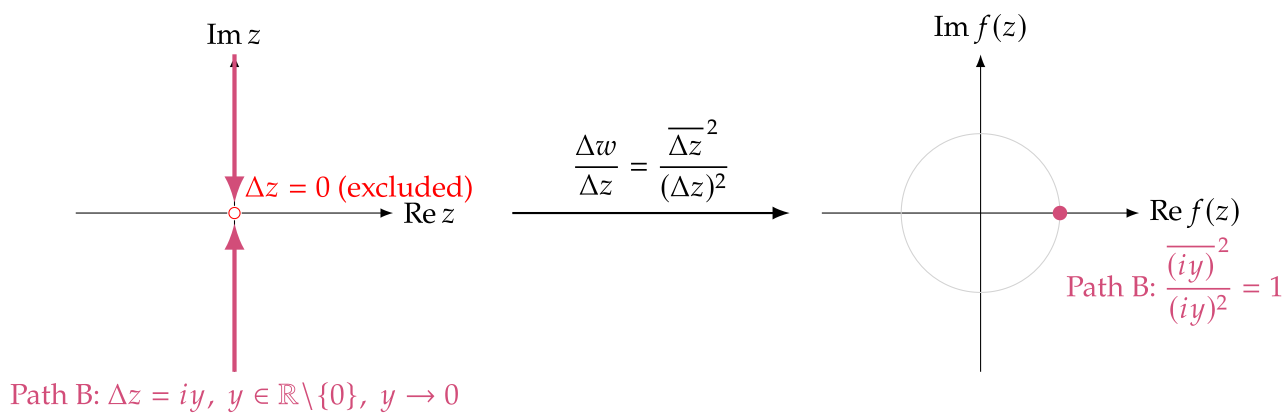
Proof. Let $\frac{\Delta w}{\Delta z} = \frac{f(\Delta z) - f(0)}{\Delta z}$ $(\Delta z \neq 0)$. Since $f(0) = 0$, for $\Delta z \neq 0$, $\dfrac{\Delta w}{\Delta z} = \dfrac{f(\Delta z)}{\Delta z} = \dfrac{\overline{\Delta z}^2}{(\Delta z)^2}$.

(1) Real and imaginary axes.

- Real axis: $\Delta z = x$ with $x \in \mathbb{R} \setminus \{0\}$, $\dfrac{\Delta w}{\Delta z} = \dfrac{\overline{x}^2}{x^2} = \dfrac{x^2}{x^2} = 1$.

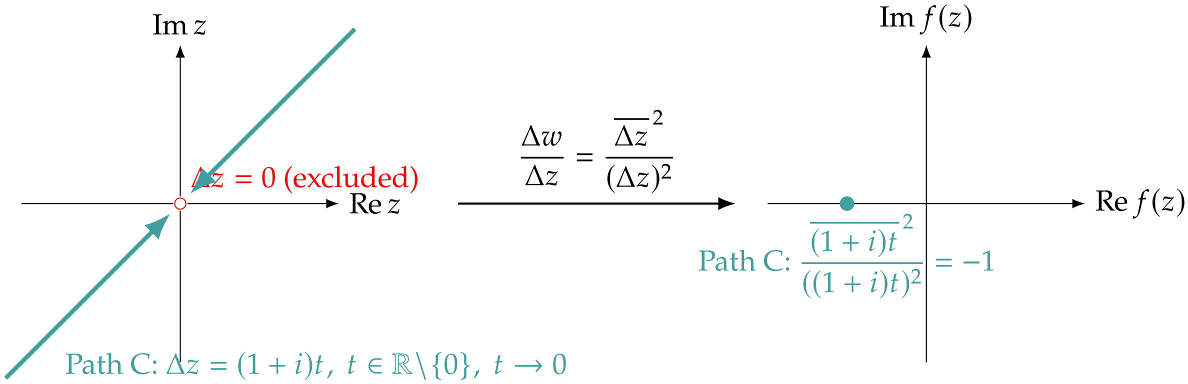


- Imaginary axis: $\Delta z = iy$ with $y \in \mathbb{R} \setminus \{0\}$, $\dfrac{\Delta w}{\Delta z} = \dfrac{\overline{iy}^2}{(iy)^2} = \dfrac{(-iy)^2}{(iy)^2} = \dfrac{-y^2}{-y^2} = 1$.



(2) Line $\Delta y = \Delta x$. Let $\Delta z = (1+i)x$ with $x \in \mathbb{R} \setminus \{0\}$. Then

$$\frac{\Delta w}{\Delta z} = \frac{\overline{(1+i)x}^2}{((1+i)x)^2} = \frac{((1-i)x)^2}{((1+i)x)^2} = \frac{(1-i)^2}{(1+i)^2} = \frac{-2i}{2i} = -1.$$



(3) Conclusion. Since the difference quotient equals 1 along the axes but -1 along the line $\Delta y = \Delta x$, the limit

$$\lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z}$$

depends on the path and therefore does not exist. Consequently, $f'(0)$ does not exist. □

3. Let
$$f(z) = \bar{z}, \quad f(z) = 2x + ixy^2, \quad f(z) = e^{\bar{z}}$$
Then show that $f'(z)$ does not exists at any point.

Sol. Let $z = x + iy$ and $f(z) = u + iv$ with $x, y, u, v \in \mathbb{R}$. The Cauchy–Riemann equations

$$u_x = v_y, \qquad u_y = -v_x$$

are necessary and sufficient for complex differentiability.

(1) $f_1(z) = \overline{z} = \overline{x + iy} = x - iy$**.**

Here $u(x, y) = x,\ v(x, y) = -y$. Thus

$$u_x = 1, \quad u_y = 0, \quad v_x = 0, \quad v_y = -1.$$

The CR require $u_x = v_y$, i.e. $1 = -1$, which is impossible. Hence f_1' does not exist anywhere.

(2) $f_2(z) = 2x + i\,xy^2$**.**

Here $u(x, y) = 2x,\ v(x, y) = xy^2$. Thus

$$u_x = 2, \quad u_y = 0, \quad v_x = y^2, \quad v_y = 2xy.$$

The CR demand
$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \implies \begin{cases} 2 = 2xy \\ 0 = y^2 \end{cases} \implies \begin{cases} xy = 1 \\ y = 0 \end{cases}.$$
These cannot hold simultaneously for any x. Hence CR fail at every point, so f_2' exists nowhere.

(3) $f_3(z) = e^{\overline{z}}$**.**

Let $\overline{z} = x - iy$. Then $f_3(x, y) = e^{x - iy} = e^x(\cos y - i \sin y)$, so

$$u(x, y) = e^x \cos y, \qquad v(x, y) = -\,e^x \sin y.$$

Compute
$$u_x = e^x \cos y, \quad u_y = -e^x \sin y, \quad v_x = -e^x \sin y, \quad v_y = -e^x \cos y.$$

The CR give

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \implies \begin{cases} e^x \cos y = -e^x \cos y \\ -e^x \sin y = +e^x \sin y \end{cases} \implies \begin{cases} \cos y = 0 \\ \sin y = 0 \end{cases}.$$

These cannot hold simultaneously for any y. Hence CR fail everywhere and f_3' exists nowhere. $\square$

4. Let $f(z) = u(x, y) + iv(x, y)$ be given by

$$f(z) = \begin{cases} \bar{z}^2/z & : z \neq 0 \\ 0 & : z = 0. \end{cases}$$

Verify that the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at the origin $z = (0, 0)$.

Proof. Write $z = x + iy$ and define

$$f(z) = \begin{cases} \dfrac{\overline{z}^2}{z}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Let $f = u + iv$. We compute the first-order partials of u, v at $(0, 0)$ by restricting to the coordinate axes.

Along the x-axis ($y = 0$): For $x \neq 0$,

$$f(x, 0) = \frac{\overline{x}^2}{x} = x,$$

hence $u(x, 0) = x$ and $v(x, 0) = 0$. Therefore

$$u_x(0, 0) = \lim_{h \to 0} \frac{u(h, 0) - u(0, 0)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1, \quad v_x(0, 0) = \lim_{h \to 0} \frac{v(h, 0) - v(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0.$$

Along the y-axis ($x = 0$): For $y \neq 0$,

$$f(0, y) = \frac{\overline{iy}^2}{iy} = \frac{(-iy)^2}{iy} = \frac{-y^2}{iy} = iy,$$

so $u(0, y) = 0$ and $v(0, y) = y$. Hence

$$u_y(0, 0) = \lim_{k \to 0} \frac{u(0, k) - u(0, 0)}{k} = \lim_{k \to 0} \frac{0 - 0}{k} = 0, \quad v_y(0, 0) = \lim_{k \to 0} \frac{v(0, k) - v(0, 0)}{k} = \lim_{k \to 0} \frac{k - 0}{k} = 1.$$

Thus at $(0, 0)$ we have

$$u_x(0, 0) = 1, \qquad v_y(0, 0) = 1, \qquad u_y(0, 0) = 0, \qquad v_x(0, 0) = 0,$$

and consequently the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ hold at the origin.

Remark. Although the Cauchy–Riemann equations hold at $(0,0)$, the complex derivative $f'(0)$ does not exist (since $\frac{f(\Delta z)-f(0)}{\Delta z} = \frac{\overline{\Delta z}^2}{(\Delta z)^2}$ takes different values along different approach directions).

□

5. Let

$$f(z) = \sin x \cosh y + i \cos x \sinh y \quad \text{and} \quad f(z) = e^{-y}(\sin x - i \cos x).$$

Then show that all f are entire.

Sol. Let

$$\begin{aligned} f_1(z) &= \sin x \cosh y + i \cos x \sinh y \quad \text{and} \\ f_2(z) &= e^{-y}(\sin x - i \cos x). \end{aligned}$$

(a) Let $z = x + iy$ with $x, y \in \mathbb{R}$. Note that

$$\begin{matrix} e^{iz} = \cos z + i \sin z & & \cos z = \dfrac{e^{iz} + e^{-iz}}{2} \\ & \longleftrightarrow & \\ e^{-iz} = \cos z - i \sin z & & \sin z = \dfrac{e^{iz} - e^{-iz}}{2i} \end{matrix}$$

By definition,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Substitute $z = x + iy$:

$$iz = i(x + iy) = ix - y, \qquad -iz = -ix + y,$$

so

$$e^{iz} = e^{ix-y} = e^{-y}e^{ix}, \qquad e^{-iz} = e^{-ix+y} = e^{y}e^{-ix}.$$

Using Euler's formula $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$, we get

$$e^{iz} = e^{-y}(\cos x + i \sin x), \qquad e^{-iz} = e^{y}(\cos x - i \sin x).$$

Then

$$\begin{aligned} e^{iz} - e^{-iz} &= e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x) \\ &= \cos x(e^{-y} - e^{y}) + i \sin x(e^{-y} + e^{y}). \end{aligned}$$

Recall the hyperbolic functions

$$\cosh y = \frac{e^y + e^{-y}}{2}, \qquad \sinh y = \frac{e^y - e^{-y}}{2},$$

so that

$$e^{-y} + e^y = 2\cosh y, \qquad e^{-y} - e^y = -2\sinh y.$$

Hence

$$\begin{aligned} e^{iz} - e^{-iz} &= \cos x(-2\sinh y) + i\sin x(2\cosh y) \\ &= 2\big(i\sin x\cosh y - \cos x\sinh y\big). \end{aligned}$$

Therefore

$$\begin{aligned} \sin z = \frac{e^{iz} - e^{-iz}}{2i} &= \frac{2\big(i\sin x\cosh y - \cos x\sinh y\big)}{2i} \\ &= \frac{i\sin x\cosh y}{i} - \frac{\cos x\sinh y}{i} \\ &= \sin x\cosh y + i\cos x\sinh y, \end{aligned}$$

since $\frac{1}{i} = -i$.

Thus

$$\boxed{\sin(x+iy) = \sin x\cosh y + i\cos x\sinh y.}$$

Using the standard identity for the complex sine,

$$\sin z = \sin(x+iy) = \sin x\cosh y + i\cos x\sinh y,$$

we see immediately that $f_1(z) = \sin z$. Since $\sin z$ is an entire function (power series with infinite radius of convergence), f_1 is entire.

(b) Note that

$$e^{iz} = e^{i(x+iy)} = e^{ix-y} = e^{-y}\big(\cos x + i\sin x\big).$$

Multiplying by $-i$ gives

$$-i\,e^{iz} = e^{-y}\big(\sin x - i\cos x\big) = f_2(z).$$

Thus $f_2(z) = -i\,e^{iz}$. Since the exponential is entire and multiplication by a constant preserves holomorphy, f_2 is entire.

Therefore both functions are entire. □

6. Show that the function

$$f(z) = \ln r + i\theta \quad (r > 0,\ 0 < \theta < 2\pi)$$

is analytic in the indicated domain of definition, with derivative $f'(z) = 1/z$. Then show that th e composite function $g(z) = f(z^2 + 1)$ is analytic in the quadratic $x > 0, y > 0$ with derivative

$$g'(z) = \frac{2z}{z^2 + 1}.$$

(Suggestion: Observe that $\text{Im}(z^2 + 1) > 0$ when $x > 0, y > 0$)

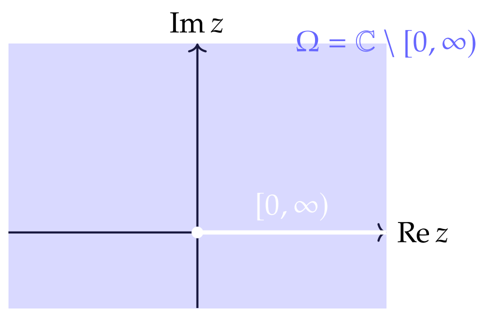
Sol. Let $z = x + iy = re^{i\theta}$ with $r = \sqrt{x^2 + y^2} > 0$ and $0 < \theta < 2\pi$. Define

$$f(z) = \ln r + i\theta.$$

Then f is analytic on the slit plane

$$\Omega := \{\, z \in \mathbb{C} : r > 0,\ 0 < \theta < 2\pi \,\} = \mathbb{C} \setminus [0, \infty),$$

and $f'(z) = \frac{1}{z}$ $(z \in \Omega)$.



Write $f = u + iv$ with

$$u(x, y) = \ln r = \frac{1}{2}\ln\left(x^2 + y^2\right), \qquad v(x, y) = \theta = \text{Arg}(z) \in (0, 2\pi).$$

On Ω the functions u, v are C^1 and their partials are:

$$u_x = \frac{x}{x^2 + y^2}, \quad u_y = \frac{y}{x^2 + y^2}, \qquad v_x = -\frac{y}{x^2 + y^2}, \quad v_y = \frac{x}{x^2 + y^2}.$$

Hence the Cauchy–Riemann equations hold on Ω:

$$u_x = v_y = \frac{x}{x^2 + y^2}, \qquad u_y = -v_x = \frac{y}{x^2 + y^2}.$$

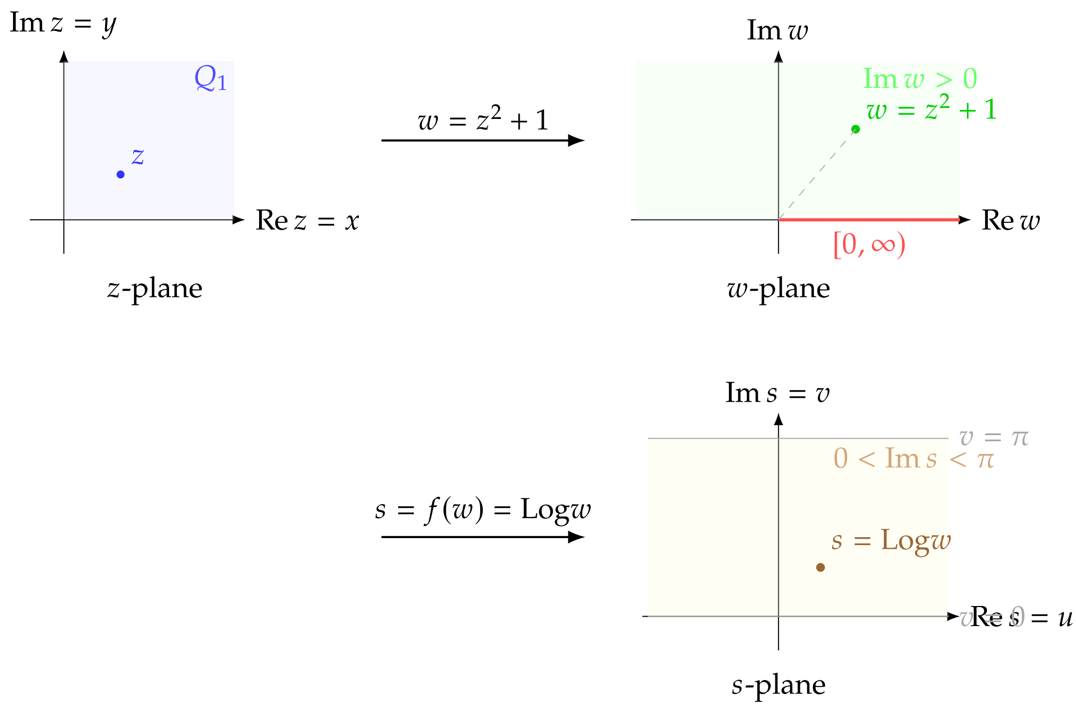
Since these partials are continuous on Ω, f is analytic there. Its complex derivative is

$$f'(z) = u_x + iv_x = \frac{x}{x^2 + y^2} + i\left(-\frac{y}{x^2 + y^2}\right) = \frac{x - iy}{x^2 + y^2} = \frac{1}{x + iy} = \frac{1}{z}.$$

For $g(z) = f(z^2+1)$, compute

$$\begin{aligned} z^2 + 1 &= (x+iy)^2 + 1 \\ &= (x^2 - y^2 + 2ixy) + 1 \\ &= (x^2 - y^2 + 1) + i(2xy). \end{aligned}$$

If $x > 0$ and $y > 0$, then $\operatorname{Im}(z^2+1) = 2xy > 0$, so z^2+1 lies in the open upper half-plane $\mathbb{H}$, in particular in Ω (its argument lies in $(0,\pi) \subset (0, 2\pi)$).



