# Set Theory II

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We cover the following topics in this note.

- Relations
- Equivalence Relations
- Partitions

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## Relation

**Definition.** Let $A \times B$ be the cartesian product of two sets $A$ and $B$. A **(binary) relation** on $A \times B$ is a subset $\mathcal{R}$ of $A \times B$. That is,

$$\mathcal{R} \text{ is a relation on } A \times B \iff \mathcal{R} \subseteq A \times B.$$

**Remark.** $\mathcal{R}$ is a relation on $A \iff \mathcal{R} \subseteq A \times A$.

**Note** (Notation). Let $(s,t) \in \mathcal{R}$. We use the notation

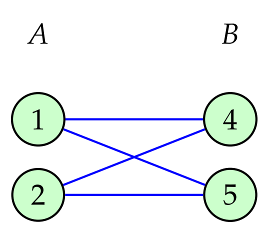
$$s \; \mathcal{R} \; t$$

and we can say "$s$ **is related to** $t$ **by** $R$". If $(s,t) \notin \mathcal{R}$, we denote as: $s \not\mathcal{R} \; t$.

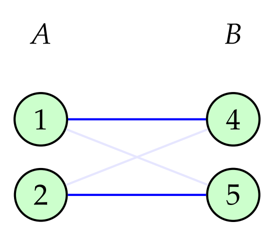
**Example.** Let $A = \{1,2\}$ and $B = \{4,5\}$. Then

$$A \times B = \left\{(1,4),(1,5),(2,4),(2,5)\right\}.$$

Here, $\mathcal{R} = \left\{(1,4),(2,5)\right\} \subseteq A \times B$ be a relation.




$A \times B$




$\mathcal{R} \subseteq A \times B$

**Example.** Let $A$ and $B$ are sets, and let $f : A \to B$ be a function form $A$ to $B$. Then

$$(a,b) \in f \iff a \; f \; b \iff b = f(a).$$

**⋆ Equivalence Relation ⋆**

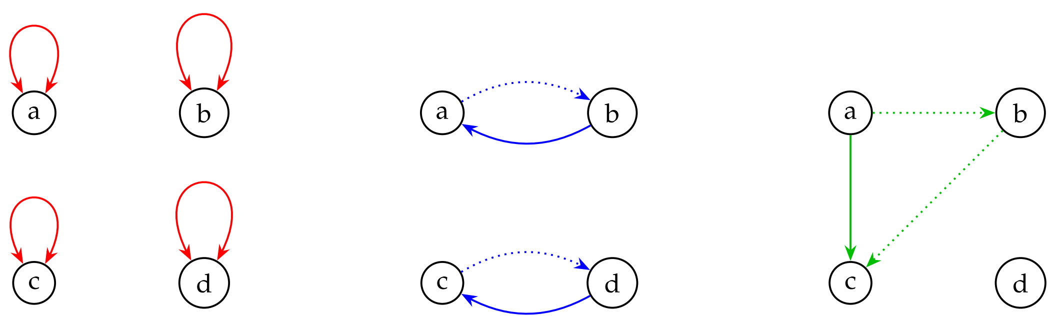
**Definition.** A binary relation $\mathcal{R}$ on a set $S$ is called an **equivalence relation** if it satisfies the following three properties: for all $a, b, c \in S$,

(i) (Reflexivity) $(a, a) \in \mathcal{R}$;

(ii) (Symmetry) $(a, b) \in \mathcal{R} \implies (b, a) \in \mathcal{R}$;

(iii) (Transitivity) $(a, b) \in \mathcal{R} \wedge (b, c) \in \mathcal{R} \implies (a, c) \in \mathcal{R}$.

**Remark.**




Reflexivity
(each element is related to itself)

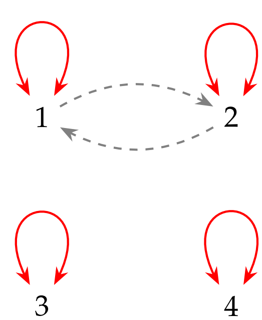
Symmetry
(if $a$ is related to $b$, then $b$ is related to $a$)

Transitivity
(if $a$ is related to $b$ and $b$ is related to $c$, then $a$ is related to $c$)

**Example.** Let $A = \{1, 2, 3, 4\}$. Then

$$\mathcal{R} = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1)\}$$

is an equivalence relation on $A$.



