Why Are the Tests for Conservative Fields Named That Way?

An Intuitive Explanation

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Each test for a conservative vector field is named after the fundamental question it answers about the field's structure.

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1 The Local Test: Equality of Mixed Partials

This is called the **"Local Test"** because it checks a property of the vector field at **one single point at a time**, independent of any other point in space.

The calculation of the partial derivatives, $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$, only requires information about the field in an infinitesimally small neighborhood around a point (x, y). You can confirm that the "no-swirl" condition holds at $(1, 2)$ without knowing anything about the field at $(5, 10)$.

It's like inspecting a tiled floor by checking each individual tile for cracks. You can declare one tile "good" or "bad" based only on a local inspection of that single tile.

Example For the vector field $\mathbf{F} = \langle -y, x \rangle$, the mixed partials are $\frac{\partial P}{\partial y} = -1$ and $\frac{\partial Q}{\partial x} = 1$. Because these are not equal, this test tells us that at *every local point*, there is a rotational component. This local failure everywhere guarantees the field is not conservative.

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2 The Global Test: Path Independence

This is called the **"Global Test"** because it checks a property that depends on the vector field along an **entire path**, not just a single point.

You cannot determine if an integral is path-independent by looking at the start point, the end point, or any single point in between. You must integrate the field's influence over the whole "global" journey. The result is a property of the path as it moves through the entire space (the "globe").

This is like checking if it's possible to walk from the ground floor to the roof of a building. You can't know the answer by just looking at the lobby; you have to check the entire global path, including all stairways and hallways, to see if it's connected.

Example To confirm that $\mathbf{F} = \langle 2x, 2y \rangle$ is conservative, we could show that the line integral from $(0,0)$ to $(1,1)$ is the same for the straight path $\gamma_1(t) = \langle t, t \rangle$ and the parabolic path $\gamma_2(t) = \langle t, t^2 \rangle$. Each calculation requires integrating over the *entire* path.

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3 The Constructive Test: Potential Recovery

This is called the **"Constructive Test"** because you literally **construct**, or build, the potential function f piece by piece.

Unlike the other tests that just give a "yes" or "no" answer, this is a procedure that results in a finished product: the potential function itself. The process is a step-by-step construction.

This is like an architect reverse-engineering a blueprint from an existing building. They start by measuring one room (integrating P), then use the connecting hallways to inform the layout of the next room (differentiating and comparing to Q), continuing until they have constructed the blueprint for the entire floor (f).

Example For $\mathbf{F} = \langle 2xy, x^2 \rangle$, the procedure is a construction:

1. **Construct a candidate:** Start by integrating P:

$$f(x,y) = \int 2xy\,dx = x^2y + h(y)$$

2. **Refine the construction:** Use Q to determine the unknown part. Differentiating our candidate gives $\frac{\partial f}{\partial y} = x^2 + h'(y)$. We set this equal to $Q = x^2$.

3. **Finalize the construction:** This shows $h'(y) = 0$, so $h(y)$ is a constant. The finished object is the potential function $f(x,y) = x^2y + C$.