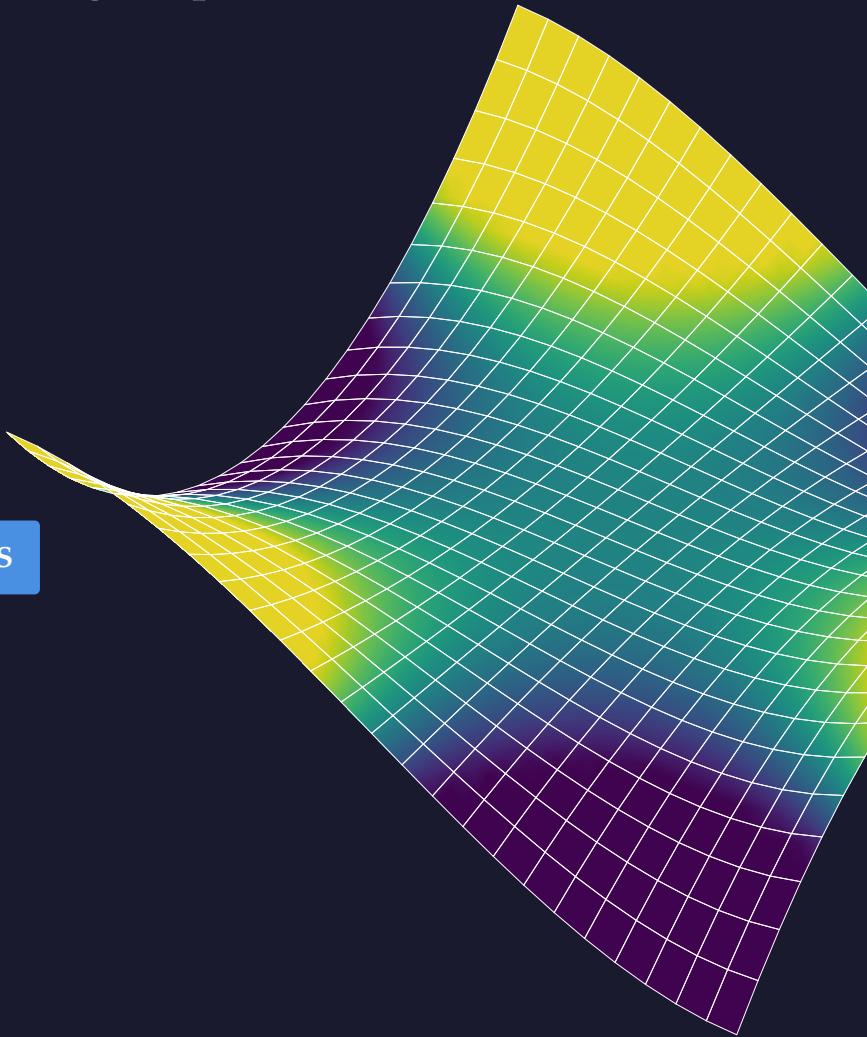


Riemann Surfaces and Algebraic Curves

A framework for understanding Elliptic Curves

Ji, Yonghyeon

PART I — MULTIVARIABLE CALCULUS



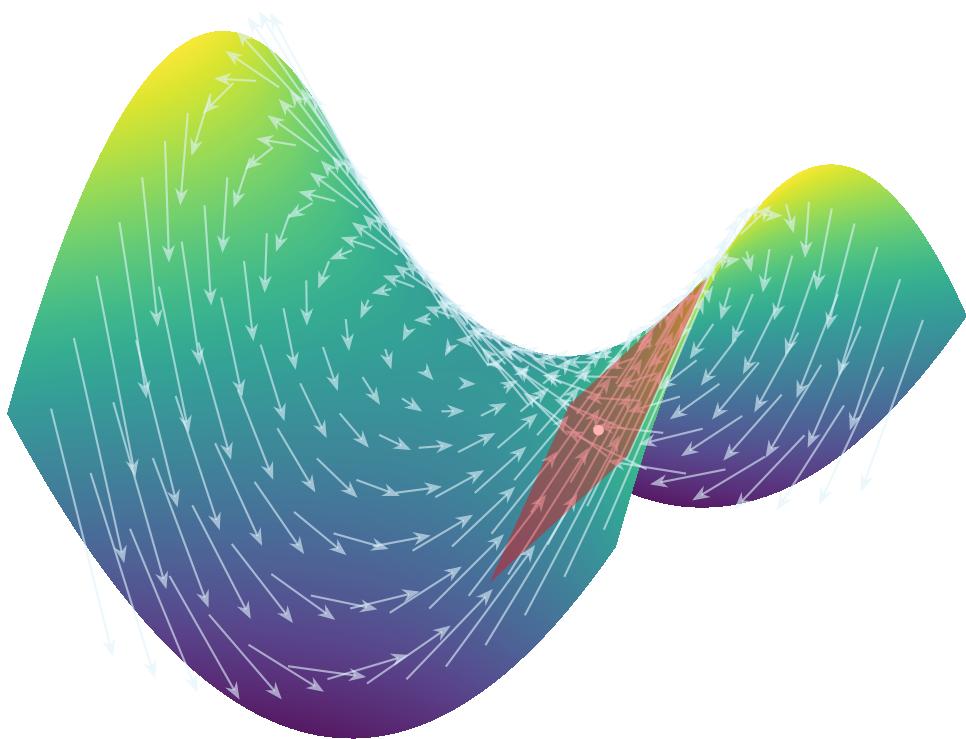
Riemann Surfaces and Algebraic Curves

A Framework for Understanding Elliptic Curves

Part I — Multivariable Calculus

Ji, Yonghyeon

February 11, 2026



WINTER 2026

The FTC hierarchy

Name	Formula
FTC I (Accumulation)	$\frac{d}{dx} \left(\int_a^x f(t) dt \right) = f(x).$
FTC II (Evaluation)	$\int_a^b f'(x) dx = f(b) - f(a).$
Fundamental Theorem of Line Integrals	$\int_C \nabla \phi \cdot d\mathbf{r} = \phi(B) - \phi(A).$
Green's Theorem	$\oint_{\partial R} P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$
Stokes' Theorem (3D)	$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS.$
Divergence Theorem	$\iint_{\partial V} \mathbf{F} \cdot \mathbf{n} dS = \iiint_V (\nabla \cdot \mathbf{F}) dV.$
Generalized Stokes	$\int_{\partial \Omega} \omega = \int_{\Omega} d\omega.$

Copyright

Copyright © 2025 by Ji, Yonghyeon All rights reserved.

No part of this publication may be reproduced, distributed, or transmitted in any form or by any means, including photocopying, recording, or other electronic or mechanical methods, without the prior written permission of the publisher, except in the case of brief quotations embodied in critical reviews and certain other noncommercial uses permitted by copyright law.

