

# Advanced Calculus III

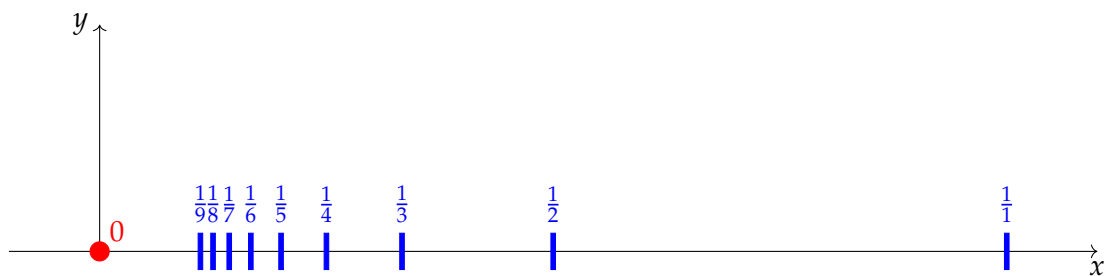
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We cover the following topics in this note.

- Limit of a Function
- Continuity of a Function
- Monotone Convergent Theorem (MCT)
- Nested Interval Property (NIP)
- TBA

What is  $0$  for the set  $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ ?



**Note** (Open  $\varepsilon$ -ball). The open  $\varepsilon$ -ball of  $x$  in  $S$  is  $B_\varepsilon(x) := \{y \in S : d(x, y) < \varepsilon\}$ .

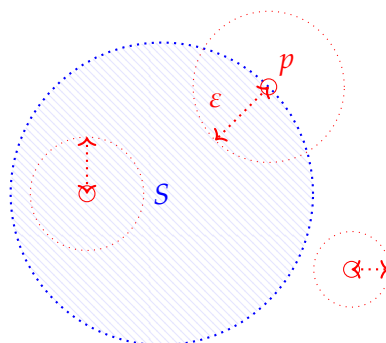
## Limit Point (Metric Space)

**Definition.** Let  $(X, d)$  be a metric space. Let  $S \subseteq X$ . A point  $p \in X$  is a **limit point** of  $S$  if and only if

$$\forall \varepsilon > 0, B_\varepsilon(p) \cap (S \setminus \{p\}) \neq \emptyset.$$

That is,

$$\forall \varepsilon > 0, \{x \in S : 0 < d(x, p) < \varepsilon\} \neq \emptyset.$$



**Remark.** Note that a limit point  $p$  may NOT belong to  $S$ .

**Note (Limit Point (Topology)).** Let  $(X, \tau)$  be a topological space. For a subset  $S \subseteq X$ . A point  $p \in X$  is a limit point of  $S$  if and only if

$$\forall U \in \tau \text{ with } p \in U, U \cap (S \setminus \{p\}) \neq \emptyset.$$

**Example.** Let  $S = (a, b) \subseteq \mathbb{R}$ :



(i) Consider  $p$  with  $p < a$ :



Let  $\varepsilon := \frac{a-p}{2} > 0$ . Then  $B_\varepsilon(p) \cap (S \setminus \{p\}) = \emptyset$ . Thus,  $p < a$  is NOT a limit point.

(ii) Consider  $p = a$ :



Let  $\varepsilon > 0$ . Then  $B_\varepsilon(p) \cap (S \setminus \{p\}) \neq \emptyset$ . Thus,  $p = a$  is a limit point of  $S = (a, b)$ .

By (i) and (ii), the set of all limit points of  $(a, b)$  is  $[a, b]$ .

**Example.** Let  $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ :



- Consider  $p = \frac{1}{n} \in S$ . No point of  $S$  is a limit point.
- Consider  $p = 0$ .



Let  $\varepsilon > 0$ . By Archimedian property,  $\exists n \in \mathbb{N}$  such that  $n > \frac{1}{\varepsilon}$ , and so  $1/n \in B_\varepsilon(0) \cap S$ . Thus,  $p = 0$  is a limit point of  $S = \{1/n : n \in \mathbb{N}\}$ .

**Example.** Let  $S = \mathbb{Q}$ .

- Consider  $p \in \mathbb{R}$ . Let  $\varepsilon > 0$ . By density of rationals,

$$\exists r \in \mathbb{Q} \text{ such that } p < r < p + \varepsilon.$$

Then  $r \in B_\varepsilon(p) \cap S$  with  $r \neq p$ , i.e.,  $r$  is a limit point. Thus, all reals are limit points of  $\mathbb{Q}$ .