Thus g is the composition of analytic functions on the first quadrant Q_1, hence analytic on Q_1. By the chain rule,

$$g'(z) = f'(z^2+1) \cdot (2z) = \frac{2z}{z^2+1} \qquad (z \in Q_1).$$

$\square$

3 Elementary Functions

3.1 The Exponential Function

Exponential Function

Definition 3.1. For $z = x + iy \in \mathbb{C}$, define

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y),$$

where y is in radians. We also write $\exp z$ for e^z.

Theorem 3.2. *For $z_1, z_2 \in \mathbb{C}$,*

$$e^{z_1+z_2} = e^{z_1}e^{z_2}, \qquad \frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}.$$

Moreover, e^z is entire, satisfies

$$\frac{d}{dz}e^z = e^z$$

for all z, and $e^z \neq 0$ for all $z \in \mathbb{C}$.

Observation. Writing $e^z = \rho e^{i\theta}$ gives $\rho = e^x$ and $\theta = y$, hence

$$|e^z| = e^x, \qquad \arg(e^z) = y + 2\pi n \ (n \in \mathbb{Z}).$$

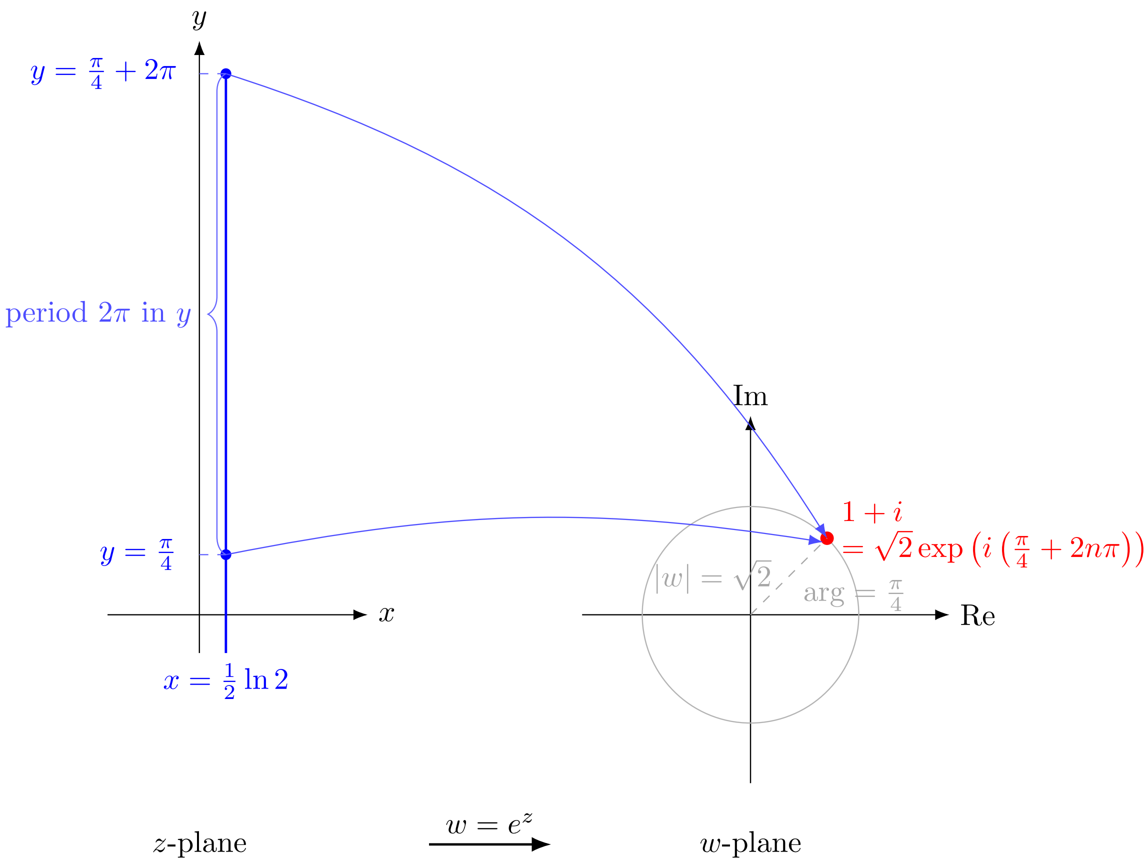
Thus $e^{z+2\pi i} = e^z$, so e^z is periodic with pure imaginary period $2\pi i$.

Euler's Identity

Corollary 3.3. *Euler's identity is given by*

$$e^{i\pi} = -1 \quad \textit{equivalently,} \quad e^{i\pi} + 1 = 0.$$

Example 3.4. Solve $e^z = 1 + i$ for $z = x + iy$.



Sol. Since $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$ and $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, we have

$$e^x = \sqrt{2} \quad \text{and} \quad y = \frac{\pi}{4} + 2n\pi \quad (n = 0, \pm 1, \pm 2, \cdots).$$

Thus,

$$x = \frac{1}{2}\ln 2 \quad \text{and} \quad y = \left(2n + \frac{1}{4}\right)\pi \quad (n = 0, \pm 1, \pm 2, \cdots)$$

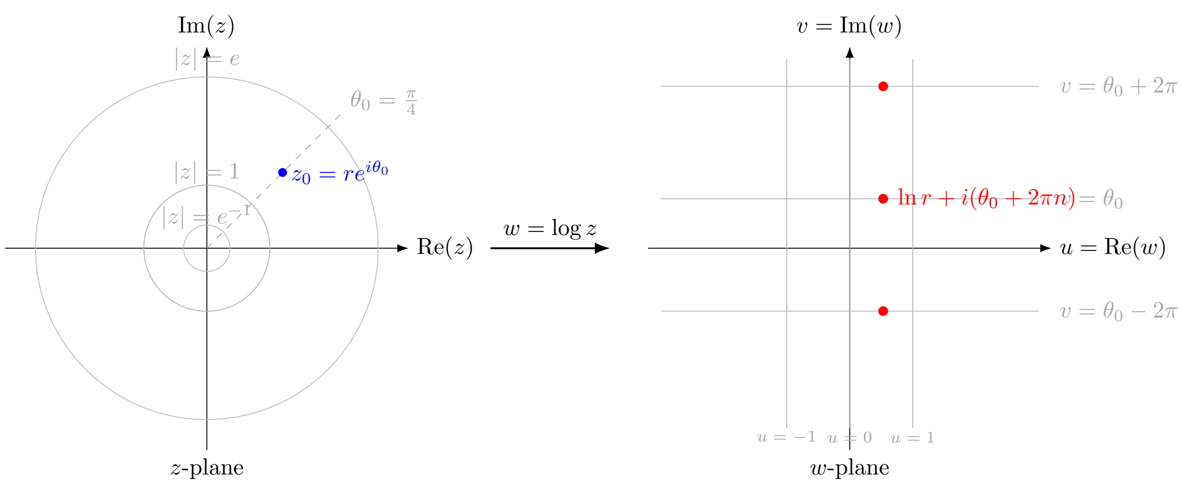
and so

$$z = \frac{1}{2}\ln 2 + i\left(2n + \frac{1}{4}\right)\pi \quad (n = 0, \pm 1, \pm 2, \cdots).$$

□

3.2 The Logarithmic Function

Observation. To solve
$$z = e^w$$
for w when $z \neq 0$, write $z = re^{i\theta}$, $w = u + iv$. Then $e^u = r$ and $v = \theta + 2\pi n$, hence
$$\log z = \ln r + i(\theta + 2\pi n), \qquad n \in \mathbb{Z},$$
a multiple-valued function with
$$e^{\log z} = z$$
for $z \neq 0$.



Example 3.5. If $z = -1 - \sqrt{3}\,i$, then $r = 2$ and $\theta = -\frac{2\pi}{3}$, so
$$\log z = \ln 2 + i\left(-\frac{2\pi}{3} + 2\pi n\right), \quad n \in \mathbb{Z}.$$

Argument and Principal Value

Definition 3.6. For $z \neq 0$, the set of all arguments is $\arg z = \{\theta + 2\pi n : n \in \mathbb{Z}\}$ when $z = re^{i\theta}$. The principal value $\mathrm{Arg}z$ is the unique θ with $-\pi < \theta \leq \pi$.

Observation. In general,

$$\log\left(e^{z}\right) = z + 2\pi i\, n, \qquad n \in \mathbb{Z}.$$

Definition 3.7 (Principal Value of the Logarithm)**.** The principal value is

$$\mathrm{Log}z = \ln r + i\theta \quad (z = re^{i\theta},\ r > 0,\ -\pi < \theta < \pi).$$

Then $\log z = \mathrm{Log}z + 2\pi i\, n$ for $n \in \mathbb{Z}$.

Example 3.8. $\log 1 = 2\pi i\, n$ with $\mathrm{Log}1 = 0$; and $\log(-1) = (2n+1)\pi i$ with $\mathrm{Log}(-1) = \pi i$. The function $\mathrm{Log}z$ is not continuous along the negative real axis.

3.3 Branches and Derivatives of Logarithms

Observation. Let $\alpha \in \mathbb{R}$. Restrict θ in

$$\log z = \ln r + i\theta \qquad (r > 0,\ \alpha < \theta < \alpha + 2\pi)$$

to obtain a single-valued continuous branch on that domain; it is in fact analytic there.

Theorem 3.9. *For a branch as above,*

$$\frac{d}{dz}\log z = \frac{1}{z} \qquad (|z| > 0,\ \alpha < \arg z < \alpha + 2\pi).$$

In particular, on the principal branch,

$$\frac{d}{dz}\mathrm{Log}z = \frac{1}{z} \qquad (|z| > 0,\ -\pi < \mathrm{Arg}z < \pi).$$

Definition 3.10 (Branch, Principal Branch, Branch Cut/Point)**.** A **branch** of a multiple-valued f is any single-valued analytic function F whose values are among those of f. The **principal branch** of log is $\mathrm{Log}z$ on $r > 0$, $-\pi < \theta < \pi$. A **branch cut** is a curve removed to render a single-valued branch; points on it are singular for that branch. The origin is a branch point for log.

Example 3.11. $\mathrm{Log}(i^3) = \mathrm{Log}(-i) = \ln 1 - i\frac{\pi}{2} = -\frac{\pi i}{2}$, while $3\mathrm{Log}i = 3 \cdot i\frac{\pi}{2} = \frac{3\pi i}{2}$. Hence $\mathrm{Log}(i^3) \neq 3\mathrm{Log}i$.

Theorem 3.12. *For nonzero z_1, z_2,*

$$\log(z_1 z_2) = \log z_1 + \log z_2, \qquad \arg(z_1 z_2) = \arg z_1 + \arg z_2,$$

and thus $\ln|z_1z_2| + i\arg(z_1z_2) = (\ln|z_1| + i\arg z_1) + (\ln|z_2| + i\arg z_2)$.

Example 3.13. Let $z_1 = z_2 = -1$. Then $\log 1 = 0$, while $\log(-1) = (2n+1)\pi i$. Equality can require compatible choices of values. Using principal values everywhere may fail: $\mathrm{Log}(z_1z_2) = 0$ but $\mathrm{Log} z_1 + \mathrm{Log} z_2 = 2\pi i$.

Theorem 3.14. *For nonzero* z_1, z_2,

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2.$$

Observation (Roots via Logarithm)**.** For $z \neq 0$ and $n \in \mathbb{N}$,

$$z^{1/n} = \exp\left(\tfrac{1}{n}\log z\right),$$

which gives exactly the n distinct nth roots when $k = 0, 1, \ldots, n-1$ are taken in the angles.

3.4 Complex Exponents

Definition 3.15 (Complex Power)**.** For $z \neq 0$ and $c \in \mathbb{C}$,

$$z^c = e^{c\log z},$$

a multiple-valued function in general.

Example 3.16.

$$i^{-2i} = e^{-2i\log i}, \qquad \log i = \ln 1 + i\left(\frac{\pi}{2} + 2\pi n\right) = \left(2n + \tfrac{1}{2}\right)\pi i.$$

Hence $i^{-2i} = \exp\big((4n+1)\pi\big)$, which are real numbers.

Observation. Since $1/e^z = e^{-z}$, we have $z^{-c} = \exp(-c\log z)$ and in particular $1/i^{2i} = i^{-2i} = \exp\big((4n+1)\pi\big)$.

Observation. Fix a branch $\log z = \ln r + i\theta$ on $\alpha < \theta < \alpha + 2\pi$. Then $z^c = \exp(c\log z)$ is single-valued and analytic there, and

$$\frac{d}{dz}z^c = c\,z^{c-1} \qquad (|z| > 0,\ \alpha < \arg z < \alpha + 2\pi).$$

The principal value is P.V. $z^c = \exp(c\,\mathrm{Log} z)$.

Example 3.17.

$$\text{P.V. } (-i)^i = \exp\big(i\,\mathrm{Log}(-i)\big) = \exp\left(i\left[\ln 1 - i\frac{\pi}{2}\right]\right) = e^{\pi/2}.$$

For $z^{2/3}$ on the principal branch ($-\pi < \mathrm{Arg} z < \pi$),

$$\text{P.V. } z^{2/3} = r^{2/3}\left(\cos\frac{2\varphi}{3} + i\sin\frac{2\varphi}{3}\right) \quad (z = re^{i\varphi}).$$

Example 3.18. Let $z_1 = 1+i$, $z_2 = 1-i$, $z_3 = -1-i$. Then

$$(z_1 z_2)^i = e^{i \ln 2}, \qquad z_1^i = e^{-\pi/4}\, e^{i(\ln 2)/2}, \qquad z_2^i = e^{\pi/4}\, e^{i(\ln 2)/2},$$

so $(z_1 z_2)^i = z_1^i z_2^i$. But

$$(z_2 z_3)^i = e^{-\pi}\, e^{i \ln 2}, \qquad z_3^i = e^{3\pi/4}\, e^{i(\ln 2)/2},$$

whence $(z_2 z_3)^i = z_2^i z_3^i e^{-2i}$, showing branch subtleties.

Definition 3.19 (Exponential with Base $c \neq 0$)**.** For fixed $c \in \mathbb{C} \setminus \{0\}$ and a chosen value of $\log c$, define

$$c^z = e^{z \log c}.$$

Then c^z is entire and $\dfrac{d}{dz} c^z = c^z \log c$.

3.5 Trigonometric Functions

Definition 3.20. For $z \in \mathbb{C}$,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Theorem 3.21. *The functions* $\sin z$ *and* $\cos z$ *are entire and satisfy*

$$\frac{d}{dz} \sin z = \cos z, \qquad \frac{d}{dz} \cos z = -\sin z,$$

and remain odd/even respectively: $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$. *Moreover* $e^{iz} = \cos z + i \sin z$.

Theorem 3.22 (Formulas)**.** *For* $z, z_1, z_2 \in \mathbb{C}$,

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2, \quad \cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2,$$
$$\sin 2z = 2 \sin z \cos z, \quad \cos 2z = \cos^2 z - \sin^2 z,$$
$$\sin\left(z + \tfrac{\pi}{2}\right) = \cos z, \quad \cos\left(z - \tfrac{\pi}{2}\right) = -\cos z,$$
$$\sin^2 z + \cos^2 z = 1, \quad \sin(z + \pi) = -\sin z,\ \cos(z + \pi) = -\cos z,$$
$$\sin(z + 2\pi) = \sin z, \quad \cos(z + 2\pi) = \cos z.$$

Observation. For real y,

$$\cos(iy) = \cosh y, \qquad \sin(iy) = i \sinh y.$$

Writing $z = x + iy$,

$$\sin z = \sin x\, \cosh y + i \cos x\, \sinh y, \quad \cos z = \cos x\, \cosh y - i \sin x\, \sinh y.$$

Remark. $\sin z$ and $\cos z$ are unbounded on $\mathbb{C}$.

Observation (Zeros)**.** $\sin z = 0$ iff $z = n\pi$; $\cos z = 0$ iff $z = \frac{\pi}{2} + n\pi$ for $n \in \mathbb{Z}$.

Definition 3.23. Define

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}.$$

Theorem 3.24.

$$\frac{d}{dz}\tan z = \sec^2 z, \quad \frac{d}{dz}\sec z = \sec z\ \tan z, \quad \frac{d}{dz}\cot z = -\csc^2 z, \quad \frac{d}{dz}\csc z = -\csc z\ \cot z.$$

Observation. $\tan z$ and $\sec z$ are analytic off $z = \frac{\pi}{2} + n\pi$; $\cot z$ and $\csc z$ are analytic off $z = n\pi$.

3.6 Hyperbolic Functions

Definition 3.25.

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2}.$$

Theorem 3.26. $\sinh z$ *and* $\cosh z$ *are entire and* $\frac{d}{dz}\sinh z = \cosh z$, $\frac{d}{dz}\cosh z = \sinh z$.

Theorem 3.27. *For* $z = x + iy$ *and* $z_1, z_2 \in \mathbb{C}$,

$$-i\ \sinh(iz) = \sin z, \quad \cosh(iz) = \cos z, \quad -i\ \sin(iz) = \sinh z, \quad \cos(iz) = \cosh z,$$
$$\sinh(-z) = -\sinh z, \quad \cosh(-z) = \cosh z, \quad \cosh^2 z - \sinh^2 z = 1,$$
$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2,$$
$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2,$$
$$\sinh z = \sinh x \cos y + i \cosh x \sin y,$$
$$\cosh z = \cosh x \cos y + i \sinh x \sin y,$$
$$|\sinh z|^2 = \sinh^2 x + \sin^2 y, \quad |\cosh z|^2 = \cosh^2 x + \cos^2 y.$$

Remark. $\sinh z$ and $\cosh z$ are periodic with period $2\pi i$.

Observation (Zeros)**.** $\sinh z = 0$ iff $z = n\pi i$; $\cosh z = 0$ iff $z = (\frac{\pi}{2} + n\pi)i$ $(n \in \mathbb{Z})$.

Definition 3.28. Define $\tanh z = \dfrac{\sinh z}{\cosh z}$ (analytic where $\cosh z \neq 0$). Set $\coth z = 1/\tanh z$, $\operatorname{sech} z = 1/\cosh z$, $\operatorname{csch} z = 1/\sinh z$.

Theorem 3.29.

$$\frac{d}{dz}\tanh z = \operatorname{sech}^2 z, \quad \frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z\ \tanh z, \quad \frac{d}{dz}\coth z = -\operatorname{csch}^2 z, \quad \frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z\ \coth z.$$

3.7 Inverse Trigonometric and Hyperbolic Functions

Observation. To define $\sin^{-1} z$, write

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}.$$

Then

$$\begin{aligned} e^{iw} - e^{-iw} &= 2iz \\ (e^{iw})^2 - 2iz - 1 &= 0 \end{aligned}$$

Solving the quadratic in e^{iw} yields

$$e^{iw} = iz + (1 - z^2)^{1/2},$$

where $(1 - z^2)^{1/2}$ is double-valued.

