Notes on Complex Analysis and Riemann Surface Theory toward Algebraic Geometry

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Contents

Chapter 1

De Rham Complex, Short Exact Sequence, and Mayer-Vietoris

1.1 Grad / Curl / Div

On $M = \mathbb{R}^3$ (with its standard Euclidean metric and orientation), the exterior derivative packages the familiar vector calculus operators:

Degree	**Differential Form**	**Exterior derivative d**	**Vector calculus**
0-forms	$f \in \Omega^0(\mathbb{R}^3)$ (scalar field)	$d : \Omega^0(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$, $df = f_x\,dx + f_y\,dy + f_z\,dz$	Gradient $df \;\leftrightarrow\; \nabla f$
1-forms	$\alpha \in \Omega^1(\mathbb{R}^3)$, $\alpha = P\,dx + Q\,dy + R\,dz$	$d : \Omega^1(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$, $d\alpha = \left(R_y - Q_z\right) dy \wedge dz + \left(P_z - R_x\right) dz \wedge dx + \left(Q_x - P_y\right) dx \wedge dy$	Curl (via Hodge star) $d\alpha \;\leftrightarrow\; \nabla \times (P, Q, R)$
2-forms	$\beta \in \Omega^2(\mathbb{R}^3)$, $\beta = A\,dy \wedge dz + B\,dz \wedge dx + C\,dx \wedge dy$	$d : \Omega^2(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3)$, $d\beta = \left(A_x + B_y + C_z\right) dx \wedge dy \wedge dz$	Divergence $d\beta \;\leftrightarrow\; \nabla \cdot (A, B, C)$

Table 1.1: On $\mathbb{R}^3$ (with Euclidean metric and orientation), the exterior derivative packages grad/curl/div (using the Hodge star to identify 2-forms with vector fields).

- **Functions** $f \in \Omega^0(\mathbb{R}^3)$ correspond to scalar fields. The map

  $$d : \Omega^0(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$$

  corresponds to the **gradient**: if $f = f(x, y, z)$ then

  $$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz.$$

  Viewing 1-forms as vector fields via the Euclidean metric, df corresponds to ∇f.

- **1-forms** $\alpha \in \Omega^1(\mathbb{R}^3)$ correspond (after identifying 1-forms with vector fields) to vector fields $\mathbf{F} = (P, Q, R)$ via

  $$\alpha = P\,dx + Q\,dy + R\,dz.$$

Then

$$d : \Omega^1(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$$

corresponds to **curl**. Indeed,

$$d\alpha = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy,$$

and (using the Hodge star to identify 2-forms with vector fields) this corresponds to $\nabla \times \mathbf{F}$.

- **2-forms** $\beta \in \Omega^2(\mathbb{R}^3)$ correspond (via the Hodge star) to vector fields $\mathbf{G} = (A, B, C)$ by writing

  $$\beta = A\, dy \wedge dz + B\, dz \wedge dx + C\, dx \wedge dy.$$

  Then

  $$d : \Omega^2(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3)$$

  corresponds to **divergence**:

  $$d\beta = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dx \wedge dy \wedge dz,$$

  which corresponds to $\nabla \cdot \mathbf{G}$.

Operator	**Differential form side**	**Vector field side (via $\flat, \sharp, *$)**
Gradient	For $f \in C^\infty(\mathbb{R}^3) = \Omega^0(\mathbb{R}^3)$, $$df \in \Omega^1(\mathbb{R}^3)$$	Define $\nabla f \in \mathfrak{X}(\mathbb{R}^3)$ by $$(\nabla f)^\flat = df, \qquad \nabla f = (df)^\sharp.$$
Curl	For a vector field $X \in \mathfrak{X}(\mathbb{R}^3)$ with 1-form $X^\flat \in \Omega^1(\mathbb{R}^3)$, $$d(X^\flat) \in \Omega^2(\mathbb{R}^3)$$	Define curl $X \in \mathfrak{X}(\mathbb{R}^3)$ by $$(\operatorname{curl} X)^\flat = *\,d(X^\flat), \qquad \operatorname{curl} X = \big(*\,d(X^\flat)\big)^\sharp.$$
Divergence	For a vector field $X \in \mathfrak{X}(\mathbb{R}^3)$, consider the 2-form $*X^\flat \in \Omega^2(\mathbb{R}^3)$ and its derivative $$d(*X^\flat) \in \Omega^3(\mathbb{R}^3)$$	Let vol be the Euclidean volume form. Define div $X \in C^\infty(\mathbb{R}^3)$ by $$d(*X^\flat) = (\operatorname{div} X)\ \mathrm{vol}, \quad \operatorname{div} X = *\,d(*X^\flat).$$

Table 1.2: Coordinate-free grad/curl/div on $(\mathbb{R}^3, g)$ using musical isomorphisms $\flat, \sharp$ and the Hodge star $*$.

$$\operatorname{curl}(\nabla f) = \big(*\,d((df)^{\sharp\flat})\big)^\sharp = \big(*\,d(df)\big)^\sharp = 0, \qquad \operatorname{div}(\operatorname{curl} X) = *\,d\big(*(\operatorname{curl} X)^\flat\big) = *\,d\big(*(*\,d(X^\flat))\big) = 0,$$

where we used $d^2 = 0$ and $\ast^2 = \pm 1$ on forms in dimension 3.

The cochain condition $d^{k+1} \circ d^k = 0$ becomes, under the above identifications,

$$\nabla \times (\nabla f) = 0 \qquad \text{and} \qquad \nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

In general, the de Rham complex is the coordinate-free framework that explains these vector calculus identities.

Cohomology as "obstructions to being a gradient/curl"

In $\mathbb{R}^3$, "closed" and "exact" specialize as follows:

- A 1-form α is **closed** iff $d\alpha = 0$, i.e. $\nabla \times \mathbf{F} = 0$ (irrotational field). It is **exact** iff $\alpha = df$, i.e. $\mathbf{F} = \nabla f$. Thus H^1_{dR} measures irrotational fields that are not global gradients.
- A 2-form β is **closed** iff $d\beta = 0$, i.e. $\nabla \cdot \mathbf{G} = 0$ (sourceless field). It is **exact** iff $\beta = d\alpha$, i.e. $\mathbf{G} = \nabla \times \mathbf{F}$. Thus H^2_{dR} measures divergence-free fields that are not global curls.

On contractible domains (like $\mathbb{R}^3$), the Poincaré lemma implies these obstructions vanish: $H^k_{\mathrm{dR}}(\mathbb{R}^3) = 0$ for $k \geq 1$.

$$\begin{tikzcd}
\Omega^0(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, equal] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] & \Omega^3(\mathbb{R}^3) \\
C^\infty(\mathbb{R}^3) \arrow[r, "\nabla"'] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{curl}"'] \arrow[u, shift left, "\sharp"'] \arrow[u, shift right, "\flat"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{div}"'] \arrow[u, shift left, "(\ast\,\cdot)^\sharp"'] \arrow[u, shift right, "(\ast\,\cdot)^\flat"] & C^\infty(\mathbb{R}^3) \arrow[u, "\ast^{-1} = \pm\ast"']
\end{tikzcd}$$

Figure 1.1: Vector calculus operators as conjugates of d via $\flat, \sharp, \ast$ in $\mathbb{R}^3$.

$$\begin{tikzcd}
C^\infty(\mathbb{R}^3) \arrow[r, "\nabla"] \arrow[d, equal] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{curl}"] \arrow[d, "\flat"'] & \mathfrak{X}(\mathbb{R}^3) \arrow[d, "\flat"'] \\
\Omega^0(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^1(\mathbb{R}^3) \arrow[r, "\ast d"'] & \Omega^1(\mathbb{R}^3)
\end{tikzcd} \qquad \Rightarrow \qquad \mathrm{curl}(\nabla f) = \big(\ast\, d(df)\big)^\sharp = \big(\ast\, d^2 f\big)^\sharp = 0.$$

$$\begin{tikzcd}
\mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{curl}"] \arrow[d, "\flat"'] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{div}"] \arrow[d, "\flat"'] & C^\infty(\mathbb{R}^3) \arrow[d, equal] \\
\Omega^1(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^3(\mathbb{R}^3)
\end{tikzcd} \qquad \text{with} \qquad \mathrm{curl}\, X = \big(\ast\, d(X^\flat)\big)^\sharp, \quad \mathrm{div}\, Y = \ast\, d\big(\ast\, Y^\flat\big).$$

$$\Rightarrow \quad \mathrm{div}(\mathrm{curl}\, X) = \ast\, d\Big(\ast\,(\mathrm{curl}\, X)^\flat\Big) = \ast\, d\Big(\ast\,(\ast\, d(X^\flat))\Big) = \pm \ast\, d(d(X^\flat)) = 0,$$

where the sign comes from $\ast^2 = \pm 1$ on k-forms in dimension 3 (in Euclidean $\mathbb{R}^3$, $\ast^2 = +1$ for all k).

de Rham Cohomology

Cochain Complex

Let M be a smooth manifold. A **cochain complex** $(C^\bullet, d^\bullet)$ in $\mathbb{R}$-vector spaces consists of

- $\mathbb{R}$-vector spaces C^k for each $k \in \mathbb{Z}$,
- $\mathbb{R}$-linear maps (coboundary maps) $d^k : C^k \to C^{k+1}$,

such that $d^{k+1} \circ d^k = 0$ for all k.

de Rham cochain complex

For a smooth manifold M, the de Rham cochain complex $(\Omega^\bullet(M), d)$ is

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots,$$

where

- $\Omega^k(M)$ is the $\mathbb{R}$-vector space of smooth differential k-forms on M.
- $d^k := d : \Omega^k(M) \to \Omega^{k+1}(M)$ is the exterior derivative such that

$$d^{k+1} \circ d^k = 0 \quad \text{for all } k \geq 0.$$

The de Rham complex as a cochain complex

For a smooth manifold M, the de Rham cochain complex is

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots, \qquad d \circ d = 0.$$

In $\mathbb{R}^3$ (with a metric), d corresponds to grad/curl/div after identifying vector fields with differential forms via the Hodge star.

De Rham cohomology

$$H^k_{\mathrm{dR}}(M) := \frac{Z^k(M)}{B^k(M)} = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

This measures global obstructions to writing a closed form as d(potential).

Short exact sequence (SES) of complexes for a cover $M = U \cup V$

For each k define

$$\alpha^k : \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V), \quad \alpha^k(\omega) = (\omega|_U, \omega|_V),$$
$$\beta^k : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V), \quad \beta^k(\eta, \theta) = \eta|_{U\cap V} - \theta|_{U\cap V}.$$

Then (for smooth manifolds) one has a short exact sequence

$$0 \to \Omega^k(M) \xrightarrow{\alpha^k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\beta^k} \Omega^k(U \cap V) \to 0.$$

Middle exactness $\operatorname{im}(\alpha^k) = \ker(\beta^k)$ is the gluing property (sheaf property); surjectivity of β^k uses partition of unity. Since d commutes with restriction, these assemble into an SES of cochain complexes:

$$0 \to \Omega^*(M) \xrightarrow{\alpha} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\beta} \Omega^*(U \cap V) \to 0.$$

Mayer–Vietoris long exact sequence

Any SES of cochain complexes yields a long exact sequence in cohomology:

$$\cdots \to H^{k-1}(U \cap V) \xrightarrow{\delta} H^k(M) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(M) \to \cdots$$

The connecting map δ may be written explicitly using a partition of unity $\rho_U + \rho_V = 1$: for $d\eta = 0$ on $U \cap V$,

$$\delta([\eta]) = \big[\, d(\rho_V \eta) \,\big] = \big[\, d\rho_V \wedge \eta \,\big] \in H^k(M).$$

$$0 \longrightarrow \Omega^*(M) \xrightarrow{\ \alpha\ } \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\ \beta\ } \Omega^*(U \cap V) \longrightarrow 0$$

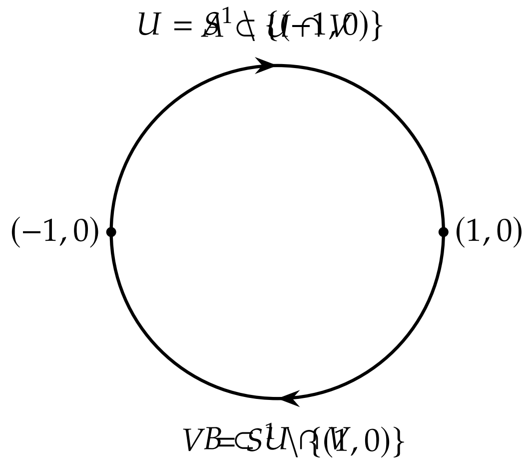
1.2 Example: The circle S^1

1.2.1 Cover and overlap

Let

$$U = S^1 \setminus \{(-1,0)\}, \qquad V = S^1 \setminus \{(1,0)\}.$$

Then U, V are contractible; $U \cap V$ has two connected components A (upper arc) and B (lower arc).



Why stereographic charts? A detailed explanation on S^1

1. Goal: exhibit U, V as coordinate domains

Let

$$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}, \qquad U = S^1 \setminus \{(-1,0)\}, \qquad V = S^1 \setminus \{(1,0)\}.$$

A **(smooth) chart** on the 1–manifold S^1 is a pair (U, φ) where

$$\varphi : U \longrightarrow \widetilde{U} \subset \mathbb{R}$$

is a homeomorphism (indeed, a diffeomorphism in the smooth category) onto an open set $\widetilde{U} \subset \mathbb{R}$.

Since removing one point from a circle "cuts it open", we expect U and V to be diffeomorphic to $\mathbb{R}$. Stereographic projection gives an explicit, canonical diffeomorphism

$$U \xrightarrow{\varphi} \mathbb{R}, \qquad V \xrightarrow{\psi} \mathbb{R},$$

with a very simple transition map on $U \cap V$.

2. Geometric definition of stereographic projection on S^1

Fix the vertical line

$$L = \{(0,t) \in \mathbb{R}^2 \mid t \in \mathbb{R}\} \cong \mathbb{R}, \qquad (0,t) \leftrightarrow t.$$

- **Chart on U.** For $p = (x,y) \in U$, draw the straight line through the **deleted** point $(-1,0)$ and p, and let it intersect L. The t–coordinate of the intersection is defined to be $\varphi(p)$.
- **Chart on V.** For $p = (x,y) \in V$, draw the straight line through the **deleted** point $(1,0)$ and p, intersect with L, and define its t–coordinate to be $\psi(p)$.

