

Advanced Calculus III

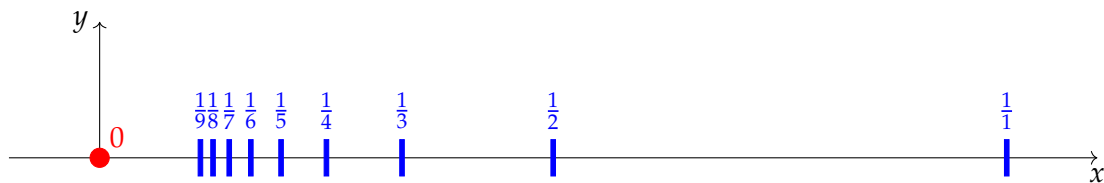
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We cover the following topics in this note.

- Limit of a Function ($\varepsilon - \delta$)
- Continuity of a Function
- Monotone Convergent Theorem (MCT)
- Nested Interval Property (NIP)
- Limit Superior and Limit Inferior

What is 0 for the set $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$?



Note (Open ε -ball). The open ε -ball of x in S is $B_\varepsilon(x) := \{y \in S : d(x, y) < \varepsilon\}$.

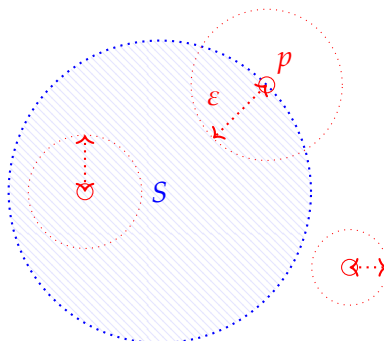
Limit Point (Metric Space)

Definition. Let (X, d) be a metric space. Let $S \subseteq X$. A point $p \in X$ is a **limit point** of S if and only if

$$\forall \varepsilon > 0, B_\varepsilon(p) \cap (S \setminus \{p\}) \neq \emptyset.$$

That is,

$$\forall \varepsilon > 0, \{x \in S : 0 < d(x, p) < \varepsilon\} \neq \emptyset.$$



Remark. Note that a limit point p may NOT belong to S .

Note (Limit Point (Topology)). Let (X, τ) be a topological space. For a subset $S \subseteq X$. A point $p \in X$ is a limit point of S if and only if

$$\forall U \in \tau \text{ with } p \in U, U \cap (S \setminus \{p\}) \neq \emptyset.$$

Example. Let $S = (a, b) \subseteq \mathbb{R}$:



(i) Consider p with $p < a$:



Let $\varepsilon := \frac{a-p}{2} > 0$. Then $B_\varepsilon(p) \cap (S \setminus \{p\}) = \emptyset$. Thus, $p < a$ is NOT a limit point.

(ii) Consider $p = a$:



Let $\varepsilon > 0$. Then $B_\varepsilon(p) \cap (S \setminus \{p\}) \neq \emptyset$. Thus, $p = a$ is a limit point of $S = (a, b)$.

By (i) and (ii), the set of all limit points of (a, b) is $[a, b]$.

Example. Let $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$:



- Consider $p = \frac{1}{n} \in S$. No point of S is a limit point.
- Consider $p = 0$.



Let $\varepsilon > 0$. By Archimedian property, $\exists n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$, and so $1/n \in B_\varepsilon(0) \cap S$. Thus, $p = 0$ is a limit point of $S = \{1/n : n \in \mathbb{N}\}$.

Example. Let $S = \mathbb{Q}$.

- Consider $p \in \mathbb{R}$. Let $\varepsilon > 0$. By density of rationals,

$$\exists r \in \mathbb{Q} \text{ such that } p < r < p + \varepsilon.$$

Then $r \in B_\varepsilon(p) \cap S$ with $r \neq p$, i.e., r is a limit point. Thus, all reals are limit points of \mathbb{Q} .

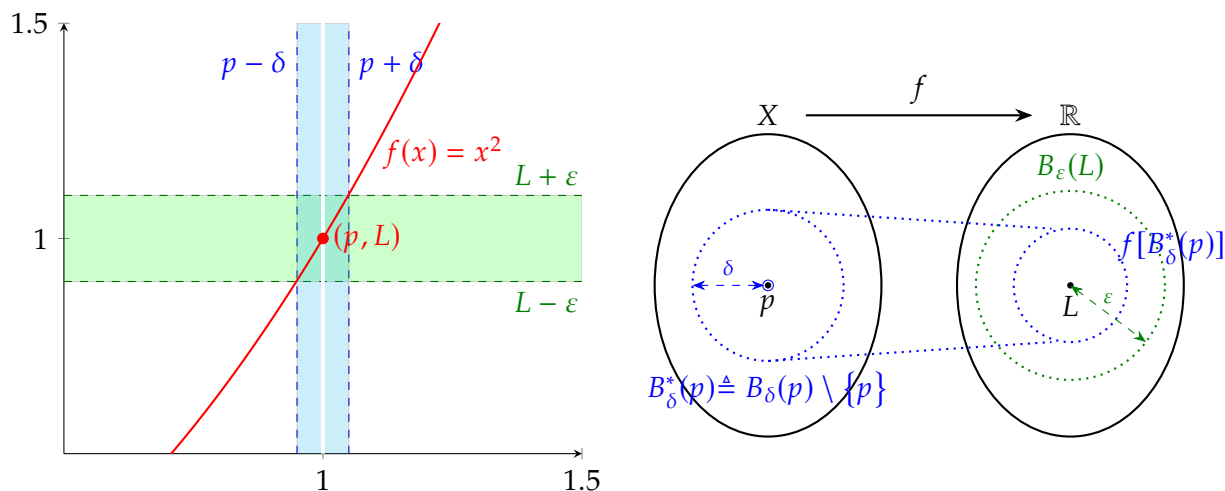
★ Limit of a Function ($\varepsilon - \delta$) ★

