Linear Algebra II

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We cover the following topics in this note.

- Uniqueness of Representation with respect to a Basis; Coordinate
- Linear Transformation
- Vector Space Isomorphism (Linear Isomorphism)
- Classification of Vector Space (up to Isomorphism)
- ~~Matrix Representation of a Linear Transformation~~
- TBA

Uniqueness of Representation with respect to a Basis

Proposition. *Let V be a vector space over a field F and let $\dim V = n < \infty$. Let*

$$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\} \subseteq V$$

be a basis of V. Then for every vector $\mathbf{v} \in V$ there exists a unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that

$$\mathbf{v} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2 + \cdots + \alpha_n \mathbf{b}_n = \sum_{i=1}^{n} \alpha_i \, \mathbf{b}_i.$$

Proof. Suppose, for contradiction, that there exist two distinct representations of some vector $\mathbf{v} \in V$ in terms of the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$:

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \, \mathbf{b}_i \quad \text{and} \quad \mathbf{v} = \sum_{j=1}^{n} \beta_j \, \mathbf{b}_j,$$

where $\alpha_i, \beta_j \in F$ for all i, j. Then

$$\sum_{i=1}^{n} \alpha_i \, \mathbf{b}_i - \sum_{j=1}^{n} \beta_j \, \mathbf{b}_j = \mathbf{0} \implies \sum_{i=1}^{n} (\alpha_i - \beta_i) \, \mathbf{b}_i = \mathbf{0}.$$

Since a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is linearly independent, we have

$$\alpha_i - \beta_i = 0, \quad \text{i.e.,} \quad \alpha_i = \beta_i$$

for all $i = 1, 2, \ldots, n$. Therefore, the representation of any $\mathbf{v} \in V$ as a finite linear combination of elements of the basis $\mathcal{B}$ is unique. □

Coordinate in a Finite-Dimensional Vector Space

Definition. Let V be a vector space over a field F with $\dim V = n < \infty$, and let

$$\mathcal{B} = \{\mathbf{b}_i\}_{i=1}^{n} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$$

be a basis of V. The **coordinate of $\mathbf{v} \in V$ with respect to $\mathcal{B}$**, denoted by $[\mathbf{v}]_{\mathcal{B}}$, is the n-tuple

$$[\mathbf{v}]_{\mathcal{B}} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \quad \text{where } \mathbf{v} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2 + \cdots + \alpha_n \mathbf{b}_n.$$

Remark (Coordinate Function)**.** Let V be a vector space over a field F and let $\mathcal{B} = \{\mathbf{b}_i\}_{i\in I}$ be a (Hamel) basis for V. Then for every vector $\mathbf{v} \in V$, there exists a unique function

$$[\mathbf{v}]_{\mathcal{B}} : \mathcal{B} \longrightarrow F$$

with the finite set $\{\, \mathbf{b} \in \mathcal{B} : [\mathbf{v}]_{\mathcal{B}}(\mathbf{b}) \neq 0 \,\}$ such that

$$\mathbf{v} = \sum_{\mathbf{b}\in\mathcal{B}} [\mathbf{v}]_{\mathcal{B}}(\mathbf{b})\, \mathbf{b}.$$

The function $[\mathbf{v}]_{\mathcal{B}}$ is called the *coordinates of* $\mathbf{v}$ *with respect to the basis* $\mathcal{B}$. In the finite-dimensional case where $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$, the coordinate function $[\mathbf{v}]_{\mathcal{B}}$ is naturally identified with the n-tuple

$$[\mathbf{v}]_{\mathcal{B}} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \quad \text{where} \quad \mathbf{v} = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \cdots + \alpha_n\mathbf{b}_n.$$

Furthermore, the mapping

$$\Phi : V \longrightarrow F^{\mathcal{B}}, \quad \mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}.$$

is a vector space isomorphism, which assigns to each $\mathbf{v} \in V$ its coordinate vector w.r.t. the basis $\mathcal{B}$.