Changelog

v1.0 2025-12-29 Initial release.

Contents

1	Fundamental Theorem of Calculus	3
1.1	Gradient Vector Fields	4
1.2	Green's Theorem	19
1.3	Divergence Theorem	23
1.4	Stokes' Theorem	27
2	Differential Forms	31
3	Winding Numbers and Complexification	32

1 Fundamental Theorem of Calculus

Fundamental Theorem for Gradient Fields

If $\mathbf{F} = \nabla f$ is a conservative vector field and C is a smooth curve from A to B , then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Green's Theorem

For a positively oriented, simple closed curve C bounding a region R in the plane,

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Divergence Theorem

Let \mathbf{F} be a vector field defined on a region E with closed boundary surface S (outward-oriented). Then

$$\iiint_E \nabla \cdot \mathbf{F} dV = \iint_S \mathbf{F} \cdot \mathbf{n} dS.$$

Stokes' Theorem

Let S be an oriented surface with boundary curve C , and let \mathbf{F} be a vector field. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS.$$

Triple Integral

To integrate a scalar function $f(x, y, z)$ over a region E in \mathbb{R}^3 ,

$$\iiint_E f(x, y, z) dV.$$

1.1 Gradient Vector Fields

Scalar field

Definition 1.1. Let $U \subseteq \mathbb{R}^n$ be an open set. A scalar field on U is a function

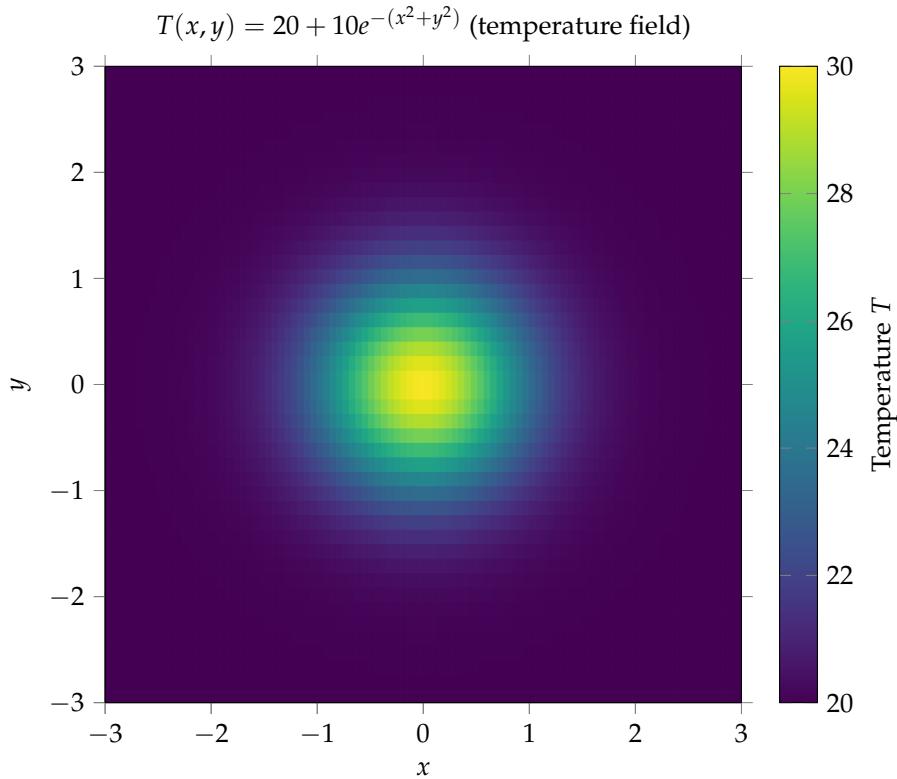
$$\begin{aligned} f &: U \longrightarrow \mathbb{R} \\ \mathbf{x} &\longmapsto f(\mathbf{x}) . \end{aligned}$$