Definition. Let $f : X \rightarrow \mathbb{R}$ be a function defined on a subset $X(\subseteq \mathbb{R})$ of a metric space, and let $p \in X$ be a limit point of X . We say that $L \in \mathbb{R}$ is the **limit of the function f as x approaches p** if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in X, 0 < |x - p| < \delta \implies |f(x) - L| < \varepsilon.$$

We write

$$\lim_{x \rightarrow p} f(x) = L.$$



Remark.

$$\lim_{x \rightarrow p} f(x) \neq L \iff \exists \varepsilon > 0 : [\forall \delta > 0 : \exists x \in X : 0 < |x - p| < \delta \text{ but } |f(x) - L| > \varepsilon].$$

Continuity of a Function

Definition. Let $f : X \rightarrow \mathbb{R}$ be a function defined on a subset $X \subseteq \mathbb{R}$ of a metric space, and let $p \in X$. The function f is **continuous at p** if and only if

$$\lim_{x \rightarrow p} f(x) = f(p).$$

That is,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |x - p| < \delta \implies |f(x) - f(p)| < \varepsilon.$$

Remark (Continuity of a Set). The function f is continuous on subset $S \subseteq X$ if it is continuous at every point $p \in S$.

Remark (Continuity in a Topological Space). Let (X, τ_X) and (Y, τ_Y) are topological spaces. $f : X \rightarrow Y$ is **continuous** if and only if

$$U_Y \in \tau_Y \implies f^{-1}[U_Y] \in \tau_X,$$

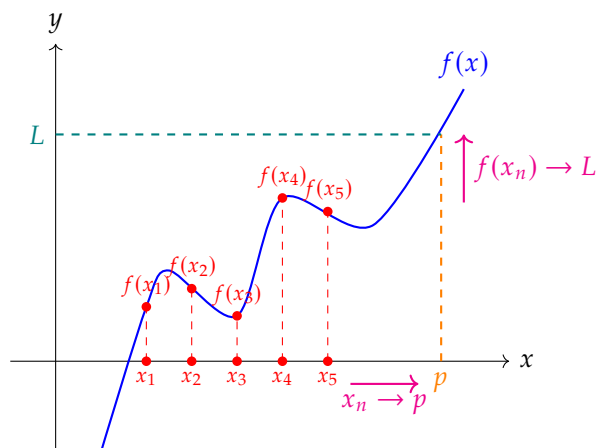
where $f^{-1}[U_Y] = \{x \in X : f(x) \in U_Y\}$ is the preimage of U_Y under f .

Note. $[p \implies (q \implies r)] \equiv [p \implies (\neg q \vee r)] \equiv [\neg p \vee (\neg q \vee r)] \equiv [\neg(p \wedge q) \vee r] \equiv [(p \wedge q) \implies r]$.

Limit of Function by Convergent Sequences

Theorem. Let $f : X \rightarrow \mathbb{R}$ be a function defined on a subset $\emptyset \neq X \subseteq \mathbb{R}$ of a metric space, and let p is a limit point of X . Then

$$\lim_{x \rightarrow p} f(x) = L \iff \left[\forall \{x_n\} \subseteq X \setminus \{p\}, \left(\lim_{n \rightarrow \infty} x_n = p \implies \lim_{n \rightarrow \infty} f(x_n) = L \right) \right].$$



Proof. (\Rightarrow) Suppose that $\lim_{x \rightarrow p} f(x) = L$. Let $\{x_n\} \subseteq X \setminus \{p\}$ be a sequence, and let $\lim_{n \rightarrow \infty} x_n = p$. We NTS that

$$\lim_{n \rightarrow \infty} f(x_n) = L, \quad \text{i.e.,} \quad \forall \varepsilon > 0 : \exists N \in \mathbb{N} : n \geq N \Rightarrow |f(x_n) - L| < \varepsilon.$$

Let $\varepsilon > 0$. Since $\lim_{x \rightarrow p} f(x) = L$, we know

$$\exists \delta > 0 \text{ such that } 0 < |x - p| < \delta \Rightarrow |f(x) - L| < \varepsilon. \quad (*)$$

Since $\lim_{n \rightarrow \infty} x_n = p$, we obtain

$$\exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow |x_n - p| < \delta.$$

Thus, if $n \geq N$ then,

$$\begin{aligned} |x_n - p| < \delta &\Rightarrow 0 < |x_n - p| < \delta \quad \because x_n \neq p \\ &\Rightarrow |f(x_n) - L| < \varepsilon \quad \text{by } (*) \end{aligned}$$

Thus, $\lim_{n \rightarrow \infty} f(x_n) = L$.

