Topology I

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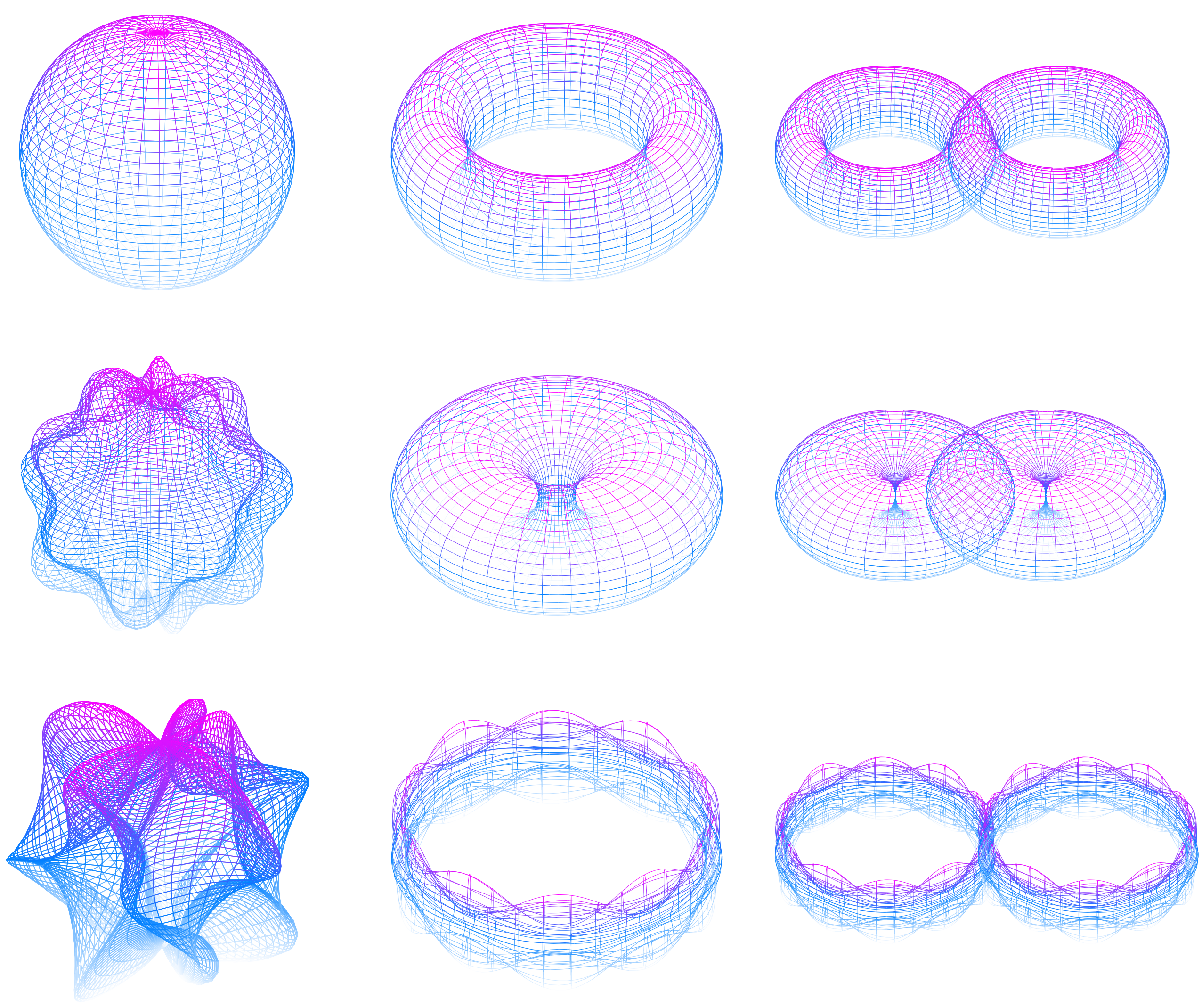
We cover the following topics in this note.

- Topology and Topological Space
- Open Set
- Continuous Mapping
- Distance Function and Metric Space
- Continuity of Function



Topology

Definition. Let S be a non-empty set. A **topology**[a] on S is a subset

$$\mathscr{T} \subseteq 2^S = \{U : U \subseteq S\}$$

that satisfies the axioms:

(O1) S and $\varnothing$ are elements of $\mathscr{T}$: $S \in \mathscr{T}$ and $\varnothing \in \mathscr{T}$.

