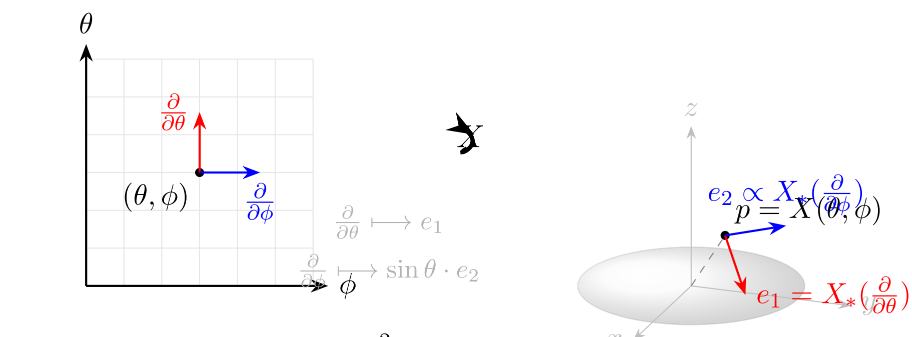
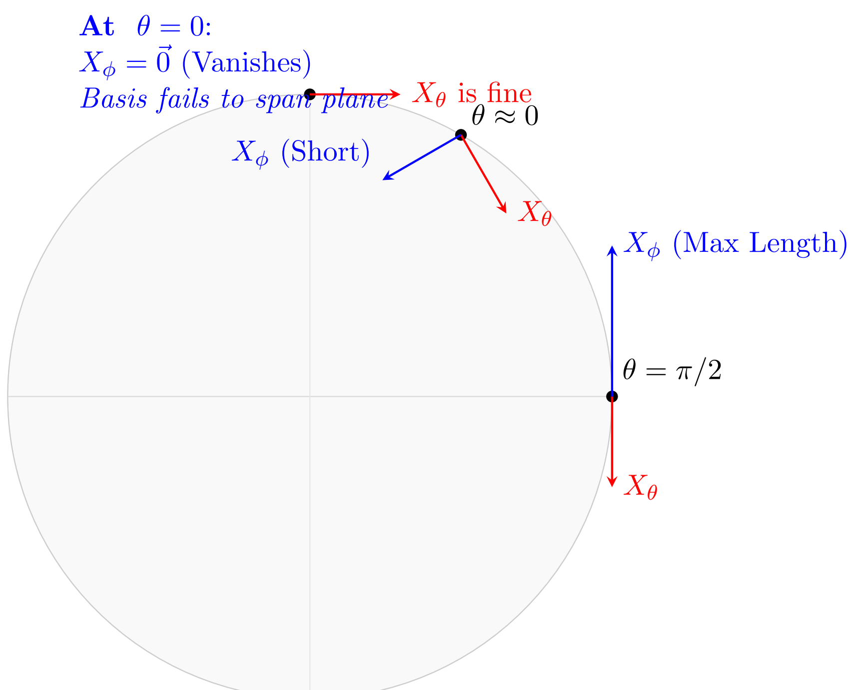
# Shape Operator on $S^2$ in Spherical Coordinates $(\theta, \phi)$




**Parameter Domain** $U \subset \mathbb{R}^2$

**Manifold** $S^2 \subset \mathbb{R}^3$




Condition $0 < \theta < \pi$ ensures
$\sin\theta \neq 0$, so $X_\phi$ exists.

## Goal

Give an explicit $(\theta, \phi)$-based example of the shape operator on $S^2$ that mirrors the linear algebra observation: *choosing a good basis makes a linear map diagonal.*

## 1 Parametrize $S^2$ by $(\theta, \phi)$

Use the standard spherical parametrization

$$X(\theta, \phi) = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \qquad 0 < \theta < \pi, \quad 0 < \phi < 2\pi.$$

The outward unit normal is

$$N(\theta, \phi) = X(\theta, \phi),$$

because

$$X_\theta \times X_\phi = \sin\theta\, X(\theta, \phi)$$

points outward (and $\|X(\theta, \phi)\| = 1$).

## 2 A basis of $T_pS^2$ from the coordinates

At $p = X(\theta, \phi)$, the coordinate tangent vectors are

$$X_\theta(\theta, \phi) = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \qquad X_\phi(\theta, \phi) = \begin{pmatrix} -\sin\theta\sin\phi \\ \sin\theta\cos\phi \\ 0 \end{pmatrix}.$$

They span $T_pS^2$ for $0 < \theta < \pi$.

Their lengths and orthogonality:

$$\|X_\theta\| = 1, \qquad \|X_\phi\| = \sin\theta, \qquad X_\theta \cdot X_\phi = 0.$$

Hence an **orthonormal basis** of $T_pS^2$ is

$$e_1 := X_\theta, \qquad e_2 := \frac{1}{\sin\theta} X_\phi.$$

## 3 Compute $(dN)_p$ explicitly

Since $N(\theta, \phi) = X(\theta, \phi)$, we have

$$N_\theta = X_\theta, \qquad N_\phi = X_\phi.$$

Interpret $(dN)_p$ on the basis vectors using curves.

### Along $e_1 = X_\theta$

Let

$$\gamma_1(t) = X(\theta + t, \phi).$$

Then $\gamma_1(0) = p$ and $\gamma_1'(0) = X_\theta = e_1$. Therefore

$$(dN)_p(e_1) = \frac{d}{dt} N(\gamma_1(t))\Big|_{t=0} = \frac{d}{dt} X(\theta + t, \phi)\Big|_{t=0} = X_\theta = e_1.$$

**Along $e_2 = \frac{1}{\sin\theta} X_\phi$**

Let

$$\gamma_2(t) = X(\theta, \phi + t).$$

Then $\gamma_2(0) = p$ and $\gamma_2'(0) = X_\phi = \sin\theta\, e_2$. Since

$$(dN)_p(X_\phi) = N_\phi = X_\phi,$$

we get

$$(dN)_p(e_2) = \frac{1}{\sin\theta}(dN)_p(X_\phi) = \frac{1}{\sin\theta} X_\phi = e_2.$$

**Matrix of $(dN)_p$ in the orthonormal basis $\{e_1, e_2\}$**

Thus

$$(dN)_p(e_1) = e_1, \qquad (dN)_p(e_2) = e_2,$$

so

$$\boxed{[(dN)_p]_{\{e_1,e_2\}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.}$$

## 4 Shape operator and diagonalization

With the common convention (Weingarten map)

$$S_p = -\,(dN)_p,$$

we obtain

$$S_p(e_1) = -e_1, \qquad S_p(e_2) = -e_2,$$

hence

$$\boxed{[S_p]_{\{e_1,e_2\}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.}$$

This matches the linear algebra observation:

- $S_p = -\mathrm{Id}$ on $T_pS^2$,
- therefore *every* orthonormal basis diagonalizes $S_p$,
- the principal curvatures are $\kappa_1 = \kappa_2 = -1$ (outward normal).

## 5 Intuition in $(\theta, \phi)$-language

Moving in the $\theta$-direction (changing latitude) or the $\phi$-direction (changing longitude), the normal vector $N(\theta, \phi)$ changes at the same rate as the position vector, because $N = X$. So the "normal variation map" $(dN)_p$ acts like the identity, and the shape operator acts like $-I$.