Equivalently, to each point $\mathbf{x} \in U$ the scalar field assigns a real number $f(\mathbf{x})$.

Example 1 (Temperature distribution). Let $U = \mathbb{R}^2$. Define

$$T(x, y) = 20 + 10e^{-(x^2+y^2)}.$$

Then $T : U \rightarrow \mathbb{R}$ is a scalar field. One may interpret $T(x, y)$ as the temperature (in degrees) at the point (x, y) . Notice that $T(0, 0) = 30$ and $T(x, y) \rightarrow 20$ as $x^2 + y^2 \rightarrow \infty$, so the temperature is highest at the origin and decays outward.



Vector field

Definition 1.2. Let $U \subseteq \mathbb{R}^n$ be an open set. A vector field on U is a function

$$\begin{aligned}\mathbf{F} : U &\longrightarrow \mathbb{R}^n \\ \mathbf{x} &\longmapsto \mathbf{F}(\mathbf{x})\end{aligned}$$

Equivalently, to each point $\mathbf{x} \in U$ the vector field assigns a vector $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^n$. In coordinates, one often writes

$$\mathbf{F}(\mathbf{x}) = \langle F_1(\mathbf{x}), F_2(\mathbf{x}), \dots, F_n(\mathbf{x}) \rangle,$$

where each component function $F_i : U \rightarrow \mathbb{R}$ is a scalar field on U .

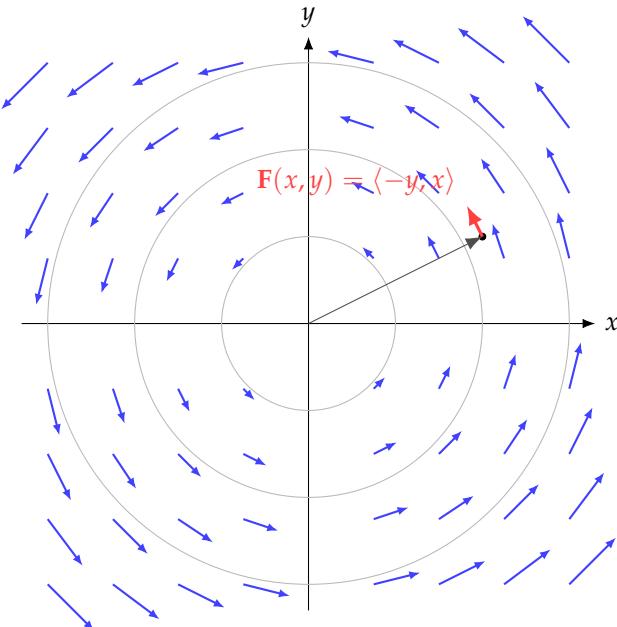
Example 2 (Rotation field). Let $U = \mathbb{R}^2$. Define

$$\mathbf{F}(x, y) = \langle -y, x \rangle.$$

Then $\mathbf{F} : U \rightarrow \mathbb{R}^2$ is a vector field. At each point (x, y) it assigns the vector $\langle -y, x \rangle$, which is perpendicular to $\langle x, y \rangle$ and tangent to the circle $x^2 + y^2 = \text{constant}$. Moreover,

$$\|\mathbf{F}(x, y)\| = \sqrt{x^2 + y^2},$$

so the magnitude increases linearly with the distance from the origin. This field models a rigid counterclockwise rotational flow about the origin.



Remark.

Definition 1.3 (Conservative vector field). Let $U \subseteq \mathbb{R}^n$ be an open set and let $\mathbf{F} : U \rightarrow \mathbb{R}^n$ be a vector field. We say that \mathbf{F} is conservative on U if there exists a scalar field $f : U \rightarrow \mathbb{R}$ of class C^1 such that

$$\mathbf{F} = \nabla f \quad \text{on } U.$$

In this case, f is called a potential function for \mathbf{F} .

Remark (Equivalent characterization). A vector field \mathbf{F} on U is conservative if and only if for every piecewise C^1 curve C in U with endpoints \mathbf{A}, \mathbf{B} , the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on \mathbf{A} and \mathbf{B} (path independence). Equivalently,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

for every piecewise C^1 closed curve C in U .