This is called **stereographic projection** (here, from the missing point onto the line L).

3. Derivation of the explicit formula for φ on U

Take a point $(x, y) \in S^1$ with $(x, y) \neq (-1, 0)$. Consider the line through $(-1, 0)$ and (x, y):

$$\ell(\lambda) = (-1,0) + \lambda\big((x,y) - (-1,0)\big) = (-1,0) + \lambda(x+1,\ y) = \big(-1 + \lambda(x+1),\ \lambda y\big).$$

We find the intersection with L by imposing x–coordinate = 0:

$$-1 + \lambda(x+1) = 0 \quad \Longrightarrow \quad \lambda = \frac{1}{x+1}.$$

Then the y–coordinate of the intersection point is

$$\lambda y = \frac{y}{x+1}.$$

Identifying $L \cong \mathbb{R}$ by $(0, t) \leftrightarrow t$, we obtain the chart map

$$\boxed{\varphi : U \longrightarrow \mathbb{R}, \qquad \varphi(x, y) = \frac{y}{1+x}.}$$

Note that φ is well-defined on U because $1 + x \neq 0$ precisely when $(x, y) \neq (-1, 0)$.

4. Derivation of the explicit formula for ψ on V

Similarly, for $(x, y) \in V$ consider the line through $(1, 0)$ and (x, y):

$$\ell(\lambda) = (1,0) + \lambda\big((x,y) - (1,0)\big) = (1,0) + \lambda(x-1,\ y) = \big(1 + \lambda(x-1),\ \lambda y\big).$$

Intersect with L by setting the x–coordinate equal to 0:

$$1 + \lambda(x-1) = 0 \quad \Longrightarrow \quad \lambda = -\frac{1}{x-1} = \frac{1}{1-x}.$$

Hence the y–coordinate of the intersection is

$$\lambda y = \frac{y}{1-x}.$$

Thus

$$\boxed{\psi : V \longrightarrow \mathbb{R}, \qquad \psi(x, y) = \frac{y}{1-x}.}$$

This is well-defined on V because $1 - x \neq 0$ precisely when $(x, y) \neq (1, 0)$.

5. Showing φ is a diffeomorphism (by writing an inverse)

Let $t \in \mathbb{R}$. We solve for $(x, y) \in S^1$ such that

$$t = \varphi(x, y) = \frac{y}{1+x}.$$

This gives $y = t(1 + x)$. Impose the circle equation $x^2 + y^2 = 1$:

$$x^2 + t^2(1+x)^2 = 1.$$

Expand and collect terms:

(B) MV computation of $H^*(S^1)$

Since U, V are contractible,

$$H^0(U) \cong H^0(V) \cong \mathbb{R}, \qquad H^1(U) = H^1(V) = 0.$$

Since $U \cap V = A \sqcup B$ has two components,

$$H^0(U \cap V) \cong \mathbb{R} \oplus \mathbb{R}, \qquad H^1(U \cap V) = 0.$$

The relevant MV segment is

$$0 \to H^0(S^1) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \xrightarrow{\delta} H^1(S^1) \to 0.$$

The map $H^0(U) \oplus H^0(V) \to H^0(U \cap V)$ sends $(a, b) \mapsto (a - b, a - b)$, whose image is the diagonal. Hence

$$H^0(S^1) \cong \mathbb{R}, \qquad H^1(S^1) \cong (\mathbb{R} \oplus \mathbb{R})/\Delta \cong \mathbb{R}.$$

(C) Generator and FTLI (grad) obstruction

Choose local angle branches θ_U on U and θ_V on V. Then $d\theta_U = d\theta_V$ on $U \cap V$, hence they glue to a global 1-form $d\theta$ on S^1. Its period is

$$\int_{S^1} d\theta = 2\pi \neq 0,$$

so $d\theta$ cannot be exact ($\int_\gamma df = 0$ for any loop γ by the fundamental theorem of line integrals). Thus $[d\theta]$ generates $H^1(S^1) \cong \mathbb{R}$.

$$0 \longrightarrow H^0(S^1) \longrightarrow H^0(U) \oplus H^0(V) \longrightarrow H^0(U \cap V) \xrightarrow{\ \delta\ } H^1(S^1) \longrightarrow 0$$

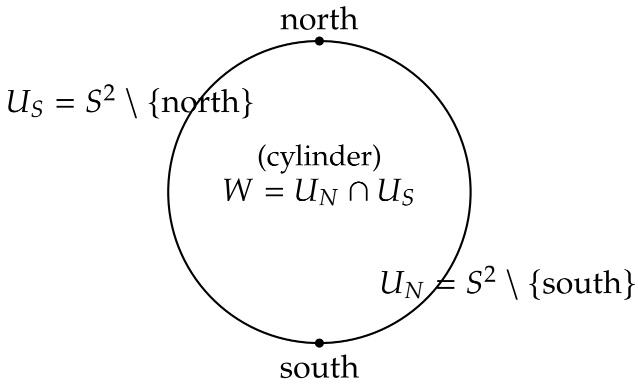
2. The sphere $S^2 \simeq \mathbb{CP}^1$

(A) Cover and overlap

Let

$$U_N = S^2 \setminus \{\text{south pole}\}, \qquad U_S = S^2 \setminus \{\text{north pole}\}, \qquad W = U_N \cap U_S.$$

Then $U_N \simeq \mathbb{R}^2$, $U_S \simeq \mathbb{R}^2$, and $W \simeq S^1 \times (0,1)$ (a cylinder).



(B) MV computation of $H^*(S^2)$

Contractibility of U_N, U_S gives

$$H^1(U_N) = H^1(U_S) = 0, \qquad H^2(U_N) = H^2(U_S) = 0.$$

Since $W \simeq S^1 \times (0,1)$, we have

$$H^1(W) \cong \mathbb{R}, \qquad H^2(W) = 0.$$

The key MV segment is

$$H^1(U_N) \oplus H^1(U_S) \to H^1(W) \xrightarrow{\delta} H^2(S^2) \to H^2(U_N) \oplus H^2(U_S),$$

which collapses to

$$0 \to \mathbb{R} \xrightarrow{\delta} H^2(S^2) \to 0,$$

hence

$$H^0(S^2) \cong \mathbb{R}, \qquad H^1(S^2) = 0, \qquad H^2(S^2) \cong \mathbb{R}.$$

(C) Generator via area form and Stokes (div/curl-type) obstruction

In spherical coordinates (ϑ, φ) on W,

$$\Omega := \sin\vartheta \, d\vartheta \wedge d\varphi$$

is the standard area form on S^2, and

$$\int_{S^2} \Omega = 4\pi \neq 0,$$

so Ω is not exact (if $\Omega = dA$ globally then $\int_{S^2} \Omega = \int_{S^2} dA = 0$ by Stokes).

Moreover, on the two charts one has explicit local potentials

$$A_N = (1 - \cos\vartheta)\, d\varphi \ \text{ on } U_N, \qquad A_S = -(1 + \cos\vartheta)\, d\varphi \ \text{ on } U_S,$$

satisfying $dA_N = dA_S = \Omega$. On the overlap,

$$A_N - A_S = 2\, d\varphi,$$

and $[d\varphi]$ generates $H^1(W) \cong \mathbb{R}$. The connecting map sends

$$\delta\big([2\, d\varphi]\big) = [\Omega] \in H^2(S^2).$$

$$0 \longrightarrow H^1(W) \xrightarrow[\cong]{\delta} H^2(S^2) \longrightarrow 0$$

3. The torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ (complex torus)

(A) Global closed forms, periods, and the expected answer

The forms dx, dy descend to T^2 and are closed. For the fundamental loops $\gamma_x(t) = (t, 0)$ and $\gamma_y(t) = (0, t)$ (mod $\mathbb{Z}^2$),

$$\int_{\gamma_x} dx = 1, \quad \int_{\gamma_y} dx = 0, \qquad \int_{\gamma_x} dy = 0, \quad \int_{\gamma_y} dy = 1.$$

Hence $[dx], [dy] \neq 0$ in $H^1(T^2)$ (exact 1-forms have zero periods on loops). Also $dx \wedge dy$ is closed and

$$\int_{T^2} dx \wedge dy = 1 \neq 0,$$

so $[dx \wedge dy] \neq 0$ in $H^2(T^2)$. Thus one expects

$$H^0(T^2) \cong \mathbb{R}, \qquad H^1(T^2) \cong \mathbb{R}^2\langle [dx], [dy] \rangle, \qquad H^2(T^2) \cong \mathbb{R}\langle [dx \wedge dy] \rangle.$$

(B) MV cover that recovers these classes

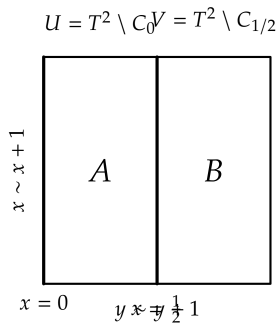
Let

$$C_0 = \{x \equiv 0 \ (\mathrm{mod}\ 1)\}, \qquad C_{1/2} = \{x \equiv \tfrac{1}{2} \ (\mathrm{mod}\ 1)\},$$

and define

$$U := T^2 \setminus C_0, \qquad V := T^2 \setminus C_{1/2}.$$

Then $U \simeq S^1$, $V \simeq S^1$, and $U \cap V$ is a disjoint union of two cylinders $A \sqcup B$ (two vertical open strips in the fundamental square).



(C) MV segments (what they do)

Cohomology of pieces:

$$H^0(U) \cong H^0(V) \cong \mathbb{R}, \qquad H^1(U) \cong H^1(V) \cong \mathbb{R},$$

$$H^0(U \cap V) \cong \mathbb{R} \oplus \mathbb{R}, \qquad H^1(U \cap V) \cong \mathbb{R} \oplus \mathbb{R}, \qquad H^2(U) = H^2(V) = H^2(U \cap V) = 0.$$

Degree 1: producing $[dx]$ and $[dy]$. The MV segment

$$H^0(U \cap V) \xrightarrow{\delta} H^1(T^2) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V)$$

shows:

- δ injects a copy of $\mathbb{R}$ (quotient of $H^0(U \cap V)$ by the diagonal) into $H^1(T^2)$; concretely it produces $[dx]$ from the "difference of constants" on the two components A, B.
- The kernel of $H^1(U) \oplus H^1(V) \to H^1(U \cap V)$ contributes the second independent class, represented globally by $[dy]$.

Hence $H^1(T^2) \cong \mathbb{R}^2$.

Degree 2: producing $[dx \wedge dy]$. The MV segment

$$H^1(U \cap V) \xrightarrow{\delta} H^2(T^2) \to 0$$

shows $H^2(T^2)$ is a quotient of $H^1(U \cap V) \cong \mathbb{R}^2$, killing the diagonal; the resulting 1-dimensional quotient maps onto $H^2(T^2)$, and the nonzero integral

$$\int_{T^2} dx \wedge dy = 1$$

identifies the generator as $[dx \wedge dy]$ (up to sign).

$$H^0(U \cap V) \xrightarrow{\ \delta\ } H^1(T^2) \longrightarrow H^1(U) \oplus H^1(V) \longrightarrow H^1(U \cap V)$$

$$H^1(U \cap V) \xrightarrow{\ \delta\ } H^2(T^2) \longrightarrow 0$$

Quick reference: generators and "potential" obstructions

- S^1: generator $[d\theta] \in H^1(S^1)$ detected by $\int_{S^1} d\theta = 2\pi$ (FTLI obstruction to being a global gradient).
- S^2: generator $[\Omega] \in H^2(S^2)$ detected by $\int_{S^2} \Omega = 4\pi$ (Stokes obstruction to being dA globally).
- T^2: generators $[dx], [dy] \in H^1(T^2)$ detected by loop periods; generator $[dx \wedge dy] \in H^2(T^2)$ detected by $\int_{T^2} dx \wedge dy = 1$.

Chapter 2

Mayer–Vietoris Sequence and de Rham Cohomology

Choose $\left\{V^k\right\}_{k=0}^{3}$ and isomorphisms $\Phi^k : V^k \to \Omega^k(U)$ such that

$$\Phi^{k+1} \circ d^k = d \circ \Phi^k$$

where $d^0 = \nabla$, $d^1 = \nabla\times$, $d^2 = \nabla\cdot$, and d is exterior derivative.

2.1 Why the spaces V^0, V^1, V^2, V^3 are chosen as scalar and vector fields

2.1.1 Axiomatic goal

Definition 2.1.1 (Design requirement: transport of the de Rham differential)**.** Let $U \subseteq \mathbb{R}^3$ be open and $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$. Let $(V^\bullet, d_V)$ be a cochain complex of $\Bbbk$-vector spaces concentrated in degrees $0, 1, 2, 3$, i.e. $V^n = 0$ for $n \notin \{0, 1, 2, 3\}$. We say that $(V^\bullet, d_V)$ **models the de Rham complex on U via identifications** if there exist $\Bbbk$-linear isomorphisms

$$\Phi^k : V^k \xrightarrow{\cong} \Omega^k(U) \qquad (k = 0, 1, 2, 3)$$

such that for all $k \in \{0, 1, 2\}$ the following diagram commutes:

$$\begin{tikzcd} V^k \arrow[r, "d_V^k"] \arrow[d, "\Phi^k"', "\cong"] & V^{k+1} \arrow[d, "\Phi^{k+1}"', "\cong"] \\ \Omega^k(U) \arrow[r, "d"] & \Omega^{k+1}(U). \end{tikzcd}$$

Equivalently,

$$\Phi^{k+1} \circ d_V^k = d \circ \Phi^k \qquad (k = 0, 1, 2).$$

2.1.2 Canonical identifications in Euclidean $\mathbb{R}^3$

Definition 2.1.2 (Scalar fields)**.** Define

$$V^0 := C^\infty(U; \Bbbk), \qquad V^3 := C^\infty(U; \Bbbk).$$

Remark 2.1.3. By definition of differential forms, $\Omega^0(U) = C^\infty(U; \Bbbk)$. Moreover, fixing the standard orientation with volume form

$$\mathrm{vol} := dx_1 \wedge dx_2 \wedge dx_3,$$

every 3-form is uniquely of the form $h\,\mathrm{vol}$ with $h \in C^\infty(U; \Bbbk)$, hence

$$\Omega^3(U) \cong C^\infty(U; \Bbbk)$$

via $h \mapsto h\,\mathrm{vol}$.