Definition 3.30. Multiple-valued inverses:

$$\sin^{-1} z = -i \log\big[\, iz + (1 - z^2)^{1/2} \big],$$

$$\cos^{-1} z = -i \log\big[\, z + i(1 - z^2)^{1/2} \big],$$

$$\tan^{-1} z = \frac{i}{2} \log\left(\frac{i + z}{i - z}\right).$$

With specific branches of $\sqrt{\cdot}$ and log, these become single-valued and analytic on suitable domains.

Theorem 3.31 (Derivatives)**.**

$$\frac{d}{dz} \sin^{-1} z = \frac{1}{(1 - z^2)^{1/2}}, \qquad \frac{d}{dz} \cos^{-1} z = -\frac{1}{(1 - z^2)^{1/2}}, \qquad \frac{d}{dz} \tan^{-1} z = \frac{1}{1 + z^2}.$$

Example 3.32.

$$\sin^{-1}(-i) = -i\ \log\left(1 \pm \sqrt{2}\right).$$

Since $\log\left(1+\sqrt{2}\right) = \ln\left(1+\sqrt{2}\right) + 2\pi i\, n$ and $\log\left(1-\sqrt{2}\right) = \ln\left(\sqrt{2}-1\right) + (2n+1)\pi i$ with $\ln\left(\sqrt{2}-1\right) = -\ln\left(1+\sqrt{2}\right)$, the values of $\sin^{-1}(-i)$ are

$$n\pi + i(-1)^{n+1}\ln\left(1+\sqrt{2}\right), \qquad n \in \mathbb{Z}.$$

Observation (Inverse Hyperbolic Functions)**.**

$$\sinh^{-1} z = \log\left[z + (z^2+1)^{1/2}\right], \quad \cosh^{-1} z = \log\left[z + (z^2-1)^{1/2}\right], \quad \tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

3.8 Exercises

1. Show that $f(z) = \exp(\overline{z})$ is not analytic anywhere.

(Hint: use the Cauchy–Riemann equations.)

Sol. (**Proof via Cauchy–Riemann equations**) Write $z = x + iy$. Then

$$f(z) = e^{\overline{z}} = e^{x-iy} = e^x(\cos y - i \sin y),$$

so

$$u(x, y) = e^x \cos y, \qquad v(x, y) = -e^x \sin y.$$

Then

$$u_x = e^x \cos y, \quad u_y = -e^x \sin y, \qquad v_x = -e^x \sin y, \quad v_y = -e^x \cos y.$$

If f is complex differentiable at (x, y), the Cauchy–Riemann equations would hold:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

That is,

$$\begin{aligned} u_x = v_y &\implies e^x \cos y = -e^x \cos y &&\implies \cos y = 0, \\ u_y = -v_x &\implies -e^x \sin y = e^x \sin y &&\implies \sin y = 0. \end{aligned}$$

There is no $y \in \mathbb{R}$ with $\cos y = 0$ and $\sin y = 0$ simultaneously. Hence the Cauchy–Riemann equations fail at every point, so f is nowhere analytic.

(**Proof via Wirtinger derivatives**) Using $\partial/\partial z = \frac{1}{2}(\partial_x - i\,\partial_y)$ and $\partial/\partial\overline{z} = \frac{1}{2}(\partial_x + i\,\partial_y)$, one checks directly that

$$\frac{\partial f}{\partial z} = 0, \qquad \frac{\partial f}{\partial \overline{z}} = e^{\overline{z}} \neq 0 \ \text{ for all } z.$$

A function is holomorphic iff $\partial f/\partial\overline{z} \equiv 0$ on its domain. Since this is not the case, f is nowhere holomorphic. □

2. Let $f(z) = u(x,y) + iv(x,y)$ be analytic in a domain D. Show that

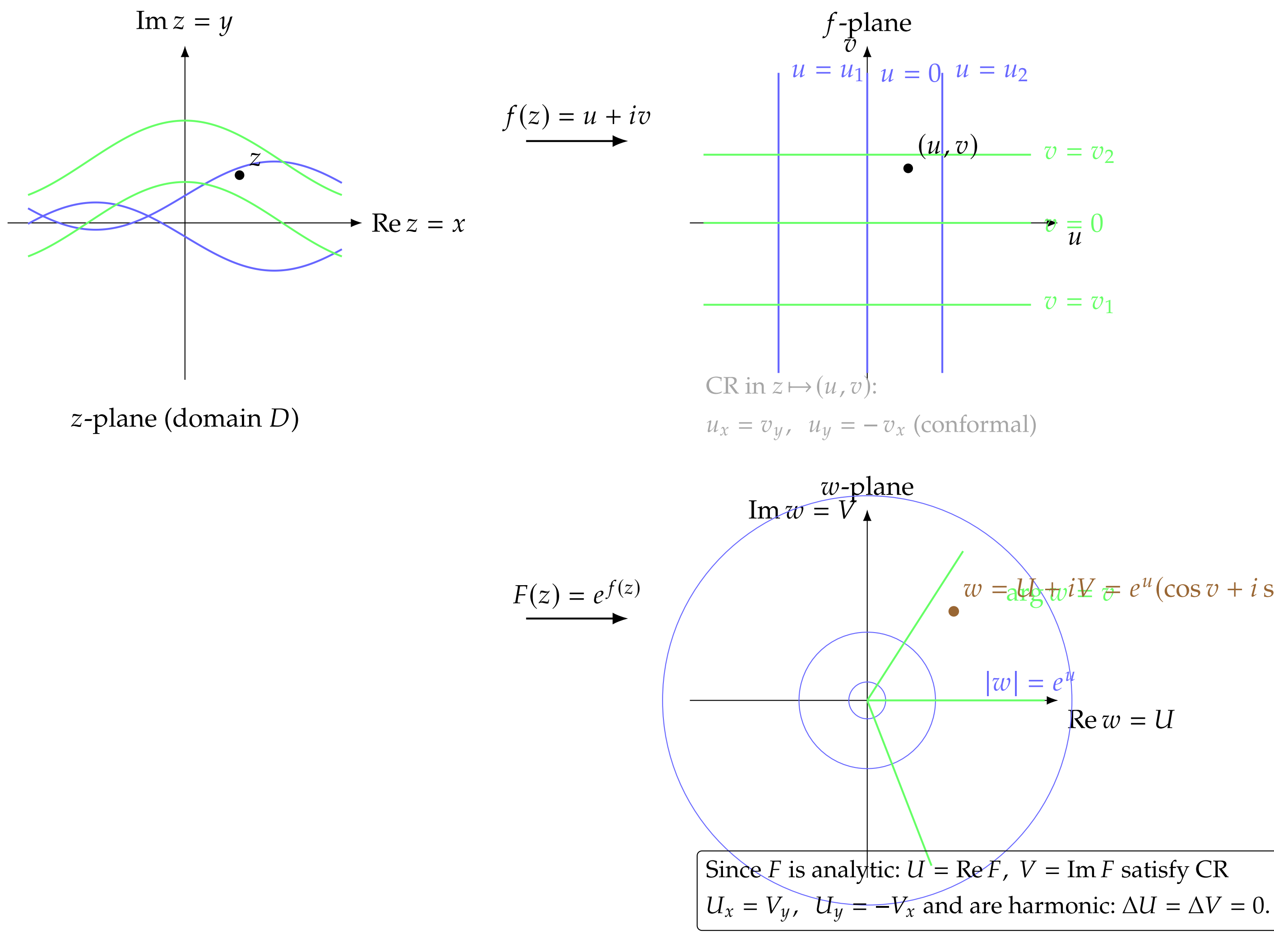
$$U(x,y) = e^{u(x,y)} \cos v(x,y), \qquad V(x,y) = e^{u(x,y)} \sin v(x,y)$$

are harmonic in D, and that V is a harmonic conjugate of U.

Sol. Since f is analytic on D, the composition

$$F(z) := e^{f(z)} = e^{u(x,y)}\big(\cos v(x,y) + i \sin v(x,y)\big) = U(x,y) + iV(x,y)$$

is analytic on D (composition of analytic maps). It follows that $U = \operatorname{Re} F$ and $V = \operatorname{Im} F$ are harmonic and satisfy the Cauchy–Riemann equations.



Note that

$$\begin{aligned} U_x &= e^{u(x,y)}\,(u_x \cos v - v_x \sin v), & U_y &= e^{u(x,y)}\,(u_y \cos v - v_y \sin v),\\ V_x &= e^{u(x,y)}\,(u_x \sin v + v_x \cos v), & V_y &= e^{u(x,y)}\,(u_y \sin v + v_y \cos v). \end{aligned}$$

Because f is analytic, u, v satisfy the CR equations $u_x = v_y$ and $u_y = -v_x$. Substituting,

$$\begin{aligned} U_x &= e^{u(x,y)}\,(v_y \cos v - v_x \sin v) = V_y,\\ U_y &= e^{u(x,y)}\,(-v_x \cos v - v_y \sin v) = -V_x. \end{aligned}$$

Thus $U_x = V_y$ and $U_y = -V_x$, i.e. V is a harmonic conjugate of U.

To show harmonicity, differentiate the CR relations and use equality of mixed partials:

$$U_{xx} = (V_y)_x = V_{yx}, \qquad U_{yy} = (-V_x)_y = -V_{xy}.$$

Hence $\Delta U := U_{xx} + U_{yy} = V_{yx} - V_{xy} = 0$. Similarly,

$$V_{xx} = (-U_y)_x = -U_{yx}, \qquad V_{yy} = (U_x)_y = U_{xy},$$

so $\Delta V := V_{xx} + V_{yy} = -U_{yx} + U_{xy} = 0$. Therefore U and V are harmonic on D, and V is a harmonic conjugate of U. □

3. Show that $f(z) = \operatorname{Log}(z - i)$ is analytic except on portion $x \le 0$ of the line $y = 1$ and that the function

$$f(z) = \frac{\operatorname{Log}(z+4)}{z^2 + i}$$

is analytic everywhere except at the points $\pm(1-i)/\sqrt{2}$ and on the portion $x \le -4$ of the real axis.

Sol. Consider $\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z$, the principal branch of the complex logarithm, with $\operatorname{Arg} z \in (-\pi, \pi)$, so that Log is analytic on

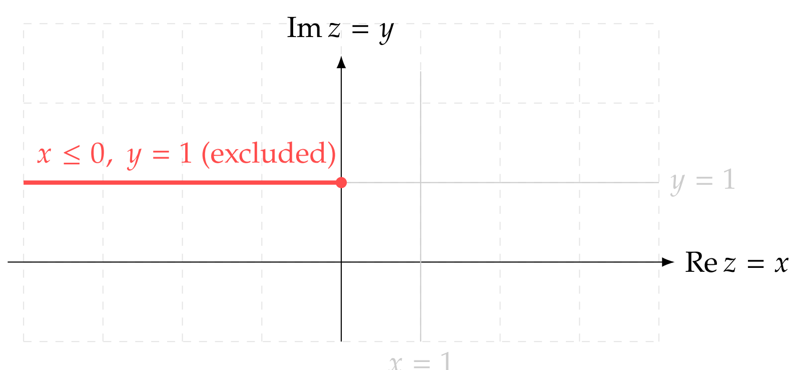
$$\begin{aligned}\mathbb{C} \setminus (-\infty, 0] &= \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Re} z \le 0 \wedge \operatorname{Im} z = 0\} \\ &= \{\, z \in \mathbb{C} : \operatorname{Re} z > 0 \vee \operatorname{Im} z \neq 0 \,\}.\end{aligned}$$

Then

(1) Since Log is analytic on $\mathbb{C} \setminus (-\infty, 0]$ and the map $z \mapsto z - i$ is entire, the composition $z \mapsto \operatorname{Log}(z - i)$ is analytic precisely where $z - i \notin (-\infty, 0]$. Equivalently,

$$\begin{aligned} z - i \in (-\infty, 0] &\iff \operatorname{Re}(z-i) \le 0 \text{ and } \operatorname{Im}(z-i) = 0 \\ &\iff \operatorname{Re}(x + i(y-1)) = x \le 0 \text{ and } \operatorname{Im}(x + i(y-1)) = y - 1 = 0. \end{aligned}$$

That is, $f(z) := \operatorname{Log}(z - i)$ is analytic on $\mathbb{C} \setminus \{\, x + iy : \ x \le 0, \ y = 1 \}$.



$f(z) = \operatorname{Log}(z - i)$

Analytic domain: $\mathbb{C} \setminus \{(x, y) \mid y = 1, \ x \le 0\}$.

(2) The numerator $z \mapsto \operatorname{Log}(z + 4)$ is analytic wherever $z + 4 \notin (-\infty, 0]$, i.e., $z \notin (-\infty, -4]$. In other words, $\operatorname{Log}(z+4)$ is analytic on

$$\mathbb{C} \setminus (-\infty, -4] = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Re} z \le -4 \wedge \operatorname{Im} z = 0\} = \{z \in \mathbb{C} : \operatorname{Re} z > -4 \vee \operatorname{Im} z \neq 0\}$$

$\mathbb{C} \setminus (-\infty]$ for $z \notin (-\infty, -4]$, which is the portion $x \le -4$ of the real axis. The denominator

$z^2 + i$ vanishes exactly at the zeros of $z^2 = -i$, namely

$$z = \pm(-i)^{1/2} = \pm e^{-i\pi/4} = \pm\frac{1-i}{\sqrt{2}}.$$

Therefore g is analytic on the domain where the numerator is analytic and the denominator is nonzero, i.e.

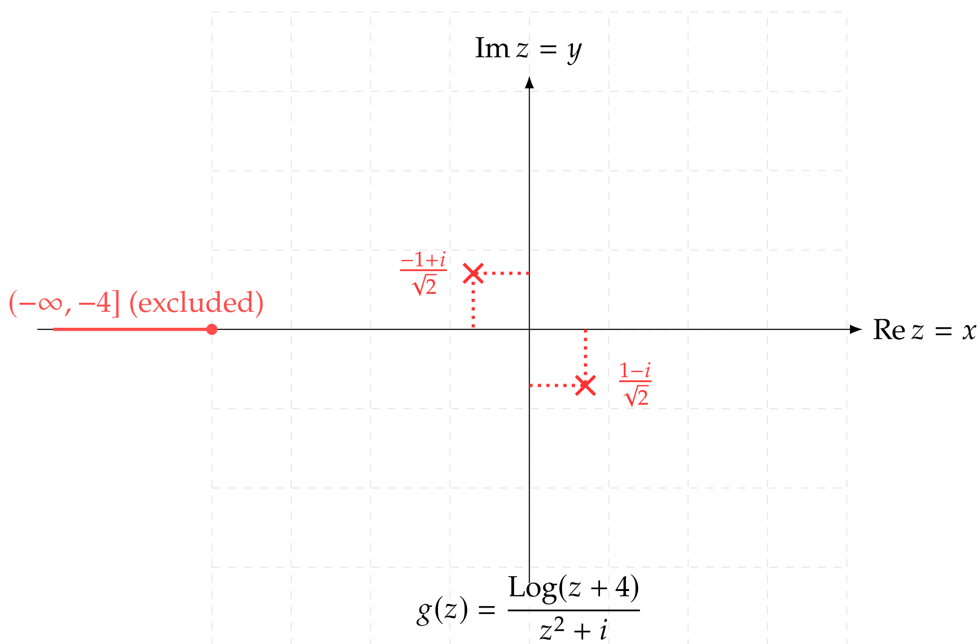
$$\mathbb{C} \setminus \Big((-\infty, -4] \cup \{\pm\tfrac{1-i}{\sqrt{2}}\}\Big),$$

which is exactly the stated set.

$g(z) := \dfrac{\mathrm{Log}(z+4)}{z^2+i}$ is analytic on

$$\mathbb{C} \setminus \Big(\{\, x+iy :\ y=0,\ x \le -4\,\} \cup \{\pm\tfrac{1-i}{\sqrt{2}}\}\Big),$$

i.e. everywhere except at the branch cut $x \le -4$ on the real axis and at the two points $\pm(1-i)/\sqrt{2}$.



$\square$

4. Show that the function $\ln\left(x^2+y^2\right)$ is harmonic in every domain that does not contain the origin.

Sol. For $(x,y)\neq(0,0)$, we can differentiate:

$$u_x=\frac{\partial}{\partial x}\ln\left(x^2+y^2\right)=\frac{2x}{x^2+y^2},$$
$$u_y=\frac{\partial}{\partial y}\ln\left(x^2+y^2\right)=\frac{2y}{x^2+y^2}.$$

And then

$$u_{xx}=\frac{2(x^2+y^2)-2x\cdot 2x}{(x^2+y^2)^2}=\frac{2(x^2+y^2)-4x^2}{(x^2+y^2)^2}=\frac{-2x^2+2y^2}{(x^2+y^2)^2}$$
$$u_{yy}=\frac{2(x^2+y^2)-2y\cdot 2y}{(x^2+y^2)^2}=\frac{2(x^2+y^2)-4y^2}{(x^2+y^2)^2}=\frac{2x^2-2y^2}{(x^2+y^2)^2}.$$

Now compute the Laplacian:

$$u_{xx}+u_{yy}=\frac{-2x^2+2y^2}{(x^2+y^2)^2}+\frac{2x^2-2y^2}{(x^2+y^2)^2}=\frac{(-2x^2+2y^2)+(2x^2-2y^2)}{(x^2+y^2)^2}=\frac{0}{(x^2+y^2)^2}=0$$

for all $(x,y)\neq(0,0)$.