(\Leftarrow) Let the RHS holds. Assume, for the contradiction, that $\lim_{x \rightarrow p} f(x) \neq L$, i.e.,

$$\exists \varepsilon > 0 : \forall \delta > 0 : \exists x_\delta \in X : 0 < |x_\delta - p| < \delta \text{ but } |f(x_\delta) - L| \geq \varepsilon.$$

Take $\delta = 1/n$ for $n \in \mathbb{N}$. Then

$$\exists x_n \in X \text{ such that } 0 < |x_n - p| < \frac{1}{n} \text{ but } |f(x_n) - L| \geq \varepsilon.$$

(Axiom of Countable Choice) This means that

$$\forall n \in \mathbb{N} : \exists \{x_n\} \subseteq X \setminus \{p\} \text{ such that } 0 < |x_n - p| < \frac{1}{n} \text{ but } |f(x_n) - L| \geq \varepsilon.$$

By Squeeze Theorem, we have $\lim_{n \rightarrow \infty} x_n = p$ since $0 < |x_n - p| < 1/n$. Since the RHS holds, we obtain $\lim_{n \rightarrow \infty} f(x_n) = L$. Then, for some $\varepsilon > 0$,

$$\exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow |f(x_n) - L| < \varepsilon \frac{1}{2}.$$

Hence it is proved. □

Continuity of Function by Convergent Sequences

Corollary. Let $f : X \rightarrow \mathbb{R}$ be a function defined on a subset $\emptyset \neq X \subseteq \mathbb{R}$ of a metric space, and let p is a limit point of X . Then

$$\lim_{x \rightarrow p} f(x) = f(p) \iff \left[\forall \{x_n\} \subseteq X, \left(\lim_{n \rightarrow \infty} x_n = p \implies \lim_{n \rightarrow \infty} f(x_n) = f(p) \right) \right].$$

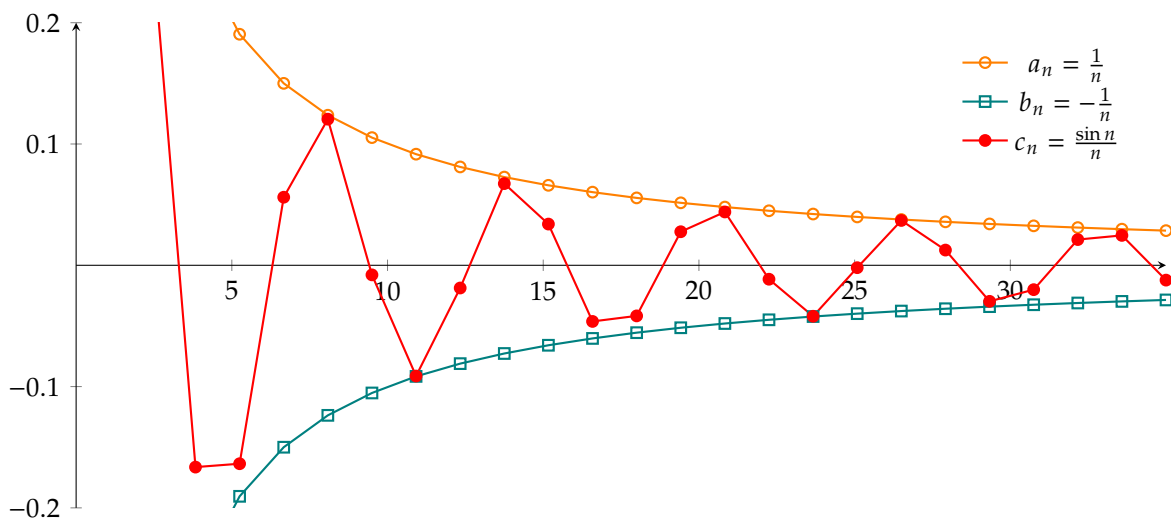
Squeeze Theorem; Sandwich Theorem

Theorem. Let

$$(i) \lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} b_n;$$

$$(ii) \exists n_0 \in \mathbb{N} \text{ such that } a_n \leq c_n \leq b_n \text{ for all } n \geq n_0.$$

Then $\lim_{n \rightarrow \infty} c_n = L$.