Definition 2.1.4 (Vector fields and the Euclidean musical isomorphism)**.** Define the $\Bbbk$-vector space of (smooth) vector fields

$$\mathfrak{X}(U; \Bbbk) := C^\infty(U; \Bbbk^3).$$

Endow U with the standard Euclidean metric $g = \sum_{i=1}^3 dx_i \otimes dx_i$. Define the $\Bbbk$-linear isomorphism

$$\flat : \mathfrak{X}(U; \Bbbk) \xrightarrow{\cong} \Omega^1(U)$$

by the coordinate formula

$$(P, Q, R)^\flat := P\,dx_1 + Q\,dx_2 + R\,dx_3.$$

Define

$$V^1 := \mathfrak{X}(U; \Bbbk) = C^\infty(U; \Bbbk^3).$$

Definition 2.1.5 (Hodge star and the identification $\Omega^2 \cong \mathfrak{X}$)**.** With the Euclidean metric and orientation, let

$$* : \Omega^k(U) \to \Omega^{3-k}(U)$$

be the Hodge star. Define the $\Bbbk$-linear isomorphism

$$\Psi : \mathfrak{X}(U;\Bbbk) \xrightarrow{\cong} \Omega^2(U), \qquad \Psi(G) := *(G^\flat).$$

In coordinates, for $G = (A, B, C)$ one has

$$\Psi(A, B, C) = A\,dx_2 \wedge dx_3 + B\,dx_3 \wedge dx_1 + C\,dx_1 \wedge dx_2.$$

Define

$$V^2 := \mathfrak{X}(U;\Bbbk) = C^\infty(U;\Bbbk^3).$$

2.1.3 Compatibility with grad, curl, div

Definition 2.1.6 (The grad–curl–div differentials)**.** Define $\Bbbk$-linear maps

$$\nabla : V^0 \to V^1, \qquad \nabla\times : V^1 \to V^2, \qquad \nabla\cdot : V^2 \to V^3$$

by the standard coordinate formulas

$$\nabla f = (\partial_1 f, \partial_2 f, \partial_3 f),$$
$$\nabla \times (P, Q, R) = (\partial_2 R - \partial_3 Q,\ \partial_3 P - \partial_1 R,\ \partial_1 Q - \partial_2 P),$$
$$\nabla \cdot (A, B, C) = \partial_1 A + \partial_2 B + \partial_3 C.$$

Proposition 2.1.7 (Commuting transport and forced shapes of V^k)**.** *Let $\Phi^0, \Phi^1, \Phi^2, \Phi^3$ be defined by*

$$\Phi^0 = \mathrm{id}_{C^\infty(U;\Bbbk)}, \qquad \Phi^1 = \flat, \qquad \Phi^2 = \Psi, \qquad \Phi^3(h) = h\,\mathrm{vol}.$$

Then

$$\Phi^1 \circ \nabla = d \circ \Phi^0, \qquad \Phi^2 \circ (\nabla\times) = d \circ \Phi^1, \qquad \Phi^3 \circ (\nabla\cdot) = d \circ \Phi^2.$$

Consequently, the grad–curl–div complex

$$0 \to V^0 \xrightarrow{\nabla} V^1 \xrightarrow{\nabla\times} V^2 \xrightarrow{\nabla\cdot} V^3 \to 0$$

is (via $\Phi^\bullet$) a transported model of the de Rham complex

$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \Omega^3(U) \to 0.$$

Proof. The equalities are verified by direct coordinate computation. Explicitly, for $f \in C^\infty(U;\Bbbk)$,

$$d(f) = \sum_{i=1}^{3} \partial_i f\,dx_i = (\nabla f)^\flat = \Phi^1(\nabla f).$$

For $F = (P, Q, R) \in V^1$ one computes

$$d(F^\flat) = (\partial_2 R - \partial_3 Q)\,dx_2 \wedge dx_3 + (\partial_3 P - \partial_1 R)\,dx_3 \wedge dx_1 + (\partial_1 Q - \partial_2 P)\,dx_1 \wedge dx_2 = \Psi(\nabla\times F) = \Phi^2(\nabla\times F),$$

and for $G = (A, B, C) \in V^2$ one computes

$$d(\Psi(G)) = (\partial_1 A + \partial_2 B + \partial_3 C)\,dx_1 \wedge dx_2 \wedge dx_3 = (\nabla \cdot G)\,\mathrm{vol} = \Phi^3(\nabla \cdot G).$$

□

2.1.4 Uniqueness up to constant changes of basis

Theorem 2.1.8 (Uniqueness up to $\mathrm{GL}_3(\Bbbk)$ in degrees 1 and 2)**.** *Let $\widetilde{\Phi}^0, \widetilde{\Phi}^1, \widetilde{\Phi}^2, \widetilde{\Phi}^3$ be any linear isomorphisms*

$$\widetilde{\Phi}^k : V^k \xrightarrow{\cong} \Omega^k(U) \qquad (k = 0, 1, 2, 3)$$

such that

$$\widetilde{\Phi}^1 \circ \nabla = d \circ \widetilde{\Phi}^0, \qquad \widetilde{\Phi}^2 \circ (\nabla\times) = d \circ \widetilde{\Phi}^1, \qquad \widetilde{\Phi}^3 \circ (\nabla\cdot) = d \circ \widetilde{\Phi}^2,$$

and assume $\widetilde{\Phi}^0 = \mathrm{id}$ and $\widetilde{\Phi}^3(h) = h\,\mathrm{vol}$. Then there exists a constant matrix $A \in \mathrm{GL}_3(\Bbbk)$ such that, after identifying $V^1 = V^2 = C^\infty(U; \Bbbk^3)$, one has

$$\widetilde{\Phi}^1 = \flat \circ A, \qquad \widetilde{\Phi}^2 = \Psi \circ A,$$

where A acts pointwise on $C^\infty(U; \Bbbk^3)$ by $(AF)(x) = A(F(x))$.

Proof. Define linear automorphisms $T^1 := \flat^{-1} \circ \widetilde{\Phi}^1$ and $T^2 := \Psi^{-1} \circ \widetilde{\Phi}^2$ of $C^\infty(U; \Bbbk^3)$. The relations $\widetilde{\Phi}^1 \circ \nabla = d \circ \mathrm{id} = \flat \circ \nabla$ and $\widetilde{\Phi}^2 \circ (\nabla\times) = d \circ \widetilde{\Phi}^1 = \Psi \circ (\nabla\times) \circ T^1$ imply

$$T^1 \circ \nabla = \nabla, \qquad T^2 \circ (\nabla\times) = (\nabla\times) \circ T^1.$$

A standard linear-algebra/analysis argument shows that any $\Bbbk$-linear endomorphism of $C^\infty(U; \Bbbk^3)$ commuting with all partial derivatives must be given by pointwise multiplication by a constant matrix in $\mathrm{GL}_3(\Bbbk)$; denote this matrix by A. Then $T^1 = A$ and the second commutation forces $T^2 = A$ as well. Hence $\widetilde{\Phi}^1 = \flat \circ A$ and $\widetilde{\Phi}^2 = \Psi \circ A$. □

Remark 2.1.9. The theorem formalizes the statement that, once one fixes the canonical identifications in degrees 0 and 3, the identifications in degrees 1 and 2 are unique up to an invertible constant change of basis of $\Bbbk^3$.

2.2 The grad–curl–div cochain complex and its cohomology

2.2.1 Vector spaces and linear maps

Definition 2.2.1 (Spaces of smooth fields)**.** Let $U \subseteq \mathbb{R}^3$ be an open set and fix a field $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$. Define $\Bbbk$-vector spaces

$$V^0 := C^\infty(U;\Bbbk), \qquad V^1 := C^\infty(U;\Bbbk^3), \qquad V^2 := C^\infty(U;\Bbbk^3), \qquad V^3 := C^\infty(U;\Bbbk),$$

with pointwise addition and scalar multiplication. For all $n \in \mathbb{Z} \setminus \{0,1,2,3\}$ set $V^n := 0$.

Definition 2.2.2 (Differentials: ∇, $\nabla\times$, $\nabla\cdot$)**.** Write (x_1, x_2, x_3) for the standard coordinates on $\mathbb{R}^3$ and $\partial_i := \frac{\partial}{\partial x_i}$. Define $\Bbbk$-linear maps

$$d^0 : V^0 \to V^1, \qquad d^1 : V^1 \to V^2, \qquad d^2 : V^2 \to V^3$$

by the following formulas:

$$\begin{aligned} d^0(f) &:= \nabla f := (\partial_1 f, \partial_2 f, \partial_3 f), \\ d^1(P,Q,R) &:= \nabla \times (P,Q,R) := \big(\partial_2 R - \partial_3 Q,\ \partial_3 P - \partial_1 R,\ \partial_1 Q - \partial_2 P\big), \\ d^2(A,B,C) &:= \nabla \cdot (A,B,C) := \partial_1 A + \partial_2 B + \partial_3 C. \end{aligned}$$

For all $n \in \mathbb{Z} \setminus \{0,1,2\}$ define $d^n : V^n \to V^{n+1}$ to be the zero map.

Proposition 2.2.3 (The grad–curl–div complex)**.** *The sequence*

$$0 \longrightarrow V^0 \xrightarrow{d^0=\nabla} V^1 \xrightarrow{d^1=\nabla\times} V^2 \xrightarrow{d^2=\nabla\cdot} V^3 \longrightarrow 0$$

is a cochain complex, i.e. $d^1 \circ d^0 = 0$ *and* $d^2 \circ d^1 = 0$*. Equivalently,*

$$\nabla \times (\nabla f) = 0 \quad \forall f \in C^\infty(U;\Bbbk), \qquad \nabla \cdot (\nabla \times F) = 0 \quad \forall F \in C^\infty(U;\Bbbk^3).$$

Proof. Let $f \in C^\infty(U;\Bbbk)$. Then

$$(\nabla \times \nabla f)_1 = \partial_2(\partial_3 f) - \partial_3(\partial_2 f) = 0$$

by commutativity of mixed partials; similarly $(\nabla \times \nabla f)_2 = (\nabla \times \nabla f)_3 = 0$. Hence $d^1 d^0 = 0$.

Let $F = (P,Q,R) \in C^\infty(U;\Bbbk^3)$. Then

$$\begin{aligned} \nabla \cdot (\nabla \times F) &= \partial_1(\partial_2 R - \partial_3 Q) + \partial_2(\partial_3 P - \partial_1 R) + \partial_3(\partial_1 Q - \partial_2 P) \\ &= \partial_1\partial_2 R - \partial_1\partial_3 Q + \partial_2\partial_3 P - \partial_2\partial_1 R + \partial_3\partial_1 Q - \partial_3\partial_2 P \\ &= 0 \end{aligned}$$

again by commutativity of mixed partial derivatives and cancellation. Thus $d^2 d^1 = 0$. □

2.2.2 Cohomology and interpretation

Definition 2.2.4 (Cocycles, coboundaries, cohomology)**.** Let $(V^\bullet, d)$ be the grad–curl–div cochain complex above. For each $n \in \mathbb{Z}$ define

$$Z^n := \ker(d^n) \subseteq V^n, \qquad B^n := \operatorname{im}(d^{n-1}) \subseteq V^n, \qquad H^n(V^\bullet) := Z^n / B^n.$$

Proposition 2.2.5 (Cohomology groups of the grad–curl–div complex). *With the conventions $d^{-1} = 0$ and $d^3 = 0$ one has:*

$$H^0(V^\bullet) \cong \ker(\nabla) = \{f \in C^\infty(U;\Bbbk) : \nabla f = 0\},$$

$$H^1(V^\bullet) \cong \ker(\nabla\times)/\mathrm{im}(\nabla) = \frac{\{F \in C^\infty(U;\Bbbk^3) : \nabla \times F = 0\}}{\{\nabla f : f \in C^\infty(U;\Bbbk)\}},$$

$$H^2(V^\bullet) \cong \ker(\nabla\cdot)/\mathrm{im}(\nabla\times) = \frac{\{G \in C^\infty(U;\Bbbk^3) : \nabla \cdot G = 0\}}{\{\nabla \times F : F \in C^\infty(U;\Bbbk^3)\}},$$

$$H^3(V^\bullet) \cong V^3/\mathrm{im}(\nabla\cdot) = \frac{C^\infty(U;\Bbbk)}{\{\nabla \cdot G : G \in C^\infty(U;\Bbbk^3)\}}.$$

Remark 2.2.6 (Interpretation). H^1 measures curl-free vector fields modulo gradients (obstructions to global scalar potentials). H^2 measures divergence-free vector fields modulo curls (obstructions to global vector potentials). H^3 measures functions modulo divergences.

2.2.3 Identification with the de Rham complex (formal transport)

Definition 2.2.7 (de Rham complex). Let $\Omega^k(U)$ denote the $\Bbbk$-vector space of smooth differential k-forms on U. The exterior derivative is a $\Bbbk$-linear map

$$d : \Omega^k(U) \to \Omega^{k+1}(U)$$

satisfying $d \circ d = 0$. The associated cohomology spaces are

$$H^k_{\mathrm{dR}}(U) := \ker\big(d : \Omega^k(U) \to \Omega^{k+1}(U)\big) \Big/ \mathrm{im}\big(d : \Omega^{k-1}(U) \to \Omega^k(U)\big).$$

Definition 2.2.8 (Musical isomorphism and Hodge star (Euclidean)). Equip $U \subseteq \mathbb{R}^3$ with the standard Euclidean metric and orientation. Let $\flat : C^\infty(U;\Bbbk^3) \to \Omega^1(U)$ denote the metric identification ("lowering an index"). Let $* : \Omega^k(U) \to \Omega^{3-k}(U)$ denote the Hodge star operator.