(**Proof via Wirtinger-operator**) Let $z=x+iy$ and

$$u(x,y)=\ln\left(x^2+y^2\right)=\ln\left(|z|^2\right)=\ln(z\bar{z}).$$

Recall the Wirtinger operators $\partial:=\frac{1}{2}(\partial_x-i\,\partial_y)$ and $\bar{\partial}:=\frac{1}{2}(\partial_x+i\,\partial_y)$, so that the Laplacian satisfies

$$\Delta=\partial_{xx}+\partial_{yy}=4\,\partial\bar{\partial}=4\,\bar{\partial}\partial.$$

On $\mathbb{C}\setminus\{0\}$ the chain rule gives

$$\partial u=\partial\big(\ln(z\bar{z})\big)=\frac{1}{z\bar{z}}\,\partial(z\bar{z})=\frac{1}{z\bar{z}}\,\bar{z}=\frac{1}{z},$$
$$\bar{\partial}u=\bar{\partial}\big(\ln(z\bar{z})\big)=\frac{1}{z\bar{z}}\,\bar{\partial}(z\bar{z})=\frac{1}{z\bar{z}}\,z=\frac{1}{\bar{z}}.$$

Therefore,

$$\Delta u=4\,\partial\bar{\partial}u=4\,\partial\left(\frac{1}{\bar{z}}\right)=0\qquad\text{on }\mathbb{C}\setminus\{0\}.$$

$\square$

5. Show that neither $\sin\bar{z}$ nor $\cos\bar{z}$ is an analytic function of z anywhere.

(Hint: use the Cauchy–Riemann equation.)

Sol. (**Proof via CR-equations**) Write $z = x + iy$.

(1) Let

$$f(z) = \sin(\overline{z}) = \sin(x - iy) = \sin(x)\cos(iy) - \cos x \sin(iy) = \sin x \cosh y - i\ \cos x \sinh y$$

then

$$u(x, y) = \sin x \cosh y, \qquad v(x, y) = -\cos x \sinh y.$$

Compute the partials:

$$u_x = \cos x \cosh y, \quad u_y = \sin x \sinh y, \qquad v_x = \sin x \sinh y, \quad v_y = -\cos x \cosh y.$$

The Cauchy–Riemann (CR) equations $u_x = v_y$ and $u_y = -v_x$ become

$$\cos x \cosh y = -\cos x \cosh y \quad \Longrightarrow \quad \cos x \cosh y = 0,$$

$$\sin x \sinh y = -\sin x \sinh y \quad \Longrightarrow \quad \sin x \sinh y = 0.$$

Since $\cosh y \neq 0$ for all y, the first forces $\cos x = 0$. Then the second gives either $\sin x = 0$ (impossible simultaneously with $\cos x = 0$) or $\sinh y = 0$, i.e. $y = 0$. Hence the CR equations can hold only at isolated points with $y = 0$ and $\cos x = 0$ (i.e. $x = \frac{\pi}{2} + k\pi$). They **cannot** hold on any open set. Therefore f is not analytic anywhere.

(2) $g(z) = \cos(\overline{z}) = \cos(x - iy) = \cos x \cosh y + i\ \sin x \sinh y$. Thus

$$u(x, y) = \cos x \cosh y, \qquad v(x, y) = \sin x \sinh y.$$

Compute the partials:

$$u_x = -\sin x \cosh y, \quad u_y = \cos x \sinh y, \qquad v_x = \cos x \sinh y, \quad v_y = \sin x \cosh y.$$

The CR equations give

$$u_x = v_y \implies -\sin x \cosh y = \sin x \cosh y \implies \sin x \cosh y = 0,$$

$$u_y = -v_x \implies \cos x \sinh y = -\cos x \sinh y \implies \cos x \sinh y = 0.$$

Again, since $\cosh y \neq 0$, the first forces $\sin x = 0$; then the second forces either $\cos x = 0$ (incompatible) or $\sinh y = 0$, i.e. $y = 0$. Thus CR can hold only at isolated points with $y = 0$ and $\sin x = 0$ (i.e. $x = k\pi$), and not on any open set. Therefore g is not analytic anywhere.

(**Proof via Wirtinger-operators**) Recall the Wirtinger operators

$$\partial = \tfrac{1}{2}(\partial_x - i\,\partial_y), \qquad \bar{\partial} = \tfrac{1}{2}(\partial_x + i\,\partial_y),$$

and the criterion: a C^1 function F is holomorphic on an open set iff $\bar{\partial}F \equiv 0$ there.

Let $h(w) = \sin w$. Then $f_1(z) := \sin(\overline{z}) = h(\overline{z})$ satisfies

$$\partial f_1(z) = 0, \qquad \bar{\partial} f_1(z) = h'(\overline{z}) = \cos(\overline{z}).$$

Similarly, with $h(w) = \cos w$, $f_2(z) := \cos(\overline{z})$ satisfies

$$\partial f_2(z) = 0, \qquad \bar{\partial} f_2(z) = h'(\overline{z}) = -\sin(\overline{z}).$$

In each case, $\bar{\partial} f_j$ is **not** identically zero on any open set (its zero set is discrete). Hence neither f_1 nor f_2 is holomorphic on any nonempty open set; i.e. they are nowhere analytic.

Remark. At isolated points where $\cos(\overline{z}) = 0$ (respectively $\sin(\overline{z}) = 0$), the complex difference quotient may happen to have limit 0; however, analyticity requires $\bar{\partial} f \equiv 0$ on a neighborhood, which fails here. □

6. Show that $\cosh^2 z - \sinh^2 z = 1$ and $\sinh z + \cosh z = e^z$.

Sol. Recall the exponential definitions (valid for all $z \in \mathbb{C}$):

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}.$$

(1) $\cosh^2 z - \sinh^2 z = 1$**.**

$$\cosh^2 z - \sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2 = \frac{(e^z + e^{-z})^2 - (e^z - e^{-z})^2}{4}.$$

Expanding,

$$(e^z + e^{-z})^2 - (e^z - e^{-z})^2 = e^{2z} + 2 + e^{-2z} - (e^{2z} - 2 + e^{-2z}) = 4,$$

so $\cosh^2 z - \sinh^2 z = \frac{4}{4} = 1$.

(2) $\sinh z + \cosh z = e^z$**.**

$$\sinh z + \cosh z = \frac{e^z - e^{-z}}{2} + \frac{e^z + e^{-z}}{2} = e^z.$$

□

4 Integrals

4.1 Definite Integrals of Complex-Valued Functions

Derivative

Definition 4.1. If $w(t) = u(t) + iv(t)$ with real-valued u, v, the derivative is

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t) = w'(t) = u'(t) + iv'(t),$$

whenever u' and v' exist. If $z_0 = x_0 + iy_0$ is constant, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[z_0 w(t)\big] = z_0\, w'(t), \qquad \frac{\mathrm{d}}{\mathrm{d}t} e^{z_0 t} = z_0\, e^{z_0 t}.$$

Observation (No mean value theorem for derivatives)**.** If $w(t)$ is continuous on $[a, b]$ and differentiable on (a, b), there need **not** exist $c \in (a, b)$ with

$$w'(c) = \frac{w(b) - w(a)}{b - a}.$$

For $w(t) = e^{it}$ on $[0, 2\pi]$, we have $\left|w'(t)\right| = 1$ but $[w(2\pi) - w(0)]/(2\pi) = 0$.

Definition 4.2 (Definite integral)**.** For $w(t) = u(t) + iv(t)$,

$$\int_a^b w(t)\, dt = \int_a^b u(t)\, dt \; + \; i \int_a^b v(t)\, dt,$$

with analogous definitions for improper integrals.

Example 4.3.

$$\int_0^1 (1 + it)^2\, dt = \int_0^1 (1 - t^2)\, dt + i \int_0^1 2t\, dt = \frac{2}{3} + i.$$

Theorem 4.4 (Additivity)**.** *For $a \le c \le b$,*

$$\int_a^b w(t)\, dt = \int_a^c w(t)\, dt + \int_c^b w(t)\, dt.$$

Fundamental Theorem of Calculus

Theorem 4.5. *If $W'(t) = w(t)$ and W, w are continuous on $[a, b]$, then*

$$\int_a^b w(t)\,dt = W(b) - W(a).$$

Example 4.6. Since $\frac{d}{dt}(e^{it}/i) = e^{it}$,

$$\int_0^{\pi/4} e^{it}\,dt = \left[\frac{e^{it}}{i}\right]_0^{\pi/4} = \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right).$$

Remark (No mean value theorem for integrals). There need not be $c \in (a, b)$ with

$$w(c) = \frac{1}{b-a}\int_a^b w(t)\,dt$$

when w is complex-valued.

4.2 Contours

Arc

Definition 4.7. An **arc** C is a set $z(t) = x(t) + iy(t)$, $a \le t \le b$, where x, y are continuous.

Definition 4.8 (Simple arc / Jordan curve)**.** C is **simple** if $z(t_1) \ne z(t_2)$ for $t_1 \ne t_2$. If C is simple with $z(a) = z(b)$, it is a **simple closed curve** (Jordan curve). Positive orientation is counterclockwise.

Example 4.9. The polygonal line from 0 to $1 + i$ to $2 + i$ is a simple arc; $z = e^{i\theta}$, $0 \le \theta \le 2\pi$, is a positively oriented unit circle; $z = z_0 + Re^{i\theta}$ is a circle centered at z_0 of radius R. Traversing $z = e^{-i\theta}$ reverses orientation; $z = e^{i2\theta}$ traverses the unit circle twice.

Observation (Arc length)**.** If $z'(t) = x'(t) + iy'(t)$ is continuous on $[a, b]$, then

$$L(C) = \int_a^b \left|z'(t)\right| dt, \qquad \left|z'(t)\right| = \left([x'(t)]^2 + [y'(t)]^2\right)^{1/2}.$$

The unit tangent is $T = z'(t)/\left|z'(t)\right|$ where $z'(t) \ne 0$; such an arc is **smooth**.

Smooth arc and contour

Definition 4.10. An arc is **smooth** if $z'(t)$ is continuous on $[a, b]$ and nonzero on (a, b). A **contour** (piecewise smooth arc) is a finite concatenation of smooth arcs. A contour with identical initial and final points is a **simple closed contour**.

Jordan Curve Theorem

Theorem 4.11. *A simple closed curve C is the boundary of exactly two domains: a bounded interior and an unbounded exterior.*

4.3 Contour Integrals

Contour integral

Definition 4.12. If C is given by $z = z(t)$, $a \le t \le b$, and f is (piecewise) continuous on C, define

$$\int_C f(z)\,dz \;=\; \int_a^b f(z(t))\,z'(t)\,dt.$$

This is invariant under reparametrization of C.

Linearity

Proposition 4.13. *For a contour C and constant z_0,*

$$\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz, \qquad \int_C [f(z) + g(z)]\,dz = \int_C f + \int_C g.$$

Orientation reversal

Proposition 4.14. *If $-C$ is C with reversed direction, then*

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz.$$

Additivity over legs

Proposition 4.15. *If $C = C_1 + C_2$ (concatenation), then*

$$\int_C f(z)\,dz = \int_{C_1} f + \int_{C_2} f.$$

Example 4.16 (Half-circle integral)**.** Let $C:\ z = 2e^{i\theta}$, $-\pi/2 \le \theta \le \pi/2$ (right half of $|z| = 2$). Then

$$\int_C z\,dz = \int_{-\pi/2}^{\pi/2} 2e^{i\theta}(2ie^{i\theta})\,d\theta = 4i\int_{-\pi/2}^{\pi/2} e^{2i\theta}\,d\theta = 4\pi i.$$

Example 4.17 (Polygonal and diagonal paths)**.** Let $f(z) = y - x - i\,3x^2$ with $z = x + iy$. With C_1 :

$O \to A \to B$ (up then right) and $C_2 : O \to B$ along $y = x$, one finds

$$\int_{C_1} f(z)\,dz = \frac{1-i}{2}, \qquad \int_{C_2} f(z)\,dz = 1-i, \qquad \int_{C_1-C_2} f(z)\,dz = \frac{-1+i}{2}.$$

Example 4.18 (Path-independence for $f(z) = z$)**.** For any smooth arc C from z_1 to z_2,

$$\int_C z\,dz = \frac{z_2^2 - z_1^2}{2}.$$

Hence the value depends only on endpoints, so the same holds for any piecewise smooth contour by telescoping the legs.

Example 4.19 (Square-root branch on a semicircle)**.** Let $C: \; z = 3e^{i\theta}$, $0 \le \theta \le \pi$ and take the branch $z^{1/2} = \exp\left(\frac{1}{2}\log z\right)$ on $|z| > 0$, $0 < \arg z < 2\pi$. Then $z^{1/2}$ is piecewise continuous on C and

$$\int_C z^{1/2}\,dz = 3\sqrt{3}\,i \int_0^{\pi} e^{i(3\theta/2)}\,d\theta = -2\sqrt{3}\,(1+i).$$

Example 4.20 (Power integral on a circle)**.** On the principal branch $z^{a-1} = \exp\left[(a-1)\mathrm{Log} z\right]$ with $|z| > 0$, $-\pi < \mathrm{Arg} z < \pi$, for $C: \; z = Re^{i\theta}$, $-\pi < \theta < \pi$,

$$\int_C z^{a-1}\,dz = iR^a \int_{-\pi}^{\pi} e^{ia\theta}\,d\theta = \frac{i\,2R^a}{a}\,\sin(a\pi).$$

If $a = n \in \mathbb{Z} \setminus \{0\}$, this vanishes; for $a = 0$ it yields

$$\int_C \frac{1}{z}\,dz = \int_{-\pi}^{\pi} \frac{i\,Re^{i\theta}}{Re^{i\theta}}\,d\theta = 2\pi i.$$

4.4 Upper Bounds for Moduli of Contour Integrals

Lemma 4.21. *If $w(t)$ is piecewise continuous on $[a, b]$, then*

$$\left|\int_a^b w(t)\,dt\right| \le \int_a^b \left|w(t)\right| dt.$$

ML-inequality

Theorem 4.22. *Let C be a contour of length $L = b - a$, and suppose f is piecewise continuous on C with $\left|f(z)\right| \le M$ on C. Then*

$$\left|\int_C f(z)\,dz\right| \le ML(= M(b-a)).$$

Example 4.23. On the quarter-circle $C : |z| = 2$ from 2 to $2i$,

$$\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \le \frac{6}{7} \cdot \frac{\pi}{2} \cdot 2 = \frac{6\pi}{7},$$

since $|z+4| \le 6$, $\left|z^3 - 1\right| \ge 7$, and $L = \pi$.

Example 4.24 (Large semicircle vanishing)**.** Let $C_R : \ z = Re^{i\theta}$, $0 \le \theta \le \pi$, and take $z^{1/2} = \exp\left(\frac{1}{2}\log z\right)$ on $|z| > 0$, $-\pi/2 < \theta < 3\pi/2$. Then

$$\left|\int_{C_R} \frac{z^{1/2}}{z^2+1}\,dz\right| \le \max_{C_R} \frac{\sqrt{R}}{R^2-1} \cdot (\pi R) = \frac{\pi R\sqrt{R}}{R^2-1} \xrightarrow[R\to\infty]{} 0.$$

Hence $\displaystyle\lim_{R\to\infty} \int_{C_R} \frac{z^{1/2}}{z^2+1}\,dz = 0$.

4.5 Antiderivatives and Path Independence

Theorem 4.25. *Let f be continuous on a domain $D \subset \mathbb{C}$. The following are equivalent:*

(1) f has an antiderivative F on D;

(2) For any $z_1, z_2 \in D$ and any contour C in D from z_1 to z_2,

$$\int_C f(z)\,dz = F(z_2) - F(z_1);$$

(3) $\displaystyle\int_C f(z)\,dz = 0$ for every closed contour C in D.

Example 4.26. $f(z) = z^2$ has antiderivative $F(z) = z^3/3$ on $\mathbb{C}$; thus for any contour $0 \to 1 + i$,

$$\int_0^{1+i} z^2\,dz = \left[\frac{z^3}{3}\right]_0^{1+i} = \frac{2}{3}(-1+i).$$

Example 4.27. $f(z) = z^{-2}$ is continuous on $\mathbb{C} \setminus \{0\}$ with antiderivative $F(z) = -1/z$ on $|z| > 0$. Therefore for the circle $z = 2e^{i\theta}$,

$$\int_{|z|=2} \frac{1}{z^2}\,dz = 0.$$

Example 4.28 (Using branches of log for $1/z$)**.** On the right semicircle $C_1 : \ z = 2e^{i\theta}, -\pi/2 \le \theta \le \pi/2$, the principal branch

$$\operatorname{Log} z = \ln r + i\varphi \quad (r > 0,\ -\pi < \varphi < \pi)$$

is an antiderivative of $1/z$, hence

$$\int_{C_1} \frac{1}{z}\,dz = \operatorname{Log}(2i) - \operatorname{Log}(-2i) = \pi i.$$

On the left semicircle $C_2 : \ \pi/2 \le \theta \le 3\pi/2$ using the branch

$$\log z = \ln r + i\theta \quad (r > 0,\ 0 < \theta < 2\pi),$$

we likewise obtain

$$\int_{C_2} \frac{1}{z}\,dz = \log(-2i) - \log(2i) = \pi i.$$

Therefore $\displaystyle\oint_{|z|=2} \frac{1}{z}\,dz = 2\pi i.$

4.6 Cauchy–Goursat Theorem

We begin with the real-variable result which motivates Cauchy's theorem.

Green's Theorem

Theorem 4.29. *Let $C(=\partial R)$ be a positively oriented simple closed contour in the plane, and let R be the region it encloses. Suppose $P(x,y), Q(x,y)$ are continuous on $C \cup R$ and have continuous first partial derivatives P_x, P_y, Q_x, Q_y there. Then*

$$\int_{C=\partial R} P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A.$$

This gives rise to one of the central theorems of complex integration.

Cauchy's Theorem (elementary form)

Theorem 4.30. *Let f be analytic and f' continuous in a simply connected domain $D \subset \mathbb{C}$. If C is a positively oriented simple closed contour in D, then*

$$\int_C f(z)\,dz = 0.$$

Remark. Let $f = u + iv$ and $z = z(t) = x(t) + iy(t)$ $(\mathrm{d}z = dx + idy)$. Then

$$\begin{aligned}
\int f(z)\mathrm{d}z &= \int (u\mathrm{d}x - v\mathrm{d}y + i(vdx + u\mathrm{d}y)) \\
&= \int (udx - vdy) + i\int (vdx + udy) \\
&= \iint_D
\end{aligned}$$

Example 4.31. Let C be any simple closed contour. The function $f(z) = e^{z^3}$ is entire. Hence

$$\int_C e^{z^3}\,dz = 0.$$

To remove the hypothesis "f' is continuous", we use a covering lemma.

Lemma 4.32. *Let f be analytic throughout a closed region R consisting of the interior of a positively oriented simple closed contour C together with the points of C itself. For any $\varepsilon > 0$, the region R can be covered by finitely many (possibly partial) squares indexed by $j = 1, \ldots, n$ such that in each square there is a point z_j with*

$$\left|\frac{f(z) - f(z_j)}{z - z_j} - f'(z_j)\right| < \varepsilon$$

for all z in that square distinct from z_j.