**Note.** Let $A, B, C$ are sets, and let $f : A \to B$ and $g : B \to C$ are functions.

- We claim that $\boxed{(g \circ f)[A] = g[f[A]]}$:

$$(g \circ f)[A] = \{(g \circ f)(a) : a \in A\} = \{g(f(a)) : a \in A\} = \{g(b) : b = f(a) \in f[A]\} = g[f[A]].$$

- We claim that $\boxed{f \text{ is surjective} \iff \mathrm{Img}(f) = f[A] = B}$:

$$f : A \twoheadrightarrow B \iff \forall b \in B, \exists a \in A \text{ s.t. } f(a) = b \iff f[A] = \{f(a) \in B : a \in A\} = B.$$

**Lemma 1** *Let $A, B$ and $C$ are sets, and let $f : A \to B$ and $g : B \to C$ are functions.*

*(1) If $f$ and $g$ are both one-to-one, then $(g \circ f) : A \to C$ is one-to-one.*

*(2) If $f$ and $g$ are both onto, then $(g \circ f) : A \to C$ is onto.*

*Proof.* (1) Let $f$ and $g$ are both one-to-one. We must show that $(g \circ f) : A \to C$ is one-to-one. Suppose that $(g \circ f)(a) = (g \circ f)(a')$. Then

$$\begin{aligned} (g \circ f)(a) = (g \circ f)(a') &\implies g(f(a)) = g(f(a')) && \text{by def. of composition} \\ &\implies f(a) = f(a') && \because g \text{ is injective} \\ &\implies a = a'. && \because f \text{ is injective} \end{aligned}$$

(2) Let $f$ and $g$ are both onto. We must show that $(g \circ f) : A \to C$ is onto, i.e., $(g \circ f)[A] = C$.

$$\begin{aligned} (g \circ f)[A] &= g[f[A]] \\ &= g[B] && \because f : A \to B \text{ is surjective, i.e., } f[A] = B \\ &= C. && \because g : B \to C \text{ is surjective, i.e., } g[B] = C \end{aligned}$$

□

**Lemma 2** *Let $A, B$ and $C$ are sets, and let $f : A \to B$ and $g : B \to C$ are functions.*

*(1) If $(g \circ f) : A \to C$ is one-to-one, then $f$ is one-to-one.*

*(2) If $(g \circ f) : A \to C$ is onto, then $g$ is onto.*

*Proof.* (1) Let $g \circ f$ is one-to-one. We must show that $f$ is one-to-one. Suppose that $f(a) = f(a')$. Then

$$\begin{aligned} f(a) = f(a') &\implies g(f(a)) = g(f(a')) && \because g \text{ is a function} \\ &\implies (g \circ f)(a) = (g \circ f)(a') && \text{by the def. of composition} \\ &\implies a = a'. && \because g \circ f \text{ is injective} \end{aligned}$$

(2) Let $g \circ f$ is onto, i.e., $(g \circ f)[A] = C$. We must show that $g$ is onto, i.e., $g[B] = C$:

($\subseteq$) $g[B] = \left\{g(b) \in C : b \in B\right\} \subseteq C$;

($\supseteq$) $C = (g \circ f)[A] = g[f[A]] = \left\{g(b) \in C : b \in f[A]\right\} \subseteq g[B]$.

$\square$

**Equivalence Relation on $2^A$ Based on Bijection**

**Proposition 3** *Let $A$ be a set, and $2^A$ be its power set. Define a relation $\sim_{\mathcal{R}}$ on $2^A$ as follows:*

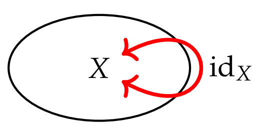
$$X \sim_{\mathcal{R}} Y \iff \exists (f : X \to Y) \text{ such that } f \text{ is bijective},$$

*for $X, Y \in 2^A$. In other words,*

$$\sim_{\mathcal{R}} := \left\{(X, Y) \in 2^A \times 2^A : \exists \text{ a bijection } (f : X \to Y)\right\}.$$

*Proof.* Let $X, Y, Z \in 2^A$. We must show that $\sim_{\mathcal{R}}$ is reflexive, symmetric and transitive:

(i) (Reflexivity) We NTS[1] that $X \sim_{\mathcal{R}} X$. In other words, we need to find a bijection from $X$ it self.