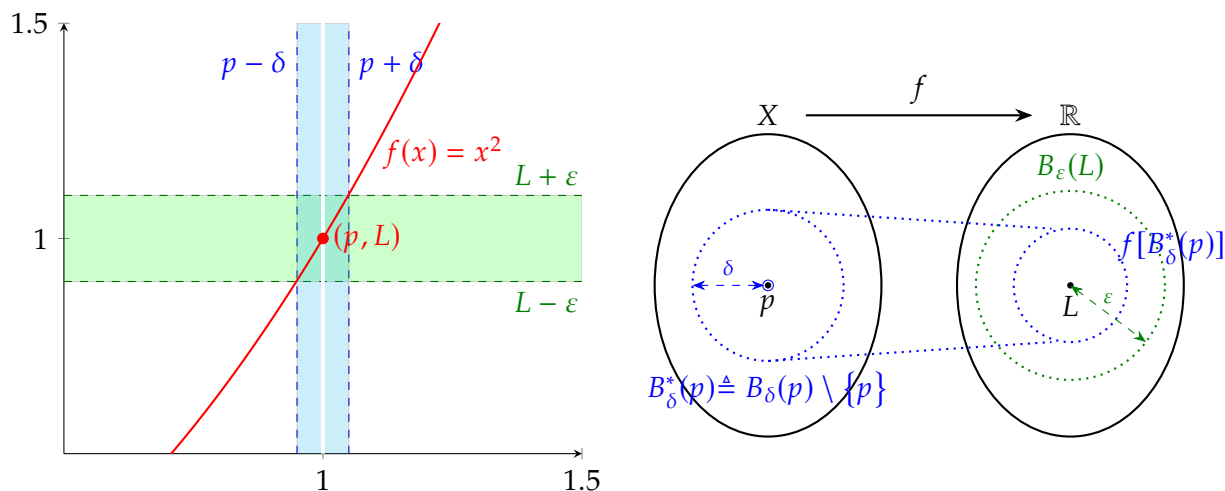
★ Limit of a Function ( $\varepsilon - \delta$ ) ★

**Definition.** Let  $f : X \rightarrow \mathbb{R}$  be a function defined on a subset  $X(\subseteq \mathbb{R})$  of a metric space, and let  $p \in X$  be a limit point of  $X$ . We say that  $L \in \mathbb{R}$  is the **limit of the function  $f$  as  $x$  approaches  $p$**  if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in X, 0 < |x - p| < \delta \implies |f(x) - L| < \varepsilon.$$

We write

$$\lim_{x \rightarrow p} f(x) = L.$$



**Remark.**

$$\lim_{x \rightarrow p} f(x) \neq L \iff \exists \varepsilon > 0 : [\forall \delta > 0 : \exists x \in X : 0 < |x - p| < \delta \text{ but } |f(x) - L| > \varepsilon].$$

### Continuity of a Function

**Definition.** Let  $f : X \rightarrow \mathbb{R}$  be a function defined on a subset  $X \subseteq \mathbb{R}$  of a metric space, and let  $p \in X$ . The function  $f$  is **continuous at  $p$**  if and only if

$$\lim_{x \rightarrow p} f(x) = f(p).$$

That is,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |x - p| < \delta \implies |f(x) - f(p)| < \varepsilon.$$

**Remark (Continuity of a Set).** The function  $f$  is continuous on subset  $S \subseteq X$  if it is continuous at every point  $p \in S$ .

**Remark (Continuity in a Topological Space).** Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  are topological spaces.  $f : X \rightarrow Y$  is **continuous** if and only if

$$U_Y \in \tau_Y \implies f^{-1}[U_Y] \in \tau_X,$$

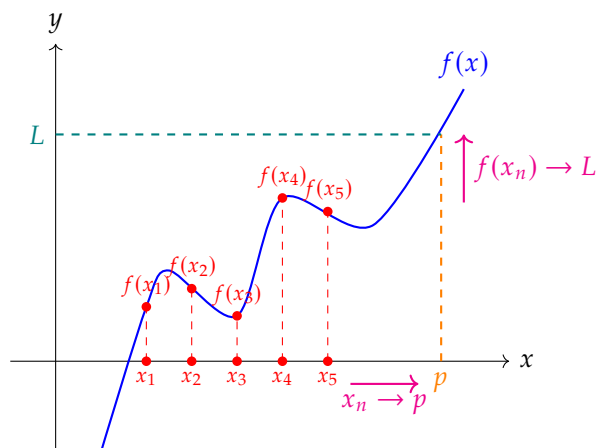
where  $f^{-1}[U_Y] = \{x \in X : f(x) \in U_Y\}$  is the preimage of  $U_Y$  under  $f$ .

**Note.**  $[p \implies (q \implies r)] \equiv [p \implies (\neg q \vee r)] \equiv [\neg p \vee (\neg q \vee r)] \equiv [\neg(p \wedge q) \vee r] \equiv [(p \wedge q) \implies r]$ .

