

Set Theory II

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We cover the following topics in this note.

- Relations
- Equivalence Relations
- Equivalence Classes
- Partitions

Relation

Definition. Let $A \times B$ be the cartesian product of two sets A and B . A **(binary) relation** on $A \times B$ is a subset \mathcal{R} of $A \times B$. That is,

$$\mathcal{R} \text{ is a relation on } A \times B \iff \mathcal{R} \subseteq A \times B.$$

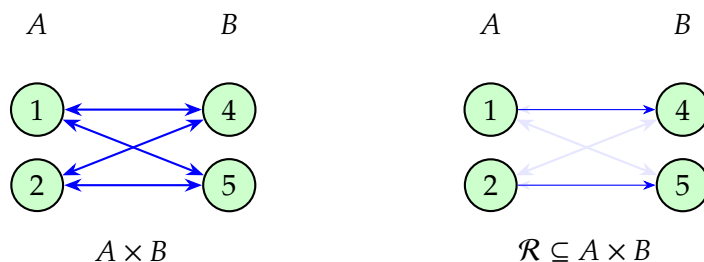
Remark. \mathcal{R} is a relation on $A \iff \mathcal{R} \subseteq A \times A$.

Note (Notation). Let $(s, t) \in \mathcal{R}$. We use the notation $s \mathcal{R} t$ and we can say “ s is related to t by \mathcal{R} ”. If $(s, t) \notin \mathcal{R}$, we denote as: $s \not\mathcal{R} t$.

Example. Let $A = \{1, 2\}$ and $B = \{4, 5\}$. Then

$$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5)\}.$$

Here, $\mathcal{R} = \{(1, 4), (2, 5)\} \subseteq A \times B$ be a relation.



Example. Let A and B are sets, and let $f : A \rightarrow B$ be a function from A to B . Then

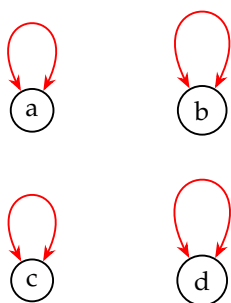
$$(a, b) \in f \iff a f b \iff b = f(a).$$

★ Equivalence Relation ★

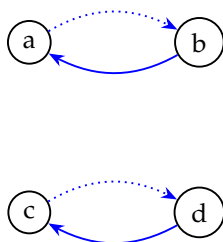
Definition. A binary relation \mathcal{R} on a set S is called an **equivalence relation** if it satisfies the following three properties: for all $a, b, c \in S$,

- (i) (Reflexivity) $(a, a) \in \mathcal{R}$;
- (ii) (Symmetry) $(a, b) \in \mathcal{R} \implies (b, a) \in \mathcal{R}$;
- (iii) (Transitivity) $(a, b) \in \mathcal{R} \wedge (b, c) \in \mathcal{R} \implies (a, c) \in \mathcal{R}$.

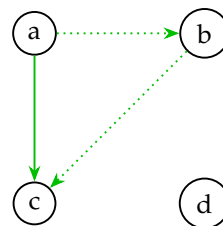
Remark.



Reflexivity
(each element is related to itself)



Symmetry
(if a is related to b ,
then b is related to a)

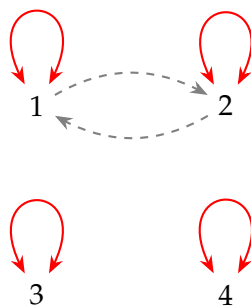


Transitivity
(if a is related to b and b is related to c ,
then a is related to c)

Example. Let $A = \{1, 2, 3, 4\}$. Then

$$\mathcal{R} = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1)\}$$

is an equivalence relation on A .



Note. Let A, B, C are sets, and let $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions.

- We claim that $(g \circ f)[A] = g[f[A]]$:

$$(g \circ f)[A] = \{(g \circ f)(a) : a \in A\} = \{g(f(a)) : a \in A\} = \{g(b) : b = f(a) \in f[A]\} = g[f[A]].$$

- We claim that $f \text{ is surjective} \iff \text{Img}(f) = f[A] = B$:

$$f : A \twoheadrightarrow B \iff \forall b \in B, \exists a \in A \text{ s.t. } f(a) = b \iff f[A] = \{f(a) \in B : a \in A\} = B.$$

Lemma 1 Let A, B and C are sets, and let $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions.

