Abstract Algebra II

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We cover the following topics in this note.

- Group Action
- Cayley Theorem
- Normal Subgroups
- Normality of the Kernel

Group Action

Definition. Let $(G, *)$ be a group and let $X \neq \emptyset$. A **(left) group action** of G on X is a function

$$\cdot : G \times X \to X, \quad (g, x) \mapsto g \cdot x$$

satisfying the followings: for all $g, h \in G$ and all $x \in X$,

(i) (Identity) $e \cdot x = x$, where $e \in G$ is the identity element of G;

(ii) (Compatibility) $(g * h) \cdot x = g \cdot (h \cdot x)$.

The pair $(X, \cdot)$ (or simply X) is then called a G-set.

Note (Notation)**.** If a group G acts on a set X, one commonly writes: $G \curvearrowright X$.

Remark. A right group action of G on X is a function $\cdot : X \times G \to X, \quad (x, g) \mapsto x \cdot g$ satisfying:

(i) $x \cdot e = x$ for all $x \in X$;

(ii) $(x \cdot g) \cdot h = x \cdot (gh)$ for all $g, h \in G, x \in X$.

Example (Scalar Multiplication on a Vector Space)**.** Let $\mathbb{F}$ be a field, and let $X = \mathbb{F}^n$ be the n-dimensional vector space over $\mathbb{F}$. Consider the multiplicative group of nonzero scalars in $\mathbb{F}$:

$$G = (\mathbb{F}^\times, \times), \quad \text{where } \mathbb{F}^\times = \mathbb{F} \setminus \{0\}.$$

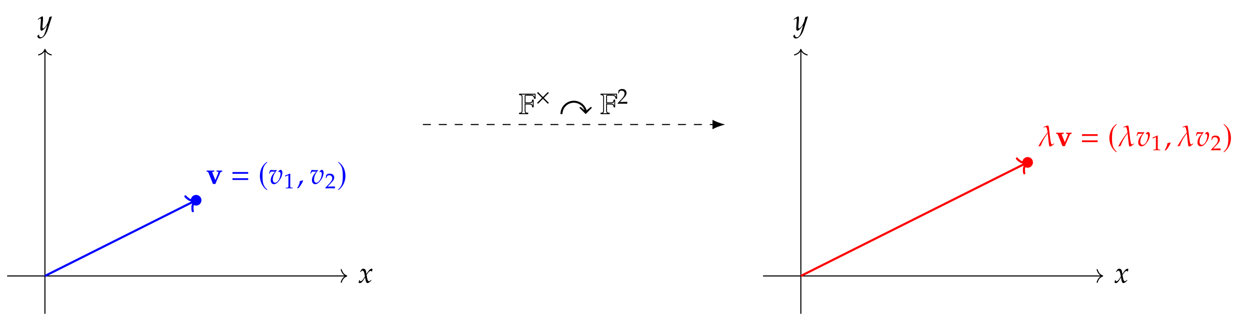
We define an action $G \curvearrowright X$ by scalar multiplication:

$$\begin{array}{cccc} \cdot & : & \mathbb{F}^\times \times \mathbb{F}^n & \longrightarrow & \mathbb{F}^n \\ & & (\lambda, \mathbf{v}) & \longmapsto & \lambda \cdot \mathbf{v} \end{array},$$

where the product $\lambda \cdot \mathbf{v}$ is defined componentwise. Then

(i) $1 \cdot \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{F}^n$.

(ii) $(\lambda\mu) \cdot \mathbf{v} = \lambda \cdot (\mu \cdot \mathbf{v})$ for all $\lambda, \mu \in \mathbb{F}^\times$, $\mathbf{v} \in \mathbb{F}^n$.



Example (Conjugation Action on the Group Itself)**.** Let G be any group, and consider $X = G$. Define an action of G on itself by conjugation:

$$G \curvearrowright G, \quad (g, x) \mapsto g \cdot x := g * x * g^{-1}.$$

Then

(i) $e \cdot x = e * x * e^{-1} = x$ for all $x \in G$.

(ii) Note that

$$\begin{aligned} (g * h) \cdot x &= (g * h) * x * (g * h)^{-1} \\ &= (g * h) * x * (h^{-1} * g^{-1}) \\ &= g * (h * x * h^{-1}) * g^{-1} \\ &= g * (h \cdot x) * g^{-1} \\ &= g \cdot (h \cdot x). \end{aligned}$$

Thus, this is a left group action.