Cauchy–Goursat Theorem

Theorem 4.33. *If f is analytic at all points on and inside a positively oriented simple closed contour C, then*

$$\int_C f(z)\, dz = 0.$$

4.7 Integrals on Simply and Multiply Connected Domains

Simply connected domain

Definition 4.34. A domain $D \subset \mathbb{C}$ is **simply connected** if every simple closed contour contained in D encloses only points of D (equivalently: any closed contour in D can be continuously deformed to a point while remaining in D).

Example 4.35. The interior of a simple closed curve is simply connected. The annulus

$$\{z : r < |z| < R\}$$

is not simply connected because closed contours can wind around the missing center point.

Definition 4.36 (Multiply connected domain)**.** A domain that is not simply connected is called **multiply connected**.

Cauchy on simply connected domains

Theorem 4.37. *If f is analytic throughout a simply connected domain D, then*

$$\int_C f(z)\, dz = 0$$

for every closed contour C contained in D.

Example 4.38. Let C be any closed contour in the disk $\{z : |z| < 2\}$. Consider

$$f(z) = \frac{ze^z}{(z^2+9)^5}.$$

The poles at $z = \pm 3i$ lie outside $|z| < 2$. On $|z| < 2$ the function f is analytic. Hence

$$\int_C \frac{ze^z}{(z^2+9)^5}\, dz = 0.$$

The following result ties together antiderivatives, path-independence, and zero integral over closed contours.

Theorem 4.39 (Equivalence)**.** *Let f be continuous on a domain D. The following are equivalent:*

1. *f has an antiderivative F on D;*
2. *For any $z_1, z_2 \in D$, and any contour C in D from z_1 to z_2,*

$$\int_C f(z)\, dz = F(z_2) - F(z_1);$$

3. *For every closed contour C in D,*

$$\int_C f(z)\, dz = 0.$$

Corollary 4.40. *If f is analytic throughout a simply connected domain D, then f has an antiderivative on D.*

Remark. Since the entire plane $\mathbb{C}$ is simply connected, every entire function possesses an entire antiderivative.

Multiply connected case

Cauchy for multiply connected regions

Theorem 4.41. *Suppose*

1. *C is a positively oriented simple closed contour;*
2. *$C_1, \dots, C_n$ are negatively oriented (clockwise) simple closed contours interior to C, pairwise disjoint, and their interiors do not intersect;*
3. *f is analytic on C, on each C_k, and on the region consisting of points inside C and outside all C_k.*

Then

$$\int_C f(z)\,dz + \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0.$$

Deformation of Paths

Corollary 4.42. *Let C_1 and C_2 be positively oriented simple closed contours with C_1 inside C_2. If f is analytic on and between these contours, then*

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Example 4.43. Let C be any positively oriented simple closed contour around the origin. Then

$$\int_C \frac{1}{z}\,dz = 2\pi i.$$

Indeed, by deformation we may replace C by the unit circle.

4.8 Cauchy Integral Formula

We now reach one of the most powerful formulas in complex analysis.

Cauchy Integral Formula

Theorem 4.44. *Let f be analytic on and inside a positively oriented simple closed contour C, and let z_0 be a point interior to C. Then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz.$$

Remark.

$$\begin{aligned}\int_C \frac{f(z)}{z - z_0}\, dz &= \int_C \frac{f(z) - f(z_0) + f(z_0)}{z - z_0} \\ &= \int_C \frac{f(z) - f(z_0)}{z - z_0}\, dz + \int_C \frac{f(z_0)}{z - z_0}\, dz\end{aligned}$$

Remark. This formula shows that the values of f inside C are **completely determined** by the values of f on C.

Observation. Written as

$$\int_C \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0),$$

the formula is very convenient for evaluating contour integrals.

Example 4.45. Let C be the positively oriented circle $|z| = 2$. Consider

$$\int_C \frac{z}{(9 - z^2)(z + i)}\, dz.$$

Write $f(z) = \frac{z}{9 - z^2}$, which is analytic on $|z| \le 2$, and $z_0 = -i$ lies inside C. Then

$$\int_C \frac{f(z)}{z - (-i)}\, dz = 2\pi i\, f(-i) = 2\pi i \cdot \frac{-i}{9 - (-i)^2} = 2\pi i \cdot \frac{-i}{9 + 1} = \frac{\pi}{5}.$$

4.8.1 Cauchy formulas for derivatives

Cauchy formula for f'

Theorem 4.46. *Under the hypotheses of Theorem 4.44, for z interior to C,*

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z)^2}\, ds.$$

Remark.

$$f(z) - \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}dz \implies f'(z) = \frac{df}{dz} = \frac{d}{dz}\left(\frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}\, ds\right)$$

Cauchy formula for higher derivatives

Corollary 4.47. *If f is analytic on and inside C, then for $n = 1, 2, \ldots,$*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)}{(s-z)^{n+1}}\, ds.$$

Equivalently,

$$\int_C \frac{f(s)}{(s-z)^{n+1}}\, ds = \frac{2\pi i}{n!} f^{(n)}(z).$$

Example 4.48. Let C be the positively oriented unit circle $|z| = 1$. Evaluate

$$\int_C \frac{e^{2z}}{z^4}\, dz.$$

Here $f(z) = e^{2z}$ is analytic everywhere and we want the coefficient corresponding to $(s-0)^{-4}$. Take $z = 0$ and $n = 3$ in the corollary:

$$\int_C \frac{f(s)}{s^4}\, ds = \frac{2\pi i}{3!} f^{(3)}(0).$$

But $f^{(3)}(z) = 2^3 e^{2z} = 8e^{2z}$, so $f^{(3)}(0) = 8$. Hence

$$\int_C \frac{e^{2z}}{z^4}\, dz = \frac{2\pi i}{6}\cdot 8 = \frac{8\pi i}{3}.$$

Example 4.49. Let $f(z) = 1$. Then

$$\int_C \frac{1}{z-z_0}\, dz = 2\pi i, \qquad \int_C \frac{1}{(z-z_0)^{n+1}}\, dz = 0, \quad n = 1, 2, \ldots$$

whenever z_0 is inside C.

4.9 Consequences of the Cauchy Integral Formula

4.9.1 Analyticity of derivatives

Theorem 4.50. *If f is analytic at a point z_0, then all derivatives $f^{(n)}$ exist and are analytic at z_0.*

Corollary 4.51. *If $f(z) = u(x, y) + iv(x, y)$ is analytic at z_0, then u and v have continuous partial derivatives of all orders in a neighborhood of z_0.*

4.10 Morera's Theorem

Theorem 4.52 (Morera)**.** *Let f be continuous on a domain D. If*

$$\int_C f(z)\,dz = 0$$

for every closed contour C in D, then f is analytic throughout D.

4.10.1 Cauchy's Inequalities

Theorem 4.53 (Cauchy's inequality)**.** *Suppose f is analytic on and inside the circle $C_R = \{z : |z - z_0| = R\}$, and let*

$$M_R = \max_{|z-z_0|=R} |f(z)|.$$

Then for $n = 0, 1, 2, \ldots,$

$$|f^{(n)}(z_0)| \le \frac{n!\,M_R}{R^n}.$$

4.11 Liouville's Theorem and the Fundamental Theorem of Algebra

Liouville

Theorem 4.54. *If f is entire and bounded in the whole complex plane, then f is constant.*

Fundamental Theorem of Algebra

Theorem 4.55. *Let*

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n, \qquad a_n \neq 0,\ n \geq 1,$$

be a complex polynomial. Then P has at least one zero in $\mathbb{C}$.

Remark. It follows that any polynomial of degree n can be factored into linear factors:

$$P(z) = c(z - z_1)(z - z_2)\cdots(z - z_n), \qquad c, z_k \in \mathbb{C}.$$

4.12 Maximum Modulus Principle

Lemma 4.56. *Suppose f is analytic in a disk $|z - z_0| < \varepsilon$ and $|f(z)| \leq |f(z_0)|$ for all such z. Then f is constant in that disk.*

Theorem 4.57 (Maximum Modulus Principle)**.** *If f is analytic and non-constant in a domain D, then $|f(z)|$ has no maximum value in D.*

Corollary 4.58. *Let f be continuous on a closed bounded region R, analytic and non-constant on the interior of R. Then $\max_{z\in R} |f(z)|$ is attained on the boundary of R, not in the interior.*

Remark. If $f = u + iv$ is analytic, then u is harmonic. The corollary implies a maximum principle for u as well.

Example 4.59. Let $R = \{0 \leq x \leq \pi,\ 0 \leq y \leq 1\}$ and $f(z) = \sin z$. Since

$$\sin z = \sin x \cosh y + i \cos x \sinh y,$$

we have

$$|f(z)|^2 = \sin^2 x + \sinh^2 y.$$

On R, $\sin^2 x$ is largest at $x = \pi/2$ and $\sinh^2 y$ is largest at $y = 1$, so the maximum of $|f(z)|$ on R is attained at $z = \pi/2 + i$ and nowhere in the interior.

4.13 Exercises

1. Let C_0 be the positively oriented circle $|z - z_0| = R$. Show that

$$\int_{C_0} (z - z_0)^{n-1}\, dz = \begin{cases} 0, & n = \pm 1, \pm 2, \ldots \\ 2\pi i, & n = 0. \end{cases}$$

Sol. Parametrize C_0 by $z(t) = z_0 + Re^{it}$ $(t \in [0, 2\pi])$ then $dz = iRe^{it}\, dt$ and

$$\int_{C_0} (z - z_0)^{n-1}\, dz = \int_0^{2\pi} (Re^{it})^{n-1}\, iRe^{it}\, dt = iR^n \int_0^{2\pi} e^{int}\, dt.$$

(1) If $n \neq 0$, then

$$\int_0^{2\pi} e^{int}\, dt = \left[\frac{1}{in} e^{int}\right]_0^{2\pi} = \frac{e^{in2\pi} - 1}{in} = \frac{1 - 1}{in} = 0, \qquad \text{so the integral is 0.}$$

(2) If $n = 0$, then $e^{int} \equiv 1$ and the integral equals $iR^0 \int_0^{2\pi} 1\, dt = 2\pi i$.

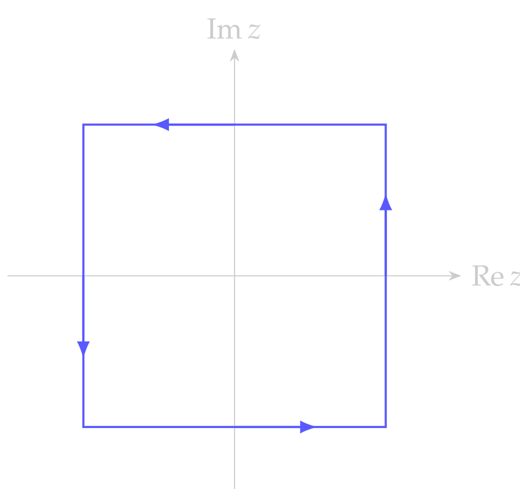
□

2. Let C be the boundary of the square with sides $x = \pm 2$, $y = \pm 2$, oriented positively. Show that

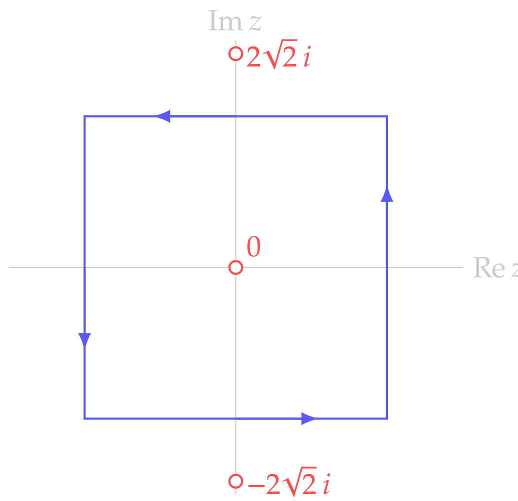
$$\int_C \frac{\cos z}{z(z^2 + 8)}\, dz = \frac{i\pi}{4}, \qquad \int_C \frac{\cosh z}{z^4}\, dz = 0, \qquad \int_C \frac{\tan(z/2)}{(z - x_0)^2}\, dz = i\pi \sec^2\left(\frac{x_0}{2}\right),$$

where $-2 < x_0 < 2$.

Sol. Let C be the positively oriented boundary of the square $\{x + iy : |x| \leq 2, |y| \leq 2\}$.



(1) $\displaystyle\int_C \frac{\cos z}{z(z^2 + 8)}\, dz$. The integrand is meromorphic with simple poles at $z = 0$ and $z = \pm 2\sqrt{2}\, i$.



Only $z = 0$ is inside C. Around $z = 0$,

$$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \cdots, \qquad \frac{1}{z(z^2+8)} = \frac{1}{z^3+8z} = \frac{1}{8z}\,\frac{1}{1+z^2/8} = \frac{1}{8z}\left(1 - \frac{z^2}{8} + \frac{z^4}{8^2} - \cdots\right).$$

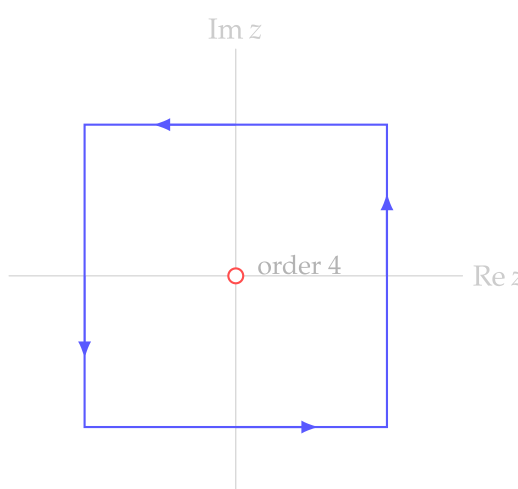
Thus,

$$\mathrm{Res}_{z=0}\frac{\cos z}{z(z^2+8)} = \mathrm{Res}_{z=0}\left(\frac{1}{8z}\left(1 - \frac{z^2}{8} + \frac{z^4}{8^2} - \cdots\right)\left(1 - \frac{z^2}{2} + \frac{z^4}{4!} - \cdots\right)\right) = \frac{1}{8}.$$

By the residue theorem,

$$\int_C \frac{\cos z}{z(z^2+8)}\,dz = 2\pi i \cdot \frac{1}{8} = \frac{i\pi}{4}.$$

(2) $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz.$ Here the only singularity is at $z = 0$ (order 4).



Using

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots, \qquad \frac{\cosh z}{z^4} = \frac{1}{z^4} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{4!} + \cdots,$$

there is no $1/z$ term; hence $\mathrm{Res}_{z=0}(\cosh z/z^4) = 0$, and therefore

$$\int_C \frac{\cosh z}{z^4}\, dz = 0.$$

(3) $\displaystyle\int_C \frac{\tan(z/2)}{(z-x_0)^2}\, dz$ with $-2 < x_0 < 2$. We know that

$$\tan w = \frac{\sin w}{\cos w},$$

so the poles of $\tan w$ occur exactly where $\cos w = 0$ and $\sin w \neq 0$. The zeros of $\cos w$ are

$$w = \frac{\pi}{2} + k\pi = \frac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z}.$$

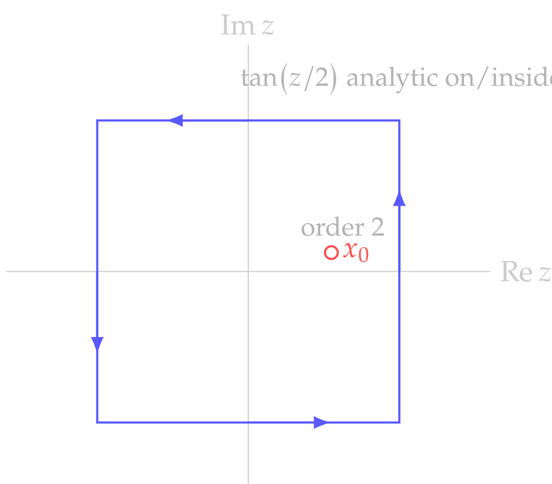
Now consider $\tan\left(\frac{z}{2}\right)$. Let $w = \frac{z}{2}$. The poles of $\tan(z/2)$ occur where w is a pole of $\tan w$, i.e. where

$$\frac{z}{2} = \frac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z}.$$

So the poles of $\tan(z/2)$ are precisely at $z = (2k+1)\pi$ with $k \in \mathbb{Z}$. Since

$$|(2k+1)\pi| \geq \pi > 2,$$

none of these poles lie inside or on C. Hence $\tan(z/2)$ is analytic on and inside C. The only singularity of the integrand $\tan(z/2)/(z-x_0)^2$ inside C is at $z = x_0$.



Since $\tan(z/2)$ is analytic at $z = x_0$, we may expand it in a Taylor series about x_0:

$$\begin{aligned}\tan\left(\frac{z}{2}\right) &= \tan\left(\frac{x_0}{2}\right) + \frac{\mathrm{d}}{\mathrm{d}z}\tan\left(\frac{z}{2}\right)\bigg|_{z=x_0}(z-x_0) + \cdots \\ &= \tan\left(\frac{x_0}{2}\right) + \frac{1}{2}\sec^2\left(\frac{x_0}{2}\right)(z-x_0) + \cdots.\end{aligned}$$

Dividing by $(z - x_0)^2$ gives the Laurent series

$$\frac{\tan(z/2)}{(z-x_0)^2} = \frac{\tan(x_0/2)}{(z-x_0)^2} + \frac{1}{2}\sec^2\left(\frac{x_0}{2}\right)\frac{1}{z-x_0} + \cdots.$$

Thus, we obtain

$$\operatorname{Res}_{z=x_0}\left(\frac{\tan(z/2)}{(z-x_0)^2}\right) = \frac{1}{2}\sec^2\left(\frac{x_0}{2}\right).$$

By the residue theorem,

$$\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz = 2\pi i \cdot \frac{1}{2}\sec^2\left(\frac{x_0}{2}\right) = i\pi\sec^2\left(\frac{x_0}{2}\right).$$

□

3. Let C be the circle $|z| = 3$, positively oriented, and define

$$f(z) = \int_C \frac{2s^2 - s - 2}{s - z}\, ds, \qquad |z| \neq 3.$$

Show that $f(2) = 8\pi i$.

