

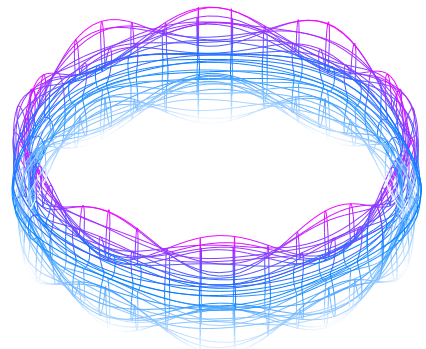
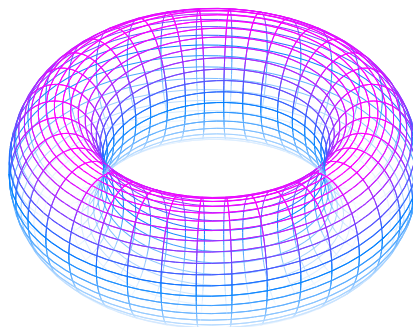
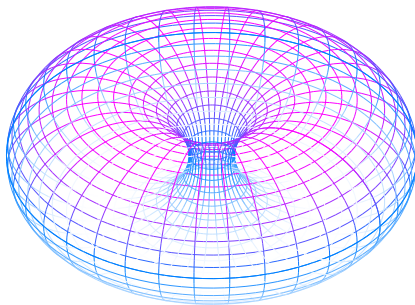
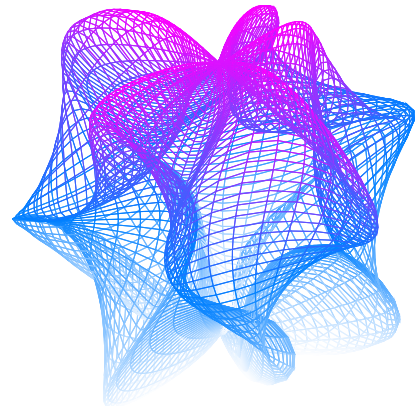
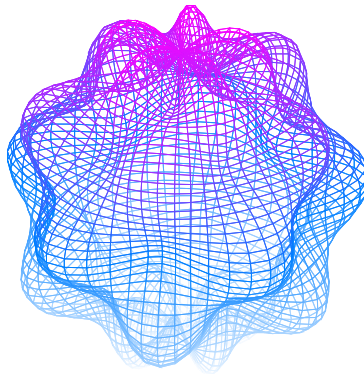
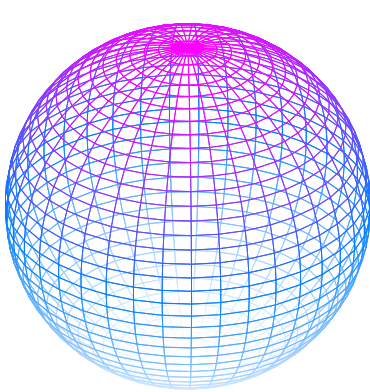
Topology I

Ji, Yong-hyeon

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We cover the following topics in this note.

- Topology and Topological Space
 - Open Set
 - Continuous Mapping
 - Distance Function and Metric Space
 - Convergence of Sequences; Continuity of Functions
 - TBA
-



Topology; Topological Space

Definition. Let S be a non-empty set. A **topology**^a on S is a subset $\mathcal{T} \subseteq 2^S$, where 2^S denotes the power set of S , that satisfies the following axioms:

(O1)^b The empty set and the entire set S belong to \mathcal{T} : $S \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.

(O2)^c The union of any collection of elements in \mathcal{T} is also an element of \mathcal{T} :

$$\{U_i\}_{i \in I} \subseteq \mathcal{T} \implies \bigcup_{i \in I} U_i \in \mathcal{T}.$$

(O3)^d The intersection of any finite number of elements in \mathcal{T} is also an element of \mathcal{T} :

$$\{U_i\}_{i=1}^n \subseteq \mathcal{T} \implies \bigcap_{i=1}^n U_i \in \mathcal{T}.$$

The pair (S, \mathcal{T}) is called a **topological space**.

^aThe word “topology” comes from the Greek roots “topos” meaning “place” and “logos” meaning “study”.

