# Why $\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z)$ via Differential Forms and Integrals

## 1 Setup: $\mathbb{CP}^1$ as the Riemann sphere

We identify the complex projective line with the Riemann sphere:

$$\mathbb{CP}^1 \cong \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}.$$

On $\mathbb{C} \subset \widehat{\mathbb{C}}$ we use the usual complex coordinate $z$. The point $\infty$ is the "point at infinity".

A *meromorphic function* on $\mathbb{CP}^1$ is then a function

$$f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

that is holomorphic on $\widehat{\mathbb{C}}$ except for isolated poles (including possibly a pole at $\infty$).

Our goal is to prove:

$$\mathcal{M}(\mathbb{CP}^1) = \{\text{meromorphic functions on } \mathbb{CP}^1\} \cong \mathbb{C}(z),$$

that is, every meromorphic $f$ on $\mathbb{CP}^1$ is a rational function in the coordinate $z$.

We will base the proof on:

- meromorphic 1-forms $\omega = f(z)\,dz$,
- residues and contour integrals,
- Laurent expansions defined by integrals.

## 2 Meromorphic 1-forms and residues

Let $f$ be a meromorphic function on $\widehat{\mathbb{C}}$, and consider the meromorphic 1-form

$$\omega = f(z)\,dz.$$

For any piecewise smooth closed loop $\gamma$ in $\mathbb{C}$ avoiding the poles of $f$, we can form the integral

$$\oint_\gamma \omega = \oint_\gamma f(z)\,dz.$$

### 2.1 Residues at finite poles (Cauchy point of view)

Let $a \in \mathbb{C}$ be a pole of $f$. Take a small positively oriented circle

$$\gamma_a : z = a + re^{it}, \quad 0 \le t \le 2\pi,$$

small enough that it encloses no other poles of $f$. The *residue* of $\omega$ at $a$ is defined by

$$\mathrm{Res}_{z=a}(f(z)\,dz) := \frac{1}{2\pi i} \oint_{\gamma_a} f(z)\,dz.$$

Equivalently, on an annulus $0 < |z-a| < \varepsilon$, $f$ has a Laurent expansion

$$f(z) = \sum_{n=-m}^{\infty} c_n (z-a)^n,$$

then the coefficient $c_{-1}$ of $(z-a)^{-1}$ is the residue:

$$\mathrm{Res}_{z=a}(f(z)\,dz) = c_{-1}.$$

These coefficients $c_n$ can be obtained from integrals. For each integer $n$,

$$c_n = \frac{1}{2\pi i} \oint_{\gamma_a} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\,d\zeta,$$

so in particular

$$c_{-1} = \frac{1}{2\pi i} \oint_{\gamma_a} f(\zeta)\,d\zeta.$$

Thus the principal part of $f$ at a finite pole is determined entirely by integrals of the 1-form $f(\zeta)\,d\zeta$.

### 2.2 Residue at infinity

We also need the residue at $\infty$. There are two equivalent ways to define it.

**Method A: change variable $w = 1/z$**

Define $w = 1/z$, so $z = 1/w$ and $dz = -w^{-2}\,dw$. Let

$$F(w) := f\left(\frac{1}{w}\right).$$

Then the 1-form $\omega$ in terms of $w$ is

$$\omega = f(z)\,dz = f\left(\frac{1}{w}\right)\left(-\frac{1}{w^2}\,dw\right) = -F(w)w^{-2}\,dw.$$

Since $f$ is meromorphic at $\infty$, the function $F(w)w^{-2}$ has a Laurent expansion near $w = 0$:

$$F(w)w^{-2} = \sum_{n=-M}^{\infty} a_n w^n$$

with finitely many negative powers. Then the residue at $\infty$ is defined as

$$\mathrm{Res}_{z=\infty}(f(z)\,dz) := -\,\mathrm{Res}_{w=0}(F(w)w^{-2}\,dw).$$

If

$$F(w)w^{-2} = \cdots + a_{-1}w^{-1} + a_0 + a_1 w + \cdots,$$

then

$$\mathrm{Res}_{z=\infty}(f(z)\,dz) = -a_{-1}.$$

**Method B: big circle and global residue theorem**

Let $R$ be so large that the circle

$$\Gamma_R : z = Re^{it}, \quad 0 \leq t \leq 2\pi,$$

encloses all finite poles of $f$. Then the residue theorem on $\mathbb{C}$ gives

$$\oint_{\Gamma_R} f(z)\, dz = 2\pi i \sum_{a_j \in \mathbb{C}} \mathrm{Res}_{z=a_j}(f(z)\, dz),$$

where the sum is over all finite poles $a_j$ of $f$.

On the Riemann sphere $\widehat{\mathbb{C}}$, the *global residue theorem* says:

$$\sum_{p \in \widehat{\mathbb{C}}} \mathrm{Res}_p(f(z)\, dz) = 0.$$

Thus

$$\mathrm{Res}_{z=\infty}(f(z)\, dz) = -\sum_{a_j \in \mathbb{C}} \mathrm{Res}_{z=a_j}(f(z)\, dz).$$

So the residue at infinity is completely determined by the residues at finite poles (and vice versa).

# 3 Meromorphic $f$ on $\widehat{\mathbb{C}}$: poles and Laurent expansions

Let $f$ be a meromorphic function on $\widehat{\mathbb{C}}$.

## 3.1 Finitely many poles

Since $\widehat{\mathbb{C}}$ is compact and poles are isolated, $f$ has only finitely many poles. Thus there exist points

$$a_1, \ldots, a_N \in \mathbb{C} \cup \{\infty\}$$

such that all poles of $f$ are among these $a_j$.