Linear Transformation

Definition. Let V and W be vector spaces over a field F. A function

$$T : V \longrightarrow W$$

is called a **linear transformation** if for all vectors $\mathbf{v}, \mathbf{w} \in V$ and for all scalars $\alpha, \beta \in F$, the following condition holds:

$$T(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha T(\mathbf{v}) + \beta T(\mathbf{v}).$$

Remark. Equivalently, a function $T : V \longrightarrow W$ is linear if it satisfies

(i) (*Additivity*) For all $\mathbf{v}, \mathbf{w} \in V$,

$$T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w});$$

(ii) (*Homogeneity*) For all $\alpha \in F$ and $\mathbf{v} \in V$,

$$T(\alpha\mathbf{v}) = \alpha T(\mathbf{v}).$$

Remark. This definition ensures T preserves the vector space structure of V in its image in W.

Vector Space Isomorphism

Definition. Let V and W be vector spaces over a field F. A mapping

$$T : V \to W$$

is called a **vector space isomorphism** if the followings are satisfied:

(i) (*Linearity*) For any vectors $\mathbf{v}, \mathbf{w} \in V$ and any scalars $\alpha, \beta \in F$,

$$T(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha T(\mathbf{v}) + \beta T(\mathbf{w}).$$

(ii) (*Bijectivity*)

- (*Injectivity*) $\forall \mathbf{v}, \mathbf{w} \in V, T(\mathbf{v}) = T(\mathbf{w}) \implies \mathbf{v} = \mathbf{w}$;
- (*Surjectivity*) $\forall \mathbf{w} \in W, \exists \mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$.

The bijectivity of T guarantees the existence of an inverse mapping $T^{-1} : W \to V$, which satisfies

$$(\forall \mathbf{v} \in V,\ T^{-1}(T(\mathbf{v})) = \mathbf{v}), \quad \text{and} \quad (\forall \mathbf{w} \in W,\ T(T^{-1}(\mathbf{w})) = \mathbf{w}).$$

Remark. The inverse mapping $T^{-1} : W \to V$ is also a linear transformation.

Proof. Let $\mathbf{w}_1, \mathbf{w}_2 \in W$ and let $\alpha, \beta \in F$. Since T is bijective, for each $\mathbf{w} \in W$, there exists a unique $\mathbf{v} \in V$ such that $\mathbf{w} = T(\mathbf{v})$. Define

$$\mathbf{v}_1 = T^{-1}(\mathbf{w}_1) \in V \quad \text{and} \quad \mathbf{v}_2 = T^{-1}(\mathbf{w}_2) \in V.$$

Since T is linear, we have

$$T(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = \alpha T(\mathbf{v}_1) + \beta T(\mathbf{v}_2) = \alpha\mathbf{w}_1 + \beta\mathbf{w}_2.$$

Thus,

$$\begin{aligned} T^{-1}(\alpha\mathbf{w}_1 + \beta\mathbf{w}_2) &= T^{-1}(T(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2)) \\ &= \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 \\ &= \alpha T^{-1}(\mathbf{w}_1) + \beta T^{-1}(\mathbf{w}_2). \end{aligned}$$

□

Remark. When a vector space isomorphism $T : V \to W$ exists, the vector spaces V and W are said to be **isomorphic**, denoted by $V \simeq W$.

Proposition. *Let V and W be vector spaces over a field F with $\dim V < \infty$ and $\dim W < \infty$. The following are equivalent:*

(1) $\dim V = \dim W$

(2) There exists a vector space isomorphism T from V to W

Proof. ((2) $\Rightarrow$ (1)) Assume that there exists a vector space isomorphism $T : V \to W$. Let $\mathcal{B}_V = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be any basis of V. Consider the set

$$\operatorname{Img}(\mathcal{B}_V) = T[\mathcal{B}_V] = \left\{T(\mathbf{v}) : \mathbf{v} \in \mathcal{B}_V\right\} = \left\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\right\} \subseteq W.$$

We claim that $T[\mathcal{B}_V]$ is a basis of W:

- (*Linear Independence*) Suppose that for some finite scalars $\{\alpha_i\}_{i=1}^n \subseteq F$ we have

  $$\alpha_1 T(\mathbf{v}_1) + \alpha_2 T(\mathbf{v}_2) + \cdots + \alpha_n T(\mathbf{v}_n) = \mathbf{0}_W.$$