^bEmpty set and Whole space

^cClosure under *arbitrary* unions

^dClosure under *finite* intersections

Remark. By mathematical induction, we have

$$O3 \iff [\{U_1, U_2\} \subseteq \mathcal{T} \Rightarrow U_1 \cap U_2 \in \mathcal{T}].$$

Open Set (Topology)

Definition. Let (S, \mathcal{T}) be a topological space. $U \subseteq S$ is an **open set**, or **open** (in S) iff $U \in \mathcal{T}$.

Remark. A subset \mathcal{T} of power set 2^S is a topology on S if and only if

(i) \emptyset and S are open;

(ii) Let $U_1, U_2, \dots \in \mathcal{T}$, i.e., $\{U_i\}_{i \in I} \subseteq \mathcal{T}$. Then $\bigcup_{i \in I} U_i$ is open.

(iii) Let $U_1, U_2, \dots, U_n \in \mathcal{T}$, i.e., $\{U_i\}_{i=1}^n \subseteq \mathcal{T}$. Then $\bigcap_{i=1}^n U_i$ is open.

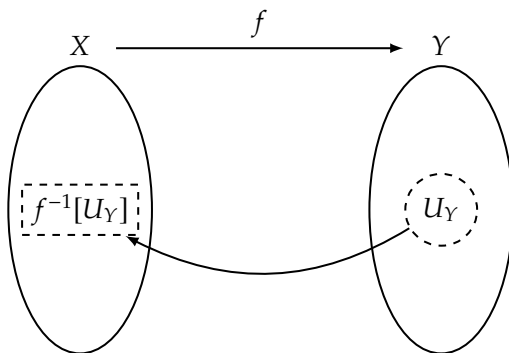
Continuous Mapping by Open Sets

Definition. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are topological spaces. Let $f : X \rightarrow Y$ be a mapping from X to Y .

(1) (Continuous Everywhere) The mapping f is **continuous on X** if and only if

$$U_Y \in \mathcal{T}_Y \implies f^{-1}[U_Y] \in \mathcal{T}_X,$$

where $f^{-1}[U_Y] = \{x \in X : f(x) \in U_Y\}$ is the preimage of U_Y under f .



Note (Preparation for **Example 1**). Let $S \neq \emptyset$ be a set, and let $\{A_\alpha\}_{\alpha \in \Lambda} \subseteq S$. Then

$$\begin{aligned} S \setminus \bigcup_{\alpha \in \Lambda} A_\alpha &= S \setminus \{x \in S : \exists \alpha \in \Lambda \text{ s.t. } x \in A_\alpha\} = \{x \in S : \neg[\exists \alpha \in \Lambda \text{ s.t. } x \in A_\alpha]\} \\ &= \{x \in S : \forall \alpha \in \Lambda, x \notin A_\alpha\} \\ &= \{x \in S : \forall \alpha \in \Lambda, x \in S \setminus A_\alpha\} \\ &= \bigcap_{\alpha \in \Lambda} (S \setminus A_\alpha). \end{aligned}$$

$$\begin{aligned} S \setminus \bigcap_{\alpha \in \Lambda} A_\alpha &= S \setminus \{x \in S : \forall \alpha \in \Lambda, x \in A_\alpha\} = \{x \in S : \neg[\forall \alpha \in \Lambda, x \in A_\alpha]\} \\ &= \{x \in S : \exists \alpha \in \Lambda \text{ s.t. } x \notin A_\alpha\} \\ &= \{x \in S : \exists \alpha \in \Lambda \text{ s.t. } x \in S \setminus A_\alpha\} \\ &= \bigcup_{\alpha \in \Lambda} (S \setminus A_\alpha). \end{aligned}$$

Note (Preparation for **Example 1**).

- (1) A Subset of a Finite Set is Finite.
- (2) The Intersection of Finite Sets is Finite.