- (1) If f and g are both one-to-one, then $(g \circ f) : A \rightarrow C$ is one-to-one.
- (2) If f and g are both onto, then $(g \circ f) : A \rightarrow C$ is onto.

Proof. (1) Let f and g are both one-to-one. We must show that $(g \circ f) : A \rightarrow C$ is one-to-one.

Suppose that $(g \circ f)(a) = (g \circ f)(a')$. Then

$$\begin{aligned} (g \circ f)(a) = (g \circ f)(a') &\implies g(f(a)) = g(f(a')) && \text{by def. of composition} \\ &\implies f(a) = f(a') && \because g \text{ is injective} \\ &\implies a = a'. && \because f \text{ is injective} \end{aligned}$$

(2) Let f and g are both onto. We must show that $(g \circ f) : A \rightarrow C$ is onto, i.e., $(g \circ f)[A] = C$.

$$\begin{aligned} (g \circ f)[A] &= g[f[A]] \\ &= g[B] && \because f : A \rightarrow B \text{ is surjective, i.e., } f[A] = B \\ &= C. && \because g : B \rightarrow C \text{ is surjective, i.e., } g[B] = C \end{aligned}$$

□

Lemma 2 Let A, B and C are sets, and let $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions.

- (1) If $(g \circ f) : A \rightarrow C$ is one-to-one, then f is one-to-one.
- (2) If $(g \circ f) : A \rightarrow C$ is onto, then g is onto.

Proof. (1) Let $g \circ f$ is one-to-one. We must show that f is one-to-one. Suppose that $f(a) = f(a')$.
Then

$$\begin{aligned} f(a) = f(a') &\implies g(f(a)) = g(f(a')) && \because g \text{ is a function} \\ &\implies (g \circ f)(a) = (g \circ f)(a') && \text{by the def. of composition} \\ &\implies a = a'. && \because g \circ f \text{ is injective} \end{aligned}$$

(2) Let $g \circ f$ is onto, i.e., $(g \circ f)[A] = C$. We must show that g is onto, i.e., $g[B] = C$:

$$\begin{aligned} (\subseteq) \quad g[B] &= \{g(b) \in C : b \in B\} \subseteq C; \\ (\supseteq) \quad C &= (g \circ f)[A] = g[f[A]] = \{g(b) \in C : b \in f[A]\} \subseteq g[B]. \end{aligned}$$

□

Equivalence Relation on 2^A Based on Bijection

Proposition 3 Let A be a set, and 2^A be its power set. Define a relation \mathcal{R} on 2^A as follows:

$$X \sim_{\mathcal{R}} Y \iff \exists f \in Y^X \text{ such that } f \text{ is bijective,}$$

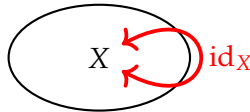
for $X, Y \in 2^A$. In other words,

$$\mathcal{R} := \{(X, Y) \in 2^A \times 2^A : \exists \text{ a bijection } f \in Y^X\}.$$

Then \mathcal{R} is an equivalence relation on 2^A .

Proof. Let $X, Y, Z \in 2^A$. We must show that \mathcal{R} is reflexive, symmetric and transitive:

(i) (Reflexivity) We NTS¹ that $X \sim_{\mathcal{R}} X$. In other words, we need to find a bijection from X to itself.



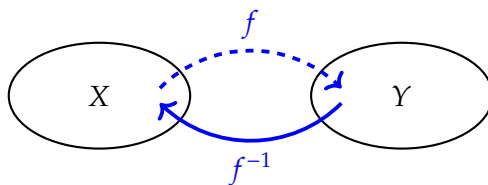
Consider the identity function

$$\begin{aligned} \text{id}_X &: X \longrightarrow X \\ x &\longmapsto x = \text{id}_X(x) \end{aligned}$$

for all $x \in X$. Clearly, id_X is a bijection. Thus, $X \sim_{\mathcal{R}} X$.

¹'NTS' means that "need to show".