Sol. Let $F(s) = 2s^2 - s - 2$, which is entire. By the Cauchy integral formula, for $|z| < 3$,

$$\int_{|s|=3} \frac{F(s)}{s - z}\, ds = 2\pi i\, F(z).$$

Since 2 lies inside the circle $|s| = 3$, we have

$$f(2) = 2\pi i\, F(2) = 2\pi i \left(2 \cdot 2^2 - 2 - 2\right) = 2\pi i\, (8 - 2 - 2) = 2\pi i \cdot 4 = 8\pi i.$$

□

4. Let C be any positively oriented simple closed contour and

$$f(z) = \int_C \frac{s^3 + 2s}{(s - z)^3}\, ds.$$

Show that $f(z) = 6\pi i\, z$ when z is inside C, and $f(z) = 0$ when z is outside.

Sol. Let $F(s) = s^3 + 2s$, an entire function. By the generalized Cauchy integral formula,

$$\int_C \frac{F(s)}{(s - z)^{n+1}}\, ds = \frac{2\pi i}{n!} F^{(n)}(z),$$

for z inside C. Here $\dfrac{F(s)}{(s - z)^3}$ corresponds to $n = 2$, so

$$f(z) = \int_C \frac{s^3 + 2s}{(s - z)^3}\, ds = \frac{2\pi i}{2!} F''(z).$$

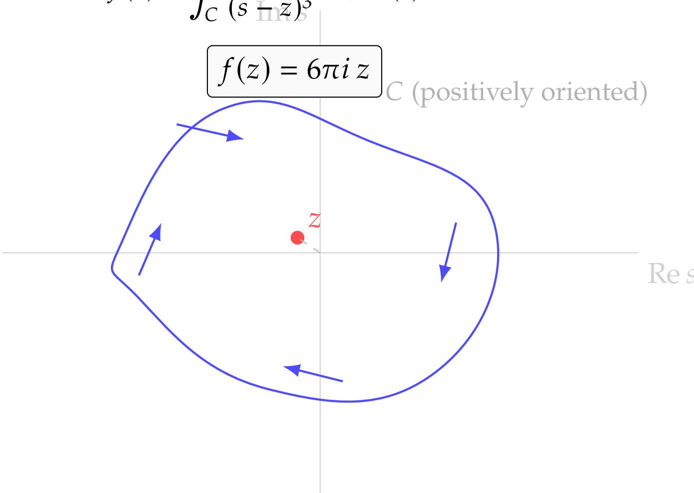
Compute $F'(s) = 3s^2 + 2$ and $F''(s) = 6s$, hence for z inside C,

$$f(z) = \frac{2\pi i}{2} \cdot 6z = 6\pi i\, z.$$

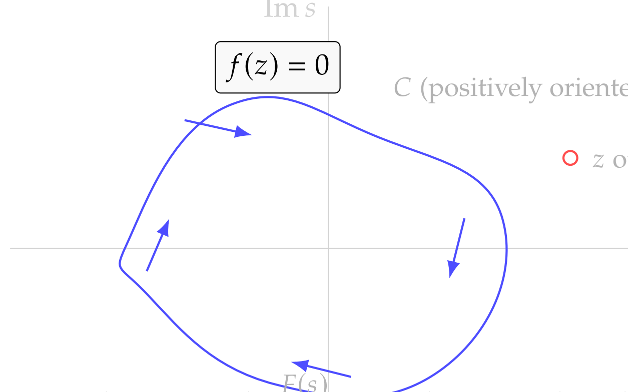
If z is outside C, then $s \mapsto \dfrac{s^3 + 2s}{(s - z)^3}$ is analytic on and inside C (the only singularity is at $s = z$, which lies outside). By Cauchy's theorem,

$$f(z) = \int_C \frac{s^3 + 2s}{(s - z)^3}\, ds = 0.$$

$$f(z) = \int_C \frac{s^3 + 2s}{(s-z)^3}\, ds, \quad F(s) = s^3 + 2s$$



Exterior case: z outside C



Here the integrand $\dfrac{F(s)}{(s-z)^3}$ is analytic on and insid
(its only singularity is at $s = z$, which lies outside).
By Cauchy's theorem, $f(z) = \displaystyle\int_C \frac{F(s)}{(s-z)^3}\, ds = 0$.

$\square$

5. Let C be the unit circle $z = e^{i\theta}$, $-\pi \le \theta \le \pi$. Show that for any constant a,

$$\int_C \frac{e^{az}}{z}\,dz = 2\pi i.$$

Then writethis integral in term of θ to derive the integration formula

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi.$$

Sol. Let C be the unit circle oriented positively. The integrand

$$\frac{e^{az}}{z}$$

has a simple pole at $z = 0$ with residue $\operatorname{Res}_{z=0}\frac{e^{az}}{z} = e^{a\cdot 0} = 1$. By the residue theorem,

$$\int_C \frac{e^{az}}{z}\,dz \;=\; 2\pi i.$$

Now parametrize C by $z = e^{i\theta}$, $-\pi \le \theta \le \pi$. Then $dz = ie^{i\theta}\,d\theta$ and

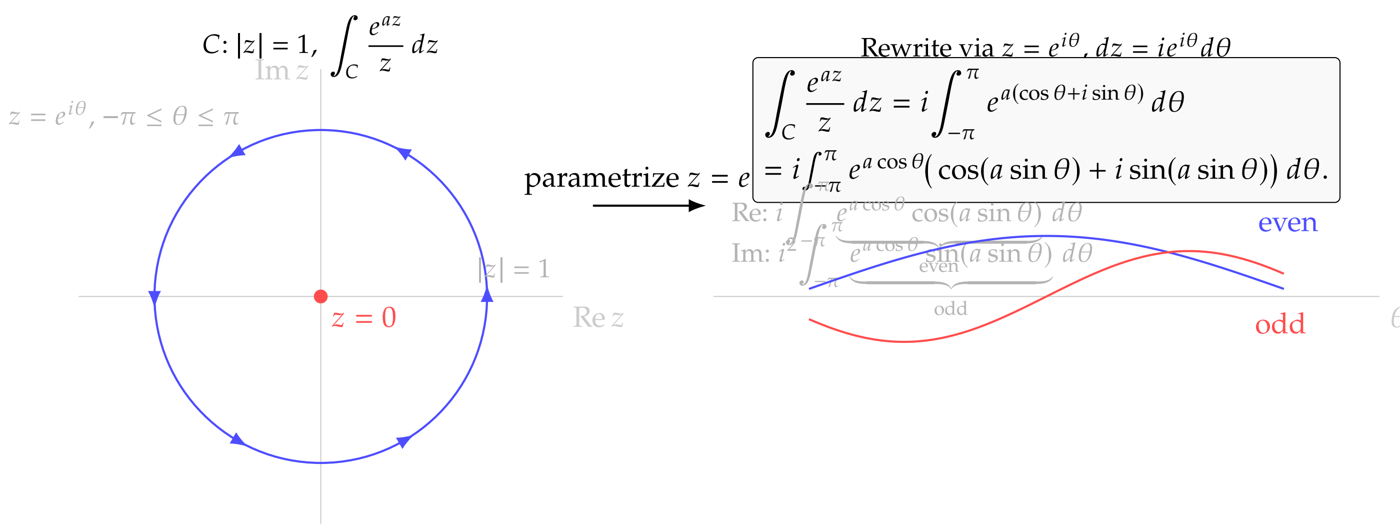
$$\int_C \frac{e^{az}}{z}\,dz = \int_{-\pi}^{\pi} \frac{e^{ae^{i\theta}}}{e^{i\theta}}\,ie^{i\theta}\,d\theta = i\int_{-\pi}^{\pi} e^{a(\cos\theta+i\sin\theta)}\,d\theta = i\int_{-\pi}^{\pi} e^{a\cos\theta}\big(\cos(a\sin\theta)+i\sin(a\sin\theta)\big)\,d\theta.$$

Equating real and imaginary parts with $2\pi i$ gives

$$\int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)\,d\theta = 0, \qquad \int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = 2\pi.$$

Since the integrand $e^{a\cos\theta}\cos(a\sin\theta)$ is even in θ, we obtain

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi.$$



$C\colon |z| = 1,\ \int_C \frac{e^{az}}{z}\, dz$
Im z
$z = e^{i\theta},\ -\pi \le \theta \le \pi$
$|z| = 1$
$z = 0$
Re z
parametrize $z = e$
Rewrite via $z = e^{i\theta},\ dz = ie^{i\theta} d\theta$
$\int_C \frac{e^{az}}{z}\, dz = i\int_{-\pi}^{\pi} e^{a(\cos\theta + i\sin\theta)}\, d\theta$
$= i\int_{-\pi}^{\pi} e^{a\cos\theta}\big(\cos(a\sin\theta) + i\sin(a\sin\theta)\big)\, d\theta.$
even
odd

□

5 Series

5.1 Convergence of Sequences

Definition 5.1 (Limit of a sequence)**.** A sequence (z_n) of complex numbers converges to $z \in \mathbb{C}$ if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon \qquad (n > N).$$

We write $\lim_{n\to\infty} z_n = z$. If no such z exists, the sequence **diverges**.

Remark (Uniqueness)*.* A complex sequence has at most one limit.

Theorem 5.2 (Componentwise convergence)**.** *Let $z_n = x_n + iy_n$ and $z = x + iy$. Then*

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y.$$

Example 5.3. (a) $z_n = \dfrac{1}{n^3} + i \Rightarrow \lim z_n = i$. (b) $z_n = -2 + i\dfrac{(-1)^n}{n^2} \Rightarrow \lim z_n = -2$.

Observation (Polar view)**.** Writing $z_n = r_n e^{i\theta_n}$ with $r_n = |z_n|$ and $\theta_n = \operatorname{Arg} z_n$, one may have $r_n \to r$ while (θ_n) fails to converge (e.g. even/odd subsequences approaching $\pm\pi$).

5.2 Convergence of Series

Series and sum

Definition 5.4. A series $\sum_{n=1}^{\infty} z_n$ converges to S if the partial sums $S_N = \sum_{n=1}^{N} z_n$ satisfy $S_N \to S$. Then $\sum_{n=1}^{\infty} z_n = S$.

Componentwise

Theorem 5.5. *If $z_n = x_n + iy_n$ and $S = X + iY$, then*

$$\sum_{n=1}^{\infty} z_n = S \iff \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y.$$

Remark (Necessary test and boundedness). If $\sum z_n$ converges, then $z_n \to 0$ (the nth-term test). In particular, the terms are bounded: there exists M with $|z_n| \le M$ for all n.

Definition 5.6 (Absolute convergence)**.** $\sum z_n$ is **absolutely convergent** if $\sum |z_n|$ converges. Absolute convergence implies convergence.

Remark (Remainders). If $S = \sum_{n=1}^{\infty} z_n$, the remainder after N terms is $\rho_N = S - S_N$. Then $S_N \to S$ iff $\rho_N \to 0$.

5.3 Power Series and Taylor Series

Power series centered at z_0

Definition 5.7.

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots,$$

with $a_n, z_0 \in \mathbb{C}$.

Taylor series

Theorem 5.8. *If f is analytic on $|z - z_0| < R_0$, then for $|z - z_0| < R_0$,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}.$$

*For $z_0 = 0$ this is the **Maclaurin series**.*

Example 5.9. Since e^z is entire,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad z^2 e^{3z} = \sum_{n=0}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n \quad \text{(interpreting } (n-2)! = \infty \text{ for } n < 2 \text{ gives zero terms).}$$

Also

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!},$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \quad \cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.$$

Example 5.10 (Geometric series)**.** For $f(z) = \dfrac{1}{1-z}$ we have

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad |z| < 1,$$

and similarly $\dfrac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$ for $|z| < 1$.

5.4 Laurent Series

Remark. At a point z_0 where f is not analytic, Taylor series may fail; on an annulus $R_1 < |z - z_0| < R_2$ one often has a two-sided power expansion (Laurent series).

Theorem 5.11 (Laurent)**.** *If f is analytic on the annulus $R_1 < |z - z_0| < R_2$ and C is any positively oriented simple closed contour around z_0 lying in that annulus, then on $R_1 < |z - z_0| < R_2$,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

with

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \qquad (n \in \mathbb{Z}).$$

If f is analytic on $|z - z_0| < R_2$ then $b_n = 0$ and Laurent reduces to Taylor.

Example 5.12. Since $e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$ for all z, we get

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}, \qquad 0 < |z| < \infty.$$

The coefficient of (z^{-1}) is 1, hence for any positively oriented simple closed contour C around 0,

$$\int_C e^{1/z}\, dz = 2\pi i.$$

Example 5.13 (Partial fractions across annuli)**.** Let

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}.$$

Three Laurent expansions in powers of z arise:

$$|z| < 1: \quad f(z) = -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - 1\right) z^n,$$

$$1 < |z| < 2: \quad f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}},$$

$$|z| > 2: \quad f(z) = \sum_{n=1}^{\infty} \frac{1 - 2^{n-1}}{z^n}.$$

5.5 Absolute and Uniform Convergence of Power Series

Theorem 5.14 (Absolute convergence inside any interior circle)**.** *If a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges at some $z_1 \neq z_0$, then it converges absolutely for all $|z - z_0| < |z_1 - z_0|$.*

Definition 5.15 (Circle of convergence)**.** The largest open disk centered at z_0 on which the series converges is the **circle of convergence**. Its radius is the **radius of convergence**.

Theorem 5.16 (Uniform convergence on closed interior disks)**.** *If $|z_1 - z_0| = R_1$ lies strictly inside the circle $|z - z_0| = R$, then the series is uniformly convergent on the closed disk $|z - z_0| \leq R_1$.*

5.6 Consequences for Sums of Power/Laurent Series

$\wedge$

Theorem 5.17 (Continuity and analyticity)**.** *Inside the circle of convergence, the sum $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is continuous and analytic.*

Remark (Exterior series). If $\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ converges at $z_1 \neq z_0$, then it converges absolutely to a continuous function on $\{|z - z_0| > |z_1 - z_0|\}$ (the exterior of the circle through z_1).

Remark (Laurent on annuli). If

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

is valid on $R_1 < |z - z_0| < R_2$, then both series converge uniformly on any closed annulus $R_1 + \varepsilon \leq |z - z_0| \leq R_2 - \varepsilon$ ($\varepsilon > 0$).

Theorem 5.18 (Termwise integration on interior contours)**.** *Let C be a contour inside the circle of convergence of $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and f continuous on C. Then*

$$\int_C f(z) \sum_{n=0}^{\infty} a_n (z - z_0)^n \, dz = \sum_{n=0}^{\infty} a_n \int_C f(z)(z - z_0)^n \, dz.$$

Corollary 5.19. *The sum $S(z)$ is analytic inside its circle of convergence and may be integrated term by term on interior contours.*

Example 5.20. Define

$$f(z) = \begin{cases} \dfrac{e^z - 1}{z}, & z \neq 0, \\ 1, & z = 0. \end{cases}$$

Since $e^z - 1 = \sum_{n=1}^{\infty} \dfrac{z^n}{n!}$, we obtain $f(z) = \sum_{n=1}^{\infty} \dfrac{z^{n-1}}{n!}$ for all z with the limit at 0 equal to 1. Thus f is entire and continuous at 0.

Theorem 5.21 (Termwise differentiation)**.** *Inside the circle of convergence,*

$$\frac{d}{dz} \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

Theorem 5.22 (Uniqueness of Taylor/Laurent expansions)**.** *If a power series in $(z - z_0)$ equals $f(z)$ on a disk, it is the Taylor series of f. If a doubly-infinite series $\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$ equals f on an annulus, it is the Laurent expansion of f on that annulus.*

Corollary 5.23 (Cauchy product)**.** *If*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

converge on $|z - z_0| < R$, then

$$f(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) (z - z_0)^n, \qquad |z - z_0| < R.$$

5.7 Exercises

1. Show that the limit of a convergent complex sequence is unique by appealing to the corresponding result for a sequence of real numbers.

Sol. We want to show that

"If a complex sequence $\{z_n\}$ converges to both L and M in $\mathbb{C}$, then $L = M$."

