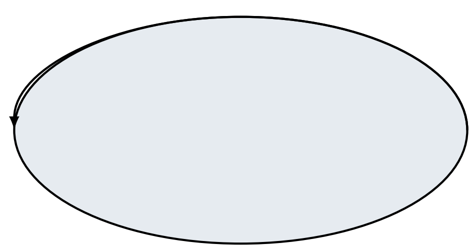
Mathematical Methods

From Vector Calculus to Differential Forms

Lecture Notes & Practice

December 25, 2025

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$



Summary of HW 1, HW 2, and Complex Analysis

Topic 1: Line Integrals of Vector Fields

Given a curve $\gamma : [a, b] \to \mathbb{R}^2$ and a vector field $\mathbf{F}$ defined on a neighborhood of γ, the line integral is defined as:

$$\int_\gamma \mathbf{F} \cdot \mathrm{d}\gamma = \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t$$

Example (The Vortex Field): Consider $\mathbf{F}(x, y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$. Calculate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where C is the unit circle traversed counter-clockwise. *(Space below for derivation showing the result is 2π)*

Topic 2: Surface Integrals and Flux

Let S be a surface parametrized by $T(u, v)$. The surface integral of a vector field $\mathbf{F}$ uses the outward normal vector $\mathbf{N} = T_u \times T_v$:

$$\iint_S \mathbf{F} \cdot \mathrm{d}S := \iint_D \mathbf{F}(T(u, v)) \cdot \left(\frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v} \right) \mathrm{d}A$$

Exercise: Compute the flux for $\mathbf{F} = (x, y, -z)$ over a parametrized surface S.

Topic 3: Stokes' Theorem & Surface Independence

Consider the vector field $\mathbf{F} = (y, xz, 1)$. We observe that $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}S$ yields the **same result** for:

1. The Disk ($x^2 + y^2 \leq 1, z = 0$)
2. The Hemisphere ($x^2 + y^2 + z^2 = 1, z \geq 0$)
3. The Paraboloid ($z = 1 - x^2 - y^2, z \geq 0$)

Why? Because they share the same boundary ∂S. This is the essence of Stokes' Theorem:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}S = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Topic 4: Exterior Derivative & Differential Forms

A k-form is an object that can be integrated over a k-dimensional manifold.

- **0-form:** Smooth function f. $df = \sum \frac{\partial f}{\partial x_i} \mathrm{d}x_i$.
- **1-form:** $\omega = \sum a_i \mathrm{d}x_i$.
- **Exterior Derivative:** $d\omega$ maps k-forms to $(k+1)$-forms.

Property: $d(d\omega) = 0$. This generalizes $\mathrm{curl}(\nabla f) = 0$ and $\mathrm{div}(\mathrm{curl}\,\mathbf{F}) = 0$.
Exercise: Compute $d\eta$ for $\eta = P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$.

Topic 5: Complex Forms & The Cauchy-Green Formula

We can relate real 1-forms to complex differentials ($z = x + iy, \mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$).
The "Vortex Field" $\mathbf{F} = (-y/r^2, x/r^2)$ corresponds to the 1-form:

$$\omega = \frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2 + y^2} = \mathrm{Im}\left(\frac{\mathrm{d}z}{z}\right)$$

This connects vector calculus to the Cauchy Integral Formula:

$$\int_C \frac{1}{z}\mathrm{d}z = 2\pi i$$

(Derive the relationship between the real line integral and the complex contour integral below)