- (ii) (Symmetry) We NTS that $X \sim_{\mathcal{R}} Y \implies Y \sim_{\mathcal{R}} X$. In other words, if there exists a bijection $f : X \rightarrow Y$, then there must exist a bijection $g : Y \rightarrow X$.

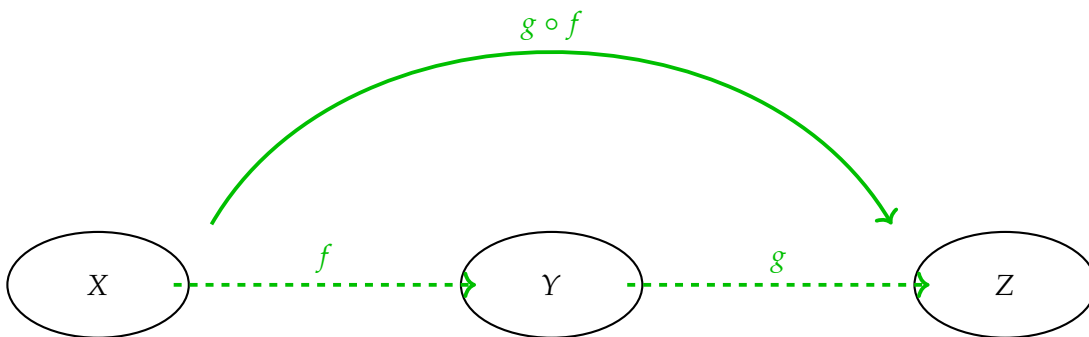


Suppose that $f : X \rightarrow Y$ is a bijection. Then it has an inverse function $f^{-1} : Y \rightarrow X$, which satisfies:

$$\forall x \in X : f^{-1}(f(x)) = x = \text{id}_X(x) \quad \text{and} \quad \forall y \in Y : f(f^{-1}(y)) = y = \text{id}_Y(y).$$

That is, $f^{-1} \circ f = \text{id}_X$ and $f \circ f^{-1} = \text{id}_Y$. By **Lemma 2**, f^{-1} must be a bijection since f , id_X and id_Y are bijections. Thus, there is a bijection $g = f^{-1}$.

- (iii) We NTS that $X \sim_{\mathcal{R}} Y \wedge Y \sim_{\mathcal{R}} Z \implies X \sim_{\mathcal{R}} Z$. In other words, if there exist two bijections $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, then there must exist a bijection $h : X \rightarrow Z$.



Suppose that $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ both are bijective. Define the function

$$\begin{aligned} h &: X \longrightarrow Z \\ x &\longmapsto (g \circ f)(x) = h(x) \end{aligned}$$

for all $x \in X$. By **Lemma 1**, $h = g \circ f$ must be a bijection since f and g are both bijective.

Hence it is proved. □

Indexed Family

Definition. Let I and S are sets. Consider a function $A : I \rightarrow S$ defined by $i \mapsto A(i) =: A_i$. The image $\text{Img}(A)$ is called an **indexed family** of elements in S indexed by I . We write this indexed family as: $\langle A_i \rangle_{i \in I}$. Note that

$$\text{Img}(A) = \{A(i) : i \in I\} = \{A_i : i \in I\} = \langle A_i \rangle_{i \in I}.$$

Example (Sequence). Let $I = \mathbb{N}$ be an indexing set. Then

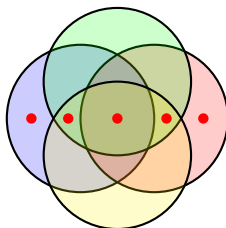
$$S := \{A_1, A_2, A_3, A_4, \dots\} = \{A_i : i \in \mathbb{N}\} = \langle A_i \rangle_{i \in \mathbb{N}}$$

is an indexed family of elements in S indexed by \mathbb{N} .

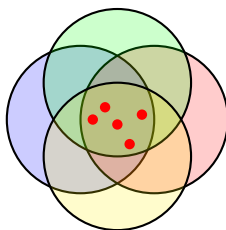
Union and Intersection of an Indexed Family

Definition. Let I and S are sets, and let $\langle A_i \rangle_{i \in I}$ be an indexed family in S .