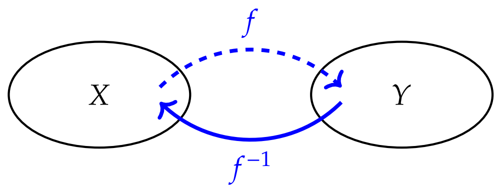

Consider the identity function

$$\begin{aligned} \mathrm{id}_X \ : \ X &\longrightarrow X \\ x &\longmapsto x = \mathrm{id}_X(x) \end{aligned}$$

for all $x \in X$. Clearly, $\mathrm{id}_X$ is a bijection. Thus, $X \sim_{\mathcal{R}} X$.

[1]'NTS' means that "need to show".

(ii) (Symmetry) We NTS that $X \sim_{\mathcal{R}} Y \implies Y \sim_{\mathcal{R}} X$. In other words, if there exists a bijection $f : X \to Y$, then there must exists a bijection $g : Y \to X$.

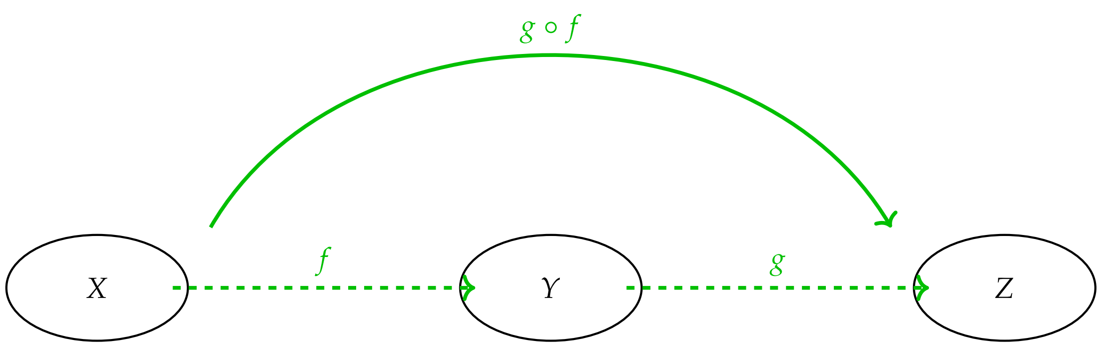



Suppose that $f : X \to Y$ is a bijection. Then it has an inverse function $f^{-1} : Y \to X$, which satisfies:

$$\forall x \in X : f^{-1}(f(x)) = x = \mathrm{id}_X(x) \quad \text{and} \quad \forall y \in X : f(f^{-1}(y)) = y = \mathrm{id}_Y(y).$$

That is, $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$. By **Lemma 2**, $f^{-1}$ must be a bijection since $f$, $\mathrm{id}_X$ and $\mathrm{id}_Y$ are bijections. Thus, there is a bijection $g = f^{-1}$.

(iii) We NTS that $X \sim_{\mathcal{R}} Y \wedge Y \sim_{\mathcal{R}} Z \implies X \sim_{\mathcal{R}} Z$. In other words, if there exists two bijection $f : X \to Y$ and $g : Y \to Z$, then there must exists a bijection $h : X \to Z$.




Suppose that $f : X \to Y$ and $g : Y \to Z$ both are bijective. Define the function

$$\begin{array}{rccl} h & : & X & \longrightarrow & Z \\ & & x & \longmapsto & (g \circ f)(x) = h(x) \end{array}$$

for all $x \in X$. By **Lemma 1**, $h = g \circ f$ must be a bijection since $f$ and $g$ are both bijective.

