Topology I

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We cover the following topics in this note.

- Topology; Topological Space

- Open Set

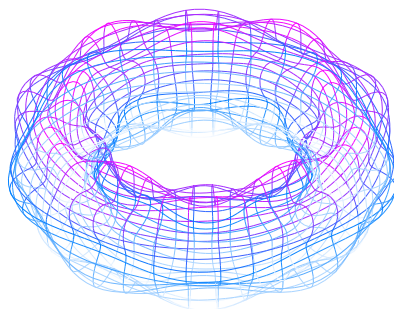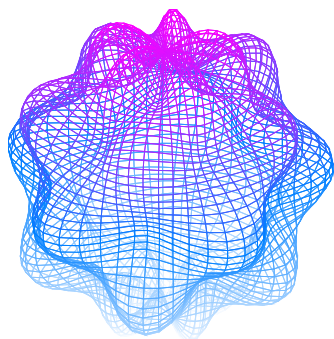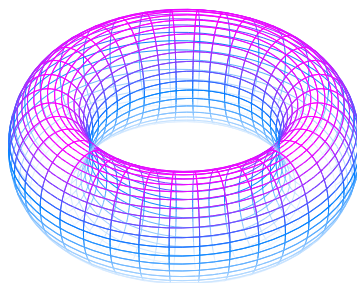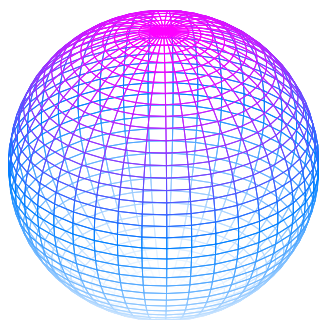- Continuous Mapping

- ~~Distance Function; Metric Topology~~
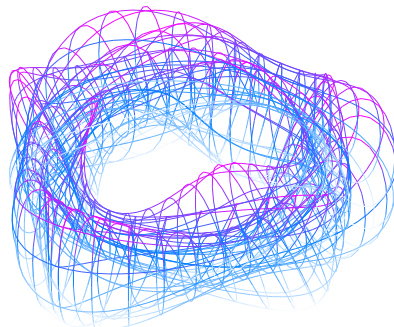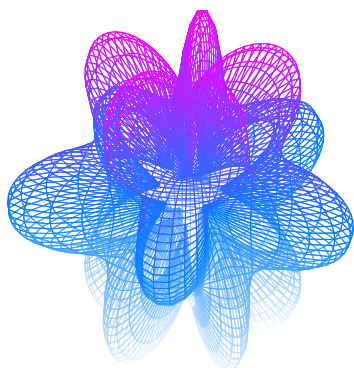
Topology

Definition. Let S be a non-empty set. A **topology** on S is a subset

$$\mathcal{T} = \{U : U \subseteq S\} \subseteq 2^S$$

that satisfies the axioms:

(O1) S and $\varnothing$ are elements of $\mathcal{T}$: $S \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$.

(O2)[a] The union of an arbitrary subset of $\mathcal{T}$ is an element of $\mathcal{T}$:

$$\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \mathcal{T} \implies \bigcup_{\alpha \in \Lambda} U_\alpha \in \mathcal{T}.$$

(O3)[b] The intersection of any finite subset of $\mathcal{T}$ is an element of $\mathcal{T}$:

$$\{U_i\}_{i=1}^n \subseteq \mathcal{T} \implies \bigcap_{i=1}^n U_i \in \mathcal{T}.$$

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[a] $\mathcal{T}$ is closed under *arbitrary* unions
[b] $\mathcal{T}$ is closed under *finite* intersection

Remark. By mathematical induction, we have

$$\text{O3} \iff [\{U_1, U_2\} \subseteq \mathcal{T} \Rightarrow U_1 \cap U_2 \in \mathcal{T}].$$

Topological Space

Definition. Let S be a set. Let $\mathcal{T}$ be a topology on S. Then the ordered pair $(S, \mathcal{T})$ is called a **topological space**.