Definition 1.4 (Gradient operator). Let $U \subseteq \mathbb{R}^n$ be open. The gradient operator (or nabla operator) is the map

$$\nabla : C^1(U) \longrightarrow C^0(U, \mathbb{R}^n)$$

defined by

$$(\nabla f)(\mathbf{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix}, \quad \mathbf{x} = (x_1, \dots, x_n) \in U.$$

Remark (Directional derivative characterization). For $f \in C^1(U)$ and $\mathbf{x} \in U$, the vector $\nabla f(\mathbf{x}) \in \mathbb{R}^n$ is uniquely characterized by the property that

$$D_{\mathbf{v}}f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{v} \quad \text{for all } \mathbf{v} \in \mathbb{R}^n,$$

where $D_{\mathbf{v}}f(\mathbf{x})$ denotes the directional derivative of f at \mathbf{x} in the direction \mathbf{v} .

[Linearity of ∇] Let $f, g \in C^1(U)$ and $a, b \in \mathbb{R}$. Then

$$\nabla(af + bg) = a \nabla f + b \nabla g.$$

Remark (Jacobian transpose viewpoint). For $f \in C^1(U)$, the total derivative at $\mathbf{x} \in U$ is the linear map

$$Df(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}, \quad Df(\mathbf{x}) \mathbf{v} = D_{\mathbf{v}}f(\mathbf{x}).$$

In coordinates, $Df(\mathbf{x})$ is represented by the $1 \times n$ Jacobian row matrix

$$Df(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}) \quad \cdots \quad \frac{\partial f}{\partial x_n}(\mathbf{x}) \right),$$

and the gradient is its transpose:

$$\nabla f(\mathbf{x}) = \left(Df(\mathbf{x}) \right)^T.$$

Consequently, for any $\mathbf{v} \in \mathbb{R}^n$,

$$D_{\mathbf{v}}f(\mathbf{x}) = Df(\mathbf{x}) \mathbf{v} = \left(\nabla f(\mathbf{x}) \right)^T \mathbf{v} = \nabla f(\mathbf{x}) \cdot \mathbf{v}.$$

Fundamental Theorem for Gradient Fields

Theorem 1.5. Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \rightarrow \mathbb{R}$ be continuously differentiable ($f \in C^1(U)$). Let C be a piecewise C^1 curve in U with a piecewise C^1 parametrization

$$\mathbf{r} : [a, b] \rightarrow U.$$

If $\mathbf{r}(a) = \mathbf{A}$ and $\mathbf{r}(b) = \mathbf{B}$, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{B}) - f(\mathbf{A}).$$

In particular, the line integral of a gradient field depends only on the endpoints of the curve.

Proof. By definition of the line integral of a vector field along a parametrized curve,

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

Define a scalar-valued function $g : [a, b] \rightarrow \mathbb{R}$ by $g(t) = f(\mathbf{r}(t))$. Since $f \in C^1(U)$ and \mathbf{r} is piecewise C^1 , the composition g is piecewise C^1 . On any subinterval where \mathbf{r} is C^1 , the multivariable chain rule gives

$$g'(t) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

Hence on each such subinterval we have $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = g'(t)$, and summing over the finitely many smooth pieces yields

$$\int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_a^b g'(t) dt.$$

By the (single-variable) Fundamental Theorem of Calculus,

$$\int_a^b g'(t) dt = g(b) - g(a).$$

Substituting back $g(t) = f(\mathbf{r}(t))$ gives

$$g(b) - g(a) = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(\mathbf{B}) - f(\mathbf{A}).$$

Therefore,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{B}) - f(\mathbf{A}),$$

as claimed. □

Consider a small rectangle centered at (x_0, y_0) with side lengths $\Delta x, \Delta y$.

