Riemann-Roch Theorem and its Application

- From de Rham to sheaf cohomology, Serre duality -

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Preliminaries I: Vector calculus as a cochain complex

On $\mathbb{R}^3$, the familiar operators form a complex:

$$C^\infty \xrightarrow{\nabla} \Gamma(T^*) \xrightarrow{\nabla\times} \Gamma(\Lambda^2 T^*) \xrightarrow{\nabla\cdot} C^\infty,$$

with $(\nabla\times)\circ\nabla = 0,\ (\nabla\cdot)\circ(\nabla\times) = 0$. The de Rham complex is $(\Omega^\bullet(M), d)$:

$$\boxed{0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots}$$

Cohomology measures *global obstructions* to solving $d\eta = \omega$.

Preliminaries II: Local-to-global glueing: Mayer–Vietoris

For $M = U \cup V$, we uses a partition of unity to build the short exact sequence

$$0 \to \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U\cap V) \to 0,$$

and then the long exact Mayer–Vietoris sequence in cohomology.

This means that "curl-free locally" does not imply "gradient globally" when topology obstructs global potentials.

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Given $M = U \cup V$, we uses the short exact sequence of de Rham complexes

$$0 \to \Omega^k(M) \xrightarrow{\alpha^k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\beta^k} \Omega^k(U\cap V) \to 0$$

with $\beta^k(\omega_U, \omega_V) = \omega_U|_{U\cap V} - \omega_V|_{U\cap V}$, and surjectivity is built using a partition of unity $\{\rho_U, \rho_V\}$.

Why this belongs on a Riemann–Roch:

- Fine sheaves (built from partitions of unity) let we compute sheaf cohomology via resolutions.
- This is the conceptual bridge from `grad/curl/div` (de Rham) to $\bar\partial$- and sheaf-cohomological formulas that culminate in Riemann–Roch.

Preliminaries III: From de Rham to Complex Geometry

On a compact Riemann surface M, complex structure refines $d = \partial + \bar\partial$. Holomorphic data live in sheaves:

$$\mathcal{O}\ \text{(holomorphic functions)}, \qquad \Omega\ \text{(holomorphic 1-forms)}.$$

Divisors $D = \sum_p n_p p$ encode zeros/poles, and determine a line bundle $\mathcal{O}(D)$:

$$\mathcal{O}(D)(V) = \{TBA\}$$

Core I: Riemann–Roch (Curves)

Let M be a compact Riemann surface of genus g, and D a divisor. Define

$$\ell(D) = \dim H^0(M, \mathcal{O}(D)), \qquad i(D) = \dim H^0(M, \Omega(-D)).$$

Riemann–Roch:

$$\boxed{\ell(D) - i(D) = 1 - g + \deg D} \qquad \text{equivalently} \quad i(D) = \ell(K - D).$$

Core II: Cohomological mechanism: Euler characteristic + Serre duality

We derives Riemann–Roch from the Euler characteristic of $\mathcal{O}(D)$:

$$\chi(\mathcal{O}(D)) := \dim H^0(M, \mathcal{O}(D)) - \dim H^1(M, \mathcal{O}(D)) = \ell(D) - i(D),$$

using Serre duality to identify $H^1(M, \mathcal{O}(D))^* \simeq I(D)$.

Key principle:

$$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}_M) + \deg D, \qquad \chi(\mathcal{O}_M) = 1 - g.$$

So χ is "topology + degree," while $\ell(K - D)$ is the global obstruction.

Examples

Sphere $\mathbb{P}^1$ ($g = 0$, $K = -2[\infty]$). For $D = n[\infty]$, $\ell(D) = n{+}1$ for $n \geq -1$ (polynomials of degree $\leq n$).

Elliptic curve ($g = 1$, $\deg K = 0$). Riemann–Roch becomes $\ell(D) - i(D) = \deg D$, and for $\deg D > 0$ one has $\ell(D) = \deg D$.

Topology link: Gauss–Bonnet parallel

We emphasizes the analogy with Gauss–Bonnet:

$$\int K_{\text{Gauss}}\, \omega = 2\pi\chi(M) = 2\pi(2 - 2g).$$

Applications I: Algebraic–geometric (AG) codes from Riemann–Roch

Let $X/\mathbb{F}_q$ be a smooth projective curve of genus g. Choose:

$$D = P_1 + \cdots + P_n \ \text{(distinct rational points)},$$
$$G \ \text{(divisor, } \operatorname{supp}(G) \cap \operatorname{supp}(D) = \emptyset).$$

Define the evaluation code

$$C_L(D, G) = \{(f(P_1), \ldots, f(P_n)) :\ f \in L(G)\} \subseteq \mathbb{F}_q^n,$$
$$L(G) = H^0(X, \mathcal{O}(G)).$$

Dimension: $k = \dim C_L(D, G) = \ell(G) - \ell(G - D)$, and for $\deg G$ sufficiently large,

$$\ell(G) = \deg G + 1 - g,$$

as large-degree/embedding discussion.

Designed distance: $d \geq n - \deg G$. (Standard AG-code bound.)

Example for Applications I: Classic McElice and FALOMA

Applications II: "Surface direction" via Riemann–Roch for surfaces

We states the surface analogue (for a smooth projective surface X and divisor D):

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2}(D \cdot D - D \cdot K_X),$$

where $\cdot$ is the intersection pairing and K_X is the canonical divisor.

- *AG surface codes:* evaluate $H^0(X, \mathcal{O}_X(D))$ on rational points/curves; use the above χ plus vanishing theorems to estimate $\dim H^0$.
- *Topological/surface codes (quantum):* homological dimension depends on H_1 (genus); the same genus parameter appears in curve Riemann–Roch and in the obstruction term.

The pipeline: Calculus (grad/curl/div) $\to$ Riemann-Roch

$$\begin{array}{ccccc}
 & \begin{array}{c}\textbf{Vector calculus}\\ \nabla, \nabla\times, \nabla\cdot\end{array} \to & \begin{array}{c}\textbf{de Rham}\\ (\Omega^\bullet, d)\end{array} \to & \begin{array}{c}\textbf{Mayer–Vietoris}\\ \text{PoU glueing}\end{array} & \\
 & & & \downarrow & \\
\begin{array}{c}\textbf{Riemann–Roch}\\ \ell(D) - i(D) = 1 - g + \deg D\end{array} & \leftarrow & \begin{array}{c}\textbf{Serre duality}\\ H^1(\mathcal{O}(D))^*\end{array} & \leftarrow \begin{array}{c}\textbf{Sheaves}\\ \mathcal{O}(D), \Omega\end{array} &
\end{array}$$

Vector calculus as a cochain complex

On an oriented surface M, smooth differential forms package familiar operators:

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to 0$$

- $d : \Omega^0 \to \Omega^1$ is the *gradient* operator in disguise.
- $d : \Omega^1 \to \Omega^2$ is the *curl* (scalar curl on a surface).
- The *divergence* is the codifferential d^* (adjoint of d via a metric).
- The Laplacian $\Delta = dd^* + d^*d$ yields **Hodge decomposition**:

$$\Omega^1(M) \ = \ \underbrace{d\Omega^0(M)}_{\text{grad part}} \oplus \underbrace{d^*\Omega^2(M)}_{\text{div part}} \oplus \underbrace{\mathcal{H}^1(M)}_{\text{harmonic}}$$

  and $\mathcal{H}^1(M) \cong H^1_{\text{dR}}(M)$ (closed mod exact).

Index-counting slogan: "#(global solutions) $-$ #(global obstructions)" is an Euler characteristic.

Riemann–Roch is precisely such an index formula for $\mathcal{O}(D)$.