Example 1 (Cofinite Topology). Let $S \neq \emptyset$ be a set. Define the cofinite topology $\mathcal{T}_C \subseteq 2^S$ by

$$\begin{aligned}\mathcal{T}_C &:= \{U \subseteq S : S \setminus U \text{ is finite}\} \cup \{\emptyset\} \\ &= \{U \subseteq S : U = \emptyset \text{ or } S \setminus U \text{ is finite}\}.\end{aligned}$$

In other words, U is open in the cofinite topology if U is the empty, or if the complement $S \setminus U$ is a finite set. We claim that \mathcal{T}_C be a topology on S :

(O1) By definition, $\emptyset \in \mathcal{T}_C$. For $U = S$, the complement $S \setminus S = \emptyset$, which is finite, so $S \in \mathcal{T}_C$. Hence, both \emptyset and S are elements of \mathcal{T}_C .

(O2) Let $\{U_i\}_{i \in I} \subseteq \mathcal{T}_C$.

(Case 1) If $U_i = \emptyset$ for all $i \in I$, then $\bigcup_{i \in I} U_i = \emptyset \in \mathcal{T}_C$.

(Case 2) Suppose that there exists $i_0 \in I$ such that $U_{i_0} \neq \emptyset$. Then

$$S \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (S \setminus U_i) \subseteq (S \setminus U_{i_0}).$$

Since $S \setminus U_{i_0}$ is finite, $S \setminus \bigcup_{i \in I} U_i$ is finite, so $\bigcup_{i \in I} U_i \in \mathcal{T}_C$.

(O3) Let $U_1 \in \mathcal{T}_C$ and $U_2 \in \mathcal{T}_C$.

(Case 1) If $U_1 = \emptyset$ or $U_2 = \emptyset$, then $U_1 \cap U_2 = \emptyset \in \mathcal{T}_C$.

(Case 2) Suppose that $U_1 \neq \emptyset$ and $U_2 \neq \emptyset$. Then $S \setminus U_1$ and $S \setminus U_2$ are finite. By the De Morgan law, we have

$$S \setminus (U_1 \cap U_2) = (S \setminus U_1) \cup (S \setminus U_2),$$

which is a finite set. Thus, $U_1 \cap U_2 \in \mathcal{T}_C$.

Example 2 (Discrete Topology). Let $S \neq \emptyset$ be a set, and let $\mathcal{T} = 2^S$ be the power set of S . Then \mathcal{T} is called the **discrete topology** on S and $(S, \mathcal{T}) = (S, 2^S)$ the **discrete (topological) space** on S .

Example 3 (Indiscrete Topology). Let $S \neq \emptyset$ be a set, and let $\mathcal{T} = \{S, \emptyset\}$. Then \mathcal{T} is called the **indiscrete topology** on S and $(S, \mathcal{T}) = (S, \{S, \emptyset\})$ the **indiscrete (topological) space** on S .

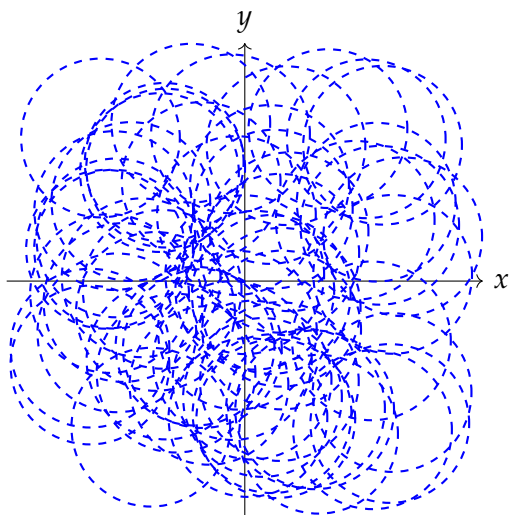
Note.

- (1) Discrete Topology is Finest Topology.
- (2) Indiscrete Topology is Coarsest Topology.

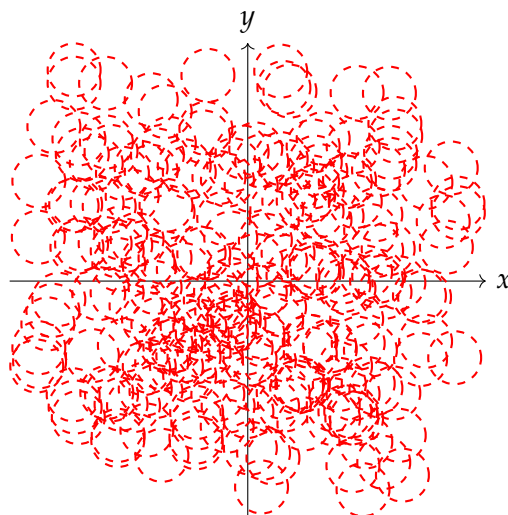
Coarser Topology and Finer Topology

Definition. Let $S \neq \emptyset$ be a set. Let \mathcal{T}_1 and \mathcal{T}_2 be topologies on S .