- The **union** of $\langle A_i \rangle_{i \in I}$ is defined by $\bigcup_{i \in I} A_i := \{x \in S : \exists i \in I \text{ such that } x \in A_i\}.$



- The **intersection** of $\langle A_i \rangle_{i \in I}$ is defined by $\bigcap_{i \in I} A_i := \{x \in S : \forall i \in I, x \in A_i\}.$



Remark. Let $I = \{1, \dots, n\}$. Then

- $\bigcup_{i \in I} S_i = \bigcup_{i=1}^n S_i = S_1 \cup S_2 \cup \dots \cup S_n.$
- $\bigcap_{i \in I} S_i = \bigcap_{i=1}^n S_i = S_1 \cap S_2 \cap \dots \cap S_n.$

★ Partitions ★

Definition. Let S be a set, and let the function $A : I \rightarrow 2^S$ as $i \mapsto A_i := A(i) \subseteq S$, for all $i \in I$. Consider a family of subsets $\langle A_i \rangle_{i \in I}$, where $A_i \subseteq S$ for every index $i \in I$. The family $\langle A_i \rangle_{i \in I}$ is called a **partition** of S if the following conditions are satisfied:

- (i) **(Non-empty Subsets)** Each subset A_i is non-empty, i.e., $\forall i \in I, A_i \neq \emptyset$.
- (ii) **(Pairwise disjoint)** For all distinct $i, j \in I$, the subsets A_i and A_j are disjoint, i.e.,

$$\forall i, j \in I, i \neq j \implies A_i \cap A_j = \emptyset.$$

- (iii) **(Union covers the entire set)** The union of all subsets A_i covers the whole set S , i.e.,

$$\bigcup_{i \in I} A_i = S.$$

Example. Let \mathbb{Z} be a set of integers. We define an indexed family $\langle A_i \rangle_{i \in \{0,1,2\}}$ of subsets of \mathbb{Z} as follows:

$$A_0 = \{n \in \mathbb{Z} : n \equiv 0 \pmod{3}\} = \{n \in \mathbb{Z} : n = 3k + 0 \text{ for some } k \in \mathbb{Z}\} =: [0],$$

$$A_1 = \{n \in \mathbb{Z} : n \equiv 1 \pmod{3}\} = \{n \in \mathbb{Z} : n = 3k + 1 \text{ for some } k \in \mathbb{Z}\} =: [1],$$

$$A_2 = \{n \in \mathbb{Z} : n \equiv 2 \pmod{3}\} = \{n \in \mathbb{Z} : n = 3k + 2 \text{ for some } k \in \mathbb{Z}\} =: [2].$$

Then

- (i) $[0] \neq \emptyset, [1] \neq \emptyset$ and $[2] \neq \emptyset$.

- (ii)

$$[0] \cap [1] = \emptyset,$$

$$[1] \cap [2] = \emptyset,$$

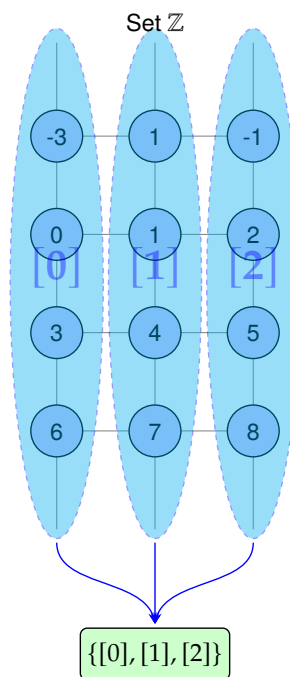
$$[2] \cap [0] = \emptyset.$$

- (iii) $[0] \cup [1] \cup [2] = \mathbb{Z}$.

Thus,

$$\{A_1, A_2, A_3\} = \{[0], [1], [2]\}$$

is a partition of \mathbb{Z} .

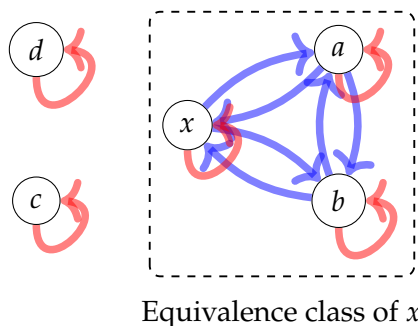


★ Equivalence Class ★

Definition. Let $\mathcal{R} \subseteq S \times S$ be an equivalence relation on S . The **equivalence class** of $x \in S$ under \mathcal{R} is the set

$$[x]_{\mathcal{R}} = \{y \in S : x \mathcal{R} y\}.$$

Note. Note that $\alpha \mathcal{R} x \iff \alpha \in [x]_{\mathcal{R}} \iff x \mathcal{R} \alpha$.