Example (Trivial G-Set)**.** Let G be any group and define the set $X = \{x\}$, a singleton. Define the action

$$G \curvearrowright X, \quad (g, x) \mapsto g \cdot x := x \quad \text{for all } g \in G.$$

This is the **trivial action**, where every group element acts as the identity on X:

(i) $e \cdot x = x$.

(ii) $(g * h) \cdot x = x = g \cdot (h \cdot x)$.

Example (Action on Coset Space G/H)**.** Let $(G, *)$ be a group, and let $H \leq G$. Let $X = G/H$ be the set of left cosets of H in G, i.e.,

$$X = G/H = \{gH \mid g \in G\}.$$

Define an action

$$G \curvearrowright G/H, \quad (g, aH) \mapsto (ga)H.$$

This is well-defined because if $a_1H = a_2H$, then $a_1^{-1}a_2 \in H$, so: $ga_1H = ga_2H$.. Since

(i) $e \cdot aH = aH$;

(ii) $(gh) \cdot aH = g \cdot (h \cdot aH)$.

Group Elements Act as Permutations

Proposition. *Let G be a group action on a set X via a left action $G \curvearrowright X$, given by $(g, x) \mapsto g \cdot x$. Then for each $g \in G$, the map*

$$\sigma_g : X \to X, \quad x \mapsto g \cdot x$$

is one-to-one and onto. That is, $\sigma_g \in \mathrm{Sym}(X)$, the group of all permutations of X.

Proof. TBA ◻

Group Actions Induce Permutation Representations

Theorem. *Let G be a group action on a set X via a left group action $G \curvearrowright X$, $(g, x) \mapsto g \cdot x$. For each $g \in G$, define the bijection $\sigma_g : X \to X$ by $\sigma_g(x) := g \cdot x$. Then the map*

$$\phi : G \to \mathrm{Sym}(X), \quad g \mapsto \sigma_g,$$

is a ***group homomorphism*** *from G to the symmetric group $\mathrm{Sym}(X)$. In other words, for all $g, h \in G$,*

$$\phi(g * h) = \sigma_{g*h} = \sigma_g \circ \sigma_h = \phi(g) \circ \phi(h).$$

Remark. A group action $G \curvearrowright X$ is equivalent to a group homomorphism $G \to \mathrm{Sym}(X)$, i.e., a **permutation representation** of G.

Proof. TBA □

Cayley Theorem

Theorem. *Let G be a group. Consider the action of G on itself by left multiplication. For each $g \in G$, define*

$$\sigma_g \;:\; G \longrightarrow G, \qquad x \;\mapsto\; g \cdot x.$$

Then the map

$$\phi \;:\; G \longrightarrow \mathrm{Sym}(G), \qquad g \;\mapsto\; \sigma_g$$

is an injective group homomorphism (group monomorphism). In particular,

$$\phi(G) \simeq G \quad \textit{and} \quad \phi(G) \leq \mathrm{Sym}(G).$$

Proof. TBA □

Normal Subgroups

Observation. Consider $4\mathbb{Z} \leq \mathbb{Z}$. Then

$$\mathbb{Z}/4\mathbb{Z} = \{0+4\mathbb{Z}, 1+4\mathbb{Z}, 2+4\mathbb{Z}, 3+4\mathbb{Z}\} = \left\{[0],[1],[2],[3]\right\}.$$

- $[0]+[1] = (0+4\mathbb{Z})+(1+4\mathbb{Z}) \overset{\text{def.}}{=} (0+1)+4\mathbb{Z} = 1+4\mathbb{Z} = [1]$.
- $[1]+[2] = (1+4\mathbb{Z})+(2+4\mathbb{Z}) \overset{\text{def.}}{=} (1+2)+4\mathbb{Z} = 3+4\mathbb{Z} = [3]$.
- $[1]+[3] = (2+4\mathbb{Z})+(3+4\mathbb{Z}) \overset{\text{def.}}{=} (1+3)+4\mathbb{Z} = 4+4\mathbb{Z} = 0+4\mathbb{Z} = [0]$.

Existence of the Quotient Group

Proposition. *Let $(G,*)$ be a group and let $H \leq G$ be a subgroup. Define a binary operation $\boxtimes$ on the set of left cosets G/H by*

$$(g*H)\boxtimes(g'*H) \;=\; (g*g')*H$$

where $g, g' \in G$. Then this operation is well-defined if and only if

$$g*h*g^{-1} \;\in\; H.$$

for all $g \in G$, $h \in H$.

Proof. TBA □

Normal Subgroup

Definition. Let $(G, *)$ be a group and let $H \leq G$. We say that H is **normal** in G, written

$$H \trianglelefteq G,$$

if $g * h * g^{-1} \in H$ for any $g \in G$ and $h \in H$.