- (1) \mathcal{T}_1 is said to be **coarser** than \mathcal{T}_2 if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.
- (2) \mathcal{T}_1 is said to be **finer** than \mathcal{T}_2 if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.



Coarser Topology



Finer Topology

Distance Function

Definition. Let S be a set. The real-valued function of two variable

$$d : S \times S \rightarrow \mathbb{R}$$

is called a **distance function** (or **metric**) if it satisfies the following properties:

- (i)^a $\forall x, y \in S, d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$.
- (ii)^b $\forall x, y \in S, d(x, y) = d(y, x)$.
- (iii)^c $\forall x, y, z \in S, d(x, z) \leq d(x, y) + d(y, z)$.

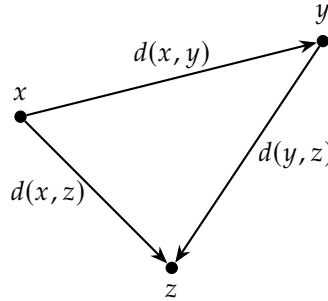
The pair (S, d) is called a **metric space**.

^aNon-negativity and Zero only for identical points

^bSymmetry

^cTriangle inequality

Remark.



Example 4.

- Let $S = \mathbb{R}$, the set of real numbers. Define the function $d : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$d(x, y) = |x - y|$$

for $x, y \in \mathbb{R}$.

- Let $S = \mathbb{R}^n$, the n -dimensional Euclidean space. Define the function $d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=0}^{n-1} |x_i - y_i|^2},$$

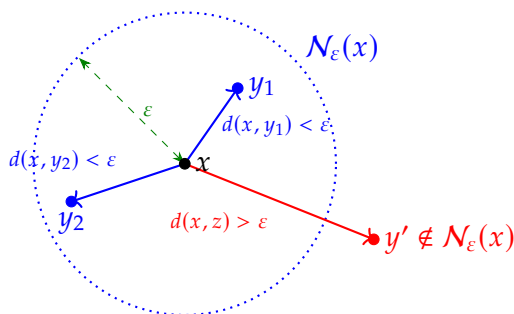
where $\mathbf{x} = (x_0, x_1, \dots, x_{n-1})$ and $\mathbf{y} = (y_0, \dots, y_{n-1})$ are vectors in \mathbb{R}^n .

Neighborhood (Metric Space)

Definition. Let (S, d) be a metric space, where S is a set and $d : S \times S \rightarrow \mathbb{R}$ is a metric. For $x \in S$ and $\varepsilon > 0$, the **ε -neighborhood of x** , denoted by $N_\varepsilon(x)$, is defined as

$$N_\varepsilon(x) := \{y \in S : d(x, y) < \varepsilon\}.$$

Remark.