Proof. Let $\varepsilon > 0$. Since $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} b_n = L$, we have

$$\exists n_1 \in \mathbb{N} \text{ such that } n \geq n_1 \implies L - \varepsilon < a_n < L + \varepsilon,$$

$$\exists n_2 \in \mathbb{N} \text{ such that } n \geq n_2 \implies L - \varepsilon < b_n < L + \varepsilon.$$

Let $N := \max \{n_0, n_1, n_2\}$. If $n \geq N$ then

$$L - \varepsilon < a_n \leq c_n \leq b_n < L + \varepsilon,$$

and so $|c_n - L| < \varepsilon$. □

Note. Recall that

“A convergent sequence is bounded.”

Formally,

$$\exists A \in \mathbb{R} \text{ s.t. } A = \lim_{n \rightarrow \infty} a_n \implies \exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, |a_n| \leq M.$$

However, the converse is not necessarily true:

$$\exists A \in \mathbb{R} \text{ s.t. } A = \lim_{n \rightarrow \infty} a_n \not\Leftarrow \exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, |a_n| \leq M.$$

To illustrate, consider the sequence $\{a_n\} = 1 - (-1)^n$ that is bounded, yet it does not converge.

Monotone Sequence

Definition. A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be **monotone** if it is either **monotone increasing** or **monotone decreasing**.

(1) A sequence $\{a_n\}_{n=1}^{\infty}$ is **monotone increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

Alternatively, it is **strictly increasing** if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.

(2) A sequence $\{a_n\}_{n=1}^{\infty}$ is **monotone decreasing** if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$.

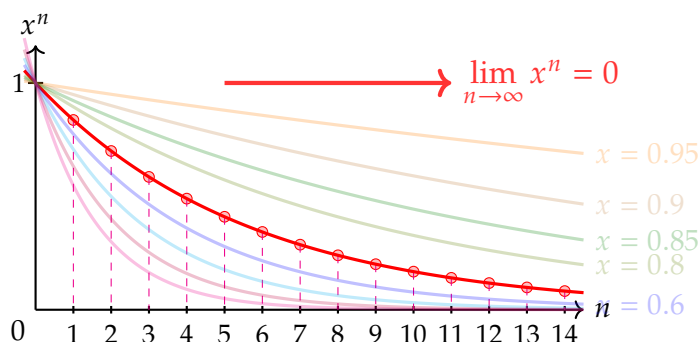
Alternatively, it is **strictly decreasing** if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$.

Remark. A sequence $\{a_n\}$ is monotone if $\begin{cases} a_n \leq a_{n+1} & (\text{monotone increasing}) \\ a_{n+1} \leq a_n & (\text{monotone decreasing}) \end{cases}$.

Example.

- $\{n\}_{n=1}^{\infty}$ is monotone increasing.
- $\{1/n\}_{n=1}^{\infty}$ is monotone decreasing.

Example. Let $0 < x < 1$.



Monotone Convergence Theorem (MCT)

Theorem. A monotone sequence of real numbers $\{a_n\}$ is convergent if and only if it is bounded.

(1) Let $\{a_n\}$ be an monotone increasing sequence of real numbers that is bounded above. Then

$$\lim_{n \rightarrow \infty} a_n = \sup \{a_n : n \in \mathbb{N}\}.$$

(2) Let $\{b_n\}$ be an monotone decreasing sequence of real numbers that is bounded below. Then

$$\lim_{n \rightarrow \infty} b_n = \inf \{b_n : n \in \mathbb{N}\}.$$

Proof.

(1) Suppose that a sequence $\{a_n\}$ is monotone increasing and bounded above. Consider the set $\{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$, which is non-empty and bounded above by assumption. By **Least Upper Bound Property**¹,

$$\exists \alpha \in \mathbb{R} \text{ such that } \alpha = \sup \{a_n : n \in \mathbb{N}\}.$$

We claim that:

$$\lim_{n \rightarrow \infty} a_n = \alpha = \sup \{a_n : n \in \mathbb{N}\}.$$

Let $\varepsilon > 0$. Since α is the supremum (*least* upper bound) of $\{a_n : n \in \mathbb{N}\}$, it follows that $\alpha - \varepsilon$ is not an upper bound of $\{a_n : n \in \mathbb{N}\}$. Thus, $\neg[\forall N \in \mathbb{N}, a_N \leq \alpha - \varepsilon]$, i.e.,

$$\exists N \in \mathbb{N} \text{ such that } \alpha - \varepsilon < a_N.$$

Since $\{a_n\}$ is monotone increasing,

$$\alpha - \varepsilon < a_N \leq a_n$$

for all $n \geq N$. Therefore,

$$\alpha - \varepsilon \underset{\substack{\alpha = \sup \{a_n\} \\ \varepsilon > 0}}{<} a_N \underset{\substack{\{a_n\} \text{ is monotone increasing} \\ n \geq N}}{\leq} a_n \underset{\substack{\alpha \text{ is an upper bound} \\ \varepsilon > 0}}{\leq} \alpha < \alpha + \varepsilon.$$

This implies that $|a_n - \alpha| < \varepsilon$ for all $n \geq N$.

¹Every non-empty subset of \mathbb{R} that is bounded above has the supremum in \mathbb{R} .