Write $z_n = x_n + iy_n$, $L = a + ib$, $M = c + id$ with $x_n, y_n, a, b, c, d \in \mathbb{R}$. Assume that

$$z_n \longrightarrow L \quad \text{and} \quad z_n \longrightarrow M$$

as $n \longrightarrow \infty$. Taking real and imaginary parts,

$$x_n = \operatorname{Re} z_n \longrightarrow \operatorname{Re} L = a \quad \text{and} \quad x_n = \operatorname{Re} z_n \longrightarrow \operatorname{Re} M = c,$$

$$y_n = \operatorname{Im} z_n \longrightarrow \operatorname{Im} L = b \quad \text{and} \quad y_n = \operatorname{Im} z_n \longrightarrow \operatorname{Im} M = d.$$

By the **uniqueness of limits for real sequences**, these imply $a = c$ and $b = d$. Hence

$$L = a + ib = c + id = M.$$

□

2. Show that

$$\sum_{n=1}^{\infty} z_n = S \implies \sum_{n=1}^{\infty} \overline{z_n} = \overline{S}.$$

Sol. Let $s_N := \sum_{n=1}^{N} z_n$ be the partial sums. By hypothesis $s_N \longrightarrow S$ as $N \longrightarrow \infty$. Consider the conjugated partial sums

$$\overline{s_N} = \overline{\sum_{n=1}^{N} z_n} = \sum_{n=1}^{N} \overline{z_n},$$

so $\{\overline{s_N}\}$ are the partial sums of $\sum_{n=1}^{\infty} \overline{z_n}$. Since complex conjugation is continuous (indeed, an isometry: $|\overline{w} - \overline{z}| = |w - z|$), we have $\overline{s_N} \longrightarrow \overline{S}$. Therefore the series $\sum_{n=1}^{\infty} \overline{z_n}$ converges and

$$\sum_{n=1}^{\infty} \overline{z_n} = \lim_{N \to \infty} \sum_{n=1}^{N} \overline{z_n} = \lim_{N \to \infty} \overline{s_N} = \overline{S}.$$

□

3. Derive the Taylor series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}, \qquad |z-i| < \sqrt{2}.$$

Sol. Note that

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1-\left(\frac{z-i}{1-i}\right)}.$$

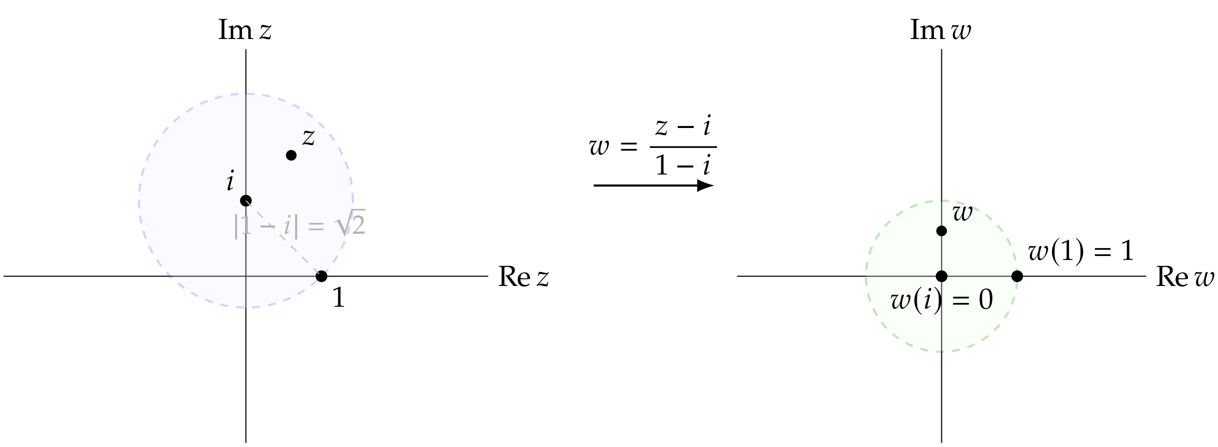
For $\left|\dfrac{z-i}{1-i}\right| < 1$ (i.e. $|z-i| < |1-i| = \sqrt{2}$), expand the geometric series:

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n \quad (|w| < 1), \qquad w = \frac{z-i}{1-i}.$$

Hence

$$\frac{1}{1-z} = \frac{1}{1-i} \sum_{n=0}^{\infty} \left(\frac{z-i}{1-i}\right)^n = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}},$$

which converges for $|z-i| < \sqrt{2}$.



z-plane: expansion about $z = i$ — *w*-plane: geometric series region

□

4. Show that the two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

are

$$\sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1} + \frac{1}{z} \quad (0 < |z| < 1), \qquad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}} \quad (1 < |z| < \infty).$$

Sol. (1) $(0 < |z| < 1)$ Since

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n} \qquad (|z| < 1),$$

we have

$$\begin{aligned}\frac{1}{z(1+z^2)} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n z^{2n} &= \sum_{n=0}^{\infty}(-1)^n z^{2n-1} \\ &= \frac{1}{z} + (-z) + z^3 + (-z^5) + z^7 + \cdots \\ &= \frac{1}{z} + \sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1}.\end{aligned}$$

Therefore the Laurent series on $0 < |z| < 1$ is $\sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1} + \frac{1}{z}$.

(2) $(1 < |z| < \infty)$ Since

$$\frac{1}{1+z^2} = \frac{1}{z^2}\frac{1}{1+z^{-2}} = \frac{1}{z^2}\frac{1}{1-(-z^{-2})} = \frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n z^{-2n} \qquad (|z| > 1),$$

we obtain

$$\begin{aligned}\frac{1}{z(1+z^2)} = \frac{1}{z}\cdot\frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n z^{-2n} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+3}} \\ &= \frac{1}{z^3} + \frac{-1}{z^5} + \frac{1}{z^7} + \frac{-1}{z^9} + \cdots \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}},\end{aligned}$$

Hence the Laurent series is $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}$ on $1 < |z| < \infty$.

□

5. Let $a \in \mathbb{R}$, where $-1 < a < 1$. Then the Laurent series representation $a/(z-a)$ is

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n}, \qquad |a| < |z| < \infty.$$

After writing $z = e^{i\theta}$ in the above equation, equate real parts and then imaginary parts on each side of the result to derive the summation formulas:

$$\sum_{n=1}^{\infty} a^n \cos(n\theta) = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin(n\theta) = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2}.$$

Sol. For $|a| < |z|$, we know that

$$\frac{a}{z-a} = \frac{a}{z}\frac{1}{1-a/z} = \frac{a}{z}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n = \sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^{n+1} = \sum_{n=1}^{\infty}\frac{a^n}{z^n}.$$

Set $z = e^{i\theta}$ (so $|a| < |z| = 1$). Then

$$\frac{a}{e^{i\theta} - a} = \sum_{n=1}^{\infty} a^n e^{-in\theta} = \sum_{n=1}^{\infty} a^n \left(\cos(n\theta) - i\sin(n\theta)\right).$$

Note that

$$\begin{aligned}
\frac{a}{e^{i\theta}-a} = \frac{e^{-i\theta}}{e^{-i\theta}} \cdot \frac{a}{e^{i\theta}-a} = \frac{ae^{-i\theta}}{1-ae^{-i\theta}} = \frac{ae^{-i\theta}(1-ae^{i\theta})}{(1-ae^{i\theta})(1-ae^{-i\theta})} &= \frac{ae^{-i\theta}(1-ae^{i\theta})}{1-a(e^{i\theta}+e^{-i\theta})+a^2e^{i\theta-i\theta}} \\
&= \frac{a\left(e^{-i\theta}-a\right)}{1-2a\cos\theta+a^2} \\
&= \frac{a\left(\cos\theta - i\sin\theta - a\right)}{1-2a\cos\theta+a^2} \\
&= \frac{a(\cos\theta - a) - i\,a\sin\theta}{1-2a\cos\theta+a^2}.
\end{aligned}$$

Thus, we obtain

$$\sum_{n=1}^{\infty} a^n \left(\cos(n\theta) - i\sin(n\theta)\right) = \frac{a}{e^{i\theta}-a} = \frac{a(\cos\theta - a) - i\,a\sin\theta}{1-2a\cos\theta+a^2}.$$

Therefore

$$\boxed{\sum_{n=1}^{\infty} a^n \cos(n\theta) = \frac{a\cos\theta - a^2}{1-2a\cos\theta+a^2}, \qquad \sum_{n=1}^{\infty} a^n \sin(n\theta) = \frac{a\sin\theta}{1-2a\cos\theta+a^2}},$$

valid for $-1 < a < 1$ (indeed $1 - 2a\cos\theta + a^2 = (1-ae^{i\theta})\overline{(1-ae^{i\theta})} = |1-ae^{i\theta}|^2 > 0$). $\square$

6. With the aid of series, show that the function f defined by means of the equations

$$f(z) = \begin{cases} (\sin z)/z & : z \neq 0 \\ 1 & : z = 0 \end{cases}$$

is entire. Use this result to establish the limit

$$\lim_{z \to 0} \frac{\sin z}{z} = 1.$$

Sol. The Maclaurin series of $\sin z$ (entire) is

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} .$$

For $z \neq 0$, divide by z:

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots .$$

This is a power series with infinite radius of convergence, hence defines an entire function

$$F(z) := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}.$$

Note that $F(0) = 1$, and for $z \neq 0$ we have $F(z) = \sin z/z$. Therefore $f \equiv F$ on $\mathbb{C}$; in particular, f is entire (the singularity at 0 is removable). By continuity of F at 0,

$$\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} F(z) = F(0) = 1.$$

□

6 Residues and Poles

6.1 Isolated Singular Points

Singular and isolated singular points

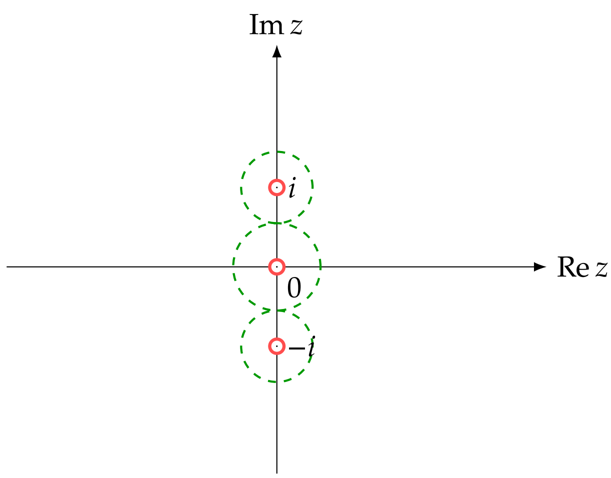
Definition 6.1.

- A point $z_0 \in \mathbb{C}$ is a **singular point** of a function f if f fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0.
- A singular point $z_0 \in \mathbb{C}$ is said to be **isolated** if there exists $\varepsilon > 0$ such that f is analytic on the punctured disk (deleted neighborhood) $0 < |z - z_0| < \varepsilon$.

Example 6.2. The function

$$\frac{z+1}{z^3(z^2+1)} = \frac{z+1}{z^3(z+i)(z-i)}$$

has three isolated singular points at $z = 0$ and $z = \pm i$.



Example 6.3. The principal branch of

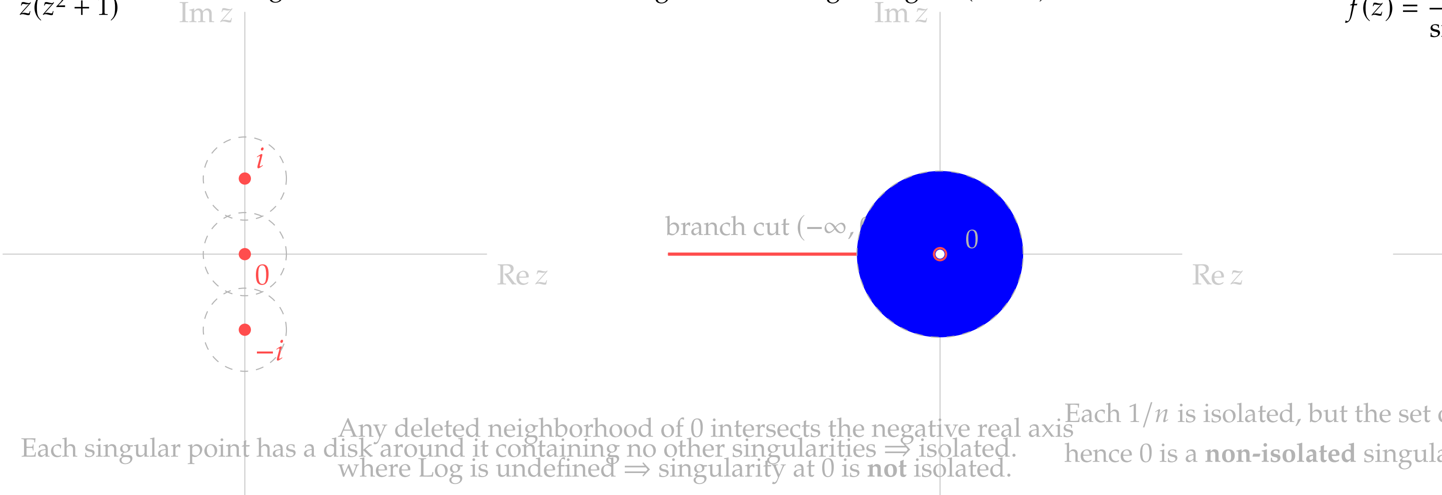
$$\mathrm{Log} z = \ln r + i\theta \qquad (r > 0,\ -\pi < \theta < \pi)$$

has a singularity at 0 that is **not** isolated because any deleted neighborhood intersects the negative real axis where the branch is undefined. Also, $f(z) = \dfrac{1}{\sin(\pi/z)}$ has singularities at 0 and $z = 1/n$ $(n = \pm 1, \pm 2, \dots)$; each $1/n$ is isolated, but 0 is not.

$\dfrac{z+1}{z(z^2+1)}$: isolated singularities at $0,\ \pm i$

$\operatorname{Log} z = \ln r + i \operatorname{Arg} z,\ \operatorname{Arg} z \in (-\pi, \pi)$: non-isolated at 0

$f(z) = $



6.2 Exercises

1. Use Cauchy's residue theorem to evaluate integral of each these functions around the circle $|z| = 3$ in the positive sense:

$$\frac{e^{-z}}{z^2}, \qquad \frac{e^{-z}}{(z-1)^2}, \qquad z^2 \exp\left(\frac{1}{z}\right), \qquad \frac{z+1}{z^2-2z}.$$

(Answers: $-2\pi i$, $-2\pi i/e$, $\pi i/3$, $2\pi i$.)

Sol. All integrals are $\displaystyle\int_{|z|=3} (\cdot)\, dz$ with positive orientation.

(1) $\displaystyle\int_{|z|=3} \frac{e^{-z}}{z^2}\,\mathrm{d}z$. Let $f(z) = e^{-z}$. Then f is entire (analytic everywhere), and the only singularity of the integrand inside $|z| = 3$ is a pole of order 2 at $z = 0$. By Cauchy's integral formula for the first derivative,

$$f'(z_0) = \frac{1}{2\pi i}\int_{|z|=3} \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z. \implies \int_{|z|=3} \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z = 2\pi i\, f'(z_0).$$

With $z_0 = 0$,

$$\int_{|z|=3} \frac{e^{-z}}{z^2}\,\mathrm{d}z = 2\pi i\, f'(0) = 2\pi i \cdot \frac{\mathrm{d}}{\mathrm{d}z} e^{-z}\Big|_{z=0} = 2\pi i \cdot -e^{-z}\Big|_{z=0} = 2\pi i \cdot (-1) = -2\pi i.$$

(2) $\displaystyle\int \frac{e^{-z}}{(z-1)^2}\,\mathrm{d}z$. Let $f(z) = e^{-z}$, which is entire. The integrand has a pole of order 2 at $z = 1$. Since $|1| < 3$, this singularity lies inside the circle $|z| = 3$, and there are no other singularities inside the contour. By Cauchy's integral formula for the first derivative, with $z_0 = 1$,

$$\int_{|z|=3} \frac{f(z)}{(z-1)^2}\,\mathrm{d}z = 2\pi i \cdot f'(1) = 2\pi i \cdot (-e^{-z})\Big|_{z=1} = 2\pi i \cdot \left(-\frac{1}{e}\right) = \frac{-2\pi i}{e}.$$

(3) $\displaystyle\int_{|z|=3} z^2 \exp\left(\frac{1}{z}\right)\,\mathrm{d}z$. The only singularity of the integrand is at $z = 0$, due to the factor $e^{1/z}$. This is an essential singularity at $z = 0$, which lies inside the contour $|z| = 3$. We use the residue theorem:

$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k).$$

Here there is only one singularity at $z = 0$, so

$$\int_{|z|=3} z^2 \exp\left(\frac{1}{z}\right)\,\mathrm{d}z = 2\pi i\, \operatorname{Res}\left(z^2 e^{1/z}, 0\right).$$

To find the residue, expand $\exp\left(\frac{1}{z}\right)$ in a Laurent series around $z = 0$:

$$\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots,$$

$$z^2 \exp\left(\frac{1}{z}\right) = z^2\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right)$$

$$= z^2 + z + \frac{1}{2!} + \frac{1}{3!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^2} + \cdots.$$

The residue at $z = 0$ is $\mathrm{Res}\left(z^2 e^{1/z}, 0\right) = \frac{1}{3!} = \frac{1}{6}$. Therefore

$$\int_{|z|=3} z^2 \exp\left(\frac{1}{z}\right) \mathrm{d}z = 2\pi i \cdot \frac{1}{6} = \frac{\pi i}{3}.$$

(4) $\displaystyle\int_{|z|=3} \frac{z+1}{z^2-2z}\,\mathrm{d}z = \int \frac{z+1}{z(z-2)}\,\mathrm{d}z$. The singularities are simple poles at $z = 0$ and $z = 2$. We can either compute residues directly or use partial fraction decomposition:

$$\begin{aligned} \frac{z+1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2} &\implies z+1 = A(z-2) + Bz \\ &\implies z+1 = (A+B)z - 2A \\ &\implies \begin{cases} A + B = 1, \\ -2A = 1. \end{cases} \\ &\implies A = -1/2 \quad \text{and} \quad B = 3/2. \end{aligned}$$

Thus

$$\frac{z+1}{z^2-2z} = -\frac{1}{2z} + \frac{3}{2(z-2)}.$$

Now the integral becomes

$$\int \frac{z+1}{z(z-2)}\,\mathrm{d}z = \int_{|z|=3}\left(-\frac{1}{2z} + \frac{3}{2(z-2)}\right) dz = -\frac{1}{2}\int_{|z|=3} \frac{1}{z}\,dz + \frac{3}{2}\int_{|z|=3} \frac{1}{z-2}\,dz.$$

Note that

$$\int_{|z|=3} \frac{1}{z}\,dz = 2\pi i \quad \text{and} \quad \int_{|z|=3} \frac{1}{z-2}\,dz = 2\pi i,$$

since both $z = 0$ and $z = 2$ lie inside $|z| = 3$. Therefore

$$\int \frac{z+1}{z(z-2)}\,\mathrm{d}z = -\frac{1}{2}\cdot 2\pi i + \frac{3}{2}\cdot 2\pi i = -\pi i + 3\pi i = 2\pi i.$$

$\square$

2. Show that the singular point of each of the following functions is a pole.

$$f(z) = \frac{1-\cosh z}{z^3}, \quad g(z) = \frac{1-\exp(2z)}{z^4}, \quad h(z) = \frac{\exp(2z)}{(z-1)^2}.$$

Determine the order m of that pole and the corresponding residue B.

(Answers: $f(z)$: $m = 1$, $B = -1/2$; $g(z)$: $m = 3$, $B = -4/3$; $h(z)$: $m = 2$, $B = 2e^2$.)

Sol.

(1) $f(z) = \dfrac{1-\cosh z}{z^3}$ at $z = 0$. Note that

$$\begin{aligned}
\cosh z &= 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \frac{1}{6!}z^6 + \cdots, \\
1 - \cosh z &= -\frac{1}{2}z^2 - \frac{1}{4!}z^4 - \frac{1}{6!}z^6 - \cdots, \\
\frac{1-\cosh z}{z^3} &= -\frac{1}{2}\frac{1}{z} - \frac{1}{4!}z + \frac{1}{6!}z^3 + \cdots.
\end{aligned}$$

Hence the singularity is a **simple pole** ($m = 1$) with residue $B = \operatorname{Res}_{z=0} f = -\frac{1}{2}$.