Proposition 2.2.9 (Commuting diagram with de Rham). *Define linear isomorphisms*

$$\Phi^0 : V^0 \xrightarrow{\cong} \Omega^0(U), \quad \Phi^0(f) = f,$$

$$\Phi^1 : V^1 \xrightarrow{\cong} \Omega^1(U), \quad \Phi^1(F) = F^\flat,$$

$$\Phi^2 : V^2 \xrightarrow{\cong} \Omega^2(U), \quad \Phi^2(G) = *(G^\flat),$$

$$\Phi^3 : V^3 \xrightarrow{\cong} \Omega^3(U), \quad \Phi^3(h) = h\, dx_1 \wedge dx_2 \wedge dx_3.$$

Then the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & V^0 \arrow[r, "\nabla"] \arrow[d, "\Phi^0"', "\cong"] & V^1 \arrow[r, "\nabla\times"] \arrow[d, "\Phi^1"', "\cong"] & V^2 \arrow[r, "\nabla\cdot"] \arrow[d, "\Phi^2"', "\cong"] & V^3 \arrow[r] \arrow[d, "\Phi^3"', "\cong"] & 0 \\
0 \arrow[r] & \Omega^0(U) \arrow[r, "d"] & \Omega^1(U) \arrow[r, "d"] & \Omega^2(U) \arrow[r, "d"] & \Omega^3(U) \arrow[r] & 0
\end{tikzcd}$$

Consequently, for each $k \in \{0,1,2,3\}$ there is an induced isomorphism

$$H^k(V^\bullet) \cong H^k_{\mathrm{dR}}(U).$$

Corollary 2.2.10 (Contractible case). *If U is contractible (e.g. U is star-shaped), then*

$$H^k(V^\bullet) = 0 \text{ for all } k \in \{1, 2, 3\},$$

and if U is connected then $H^0(V^\bullet) \cong \mathbb{k}$.

2.3 The grad–curl–div cochain complex and its identification with the de Rham complex

2.3.1 The grad–curl–div cochain complex

Definition 2.3.1 (Spaces and differentials)**.** Let $U \subseteq \mathbb{R}^3$ be open and fix $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$. Define $\Bbbk$-vector spaces

$$V^0 := C^\infty(U;\Bbbk), \quad V^1 := C^\infty(U;\Bbbk^3), \quad V^2 := C^\infty(U;\Bbbk^3), \quad V^3 := C^\infty(U;\Bbbk).$$

Write (x_1, x_2, x_3) for the standard coordinates and $\partial_i := \frac{\partial}{\partial x_i}$. Define $\Bbbk$-linear maps

$$d^0 : V^0 \to V^1, \qquad d^1 : V^1 \to V^2, \qquad d^2 : V^2 \to V^3$$

by

$$\begin{aligned} d^0(f) &:= \nabla f := (\partial_1 f, \partial_2 f, \partial_3 f), \\ d^1(P,Q,R) &:= \nabla \times (P,Q,R) := (\partial_2 R - \partial_3 Q,\ \partial_3 P - \partial_1 R,\ \partial_1 Q - \partial_2 P), \\ d^2(A,B,C) &:= \nabla \cdot (A,B,C) := \partial_1 A + \partial_2 B + \partial_3 C. \end{aligned}$$

Proposition 2.3.2 (Cochain complex condition)**.** *One has $d^1 \circ d^0 = 0$ and $d^2 \circ d^1 = 0$. Hence*

$$0 \longrightarrow V^0 \xrightarrow{\nabla} V^1 \xrightarrow{\nabla\times} V^2 \xrightarrow{\nabla\cdot} V^3 \longrightarrow 0$$

is a cochain complex.

Proof. This follows immediately from the computations

$$\nabla \times (\nabla f) = 0, \qquad \nabla \cdot (\nabla \times F) = 0,$$

which are verified componentwise using commutativity of mixed partial derivatives. □

2.3.2 Cohomology of the grad–curl–div complex

Definition 2.3.3 (Cohomology)**.** For each $n \in \{0,1,2,3\}$ define

$$Z^n := \operatorname{Ker}(d^n) \subseteq V^n, \qquad B^n := \operatorname{im}(d^{n-1}) \subseteq V^n, \qquad H^n(V^\bullet) := Z^n / B^n,$$

with the conventions $d^{-1} = 0$ and $d^3 = 0$.

Proposition 2.3.4 (Concrete description)**.** *One has canonical identifications*

$$\begin{aligned} H^0(V^\bullet) &\cong \operatorname{Ker}(\nabla), \\ H^1(V^\bullet) &\cong \operatorname{Ker}(\nabla\times) / \operatorname{im}(\nabla), \\ H^2(V^\bullet) &\cong \operatorname{Ker}(\nabla\cdot) / \operatorname{im}(\nabla\times), \\ H^3(V^\bullet) &\cong V^3 / \operatorname{im}(\nabla\cdot). \end{aligned}$$

2.3.3 Differential forms on $U \subseteq \mathbb{R}^3$

Definition 2.3.5 (de Rham complex)**.** Let $\Omega^k(U)$ be the $\Bbbk$-vector space of smooth k-forms on U. The exterior derivative is the $\Bbbk$-linear map

$$d : \Omega^k(U) \to \Omega^{k+1}(U)$$

characterized in coordinates by the usual rules (graded Leibniz rule and $d(dx_i) = 0$), and satisfies $d \circ d = 0$. The k-th de Rham cohomology is

$$H^k_{\mathrm{dR}}(U) := \mathrm{Ker}\big(d : \Omega^k(U) \to \Omega^{k+1}(U)\big) \Big/ \mathrm{im}\big(d : \Omega^{k-1}(U) \to \Omega^k(U)\big).$$

2.3.4 Explicit formulas for $\flat$ and $*$

Definition 2.3.6 (Euclidean musical isomorphisms)**.** Endow $U \subseteq \mathbb{R}^3$ with the standard Euclidean metric $g = \sum_{i=1}^3 dx_i \otimes dx_i$. Define the $\Bbbk$-linear map ("lowering an index")

$$\flat : C^\infty(U; \Bbbk^3) \to \Omega^1(U)$$

by the coordinate formula

$$(P, Q, R)^\flat := P\, dx_1 + Q\, dx_2 + R\, dx_3.$$

Its inverse $\sharp : \Omega^1(U) \to C^\infty(U; \Bbbk^3)$ is given by

$$(a_1 dx_1 + a_2 dx_2 + a_3 dx_3)^\sharp := (a_1, a_2, a_3).$$

Definition 2.3.7 (Hodge star in $\mathbb{R}^3$)**.** Fix the standard orientation, with volume form

$$\mathrm{vol} := dx_1 \wedge dx_2 \wedge dx_3 \in \Omega^3(U).$$

Define the Hodge star operator $* : \Omega^k(U) \to \Omega^{3-k}(U)$ by specifying its values on the standard basis:

$$\begin{aligned}
&*1 = \mathrm{vol}, \\
&*\, dx_1 = dx_2 \wedge dx_3, \quad *dx_2 = dx_3 \wedge dx_1, \quad *dx_3 = dx_1 \wedge dx_2, \\
&*\, (dx_2 \wedge dx_3) = dx_1, \quad *(dx_3 \wedge dx_1) = dx_2, \quad *(dx_1 \wedge dx_2) = dx_3, \\
&*\, \mathrm{vol} = 1,
\end{aligned}$$

and extending $\Bbbk$-linearly.

2.3.5 Transport of the de Rham differential to grad–curl–div

Definition 2.3.8 (The comparison isomorphisms Φ^k)**.** Define $\Bbbk$-linear isomorphisms

$$\Phi^0 : V^0 \xrightarrow{\cong} \Omega^0(U), \quad \Phi^0(f) := f,$$

$$\Phi^1 : V^1 \xrightarrow{\cong} \Omega^1(U), \quad \Phi^1(F) := F^\flat,$$

$$\Phi^2 : V^2 \xrightarrow{\cong} \Omega^2(U), \quad \Phi^2(G) := *(G^\flat),$$

$$\Phi^3 : V^3 \xrightarrow{\cong} \Omega^3(U), \quad \Phi^3(h) := h\, \mathrm{vol}.$$

Proposition 2.3.9 (Commutativity of the comparison diagram). *For all $f \in V^0$, $F \in V^1$, $G \in V^2$, one has*

$$\Phi^1(\nabla f) = d(\Phi^0(f)), \qquad \Phi^2(\nabla \times F) = d(\Phi^1(F)), \qquad \Phi^3(\nabla \cdot G) = d(\Phi^2(G)).$$

Equivalently, the diagram of cochain complexes commutes:

$$\begin{tikzcd}
0 \arrow[r] & V^0 \arrow[r, "\nabla"] \arrow[d, "\Phi^0"', "\cong"] & V^1 \arrow[r, "\nabla\times"] \arrow[d, "\Phi^1"', "\cong"] & V^2 \arrow[r, "\nabla\cdot"] \arrow[d, "\Phi^2"', "\cong"] & V^3 \arrow[r] \arrow[d, "\Phi^3"', "\cong"] & 0 \\
0 \arrow[r] & \Omega^0(U) \arrow[r, "d"] & \Omega^1(U) \arrow[r, "d"] & \Omega^2(U) \arrow[r, "d"] & \Omega^3(U) \arrow[r] & 0.
\end{tikzcd}$$

Proof. **Step 1: grad.** Let $f \in V^0 = C^\infty(U;\Bbbk)$. Then

$$d(\Phi^0(f)) = d(f) = \partial_1 f\, dx_1 + \partial_2 f\, dx_2 + \partial_3 f\, dx_3 = (\nabla f)^\flat = \Phi^1(\nabla f).$$

Step 2: curl. Let $F = (P, Q, R) \in V^1$. Then $\Phi^1(F) = F^\flat = P\, dx_1 + Q\, dx_2 + R\, dx_3$, hence

$$\begin{aligned}
d(\Phi^1(F)) &= d(P) \wedge dx_1 + d(Q) \wedge dx_2 + d(R) \wedge dx_3 \\
&= (\partial_1 P\, dx_1 + \partial_2 P\, dx_2 + \partial_3 P\, dx_3) \wedge dx_1 \\
&\qquad + (\partial_1 Q\, dx_1 + \partial_2 Q\, dx_2 + \partial_3 Q\, dx_3) \wedge dx_2 \\
&\qquad + (\partial_1 R\, dx_1 + \partial_2 R\, dx_2 + \partial_3 R\, dx_3) \wedge dx_3.
\end{aligned}$$

Using $dx_i \wedge dx_i = 0$ and $dx_j \wedge dx_i = -dx_i \wedge dx_j$, this simplifies to

$$\begin{aligned}
d(\Phi^1(F)) &= (\partial_2 P)\, dx_2 \wedge dx_1 + (\partial_3 P)\, dx_3 \wedge dx_1 \\
&\quad + (\partial_1 Q)\, dx_1 \wedge dx_2 + (\partial_3 Q)\, dx_3 \wedge dx_2 \\
&\quad + (\partial_1 R)\, dx_1 \wedge dx_3 + (\partial_2 R)\, dx_2 \wedge dx_3 \\
&= (\partial_2 R - \partial_3 Q)\, dx_2 \wedge dx_3 + (\partial_3 P - \partial_1 R)\, dx_3 \wedge dx_1 + (\partial_1 Q - \partial_2 P)\, dx_1 \wedge dx_2.
\end{aligned}$$

On the other hand,

$$\nabla \times F = (\partial_2 R - \partial_3 Q,\ \partial_3 P - \partial_1 R,\ \partial_1 Q - \partial_2 P),$$

so

$$\begin{aligned}
\Phi^2(\nabla \times F) &= *((\nabla \times F)^\flat) \\
&= *(\, (\partial_2 R - \partial_3 Q)\, dx_1 + (\partial_3 P - \partial_1 R)\, dx_2 + (\partial_1 Q - \partial_2 P)\, dx_3 \,) \\
&= (\partial_2 R - \partial_3 Q)\, dx_2 \wedge dx_3 + (\partial_3 P - \partial_1 R)\, dx_3 \wedge dx_1 + (\partial_1 Q - \partial_2 P)\, dx_1 \wedge dx_2.
\end{aligned}$$

Comparing, $d(\Phi^1(F)) = \Phi^2(\nabla \times F)$.

Step 3: div. Let $G = (A, B, C) \in V^2$. Then

$$\Phi^2(G) = *(G^\flat) = *(A\, dx_1 + B\, dx_2 + C\, dx_3) = A\, dx_2 \wedge dx_3 + B\, dx_3 \wedge dx_1 + C\, dx_1 \wedge dx_2.$$

Therefore

$$\begin{aligned}
d(\Phi^2(G)) &= d(A) \wedge dx_2 \wedge dx_3 + d(B) \wedge dx_3 \wedge dx_1 + d(C) \wedge dx_1 \wedge dx_2 \\
&= (\partial_1 A\, dx_1 + \partial_2 A\, dx_2 + \partial_3 A\, dx_3) \wedge dx_2 \wedge dx_3 \\
&\quad + (\partial_1 B\, dx_1 + \partial_2 B\, dx_2 + \partial_3 B\, dx_3) \wedge dx_3 \wedge dx_1 \\
&\quad + (\partial_1 C\, dx_1 + \partial_2 C\, dx_2 + \partial_3 C\, dx_3) \wedge dx_1 \wedge dx_2 \\
&= (\partial_1 A)\, dx_1 \wedge dx_2 \wedge dx_3 + (\partial_2 B)\, dx_2 \wedge dx_3 \wedge dx_1 + (\partial_3 C)\, dx_3 \wedge dx_1 \wedge dx_2 \\
&= (\partial_1 A + \partial_2 B + \partial_3 C)\, dx_1 \wedge dx_2 \wedge dx_3 \\
&= (\nabla \cdot G)\, \mathrm{vol} = \Phi^3(\nabla \cdot G).
\end{aligned}$$

This completes the proof. □

Corollary 2.3.10 (Cohomology identification)**.** *The maps Φ^k induce isomorphisms on cohomology:*

$$H^k(V^\bullet) \cong H^k_{\mathrm{dR}}(U) \qquad (k = 0, 1, 2, 3).$$

Remark 2.3.11 (Topology and "potential" obstructions)**.** Under the identification above, $H^1(V^\bullet)$ measures curl-free fields modulo gradients, and $H^2(V^\bullet)$ measures divergence-free fields modulo curls. If U is contractible (e.g. star-shaped), then $H^k_{\mathrm{dR}}(U) = 0$ for $k \geq 1$, hence $H^1(V^\bullet) = H^2(V^\bullet) = H^3(V^\bullet) = 0$.

2.4 Cochain complexes and grad–curl–div as de Rham cohomology in $\mathbb{R}^3$

2.4.1 Formal construction of differential forms on an open set of $\mathbb{R}^3$

Definition 2.4.1 (Coordinate ring of smooth functions)**.** Let $U \subseteq \mathbb{R}^3$ be open. For a field $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$ define

$$\Omega^0(U) := C^\infty(U; \Bbbk),$$

viewed as a commutative unital $\Bbbk$-algebra under pointwise operations.