- (2) Suppose that a sequence $\{b_n\}$ is monotone decreasing and bounded below. Consider the set $\{b_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$, which is non-empty and bounded below by assumption. By **Greatest Lower Bound Property**²,

$$\exists \beta \in \mathbb{R} \text{ such that } \beta = \inf \{b_n : n \in \mathbb{N}\}.$$

We claim that:

$$\lim_{n \rightarrow \infty} b_n = \beta = \inf \{b_n : n \in \mathbb{N}\}.$$

Let $\varepsilon > 0$. Since β is the infimum (*greatest* lower bound) of $\{b_n : n \in \mathbb{N}\}$, it follows that $\beta + \varepsilon$ is not a lower bound of $\{b_n : n \in \mathbb{N}\}$. Thus, $\neg[\forall n \in \mathbb{N}, \beta + \varepsilon \leq b_n]$, i.e.,

$$\exists N \in \mathbb{N} \text{ such that } b_N < \beta + \varepsilon.$$

Since $\{b_n\}$ is monotone decreasing,

$$b_n \leq b_N < \beta + \varepsilon$$

for all $n \geq N$. Therefore,

$$\beta - \varepsilon \stackrel{\varepsilon > 0}{<} \beta \stackrel{\beta \text{ is a lower bound}}{\leq} b_n \stackrel{\{b_n\} \text{ is monotone decreasing}}{\leq_{n \geq N}} b_N \stackrel{\beta = \inf \{b_n\}}{<_{\varepsilon > 0}} \beta + \varepsilon$$

This implies that $|b_n - \beta| < \varepsilon$ for all $n \geq N$.

□

Divergence of Sequence

Definition. Let $\{a_n\}$ be a sequence of real numbers.

- (1) We say that the sequence $\{a_n\}$ **diverges to infinity** (or **tends to infinity**) if

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } n \geq N \implies M < a_n,$$

and write $\lim_{n \rightarrow \infty} a_n = +\infty$.

- (2) We say that the sequence $\{a_n\}$ **diverges to minus infinity** (or **tends to infinity**) if

$$\forall M \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } n \geq N \implies a_n < M,$$

and write $\lim_{n \rightarrow \infty} a_n = -\infty$.

- (3) We say that $\{a_n\}$ is properly divergent in case we have either $\lim_{n \rightarrow \infty} a_n = +\infty$ or $\lim_{n \rightarrow \infty} a_n = -\infty$.

²Every non-empty subset of \mathbb{R} that is bounded below has the infimum in \mathbb{R} .

Note. Recall that

(Monotonicity) A sequence $\{a_n\}$ is monotone increasing if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$;

(Not Bounded Above) The sequence $\{a_n\}$ is not bounded above if

$$\neg[\exists M \in \mathbb{R}, \forall n \in \mathbb{N}, a_n \leq M] \equiv [\forall M \in \mathbb{R}, \exists n \in \mathbb{N} \text{ such that } a_n > M].$$

We claim that a sequence $\{a_n\}$ that is monotone increasing and not bounded above diverges to infinity:

Proof. Let $M \in \mathbb{R}$. Since $\{a_n\}$ is not bounded above,

$$\exists n_0 \in \mathbb{N} \text{ such that } a_{n_0} > M.$$

Since $\{a_n\}$ is monotonic increasing, it follows that

$$a_{n_0} \leq a_n, \forall n \geq n_0.$$

Thus

$$n \geq n_0 \xRightarrow{\text{monotone increasing}} a_{n_0} \leq a_n \xRightarrow{\text{Not Bounded Above}} M < a_{n_0} < a_n.$$

Hence it is proved. □

Note that

Lemma. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Then

$$[\forall n \in \mathbb{N}, a_n \leq b_n] \implies \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n.$$

Proof. Let $a = \lim_{n \rightarrow \infty} a_n$ and $b = \lim_{n \rightarrow \infty} b_n$. Suppose that $a > b$. Let $\varepsilon = a - b > 0$. Then

$$\exists N_1 \in \mathbb{N} \text{ such that } n \geq N_1 \implies |a_n - a| < \varepsilon,$$

$$\exists N_2 \in \mathbb{N} \text{ such that } n \geq N_2 \implies |b_n - b| < \varepsilon.$$

Let $N := \max\{N_1, N_2\}$. Then $b_N < b + \varepsilon < a + \varepsilon < a_N$ \nlessgtr . Hence $a \leq b$, i.e., $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$. \square

Note. Let $I_n = (0, \frac{1}{n}) \subseteq \mathbb{R}$ for all $n \in \mathbb{N}$.



Suppose that $x \in \bigcap_{n=1}^{\infty} I_n$ then $x \in I_n$ for all $n \geq 1$. That is,

$$0 < x < \frac{1}{n} \quad \text{for all } n \geq 1.$$

By Archimedian property, $\exists n_0 \in \mathbb{N}$ s.t. $n_0 x > 1$ \nlessgtr . Hence $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Note. Let $I_n = [n, \infty) \subseteq \mathbb{R}$ for all $n \in \mathbb{N}$.