## 3.2 Laurent expansions at finite poles (via integrals)

Fix a finite pole $a_j \in \mathbb{C}$. There exists a small circle $\gamma_j$ around $a_j$ enclosing no other poles. On an annulus $0 < |z - a_j| < \varepsilon$, $f$ has a Laurent expansion

$$f(z) = \sum_{n=-m_j}^{\infty} c_{j,n}(z - a_j)^n,$$

where $m_j \geq 1$ and $c_{j,-m_j} \neq 0$. Each coefficient is given by

$$c_{j,n} = \frac{1}{2\pi i} \oint_{\gamma_j} \frac{f(\zeta)}{(\zeta - a_j)^{n+1}}\, d\zeta.$$

The *principal part* of $f$ at $a_j$ is

$$\mathrm{PP}_{a_j}(f)(z) := \sum_{n=-m_j}^{-1} c_{j,n}(z - a_j)^n.$$

This is a finite sum of negative powers of $(z - a_j)$, with coefficients defined by integrals of $f(\zeta)\, d\zeta$.

### 3.3 Laurent expansion at infinity (via integrals)

At $\infty$, use $w = 1/z$ as local coordinate. Define

$$F(w) := f\left(\frac{1}{w}\right).$$

Since $f$ is meromorphic at $\infty$, there is an integer $M \geq 0$ and coefficients $b_n$ such that

$$F(w) = \sum_{n=-M}^{\infty} b_n w^n$$

for $0 < |w| < \varepsilon$. These $b_n$ are also given by Cauchy integrals:

$$b_n = \frac{1}{2\pi i} \oint_{|\xi|=\rho} \frac{F(\xi)}{\xi^{n+1}}\, d\xi,$$

for sufficiently small $\rho > 0$.

The *principal part* at $\infty$ is

$$\mathrm{PP}_\infty(f)(w) := \sum_{n=-M}^{-1} b_n w^n.$$

In terms of the original variable $z = 1/w$, note that $w^{-k} = z^k$, so $\mathrm{PP}_\infty(f)$ corresponds to a *polynomial* in $z$:

$$P(z) = \sum_{k=1}^{M} \tilde{b}_k\, z^k.$$

# 4 Constructing a rational function $R(z)$ with the same principal parts

We now build a single rational function $R(z)$ that has exactly the same principal parts as $f$ at each pole $a_j$ (including $\infty$).

## 4.1 Definition of $R(z)$

For each finite pole $a_j \in \mathbb{C}$, write the principal part as

$$\mathrm{PP}_{a_j}(f)(z) = \sum_{k=1}^{m_j} \frac{c_{j,-k}}{(z-a_j)^k}.$$

For the pole at $\infty$, we have a polynomial $P(z)$ as above.

Define

$$R(z) := P(z) + \sum_{j=1}^{N} \mathrm{PP}_{a_j}(f)(z) = P(z) + \sum_{j=1}^{N} \sum_{k=1}^{m_j} \frac{c_{j,-k}}{(z-a_j)^k}.$$

Each term is a rational function in $z$. Hence

$$R(z) \in \mathbb{C}(z).$$

By construction:

- At each finite pole $a_j$, $f$ and $R$ have the same principal part.
- At $\infty$, $f$ and $R$ also have the same principal part.

# 5 The difference $g = f - R$ is holomorphic everywhere

Define

$$g(z) := f(z) - R(z).$$

## 5.1 Behavior at finite points

At each finite pole $a_j \in \mathbb{C}$, $f$ and $R$ have the same principal part, so the negative powers in the Laurent expansion cancel. Thus near $a_j$ we have

$$g(z) = \sum_{n=0}^{\infty} d_{j,n}(z - a_j)^n,$$

i.e. $g$ is holomorphic at $a_j$.

At points where $f$ (hence $R$) is already holomorphic, clearly $g$ is also holomorphic. Therefore $g$ is holomorphic on all of $\mathbb{C}$.

## 5.2 Behavior at infinity

At $\infty$, in the coordinate $w = 1/z$, $f$ and $R$ have the same principal part in $w$. Thus the Laurent expansion of $g(1/w)$ at $w = 0$ has no negative powers:

$$g\left(\frac{1}{w}\right) = \sum_{n=0}^{\infty} d_n w^n.$$

So $g$ is holomorphic at $w = 0$, i.e. at $z = \infty$.

Therefore $g$ is holomorphic at every point of $\widehat{\mathbb{C}}$. In other words, $g$ is a global holomorphic function

$$g : \mathbb{CP}^1 \to \mathbb{C}.$$

# 6 Holomorphic on $\mathbb{CP}^1$ implies constant

The Riemann sphere $\mathbb{CP}^1$ is compact. A holomorphic function on a compact Riemann surface is bounded. By the maximum modulus principle (or Liouville's theorem), such a function must be constant.

Therefore there exists $C \in \mathbb{C}$ with

$$g(z) \equiv C,$$

i.e.

$$f(z) = R(z) + C.$$

Since $R(z) \in \mathbb{C}(z)$, we conclude that $f(z)$ is also rational:

$$f(z) \in \mathbb{C}(z).$$

This proves that *every* meromorphic function on $\mathbb{CP}^1$ is a rational function in $z$.

# 7 Conclusion

We have shown, using only:

- meromorphic 1-forms $\omega = f(z)\,dz$,
- residues, defined as contour integrals,
- Laurent coefficients extracted from integrals,
- and the maximum modulus principle on the compact surface $\mathbb{CP}^1$,

that

$$\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z).$$

Thus there is an isomorphism of fields

$$\mathcal{M}(\mathbb{CP}^1) \xrightarrow{\cong} \mathbb{C}(z),$$

sending a meromorphic function $f$ on the sphere to its expression as a rational function in the affine (stereographic) coordinate $z$.