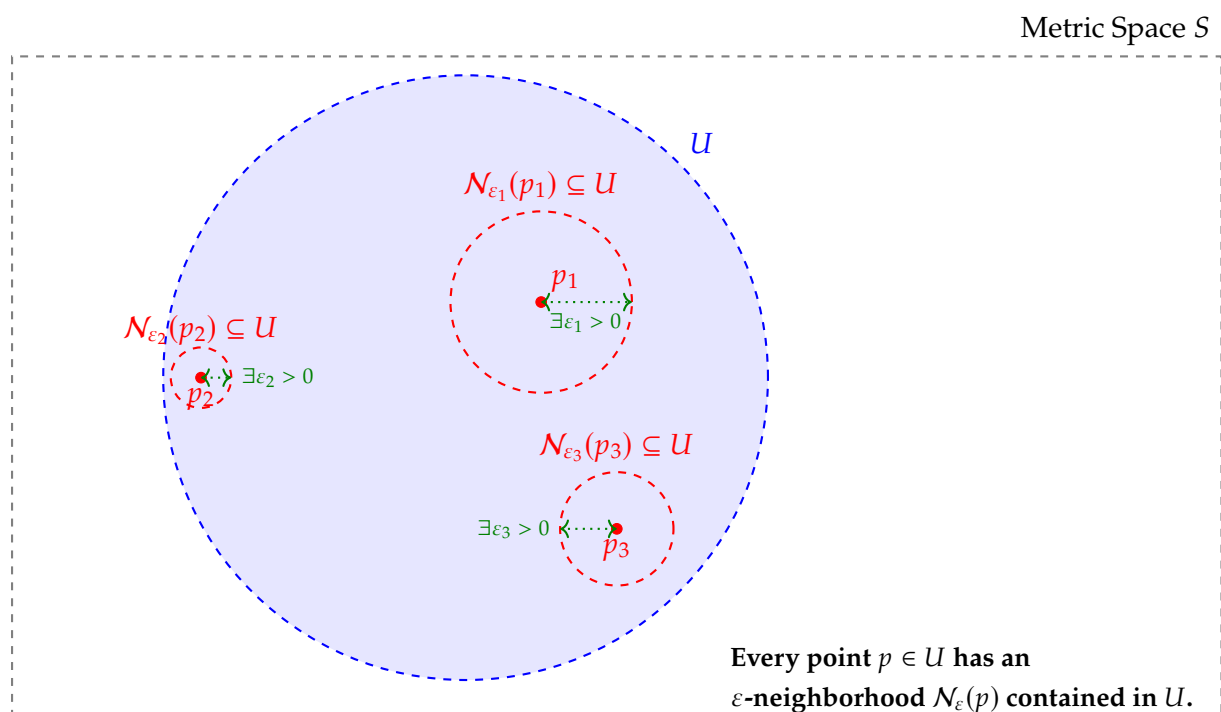


Open Set (Metric Space)

Definition. Let (S, d) be a metric space, where S is a set and $d : S \times S \rightarrow \mathbb{R}$ is a metric. Then

$$U \subseteq S \text{ is open in } S \stackrel{\text{def}}{\iff} \forall p \in U, \exists \varepsilon > 0 \text{ such that } N_\varepsilon(p) \subseteq U.$$

Remark.



Exercise (Metric Topology). Let (S, d) be a metric space, where S is a set and $d : S \times S \rightarrow \mathbb{R}$ is a metric. Consider the set τ of all open sets of S :

$$\begin{aligned}\tau &:= \{U \subseteq S : U \text{ is open in } S\} \\ &= \{U \subseteq S : \forall p \in U, \exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(p) \subseteq U\}.\end{aligned}$$

We claim that τ is the topology induced by the metric d on the space (S, d) :

(O1) $S \in \tau$ and $\emptyset \in \tau$:

($\emptyset \in \tau$) The condition

$$“\forall p \in U, \exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(p) \subseteq U”$$

is vacuously true for $U = \emptyset$. Therefore $\emptyset \in \tau$.

($S \in \tau$) For $p \in S$, the ε -neighborhood of p is defined as

$$\mathcal{N}_\varepsilon(p) = \{q \in S : d(p, q) < \varepsilon\} \subseteq S.$$

Since S is the entire space, $\mathcal{N}_\varepsilon(p) \subseteq S$ for any $\varepsilon > 0$.

(O2) τ is closed under arbitrary unions:

Let $\{U_i\}_{i \in I}$ be an arbitrary collection of sets in τ . Let $p \in \bigcup_{i \in I} U_i$. Then

$$\exists i_0 \in I \text{ such that } p \in U_{i_0}.$$

Since $U_{i_0} \in \tau$, there exists $\varepsilon > 0$ such that $\mathcal{N}_\varepsilon(p) \subseteq U_{i_0}$. Then

$$\mathcal{N}_\varepsilon(p) \subseteq U_{i_0} \subseteq \bigcup_{i \in I} U_i.$$

Thus, $\bigcup_{i \in I} U_i \in \tau$.

