$\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z)$ and $\mathcal{M}(X) \cong \mathbb{C}(z) \iff X \cong \mathbb{CP}^1$ (Algebraic and Calculus Viewpoints)

1 Setup and notation

We work over $\mathbb{C}$.

- $\mathbb{CP}^1$ is the complex projective line. As a set,

$$\mathbb{CP}^1 = \{[z_0 : z_1] \mid (z_0, z_1) \neq (0,0)\}/\sim,$$

  where $[z_0 : z_1] \sim [\lambda z_0 : \lambda z_1]$ for $\lambda \neq 0$.

- Analytically, $\mathbb{CP}^1$ is the Riemann sphere

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

  On $\mathbb{C} \subset \widehat{\mathbb{C}}$ we use the coordinate z, and ∞ is the "point at infinity".

- For any compact Riemann surface X, we denote its field of meromorphic functions by $\mathcal{M}(X)$.
- For $\mathbb{CP}^1$, we want to show

$$\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z),$$

  and then show that for a general compact Riemann surface X,

$$\mathcal{M}(X) \cong \mathbb{C}(z) \quad \iff \quad X \cong \mathbb{CP}^1.$$

2 Part A: Analytic ("calculus") proof that $\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z)$

We identify $\mathbb{CP}^1$ with $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, so a meromorphic function on $\mathbb{CP}^1$ is a meromorphic $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

2.1 Step A.1: Meromorphic 1-forms and residues

Given a meromorphic function f on $\widehat{\mathbb{C}}$, consider the 1-form

$$\omega = f(z)\, dz.$$

For any closed loop γ in $\mathbb{C}$ avoiding the poles of f, we can form

$$\oint_\gamma \omega = \oint_\gamma f(z)\, dz.$$

Residues at finite poles

Let $a \in \mathbb{C}$ be a pole of f. Take a small positively oriented circle

$$\gamma_a : z = a + re^{it}, \quad 0 \le t \le 2\pi,$$

small enough to enclose no other poles. The residue of ω at a is

$$\operatorname{Res}_{z=a}(f(z)\,dz) := \frac{1}{2\pi i} \oint_{\gamma_a} f(z)\,dz.$$

On an annulus $0 < |z-a| < \varepsilon$, f has a Laurent series

$$f(z) = \sum_{n=-m}^{\infty} c_n (z-a)^n.$$

Then the coefficient of $(z-a)^{-1}$ is

$$c_{-1} = \operatorname{Res}_{z=a}(f(z)\,dz).$$

More generally,

$$c_n = \frac{1}{2\pi i} \oint_{\gamma_a} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\,d\zeta.$$

So the principal part at a is determined by integrals of the 1-form $f(\zeta)\,d\zeta$.

Residue at infinity

At ∞, use the coordinate $w = 1/z$. Then $z = 1/w$ and $dz = -w^{-2}\,dw$. Define

$$F(w) := f\left(\frac{1}{w}\right).$$

Then

$$\omega = f(z)\,dz = f\left(\frac{1}{w}\right)\left(-\frac{1}{w^2}\,dw\right) = -F(w)w^{-2}\,dw.$$

Since f is meromorphic at ∞, $F(w)w^{-2}$ has a Laurent expansion

$$F(w)w^{-2} = \sum_{n=-M}^{\infty} a_n w^n$$

with finitely many negative powers. Define

$$\operatorname{Res}_{z=\infty}(f(z)\,dz) := -\operatorname{Res}_{w=0}\big(F(w)w^{-2}\,dw\big).$$