Hence it is proved. $\square$

## Indexed Family

**Definition.** Let $I$ and $S$ are sets. Consider a function $A : I \to S$ defined by $i \mapsto A(i) =: A_i$. The image $\mathrm{Img}(A)$ is called an **indexed family of elements in $S$ indexed by $I$**. We write this indexed family as:

$$\langle A_i \rangle_{i \in I}.$$

Note that

$$\mathrm{Img}(A) = \{A(i) : i \in I\} = \{A_i : i \in I\} = \langle A_i \rangle_{i \in I}.$$

**Example** (Sequence). Let $I = \mathbb{N}$ be an indexing set. Then

$$S := \{A_1, A_2, A_3, A_4, \cdots\} = \{A_i : i \in \mathbb{N}\} = \langle A_i \rangle_{i \in \mathbb{N}}$$

is an indexed family of elements in $S$ indexed by $\mathbb{N}$.

## Union and Intersection of an Indexed Family

**Definition.** Let $I$ and $S$ are sets, and let $\langle A_i \rangle_{i\in}$ be an indexed family in $S$.

- The **union of** $\langle A_i \rangle_{i\in}$ is defined by

$$\bigcup_{i \in I} A_i := \{x \in S : \exists i \in I \text{ such that } x \in A_i\}.$$

- The **intersection of** $\langle A_i \rangle_{i\in}$ is defined by

$$\bigcap_{i \in I} A_i := \{x \in X : \forall i \in I,\ x \in A_i\}.$$

**Remark.** Let $I = \mathbb{N}$. Then

- $\bigcup_{i \in I} S_i = \bigcup_{i=1}^{n} S_i = S_1 \cup S_2 \cup \cdots \cup S_n.$
- $\bigcap_{i \in I} S_i = \bigcap_{i=1}^{n} S_i = S_1 \cap S_2 \cap \cdots \cap S_n.$

**⋆ Partitions ⋆**

**Definition.** Let $S$ be a set. Let $\langle A_i \rangle_{i\in I}$ be a family of subsets of $S$, where $A_i \subseteq S$ for each index $i \in I$. The family $\langle A_i \rangle_{i\in I}$ is called a **partition of** $S$ if the following conditions hold:

(i) (**Non-empty Subsets**) $A_i \neq \varnothing$ for all $i \in I$. Formally

$$\boxed{\forall i \in I,\ A_i \neq \varnothing}.$$

(ii) (**Pairwise Disjoint**) For any $i, j \in I$, if $i \neq j$, then $A_i \cap A_j = \varnothing$. Formally

$$\boxed{\forall i, j \in I,\ \left[i \neq j \implies A_i \cap A_j = \varnothing\right]}.$$

(iii) (**Union Covers the Whole Set**) The union of all sets $A_i$ is the whole set $S$. Formally

$$\boxed{\bigcup_{i\in A} A_i = S}.$$

**Example.** Let $\mathbb{Z}$ be a set of integers. We define an indexed family $\langle A_i \rangle_{i\in\{0,1,2\}}$ of subsets of $\mathbb{Z}$ as follows:

$$\begin{aligned}
A_0 &= \left\{n \in \mathbb{Z} : n \equiv 0 \pmod 3\right\} = \{n \in \mathbb{Z} : n = 3k + 0 \text{ for some } k \in \mathbb{Z}\} =: [0],\\
A_1 &= \left\{n \in \mathbb{Z} : n \equiv 1 \pmod 3\right\} = \{n \in \mathbb{Z} : n = 3k + 1 \text{ for some } k \in \mathbb{Z}\} =: [1],\\
A_2 &= \left\{n \in \mathbb{Z} : n \equiv 2 \pmod 3\right\} = \{n \in \mathbb{Z} : n = 3k + 2 \text{ for some } k \in \mathbb{Z}\} =: [2].
\end{aligned}$$

Then

(i) $[0] \neq \varnothing$, $[1] \neq \varnothing$ and $[2] \neq \varnothing$.

(ii)

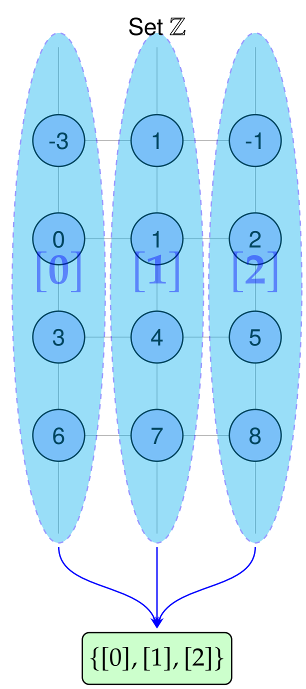
$$\begin{aligned}
[0] \cap [1] &= \varnothing,\\
[1] \cap [2] &= \varnothing,\\
[2] \cap [0] &= \varnothing.
\end{aligned}$$

(iii) $[0] \cup [1] \cup [2] = \mathbb{Z}$.