### Limit of Function by Convergent Sequences

**Theorem.** Let  $f : X \rightarrow \mathbb{R}$  be a function defined on a subset  $\emptyset \neq X \subseteq \mathbb{R}$  of a metric space, and let  $p$  is a limit point of  $X$ . Then

$$\lim_{x \rightarrow p} f(x) = L \iff \left[ \forall \{x_n\} \subseteq X \setminus \{p\}, \left( \lim_{n \rightarrow \infty} x_n = p \implies \lim_{n \rightarrow \infty} f(x_n) = L \right) \right].$$



*Proof.*  $(\Rightarrow)$  Suppose that  $\lim_{x \rightarrow p} f(x) = L$ . Let  $\{x_n\} \subseteq X \setminus \{p\}$  be a sequence, and let  $\lim_{n \rightarrow \infty} x_n = p$ . We NTS that

$$\lim_{n \rightarrow \infty} f(x_n) = L, \quad \text{i.e.,} \quad \forall \varepsilon > 0 : \exists N \in \mathbb{N} : n \geq N \Rightarrow |f(x_n) - L| < \varepsilon.$$

Let  $\varepsilon > 0$ . Since  $\lim_{x \rightarrow p} f(x) = L$ , we know

$$\exists \delta > 0 \text{ such that } 0 < |x - p| < \delta \Rightarrow |f(x) - L| < \varepsilon. \quad (*)$$

Since  $\lim_{n \rightarrow \infty} x_n = p$ , we obtain

$$\exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow |x_n - p| < \delta.$$

Thus, if  $n \geq N$  then,

$$\begin{aligned} |x_n - p| < \delta &\Rightarrow 0 < |x_n - p| < \delta \quad \because x_n \neq p \\ &\Rightarrow |f(x_n) - L| < \varepsilon \quad \text{by } (*) \end{aligned}$$

Thus,  $\lim_{n \rightarrow \infty} f(x_n) = L$ .

$(\Leftarrow)$  Let the RHS holds. Assume, for the contradiction, that  $\lim_{x \rightarrow p} f(x) \neq L$ , i.e.,

$$\exists \varepsilon > 0 : \forall \delta > 0 : \exists x_\delta \in X : 0 < |x_\delta - p| < \delta \text{ but } |f(x_\delta) - L| \geq \varepsilon.$$

Take  $\delta = 1/n$  for  $n \in \mathbb{N}$ . Then

$$\exists x_n \in X \text{ such that } 0 < |x_n - p| < \frac{1}{n} \text{ but } |f(x_n) - L| \geq \varepsilon.$$

(Axiom of Countable Choice) This means that

$$\forall n \in \mathbb{N} : \exists \{x_n\} \subseteq X \setminus \{p\} \text{ such that } 0 < |x_n - p| < \frac{1}{n} \text{ but } |f(x_n) - L| \geq \varepsilon.$$

By Squeeze Theorem, we have  $\lim_{n \rightarrow \infty} x_n = p$  since  $0 < |x_n - p| < 1/n$ . Since the RHS holds, we obtain  $\lim_{n \rightarrow \infty} f(x_n) = L$ . Then, for some  $\varepsilon > 0$ ,

$$\exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow |f(x_n) - L| < \varepsilon \frac{1}{2}.$$

Hence it is proved. □

### Continuity of Function by Convergent Sequences

**Corollary.** Let  $f : X \rightarrow \mathbb{R}$  be a function defined on a subset  $\emptyset \neq X \subseteq \mathbb{R}$  of a metric space, and let  $p$  is a limit point of  $X$ . Then

$$\lim_{x \rightarrow p} f(x) = f(p) \iff \left[ \forall \{x_n\} \subseteq X, \left( \lim_{n \rightarrow \infty} x_n = p \implies \lim_{n \rightarrow \infty} f(x_n) = f(p) \right) \right].$$

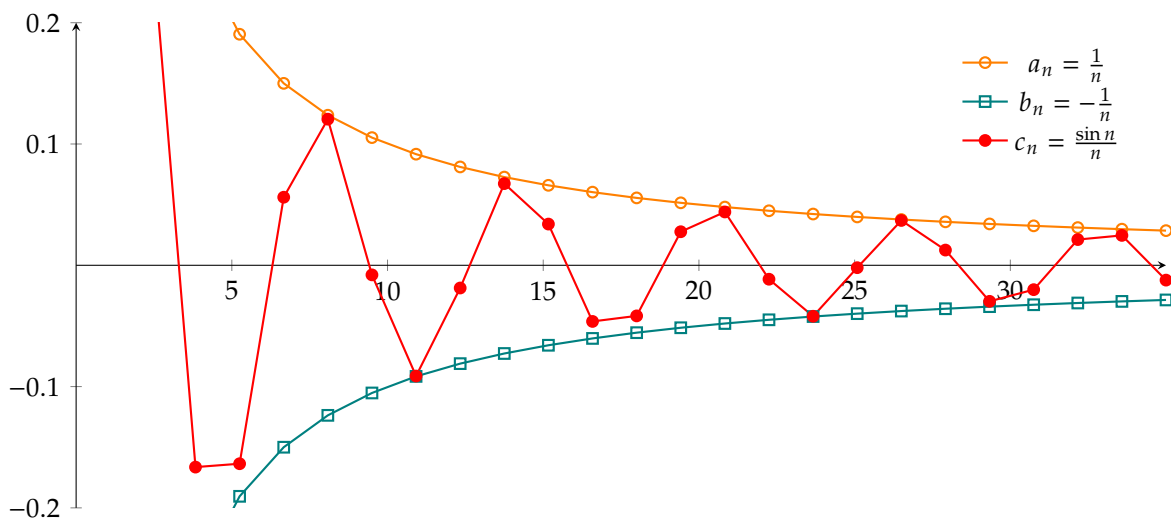
### Squeeze Theorem; Sandwich Theorem

**Theorem.** Let

$$(i) \lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} b_n;$$

$$(ii) \exists n_0 \in \mathbb{N} \text{ such that } a_n \leq c_n \leq b_n \text{ for all } n \geq n_0.$$

Then  $\lim_{n \rightarrow \infty} c_n = L$ .



