Coordinate Charts on a Plane Curve and Its Tangent Line

Let $C \subseteq \mathbb{R}^2$ be the graph of a smooth function $f : \mathbb{R} \to \mathbb{R}$,

$$C = \big\{(x, y) \in \mathbb{R}^2 : y = f(x)\big\}.$$

Fix a point $p = (a, f(a)) \in C$. Its tangent vector is $\vec{v} = (1, f'(a))^T$, and

$$T_pC = \operatorname{span}\{\, (1, f'(a))^T\} \;\subset\; T_p\mathbb{R}^2 \cong \mathbb{R}^2.$$

1. Chart on the Curve C

Define the ambient coordinate projections

$$x, y \;:\; \mathbb{R}^2 \;\longrightarrow\; \mathbb{R}, \qquad x(x, y) = x, \quad y(x, y) = y,$$

and restrict them to C:

$$x|_C : C \to \mathbb{R}, \quad y|_C : C \to \mathbb{R}.$$

These assemble into the smooth chart

$$\Phi_C \;:\; C \;\longrightarrow\; \mathbb{R}^2, \qquad \Phi_C(p) = \big(x|_C(p),\, y|_C(p)\big) = (a, f(a)).$$

Note: Φ_C records the *location* of the point $p \in C \subset \mathbb{R}^2$.

2. Chart on the Tangent Line T_pC

On the ambient tangent plane $T_p\mathbb{R}^2 \cong \mathbb{R}^2$ we have the dual projections

$$dx,\, dy \;:\; T_p\mathbb{R}^2 \;\longrightarrow\; \mathbb{R}, \qquad dx\big((v^1, v^2)^T\big) = v^1,\; dy\big((v^1, v^2)^T\big) = v^2.$$

Restrict these functionals to the line T_pC:

$$dx\big|_{T_pC},\; dy\big|_{T_pC} \;:\; T_pC \;\longrightarrow\; \mathbb{R}.$$

Stacking them gives the fiber-chart

$$\Psi_{T_pC} \;:\; T_pC \;\longrightarrow\; \mathbb{R}^2, \qquad \Psi_{T_pC}(\vec{v}) = \begin{pmatrix} dx(\vec{v}) \\ dy(\vec{v}) \end{pmatrix} = \begin{pmatrix} 1 \\ f'(a) \end{pmatrix}.$$

Note: Ψ_{T_pC} records the *components* of the tangent vector in the ambient basis $\{\partial_x, \partial_y\}$.

3. Distinguishing Points vs. Vectors

- A *point* $p = (a, f(a)) \in C$ is an element of the set C. Its chart–coordinate $\Phi_C(p) = (a, f(a)) \in \mathbb{R}^2$ tells *where* on the curve p lies.

- A *tangent vector* $\vec{v} \in T_pC$ is an element of the tangent space, encoding a *direction and speed* at p. Its chart–coordinate $\Psi_{T_pC}(\vec{v}) = (dx(\vec{v}),\, dy(\vec{v})) \in \mathbb{R}^2$ gives its components relative to $\{\partial_x, \partial_y\}$.