  By the linearity of T, we obtain $T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n) = \mathbf{0}_W$. Note that $T(\mathbf{0}_V) = T(0\cdot\mathbf{v}) = 0 \cdot T(\mathbf{v}) = \mathbf{0}_W$ for any $\mathbf{v} \in V$. Since T is injective, it follows that

  $$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}_V.$$

  As $\mathcal{B}_V = \mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ is a basis (and hence linearly independent), $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Thus, $T[\mathcal{B}_V]$ is linearly independent.
- (*Spanning Property*) Let $\mathbf{w} \in W$. Since T is surjective, there exists $\mathbf{v} \in V$ such that

  $$T(\mathbf{v}) = \mathbf{w}.$$

  By Uniqueness of Representation w.r.t. a Basis, we know that there exists a unique scalars $\{\alpha\}_{i=1}^n \subseteq F$ such that

  $$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots \alpha_n\mathbf{v}_n.$$

  Then

  $$\mathbf{w} = T(\mathbf{v}) = T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots \alpha_n\mathbf{v}_n) \overset{\text{linearity}}{=} \alpha_1 T(\mathbf{v}_1) + \alpha_2 T(\mathbf{v}_2) + \cdots + \alpha_n T(\mathbf{v}_n) \in \operatorname{span} T[\mathcal{B}_V].$$

  That is, $\mathbf{w} \in W$ is a linear combination of elements of $T[\mathcal{B}_V]$. Therefore, $\operatorname{span} T[\mathcal{B}_V] = W$.

Since $\left|\mathcal{B}_V\right| = \left|T[\mathcal{B}_V]\right| = n$, thus, we have

$$\dim V = \dim W.$$

((1) $\Rightarrow$ (2)) Conversely, assume that $\dim V = \dim W =: n$. Consider bases

$$\mathcal{B}_V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \quad \text{and} \quad \mathcal{B}_W = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$$

for V and W, respectively. By Uniqueness of Representation w.r.t. a Basis, for each vector $\mathbf{v} \in V$, there exists a unique finite scalars $\{\alpha_i\}_{i=1}^n \subseteq F$ such that

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n.$$

Define a mapping $T : V \longrightarrow W$ given by

$$T(\mathbf{v}) = T\left(\sum_{i=1}^n \alpha_i\mathbf{v}_i\right) := \sum_{j=1}^n \alpha_j\mathbf{w}_j.$$

We NTS that T be a one-to-one and onto linear transformation:

(i) (*Linearity*) Let $\mathbf{v}, \mathbf{v}' \in V$ with $\mathbf{v} = \sum_{i=1}^n \alpha_i\mathbf{v}_i$ and $\mathbf{v}' = \sum_{j=1}^n \beta_j\mathbf{v}_j$. For any $\lambda, \mu \in F$, we have

$$\begin{aligned}\lambda\mathbf{v} + \mu\mathbf{v}' = \lambda\sum_{i=1}^n \alpha_i\mathbf{v}_i + \mu\sum_{j=1}^n \beta_j\mathbf{v}_j &= \lambda(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n) + \mu(\beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_n\mathbf{v}_n) \\ &= (\lambda\alpha_1 + \mu\beta_1)\mathbf{v}_1 + (\lambda\alpha_2 + \mu\beta_2)\mathbf{v}_2 + \cdots + (\lambda\alpha_n + \mu\beta_n)\mathbf{v}_n \\ &= \sum_{i=1}^n \gamma_k\mathbf{v}_k \quad \text{where} \quad \gamma_k = \lambda\alpha_k + \mu\beta_k.\end{aligned}$$

By definition of T, we have

$$T(\lambda\mathbf{v} + \mu\mathbf{v}') = \sum_{i=1}^n \gamma_k\mathbf{w}_k = \lambda\sum_{i=1}^n \alpha_i\mathbf{w}_i + \mu\sum_{j=1}^n \beta_j\mathbf{w}_j = \lambda T(\mathbf{v}) + \mu T(\mathbf{v}').$$

(ii) (*Injectivity*) Let $\mathbf{v}, \mathbf{v}' \in V$ with $\mathbf{v} = \sum_{i=1}^n \alpha_i\mathbf{v}_i$ and $\mathbf{v}' = \sum_{j=1}^n \beta_j\mathbf{v}_j$. Suppose $T(\mathbf{v}) = T(\mathbf{v}')$. Then

$$T(\mathbf{v}) - T(\mathbf{v}') = \sum_{i=1}^n \gamma_k\mathbf{w}_k = \mathbf{0}_W, \quad \text{where } \gamma_k = \alpha_k - \beta_k.$$

Since $\mathcal{B}_W = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is a basis of W, the linear independence of $\mathcal{B}_W$ implies that

$$\alpha_k = \beta_k$$

for all $k = 1, 2, \ldots, n$. Thus $\mathbf{v} = \mathbf{v}'$, and so T is injective.