(O2)[b] The union of an arbitrary collection in $\mathscr{T}$ is an element of $\mathscr{T}$:

$$\{U_\alpha\}_{\alpha\in\Lambda} \subseteq \mathscr{T} \implies \bigcup_{\alpha\in\Lambda} U_\alpha \in \mathscr{T}.$$

(O3)[c] The intersection of any finite collection in $\mathscr{T}$ is an element of $\mathscr{T}$:

$$\{U_i\}_{i=1}^n \subseteq \mathscr{T} \implies \bigcap_{i=1}^n U_i \in \mathscr{T}.$$

[a] The word "topology" comes from the Greek roots "topos" meaning "place" and "logos" meaning "study".
[b] $\mathscr{T}$ is closed under *arbitrary* unions
[c] $\mathscr{T}$ is closed under *finite* intersection

Remark. By mathematical induction, we have

$$\text{O3} \iff [\{U_1, U_2\} \subseteq \mathscr{T} \Rightarrow U_1 \cap U_2 \in \mathscr{T}].$$

Topological Space

Definition. Let $S \neq \varnothing$ be a set. Let $\mathscr{T}$ be a topology on S. Then the ordered pair $(S, \mathscr{T})$ is called a **topological space**.

Open Set (Topology)

Definition. Let $(S, \mathscr{T})$ be a topological space. $U \subseteq S$ is an **open set**, or **open** (in S) iff $U \in \mathscr{T}$.

Remark. A subset $\mathscr{T} \subseteq 2^S$ is a topology on S if and only if

(i) $\varnothing$ and S are open;

(ii) Let $\{U_\alpha\}_{\alpha\in\Lambda} \subseteq \mathscr{T}$. Then $\displaystyle\bigcup_{\alpha\in\Lambda} U_\alpha$ is open.

(iii) Let $\{U_i\}_{i=1}^n \subseteq \mathscr{T}$. Then $\displaystyle\bigcap_{i=1}^n U_i$ is open.

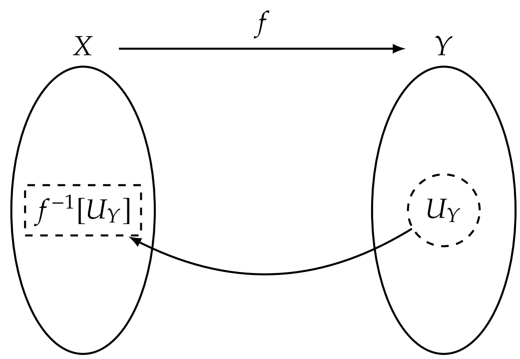
Continuous Mapping by Open Sets

Definition. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are topological spaces. Let $f : X \to Y$ be a mapping from X to Y.

(1) (Continuous Everywhere) The mapping f is **continuous on X** if and only if

$$U_Y \in \mathscr{T}_Y \implies f^{-1}[U_Y] \in \mathscr{T}_X,$$

where $f^{-1}[U_Y] = \left\{x \in X : f(x) \in U_Y\right\}$ is the preimage of U_Y under f.



Note (Preparation for **Example 1**). Let $S \neq \varnothing$ be a set, and let $\{A_\alpha\}_{\alpha \in \Lambda} \subseteq S$. Then

$$\begin{aligned} S \setminus \bigcup_{\alpha \in \Lambda} A_\alpha = S \setminus \{x \in S : \exists \alpha \in \Lambda \text{ s.t. } x \in A_\alpha\} &= \left\{x \in S : \neg[\exists \alpha \in \Lambda \text{ s.t. } x \in A_\alpha]\right\} \\ &= \{x \in S : \forall \alpha \in \Lambda,\ x \notin A_\alpha\} \\ &= \left\{x \in S : \forall \alpha \in \Lambda,\ x \in S \setminus A_\alpha\right\} \\ &= \bigcap_{\alpha \in \Lambda} (S \setminus A_\alpha). \end{aligned}$$

$$\begin{aligned} S \setminus \bigcap_{\alpha \in \Lambda} A_\alpha = S \setminus \{x \in S : \forall \alpha \in \Lambda,\ x \in A_\alpha\} &= \left\{x \in S : \neg[\forall \alpha \in \Lambda,\ x \in A_\alpha]\right\} \\ &= \{x \in S : \exists \alpha \in \Lambda \text{ s.t. } x \notin A_\alpha\} \\ &= \left\{x \in S : \exists \alpha \in \Lambda \text{ s.t. } x \in S \setminus A_\alpha\right\} \\ &= \bigcup_{\alpha \in \Lambda} (S \setminus A_\alpha). \end{aligned}$$

Note (Preparation for **Example 1**).

(1) A Subset of a Finite Set is Finite.

(2) The Intersection of Finite Sets is Finite.