This gives the global residue theorem on $\mathbb{CP}^1$:

$$\sum_{p \in \widehat{\mathbb{C}}} \operatorname{Res}_p(f(z)\,dz) = 0.$$

2.2 Step A.2: Finitely many poles and principal parts

Let f be meromorphic on $\widehat{\mathbb{C}}$. Since $\widehat{\mathbb{C}}$ is compact and poles are isolated, f has only finitely many poles:

$$\{a_1, \dots, a_N\} \subset \mathbb{C} \cup \{\infty\}.$$

At each finite pole $a_j \in \mathbb{C}$, f has a Laurent series:

$$f(z) = \sum_{n=-m_j}^{\infty} c_{j,n}(z - a_j)^n.$$

The *principal part* at a_j is

$$\mathrm{PP}_{a_j}(f)(z) := \sum_{n=-m_j}^{-1} c_{j,n}(z - a_j)^n.$$

At ∞ in coordinate $w = 1/z$,

$$F(w) := f\left(\frac{1}{w}\right) = \sum_{n=-M}^{\infty} b_n w^n.$$

The principal part at ∞ is

$$\mathrm{PP}_{\infty}(f)(w) := \sum_{n=-M}^{-1} b_n w^n,$$

which corresponds to a polynomial in z because $w^{-k} = z^k$.

2.3 Step A.3: Build a rational function $R(z)$ from principal parts

Define

$$R(z) := P(z) + \sum_{j=1}^{N} \mathrm{PP}_{a_j}(f)(z),$$

where $P(z)$ is the polynomial corresponding to the principal part at ∞.

Concretely,

$$\mathrm{PP}_{a_j}(f)(z) = \sum_{k=1}^{m_j} \frac{c_{j,-k}}{(z - a_j)^k},$$

and

$$P(z) = \sum_{k=1}^{M} \tilde{b}_k z^k.$$

So

$$R(z) = \sum_{k=1}^{M} \tilde{b}_k z^k + \sum_{j=1}^{N} \sum_{k=1}^{m_j} \frac{c_{j,-k}}{(z - a_j)^k}.$$

Each term is rational in z, so

$$R(z) \in \mathbb{C}(z).$$

By construction:

- At each finite pole a_j, R has the same principal part as f.
- At ∞, R has the same principal part as f.

2.4 Step A.4: Holomorphic difference $g = f - R$ and Liouville

Define

$$g(z) := f(z) - R(z).$$

At finite points. At each finite pole a_j, the principal parts of f and R cancel, so the Laurent expansion of g at a_j has no negative powers. Therefore g is holomorphic at a_j. At points where f is holomorphic, so is g. Hence g is holomorphic on all of $\mathbb{C}$.

At infinity. At ∞, in coordinate $w = 1/z$, f and R have the same principal part at $w = 0$, so $g(1/w)$ has a power series expansion with no negative powers. Thus g is holomorphic at $w = 0$, i.e. at $z = \infty$.

So g is holomorphic on the entire sphere $\widehat{\mathbb{C}} = \mathbb{CP}^1$ (a compact Riemann surface). By the maximum modulus principle or Liouville's theorem, g is constant:

$$g(z) \equiv C \in \mathbb{C}.$$

Thus

$$f(z) = R(z) + C.$$

Since $R(z)$ is rational in z, so is $f(z)$.

$$\boxed{\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z).}$$

This is the *analytic* / "calculus" proof: we used differential forms $f(z)\,dz$, residues, contour integrals, and Liouville.

3 Part B: Algebraic / projective viewpoint on $\mathcal{M}(\mathbb{CP}^1)$

Now we describe the same fact algebraically, using homogeneous coordinates and maps of projective varieties.

3.1 Step B.1: Affine chart and rational functions

Consider the affine chart

$$U_1 = \{[z_0 : z_1] \in \mathbb{CP}^1 \mid z_1 \neq 0\},$$

with coordinate

$$z = \frac{z_0}{z_1} : U_1 \longrightarrow \mathbb{C}.$$

This identifies $U_1 \cong \mathbb{C}$. The remaining point $[1 : 0]$ corresponds to ∞.