Suppose that $x \in \bigcap_{n=1}^{\infty} I_n$ then $x \in I_n$ for all $n \geq 1$. That is,

$$n \leq x \quad \text{for all } n \geq 1.$$

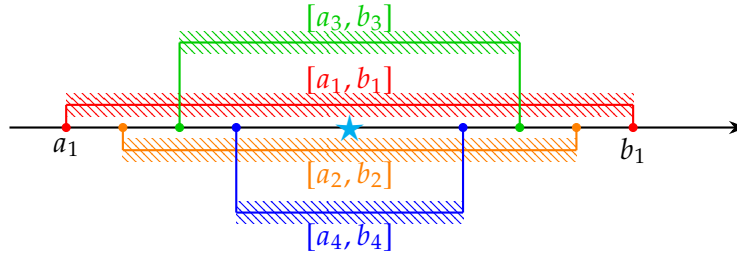
By Archimedian property, $\exists n_0 \in \mathbb{N}$ s.t. $x < n_0$ \nlessgtr . Hence $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Nested Interval Property (NIP)

Theorem. Let $a_n \leq b_n$ for all $n \in \mathbb{N}$, and let $\{[a_n, b_n]\}_{n=1}^{\infty} \subseteq \mathbb{R}$ be a sequence of bounded and closed intervals satisfying $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for all $n \in \mathbb{N}$. Then

$$\bigcap_{n=1}^{\infty} [a_n, b_n] := \{x \in \mathbb{R} : x \in [a_n, b_n] \text{ for all } n \in \mathbb{N}\} \neq \emptyset.$$

If $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$, then $|\bigcap_{n=1}^{\infty} [a_n, b_n]| = 1$.



Proof. Since $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for all $n \in \mathbb{N}$, we know the sequence $\{a_n\}$ is monotone increasing, and the sequence $\{b_n\}$ is monotone decreasing. In other words,

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1.$$

By Monotone Convergence Theorem, we obtain

$$\lim_{n \rightarrow \infty} a_n = \sup_{n \in \mathbb{N}} a_n \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n = \inf_{n \in \mathbb{N}} b_n.$$

Thus,

$$[\forall n \in \mathbb{N}, a_n \leq b_n] \implies \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n \implies \sup_{n \in \mathbb{N}} a_n \leq \inf_{n \in \mathbb{N}} b_n.$$

Then

$$\begin{aligned} x \in \bigcap_{n=1}^{\infty} [a_n, b_n] &\iff \forall n \in \mathbb{N}, a_n \leq x \leq b_n \iff \sup_{n \in \mathbb{N}} a_n \leq x \leq \inf_{n \in \mathbb{N}} b_n \\ &\iff x \in [\sup_{n \in \mathbb{N}} a_n, \inf_{n \in \mathbb{N}} b_n]. \end{aligned}$$

By Set Equality, we have

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = [\sup_{n \in \mathbb{N}} a_n, \inf_{n \in \mathbb{N}} b_n],$$

and so $[\sup_{n \in \mathbb{N}} a_n, \inf_{n \in \mathbb{N}} b_n] \neq \emptyset$ by Least Upper Bound Property. □

Monotonicity of Supremum and Infimum

Proposition. Let $\{a_n\}, \{b_n\} \subseteq \mathbb{R}$ be sequences of real numbers. Let $\{b_n\}$ is a subsequence of $\{a_n\}$, i.e., $\{b_n\} \subseteq \{a_n\}$. Then

(1) $\sup \{b_n\} \leq \sup \{a_n\}$;

(2) $\inf \{a_n\} \leq \inf \{b_n\}$.

Proof. (1) Since

$$\beta \in \{b_n\} \xRightarrow{\{b_n\} \subseteq \{a_n\}} \beta \in \{a_n\} \xRightarrow{\sup \{a_n\}} \beta \leq \sup \{a_n\},$$

$\sup \{a_n\}$ be an upper bound of $\{b_n\}$. Since $\sup \{b_n\}$ is the *least* upper bound of $\{b_n\}$, we have $\sup \{b_n\} \leq \sup \{a_n\}$.

(2) Since

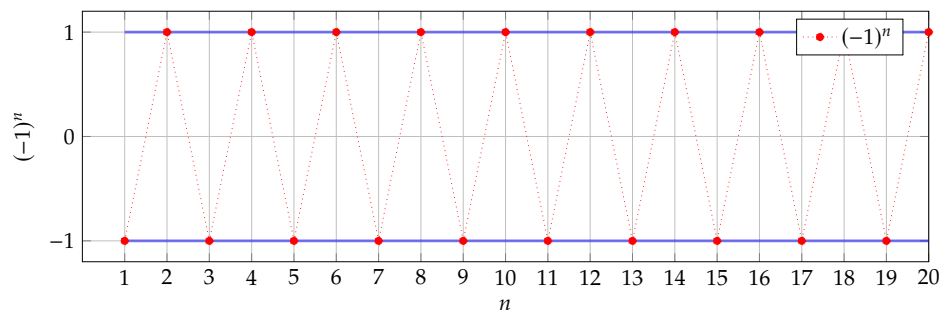
$$\beta \in \{b_n\} \xRightarrow{\{b_n\} \subseteq \{a_n\}} \beta \in \{a_n\} \xRightarrow{\inf \{a_n\}} \inf \{a_n\} \leq \beta,$$

$\inf \{a_n\}$ be a lower bound of $\{b_n\}$. Since $\inf \{b_n\}$ is the *greatest* lower bound of $\{b_n\}$, we have $\inf \{a_n\} \leq \inf \{b_n\}$.

