Advanced Calculus III

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We cover the following topics in this note.

- Limit of a Function
- Continuity of a Function
- TBA

Limit Point (Metric Space)

Definition. Let (X, d) be a metric space. Let $S \subseteq X$ and $\alpha \in X$. A point $p \in X$ is a **limit point** of S if and only if

$$\forall \varepsilon > 0,\ B_\varepsilon(\alpha) \cap (S \setminus \{p\}) \neq \varnothing.$$

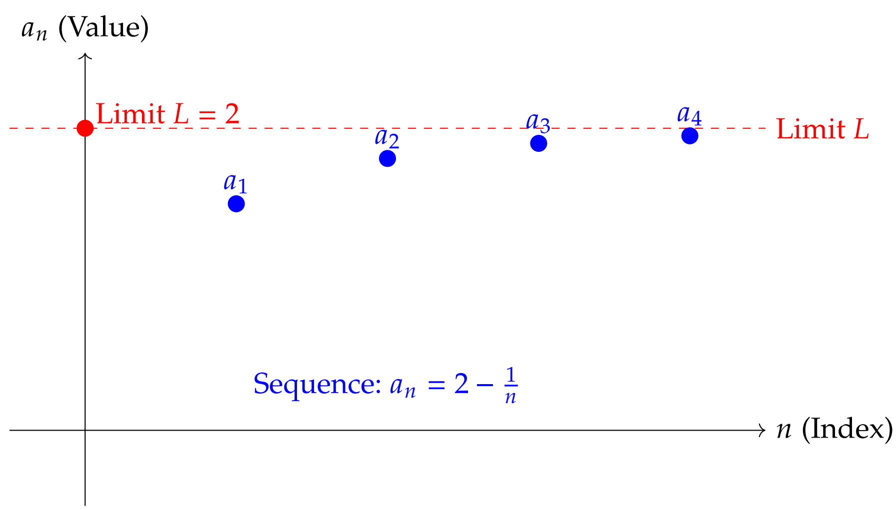
That is,

$$\forall \varepsilon > 0,\ \{x \in S : 0 < d(x, p) < \varepsilon\} \neq \varnothing.$$

Remark 1. Note that α does not have to be an element of A to be a limit point.

Note (Limit Point (Topology)). Let (X, τ) be a topological space. For a subset $S \subseteq X$. A point $p \in X$ is a limit point of S if and only if

$$\forall U \in \tau \text{ with } p \in U,\ U \cap (S \setminus \{p\}) \neq \varnothing.$$



Example 1.

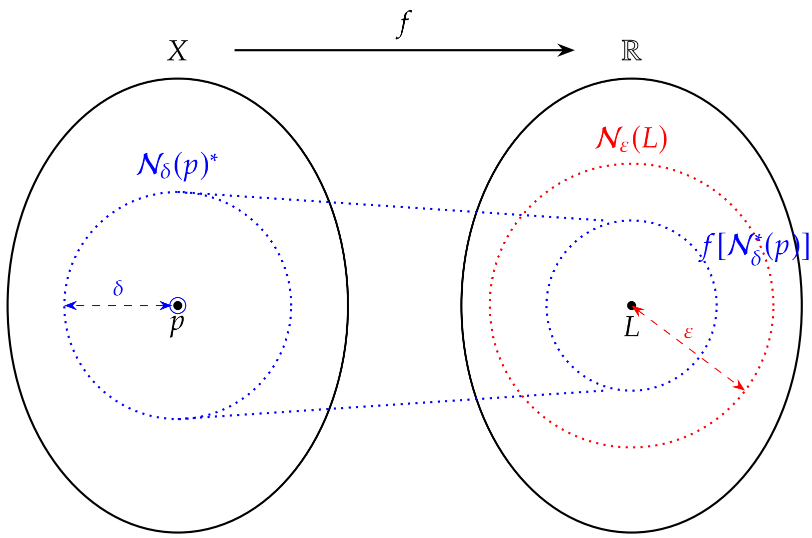
★ Limit of a Function ($\epsilon - \delta$) ★