*Proof.* Let  $\varepsilon > 0$ . Since  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} b_n = L$ , we have

$$\exists n_1 \in \mathbb{N} \text{ such that } n \geq n_1 \implies L - \varepsilon < a_n < L + \varepsilon,$$

$$\exists n_2 \in \mathbb{N} \text{ such that } n \geq n_2 \implies L - \varepsilon < b_n < L + \varepsilon.$$

Let  $N := \max \{n_0, n_1, n_2\}$ . If  $n \geq N$  then

$$L - \varepsilon < a_n \leq c_n \leq b_n < L + \varepsilon,$$

and so  $|c_n - L| < \varepsilon$ . □

**Note.** Recall that

“A convergent sequence is bounded.”

Formally,

$$\exists A \in \mathbb{R} \text{ s.t. } A = \lim_{n \rightarrow \infty} a_n \implies \exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, |a_n| \leq M.$$

However, the converse is not necessarily true:

$$\exists A \in \mathbb{R} \text{ s.t. } A = \lim_{n \rightarrow \infty} a_n \not\Leftarrow \exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, |a_n| \leq M.$$

To illustrate, consider the sequence  $\{a_n\} = 1 - (-1)^n$  that is bounded, yet it does not converge.

### Monotone Sequence

**Definition.** A sequence  $\{a_n\}_{n=1}^{\infty}$  is said to be **monotone** if it is either **monotone increasing** or **monotone decreasing**.

(1) A sequence  $\{a_n\}_{n=1}^{\infty}$  is **monotone increasing** if  $a_n \leq a_{n+1}$  for all  $n \in \mathbb{N}$ .

Alternatively, it is **strictly increasing** if  $a_n < a_{n+1}$  for all  $n \in \mathbb{N}$ .

(2) A sequence  $\{a_n\}_{n=1}^{\infty}$  is **monotone decreasing** if  $a_{n+1} \leq a_n$  for all  $n \in \mathbb{N}$ .

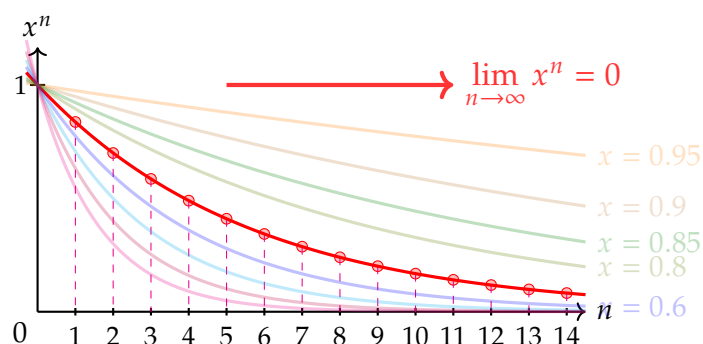
Alternatively, it is **strictly decreasing** if  $a_{n+1} < a_n$  for all  $n \in \mathbb{N}$ .

**Remark.** A sequence  $\{a_n\}$  is monotone if  $\begin{cases} a_n \leq a_{n+1} & (\text{monotone increasing}) \\ a_{n+1} \leq a_n & (\text{monotone decreasing}) \end{cases}$ .

**Example.**

- $\{n\}_{n=1}^{\infty}$  is monotone increasing.
- $\{1/n\}_{n=1}^{\infty}$  is monotone decreasing.

**Example.** Let  $0 < x < 1$ .



### Monotone Convergence Theorem (MCT)

**Theorem.** A monotone sequence of real numbers  $\{a_n\}$  is convergent if and only if it is bounded.

(1) Let  $\{a_n\}$  be an monotone increasing sequence of real numbers that is bounded above. Then

$$\lim_{n \rightarrow \infty} a_n = \sup \{a_n : n \in \mathbb{N}\}.$$

(2) Let  $\{b_n\}$  be an monotone decreasing sequence of real numbers that is bounded below. Then

$$\lim_{n \rightarrow \infty} b_n = \inf \{b_n : n \in \mathbb{N}\}.$$

*Proof.*

(1) Suppose that a sequence  $\{a_n\}$  is monotone increasing and bounded above. Consider the set  $\{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ , which is non-empty and bounded above by assumption. By **Least Upper Bound Property**<sup>1</sup>,

$$\exists \alpha \in \mathbb{R} \text{ such that } \alpha = \sup \{a_n : n \in \mathbb{N}\}.$$