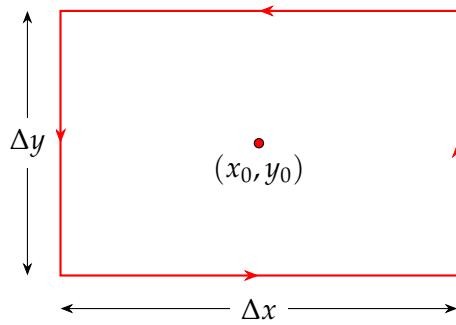


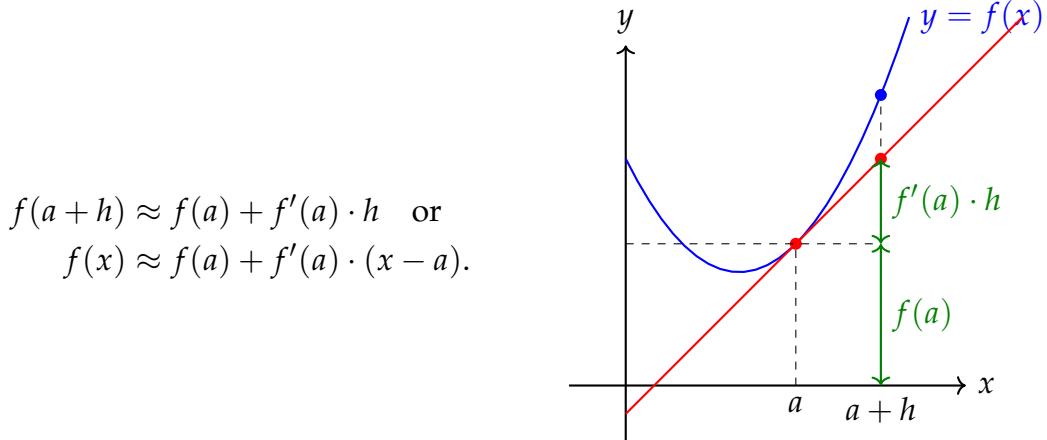
Figure 1: Circulation around an infinitesimal rectangle.

The total counterclockwise circulation is the sum of the line integrals along the four edges:

$$\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \int_{\text{bottom}} P dx + \int_{\text{right}} Q dy + \int_{\text{top}} P dx + \int_{\text{left}} Q dy.$$

We will approximate the value of P or Q along each edge as being constant, equal to its value at the midpoint of that edge. We find this value using a first-order Taylor expansion from the center point (x_0, y_0) .

For a simple function of one variable, $f(x)$, if we know its value at a point a , then we can estimate its value at a nearby point $a + h$ using the tangent line at a :



In words, “New Value \approx Old Value + (Rate of Change) \times (Small Step)”.

For a function of two variables like $P(x, y)$, the idea is identical, but the “rate of change” now has two components (one for each direction), and the “tangent line” becomes a “tangent plane”. The general first-order Taylor expansion for $P(x, y)$ around a center point (x_0, y_0) is

$$P(x_0 + a, y_0 + b) \approx P(x_0, y_0) + \frac{\partial P}{\partial x}(x_0, y_0) \cdot a + \frac{\partial P}{\partial y}(x_0, y_0) \cdot b$$

Here, a is the small step in the x -direction, and b is the small step in the y -direction.

1. The Horizontal Paths These integrals involve the horizontal component of $P(x, y)$.

- **Bottom Path (\rightarrow):**

$$\begin{aligned} P\left(x, y_0 - \frac{\Delta y}{2}\right) &\approx P(x_0, y_0) - \frac{\partial P}{\partial y}(x_0, y_0) \frac{\Delta y}{2} \\ \implies \int_{\text{bottom}} P \, dx &\approx \int_{-\Delta x/2}^{\Delta x/2} \left(P_0 + P_x s - P_y \frac{\Delta y}{2} \right) \, ds \quad (x = x_0 + s, \, dx = ds) \\ \implies \int_{\text{bottom}} P \, dx &\approx \left(P(x_0, y_0) - \frac{\partial P}{\partial y}(x_0, y_0) \frac{\Delta y}{2} \right) (\Delta x) \end{aligned}$$

Note that

$$\int_{-\Delta x/2}^{\Delta x/2} P_x s \, ds = P_x \left[\frac{s^2}{2} \right]_{-\Delta x/2}^{\Delta x/2} = P_x \left(\frac{(\Delta x/2)^2}{2} - \frac{(-\Delta x/2)^2}{2} \right) = 0.$$

- **Top Path (\leftarrow):**

$$P\left(x, y_0 + \frac{\Delta y}{2}\right) \approx P(x_0, y_0) + \frac{\partial P}{\partial y} \frac{\Delta y}{2} \implies \int_{\text{top}} P \, dx \approx - \left(P(x_0, y_0) + \frac{\partial P}{\partial y} \frac{\Delta y}{2} \right) (\Delta x)$$

Here, we are left with only the parts that describe the *change* in P with respect to y .

$$\int_{\text{bottom}} P \, dx + \int_{\text{top}} P \, dx \approx \left(-\frac{\partial P}{\partial y} \frac{\Delta y}{2} \right) \Delta x - \left(\frac{\partial P}{\partial y} \frac{\Delta y}{2} \right) \Delta x = -\frac{\partial P}{\partial y} \Delta x \Delta y$$

2. The Vertical Paths These integrals involve the vertical component of $Q(x, y)$.

- **Right Path (\uparrow):**

$$Q\left(x_0 + \frac{\Delta x}{2}, y\right) \approx Q(x_0, y_0) + \frac{\partial Q}{\partial x} \frac{\Delta x}{2} \implies \int_{\text{right}} Q \, dy \approx \left(Q(x_0, y_0) + \frac{\partial Q}{\partial x} \frac{\Delta x}{2} \right) (\Delta y)$$