Lemma 4 Let \mathcal{R} be an equivalence relation on a set S . For any $x, y \in S$, let $[x]$ and $[y]$ represent the equivalence classes of x and y , respectively, under \mathcal{R} .

- (1) $\forall x \in S, x \in [x]$.
- (2) $x \mathcal{R} y \iff [x] = [y]$.
- (3) $x \not\mathcal{R} y \iff [x] \cap [y] = \emptyset$.

Proof. (1) Let $x \in S$. Since \mathcal{R} is reflexive, we have $x \mathcal{R} x$, i.e., $x \in [x]$.

(2) (\implies) Let $x \mathcal{R} y$. We NTS that $[x] = [y]$:

(\subseteq) Let $\alpha \in [x]$, i.e., $\alpha \mathcal{R} x$. Then

$$\begin{aligned} \alpha \mathcal{R} x &\implies \alpha \mathcal{R} y && \because x \mathcal{R} y \text{ and } \mathcal{R} \text{ is transitive} \\ &\implies \alpha \in [y]. \end{aligned}$$

(\supseteq) Let $\beta \in [y]$, i.e., $y \mathcal{R} \beta$. Then

$$\begin{aligned} y \mathcal{R} \beta &\implies x \mathcal{R} \beta && \because x \mathcal{R} y \text{ and } \mathcal{R} \text{ is transitive} \\ &\implies \beta \in [x]. \end{aligned}$$

(\Leftarrow) Let $[x] = [y]$. Then

$$x \in S \xrightarrow{\text{by (1)}} x \in [x] = [y] \implies x \in [y] \implies x \mathcal{R} y.$$

(3) (\Rightarrow) Let $x \mathcal{R} y$. Suppose that $[x] \cap [y] \neq \emptyset$ then $\exists \gamma \in S$ such that $\gamma \in [x] \cap [y]$. Then

$$\gamma \in [x] \cap [y] \Rightarrow \gamma \in [x] \wedge \gamma \in [y] \Rightarrow x \mathcal{R} \gamma \wedge \gamma \mathcal{R} y \Rightarrow x \mathcal{R} y \quad \checkmark.$$

(\Leftarrow) Let $[x] \cap [y] = \emptyset$. Suppose that $x \mathcal{R} y$. By (1) and (2), we have $x \in [x] = [y] \quad \checkmark$.

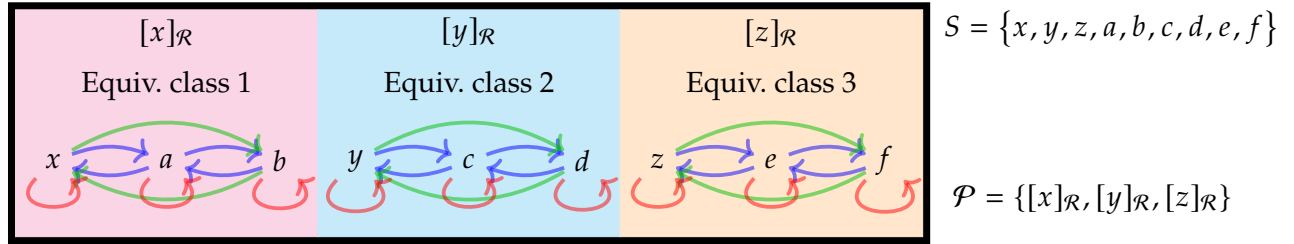
□

Fundamental Theorem on Equivalence Relations

Theorem 5 Let S be a set and let \mathcal{R} be an equivalence relation on S . Define the set of equivalence classes

$$\mathcal{P} := \{[x]_{\mathcal{R}} : x \in S\}, \text{ where } [x]_{\mathcal{R}} = \{y \in S : x \mathcal{R} y\}.$$

Then \mathcal{P} forms the partition of S .



