Linear Algebra IV

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We cover the following topics in this note.

- Eigenvectors and Diagonalization.
  - * Hessian Matrix
  - * Differential Equation
- TBA.

Observation (Choosing a basis to simplify a linear map). Consider a finite-dimensional vector space V over a field $\mathbb{F}$. Let $T : V \to V$ be a linear operator. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of V such that $[T]_{\mathcal{B}}$ is a diagonal matrix:

$$[T]_{\mathcal{B}} = [T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}_{n\times n}, \quad \text{i.e.,} \quad T(\mathbf{v}_i) = d_i \mathbf{v}_i \text{ with } 1 \le i \le n.$$

Then T may be very complicated, but with respect to $[T]_{\mathcal{B}}$ it looks nice.

Eigenvector, Eigenvalue

Definition. A nonzero vector $\mathbf{v} \in V \setminus \{0\}$ is an *eigenvector* of T if there exists $\lambda \in \mathbb{F}$ such that

$$T(\mathbf{v}) = \lambda \mathbf{v}.$$

The scalar λ is called the *eigenvalue* corresponding to $\mathbf{v}$.

Diagonalizable Linear Operator

Definition. We say $T : V \to V$ is *diagonalizable* if $\exists$ a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is diagonal. Equivalently, T is diagonalizable iff V has a basis consisting of eigenvectors of T.

Example 1 (Hessian; Quadratic form diagonalization). Note that

- Single variable Taylor series:

$$\begin{aligned} f(x) &= f(p) + \frac{1}{1!} f'(p)(x-p) + \frac{1}{2!} f''(p)(x-p)^2 + \cdots \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(p)}{k!}(x-p)^k + R. \end{aligned}$$

- Two variables Taylor series:

$$\begin{aligned} f(x,y) = f(p,q) &+ \frac{1}{1!} f_x(p,q)(x-p) + \frac{1}{1!} f_y(p,q)(y-q) \\ &+ \frac{1}{2!} f_{xx}(p,q)(x-p)^2 + \frac{1}{1!} f_{xy}(p,q)(x-p)(y-q) + \frac{1}{2!} f_{yy}(p,q)(y-q)^2 + \cdots \end{aligned}$$

  Let

$$X = \begin{bmatrix} x-p \\ y-q \end{bmatrix}, \qquad \nabla f = \begin{bmatrix} \frac{\partial}{\partial x} f \\ \frac{\partial}{\partial y} f \end{bmatrix}, \qquad H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

Then

$$\begin{aligned} f(x,y) &= \boxed{f(p,q)} + \boxed{\frac{1}{1!}\begin{bmatrix} f_x & f_y \end{bmatrix}\begin{bmatrix} x-p \\ y-q \end{bmatrix}} + \boxed{\frac{1}{2!}\begin{bmatrix} x-p & y-q \end{bmatrix}\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\begin{bmatrix} x-p \\ y-q \end{bmatrix}} + R \\ &= \boxed{f(p,q)} + \boxed{\frac{1}{1!}\nabla f^T X} + \boxed{\frac{1}{2!}X^T H X} + R. \end{aligned}$$

Consider $f(x,y) = 4x^2 + 2xy + 4y^2$. Then

$$f_{xx} = 4, \quad f_{yy} = 4, \quad f_{xy} = f_{yx} = 2 \quad \Rightarrow \quad H = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}.$$

Let

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \quad \Rightarrow \quad f = X^T H X.$$

Eigenpairs:

$$\lambda_1 = 6 \ \Rightarrow \ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 2 \ \Rightarrow \ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

So

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad D = \mathrm{diag}(\lambda_1, \lambda_2) = \mathrm{diag}(6, 2), \qquad H = Q^T D Q.$$

Hence

$$f = X^T H X = X^T (Q^T D Q) X = (QX)^T D (QX).$$

Let $QX := V = \begin{bmatrix} u \\ v \end{bmatrix}$ then

$$u = \frac{x+y}{\sqrt{2}}, \qquad v = \frac{x-y}{\sqrt{2}}.$$

Then

$$f = V^T D V = \begin{bmatrix} u & v \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 6u^2 + 2v^2.$$

Here, intersection term (xy) disappeared.

References

[1] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 31. 선형대수학 (h) 고유벡터와 행렬의 대각화 -1” YouTube Video, 29:46. Published November 06, 2019. URL: https://www.youtube.com/watch?v=RSOxa1rI_Kk.