Why $\partial Q/\partial x - \partial P/\partial y$ is the Curl in 2D

Local circulation density from a small rectangle

Let $\vec{F} = \langle P(x,y), Q(x,y)\rangle$ be C^1. Consider a small, axis-aligned rectangle centered at (x_0, y_0) with side lengths $\Delta x, \Delta y$. Its counterclockwise circulation is

$$\oint_{\partial R} \vec{F} \cdot d\vec{r} = \int_{\text{bottom}} P\,dx + \int_{\text{right}} Q\,dy + \int_{\text{top}} P\,dx + \int_{\text{left}} Q\,dy.$$

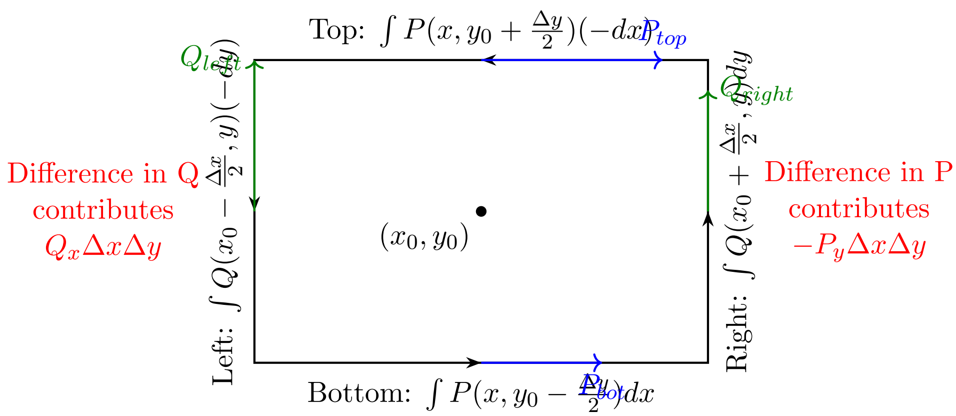


Figure 1: Circulation around an infinitesimal rectangle. The net circulation arises from the difference in the vector field components on opposite sides.

Taylor expand P, Q to first order along each edge and keep first-order terms. One obtains

$$\oint_{\partial R} \vec{F} \cdot d\vec{r} \approx \left(\frac{\partial Q}{\partial x}(x_0, y_0) - \frac{\partial P}{\partial y}(x_0, y_0) \right) \Delta x\, \Delta y.$$

Dividing by the area $\Delta A = \Delta x\, \Delta y$ and shrinking the rectangle,

$$\lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint_{\partial R} \vec{F} \cdot d\vec{r} = \frac{\partial Q}{\partial x}(x_0, y_0) - \frac{\partial P}{\partial y}(x_0, y_0).$$

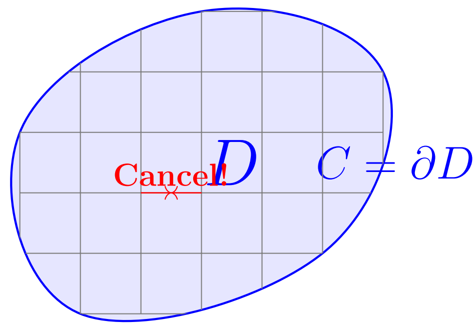
Thus the scalar $Q_x - P_y$ is the *circulation per unit area*—the 2D curl.

Green's theorem (global circulation)

If $C = \partial D$ is a positively oriented simple closed curve enclosing a region D, Green's theorem states

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

So the total circulation equals the area integral of the local circulation density.



The sum of circulations on all small rectangles cancels on interior edges, leaving only the circulation on the outer boundary C.

Figure 2: Green's theorem intuition: Total circulation is the sum of local circulations.

Example 1: Rigid rotation and angular velocity

Consider the rigid rotation field with angular speed ω:

$$\vec{F}(x, y) = \langle -\omega y, \ \omega x \rangle.$$

Then

$$\frac{\partial Q}{\partial x} = \omega, \qquad \frac{\partial P}{\partial y} = -\omega \quad \Rightarrow \quad \operatorname{curl} \vec{F} = Q_x - P_y = 2\omega.$$

This shows curl equals twice the angular velocity. For a circle of radius R, parametrize $r(t) = (R \cos t, R \sin t)$, $dr = (-R \sin t, R \cos t)\, dt$. Then

$$\oint \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \omega R^2 \, dt = 2\pi \omega R^2.$$

Meanwhile, $\iint_D (2\omega)\, dA = 2\omega \cdot \pi R^2 = 2\pi\omega R^2$, agreeing with Green's theorem.

Example 2: Curl-free but not conservative (topology matters)

On $\mathbb{R}^2 \setminus \{(0, 0)\}$, define

$$\vec{F}(x, y) = \left\langle -\frac{y}{x^2 + y^2}, \ \frac{x}{x^2 + y^2} \right\rangle.$$

Rigid Rotation $\vec{F} = \langle -y, x \rangle$ (for $\omega = 1$)

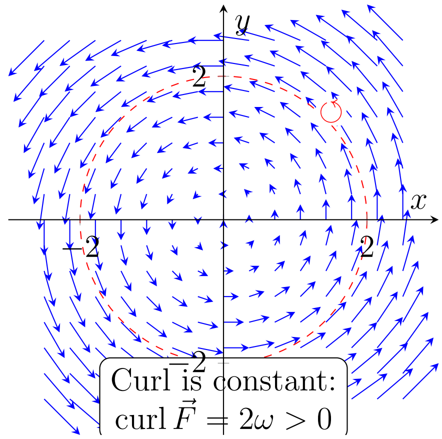


Figure 3: A rigid rotation field has a constant, positive curl, indicating a uniform counter-clockwise rotation at every point.

A direct calculation shows $Q_x - P_y = 0$ wherever defined (curl-free). However, the circulation around the unit circle is

$$\oint \vec{F} \cdot d\vec{r} = 2\pi \neq 0.$$

Hence there is no global potential function; the puncture creates a topological obstruction. This illustrates that curl $\vec{F} = 0$ captures *local* rotation, while global circulation can persist in domains with holes.

Summary checklist

- $Q_x - P_y$ is the infinitesimal (per-area) circulation density.
- Green's theorem sums local curl to give total circulation.
- Rigid rotation: curl $= 2\omega$ (twice angular velocity).
- Curl $= 0$ can still have nonzero loop integrals if the domain has holes.