Proof. We must show that the set of equivalence classes $\{[x]_{\mathcal{R}} : x \in S\}$ satisfies the three conditions of a partition:

(i) (Equivalence Class is not Empty) By (1) of **Lemma 4**, it is proved.

(ii) (Equivalence Classes are Disjoint) By (2) and (3) of **Lemma 4**, it is proved.

(iii) (Union of Equivalence Classes is Whole Set) We NTS that $\bigcup \{[x]_{\mathcal{R}} : x \in S\} = S$:

(\subseteq) Since $[x]_{\mathcal{R}} \subseteq S$, we have

$$\bigcup \{[x]_{\mathcal{R}} : x \in S\} = \bigcup_{x \in S} [x]_{\mathcal{R}} \subseteq S.$$

(\supseteq) Let $\alpha \in S$. We want to show that $\alpha \in \bigcup_{x \in S} [x]_{\mathcal{R}}$, i.e.,

$$\exists x \in S \text{ such that } \alpha \in [x]_{\mathcal{R}}.$$

By (1) of **Lemma 4**, we obtain $\alpha \in [\alpha]$. Thus, for every $\alpha \in S$, $\alpha \in \bigcup_{x \in S} [x]_{\mathcal{R}}$.

□

Relation Induced by Partition is Equivalence

Theorem 6 Let S be a set and $\mathcal{P} = \langle P_i \rangle_{i \in I}$ be a partition of S . We define a relation \mathcal{R} on S :

$$x \sim_{\mathcal{R}} y \iff \exists i \in I \text{ such that } x, y \in P_i$$

for all $x, y \in S$. That is, x is related to y under \mathcal{R} if and only if x and y belong to the same subset P_i in the partition. Then \mathcal{R} is the equivalence relation induced by a partition \mathcal{P} .

Proof. Let $\langle P_i \rangle_{i \in I}$ be a partition of S . That is,

$$(a) P_i \neq \emptyset \text{ for all } i \in I; \quad (b) P_i \cap P_j = \emptyset \text{ for } i \neq j; \quad (c) \bigcup_{i \in I} P_i = S.$$

Let $x, y \in S$. Note that

$$\mathcal{R} := \{(x, y) \in S \times S : \exists i \in I \text{ s.t. } x \in P_i \wedge y \in P_i\}.$$

We NTS that \mathcal{R} is reflexive, symmetric and transitive:

(i) (Reflexivity) We NTS that $x \sim_{\mathcal{R}} x$:

$$x \in S \xrightarrow{\text{by (c)}} x \in \bigcup_{i \in I} P_i \implies \exists i \in I \text{ s.t. } x \in P_i \implies \exists i \in I \text{ s.t. } x \in P_i \wedge x \in P_i \implies x \sim_{\mathcal{R}} x.$$

(ii) (Symmetry) We NTS that $x \sim_{\mathcal{R}} y \implies y \sim_{\mathcal{R}} x$:

$$x \sim_{\mathcal{R}} y \implies \exists i \in I \text{ s.t. } x \in P_i \wedge y \in P_i \implies \exists i \in I \text{ s.t. } y \in P_i \wedge x \in P_i \implies y \sim_{\mathcal{R}} x.$$

(iii) (Transitivity) We NTS that $x \sim_{\mathcal{R}} y \wedge y \sim_{\mathcal{R}} z \implies x \sim_{\mathcal{R}} z$:

$$\begin{cases} x \sim_{\mathcal{R}} y \\ y \sim_{\mathcal{R}} z \end{cases} \implies \begin{cases} \exists i \in I \text{ s.t. } x \in P_i \wedge y \in P_i \\ \exists j \in I \text{ s.t. } y \in P_j \wedge z \in P_j \end{cases} \xrightarrow{\text{by (b), } i=j} \exists i = j \in I \text{ s.t. } x \in P_i \wedge z \in P_{j=i} \implies x \sim_{\mathcal{R}} z.$$

□

References

- [1] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 3. 집합론 기초 (c).” YouTube Video, 35:04. Published September 07, 2019. URL: <https://www.youtube.com/watch?v=2gM-Vh8CY8I&t=1607s>.