Remark. The set of (left) cosets G/H be a well-defined group structure via

$$(g * H) \boxast (k * H) = (g * k) * H,$$

making G/H the quotient group of G by H.

Equivalent Definitions of Normal Subgroup

Proposition. *Let $(G, *)$ be a group and let $H \leq G$. The Following Are Equivalent:*

(1)[a] *H is normal in G,* i.e., *$H \trianglelefteq G$;*

(2)[b] *$g * h * g^{-1} \in H$ for all $g \in G$, $h \in H$;*

(3)[c] *$g * H * g^{-1} = H$ for all $g \in G$;*

(4)[d] *$g * H = H * g$ for all $g \in G$.*

[a] Terminology and Notation
[b] (Elementwise Conjugation)
[c] (Conjugation Invariance)
[d] (Coset Equality)

Proof. ((2)⇒(3)) TBA

((3)⇒(4)) TBA

((4)⇒(2)) TBA

□

Normality of Kernel

Theorem. *Let $\phi : (G, *) \longrightarrow (H, *')$ be a group homomorphism, and define its kernel by*

$$\ker\phi \;=\; \left\{ g \in G : \phi(g) = e_H \right\}.$$

Then $\ker\phi$ is a normal subgroup of G; that is, $\ker\phi \trianglelefteq G$.

Proof. Since ϕ is a homomorphism, for every $g \in G$ and every $k \in \ker\phi$ we have

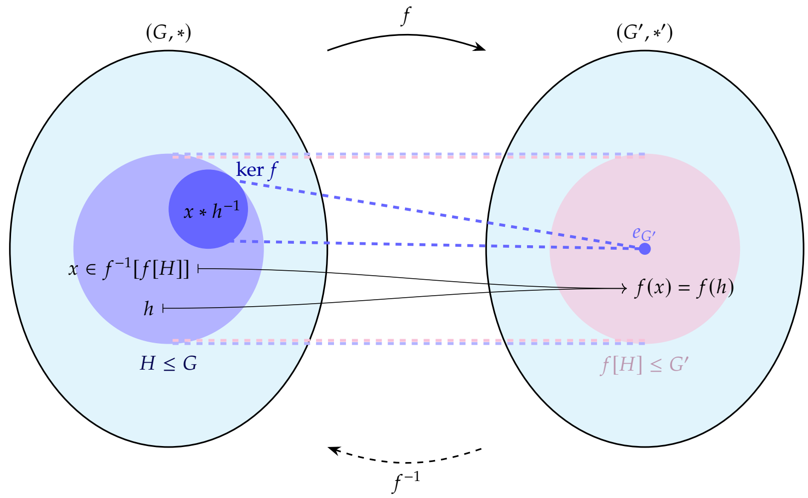
$$\phi\left(g * k * g^{-1}\right) = \phi(g) *' \phi(k) *' \phi(g)^{-1} = \phi(g) *' e_H *' \phi(g)^{-1} = e_H,$$

so $g * k * g^{-1} \in \ker\phi$. Thus,

$$g * (\ker\phi) * g^{-1} \;=\; \ker\phi \quad \forall\, g \in G,$$

i.e. $\ker\phi$ is invariant under conjugation and hence normal in G. $\square$

Illustration.



$f^{-1}[f[H]] = H.$

Preimage of the Image of a Subgroup

Theorem. *Let $f : (G,*) \longrightarrow (G',*')$ be a group homomorphism, and let $H \leq G$ such that*

$$\{\, g \in G : f(g) = e_{G'} \} = \ker f \subseteq H.$$

Then

$$f^{-1}[f[H]] = H,$$

with $f[H] = \{f(h) \mid h \in H\}$ and $f^{-1}[f[H]] = \{g \in G \mid f(g) \in f[H]\}$.

Proof. Suppose that $\ker f \subseteq H \leq G$. We NTS that $f^{-1}[f[H]] = H$:

($\supseteq$) $h \in H \implies f(h) \in f[H] \implies h \in f^{-1}[f[H]]$.

($\subseteq$) Let $x \in f^{-1}[f[H]]$. Then $f(x) \in f[H]$; that is,

$$\exists h \in H \quad \text{such that} \quad f(h) = f(x).$$

Thus,

$$f(x * h^{-1}) = f(x) *' f(h)^{-1} = f(x) *' f(x)^{-1} = e_{G'},$$

so $x * h^{-1} \in \ker f$. Since $\ker f \subseteq H$, we have

$$x = (x * h^{-1}) * h \in H,$$

and hence $f^{-1}[f[H]] \subseteq H$.