Definition 2.4.2 (The $\Bbbk$-vector spaces $\Omega^1(U), \Omega^2(U), \Omega^3(U)$)**.** Let (x_1, x_2, x_3) be the standard coordinate functions on U. Define $\Omega^1(U)$ to be the free $\Omega^0(U)$-module with basis $\{dx_1, dx_2, dx_3\}$, i.e.

$$\Omega^1(U) := \Omega^0(U)\, dx_1 \oplus \Omega^0(U)\, dx_2 \oplus \Omega^0(U)\, dx_3.$$

Define $\Omega^2(U)$ to be the free $\Omega^0(U)$-module with basis $\{dx_1 \wedge dx_2,\ dx_2 \wedge dx_3,\ dx_3 \wedge dx_1\}$, i.e.

$$\Omega^2(U) := \Omega^0(U)\, (dx_1 \wedge dx_2) \oplus \Omega^0(U)\, (dx_2 \wedge dx_3) \oplus \Omega^0(U)\, (dx_3 \wedge dx_1).$$

Define $\Omega^3(U)$ to be the free $\Omega^0(U)$-module of rank 1 with basis

$$\mathrm{vol} := dx_1 \wedge dx_2 \wedge dx_3, \qquad \Omega^3(U) := \Omega^0(U)\, \mathrm{vol}.$$

Definition 2.4.3 (Wedge product on coordinate forms)**.** Define a $\Bbbk$-bilinear map

$$\wedge : \Omega^p(U) \times \Omega^q(U) \to \Omega^{p+q}(U)$$

by imposing the following axioms:

1. $\wedge$ is $\Omega^0(U)$-bilinear in the sense that for $f \in \Omega^0(U)$ and forms α, β

$$(f\alpha) \wedge \beta = f(\alpha \wedge \beta), \qquad \alpha \wedge (f\beta) = f(\alpha \wedge \beta);$$

2. $\wedge$ is associative;

3. on basis elements it is alternating:

$$dx_i \wedge dx_i = 0, \qquad dx_i \wedge dx_j = -\,dx_j \wedge dx_i \quad (i \neq j);$$

4. $1 \in \Omega^0(U)$ acts as a unit: $1 \wedge \alpha = \alpha = \alpha \wedge 1$ for all α.

Remark 2.4.4 (Coordinate expansions)**.** Every $\alpha \in \Omega^1(U)$ has a unique expression

$$\alpha = a_1\, dx_1 + a_2\, dx_2 + a_3\, dx_3 \qquad (a_i \in \Omega^0(U)),$$

every $\beta \in \Omega^2(U)$ has a unique expression

$$\beta = b_{12}\, dx_1 \wedge dx_2 + b_{23}\, dx_2 \wedge dx_3 + b_{31}\, dx_3 \wedge dx_1 \qquad (b_{ij} \in \Omega^0(U)),$$

and every $\gamma \in \Omega^3(U)$ has a unique expression $\gamma = c\, \mathrm{vol}$ with $c \in \Omega^0(U)$.

2.4.2 Formal definition of the exterior derivative

Definition 2.4.5 (Exterior derivative in coordinates)**.** Define $\Bbbk$-linear maps

$$d : \Omega^k(U) \to \Omega^{k+1}(U) \qquad (k = 0, 1, 2)$$

by the following coordinate rules.

1. If $f \in \Omega^0(U)$, define
$$df := \partial_1 f\, dx_1 + \partial_2 f\, dx_2 + \partial_3 f\, dx_3 \in \Omega^1(U).$$
2. If $\alpha = a_1 dx_1 + a_2 dx_2 + a_3 dx_3 \in \Omega^1(U)$, define
$$\begin{aligned} d\alpha &:= da_1 \wedge dx_1 + da_2 \wedge dx_2 + da_3 \wedge dx_3 \\ &= (\partial_2 a_1 - \partial_1 a_2)\, dx_1 \wedge dx_2 + (\partial_3 a_2 - \partial_2 a_3)\, dx_2 \wedge dx_3 + (\partial_1 a_3 - \partial_3 a_1)\, dx_3 \wedge dx_1 \in \Omega^2(U). \end{aligned}$$
3. If $\beta = b_{12} dx_1 \wedge dx_2 + b_{23} dx_2 \wedge dx_3 + b_{31} dx_3 \wedge dx_1 \in \Omega^2(U)$, define
$$\begin{aligned} d\beta &:= db_{12} \wedge dx_1 \wedge dx_2 + db_{23} \wedge dx_2 \wedge dx_3 + db_{31} \wedge dx_3 \wedge dx_1 \\ &= (\partial_3 b_{12} + \partial_1 b_{23} + \partial_2 b_{31})\, dx_1 \wedge dx_2 \wedge dx_3 \in \Omega^3(U). \end{aligned}$$

Finally define $d : \Omega^3(U) \to 0$ to be the zero map.

Proposition 2.4.6 (Graded Leibniz rule)**.** *For all $p, q \geq 0$ with $p + q \leq 3$, and all $\alpha \in \Omega^p(U)$, $\beta \in \Omega^q(U)$, one has*
$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

Proposition 2.4.7 ($d^2 = 0$)**.** *The maps $d : \Omega^k(U) \to \Omega^{k+1}(U)$ satisfy $d \circ d = 0$, i.e.*
$$d^2 = 0 : \Omega^k(U) \to \Omega^{k+2}(U) \qquad \text{for } k = 0, 1, 2.$$

Proof. It suffices to check $d(df) = 0$ for $f \in \Omega^0(U)$ and $d(d\alpha) = 0$ for $\alpha \in \Omega^1(U)$. The coordinate formulas show each coefficient is a sum of mixed second derivatives which cancel by $\partial_i \partial_j = \partial_j \partial_i$. □

Definition 2.4.8 (de Rham cochain complex and cohomology)**.** The sequence

$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \Omega^3(U) \to 0$$

is a cochain complex. Its cohomology vector spaces are

$$H^k_{\mathrm{dR}}(U) := \mathrm{Ker}\big(d : \Omega^k(U) \to \Omega^{k+1}(U)\big) \Big/ \mathrm{im}\big(d : \Omega^{k-1}(U) \to \Omega^k(U)\big).$$

2.4.3 grad, curl, div as transported differentials

Definition 2.4.9 (Vector-field spaces and vector-calculus differentials)**.** Let $V^0 := C^\infty(U; \Bbbk)$, $V^1 := C^\infty(U; \Bbbk^3)$, $V^2 := C^\infty(U; \Bbbk^3)$, $V^3 := C^\infty(U; \Bbbk)$. Define

$$\nabla : V^0 \to V^1, \qquad \nabla\times : V^1 \to V^2, \qquad \nabla\cdot : V^2 \to V^3$$

by

$$\begin{aligned} \nabla f &:= (\partial_1 f, \partial_2 f, \partial_3 f), \\ \nabla \times (P, Q, R) &:= (\partial_2 R - \partial_3 Q,\ \partial_3 P - \partial_1 R,\ \partial_1 Q - \partial_2 P), \\ \nabla \cdot (A, B, C) &:= \partial_1 A + \partial_2 B + \partial_3 C. \end{aligned}$$

Definition 2.4.10 (Explicit identifications Φ^k)**.** Equip U with the Euclidean metric and standard orientation. Define linear isomorphisms

$$\Phi^0 : V^0 \xrightarrow{\cong} \Omega^0(U), \quad \Phi^0(f) = f,$$

$$\Phi^1 : V^1 \xrightarrow{\cong} \Omega^1(U), \quad \Phi^1(P,Q,R) = P\,dx_1 + Q\,dx_2 + R\,dx_3,$$

$$\Phi^2 : V^2 \xrightarrow{\cong} \Omega^2(U), \quad \Phi^2(A,B,C) = A\,dx_2 \wedge dx_3 + B\,dx_3 \wedge dx_1 + C\,dx_1 \wedge dx_2,$$

$$\Phi^3 : V^3 \xrightarrow{\cong} \Omega^3(U), \quad \Phi^3(h) = h\,dx_1 \wedge dx_2 \wedge dx_3.$$

Proposition 2.4.11 (Diagram commutativity: explicit proof)**.** *For all $f \in V^0$, $F \in V^1$, $G \in V^2$,*

$$\Phi^1(\nabla f) = d(\Phi^0(f)), \qquad \Phi^2(\nabla \times F) = d(\Phi^1(F)), \qquad \Phi^3(\nabla \cdot G) = d(\Phi^2(G)).$$

Equivalently, the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & V^0 \arrow[r, "\nabla"] \arrow[d, "\Phi^0"', "\cong"] & V^1 \arrow[r, "\nabla\times"] \arrow[d, "\Phi^1"', "\cong"] & V^2 \arrow[r, "\nabla\cdot"] \arrow[d, "\Phi^2"', "\cong"] & V^3 \arrow[r] \arrow[d, "\Phi^3"', "\cong"] & 0 \\
0 \arrow[r] & \Omega^0(U) \arrow[r, "d"] & \Omega^1(U) \arrow[r, "d"] & \Omega^2(U) \arrow[r, "d"] & \Omega^3(U) \arrow[r] & 0.
\end{tikzcd}$$

Proof. **(i) grad.** For $f \in V^0$,

$$d(\Phi^0(f)) = df = \partial_1 f\,dx_1 + \partial_2 f\,dx_2 + \partial_3 f\,dx_3 = \Phi^1(\nabla f).$$

(ii) curl. Let $F = (P,Q,R) \in V^1$. Then $\Phi^1(F) = P dx_1 + Q dx_2 + R dx_3$. Hence

$$\begin{aligned}
d(\Phi^1(F)) &= d(P) \wedge dx_1 + d(Q) \wedge dx_2 + d(R) \wedge dx_3 \\
&= (\partial_2 R - \partial_3 Q)\,dx_2 \wedge dx_3 + (\partial_3 P - \partial_1 R)\,dx_3 \wedge dx_1 + (\partial_1 Q - \partial_2 P)\,dx_1 \wedge dx_2.
\end{aligned}$$

By definition,

$$\nabla \times F = (\partial_2 R - \partial_3 Q,\ \partial_3 P - \partial_1 R,\ \partial_1 Q - \partial_2 P),$$

so

$$\Phi^2(\nabla \times F) = (\partial_2 R - \partial_3 Q)\,dx_2 \wedge dx_3 + (\partial_3 P - \partial_1 R)\,dx_3 \wedge dx_1 + (\partial_1 Q - \partial_2 P)\,dx_1 \wedge dx_2.$$

Thus $d(\Phi^1(F)) = \Phi^2(\nabla \times F)$.

(iii) div. Let $G = (A,B,C) \in V^2$. Then

$$\Phi^2(G) = A\,dx_2 \wedge dx_3 + B\,dx_3 \wedge dx_1 + C\,dx_1 \wedge dx_2.$$

Hence

$$\begin{aligned}
d(\Phi^2(G)) &= d(A) \wedge dx_2 \wedge dx_3 + d(B) \wedge dx_3 \wedge dx_1 + d(C) \wedge dx_1 \wedge dx_2 \\
&= (\partial_1 A + \partial_2 B + \partial_3 C)\,dx_1 \wedge dx_2 \wedge dx_3 = \Phi^3(\nabla \cdot G).
\end{aligned}$$

$\square$

2.4.4 Cohomology on the vector-calculus side

Definition 2.4.12 (Cohomology of grad–curl–div)**.** Define $d^0 := \nabla$, $d^1 := \nabla\times$, $d^2 := \nabla\cdot$, and extend by $d^{-1} = 0$, $d^3 = 0$. Define

$$Z^n := \operatorname{Ker}(d^n), \qquad B^n := \operatorname{im}(d^{n-1}), \qquad H^n(V^\bullet) := Z^n/B^n \quad (n = 0, 1, 2, 3).$$

Equivalently,

$$\begin{aligned} H^0(V^\bullet) &= \operatorname{Ker}(\nabla), \\ H^1(V^\bullet) &= \operatorname{Ker}(\nabla\times)\big/\operatorname{im}(\nabla), \\ H^2(V^\bullet) &= \operatorname{Ker}(\nabla\cdot)\big/\operatorname{im}(\nabla\times), \\ H^3(V^\bullet) &= C^\infty(U;\mathbb{k})\big/\operatorname{im}(\nabla\cdot). \end{aligned}$$

Corollary 2.4.13 (Identification with de Rham cohomology)**.** *For each $k \in \{0, 1, 2, 3\}$, the isomorphisms Φ^k induce canonical isomorphisms*

$$H^k(V^\bullet) \cong H^k_{\mathrm{dR}}(U).$$

Remark 2.4.14 (Contractible domains)**.** If U is contractible, then $H^k_{\mathrm{dR}}(U) = 0$ for $k \geq 1$ and (if U is connected) $H^0_{\mathrm{dR}}(U) \cong \mathbb{k}$. Consequently $H^1(V^\bullet) = H^2(V^\bullet) = H^3(V^\bullet) = 0$ and $H^0(V^\bullet) \cong \mathbb{k}$.

2.5 Cochain complexes of vector spaces

Definition 2.5.1 (Graded vector space)**.** Fix a field $\Bbbk$. A **$\mathbb{Z}$-graded $\Bbbk$-vector space** is a family $V^\bullet = \{V^n\}_{n\in\mathbb{Z}}$ of $\Bbbk$-vector spaces.

Definition 2.5.2 (Cochain complex)**.** A **cochain complex of $\Bbbk$-vector spaces** is a pair $(V^\bullet, d)$ where

1. $V^\bullet = \{V^n\}_{n\in\mathbb{Z}}$ is a $\mathbb{Z}$-graded $\Bbbk$-vector space;
2. $d = \{d^n\}_{n\in\mathbb{Z}}$ is a family of $\Bbbk$-linear maps
$$d^n : V^n \longrightarrow V^{n+1} \qquad (n \in \mathbb{Z})$$
such that
$$d^{n+1} \circ d^n = 0 \qquad \text{for all } n \in \mathbb{Z}.$$

In diagrammatic form, one writes

$$\cdots \xrightarrow{d^{n-2}} V^{n-1} \xrightarrow{d^{n-1}} V^n \xrightarrow{d^n} V^{n+1} \xrightarrow{d^{n+1}} \cdots, \qquad d^n d^{n-1} = 0.$$

Remark 2.5.3 (The condition $d^{n+1}d^n = 0$)**.** For each $n \in \mathbb{Z}$ the equality $d^{n+1} \circ d^n = 0$ is an equality of $\Bbbk$-linear maps $V^n \to V^{n+2}$. Equivalently,

$$\forall v \in V^n, \quad d^{n+1}\big(d^n(v)\big) = 0.$$

2.6 Cocycles, coboundaries, and cohomology

Definition 2.6.1 (Cocycles and coboundaries)**.** Let $(V^\bullet, d)$ be a cochain complex. For each $n \in \mathbb{Z}$ define the subspaces

$$Z^n(V^\bullet) := \ker(d^n) \subseteq V^n, \qquad B^n(V^\bullet) := \operatorname{im}(d^{n-1}) \subseteq V^n.$$

Elements of $Z^n(V^\bullet)$ are called **n-cocycles**, and elements of $B^n(V^\bullet)$ are called **n-coboundaries**.