(O3) τ is closed under finite intersections:

Let $U_1, U_2 \in \tau$, and let $p \in (U_1 \cap U_2)$. Then

$$\exists \varepsilon_1 > 0 \text{ such that } \mathcal{N}_{\varepsilon_1}(p) \subseteq U_1,$$

$$\exists \varepsilon_2 > 0 \text{ such that } \mathcal{N}_{\varepsilon_2}(p) \subseteq U_2.$$

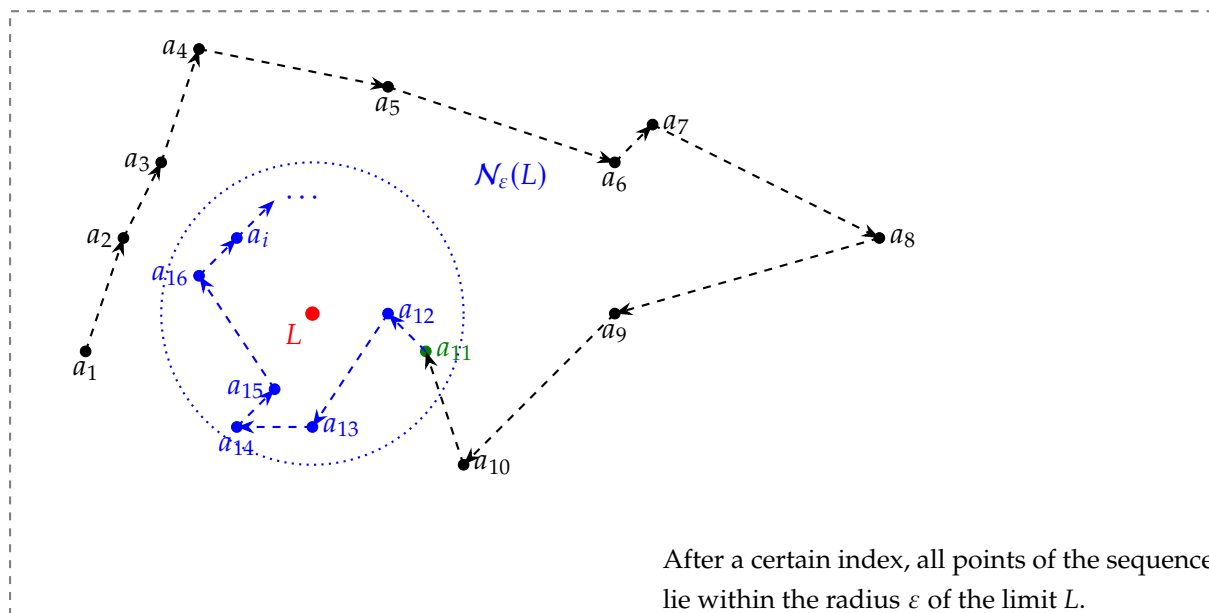
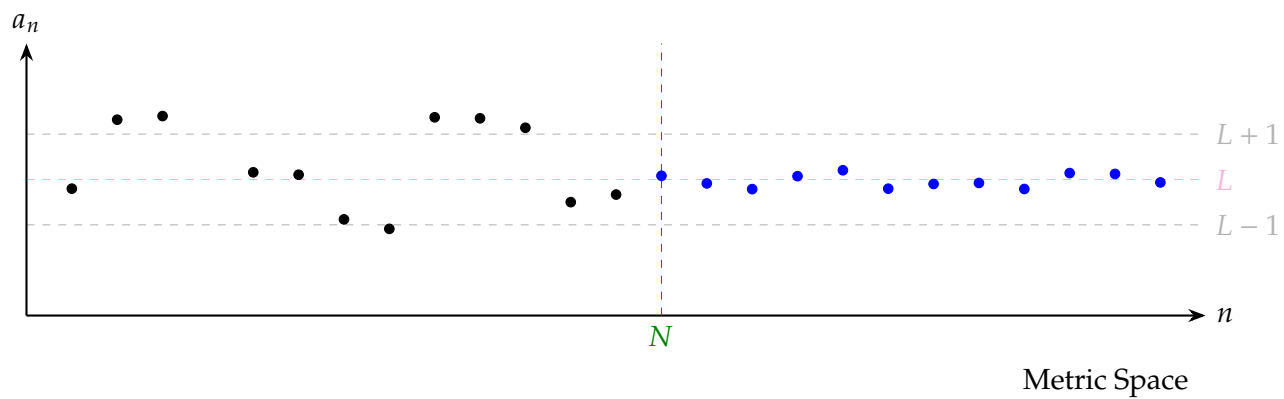
Define $\varepsilon := \min(\varepsilon_1, \varepsilon_2)$. Then

$$\mathcal{N}_\varepsilon(p) \subseteq \mathcal{N}_{\varepsilon_i}(p) \subseteq U_i \text{ for } i = 1, 2.$$