Example 1 (Cofinite Topology). Let $S \neq \varnothing$ be a set. Define the cofinite topology $\mathscr{T}_C \subseteq 2^S$ by

$$\begin{aligned}\mathscr{T}_C := & \left\{U \subseteq S : S \setminus U \text{ is finite}\right\} \cup \{\varnothing\} \\ = & \left\{U \subseteq S : U = \varnothing \text{ or } S \setminus U \text{ is finite}\right\}.\end{aligned}$$

In other words, U is open in the cofinite topology if U is the empty, or if the complement $S \setminus U$ is a finite set. We claim that $\mathscr{T}_C$ be a topology on S:

(O1) By definition, $\varnothing \in \mathscr{T}_C$. For $U = S$, the complement $S \setminus S = \varnothing$, which is finite, so $S \in \mathscr{T}_C$. Hence, both $\varnothing$ and S are elements of $\mathscr{T}_C$.

(O2) Let $\{U_\alpha\}_{\alpha\in\Lambda} \subseteq \mathscr{T}_C$.

(Case 1) If $U_\alpha = \varnothing$ for all $\alpha \in \Lambda$, then $\bigcup_{\alpha\in\Lambda} U_\alpha = \varnothing \in \mathscr{T}_C$.

(Case 2) Suppose that there exists $\alpha_0 \in \Lambda$ such that $U_{\alpha_0} \neq \varnothing$. Then

$$S \setminus \bigcup_{\alpha\in\Lambda} U_\alpha = \bigcap_{\alpha\in\Lambda} (S \setminus U_\alpha) \subseteq S \setminus U_{\alpha_0}.$$

Since $S \setminus U_{\alpha_0}$ is finite, $S \setminus \bigcup_{\alpha\in\Lambda} U_\alpha$ if finite, so $\bigcup_{\alpha\in\Lambda} U_\alpha \in \mathscr{T}_C$.

(O3) Let $U_1 \in \mathscr{T}_C$ and $U_2 \in \mathscr{T}_C$.

(Case 1) If $U_1 = \varnothing$ or $U_2 = \varnothing$, then $U_1 \cap U_2 = \varnothing \in \mathscr{T}_C$.

(Case 2) Suppose that $U_1 \neq \varnothing$ and $U_2 \neq \varnothing$. Then $S \setminus U_1$ and $S \setminus U_2$ are finite. By the De Morgan law, we have

$$S \setminus (U_1 \cap U_2) = (S \setminus U_1) \cup (S \setminus U_2),$$

which is a finite set. Thus, $U_1 \cap U_2 \in \mathscr{T}_C$.

Example 2 (Discrete Topology). Let $S \neq \varnothing$ be a set, and let $\mathscr{T} = 2^S$ be the power set of S. Then $\mathscr{T}$ is called the **discrete topology** on S and $(S, \mathscr{T}) = (S, 2^S)$ the **discrete (topological) space** on S.

Example 3 (Indiscrete Topology). Let $S \neq \varnothing$ be a set, and let $\mathscr{T} = \{S, \varnothing\}$. Then $\mathscr{T}$ is called the **indiscrete topology** on S and $(S, \mathscr{T}) = (S, \{S, \varnothing\})$ the **indiscrete (topological) space** on S.

Note.

(1) Discrete Topology is Finest Topology.

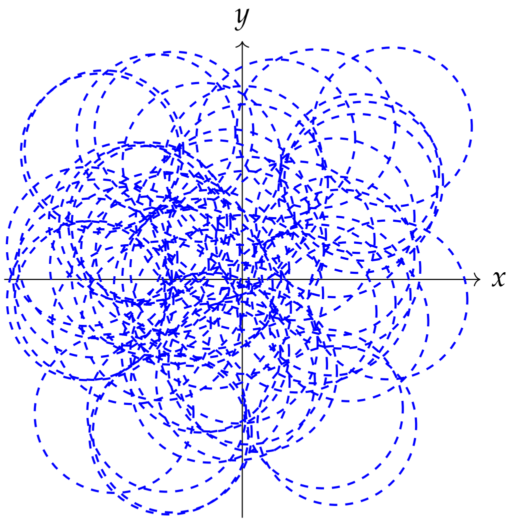
(2) Indiscrete Topology is Coarsest Topology.

