Topology I

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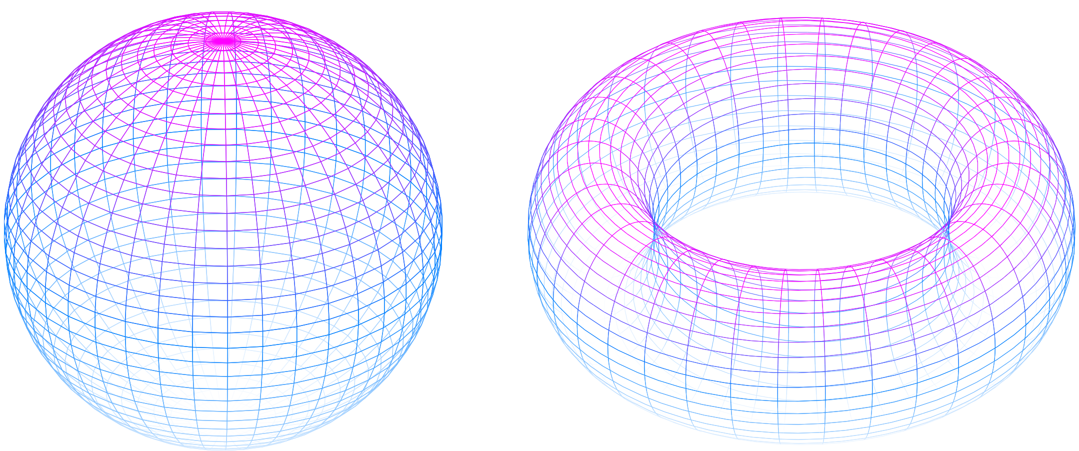
We cover the following topics in this note.

- Topology



Topology

Definition. Let S be a non-empty set. A **topology** on S is a subset

$$\mathcal{T} = \{E : E \subseteq S\} \subseteq 2^S$$

that satisfies the open set axioms:

(O1) $\varnothing$ and S are elements of $\mathcal{T}$: $\{\varnothing, S\} \subseteq \mathcal{T}$.

(O2)[a] The union of an arbitrary subset of $\mathcal{T}$ is an element of $\mathcal{T}$:

$$\{E_\alpha\}_{\alpha\in\Lambda} \subseteq \mathcal{T} \implies \bigcup_{\alpha\in\Lambda} E_\alpha \in \mathcal{T}.$$

(O3)[b] The intersection of any finite subset of $\mathcal{T}$ is an element of $\mathcal{T}$:

$$\{E_i\}_{i=1}^n \subseteq \mathcal{T} \implies \bigcap_{i=1}^n E_i \in \mathcal{T}.$$

[a] $\mathcal{T}$ is closed under *arbitrary* unions
[b] $\mathcal{T}$ is closed under *finite* intersection

Remark. By mathematical induction, we have

$$\text{O3} \iff [\{E_1, E_2\} \subseteq \mathscr{T} \Rightarrow E_1 \cap E_2 \in \mathscr{T}].$$

Example 1 (Cofinite Topology). Let S be a set. Define a subset $\mathscr{T}_C \subseteq 2^S$ by

$$\mathscr{T}_C := \left\{T \subseteq S : T^C \subseteq S \text{ is a finite set}\right\} \cup \{\varnothing\}$$

We claim that $\mathscr{T}_C$ be a topology on S:

(i) Clearly $\subseteq\in \mathscr{T}_C$. Since $S^C = \varnothing$ and $\varnothing$ is finite, $S \in \mathscr{T}$.

(ii) Let $\{E_\alpha\}_{\alpha\in\Lambda} \subseteq \mathscr{T}_C$. Then

$$\left(\bigcup_{\alpha\in\Lambda} E_\alpha\right)^C = \bigcap_{\alpha\in\Lambda} E_\alpha^C$$

and so

(iii)

Topological Space

Definition. Let S be a set. Let $\mathscr{T}$ be a topology on S. Then the ordered pair $(S, \mathscr{T})$ is called a **topological space**.

Open Set

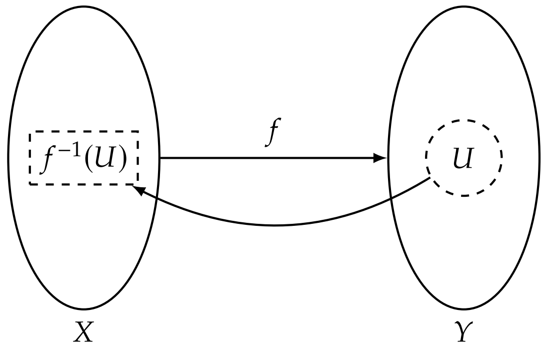
Definition. Let $(S, \mathscr{T})$ be a topological space. $E \subseteq S$ is an **open set**, or **open** (in S) iff $E \in \mathscr{T}$.

Remark. A subset $\mathscr{T} \subseteq 2^S$ is a topology on S if and only if

(i) $\varnothing$ and S are open;

(ii) Let $\{E_\alpha\}_{\alpha\in\Lambda} \subseteq \mathscr{T}$. Then $\bigcup_{\alpha\in\Lambda} E_\alpha$ is open.

(iii) Let $\{E_i\}_{i=1}^n \subseteq \mathscr{T}$. Then $\bigcap_{i=1}^n E_i$ is open.



References

[1] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 8. 위상수학 (a) 위상공간의 정의." YouTube Video, 41:25. Published September 27, 2019. URL: `https://www.youtube.com/watch?v=q8BtXIFzo2Q`.

A Complement of Family

Note.

$$\left(\bigcup_{i\in\Lambda} E_i\right)^C = \bigcap_{i\in\Lambda} (E_i)^C$$

Proof. content... □