Any rational function in the affine coordinate z,

$$R(z) = \frac{p(z)}{q(z)}, \quad p, q \in \mathbb{C}[z],\ q \not\equiv 0,$$

defines a map on U_1 by

$$[z_0 : z_1] \mapsto [R(z_0/z_1) : 1],$$

with the convention that if $R(z_0/z_1) = \infty$ we send to the point $[1 : 0]$. One checks this extends uniquely to a holomorphic map

$$F_R : \mathbb{CP}^1 \to \mathbb{CP}^1.$$

3.2 Step B.2: Homogeneous polynomial description

Writing $R(z) = p(z)/q(z)$ with $\deg p, \deg q \le m$, define homogeneous polynomials of degree m:

$$P(z_0, z_1) = z_1^m \, p\left(\frac{z_0}{z_1}\right), \quad Q(z_0, z_1) = z_1^m \, q\left(\frac{z_0}{z_1}\right).$$

Then

$$F_R([z_0 : z_1]) = \begin{cases} [P(z_0, z_1) : Q(z_0, z_1)], & Q(z_0, z_1) \neq 0, \\ [1 : 0], & Q(z_0, z_1) = 0. \end{cases}$$

This is well-defined on projective space (scaling (z_0, z_1) multiplies (P, Q) by a common factor) and holomorphic. On the affine chart U_1 this agrees with $R(z)$.

Conversely, any holomorphic map $F : \mathbb{CP}^1 \to \mathbb{CP}^1$ is given by a pair of homogeneous polynomials of the same degree, and on U_1 its affine expression is a rational function in z.

Thus algebraically:

$$\mathcal{M}(\mathbb{CP}^1) = \{\text{meromorphic functions } \mathbb{CP}^1 \to \mathbb{CP}^1\} \cong \mathbb{C}(z).$$

4 Part C: General X and the condition $\mathcal{M}(X) \cong \mathbb{C}(z)$

Now let X be an arbitrary compact Riemann surface. We consider its function field

$$\mathcal{M}(X) := \{\text{meromorphic functions on } X\}.$$

4.1 Step C.1: Non-constant meromorphic maps $X \to \mathbb{CP}^1$

Any non-constant meromorphic function $f \in \mathcal{M}(X)$ gives a holomorphic map

$$f : X \to \mathbb{CP}^1$$

by the same recipe as before:

$$f(p) = \begin{cases} [f(p) : 1], & f(p) \text{ finite}, \\ [1 : 0], & f(p) = \infty. \end{cases}$$

This map is *finite*: for a generic point $w \in \mathbb{CP}^1$, the fiber $f^{-1}(w)$ has finitely many points, counted with multiplicity. That number is called the *degree* of f, $\deg(f)$.

Analytically, around a point $p \in X$ where f is not critical, in local coordinates z on X and ζ on $\mathbb{CP}^1$, f looks like

$$\zeta = f(z) \approx z^k$$

for some $k \ge 1$; k is the local degree at p.

4.2 Step C.2: Function field extension viewpoint

From algebraic geometry / field theory:

- The map $f : X \to \mathbb{CP}^1$ induces an inclusion of function fields

  $$f^* : \mathcal{M}(\mathbb{CP}^1) \hookrightarrow \mathcal{M}(X),$$

  by pullback: $R(z) \mapsto R(f)$.

- This is an embedding of fields $\mathbb{C}(z) \hookrightarrow \mathcal{M}(X)$.
- The degree of the field extension $[\mathcal{M}(X) : \mathbb{C}(z)]$ equals the degree of the map f:

$$[\mathcal{M}(X) : \mathbb{C}(z)] = \deg(f).$$

In particular:

If $\mathcal{M}(X) \cong \mathbb{C}(z)$ as fields, any non-constant $f : X \to \mathbb{CP}^1$ must have $\deg(f) = 1$.

Because $\deg(f) = [\mathcal{M}(X) : \mathbb{C}(z)]$ and if $\mathcal{M}(X) = \mathbb{C}(z)$, the extension has degree 1.