Lemma 2.6.2 (Coboundaries are cocycles)**.** *For every $n \in \mathbb{Z}$ one has $B^n(V^\bullet) \subseteq Z^n(V^\bullet)$.*

Proof. Let $x \in B^n(V^\bullet)$. By definition, $\exists\, y \in V^{n-1}$ such that $x = d^{n-1}(y)$. Then

$$d^n(x) = d^n\big(d^{n-1}(y)\big) = \big(d^n \circ d^{n-1}\big)(y) = 0,$$

hence $x \in \ker(d^n) = Z^n(V^\bullet)$. □

Definition 2.6.3 (Cohomology)**.** Let $(V^\bullet, d)$ be a cochain complex. For each $n \in \mathbb{Z}$ the **n-th cohomology vector space** is the quotient

$$H^n(V^\bullet) := Z^n(V^\bullet)\big/B^n(V^\bullet) = \ker(d^n)\big/\operatorname{im}(d^{n-1}).$$

Remark 2.6.4 (Cohomology classes and equivalence relation)**.** Fix $n \in \mathbb{Z}$. Define a binary relation $\sim$ on $Z^n(V^\bullet)$ by

$$z \sim z' \iff z - z' \in B^n(V^\bullet).$$

Then $\sim$ is an equivalence relation on $Z^n(V^\bullet)$ (reflexive, symmetric, transitive), and the quotient set $Z^n(V^\bullet)/\!\sim$ inherits a unique $\Bbbk$-vector space structure for which the canonical projection $Z^n(V^\bullet) \to Z^n(V^\bullet)/\!\sim$ is $\Bbbk$-linear. Under this identification one has

$$Z^n(V^\bullet)/\!\sim\; \cong\; Z^n(V^\bullet)\big/B^n(V^\bullet) = H^n(V^\bullet).$$

2.7 Functoriality

Definition 2.7.1 (Morphism of cochain complexes)**.** Let $(V^\bullet, d_V)$ and $(W^\bullet, d_W)$ be cochain complexes of $\Bbbk$-vector spaces. A **morphism of cochain complexes** (or **cochain map**) $f : (V^\bullet, d_V) \to (W^\bullet, d_W)$ is a family of $\Bbbk$-linear maps

$$f^n : V^n \to W^n \qquad (n \in \mathbb{Z})$$

such that

$$d_W^n \circ f^n = f^{n+1} \circ d_V^n \qquad \text{for all } n \in \mathbb{Z}.$$

Proposition 2.7.2 (Induced map on cohomology)**.** *Let $f : (V^\bullet, d_V) \to (W^\bullet, d_W)$ be a cochain map. For each $n \in \mathbb{Z}$ there exists a unique $\Bbbk$-linear map*

$$H^n(f) : H^n(V^\bullet) \to H^n(W^\bullet)$$

such that for every $z \in Z^n(V^\bullet)$ one has

$$H^n(f)\big([z]\big) = [f^n(z)].$$

Proof. First, if $z \in Z^n(V^\bullet)$ then

$$d_W^n\big(f^n(z)\big) = (d_W^n \circ f^n)(z) = (f^{n+1} \circ d_V^n)(z) = f^{n+1}(0) = 0,$$

so $f^n(z) \in Z^n(W^\bullet)$ and $[f^n(z)]$ is defined.

To check well-definedness on cohomology classes: if $[z] = [z']$ in $H^n(V^\bullet)$ then $z - z' \in B^n(V^\bullet)$, so $\exists\, y \in V^{n-1}$ with $z - z' = d_V^{n-1}(y)$. Hence

$$f^n(z) - f^n(z') = f^n(z - z') = f^n\big(d_V^{n-1}(y)\big) = (f^n \circ d_V^{n-1})(y) = (d_W^{n-1} \circ f^{n-1})(y) \in \operatorname{im}(d_W^{n-1}) = B^n(W^\bullet).$$

Thus $[f^n(z)] = [f^n(z')]$ in $H^n(W^\bullet)$, so the formula defines a function $H^n(f)$.

Linearity follows because the quotient map $Z^n(V^\bullet) \to H^n(V^\bullet)$ is linear and f^n is linear. Uniqueness holds because every class in $H^n(V^\bullet)$ has a cocycle representative. □

2.8 Finite-dimensional dimension formula

Proposition 2.8.1 (Dimension identity)**.** *Assume each V^n is finite-dimensional. Then for all $n \in \mathbb{Z}$,*

$$\dim_\Bbbk H^n(V^\bullet) = \dim_\Bbbk \ker(d^n) - \dim_\Bbbk \operatorname{im}(d^{n-1}) = \operatorname{nullity}(d^n) - \operatorname{rank}(d^{n-1}).$$

Proof. Since $B^n(V^\bullet) \subseteq Z^n(V^\bullet)$, the quotient $H^n(V^\bullet) = Z^n/B^n$ is a vector space and

$$\dim_\Bbbk H^n(V^\bullet) = \dim_\Bbbk Z^n(V^\bullet) - \dim_\Bbbk B^n(V^\bullet).$$

By definition $Z^n = \ker(d^n)$ and $B^n = \operatorname{im}(d^{n-1})$, giving the stated formula. □

Example 2.8.2 (Two-step complex)**.** Let V^0, V^1, V^2 be $\Bbbk$-vector spaces and let $d^0 : V^0 \to V^1$, $d^1 : V^1 \to V^2$ be linear maps satisfying $d^1 d^0 = 0$. Extend by $V^n = 0$ for $n \notin \{0,1,2\}$ and $d^n = 0$ otherwise. Then

$$H^0 \cong \ker(d^0), \qquad H^1 \cong \ker(d^1)/\operatorname{im}(d^0), \qquad H^2 \cong V^2/\operatorname{im}(d^1),$$

and $H^n = 0$ for $n \notin \{0,1,2\}$.

Definition 2.8.3 (Graded object)**.** Let $\mathcal{A}$ be an abelian category. A **$\mathbb{Z}$-graded object** of $\mathcal{A}$ is a family $A^\bullet = \{A^k\}_{k\in\mathbb{Z}}$ of objects of $\mathcal{A}$.

Definition 2.8.4 (Cochain complex)**.** A **cochain complex** in $\mathcal{A}$ is a pair $(A^\bullet, d)$ consisting of a $\mathbb{Z}$-graded object $A^\bullet$ and morphisms

$$d^k : A^k \longrightarrow A^{k+1} \qquad (k \in \mathbb{Z})$$

such that

$$d^{k+1} \circ d^k = 0 \qquad \text{for all } k \in \mathbb{Z}.$$

We write the complex as

$$\cdots \xrightarrow{d^{k-2}} A^{k-1} \xrightarrow{d^{k-1}} A^k \xrightarrow{d^k} A^{k+1} \xrightarrow{d^{k+1}} \cdots .$$

Definition 2.8.5 (Morphisms of cochain complexes)**.** Let $(A^\bullet, d_A)$ and $(B^\bullet, d_B)$ be cochain complexes in $\mathcal{A}$. A **morphism of complexes** (or **cochain map**) $f : (A^\bullet, d_A) \to (B^\bullet, d_B)$ is a family of morphisms

$$f^k : A^k \to B^k \qquad (k \in \mathbb{Z})$$

such that

$$d_B^k \circ f^k = f^{k+1} \circ d_A^k \qquad \text{for all } k \in \mathbb{Z}.$$

2.9 Cocycles, coboundaries, and cohomology

Definition 2.9.1 (Cocycles and coboundaries)**.** Let $(A^\bullet, d)$ be a cochain complex in an abelian category $\mathcal{A}$. Define

$$Z^k(A^\bullet) := \ker(d^k) \subseteq A^k, \qquad B^k(A^\bullet) := \operatorname{im}(d^{k-1}) \subseteq A^k.$$

Lemma 2.9.2 (Boundaries are cycles)**.** *For every $k \in \mathbb{Z}$ one has $B^k(A^\bullet) \subseteq Z^k(A^\bullet)$.*

Proof. Let $x \in B^k(A^\bullet)$. Then $x = d^{k-1}(y)$ for some $y \in A^{k-1}$, hence

$$d^k(x) = d^k(d^{k-1}(y)) = (d^k \circ d^{k-1})(y) = 0$$

by the defining condition $d^k \circ d^{k-1} = 0$. Therefore $x \in \ker(d^k) = Z^k(A^\bullet)$. □

Definition 2.9.3 (Cohomology)**.** The **k-th cohomology object** of $(A^\bullet, d)$ is

$$H^k(A^\bullet) := Z^k(A^\bullet)/B^k(A^\bullet) = \ker(d^k)/\operatorname{im}(d^{k-1}).$$

Remark 2.9.4 (Cohomology classes)**.** If $\mathcal{A} = \mathsf{Ab}$ (or R-Mod), elements of $H^k(A^\bullet)$ are classes $[\alpha]$ with $\alpha \in Z^k(A^\bullet)$, and $[\alpha] = [\alpha']$ iff $\alpha - \alpha' \in B^k(A^\bullet)$, i.e. iff $\alpha - \alpha' = d^{k-1}\beta$ for some $\beta \in A^{k-1}$.

2.10 Exactness

Definition 2.10.1 (Exactness)**.** A cochain complex $(A^\bullet, d)$ is **exact at** A^k if

$$\operatorname{im}(d^{k-1}) = \ker(d^k).$$

It is **exact** if it is exact at every degree.

Proposition 2.10.2 (Exactness and vanishing cohomology)**.** *A cochain complex $(A^\bullet, d)$ is exact if and only if $H^k(A^\bullet) = 0$ for all $k \in \mathbb{Z}$.*

Proof. By definition,

$$H^k(A^\bullet) = 0 \iff \ker(d^k) = \operatorname{im}(d^{k-1}).$$

Thus vanishing of all cohomology objects is equivalent to exactness in every degree. □

2.11 Homotopy of cochain maps

Definition 2.11.1 (Cochain homotopy)**.** Let $f, g : (A^\bullet, d_A) \to (B^\bullet, d_B)$ be cochain maps. A **cochain homotopy** from f to g is a family of morphisms

$$h^k : A^k \longrightarrow B^{k-1} \qquad (k \in \mathbb{Z})$$

such that

$$f^k - g^k = d_B^{k-1} \circ h^k + h^{k+1} \circ d_A^k \qquad \text{for all } k \in \mathbb{Z}.$$

We write $f \simeq g$ if there exists such a homotopy.

Proposition 2.11.2 (Homotopic maps induce the same map on cohomology)**.** *If $f \simeq g$, then $H^k(f) = H^k(g)$ for all $k \in \mathbb{Z}$.*

Proof. Let $\alpha \in Z^k(A^\bullet)$, so $d_A^k(\alpha) = 0$. Then

$$(f^k - g^k)(\alpha) = d_B^{k-1}(h^k(\alpha)) + h^{k+1}(d_A^k(\alpha)) = d_B^{k-1}(h^k(\alpha)),$$

so $f^k(\alpha) - g^k(\alpha) \in \operatorname{im}(d_B^{k-1}) = B^k(B^\bullet)$. Hence $[f^k(\alpha)] = [g^k(\alpha)]$ in $H^k(B^\bullet)$. □

2.12 Mapping cone and long exact sequence

Definition 2.12.1 (Shift)**.** Given a complex $(A^\bullet, d_A)$, its **shift** $A[1]^\bullet$ is defined by

$$A[1]^k := A^{k+1}, \qquad d_{A[1]}^k := -d_A^{k+1}.$$

Definition 2.12.2 (Mapping cone)**.** Let $f : (A^\bullet, d_A) \to (B^\bullet, d_B)$ be a cochain map. The **mapping cone** $\operatorname{Cone}(f)$ is the complex with

$$\operatorname{Cone}(f)^k := B^k \oplus A^{k+1}$$

and differential

$$d_{\operatorname{Cone}(f)}^k(b, a) := \big(d_B^k(b) + f^{k+1}(a),\ -d_A^{k+1}(a)\big).$$

Lemma 2.12.3. *$\operatorname{Cone}(f)$ is a cochain complex, i.e. $d_{\operatorname{Cone}(f)}^{k+1} \circ d_{\operatorname{Cone}(f)}^k = 0$.*

Proof. A direct computation using $d_B \circ d_B = 0$, $d_A \circ d_A = 0$, and $d_B \circ f = f \circ d_A$. □

Proposition 2.12.4 (Short exact sequence of complexes)**.** *There is a natural short exact sequence of complexes*

$$0 \longrightarrow B^\bullet \xrightarrow{i} \operatorname{Cone}(f)^\bullet \xrightarrow{p} A[1]^\bullet \longrightarrow 0,$$

where $i(b) = (b, 0)$ and $p(b, a) = a$ in each degree.

Theorem 2.12.5 (Long exact sequence in cohomology)**.** *Let $0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$ be a short exact sequence of cochain complexes in an abelian category. Then there exist connecting morphisms $\delta^k : H^k(Z^\bullet) \to H^{k+1}(X^\bullet)$ such that*

$$\cdots \to H^k(X^\bullet) \to H^k(Y^\bullet) \to H^k(Z^\bullet) \xrightarrow{\delta^k} H^{k+1}(X^\bullet) \to H^{k+1}(Y^\bullet) \to \cdots$$

is exact.