Thus $\mathcal{N}_\varepsilon(p) \subseteq U_1 \cap U_2$, and so $U_1 \cap U_2 \in \tau$.

Note (Convergence of Sequences). A sequence $\{a_n\}_{n=1}^{\infty} (\subseteq \mathbb{R})$ is **converge** to $L \in \mathbb{R}$ if and only if

$$\begin{aligned} & \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } [n \geq N \implies |a_n - L| < \varepsilon] \\ \iff & \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } [n \geq N \implies d(a_n, L) < \varepsilon] \\ \iff & \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } [n \geq N \implies a_n \in \mathcal{N}_{\varepsilon}(L)] \end{aligned}$$



Continuity of Functions

Definition. Let $S \subseteq \mathbb{R}$ be a non-empty subset of \mathbb{R} . Let $f : S \rightarrow \mathbb{R}$ be a real-valued function, and let $a \in S$. We say that f is **continuous at a** if and only if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

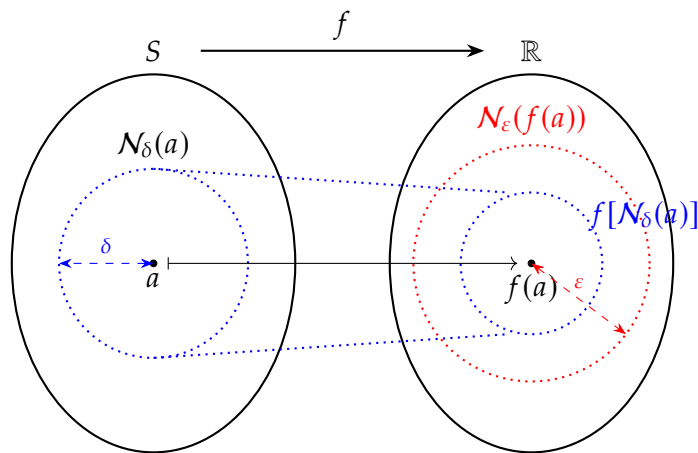
That is,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

If f is continuous on every point of S , then f is called a **continuous function on S** .

Remark.

$$\begin{aligned} & \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon \\ \iff & \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } x \in \mathcal{N}_\delta(a) \implies f(x) \in \mathcal{N}_\varepsilon(f(a)) \\ \iff & \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } f(x) \in f[\mathcal{N}_\delta(a)] \implies f(x) \in \mathcal{N}_\varepsilon(f(a)) \quad \because f[\mathcal{N}_\delta(a)] = \{f(x) : x \in \mathcal{N}_\delta(a)\} \\ \iff & \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } f[\mathcal{N}_\delta(a)] \subseteq \mathcal{N}_\varepsilon(f(a)). \end{aligned}$$



Remark. f is discontinuous at a if and only if

$$\begin{aligned} & \exists \varepsilon > 0 \text{ such that } \forall \delta > 0, |x - a| < \delta \text{ but } |f(x) - f(a)| \geq \varepsilon \\ \iff & \exists \varepsilon > 0 \text{ such that } \forall \delta > 0, \mathcal{N}_\varepsilon(f(a)) \not\subset f[\mathcal{N}_\delta(a)]. \end{aligned}$$

Note. TBA

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References

- [1] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 8. 위상수학 (a) 위상공간의 정의.” YouTube Video, 41:25. Published September 27, 2019. URL: <https://www.youtube.com/watch?v=q8BtXIFzo2Q>.
- [2] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 9. 위상수학 (b) 해석학개론과 거리위상” YouTube Video, 33:43. Published September 29, 2019. URL: <https://www.youtube.com/watch?v=uJ0Gw7Yxk7c&t=242s>.