Thus,

$$\{A_1, A_2, A_3\} = \left\{[0], [1], [2]\right\}$$

is a partition of $\mathbb{Z}$.



**⋆ Equivalence Class ⋆**

**Definition.** Let $\mathcal{R} \subseteq S \times S$ be an equivalence relation on $S$. The **equivalence class of** $x \in S$ **under** $\mathcal{R}$ is the set

$$[x]_{\mathcal{R}} = \left\{ y \in S : x \,\mathcal{R}\, y \right\}.$$

**Note.** Note that $\alpha \,\mathcal{R}\, x \iff \alpha \in [x]_{\mathcal{R}} \iff x \,\mathcal{R}\, \alpha$.

**Lemma 4** *Let $\mathcal{R}$ be an equivalence relation on a set $S$. For any $x, y \in S$, let $[x]$ and $[y]$ represent the equivalence classes of $x$ and $y$, respectively, under $\mathcal{R}$.*

*(1)* $\forall x \in S,\ x \in [x]$.

*(2)* $x \,\mathcal{R}\, y \iff [x] = [y]$.

*(3)* $x \,\not{\mathcal{R}}\, y \iff [x] \cap [y] = \varnothing$.

*Proof.* (1) Let $x \in S$. Since $\mathcal{R}$ is reflexive, we have $x \,\mathcal{R}\, x$, i.e., $x \in [x]$.

(2) $(\Rightarrow)$ Let $x \,\mathcal{R}\, y$. We NTS that $[x] = [y]$:

$(\subseteq)$ Let $\alpha \in [x]$, i.e., $\alpha \,\mathcal{R}\, x$. Then

$$\begin{aligned} \alpha \,\mathcal{R}\, x &\implies \alpha \,\mathcal{R}\, y && \because x \,\mathcal{R}\, y \text{ and } \mathcal{R} \text{ is transitive} \\ &\implies \alpha \in [y]. \end{aligned}$$

$(\supseteq)$ Let $\beta \in [y]$, i.e., $y \,\mathcal{R}\, \beta$. Then

$$\begin{aligned} y \,\mathcal{R}\, \beta &\implies x \,\mathcal{R}\, \beta && \because x \,\mathcal{R}\, y \text{ and } \mathcal{R} \text{ is transitive} \\ &\implies \beta \in [x]. \end{aligned}$$

$(\Leftarrow)$ Let $[x] = [y]$. Then

$$x \in S \overset{\text{by (1)}}{\implies} x \in [x] = [y] \implies x \in [y] \implies x \,\mathcal{R}\, y.$$

(3) $(\Rightarrow)$ Let $x \,\not{\mathcal{R}}\, y$. Suppose that $[x] \cap [y] \neq \varnothing$ then $\exists \gamma \in S$ such that $\gamma \in [x] \cap [y]$. Then

$$\gamma \in [x] \cap [y] \implies \gamma \in [x] \wedge \gamma \in [y] \implies x \,\mathcal{R}\, \gamma \wedge \gamma \,\mathcal{R}\, y \implies x \,\mathcal{R}\, y \lightning.$$

$(\Leftarrow)$ Let $[x] \cap [y] = \varnothing$. Suppose that $x \,\mathcal{R}\, y$. By (1) and (2), we have $x \in [x] = [y] \lightning$.

$\square$

**Theorem 5** *Let $S$ be a set. Let $\mathcal{R}$ be an equivalence relation on $S$. The set of equivalence classes*

$$\mathcal{P} := \left\{[x]_{\mathcal{R}} : x \in S\right\} \text{ where } [x]_{\mathcal{R}} = \left\{y \in S : x \mathcal{R} y\right\}$$

*forms a partition of $S$.*

*Proof.* We must show that the set of equivalence classes $\left\{[x]_{\mathcal{R}} : x \in S\right\}$ satisfies the three conditions of a partition:

(i) (Equivalence Class is not Empty) By (1) of **Lemma 4**, it is proved.