Remark 2.12.6 (Explicit connecting morphism in Ab or R-Mod)**.** Suppose $0 \to X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \to 0$ is degreewise exact. Given $[z] \in H^k(Z^\bullet)$ with $z \in Z^k(Z^\bullet)$, choose $y \in Y^k$ with $v(y) = z$. Then $v(d_Y^k y) = d_Z^k(v(y)) = d_Z^k(z) = 0$, hence $d_Y^k y \in \ker(v) = \operatorname{im}(u)$. Choose $x \in X^{k+1}$ with $u(x) = d_Y^k y$ and set $\delta^k([z]) := [x] \in H^{k+1}(X^\bullet)$. One checks δ^k is well-defined and yields exactness.

2.13 Examples

Example 2.13.1 (de Rham complex)**.** For a smooth manifold M, the graded $\mathbb{R}$-vector space $\Omega^\bullet(M)$ with exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfies $d \circ d = 0$, hence forms a cochain complex. Its cohomology is

$$H^k_{\mathrm{dR}}(M) := \ker\bigl(d : \Omega^k(M) \to \Omega^{k+1}(M)\bigr) \Big/ \operatorname{im}\bigl(d : \Omega^{k-1}(M) \to \Omega^k(M)\bigr).$$

Example 2.13.2 (Singular cochains)**.** Let X be a topological space and $G \in \mathsf{Ab}$. Let $C_k(X)$ be the singular chain group and define $C^k(X;G) := \mathrm{Hom}(C_k(X), G)$. The coboundary $\delta : C^k(X;G) \to C^{k+1}(X;G)$ satisfies $\delta^2 = 0$. The cohomology $H^k(C^\bullet(X;G))$ is the singular cohomology $H^k(X;G)$.

Chapter 3

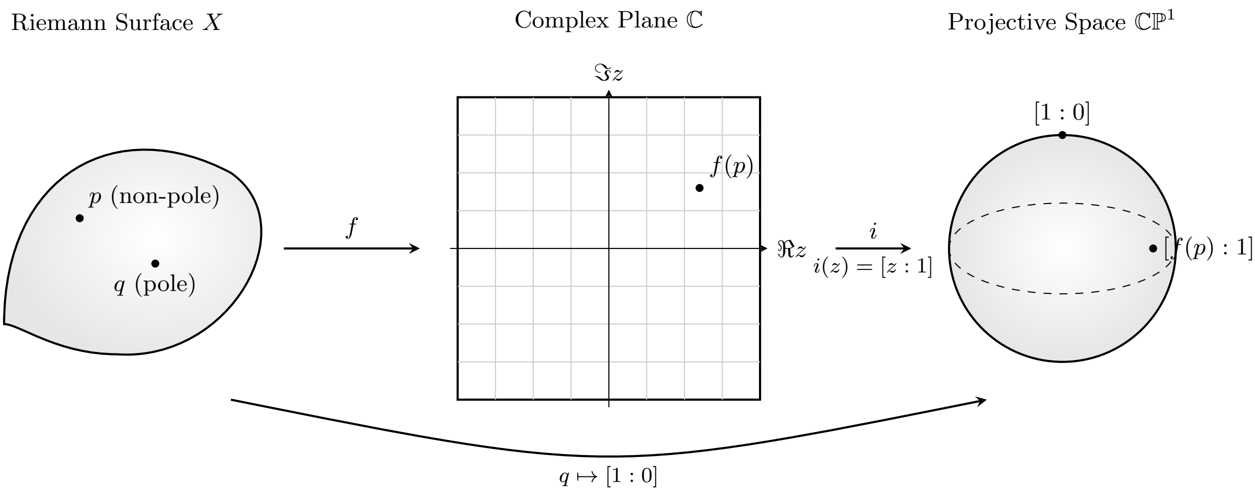
Elliptic Curve and Torus

3.1 Note 1: Meromorphic Function and Order

3.2 Note 2: Meromorphic $f \in \mathbb{C}^X$ and Holomorphic $F \in (\mathbb{CP}^1)^X$

Given a meromorphic $f : X \to \mathbb{C}$ on a Riemann surface X, we define

$$\begin{aligned} F \quad : \quad X &\longrightarrow \mathbb{C} \cup \{\infty\}\,(\simeq \mathbb{CP}^1) \\ p &\longmapsto F(p) = \begin{cases} [1 : f(p)] & \text{if } p \text{ is not a pole} \\ [0 : 1] & \text{if } p \text{ is a pole} \end{cases} \end{aligned}$$



In other word,

$$\begin{array}{ccccc} X & \xrightarrow{\quad f \quad} & \mathbb{C} & \xrightarrow{\qquad i \qquad} & \mathbb{CP}^1 \\ \\ p_{\text{non-pole}} & \longmapsto & f(p) & \longmapsto & [z_0 : z_1] = [1 : z_1/z_0] = [1 : f(p)] \\ \\ q_{\text{pole}} & & \longmapsto & & [0 : 1] = \infty \end{array}$$

3.2.1 Example 1: $X = \mathbb{CP}^1$ (Riemann sphere)

We view $\mathbb{CP}^1$ as the Riemann sphere. On the affine chart

$$U_1 = \{[z_0 : z_1] \in \mathbb{CP}^1 \mid z_1 \neq 0\},$$

we use the coordinate $z = z_0/z_1$. The point at infinity is $\infty = [1 : 0]$.

On $\mathbb{CP}^1$, a meromorphic function is the same as a rational function. Take for instance

$$f(z) = \frac{z^2 - 1}{z - 2}.$$

This is meromorphic on $\mathbb{CP}^1$, with a simple pole at $z = 2$, and (possibly) a pole at ∞.

Define

$$F : \mathbb{CP}^1 \to \mathbb{CP}^1, \qquad F(p) = \begin{cases} [f(p) : 1], & p \text{ not a pole of } f, \\ [1 : 0], & p \text{ a pole of } f. \end{cases}$$

Concretely, for $p = [z : 1]$ with $z \neq 2$,

$$F([z : 1]) = [f(z) : 1] = \left[\frac{z^2 - 1}{z - 2} : 1\right],$$

and at the pole $p = [2 : 1]$,

$$F([2 : 1]) = [1 : 0].$$

Similarly one checks the value at $\infty = [1 : 0]$ using the behavior of $f(z)$ as $|z| \to \infty$.

To see that F is holomorphic, we use the usual charts on $\mathbb{CP}^1$:

- **At a non-pole point p.** Suppose p is not a pole of f. Then f is holomorphic near p and finite there, so $F(p) = [f(p) : 1] \in U_1$. Let

  $$w = \frac{z_0}{z_1} : U_1 \to \mathbb{C}$$

  be the affine coordinate on U_1. In this chart,

  $$(w \circ F)(q) = \left.\frac{z_0}{z_1}\right|_{F(q)} = f(q),$$

  which is holomorphic in any local coordinate around p. Hence F is holomorphic at non-poles.

- **At a pole p.** Let p be a pole of order $m > 0$. Choose a local coordinate z on $\mathbb{CP}^1$ with $z(p) = 0$. Then

  $$f(z) = z^{-m} g(z), \qquad g \text{ holomorphic}, \ g(0) \neq 0.$$

  Here $F(p) = [1 : 0]$. Use the chart

  $$U_0 = \{[z_0 : z_1] \in \mathbb{CP}^1 \mid z_0 \neq 0\},$$

  with coordinate

  $$u = \frac{z_1}{z_0} : U_0 \to \mathbb{C}.$$

For $z \neq 0$ near p,

$$F(z) = [f(z) : 1] = [z^{-m} g(z) : 1].$$

Multiplying homogeneous coordinates by z^m (which does not change the point in projective space), we get

$$[z^{-m} g(z) : 1] = [g(z) : z^m].$$

Thus, in the chart U_0,

$$(u \circ F)(z) = \frac{z^m}{g(z)}.$$

Since $g(z)$ is holomorphic with $g(0) \neq 0$, the function $\dfrac{1}{g(z)}$ is holomorphic near 0, and hence

$$\frac{z^m}{g(z)}$$

is holomorphic near 0 (and vanishes to order m). Therefore F is holomorphic at the pole p.

Since we have holomorphicity in local charts at every point of $\mathbb{CP}^1$, $F : \mathbb{CP}^1 \to \mathbb{CP}^1$ is a holomorphic map.

3.2.2 Example 2: $X = \mathbb{C}/\Lambda$ (complex torus)

Let $\Lambda \subset \mathbb{C}$ be a lattice and consider the complex torus

$$X = \mathbb{C}/\Lambda.$$

The quotient map is

$$\pi : \mathbb{C} \to X, \qquad \pi(z) = [z].$$

A meromorphic function $f : X \to \mathbb{C}$ corresponds to a Λ-periodic meromorphic function $\tilde{f} : \mathbb{C} \to \mathbb{C}$ satisfying

$$\tilde{f}(z + \lambda) = \tilde{f}(z), \quad \forall \lambda \in \Lambda,$$

and

$$f([z]) = \tilde{f}(z).$$

A standard example is the Weierstrass $\wp$-function $\wp : \mathbb{C} \to \mathbb{C}$, which is Λ-periodic and meromorphic with double poles at lattice points. Thus it descends to a meromorphic

$$f : X \to \mathbb{C}, \qquad f([z]) = \wp(z).$$

We define

$$F : X \to \mathbb{CP}^1, \qquad F(p) = \begin{cases} [f(p) : 1], & p \text{ not a pole of } f, \\ [1 : 0], & p \text{ a pole of } f. \end{cases}$$

For our example $f([z]) = \wp(z)$:

- $\wp(z)$ has poles precisely at lattice points $z \in \Lambda$, which all represent the same point on the torus, usually denoted $[0]$.
- For $[z] \neq [0]$, we set $F([z]) = [\wp(z) : 1]$.
- At $[0]$, we set $F([0]) = [1 : 0]$.

Local coordinate on the torus near a pole

To get a local coordinate near $[0] \in X$, choose a small disc $D \subset \mathbb{C}$ around 0 such that $\pi|_D : D \to \pi(D)$ is a biholomorphism. Then

$$\varphi : \pi(D) \to \mathbb{C}, \qquad \varphi([z]) = z,$$

is a local coordinate on X near $[0]$.

The local behavior of $\wp(z)$ at $z = 0$ is

$$\wp(z) = \frac{1}{z^2} + \text{holomorphic terms},$$

so more precisely,

$$\wp(z) = z^{-2}g(z), \qquad g(z) \text{ holomorphic, } g(0) \neq 0.$$

Thus, for the induced f,

$$f([z]) = \wp(z) = z^{-2}g(z),$$

so f has a pole of order $m = 2$ at $[0]$.

Holomorphicity of F at the pole $[0]$

As before, we use the chart around $[1:0] \in \mathbb{CP}^1$:

$$U_0 = \{[z_0 : z_1] \mid z_0 \neq 0\}, \qquad u = \frac{z_1}{z_0} : U_0 \to \mathbb{C}.$$

For $z \neq 0$ small, we have $p = [z] \neq [0]$ and

$$F([z]) = [f([z]) : 1] = [\wp(z) : 1] = [z^{-2}g(z) : 1].$$

Multiplying the homogeneous coordinates by z^2 gives

$$[z^{-2}g(z) : 1] = [g(z) : z^2].$$

So in the chart U_0,

$$(u \circ F)([z]) = \frac{z^2}{g(z)}.$$

Since $g(z)$ is holomorphic with $g(0) \neq 0$, the function $\dfrac{1}{g(z)}$ is holomorphic near 0, and hence $\dfrac{z^2}{g(z)}$ is holomorphic near 0 and vanishes at $z = 0$. In the local coordinate $\varphi([z]) = z$ on X, the expression

$$u \circ F \circ \varphi^{-1}(z) = \frac{z^2}{g(z)}$$

is holomorphic, so F is holomorphic at the pole $[0]$.

At a non-pole point $[z_0] \in X$, the same argument as in Example 1 applies: f is holomorphic and finite, and in the affine chart

$$U_1 = \{[z_0 : z_1] \mid z_1 \neq 0\}, \qquad w = \frac{z_0}{z_1},$$

we have

$$(w \circ F)([z]) = f([z]) = \wp(z),$$

which is holomorphic in the local coordinate on X.

Conclusion

For both examples $X = \mathbb{CP}^1$ and $X = \mathbb{C}/\Lambda$, the construction

$$f : X \to \mathbb{C} \text{ meromorphic} \quad \longmapsto \quad F : X \to \mathbb{CP}^1, \quad F(p) = \begin{cases} [f(p) : 1], & p \text{ not a pole,} \\ [1 : 0], & p \text{ a pole,} \end{cases}$$

produces a holomorphic map $F : X \to \mathbb{CP}^1$. This concretely illustrates the general principle that a meromorphic function on a Riemann surface is the same as a holomorphic map to $\mathbb{CP}^1$.

We start with a meromorphic function

$$f : X \to \mathbb{C}$$

on a Riemann surface X, and define a map

$$F : X \to \mathbb{CP}^1$$

by

$$F(p) = \begin{cases} [f(p) : 1], & p \text{ not a pole of } f, \\ [1 : 0], & p \text{ a pole of } f. \end{cases}$$

You're asking: **why is this F holomorphic as a map of Riemann surfaces?**

1. Definition to remember

A map $F : X \to Y$ between Riemann surfaces is **holomorphic** if, for every point $p \in X$, you can choose local coordinates

- φ: neighborhood of $p \to \mathbb{C}$,
- ψ: neighborhood of $F(p) \to \mathbb{C}$,

such that the coordinate expression

$$\psi \circ F \circ \varphi^{-1} : (\text{open in } \mathbb{C}) \to \mathbb{C}$$

is an ordinary holomorphic function.

So we need to check this around:

1. a point where f is holomorphic (no pole),
2. a point where f has a pole.

2. Case 1: p is not a pole (easy)

If p is not a pole, then f is holomorphic near p and finite there.

- On X: choose any local coordinate z with $z(p) = 0$.
- On $\mathbb{CP}^1$: since $F(p) = [f(p) : 1]$ has second coordinate $\neq 0$, it lies in the chart

  $$U_1 = \{[z_0 : z_1] \mid z_1 \neq 0\}$$

  with coordinate

  $$w = \frac{z_0}{z_1} : U_1 \to \mathbb{C}.$$

Then on some neighborhood of p,

$$(w \circ F)(q) = \frac{z_0}{z_1}\Big|_{F(q)} = \frac{f(q)}{1} = f(q),$$

which is holomorphic in z.

So $\psi \circ F \circ \varphi^{-1} = f$ is holomorphic $\Rightarrow$ F is holomorphic at non-pole points.

3. Case 2: p is a pole of order $m > 0$

This is the interesting part.