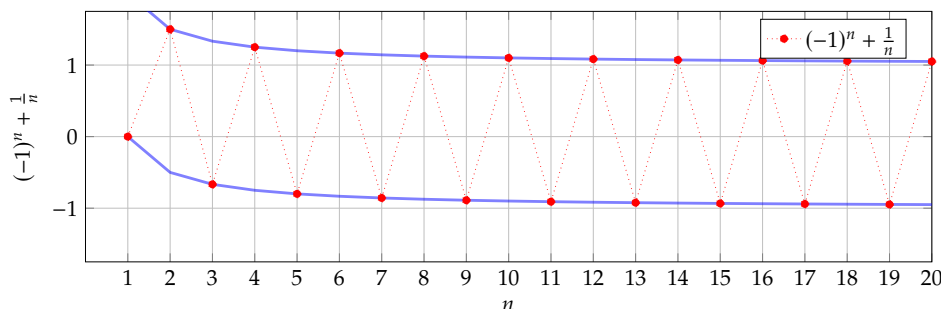
□

Observation.

- What is ± 1 for the set $S = \{(-1)^n : n \in \mathbb{N}\}$?



- What is ± 1 for the set $S = \{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\}$?



Let $\{x_n\}_{i=1}^{\infty}$ be a sequence in \mathbb{R} . Define

$$s_1 = \sup \{x_1, x_2, x_3, \dots\} = \sup \{x_k : k \geq 1\},$$

$$s_1 = \sup \{x_2, x_3, x_4, \dots\} = \sup \{x_k : k \geq 2\},$$

\vdots

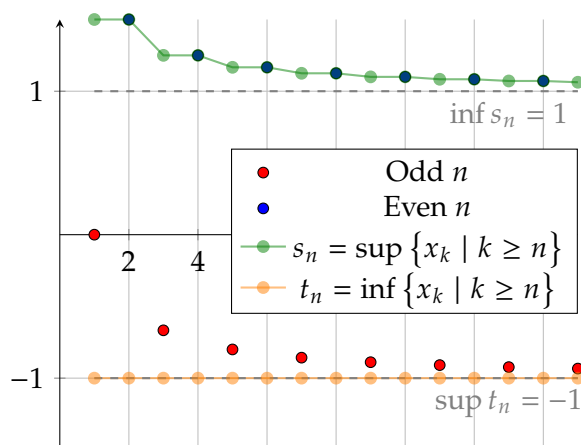
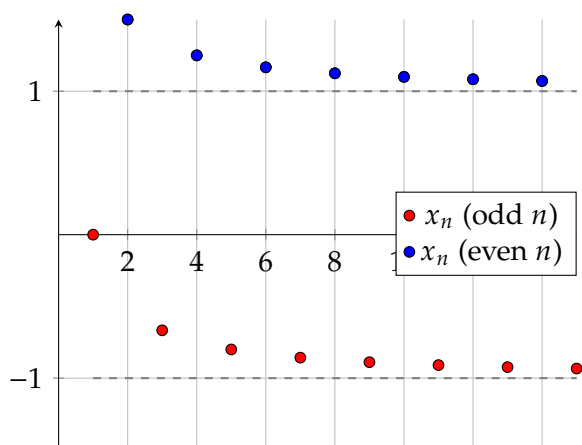
$$s_n = \sup \{x_k, x_{k+1}, \dots\} = \sup \{x_k : k \geq n\}.$$