Definition. Let $f : X \to \mathbb{R}$ be a function defined on a subset $X \subseteq \mathbb{R}$ of a topological space, and let $p \in X$ be a limit point of X. We say that $L \in \mathbb{R}$ is the **limit of the function** f **as** x **approaches** p if

$$\boxed{\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x \in X,\ 0 < |x - p| < \delta \implies |f(x) - L| < \varepsilon}.$$

We write

$$\boxed{\lim_{x \to p} f(x) = L}.$$



Remark 2.

$$\lim_{x\to p} f(x) \neq L \iff \exists \varepsilon > 0 : [\forall \delta > 0 : \exists x \in X : 0 < |x-p| < \delta \text{ but } |f(x) - L| > 0].$$

Continuity of a Function

Definition. Let $f : X \to \mathbb{R}$ be a function defined on a subset $X \subseteq \mathbb{R}$ of a topological space, and let $p \in X$. The function f is said to be f is **continuous at** p if and only if

$$\lim_{x\to p} f(x) = f(p).$$

That is,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \left|x - p\right| < \delta \implies |f(x) - f(p)| < \varepsilon.$$

Remark 3 (Continuity of a Set)**.** The function f is continuous on subset $S \subseteq X$ if it it continuous at every point $p \in S$.

Remark 4 (Continuity in a Topological Space)**.** Let (X, τ_X) and (Y, τ_Y) are topological spaces. $f : X \to Y$ is **continuous** if and only if

$$U_Y \in \tau_Y \implies f^{-1}[U_Y] \in \tau_X,$$

where $f^{-1}[U_Y] = \left\{x \in X : f(x) \in U_Y\right\}$ is the preimage of U_Y under f.

Note. $[p \Rightarrow (q \Rightarrow r)] \equiv [p \Rightarrow (\neg q \vee r)] \equiv [\neg p \vee (\neg q \vee r)] \equiv [\neg(p \wedge q) \vee r] \equiv [(p \wedge q) \Rightarrow r]$.

Limit of Function by Convergent Sequences

Theorem. *Let $f : X \to \mathbb{R}$ be a function defined on a subset $\varnothing \neq X \subseteq \mathbb{R}$ of a topological space, and let p is a limit point of X. Then*

$$\lim_{x \to p} f(x) = L \iff \left[\forall \{x_n\} \subseteq X \setminus \{p\}, \left(\lim_{n\to\infty} x_n = p \implies \lim_{n\to\infty} f(x_n) = L \right) \right].$$

Proof. $(\Rightarrow)$ Suppose that $\lim_{x\to p} f(x) = L$. Let $\{x_n\} \subseteq X \setminus \{p\}$ be a sequence, and let $\lim_{n\to\infty} x_n = p$. We NTS that

$$\lim_{n\to\infty} f(x_n) = L, \quad \text{i.e.,} \quad \forall \varepsilon > 0 : \exists N \in \mathbb{N} : n \geq N \Rightarrow |f(x_n) - L| < \varepsilon.$$

Let $\varepsilon > 0$. Since $\lim_{x\to p} f(x) = L$, we know

$$\exists \delta > 0 \text{ such that } 0 < |x - p| < \delta \implies |f(x) - L| < \varepsilon. \tag{*}$$

Since $\lim_{n\to\infty} x_n = p$, we obtain

$$\exists N \in \mathbb{N} \text{ such that } n \geq N \implies |x_n - p| < \delta.$$

Thus, if $n \geq N$ then,

$$\begin{aligned} |x_n - p| < \delta &\implies 0 < |x_n - p| < \delta \quad \because x_n \neq p \\ &\implies |f(x_n) - L| < \varepsilon \quad \text{by (*)} \end{aligned}$$

Thus, $\lim_{n\to\infty} f(x_n) = L$.