Coarser Topology and Finer Topology

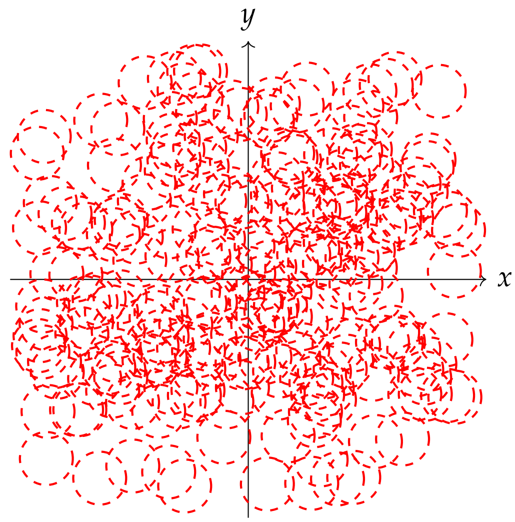
Definition. Let $S \neq \varnothing$ be a set. Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be topologies on S.

(1) $\mathscr{T}_1$ is said to be **coarser** than $\mathscr{T}_2$ if $\mathscr{T}_1 \subseteq \mathscr{T}_2$.

(2) $\mathscr{T}_1$ is said to be **finer** than $\mathscr{T}_2$ if $\mathscr{T}_2 \subseteq \mathscr{T}_1$.



Coarser Topology



Finer Topology

Distance Function

Definition. Let S be a set. The function $d : S \times S \to \mathbb{R}$ is called a **distance function** (or **metric**) if it satisfies the following properties:

(i)[a] $\forall x, y \in S,\ d(x,y) \geq 0 \quad \text{and} \quad d(x,y) = 0 \Leftrightarrow x = y.$

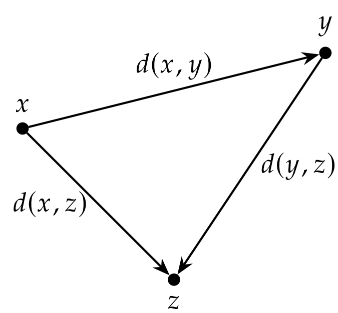
(ii)[b] $\forall x, y \in S,\ d(x,y) = d(y,x).$

(iii)[c] $\forall x, y, z \in S,\ d(x,z) \leq d(x,y) + d(y,z).$

The pair (S, d) is called a **metric space**.

[a] Non-negativity and Zero only for identical points
[b] Symmetry
[c] Triangle inequality

Remark.



Example 4.

- Let $S = \mathbb{R}$, the set of real numbers. Define the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$d(x,y) = |x - y|$$

for $x, y \in \mathbb{R}$.

- Let $S = \mathbb{R}^n$, the n-dimensional Euclidean space. Define the function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=0}^{n-1} |x_i - y_i|^2},$$

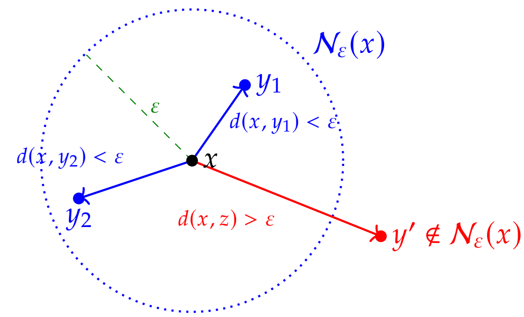
where $\mathbf{x} = (x_0, x_1, \ldots x_{n-1})$ and $\mathbf{y} = (y_0, \ldots, y_{n-1})$ are vectors in $\mathbb{R}^n$.

Neighborhood

Definition. Let (S, d) be a metric space, where S is a set and $d : S \times S \longrightarrow \mathbb{R}$ is a metric. For $x \in S$ and $\varepsilon > 0$, the **ε-neighborhood of** x , denoted by $\mathcal{N}_\varepsilon(x)$, is defined as

$$\mathcal{N}_\varepsilon(x) := \left\{ y \in S : d(x, y) < \varepsilon \right\}.$$

Remark.

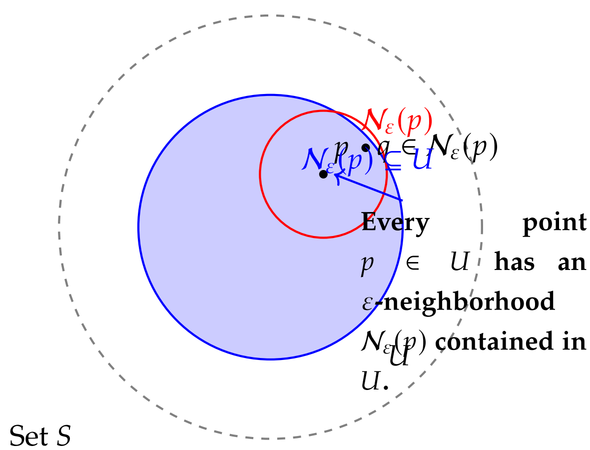


Open Set (Metric Space)

Definition. Let (S, d) be a metric space, where S is a set and $d : S \times S \longrightarrow \mathbb{R}$ is a metric. Let $U \subseteq S$. Then U is an **open set** in S (as metric space) if and only if it is a neighborhood of each of its elements, i.e.,

$$U \text{ is open in } S \overset{\text{def}}{\iff} \forall p \in U,\ \exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(p) \subseteq U.$$

Remark.