(ii) (Equivalence Classes are Disjoint) By (2) and (3) of **Lemma 4**, it is proved.

(iii) (Union of Equivalence Classes is Whole Set) We NTS that $\bigcup \left\{[x]_{\mathcal{R}} : x \in S\right\} = S$:

($\subseteq$) Since $[x]_{\mathcal{R}} \subseteq S$, we have

$$\bigcup \left\{[x]_{\mathcal{R}} : x \in S\right\} = \bigcup_{x \in S} [x]_{\mathcal{R}} \subseteq S.$$

($\supseteq$) Let $\alpha \in S$. We want to show that $\alpha \in \bigcup_{x \in S} [x]_{\mathcal{R}}$, i.e.,

$$\exists x \in S \text{ such that } \alpha \in [x].$$

By (1) of **Lemma 4**, we obtain $\alpha \in [\alpha]$. Thus, for every $\alpha \in S$, $\alpha \in \bigcup_{x \in S} [x]_{\mathcal{R}}$.

$\square$

**Theorem 6** *Let $S$ be a set and $\mathcal{P} = \langle P_i \rangle_{i \in I}$ a partition of $S$. We define a relation $\sim_{\mathcal{P}}$ be the relation defined as:*

$$x \sim_{\mathcal{P}} y \iff \exists i \in I \text{ such that } x, y \in P_i$$

*for all $x, y \in S$. That is, $x$ is related to $y$ under $\sim_{\mathcal{P}}$ if and only if $x$ and $y$ belong to the same subset $P_i$ in the partition. Then $\sim_{\mathcal{P}}$ is the equivalence relation induced by a partition $\mathcal{P}$.*

*Proof.* Let $\langle P_i \rangle_{i \in I}$ be a partition of $S$. That is,

$$\text{(a) } P_i \neq \varnothing \text{ for all } i \in I; \qquad \text{(b) } P_i \cap P_j = \varnothing \text{ for } i \neq j; \qquad \text{(c) } \bigcup_{i \in I} P_i = S.$$

Let $x, y \in S$. Note that

$$\sim_{\mathcal{P}} := \left\{ (x, y) \in S \times S : \exists i \in I \text{ s.t. } x \in P_i \wedge y \in P_i \right\}.$$

We NTS $\sim_{\mathcal{P}}$ is reflexive, symmetric and transitive:

(i) (Reflexivity) We NTS that $x \sim_{\mathcal{P}} x$:

$$x \in S \overset{\text{by (c)}}{\implies} x \in \bigcup_{i \in I} P_i \implies \exists i \in I \text{ s.t. } x \in P_i \implies \exists i \in I \text{ s.t. } x \in P_i \wedge x \in P_i \implies x \sim_{\mathcal{P}} x.$$

(ii) (Symmetry) We NTS that $x \sim_{\mathcal{P}} y \implies y \sim_{\mathcal{P}} x$:

$$x \sim_{\mathcal{P}} y \implies \exists i \in I \text{ s.t. } x \in P_i \wedge y \in P_i \implies \exists i \in I \text{ s.t. } y \in P_i \wedge x \in P_i \implies y \sim_{\mathcal{P}} x.$$

(iii) (Transitivity) We NTS that $x \sim_{\mathcal{P}} y \wedge y \sim_{\mathcal{P}} z \implies x \sim_{\mathcal{P}} z$:

$$\begin{cases} x \sim_{\mathcal{P}} y \\ y \sim_{\mathcal{P}} z \end{cases} \implies \begin{cases} \exists i \in I \text{ s.t. } x \in P_i \wedge y \in P_i \\ \exists j \in I \text{ s.t. } y \in P_j \wedge z \in P_j \end{cases} \overset{\text{by (b)}, i=j}{\implies} \exists i = j \in I \text{ s.t. } x \in P_i \wedge z \in P_i \implies x \sim_{\mathcal{P}} z.$$

□

## References

[1] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 3. 집합론 기초 (c)." YouTube Video, 35:04. Published September 07, 2019. URL: https://www.youtube.com/watch?v=2gM-Vh8CY8I&list=PL4m4z_pFWq2pLwFsWf0KJX_uMNo-jktN5&index=136.