We claim that:

$$\lim_{n \rightarrow \infty} a_n = \alpha = \sup \{a_n : n \in \mathbb{N}\}.$$

Let  $\varepsilon > 0$ . Since  $\alpha$  is the supremum (*least* upper bound) of  $\{a_n : n \in \mathbb{N}\}$ , it follows that  $\alpha - \varepsilon$  is not an upper bound of  $\{a_n : n \in \mathbb{N}\}$ . Thus,  $\neg[\forall n \in \mathbb{N}, a_n \leq \alpha - \varepsilon]$ , i.e.,

$$\exists N \in \mathbb{N} \text{ such that } \alpha - \varepsilon < a_N.$$

Since  $\{a_n\}$  is monotone increasing,

$$\alpha - \varepsilon < a_N \leq a_n$$

for all  $n \geq N$ . Therefore,

$$\alpha - \varepsilon \underset{\substack{\alpha = \sup \{a_n\} \\ \varepsilon > 0}}{<} a_N \underset{\substack{\{a_n\} \text{ is monotone increasing} \\ n \geq N}}{\leq} a_n \underset{\substack{\alpha \text{ is an upper bound} \\ \varepsilon > 0}}{\leq} \alpha < \alpha + \varepsilon.$$

This implies that  $|a_n - \alpha| < \varepsilon$  for all  $n \geq N$ .

<sup>1</sup>Every non-empty subset of  $\mathbb{R}$  that is bounded above has the supremum in  $\mathbb{R}$ .



- (2) Suppose that a sequence  $\{b_n\}$  is monotone decreasing and bounded below. Consider the set  $\{b_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ , which is non-empty and bounded below by assumption. By **Greatest Lower Bound Property**<sup>2</sup>,

$$\exists \beta \in \mathbb{R} \text{ such that } \beta = \inf \{b_n : n \in \mathbb{N}\}.$$

We claim that:

$$\lim_{n \rightarrow \infty} b_n = \beta = \inf \{b_n : n \in \mathbb{N}\}.$$

Let  $\varepsilon > 0$ . Since  $\beta$  is the infimum (*greatest* lower bound) of  $\{b_n : n \in \mathbb{N}\}$ , it follows that  $\beta + \varepsilon$  is not a lower bound of  $\{b_n : n \in \mathbb{N}\}$ . Thus,  $\neg[\forall n \in \mathbb{N}, \beta + \varepsilon \leq b_n]$ , i.e.,

$$\exists N \in \mathbb{N} \text{ such that } b_N < \beta + \varepsilon.$$

Since  $\{b_n\}$  is monotone decreasing,

$$b_n \leq b_N < \beta + \varepsilon$$

for all  $n \geq N$ . Therefore,

$$\beta - \varepsilon \stackrel{\varepsilon > 0}{<} \beta \stackrel{\beta \text{ is a lower bound}}{\leq} b_n \stackrel{\{b_n\} \text{ is monotone decreasing}}{\leq}_{n \geq N} b_N \stackrel{\beta = \inf \{b_n\}}{<}_{\varepsilon > 0} \beta + \varepsilon$$

This implies that  $|b_n - \beta| < \varepsilon$  for all  $n \geq N$ .

□

### Divergence of Sequence

**Definition.** Let  $\{a_n\}$  be a sequence of real numbers.

- (1) We say that the sequence  $\{a_n\}$  **diverges to infinity** (or **tends to infinity**) if

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } n \geq N \implies M < a_n,$$

and write  $\lim_{n \rightarrow \infty} a_n = +\infty$ .

- (2) We say that the sequence  $\{a_n\}$  **diverges to minus infinity** (or **tends to infinity**) if

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } n \geq N \implies a_n < M,$$

and write  $\lim_{n \rightarrow \infty} a_n = -\infty$ .

- (3) We say that  $\{a_n\}$  is properly divergent in case we have either  $\lim_{n \rightarrow \infty} a_n = +\infty$  or  $\lim_{n \rightarrow \infty} a_n = -\infty$ .

<sup>2</sup>Every non-empty subset of  $\mathbb{R}$  that is bounded below has the infimum in  $\mathbb{R}$ .