- **Left Path (\downarrow):**

$$Q\left(x_0 - \frac{\Delta x}{2}, y\right) \approx Q(x_0, y_0) - \frac{\partial Q}{\partial x} \frac{\Delta x}{2} \implies \int_{\text{left}} Q \, dy \approx - \left(Q(x_0, y_0) - \frac{\partial Q}{\partial x} \frac{\Delta x}{2} \right) (\Delta y)$$

Here, we are left with only the parts that describe the *change* in Q with respect to x .

$$\int_{\text{right}} Q \, dy + \int_{\text{left}} Q \, dy \approx \left(\frac{\partial Q}{\partial x} \frac{\Delta x}{2} \right) \Delta y + \left(\frac{\partial Q}{\partial x} \frac{\Delta x}{2} \right) \Delta y = \frac{\partial Q}{\partial x} \Delta x \Delta y$$

Now we sum the results from the horizontal and vertical pairs:

$$\begin{aligned} \oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} &\approx \left(-\frac{\partial P}{\partial y} \Delta x \Delta y \right) + \left(\frac{\partial Q}{\partial x} \Delta x \Delta y \right) \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \Delta x \Delta y \end{aligned}$$

This shows that the total circulation around the tiny loop is approximately the quantity $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$ multiplied by the area of the loop ($\Delta A = \Delta x \Delta y$).

To get the property *at the point* (x_0, y_0) , we find the circulation **density**. We divide by the area and take the limit as the rectangle shrinks to zero.

$$\lim_{\Delta A \rightarrow 0} \frac{1}{\Delta A} \oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \frac{\partial Q}{\partial x}(x_0, y_0) - \frac{\partial P}{\partial y}(x_0, y_0)$$

This is why we call the scalar quantity $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ the **curl**: it is the circulation per unit area at a point, which measures the local rotational tendency of the field.

If $C = \partial D$ is a positively oriented simple closed curve enclosing a region D , Green's theorem states

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\substack{\text{Line Integral} \\ (\text{Total Circulation})}} = \underbrace{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA}_{\substack{\text{Double Integral} \\ (\text{Sum of Local Curls})}}$$

1. Let $\mathbf{F} = \langle 2x, 2y \rangle$. Show that \mathbf{F} is conservative and compute

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is any path from $(0, 0)$ to $(1, 1)$.

Sol. Let $\mathbf{F} = \langle P, Q \rangle = \langle 2x, 2y \rangle$. Since

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2x) = 0 \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(2y) = 0,$$

we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere in \mathbb{R}^2 , a simply connected domain. Therefore \mathbf{F} is conservative.

To find a potential function f with $\nabla f = \mathbf{F}$, we solve

$$f_x = 2x \quad \Rightarrow \quad f(x, y) = \int 2x \, dx = x^2 + g(y),$$

for some function $g(y)$. Then

$$f_y = g'(y) = 2y \quad \Rightarrow \quad g(y) = y^2 + C.$$

Hence a potential function is

$$f(x, y) = x^2 + y^2 \quad (\text{constant irrelevant}).$$

By the Fundamental Theorem for Line Integrals, for any path C from $(0, 0)$ to $(1, 1)$,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(0, 0) = (1^2 + 1^2) - (0^2 + 0^2) = 2.$$

□

2. Determine whether the vector field $\mathbf{F} = \langle y, x \rangle$ is conservative. If so, find a potential function.

Sol. Let $\mathbf{F} = \langle P, Q \rangle = \langle y, x \rangle$. Compute the mixed partials:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y) = 1, \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x) = 1.$$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere in \mathbb{R}^2 (a simply connected domain), \mathbf{F} is conservative.

To find a potential function f such that $\nabla f = \mathbf{F}$, we solve

$$f_x = P = y.$$

Integrating with respect to x gives

$$f(x, y) = \int y \, dx = xy + g(y),$$

where g is a function of y only. Differentiate with respect to y :

$$f_y = x + g'(y).$$

But f_y must equal $Q = x$, so $g'(y) = 0$, hence $g(y) = C$.

Therefore, a potential function is

$$f(x, y) = xy \quad (\text{up to an additive constant}).$$

Let $\mathbf{F} = \langle P, Q \rangle = \langle y, x \rangle$ on an open set $U \subseteq \mathbb{R}^2$. Since $P, Q \in C^1(U)$, \mathbf{F} is conservative on U if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on U .

Compute:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y) = 1, \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x) = 1.$$

Thus $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere on U , so \mathbf{F} is conservative.