By monotonicity of supremum,

$$s_1 \geq s_2 \geq \dots \geq s_n \geq s_{n+1} \geq \dots.$$

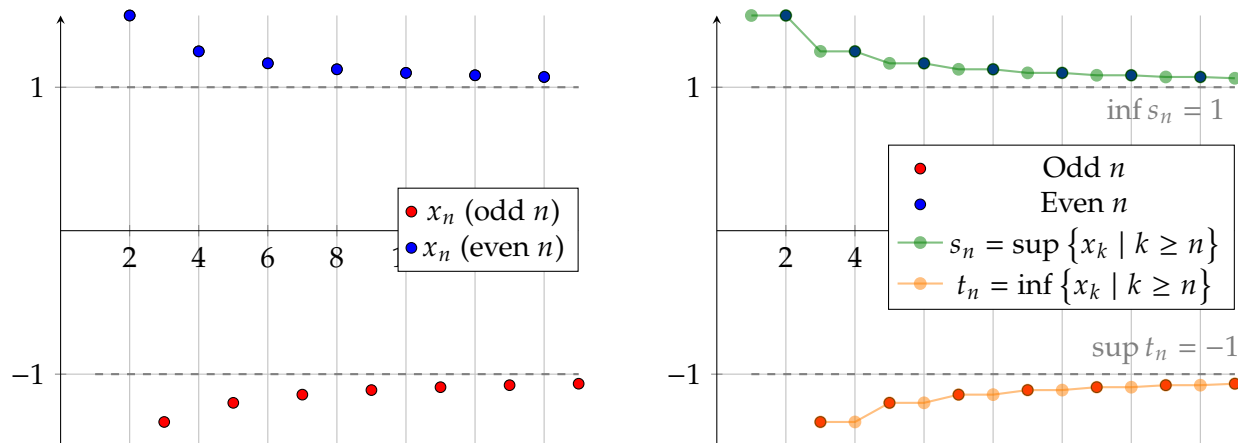
That is, $\{s_n\}_{n=1}^{\infty}$ be a monotone decreasing sequence. Similarly, for $t_n = \inf \{x_k, x_{k+1}, \dots\} = \inf \{x_k : k \geq n\}$, we have a monotone increasing sequence $\{t_n\}_{n=1}^{\infty}$. Consider the followings:

n	$(-1)^n$	$\frac{1}{n}$	$x_n = (-1)^n + \frac{1}{n}$	$\sup \{x_k : k \geq n\}$	$\inf \{x_k : k \geq n\}$
1	-1	1	0	1.5	-1
2	1	$\frac{1}{2} = 0.5$	$\frac{3}{2} = 1.5$	1.5	-1
3	-1	$\frac{1}{3} \approx 0.33$	$-\frac{2}{3} \approx -0.67$	1.25	-1
4	1	$\frac{1}{4} = 0.25$	$\frac{5}{4} = 1.25$	1.25	-1
5	-1	$\frac{1}{5} = 0.2$	$-\frac{4}{5} = -0.8$	1.17	-1
6	1	$\frac{1}{6} \approx 0.17$	$\frac{7}{6} \approx 1.17$	1.17	-1
7	-1	$\frac{1}{7} \approx 0.14$	$-\frac{6}{7} \approx -0.86$	1.125	-1
8	1	$\frac{1}{8} = 0.125$	$\frac{9}{8} = 1.125$	1.125	-1
9	-1	$\frac{1}{9} \approx 0.11$	$-\frac{8}{9} \approx -0.89$	1.1	-1
10	1	$\frac{1}{10} = 0.1$	$\frac{11}{10} = 1.1$	1.1	-1



Remark. Consider

$$x_n := (-1)^n + (-1)^n \cdot \frac{1}{n}$$



Limit Superior and Limit Inferior

Definition. Let $\{x_n\}$ be a sequence of real numbers. Suppose that $\{x_n\}$ is bounded.

- (1) The **limit superior** of $\{x_n\}$, denoted by $\limsup_{n \rightarrow \infty} x_n$ (or $\overline{\lim}_{n \rightarrow \infty} x_n$) is defined as

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(\sup_{k \geq n} x_k \right) := \inf_{n \in \mathbb{N}} \left(\sup_{k \geq n} x_k \right),$$

where $\sup_{k \geq n} x_k$ represents the supremum of the subsequence $\{x_k : k \geq n\}$.

- (2) The **limit inferior** of $\{x_n\}$, denoted by $\liminf_{n \rightarrow \infty} x_n$ (or $\underline{\lim}_{n \rightarrow \infty} x_n$) is defined as

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(\inf_{k \geq n} x_k \right) := \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} x_k \right),$$

where $\inf_{k \geq n} x_k$ represents the infimum of the subsequence $\{x_k : k \geq n\}$.

Note (Extended Real Number Line). The **extended real number line** $\overline{\mathbb{R}}$ is defined as

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}.$$

That is, the set of real numbers together with two auxiliary symbols $+\infty, -\infty$ such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty.$$

Bolzano-Weierstrass Theorem

Theorem. *A bounded sequence of real numbers has a convergent subsequence.*

Remark. $\liminf s_n \leq \limsup s_n$

Proof. content... □

Proposition. $\limsup s_n = \beta \iff \begin{cases} \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } n \geq n_0 \Rightarrow s_n < \beta + \varepsilon \\ \forall \varepsilon > 0, \forall n \in \mathbb{N} \exists k \in \mathbb{N} \text{ s.t. } k \geq n \wedge \beta - \varepsilon < s_k. \end{cases}$

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A Equivalent Statements of the Least Upper Bound Property

Least Upper Bound Property \iff Monotone Convergence Theorem
 \iff Nested Interval Property

Theorem. *Monotone Convergence Theorem \iff Nested Interval Property*

Proof. (\Rightarrow) See **Nested Interval Property**.

(\Leftarrow) TBA

□

Theorem. *Least Upper Bound Property \iff Monotone Convergence Theorem*

Proof. (\Rightarrow) See **Monotone Convergence Theorem**.

(\Leftarrow) TBA

□