**Note.** Recall that

**(Monotonicity)** A sequence  $\{a_n\}$  is monotone increasing if  $a_n \leq a_{n+1}$  for all  $n \in \mathbb{N}$ ;

**(Not Bounded Above)** The sequence  $\{a_n\}$  is not bounded above if

$$\neg[\exists M \in \mathbb{R}, \forall n \in \mathbb{N}, a_n \leq M] \equiv [\forall M \in \mathbb{R}, \exists n \in \mathbb{N} \text{ such that } a_n > M].$$

We claim that a sequence  $\{a_n\}$  that is monotone increasing and not bounded above diverges to infinity:

*Proof.* Let  $M \in \mathbb{R}$ . Since  $\{a_n\}$  is not bounded above,

$$\exists n_0 \in \mathbb{N} \text{ such that } a_{n_0} > M.$$

Since  $\{a_n\}$  is monotonic increasing, it follows that

$$a_{n_0} \leq a_n, \forall n \geq n_0.$$

Thus

$$n \geq n_0 \xRightarrow{\text{monotone increasing}} a_{n_0} \leq a_n \xRightarrow{\text{Not Bounded Above}} M < a_{n_0} < a_n.$$

Hence it is proved. □

Note that

**Lemma.** Let  $\{a_n\}$  and  $\{b_n\}$  be sequences of real numbers. Then

$$[\forall n \in \mathbb{N}, a_n \leq b_n] \implies \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n.$$

*Proof.* Let  $a = \lim_{n \rightarrow \infty} a_n$  and  $b = \lim_{n \rightarrow \infty} b_n$ . Suppose that  $a > b$ . Let  $\varepsilon = a - b > 0$ . Then

$$\exists N_1 \in \mathbb{N} \text{ such that } n \geq N_1 \implies |a_n - a| < \varepsilon,$$

$$\exists N_2 \in \mathbb{N} \text{ such that } n \geq N_2 \implies |b_n - b| < \varepsilon.$$

Let  $N := \max\{N_1, N_2\}$ . Then  $b_N < b + \varepsilon < a + \varepsilon < a_N$   $\nlessgtr$ . Hence  $a \leq b$ , i.e.,  $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$ .  $\square$

**Note.** Let  $I_n = (0, \frac{1}{n}) \subseteq \mathbb{R}$  for all  $n \in \mathbb{N}$ .



Suppose that  $x \in \bigcap_{n=1}^{\infty} I_n$  then  $x \in I_n$  for all  $n \geq 1$ . That is,

$$0 < x < \frac{1}{n} \quad \text{for all } n \geq 1.$$

By Archimedian property,  $\exists n_0 \in \mathbb{N}$  s.t.  $n_0 x > 1$   $\nlessgtr$ . Hence  $\bigcap_{n=1}^{\infty} I_n = \emptyset$ .

**Note.** Let  $I_n = [n, \infty) \subseteq \mathbb{R}$  for all  $n \in \mathbb{N}$ .



Suppose that  $x \in \bigcap_{n=1}^{\infty} I_n$  then  $x \in I_n$  for all  $n \geq 1$ . That is,

$$n \leq x \quad \text{for all } n \geq 1.$$

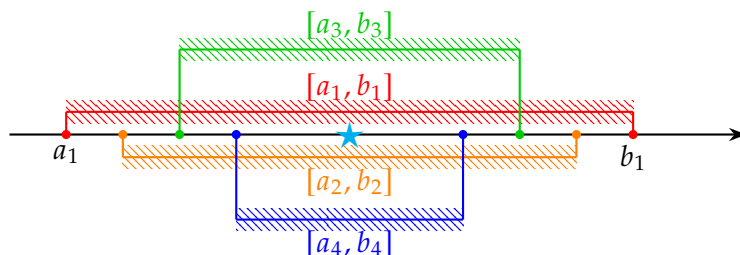
By Archimedian property,  $\exists n_0 \in \mathbb{N}$  s.t.  $x < n_0$   $\nlessgtr$ . Hence  $\bigcap_{n=1}^{\infty} I_n = \emptyset$ .

### Nested Interval Property (NIP)

**Theorem.** Let  $a_n \leq b_n$  for all  $n \in \mathbb{N}$ , and let  $\{[a_n, b_n]\}_{n=1}^{\infty} \subseteq \mathbb{R}$  be a sequence of bounded and closed intervals satisfying  $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$  for all  $n \in \mathbb{N}$ . Then

$$\bigcap_{n=1}^{\infty} [a_n, b_n] := \{x \in \mathbb{R} : x \in [a_n, b_n] \text{ for all } n \in \mathbb{N}\} \neq \emptyset.$$

If  $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$ , then  $|\bigcap_{n=1}^{\infty} [a_n, b_n]| = 1$ .



*Proof.* Since  $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$  for all  $n \in \mathbb{N}$ , we know the sequence  $\{a_n\}$  is monotone increasing, and the sequence  $\{b_n\}$  is monotone decreasing. In other words,

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1.$$

By Monotone Convergence Theorem, we obtain

$$\lim_{n \rightarrow \infty} a_n = \sup_{n \in \mathbb{N}} a_n \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n = \inf_{n \in \mathbb{N}} b_n.$$

Thus,

$$[\forall n \in \mathbb{N}, a_n \leq b_n] \implies \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n \implies \sup_{n \in \mathbb{N}} a_n \leq \inf_{n \in \mathbb{N}} b_n.$$

Then

$$\begin{aligned} x \in \bigcap_{n=1}^{\infty} [a_n, b_n] &\iff \forall n \in \mathbb{N}, a_n \leq x \leq b_n \iff \sup_{n \in \mathbb{N}} a_n \leq x \leq \inf_{n \in \mathbb{N}} b_n \\ &\iff x \in [\sup_{n \in \mathbb{N}} a_n, \inf_{n \in \mathbb{N}} b_n]. \end{aligned}$$