Exercise (Metric Topology). Let (S, d) be a metric space, where S is a set and $d : S \times S \to \mathbb{R}$ is a metric. Consider the set τ of all open sets of S:

$$\begin{aligned} \tau := & \left\{ U \subseteq S : U \text{ is open in } S \right\} \\ = & \left\{ U \subseteq S : \forall p \in U,\ \exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(p) \subseteq U \right\}. \end{aligned}$$

We claim that τ is the topology on the metric space (S, d):

(O1)

(O2)

(O3)

Note (Convergence of Sequences). A sequence $\{a_n\}_{n=1}^{\infty}$ $(\subseteq \mathbb{R})$ is **converge** to $L \in \mathbb{R}$ if and only if

$$\begin{aligned} & \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \left[n \geq N \implies |a_n - L| < \varepsilon \right] \\ \iff & \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \left[n \geq N \implies d(a_n, L) < \varepsilon \right] \\ \iff & \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \left[n \geq N \implies a_n \in \mathcal{N}_\varepsilon(L) \right] \end{aligned}$$

Continuity of Functions

Definition. Let $S \subseteq$ be a non-empty subset of $\mathbb{R}$. Let $f : S \to \mathbb{R}$ be a real-valued function, and let $a \in S$. We say that f is **continuous at** a if and only if

$$\lim_{x \to a} f(x) = f(a).$$

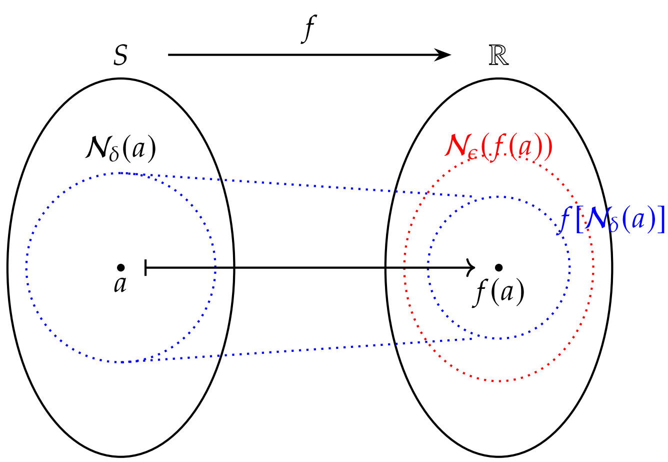
That is,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

If f is continuous on every point of S, then f is called a **continuous function on** S.

Remark.

$$\begin{aligned}
& \forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon \\
\iff & \forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad x \in \mathcal{N}_\delta(a) \implies f(x) \in \mathcal{N}_\varepsilon(f(a)) \\
\iff & \forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad f(x) \in f[\mathcal{N}_\delta(a)] \implies f(x) \in \mathcal{N}_\varepsilon(f(a)) \quad \because f[\mathcal{N}_\delta(a)] = \left\{ f(x) : x \in \mathcal{N}_\delta(a) \right\} \\
\iff & \forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad f[\mathcal{N}_\delta(a)] \subseteq \mathcal{N}_\varepsilon\left(f(a)\right).
\end{aligned}$$



Remark. f is discontinuous at a if and only if

$$\begin{aligned}
& \exists \varepsilon > 0 \quad \text{such that} \quad \forall \delta > 0,\ |x - \alpha| < \delta \text{ but } |f(x) - f(a)| \geq \varepsilon \\
\iff & \exists \varepsilon > 0 \quad \text{such that} \quad \forall \delta > 0,\ \mathcal{N}_\varepsilon\left(f(a)\right) \subset f\left[\mathcal{N}_\delta(a)\right].
\end{aligned}$$

References

[1] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 8. 위상수학 (a) 위상공간의 정의.” YouTube Video, 41:25. Published September 27, 2019. URL: https://www.youtube.com/watch?v=q8BtXIFzo2Q.

[2] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 9. 위상수학 (b) 해석학개론과 거리위상” YouTube Video, 33:43. Published September 29, 2019. URL: https://www.youtube.com/watch?v=uJOGw7Yxk7c&t=242s.