4.3 Step C.3: Degree 1 map $X \to \mathbb{CP}^1$ is an isomorphism

If $f : X \to \mathbb{CP}^1$ is a non-constant holomorphic map of compact Riemann surfaces with degree 1, then:

- f is surjective (image of compact + open implies all of $\mathbb{CP}^1$).
- $\deg(f) = 1$ means generically each point of $\mathbb{CP}^1$ has exactly one preimage.
- One can show (using local behavior and the open mapping theorem) that f is a bijection.
- A bijective holomorphic map between compact Riemann surfaces has a holomorphic inverse (by the open mapping theorem + properness), so f is a biholomorphism.

Hence:

If there exists a meromorphic $f : X \to \mathbb{CP}^1$ with $\deg(f) = 1$, then $X \cong \mathbb{CP}^1$ as Riemann surfaces.

Combining with the field-extension fact:

$$\mathcal{M}(X) \cong \mathbb{C}(z) \implies \text{there exists } f : X \to \mathbb{CP}^1 \text{ with } \deg(f) = 1 \implies X \cong \mathbb{CP}^1.$$

4.4 Step C.4: Converse: if $X \cong \mathbb{CP}^1$ then $\mathcal{M}(X) \cong \mathbb{C}(z)$

Conversely, if $X \cong \mathbb{CP}^1$ as Riemann surfaces, then by definition

$$\mathcal{M}(X) \cong \mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z).$$

So we have the equivalence:

$$X \cong \mathbb{CP}^1 \quad \iff \quad \mathcal{M}(X) \cong \mathbb{C}(z) \text{ (as fields)}.$$

5 Part D: Extra analytic structure (genus and differentials)

There is a deeper equivalence involving the *genus* $g(X)$.

- The genus $g(X)$ is the "number of holes" of X as a surface. Topologically: $g(\mathbb{CP}^1) = 0$, $g(\mathbb{C}/\Lambda) = 1$, etc.
- Analytically, $g(X)$ equals the dimension of the space of holomorphic 1-forms on X:

$$g(X) = \dim_{\mathbb{C}} H^0(X, \Omega^1_X).$$

  For $\mathbb{CP}^1$, there are no holomorphic 1-forms, so $g(\mathbb{CP}^1) = 0$.

One can prove the following classical equivalence:

For a compact Riemann surface X, the following are equivalent:

1. $X \cong \mathbb{CP}^1$.
2. $g(X) = 0$ (no holomorphic 1-forms).
3. $\mathcal{M}(X) \cong \mathbb{C}(z)$ as fields.

The equivalence (1) $\Leftrightarrow$ (3) is what we just discussed in detail. The equivalence (1) $\Leftrightarrow$ (2) can be seen via differential forms and the Riemann–Roch theorem. Intuitively:

- On $\mathbb{CP}^1$, every meromorphic 1-form has total number of zeros minus poles equal to -2, and there are no holomorphic ones (no poles).
- On higher-genus surfaces, there exist nontrivial holomorphic 1-forms, reflecting the topology of the surface (more "holes").

Summary in one sentence

- **Calculus / analytic side:** On the Riemann sphere, any meromorphic f has only finitely many poles, each with a Laurent expansion whose principal parts are given by contour integrals of $f(z)\,dz$. Subtracting a rational function $R(z)$ built from those principal parts gives an entire function on the sphere, hence constant. So every meromorphic function is rational: $\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z)$.
- **Algebraic side:** For a general compact Riemann surface X, non-constant meromorphic functions $f : X \to \mathbb{CP}^1$ induce field embeddings $\mathbb{C}(z) \hookrightarrow \mathcal{M}(X)$ and finite field extensions. If $\mathcal{M}(X) \cong \mathbb{C}(z)$, there is a degree-1 map $X \to \mathbb{CP}^1$, which must be a biholomorphism. Conversely, if $X \cong \mathbb{CP}^1$, then $\mathcal{M}(X) \cong \mathbb{C}(z)$.

So:

$$\boxed{\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z) \quad \text{and} \quad \mathcal{M}(X) \cong \mathbb{C}(z) \iff X \cong \mathbb{CP}^1.}$$