By Set Equality, we have

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = [\sup_{n \in \mathbb{N}} a_n, \inf_{n \in \mathbb{N}} b_n],$$

and so  $[\sup_{n \in \mathbb{N}} a_n, \inf_{n \in \mathbb{N}} b_n] \neq \emptyset$  by Least Upper Bound Property. □

### Monotonicity of Supremum and Infimum

**Proposition.** Let  $A, B \subseteq \mathbb{R}$  be non-empty sets. Let  $B \subseteq A$ . Then

(1)  $\sup B \leq \sup A$ ;

(2)  $\inf A \leq \inf B$ .

*Proof.* (1) Since

$$b \in B \xRightarrow{B \subseteq A} b \in A \xRightarrow{\sup A} b \leq \sup A,$$

$\sup A$  be an upper bound of  $B$ . Since  $\sup B$  is the *least* upper bound of  $B$ , we have  $\sup B \leq \sup A$ .

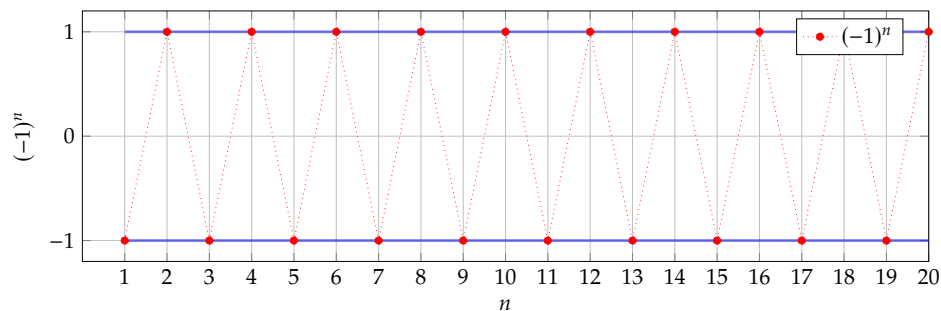
(2) Since

$$b \in B \xRightarrow{B \subseteq A} b \in A \xRightarrow{\inf A} \inf A \leq b,$$

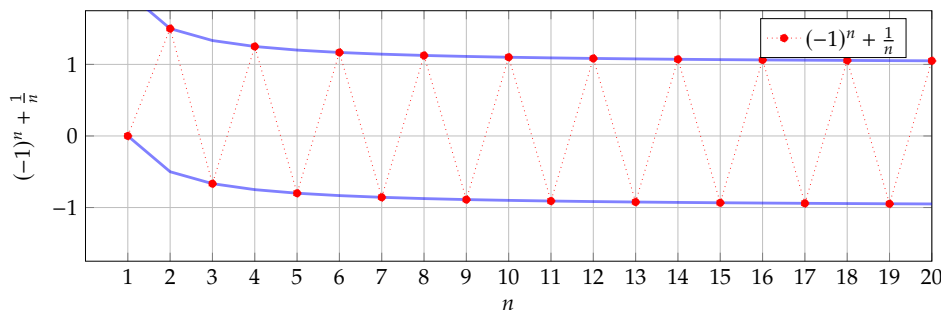
$\inf A$  be a lower bound of  $B$ . Since  $\inf B$  is the *greatest* lower bound of  $B$ , we have  $\inf A \leq \inf B$ .  $\square$

**Observation.**

- What is  $\pm 1$  for the set  $S = \{(-1)^n : n \in \mathbb{N}\}$ ?



- What is  $\pm 1$  for the set  $S = \{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\}$ ?



Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence in  $\mathbb{R}$ . Define