$(\Leftarrow)$ Let the RHS holds. Assume, for the contradiction, that $\lim_{x\to p} f(x) \neq L$, i.e.,

$$\exists \varepsilon > 0 : \forall \delta > 0 : \exists x_\delta \in X : 0 < |x_\delta - p| < \delta \text{ but } |f(x_\delta) - L| \geq \varepsilon.$$

Take $\delta = 1/n$. Then

$$\exists x_n \in X \text{ such that } 0 < |x_n - p| < \delta \text{ but } |f(x_n) - L| \geq \varepsilon.$$

~~(Axiom of Countable Choice)~~ This means that

$$\forall n \in \mathbb{N} : \exists \{x_n\} \subseteq X \setminus \{p\} \text{ such that } 0 < |x_n - p| < \frac{1}{n} \text{ but } |f(x_n) - L| \geq \varepsilon.$$

By Squeeze Theorem, we have $\lim_{n\to\infty} x_n = p$ since $0 < |x_n - p| < 1/n$. Since the RHS holds, we know $\lim_{n\to\infty} f(x_n) = L$. Then, for some $\varepsilon > 0$,

$$\exists N \in \mathbb{N} \text{ such that } n \geq N \implies \left|f(x_n) - L\right| < \varepsilon \lightning.$$

□

Continuity of Function by Convergent Sequences

Corollary. *Let $f : X \to \mathbb{R}$ be a function defined on a subset $\varnothing \neq X \subseteq \mathbb{R}$ of a topological space, and let p is a limit point of X. Then*

$$\lim_{x\to p} f(x) = f(p) \iff \left[\forall \{x_n\} \subseteq X, \left(\lim_{n\to\infty} x_n = p \implies \lim_{n\to\infty} f(x_n) = f(p)\right)\right].$$

Proof. □

Sandwitch Theorem; Squeeze Theorem

Theorem.

Proof. content... □

Monotone Convergence Theorem (MCT)

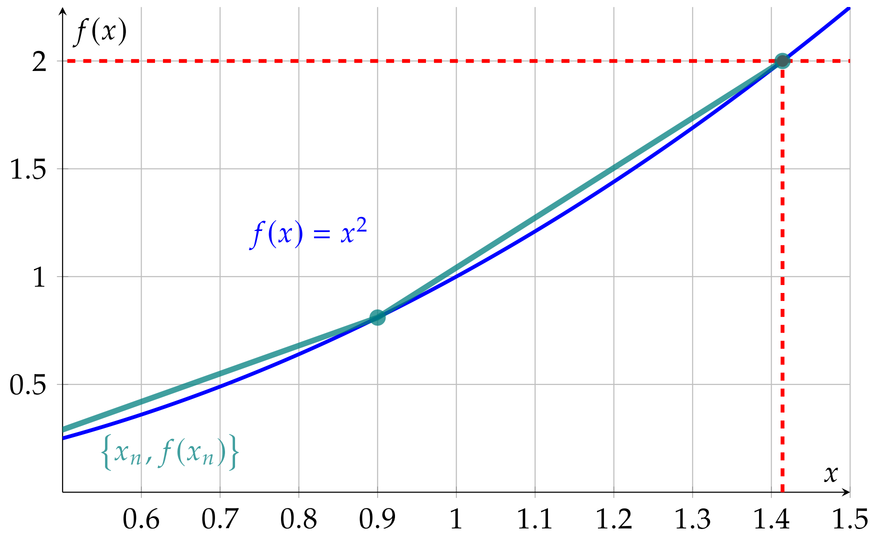
Theorem. *TBA*

Proof. TBA □

Nested Interval Property (NIP)

Theorem. *TBA*

Proof. TBA □



References

[1] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 10. 해석학 개론 (e) 엡실론-델타와 수열의 수렴성" YouTube Video, 25:57. Published September 29, 2019. URL: `https://youtu.be/2Ml3G_Duffk?si=qo-CVgW3Ukd4ADRL`.