(2) $g(z) = \dfrac{1-\exp(2z)}{z^4}$ at $z = 0$. Note that

$$\begin{aligned}
\exp(2z) &= 1 + 2z + \frac{(2z)^2}{2!} + \frac{(2z)^3}{3!} + \frac{(2z)^4}{4!} + \cdots \\
&= 1 + 2z + 2z^2 + \frac{4}{3}z^3 + \frac{2}{3}z^4 + \cdots, \\
1 - \exp(2z) &= -\left(2z + 2z^2 + \frac{4}{3}z^3 + \frac{2}{3}z^4 + \cdots\right), \\
\frac{1-\exp(2z)}{z^4} &= -\frac{2}{z^3} - \frac{2}{z^2} - \frac{4}{3}\frac{1}{z} - \frac{2}{3} + \cdots.
\end{aligned}$$

Thus we have a pole of order 3 ($m = 3$) with residue $B = \operatorname{Res}_{z=0} g = -\frac{4}{3}$.

(3) $h(z) = \dfrac{\exp(2)}{(z-1)^2}$ at $z = 1$. Let $w := z - 1$, i.e., $z = 1 + w$. Then

$$\begin{aligned}
\exp(2z) &= \exp\big(2(1+w)\big) = \exp(2)\exp(2w) \\
&= \exp(2)\left(1 + 2w + \frac{2^2}{2!}w^2 + \frac{2^3}{3!}w^3 + \cdots\right), \\
\frac{\exp(2z)}{(z-1)^2} &= \frac{\exp(2)}{w^2}\left(1 + 2w + \frac{2^2}{2!}w^2 + \cdots\right) = \exp(2)\left(\frac{1}{w^2} + \frac{2}{w} + \frac{2^2}{2!} + \cdots\right).
\end{aligned}$$

Therefore the singularity is a pole of order 2 ($m = 2$) with residue $B = \operatorname{Res}_{z=1} h = 2\exp(2)$.

□

3. Show that

$$\mathrm{Res}_{z=-1}\frac{z^{1/4}}{z+1}=\frac{1+i}{\sqrt{2}} \qquad (|z|>0,\ 0<\arg z<2\pi),$$
$$\mathrm{Res}_{z=i}\frac{\mathrm{Log}\,z}{(z^2+1)^2}=\frac{\pi+2i}{8},$$
$$\mathrm{Res}_{z=i}\frac{z^{1/2}}{(z^2+1)^2}=\frac{1-i}{8\sqrt{2}} \qquad (|z|>0,\ 0<\arg z<2\pi).$$

Sol.

(1) **(Residue of $f_1(z)=\dfrac{z^{1/4}}{z+1}$ at $z=-1$)** We work with the branch

$$|z|>0, \quad 0<\arg z<2\pi,$$

so the branch cut is along the positive real axis, and

$$z^{1/4}=\exp\left(\frac{1}{4}\mathrm{Log}z\right), \qquad \mathrm{Log}\,z=\ln|z|+i\arg z, \quad 0<\arg z<2\pi.$$

At $z=-1$ we have $|-1|=1$ and $\arg(-1)=\pi$, hence

$$\mathrm{Log}(-1)=\ln 1+i\pi=i\pi,$$

and therefore

$$z^{1/4}\big|_{z=-1}=(-1)^{1/4}=\exp\left(i\frac{\pi}{4}\right)=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}=\frac{1+i}{\sqrt{2}}.$$

The integrand $f_1(z)=\dfrac{z^{1/4}}{z+1}$ has a simple pole at $z=-1$, and $z^{1/4}$ is analytic at $z=-1$ on this branch. Consider

$$g(z):=(z+1)\,f_1(z)=(z+1)\,\frac{z^{1/4}}{z+1}=z^{1/4}.$$

Since $z^{1/4}$ is analytic at $z=-1$, the function $(z+1)f_1(z)$ is analytic at $z=-1$. Therefore it has a Taylor expansion around $z=-1$:

$$(z+1)f_1(z)=z^{1/4}=a_0+a_1(z+1)+a_2(z+1)^2+\cdots, \quad \text{with} \quad a_n=\frac{g^{(n)}(-1)}{n!}\ (n=0,1,2,\dots),$$

valid for z near -1. Dividing both sides by $(z+1)$, we get a Laurent expansion for f_1 at $z=-1$:

$$f_1(z)=\frac{a_0}{z+1}+a_1+a_2(z+1)+\cdots.$$

By definition, the residue of f_1 at $z = -1$ is $\mathrm{Res}_{z=-1} f_1(z) = a_0$. Thus

$$\mathrm{Res}_{z=-1} \frac{z^{1/4}}{z+1} = a_0 = \frac{g^{(0)}(-1)}{0!} = \lim_{z\to-1}(z+1)\frac{z^{1/4}}{z+1} = (-1)^{1/4} = \frac{1+i}{\sqrt{2}}.$$

(2) **(Residue of $f_2(z) = \dfrac{\mathrm{Log} z}{(z^2+1)^2}$ at $z = i$)** We factor $z^2+1 = (z-i)(z+i)$, so near $z = i$,

$$(z^2+1)^2 = (z-i)^2(z+i)^2.$$

Hence

$$f_2(z) = \frac{\mathrm{Log} z}{(z-i)^2(z+i)^2} = \frac{g(z)}{(z-i)^2}, \qquad g(z) := \frac{\mathrm{Log} z}{(z+i)^2}.$$

The function g is analytic at $z = i$. Thus $z = i$ is a double pole of f of the form $f_2(z) = g(z)/(z-i)^2$ with g analytic at i. Then

$$\mathrm{Res}_{z=i} f_2(z) = \mathrm{Res}_{z=i} \frac{g(z)}{(z-i)^2} = \frac{g^{(1)}(i)}{1!} = g'(i).$$

Compute $g'(i)$:

$$\begin{aligned}
g'(i) = \frac{d}{dz} g(z)\Big|_{z=i} &= \frac{d}{dz}\left((\mathrm{Log} z)(z+i)^{-2}\right)\Big|_{z=i} \\
&= \left[\frac{1}{z}(z+i)^{-2} - 2\mathrm{Log} z\,(z+i)^{-3}\right]_{z=i} \\
&= \frac{1}{i}\cdot\frac{1}{-4} - 2\mathrm{Log}(i)\cdot\frac{1}{-8i} \\
&= \frac{-1}{4i} + \frac{1}{4i}\cdot(\ln|i| + i\arg(i)) \\
&= \frac{i}{4} + \frac{-i}{4}\cdot\left(0 + \frac{\pi i}{2}\right) \\
&= \frac{2i}{8} + \frac{\pi}{8} \\
&= \frac{\pi + 2i}{8}.
\end{aligned}$$

(3) **(Residue of $f_3(z) = \dfrac{z^{1/2}}{(z^2+1)^2}$ at $z = i$)** As before, $(z^2+1)^2 = (z-i)^2(z+i)^2$, so

$$f_3(z) = \frac{z^{1/2}}{(z-i)^2(z+i)^2} = \frac{h(z)}{(z-i)^2}, \qquad h(z) := \frac{z^{1/2}}{(z+i)^2},$$

and $h(z)$ is analytic at $z = i$. Thus $z = i$ is again a double pole of f_3, and $\mathrm{Res}_{z=i} f(z) = h'(i)$.

Compute $h'(i)$:

$$\begin{aligned} h'(i) = \frac{d}{dz}h(z)\Big|_{z=i} &= \frac{d}{dz}\left(z^{1/2}(z+i)^{-2}\right)\Big|_{z=i} \\ &= \left[\frac{1}{2}z^{-1/2}(z+i)^{-2} - 2z^{1/2}(z+i)^{-3}\right]_{z=i} \\ &= \frac{1}{2}\cdot i^{-1/2}\cdot\frac{1}{-4} - 2\cdot i^{1/2}\cdot\frac{1}{-8i}. \end{aligned}$$

We need the branch values of $z^{1/2}$ and $z^{-1/2}$ at $z = i$ for $0 < \arg z < 2\pi$:

$$i^{1/2} = \exp\left(\frac{1}{2}\mathrm{Log}\, i\right) = \exp\left(\frac{1}{2}(\ln|i| + i\arg(i))\right) = \exp\left(\frac{1}{2}\left(\frac{\pi i}{2}\right)\right) = \exp(i\pi/4) = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1+i}{\sqrt{2}},$$

$$i^{-1/2} = \frac{1}{i^{1/2}} = \frac{1}{\frac{1+i}{\sqrt{2}}} = \frac{\sqrt{2}}{1+i} = \frac{\sqrt{2}(1-i)}{(1+i)(1-i)} = \frac{\sqrt{2}(1-i)}{2} = \frac{1-i}{\sqrt{2}}.$$

Therefore

$$\begin{aligned} h'(i) &= \frac{1}{2}\cdot i^{-1/2}\cdot\frac{1}{-4} - 2\cdot i^{1/2}\cdot\frac{1}{-8i} \\ &= \frac{-1}{8}\left(\frac{1-i}{\sqrt{2}}\right) + \frac{-i}{4}\left(\frac{1+i}{\sqrt{2}}\right) \\ &= \frac{-(1-i) + 2(1-i)}{8\sqrt{2}} \\ &= \frac{(1-i)}{8\sqrt{2}} \end{aligned}$$

Thus

$$\mathrm{Res}_{z=i}\frac{z^{1/2}}{(z^2+1)^2} = h'(i) = \frac{1-i}{8\sqrt{2}}.$$

□

4. Find the value of the integral

$$\int_{|z|=3} \frac{z^3\, e^{1/z}}{1+z^3}\, dz$$

taken CCW around the circle $|z| = 3$.

(Answer: $2\pi i$.)

Sol. Let

$$f(z) = \frac{z^3 e^{1/z}}{1+z^3}.$$

Set

$$w = \frac{1}{z} \quad \Longrightarrow \quad z = \frac{1}{w}, \quad dz = -\frac{1}{w^2}\, dw.$$

The circle $|z| = 3$ corresponds to $|w| = 1/3$. As z runs CCW, $w = 1/z$ runs clockwise. Therefore,

$$\begin{aligned}
\int_{|z|=3} f(z)\, dz &= \underbrace{\int_{|w|=1/3} f\left(\frac{1}{w}\right)\left(-\frac{1}{w^2}\, dw\right)}_{\text{CW}} \\
&= -\underbrace{\int_{|w|=1/3} f\left(\frac{1}{w}\right)\left(-\frac{1}{w^2}\, dw\right)}_{\text{CCW}} \\
&= \int_{|w|=1/3} \frac{f(1/w)}{w^2}\, dw.
\end{aligned}$$

Since

$$f\left(\frac{1}{w}\right) = \frac{(1/w)^3 e^w}{1+(1/w)^3} = \frac{\frac{e^w}{w^3}}{\frac{w^3+1}{w^3}} = \frac{e^w}{w^3+1},$$

we have

$$\int_{|z|=3} f(z)\, dz = \int_{|w|=1/3} \left(\frac{1}{w^2} \cdot f\left(\frac{1}{w}\right)\right) dw = \int_{|w|=1/3} \frac{e^w}{w^2(w^3+1)}\, dw.$$

The only singularity of

$$h(w) = \frac{e^w}{w^2(w^3+1)}$$

inside the circle $|w| = 1/3$ is at $w = 0$, since the roots of $w^3 + 1 = 0$ are

$$w = -1, \quad w = e^{i\pi/3}, \quad w = e^{-i\pi/3},$$

all of which satisfy $|w| = 1$. Hence, by the residue theorem,

$$\int_{|z|=3} f(z)\, dz = 2\pi i \ \mathrm{Res}_{w=0}\, h(w).$$

We compute the residue via series expansion. Write

$$h(w) = \frac{1}{w^2} \cdot \frac{e^w}{1+w^3}.$$

Since

$$\frac{1}{1+w^3} = 1 - w^3 + w^6 - w^9 + \cdots \quad (|w| < 1) \quad \text{and}$$
$$e^w = 1 + w + \frac{w^2}{2} + \frac{w^3}{6} + \cdots,$$

we have

$$\frac{e^w}{1+w^3} = (1 + w + \frac{w^2}{2} + \frac{w^3}{6} + \cdots)(1 - w^3 + w^6 - \cdots),$$

and so

$$\begin{aligned}
h(w) &= \frac{1}{w^2} \cdot \left(\left(1 + w + \frac{w^2}{2} + \frac{w^3}{6} + \cdots \right)(1 - w^3 + w^6 - \cdots) \right) \\
&= \frac{1}{w^2} \left(\left(1 + w + \frac{w^2}{2} + \frac{w^3}{6} + \cdots \right) - \left(w^3 + w^4 + \frac{w^5}{2} + \frac{w^6}{6} + \cdots \right) + \left(w^6 + w^7 + \frac{w^8}{2} + \frac{w^9}{6} + \cdots \right) - \cdots \right) \\
&= \left(\frac{1}{w^2} + \frac{1}{w} + \frac{1}{2} + \frac{w}{6} + \cdots \right) - \left(w^1 + w^2 + \frac{w^3}{2} + \frac{w^4}{6} + \cdots \right) + \left(w^4 + w^5 + \frac{w^6}{2} + \frac{w^7}{6} + \cdots \right) - \cdots.
\end{aligned}$$

Thus $\operatorname{Res}_{w=0} h(w) = 1$. By the residue theorem,

$$\int_{|z|=3} \frac{z^3 e^{1/z}}{1+z^3}\, dz = 2\pi i \operatorname{Res}_{w=0} h(w) = 2\pi i \cdot 1 = 2\pi i.$$

□

5. Let C denote the positively oriented circle $|z| = 2$. Show that

$$\int_C \tan z\, dz = -4\pi i \quad \text{and} \quad \int_C \frac{dz}{\sinh 2z} = -\pi i.$$

Sol.

(1) We have $\tan z = \dfrac{\sin z}{\cos z}$. The poles of $\tan z$ are the zeros of $\cos z$, i.e.,

$$\cos z = 0 \iff z = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}.$$

We need those with $|z| < 2$. Numerically,

$$\left|\frac{\pi}{2}\right| \approx 1.57 < 2, \quad \left|-\frac{\pi}{2}\right| \approx 1.57 < 2,$$

while $\left|\frac{3\pi}{2}\right| > 2$, etc. So the poles inside $|z| = 2$ are $z = \pm\frac{\pi}{2}$. These are simple poles. For a simple pole of a quotient f/g with $g(z_0) = 0$, $g'(z_0) \neq 0$,

$$\text{Res}_{z=z_0} \tan z = \text{Res}_{z=z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} (z - z_0)\frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f(z)}{\frac{g(z)}{(z-z_0)}} = \frac{f(z_0)}{g'(z_0)}.$$

Here $f(z) = \sin z$, $g(z) = \cos z$, so $g'(z) = -\sin z$. Then

$$\text{Res}_{z=\frac{\pi}{2}} \tan z = \frac{\sin(\pi/2)}{\cos'(\pi/2)} = \frac{1}{-\sin(\pi/2)} = \frac{1}{-1} = -1$$
$$\text{Res}_{z=-\frac{\pi}{2}} \tan z = \frac{\sin(-\pi/2)}{\cos'(-\pi/2)} = \frac{-1}{-\sin(-\pi/2)} = \frac{-1}{1} = -1.$$

So the sum of residues inside C is

$$\text{Res}_{z=\pi/2} \tan z + \text{Res}_{z=-\pi/2} \tan z = -1 + (-1) = -2.$$

By the residue theorem,

$$\int_C \tan z\, dz = 2\pi i \sum \text{Res}(\tan z) = 2\pi i \cdot (-2) = -4\pi i.$$

(2) Now consider $\displaystyle\int_C \frac{dz}{\sinh 2z}$. The poles occur where

$$\sinh 2z = 0.$$

Recall $\sinh w = 0 \iff w = n\pi i, n \in \mathbb{Z}$. Thus

$$\sinh 2z = 0 \iff 2z = n\pi i \iff z = \frac{n\pi i}{2}, \quad n \in \mathbb{Z}.$$

We need those with $|z| < 2$. We have

$$\left|\frac{n\pi i}{2}\right| = \frac{|n|\pi}{2} < 2 \implies |n| < \frac{4}{\pi} \approx 1.27,$$

so the only possibilities are $n = 0, \pm 1$. Hence the poles inside $|z| = 2$ are

$$z = 0, \quad z = \pm\frac{\pi i}{2}.$$

These are simple zeros of $\sinh 2z$, since $\frac{d}{dz}(\sinh 2z) = 2\cosh 2z$ and $\cosh 2z \neq 0$ at these points. We write

$$f(z) = \frac{1}{\sinh 2z} = \frac{1}{h(z)}, \quad h(z) = \sinh 2z.$$

For a simple zero $h(z_0) = 0$ with $h'(z_0) \neq 0$, the residue of $1/h(z)$ is

$$\operatorname{Res}_{z=z_0} \frac{1}{h(z)} = \frac{1}{h'(z_0)}.$$

Here $h'(z) = 2\cosh 2z$, so

$$\operatorname{Res}_{z=z_0} \frac{1}{\sinh 2z} = \frac{1}{2\cosh 2z_0}.$$

Residue at $z = 0$

$$\operatorname{Res}_{z=0} \frac{1}{\sinh 2z} = \frac{1}{2\cosh 0} = \frac{1}{2 \cdot 1} = \frac{1}{2}.$$

Residue at $z = \frac{\pi i}{2}$

Here $2z = \pi i$:

$$\cosh(\pi i) = \cos\pi = -1,$$

so

$$\operatorname{Res}_{z=\frac{\pi i}{2}} \frac{1}{\sinh 2z} = \frac{1}{2\cosh \pi i} = \frac{1}{2(-1)} = -\frac{1}{2}.$$

Residue at $z = -\frac{\pi i}{2}$

Here $2z = -\pi i$:

$$\cosh(-\pi i) = \cosh(\pi i) = \cos\pi = -1,$$

so

$$\mathrm{Res}_{z=-\frac{\pi i}{2}} \frac{1}{\sinh 2z} = \frac{1}{2\cosh(-\pi i)} = \frac{1}{2(-1)} = -\frac{1}{2}.$$

Sum of residues and the integral

The sum of residues inside $|z| = 2$ is

$$\frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -\frac{1}{2}.$$

By the residue theorem,

$$\int_C \frac{dz}{\sinh 2z} = 2\pi i \sum \mathrm{Res}\left(\frac{1}{\sinh 2z}\right) = 2\pi i \cdot \left(-\frac{1}{2}\right) = -\pi i.$$

$$\boxed{\int_C \frac{dz}{\sinh 2z} = -\pi i.}$$

□