Let p be a pole of f of order m. Choose a local coordinate z on X with $z(p) = 0$. By the definition of meromorphic:

$$f(z) = z^{-m} g(z),$$

where g is holomorphic and $g(0) \neq 0$.

By definition,

$$F(p) = [1 : 0] \in \mathbb{CP}^1.$$

Now we must look at a chart of $\mathbb{CP}^1$ that contains $[1 : 0]$. That is:

$$U_0 = \{[z_0 : z_1] \mid z_0 \neq 0\},$$

with coordinate

$$u = \frac{z_1}{z_0} : U_0 \to \mathbb{C},$$

and in this chart $[1 : 0]$ corresponds to $u = 0$.

For $z \neq 0$ near p,

$$F(z) = [f(z) : 1] = [z^{-m} g(z) : 1].$$

Multiply homogeneous coordinates by z^m (allowed in projective space):

$$[z^{-m} g(z) : 1] = [g(z) : z^m].$$

So in the chart U_0 we have:

$$u(F(z)) = \frac{z^m}{g(z)}.$$

Now, check holomorphicity:

- $g(z)$ is holomorphic with $g(0) \neq 0 \Rightarrow 1/g(z)$ is holomorphic near 0.
- z^m is holomorphic.
- The product $z^m \cdot \dfrac{1}{g(z)}$ is holomorphic near 0.

So

$$u \circ F(z) = \frac{z^m}{g(z)}$$

is an ordinary holomorphic function of z on a neighborhood of 0, and it extends to $z = 0$ with value 0.

Thus, in local coordinates,

$$\psi \circ F \circ \varphi^{-1} = u \circ F$$

is holomorphic at $z = 0$. Therefore, F **is holomorphic at the pole** p.

4. Conclusion

We have checked:

- At non-poles: in the chart U_1, $w \circ F = f$ is holomorphic.
- At poles: in the chart U_0, $u \circ F = z^m/g(z)$ is holomorphic.

So at **every** point $p \in X$, we can choose charts making the coordinate expression of F holomorphic. That's exactly the definition:

$$\boxed{F : X \to \mathbb{CP}^1 \text{ is holomorphic.}}$$

This is why we can safely say:

3.3 Note 3: The Isomorphism $\mathcal{M}(\mathbb{CP}^1) \simeq \mathbb{C}(x)$

We explain that the field of meromorphic functions on $\mathbb{CP}^1$ is isomorphic to the field $\mathbb{C}(x)$ of rational functions in one variable.

$$\mathcal{M}(X) = \left\{ \overline{i \circ f} \in (\mathbb{CP}^1)^X \,\middle|\, f \text{ meromorphic on } X \right\},$$

$$\mathcal{M}(X) = \{ F : X \to \mathbb{CP}^1 \mid F \text{ holomorphic} \}.$$

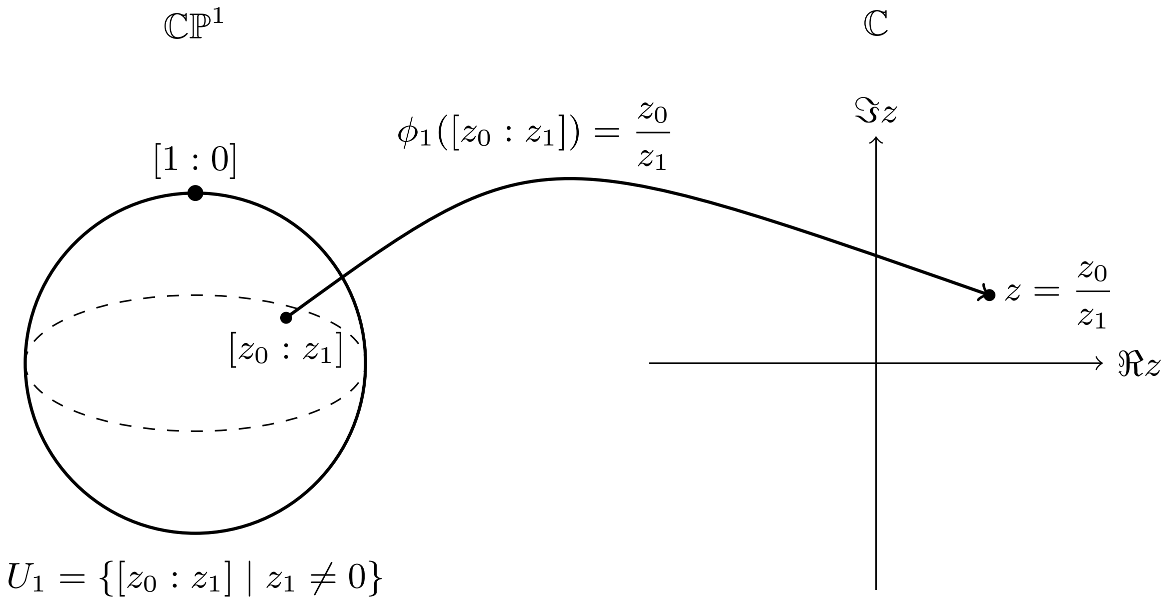
3.3.1 Charts on $\mathbb{CP}^1$ and Field of Meromorphic Functions

View $\mathbb{CP}^1$ as the Riemann sphere. Consider the standard affine chart

$$U_1 = \{[z_0 : z_1] \in \mathbb{CP}^1 \mid z_1 \neq 0\}$$

with coordinate map

$$\begin{array}{cccc} \phi_1 & : & U_1 & \longrightarrow & \mathbb{C} \\ & & [z_0 : z_1] & \longmapsto & \dfrac{z_0}{z_1} \end{array}.$$



We write

$$x := \phi_1,$$

and think of x as the **coordinate function** on U_1. This function extends meromorphically to all of $\mathbb{CP}^1$, with a simple pole at $\infty = [1 : 0]$.

We define the field of meromorphic functions on $\mathbb{CP}^1$ as

$$\mathcal{M}(\mathbb{CP}^1) = \{F : \mathbb{CP}^1 \to \mathbb{CP}^1 \mid F \text{ holomorphic}\},$$

viewing a meromorphic function as a holomorphic map into $\mathbb{CP}^1$ (via the usual convention "finite value $\mapsto [f(p) : 1]$, pole $\mapsto [1 : 0]$").

On the other hand, the field $\mathbb{C}(x)$ is

$$\mathbb{C}(x) = \left\{ \frac{p(x)}{q(x)} \,\middle|\, p, q \in \mathbb{C}[x],\ q \not\equiv 0 \right\} \Big/ \sim,$$

where $\frac{p}{q} \sim \frac{p'}{q'}$ if $p(x)q'(x) = p'(x)q(x)$.

Here ϕ_1 is a biholomorphism between U_1 and $\mathbb{C}$, its inverse is

$$\begin{array}{rccc} \phi_1^{-1} & : & \mathbb{C} & \longrightarrow & U_1 \\ & & z & \longmapsto & [z:1] \end{array}.$$

We'll write

$$x := \phi_1$$

and think of x as the **coordinate function** on U_1. It extends meromorphically to all of $\mathbb{CP}^1$ with a simple pole at $[1:0]$ (the point at infinity).

1. Describe both sides with ϕ_1

Side 1: $\mathcal{M}(\mathbb{CP}^1)$

We use the "holomorphic map to $\mathbb{CP}^1$" definition:

$$\mathcal{M}(\mathbb{CP}^1) = \left\{F : \mathbb{CP}^1 \to \mathbb{CP}^1 \,\middle|\, F \text{ holomorphic}\right\}.$$

We want to use ϕ_1, so whenever the image of F lies in U_1, we can look at

$$\phi_1 \circ F : \text{ (some open set)} \longrightarrow \mathbb{C}.$$

That's just the "affine coordinate" of the value of F.

Side 2: $\mathbb{C}(x)$

$$\mathbb{C}(x) = \left\{\frac{p(x)}{q(x)} \,\middle|\, p(x), q(x) \in \mathbb{C}[x],\ q(x) \not\equiv 0\right\} \Big/ \sim,$$

where $\frac{p}{q} \sim \frac{p'}{q'}$ iff $p(x)q'(x) = p'(x)q(x)$.

Here the symbol x is exactly your coordinate function

$$x = \phi_1 : U_1 \to \mathbb{C}.$$

2. Map $\mathbb{C}(x) \longrightarrow \mathcal{M}(\mathbb{CP}^1)$ using ϕ_1

Take a rational function

$$R(x) = \frac{p(x)}{q(x)} \in \mathbb{C}(x).$$

On the affine chart U_1:

Given a point $[z_0 : z_1] \in U_1$, write

$$x([z_0 : z_1]) = \phi_1([z_0 : z_1]) = z_0/z_1 =: z.$$

We **define** a map $F_R : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ by saying on U_1,

$$\phi_1\big(F_R([z_0 : z_1])\big) = R\big(\phi_1([z_0 : z_1])\big) = R(z).$$

In other words,

$$F_R|_{U_1} = \phi_1^{-1} \circ R \circ \phi_1.$$

Concretely:

$$F_R([z_0 : z_1]) = [R(z_0/z_1) : 1] \quad (\text{for } z_1 \neq 0,\ R(z) \neq \infty).$$

At points where $R(z) = \infty$ (i.e. $q(z) = 0$), we set

$$F_R([z_0 : z_1]) = [1 : 0].$$

This defines F_R on $U_1 \cup \{\infty\}$, but one must check it is **holomorphic at** ∞. Using homogeneous polynomials is a cleaner way:

- Let $\deg p \leq m,\ \deg q \leq m$. Define

  $$P(z_0, z_1) = z_1^m p(z_0/z_1), \quad Q(z_0, z_1) = z_1^m q(z_0/z_1),$$

  homogeneous of degree m.

- Then set

  $$F_R([z_0 : z_1]) = \begin{cases} [P(z_0, z_1) : Q(z_0, z_1)], & Q(z_0, z_1) \neq 0, \\ [1 : 0], & Q(z_0, z_1) = 0. \end{cases}$$

This is well-defined and holomorphic on all of $\mathbb{CP}^1$. In the chart U_1, this is exactly $\phi_1^{-1} \circ R \circ \phi_1$. So we get a map

$$\Phi : \mathbb{C}(x) \to \mathcal{M}(\mathbb{CP}^1), \quad R \mapsto F_R.$$

3. Use ϕ_1 to go backwards: from F to $R(x)$

Now take any

$$F \in \mathcal{M}(\mathbb{CP}^1), \quad F : \mathbb{CP}^1 \to \mathbb{CP}^1 \text{ holomorphic.}$$

We want to show: **there exists a unique rational function** $R(x) \in \mathbb{C}(x)$ **such that**

$$F = F_R.$$

Using ϕ_1:

1. Consider the open set where the image of F stays inside U_1:

   $$V := F^{-1}(U_1) \subset \mathbb{CP}^1.$$

2. On V, define

   $$f := \phi_1 \circ F : V \to \mathbb{C}.$$

   In local coordinates, f is holomorphic. So f is a holomorphic function on the Riemann surface V.

3. The complement $\mathbb{CP}^1 \setminus V = F^{-1}(\infty)$ is a **finite set** (preimages of the point $[1 : 0]$ under a holomorphic map from a compact Riemann surface). At those points, we'll see f has poles. So in the chart ϕ_1, f is a **meromorphic function on** $\mathbb{C}$ with finitely many poles.

Now, via ϕ_1, we can identify $\mathbb{CP}^1 \setminus \{\infty\}$ with $\mathbb{C}$. Under this, F becomes a meromorphic function

$$\tilde{f} : \mathbb{C} \to \mathbb{C} \cup \{\infty\},$$

which has only finitely many poles (coming from $F^{-1}(\infty)$) and maybe a pole at ∞.

From standard complex analysis:

> A meromorphic function on $\mathbb{CP}^1$ (i.e. on $\mathbb{C} \cup \{\infty\}$) is **rational**.

Concretely, we do the principal-part argument **in the coordinate** ϕ_1:

- In the x-coordinate (i.e. using ϕ_1 as your chart), $f(x)$ has Laurent expansions at each finite pole $x = a_j$.
- You build a rational function $R(x)$ whose principal parts match those of f at all finite poles and at ∞.
- Then $f(x) - R(x)$ is entire and holomorphic at ∞, so it's constant. So $f(x) = R(x) + C$, still rational.

Thus there exists some $R(x) \in \mathbb{C}(x)$ such that

$$f(x) = R(x) \quad \text{as meromorphic functions on } \mathbb{C} \cup \{\infty\}.$$

But $f = \phi_1 \circ F$ and $R \circ \phi_1$ have the same values on U_1, so

$$\phi_1 \circ F = R \circ \phi_1 \quad \text{on } U_1,$$

hence

$$F|_{U_1} = \phi_1^{-1} \circ R \circ \phi_1 = F_R|_{U_1}.$$

Both F and F_R are holomorphic maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ that agree on the nonempty open set U_1, so by the identity theorem they agree everywhere:

$$F = F_R.$$

So every $F \in \mathcal{M}(\mathbb{CP}^1)$ comes from a unique $R \in \mathbb{C}(x)$. That's surjectivity and injectivity of Φ.

4. Summary in your language

Using your chart

$$\phi_1 : U_1 \to \mathbb{C}, \quad [z_0 : z_1] \mapsto z_0/z_1,$$

we have:

- Define $x := \phi_1$. This is a meromorphic function on $\mathbb{CP}^1$ with one pole at $[1 : 0]$.
- Given $R(x) \in \mathbb{C}(x)$, we define a holomorphic map $F_R : \mathbb{CP}^1 \to \mathbb{CP}^1$ by

  $$F_R = \phi_1^{-1} \circ R \circ \phi_1 \quad \text{on } U_1,$$

  extended holomorphically to ∞.

- Given a holomorphic $F : \mathbb{CP}^1 \to \mathbb{CP}^1$, its coordinate expression

$$f = \phi_1 \circ F \circ \phi_1^{-1} : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$$

  is a meromorphic function on the sphere, hence a rational function $R(x)$. Then $F = F_R$.

So precisely:

$$\boxed{\mathcal{M}(\mathbb{CP}^1) = \{F : \mathbb{CP}^1 \to \mathbb{CP}^1 \text{ holomorphic}\} \;\cong\; \{R(x) \in \mathbb{C}(x)\}}$$

and the chart ϕ_1 is the bridge that makes this identification explicit.