(iii) (*Surjectivity*) Let $\mathbf{w} \in W$. Since $\mathcal{B}_W = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is a basis of W, there exists a unique finite scalars $\{\alpha_i\}_{i=1}^n \subseteq F$ such that

$$\mathbf{w} = \alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2 + \cdots + \alpha_n\mathbf{w}_n.$$

Define a vector

$$\mathbf{v} := \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \sum_{i=1}^n \alpha_i\mathbf{v}_i \in V.$$

Then $T(\mathbf{v}) = \sum_{i=1}^n \alpha_i\mathbf{w}_i = \mathbf{w}$. Thus, T is surjective.

□

Classification of Vector Spaces up to Isomorphism

Theorem. *Let*

$$\mathcal{V}_F := \left\{V : V \text{ is a vector space over a field } F\right\}.$$

Define a relation $\sim$ on $\mathcal{V}_F$ by

$$\forall V, W \in \mathcal{V}_F, \quad V \sim W \iff \exists T \in W^V \quad \text{such that} \quad T \text{ is a vector space isomorphism}.$$

Then

(1) $\sim$ is an equivalence relation on $\mathcal{V}_F$;

(2) For any vector spaces $V, W \in \mathcal{V}_F$, $V \simeq W \iff \dim V = \dim W$.

The isomorphism classes of vector spaces over F are completely determined by their dimensions.

Proof.

(1) We NTS that the relation $\sim$ is reflexive, symmetric, and transitive:

(i) (*Reflexivity*) For each $V \in \mathcal{V}_F$, the identity map $\mathrm{id}_V : V \to V$ is a linear isomorphism, so $V \sim V$.

(ii) (*Symmetry*) If $V \sim W$ via an isomorphism $T : V \to W$, then its inverse $T^{-1} : W \to V$ is also linear, implying $W \sim V$.

(iii) (*Transitivity*) If $V \sim W$ via $T : V \to W$ and $W \sim U$ via $S : W \to U$, then the composition $S \circ T : V \to U$ is a linear isomorphism, so $V \sim U$.

(2) See previous proposition.

□

Corollary. *Let V be a vector space over a field F with* $\dim V = n \in \mathbb{N}$, *and let*

$$F^n = \left\{(x_1, x_2, \dots, x_n) : x_i \in F,\ 1 \le i \le n\right\}$$

is the space of n-tuples over F equipped with the usual operations of vector addition and scalar multiplication. . Then there exists a vector space isomorphism

$$\Phi : V \longrightarrow F^n, \quad \text{i.e.,} \quad V \simeq F^n.$$

Example. Consider the vector space

$$\mathrm{Mat}_{n\times m}(\mathbb{R}) = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} : a_{ij} \in \mathbb{R},\ 1 \le i \le n,\ 1 \le j \le m \right\}$$

which consists of all $n \times m$ matrices with entries in $\mathbb{R}$ (and where the vector space structure is defined over the field $\mathbb{R}$). Also, let

$$\mathbb{R}^{nm} = \left\{(x_1, x_2, \dots, x_{nm}) : x_k \in \mathbb{R},\ 1 \le k \le nm\right\}$$

the vector space of nm-tuples of real numbers, with the usual coordinate-wise addition and scalar multiplication (again, over the field $\mathbb{R}$). Then there exists a vector space isomorphism

$$\Phi : \mathrm{Mat}_{n\times m}(\mathbb{R}) \longrightarrow \mathbb{R}^{nm},$$

i.e., $\mathrm{Mat}_{n\times m}(\mathbb{R}) \simeq \mathbb{R}^{nm}$.

Note. We also denote the set of all $n \times m$ matrices with real entries, namely $\mathrm{Mat}_{n\times m}(\mathbb{R})$ by $\mathbb{R}^{n\times m}$.

Matrix Representation of a Linear Transformation

Definition. TBA

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