To find a potential function f with $\nabla f = \mathbf{F}$, we require

$$f_x = y, \quad f_y = x.$$

Integrate $f_x = y$ with respect to x :

$$f(x, y) = xy + g(y),$$

for some function g of y alone. Differentiate with respect to y :

$$f_y(x, y) = x + g'(y).$$

Set this equal to x (since $f_y = x$):

$$x + g'(y) = x \Rightarrow g'(y) = 0 \Rightarrow g(y) = C.$$

Therefore a potential function is

$$f(x, y) = xy + C.$$

(Any two potential functions differ by an additive constant.) □

Theorem 1.6 (Curl test for conservativeness in \mathbb{R}^2). *Let $U \subseteq \mathbb{R}^2$ be an open and simply connected set, and let*

$$\mathbf{F} = \langle P, Q \rangle : U \rightarrow \mathbb{R}^2$$

be a C^1 vector field. Then the following are equivalent:

- (a) \mathbf{F} is conservative on U , i.e. there exists a C^1 function $f : U \rightarrow \mathbb{R}$ such that $\nabla f = \mathbf{F}$.
- (b) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere on U .

Proof. (1) \Rightarrow (2). Assume \mathbf{F} is conservative, so $\mathbf{F} = \nabla f$ for some $f \in C^1(U)$. In coordinates,

$$P = f_x, \quad Q = f_y.$$

If moreover $f \in C^2(U)$ (which holds, for instance, if $P, Q \in C^1$ and f is constructed as in the converse direction), then

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(f_x) = f_{xy}, \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(f_y) = f_{yx}.$$

By Clairaut's theorem (equality of mixed partials for C^2 functions), $f_{xy} = f_{yx}$, hence

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{on } U.$$

(2) \Rightarrow (1). Assume $P_y = Q_x$ on U . Let C be any positively oriented, piecewise C^1 , simple closed curve in U bounding a region $D \subseteq U$. By Green's Theorem,

$$\oint_C P dx + Q dy = \iint_D (Q_x - P_y) dA.$$

Since $Q_x - P_y = 0$ on U , it follows that

$$\oint_C P dx + Q dy = 0$$

for every such curve C .

Path independence. Fix two piecewise C^1 curves C_1 and C_2 in U with the same endpoints A and B . Consider the closed curve $C = C_1 \cup (-C_2)$ obtained by traversing C_1 from A to B and then C_2 from B back to A . Because U is simply connected, this closed curve can be decomposed into finitely many simple closed curves each bounding a region contained in U . Since the integral around each such simple closed curve is 0, additivity of line integrals yields

$$\oint_C P dx + Q dy = 0.$$

Therefore,

$$\int_{C_1} P dx + Q dy = \int_{C_2} P dx + Q dy,$$

so the line integral depends only on endpoints (path independence).

Construction of a potential. Fix a base point $A_0 \in U$. Define $f : U \rightarrow \mathbb{R}$ by

$$f(x, y) = \int_{C_{A_0 \rightarrow (x,y)}} P dx + Q dy,$$

where $C_{A_0 \rightarrow (x,y)}$ is any piecewise C^1 curve in U from A_0 to (x, y) . By path independence, f is well-defined.

Verification that $\nabla f = \mathbf{F}$. Let $(x, y) \in U$ and choose h small so that the horizontal segment from (x, y) to $(x + h, y)$ lies in U . Using the definition of f and path independence,

$$f(x + h, y) - f(x, y) = \int_x^{x+h} P(s, y) ds.$$

Divide by h and let $h \rightarrow 0$ to obtain

$$f_x(x, y) = P(x, y).$$

Similarly, using a vertical segment,

$$f(x, y + h) - f(x, y) = \int_y^{y+h} Q(x, t) dt,$$

so

$$f_y(x, y) = Q(x, y).$$

Hence $\nabla f = \langle f_x, f_y \rangle = \langle P, Q \rangle = \mathbf{F}$, and \mathbf{F} is conservative on U . \square

3. Let $f(x, y, z) = xyz$. Compute ∇f and evaluate the line integral of ∇f over the path from $(1, 0, 0)$ to $(1, 2, 3)$.

Sol. Given $f(x, y, z) = xyz$, its gradient is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle yz, xz, xy \rangle.$$

To evaluate the line integral of ∇f over any path C from $(1, 0, 0)$ to $(1, 2, 3)$, use the Fundamental Theorem for Line Integrals:

$$\int_C \nabla f \cdot d\mathbf{r} = f(1, 2, 3) - f(1, 0, 0).$$

Compute the endpoint values:

$$f(1, 2, 3) = (1)(2)(3) = 6, \quad f(1, 0, 0) = (1)(0)(0) = 0.$$

Hence,

$$\int_C \nabla f \cdot d\mathbf{r} = 6 - 0 = 6.$$

□

4. Let $\mathbf{F} = \nabla f$ for $f(x, y) = x^2 + y^2$. Compute the line integral over a circular path from $(1, 0)$ to $(0, 1)$ and explain the result.