$\square$

Theorem. *Let* $\phi\colon G \longrightarrow G'$ *be a surjective homomorphism of groups. Define two collections:*

$$\mathcal{S} = \{H \subseteq G : \ker\phi \subseteq H \leq G\}, \qquad \mathcal{T} = \{H' \subseteq G' : H' \leq G'\}.$$

Then the map

$$\Phi\colon \mathcal{S} \longrightarrow \mathcal{T}, \qquad \Phi(H) = \phi(H)$$

is a bijection. Its inverse is

$$\Phi^{-1}\colon \mathcal{T} \longrightarrow \mathcal{S}, \qquad \Phi^{-1}(H') = \phi^{-1}(H').$$

Moreover,

$$H \trianglelefteq G \iff \phi(H) \trianglelefteq G'.$$

Proof. $\square$

References

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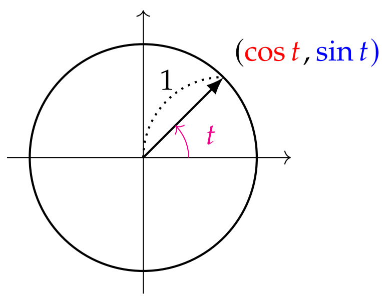
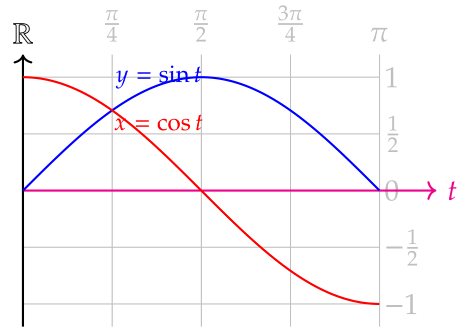
A Appendices

The unit circle $\mathbb{S}^1$ as the Rotation Group in the Plane

The set $\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is called the **unit circle**.





Geometrically, it represents the set of points at a fixed distance 1 from the origin in $\mathbb{R}^2$, while algebraically it can be seen as a group under complex multiplication. In the complex plane, we write:

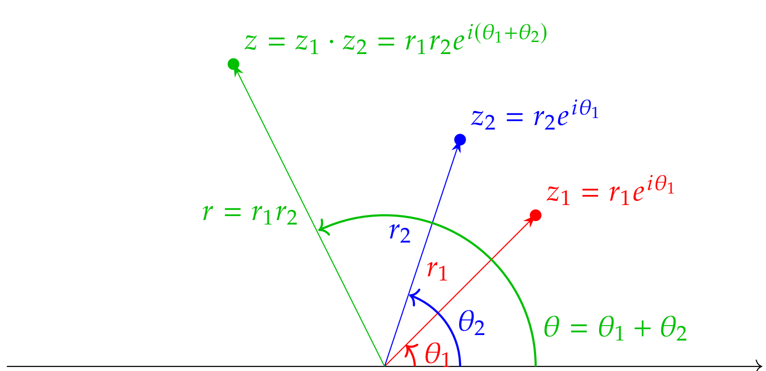
$$\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\} = \{re^{i\theta} : |r| = 1 \text{ and } \theta \in \mathbb{R}\}.$$

Let

$$\begin{aligned} z_1 &= r_1 e^{i\theta_1} = r_1 (\cos\theta_1 + i \sin\theta_1) \in \mathbb{C} \text{ and} \\ z_2 &= r_2 e^{i\theta_2} = r_2 (\cos\theta_2 + i \sin\theta_2) \in \mathbb{C}. \end{aligned}$$

Then

$$\begin{aligned} z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = &= r_1 r_2 (\cos\theta_1 + i \sin\theta_1)(\cos\theta_2 + i \sin\theta_2) \\ &= r_1 r_2 \left[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i (\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)\right] \\ &= r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\right] \\ &= r(\cos\theta + \sin\theta) \text{ with } \begin{cases} r = r_1 r_2 \\ \theta = \theta_1 + \theta_2. \end{cases} \end{aligned}$$



Multiplying a point $z = x + iy \in \mathbb{C}$ by $e^{i\theta}$ is exactly the rotation

$$(x + iy) \mapsto (x + iy)e^{i\theta} = (x + iy)(\cos\theta + i\sin\theta) = (x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta).$$

In matrix form

$$e^{i\theta} \longleftrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

B Embed "Plane of Rotation" into $\mathbb{R}^3$ by "fixing " the z-axis

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