A Gentle Note on $dz = dx + \mathrm{i}\,dy$, Dual Frames, and the Winding Form $\dfrac{\mathrm{d}z}{z}$

Goal and mindset

We view $\mathbb{C} \cong \mathbb{R}^2$ with coordinates x, y and complex coordinate $z = x + \mathrm{i}y$. One-forms act on vector fields:

$$dx(\partial_x) = 1, \quad dx(\partial_y) = 0, \qquad dy(\partial_x) = 0, \quad dy(\partial_y) = 1.$$

You can think of dx as "measuring the x-component of motion" and ignoring motion tangent to x-level sets; similarly for dy. This note packages them into the *complex* 1-form

$$dz = dx + \mathrm{i}\,dy,$$

explains what dz does to vectors, and shows why holomorphic differentials are multiples of dz. We finish with the geometric decomposition of the winding form $\mathrm{d}z/z$ into radial and angular parts.

1 What dz does to a vector

Let $v = a\,\partial_x + b\,\partial_y$. Then

$$dz(v) = dx(v) + \mathrm{i}\,dy(v) = a + \mathrm{i}b.$$

Thus dz turns the real vector (a, b) into the complex number $a + \mathrm{i}b$: its modulus is the speed, its argument is the direction angle.

2 Complex frames $\partial_z, \partial_{\bar{z}}$ and the dual coframe $dz, d\bar{z}$

Define

$$\partial_z := \tfrac{1}{2}\big(\partial_x - \mathrm{i}\,\partial_y\big), \qquad \partial_{\bar{z}} := \tfrac{1}{2}\big(\partial_x + \mathrm{i}\,\partial_y\big),$$

and

$$dz := dx + \mathrm{i}\,dy, \qquad d\bar{z} := dx - \mathrm{i}\,dy.$$

Then $\{dz, d\bar{z}\}$ is dual to $\{\partial_z, \partial_{\bar{z}}\}$:

$$dz(\partial_z) = 1, \quad dz(\partial_{\bar{z}}) = 0, \qquad d\bar{z}(\partial_{\bar{z}}) = 1, \quad d\bar{z}(\partial_z) = 0.$$

So dz detects motion in the *holomorphic* direction ∂_z and kills the anti-holomorphic direction $\partial_{\bar{z}}$.

Example 1 (How holomorphicity appears)**.** *For a complex-valued $f(x, y)$,*

$$df = f_x\,dx + f_y\,dy = f_z\,dz + f_{\bar{z}}\,d\bar{z}, \qquad f_z = \tfrac{1}{2}(f_x - \mathrm{i}f_y), \quad f_{\bar{z}} = \tfrac{1}{2}(f_x + \mathrm{i}f_y).$$

Holomorphicity is exactly the condition $f_{\bar{z}} = 0$, i.e. df is a multiple of dz only. This is the 1-form version of the Cauchy–Riemann equations.

3 Level-set intuition, revisited

Because dz is complex-valued, talking about a single "level set" is less natural. Instead:

$$\mathrm{Re}(dz) = dx \quad (\text{vertical lines } x = \text{const}), \qquad \mathrm{Im}(dz) = dy \quad (\text{horizontal lines } y = \text{const}).$$

The pair (dx, dy) forms two orthogonal foliations; dz packages both and carries orientation via its complex phase.

4 Integrating dz

For a path $\gamma : [a, b] \to \mathbb{C}$, $\gamma(t) = x(t) + \mathrm{i}y(t)$,

$$\int_\gamma dz = \int_\gamma (dx + \mathrm{i}\, dy) = \big[x(t) + \mathrm{i}y(t)\big]_{t=a}^{t=b} = z(\gamma(b)) - z(\gamma(a)).$$

Thus dz is exact with potential z. This is why on $\mathbb{C}$ the "flat holomorphic 1-form" is just dz.

5 The winding form $\dfrac{\mathrm{d}z}{z}$

Away from $z = 0$, write $z = re^{\mathrm{i}\theta}$ ($r > 0$, $\theta = \arg z$). Then

$$\frac{\mathrm{d}z}{z} = \frac{d(re^{\mathrm{i}\theta})}{re^{\mathrm{i}\theta}} = \frac{dr}{r} + \mathrm{i}\, d\theta = d(\log r) + \mathrm{i}\, d \arg z.$$

$\mathrm{Re}(\mathrm{d}z/z) = d(\log r)$ measures *radial* change; $\mathrm{Im}(\mathrm{d}z/z) = d\theta$ measures *angular* change (winding).

Example 2 (Winding number). *If γ is a closed loop avoiding 0,*

$$\frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{\mathrm{d}z}{z} = \text{winding number of } \gamma \text{ around } 0 \in \mathbb{Z}.$$

Indeed $\oint d(\log r) = 0$ (since $\log r$ is single-valued on $\mathbb{C}^\times$), and $\oint d\theta = 2\pi$ (winding).

6 Mini-computations and sanity checks

Example 3 (Action on basis vectors). *$dz(\partial_x) = 1$, $dz(\partial_y) = \mathrm{i}$. For $v = \partial_x + \partial_y$, $dz(v) = 1 + \mathrm{i}$.*

Example 4 (Directional derivative of a holomorphic function). *If f is holomorphic, $df = f_z\, dz$ with $f_z = \partial f/\partial z$. For $v = a\,\partial_x + b\,\partial_y$,*

$$df(v) = f_z\, dz(v) = f_z\, (a + \mathrm{i}b).$$

So the real 2D directional derivative is the complex derivative times the complex number representing v.

Example 5 (Line integrals). *Let $\gamma(t) = t$ on $[0, 1]$. Then $\int_\gamma dz = 1$. Let $\gamma(t) = e^{\mathrm{i}t}$ on $[0, 2\pi]$. Then $\oint \frac{\mathrm{d}z}{z} = 2\pi\mathrm{i}$.*

7 Exercises (with short solutions at the end)

Exercise 1. *Show that $dz(\partial_{\bar z}) = 0$ and $d\bar z(\partial_z) = 0$.*

Exercise 2. *Write df in the $(dz, d\bar z)$-basis and show that f holomorphic $\iff f_{\bar z} = 0$.*

Exercise 3. *Let $\gamma(t) = re^{\mathrm{i}t}$ with constant $r > 0$, $t \in [t_0, t_1]$. Compute $\int_\gamma \mathrm{d}z/z$.*

Exercise 4. *Prove that for γ closed avoiding 0, $\frac{1}{2\pi\mathrm{i}} \oint_\gamma \mathrm{d}z/z \in \mathbb{Z}$.*

Sketch solutions

1) Use definitions: $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$ and $dz = dx + \mathrm{i}dy$.
2) Compute $df = f_x\,dx + f_y\,dy$ and rewrite as $f_z\,dz + f_{\bar{z}}\,d\bar{z}$.
3) $\mathrm{d}z/z = \mathrm{i}\,dt$ along γ (since $dr = 0$), so the integral is $\mathrm{i}(t_1 - t_0)$.
4) Approximate γ by a polygon and use the argument principle, or apply homotopy invariance of $\oint \mathrm{d}z/z$ on $\mathbb{C}^\times$ together with the circle case.

Where this connects next

On a complex torus $X = \mathbb{C}/\Lambda$ the (nowhere-vanishing) holomorphic 1-form is the pushdown of dz. Choosing cycles a, b gives periods $\int_a dz = 1$, $\int_b dz = \tau$. From dz one builds the Weierstrass $\wp$-function and the elliptic curve $y^2 = 4x^3 - g_2x - g_3$. (See the companion note: constructing $\wp$ from $dt = \omega$.)