Sol. Given $f(x, y) = x^2 + y^2$, we have

$$\mathbf{F} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2x, 2y \rangle.$$

Since \mathbf{F} is a gradient field, it is conservative. Therefore, by the Fundamental Theorem for Line Integrals, for any smooth path C from $(1, 0)$ to $(0, 1)$ (including a circular arc),

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0, 1) - f(1, 0).$$

Compute:

$$f(0, 1) = 0^2 + 1^2 = 1, \quad f(1, 0) = 1^2 + 0^2 = 1.$$

Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 1 - 1 = 0.$$

Explanation. The line integral depends only on the endpoints because $\mathbf{F} = \nabla f$ is conservative. Here both endpoints lie on the same level curve of f (indeed, on the circle $x^2 + y^2 = 1$), so f has the same value at $(1, 0)$ and $(0, 1)$; thus the net change in potential is zero, and the work done by \mathbf{F} along the circular path is 0. \square

1.2 Green's Theorem

1. Use Green's Theorem to evaluate

$$\oint_C x \, dy - y \, dx$$

where C is the unit circle oriented counterclockwise.

Sol. Write the integral in Green's Theorem form:

$$\oint_C P \, dx + Q \, dy,$$

where here $P(x, y) = -y$ and $Q(x, y) = x$, since

$$x \, dy - y \, dx = (-y) \, dx + x \, dy.$$

By Green's Theorem (with C positively oriented),

$$\oint_C P \, dx + Q \, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

where D is the unit disk. Compute:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(-y) = 1, \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x) = -1.$$

Thus

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2.$$

Therefore,

$$\oint_C x \, dy - y \, dx = \iint_D 2 \, dA = 2 \cdot \text{Area}(D) = 2 \cdot \pi(1)^2 = 2\pi.$$

□

2. Let $\mathbf{F} = \langle y^2, 2xy \rangle$. Use Green's Theorem to evaluate the line integral around the boundary of the square $[0, 1] \times [0, 1]$.

Sol. Let $\mathbf{F} = \langle P, Q \rangle = \langle y^2, 2xy \rangle$, and let C be the positively oriented (counterclockwise) boundary of the square

$$D = [0, 1] \times [0, 1].$$

By Green's Theorem,

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(y^2) = 0, \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x.$$

Hence,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - 2x = 0.$$

Therefore,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy = \iint_D 0 dA = 0.$$

□

3. Evaluate

$$\oint_C (x+y)dx + (x-y)dy$$

where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$ oriented counterclockwise.

Sol. Write the line integral as $\oint_C P dx + Q dy$ with

$$P(x,y) = x+y, \quad Q(x,y) = x-y.$$

Since C is positively oriented (counterclockwise), Green's Theorem gives

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

where D is the triangular region with vertices $(0,0)$, $(1,0)$, $(1,1)$.

Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x-y) = 1, \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x+y) = 1.$$

Thus,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 1 = 0.$$

Therefore,

$$\oint_C (x+y)dx + (x-y)dy = \iint_D 0 dA = 0.$$

□

4. Determine if

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

for $\mathbf{F} = \langle y, -x \rangle$ around a circle of radius r centered at the origin.

Sol. Let $\mathbf{F} = \langle P, Q \rangle = \langle y, -x \rangle$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy = \oint_C y dx - x dy,$$

where C is the circle of radius r centered at the origin, oriented counterclockwise.

Using Green's Theorem,

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

where D is the disk of radius r . Compute

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(-x) = -1, \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y) = 1.$$

Hence

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -1 - 1 = -2.$$

Therefore,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (-2) dA = -2 \text{Area}(D) = -2 \cdot \pi r^2 = -2\pi r^2.$$

So the integral is *not* zero (except in the degenerate case $r = 0$). For clockwise orientation, the value would be $+2\pi r^2$. \square

1.3 Divergence Theorem

1. Let $\mathbf{F} = \langle x, y, z \rangle$. Use the Divergence Theorem to compute the flux across the surface of the unit sphere.

Sol. content... □

2. Let $\mathbf{F} = \langle x^2, y^2, z^2 \rangle$. Compute both the divergence and the surface integral over the unit cube $[0, 1]^3$.

Sol. content... □

3. Use the Divergence Theorem to find the outward flux of $\mathbf{F} = \langle yz, xz, xy \rangle$ through the unit cube.

Sol. content... □

4. Let $\mathbf{F} = \langle x, -y, z \rangle$. Verify the Divergence Theorem on the upper hemisphere of radius 1 centered at the origin.

Sol. content... □

1.4 Stokes' Theorem

1. Let $\mathbf{F} = \langle -y, x, 0 \rangle$. Use Stokes' Theorem to compute the circulation around the boundary of the disk $x^2 + y^2 \leq 1$ in the xy -plane.

Sol. content... □

2. Let $\mathbf{F} = \langle z, 0, x \rangle$. Use Stokes' Theorem on the triangular surface with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$.

Sol. content... □

3. Compute both sides of Stokes' Theorem for $\mathbf{F} = \langle y, z, x \rangle$ on the surface $z = 0$ bounded by the unit circle.

Sol. content... □

4. Use Stokes' Theorem to show that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

if \mathbf{F} is the gradient of some scalar field f .

Sol. content...

□

2 Differential Forms

TBA

3 Winding Numbers and Complexification

TBA