$$s_1 = \sup \{x_1, x_2, x_3, \dots\} = \sup \{x_k : k \geq 1\},$$

$$s_1 = \sup \{x_2, x_3, x_4, \dots\} = \sup \{x_k : k \geq 2\},$$

$\vdots$

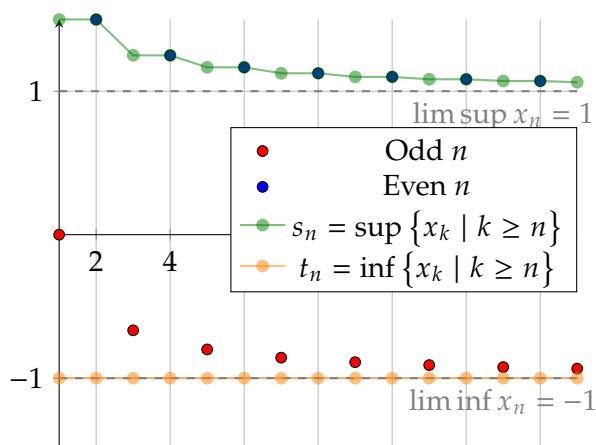
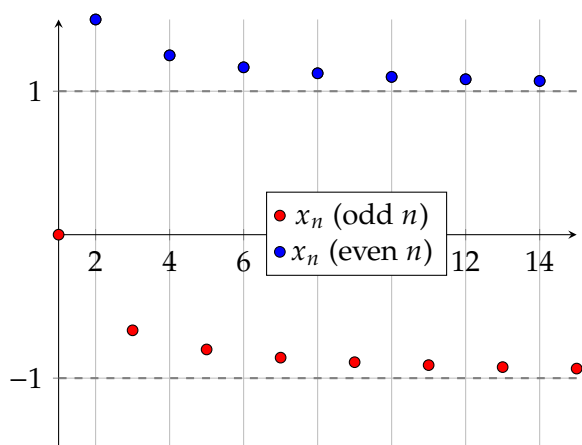
$$s_n = \sup \{x_k, x_{k+1}, \dots\} = \sup \{x_k : k \geq n\}.$$

By monotonicity of supremum,

$$s_1 \geq s_2 \geq \dots \geq s_n \geq s_{n+1} \geq \dots.$$

That is,  $\{s_n\}_{n=1}^{\infty}$  be a monotone decreasing sequence. Similarly, for  $t_n = \inf \{x_k, x_{k+1}, \dots\} = \inf \{x_k : k \geq n\}$ , we have a monotone increasing sequence  $\{t_n\}_{n=1}^{\infty}$ . Consider the followings:

$n$	$(-1)^n$	$\frac{1}{n}$	$x_n = (-1)^n + \frac{1}{n}$	$\sup \{x_k : k \geq n\}$	$\inf \{x_k : k \geq n\}$
1	-1	1	0	1.5	-1
2	1	$\frac{1}{2} = 0.5$	$\frac{3}{2} = 1.5$	1.5	-1
3	-1	$\frac{1}{3} \approx 0.33$	$-\frac{2}{3} \approx -0.67$	1.5	-1
4	1	$\frac{1}{4} = 0.25$	$\frac{5}{4} = 1.25$	1.5	-1
5	-1	$\frac{1}{5} = 0.2$	$-\frac{4}{5} = -0.8$	1.5	-1
6	1	$\frac{1}{6} \approx 0.167$	$\frac{7}{6} \approx 1.167$	1.5	-1
7	-1	$\frac{1}{7} \approx 0.143$	$-\frac{6}{7} \approx -0.857$	1.5	-1
8	1	$\frac{1}{8} = 0.125$	$\frac{9}{8} = 1.125$	1.5	-1
9	-1	$\frac{1}{9} \approx 0.111$	$-\frac{8}{9} \approx -0.889$	1.5	-1
10	1	$\frac{1}{10} = 0.1$	$\frac{11}{10} = 1.1$	1.5	-1



**Note** (Extended Real Number Line). The **extended real number line**  $\overline{\mathbb{R}}$  is defined as

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}.$$

That is, the set of real numbers together with two auxiliary symbols  $+\infty, -\infty$  such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty.$$

### Limit Superior and Limit Inferior

**Definition.** Let  $\{x_n\}$  be a sequence of real numbers.

(1) The **limit superior** of the sequence, denoted by  $\limsup_{n \rightarrow \infty} x_n$  is defined as

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \sup_{k \geq n} x_k \right) := \inf_{n \in \mathbb{N}} \left( \sup_{k \geq n} x_k \right),$$

where  $\sup_{k \geq n} x_k$  represents the supremum of the subsequence  $\{x_k : k \geq n\}$ .

### Bolzano-Weierstrass Theorem

**Theorem.** A bounded sequence of real numbers has a convergent subsequence.

**Remark.**  $\liminf s_n \leq \limsup s_n$

*Proof.* content...

□

**Proposition.**  $\limsup s_n = \beta \iff \begin{cases} \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } n \geq n_0 \Rightarrow s_n < \beta + \varepsilon \\ \forall \varepsilon > 0, \forall n \in \mathbb{N} \exists k \in \mathbb{N} \text{ s.t. } k \geq n \wedge \beta - \varepsilon < s_k. \end{cases}$

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- [2] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 11. 해석학 개론 (f) MCT and NIP” YouTube Video, 20:17. Published October 1, 2019. URL: <https://youtu.be/YdnBQaY5eDk?si=BNe0Ue4iq2P9Fxsd>.
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## A Equivalent Statements of the Least Upper Bound Property

Least Upper Bound Property  $\iff$  Monotone Convergence Theorem  
 $\iff$  Nested Interval Property

**Theorem.** *Monotone Convergence Theorem  $\iff$  Nested Interval Property*

*Proof.*  $(\Rightarrow)$  See **Nested Interval Property**.

$(\Leftarrow)$  TBA

□

**Theorem.** *Least Upper Bound Property  $\iff$  Monotone Convergence Theorem*

*Proof.*  $(\Rightarrow)$  See **Monotone Convergence Theorem**.

$(\Leftarrow)$  TBA

□