Open Set

Definition. Let $(S, \mathcal{T})$ be a topological space. $E \subseteq S$ is an **open set**, or **open** (in S) iff $E \in \mathcal{T}$.

Remark. A subset $\mathcal{T} \subseteq 2^S$ is a topology on S if and only if

(i) $\varnothing$ and S are open;

(ii) Let $\{E_\alpha\}_{\alpha \in \Lambda} \subseteq \mathcal{T}$. Then $\displaystyle\bigcup_{\alpha \in \Lambda} E_\alpha$ is open.

(iii) Let $\{E_i\}_{i=1}^n \subseteq \mathcal{T}$. Then $\displaystyle\bigcap_{i=1}^n E_i$ is open.

Continuous Mapping

Definition. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are topological spaces. Let $f : X \to Y$ be a mapping from X to Y.

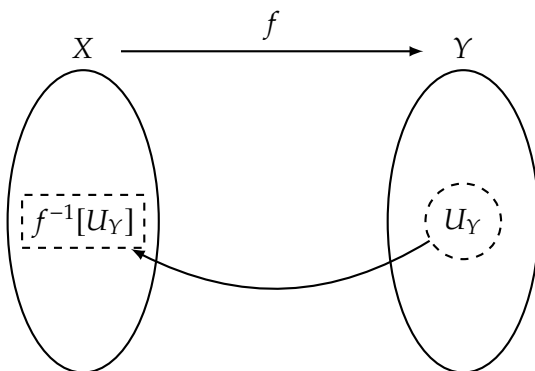(1) (Continuous at a Point) Let $x \in X$. The mapping f is **continuous at x** if and only if

$$\forall U_Y \in \mathscr{T}_Y, \ (f(x) \in U_Y \implies \exists U_X \in \mathscr{T}_X \text{ such that } x \in U_X \land f[U_X] \subseteq U_Y.)$$

(2) (Continuous on a Set) Let $S \subseteq X$. The mapping f is **continuous on S** if and only if f is continuous at every point $x \in S$.

(3) (Continuous Everywhere) The mapping f is **continuous on X** if and only if

$$U_Y \in \mathscr{T}_Y \implies f^{-1}[U_Y] \in \mathscr{T}_X,$$

where $f^{-1}[U_Y] = \{x \in X : f(x) \in U_Y\}$ is the preimage of U_Y under f.

Example 1 (Cofinite Topology). Let S be a set. Define a subset $\mathscr{T}_C \subseteq 2^S$ by

$$\mathscr{T}_C := \left\{ T \subseteq S : T^C \subseteq S \text{ is a finite set} \right\} \cup \{\varnothing\}$$

We claim that $\mathscr{T}_C$ be a topology on S:

(i) Clearly $\subseteq \in \mathscr{T}_C$. Since $S^C = \varnothing$ and $\varnothing$ is finite, $S \in \mathscr{T}$.

(ii) Let $\{E_\alpha\}_{\alpha \in \Lambda} \subseteq \mathscr{T}_C$. Then

$$\left(\bigcup_{\alpha \in \Lambda} E_\alpha \right)^C = \bigcap_{\alpha \in \Lambda} E_\alpha^C$$

and so

(iii)

Example 2 (Discrete Topology).

Example 3 (Indiscrete Topology).

Finer and Coarser

Definition.

> **Distance Function**
>
> **Definition.** TBA

> **Metric Topology**
>
> **Definition.** TBA

References

[1] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 8. 위상수학 (a) 위상공간의 정의." YouTube Video, 41:25. Published September 27, 2019. URL: `https://www.youtube.com/watch?v=q8BtXIFzo2Q`.

[2] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 9. 위상수학 (b) 해석학개론과 거리위상" YouTube Video, 33:43. Published September 29, 2019. URL: `https://www.youtube.com/watch?v=uJOGw7Yxk7c&t=242s`.

A   Complement of Family

Note.

$$\left(\bigcup_{i \in \Lambda} E_i \right)^C = \bigcap_{i \in \Lambda} (E_i)^C$$

Proof. content...   □