Step 0: What are the two objects?

- $\mathbb{C}(x)$: the field of rational functions in one variable x:

$$\mathbb{C}(x) = \left\{ \frac{p(x)}{q(x)} \;\middle|\; p, q \in \mathbb{C}[x],\ q \not\equiv 0 \right\} \Big/ \sim,$$

  where $\frac{p}{q} \sim \frac{p'}{q'}$ if $p(x)q'(x) = p'(x)q(x)$.

- $\mathcal{M}(\mathbb{CP}^1)$: the field of meromorphic functions on the Riemann sphere $\mathbb{CP}^1$. We view a meromorphic function as a holomorphic map

$$F : \mathbb{CP}^1 \to \mathbb{CP}^1$$

  (finite values in $\mathbb{C}$ are written as $[F(p) : 1]$, poles as $[1 : 0]$).

We want a *field isomorphism*

$$\Phi : \mathbb{C}(x) \xrightarrow{\simeq} \mathcal{M}(\mathbb{CP}^1).$$

Step 1: Fix the affine chart on $\mathbb{CP}^1$

Consider the standard chart

$$U_1 := \{[z_0 : z_1] \in \mathbb{CP}^1 \mid z_1 \neq 0\}, \qquad \phi_1 : U_1 \to \mathbb{C}, \quad \phi_1([z_0 : z_1]) = \frac{z_0}{z_1}.$$

We write

$$x := \phi_1,$$

and think of x as a *coordinate function* on U_1. It extends meromorphically to all of $\mathbb{CP}^1$ with a simple pole at $\infty = [1 : 0]$.

Intuitively: $U_1 \simeq \mathbb{C}$ and x is the coordinate z.

Step 2: From $\mathbb{C}(x)$ to $\mathcal{M}(\mathbb{CP}^1)$

Let

$$R(x) = \frac{p(x)}{q(x)} \in \mathbb{C}(x), \quad p, q \in \mathbb{C}[x],\ q \not\equiv 0.$$

2a. Define F_R on the affine chart U_1. Take a point $[z_0 : z_1] \in U_1$. Write

$$x([z_0 : z_1]) = \phi_1([z_0 : z_1]) = \frac{z_0}{z_1} =: z.$$

We *define* a map F_R on U_1 by

$$\phi_1\big(F_R([z_0 : z_1])\big) = R\big(\phi_1([z_0 : z_1])\big) = R(z),$$

i.e.

$$F_R|_{U_1} = \phi_1^{-1} \circ R \circ \phi_1.$$

Concretely, for $z_1 \neq 0$ and $R(z) \neq \infty$,

$$F_R([z_0 : z_1]) = [R(z_0/z_1) : 1].$$

If $R(z) = \infty$ (i.e. $q(z) = 0$), we set

$$F_R([z_0 : z_1]) = [1 : 0].$$

So on $\mathbb{CP}^1 \setminus \{\infty\} \cong \mathbb{C}$, F_R is just "apply R in the coordinate x".

2b. Extend F_R globally using homogeneous polynomials. Let

$$m = \max\{\deg p, \deg q\},$$

and define homogeneous polynomials of degree m by

$$P(z_0, z_1) = z_1^m \, p\left(\frac{z_0}{z_1}\right), \qquad Q(z_0, z_1) = z_1^m \, q\left(\frac{z_0}{z_1}\right).$$

Then for any $[z_0 : z_1] \in \mathbb{CP}^1$, define

$$F_R([z_0 : z_1]) = \begin{cases} [P(z_0, z_1) : Q(z_0, z_1)], & Q(z_0, z_1) \neq 0, \\ [1 : 0], & Q(z_0, z_1) = 0. \end{cases}$$

Facts:

- $[P : Q]$ is a well-defined point of $\mathbb{CP}^1$ (projective coordinates).
- Since P, Q are homogeneous polynomials, this gives a holomorphic map $\mathbb{CP}^1 \to \mathbb{CP}^1$.
- On U_1, this agrees with $\phi_1^{-1} \circ R \circ \phi_1$.

Thus F_R is a meromorphic function on $\mathbb{CP}^1$, i.e. $F_R \in \mathcal{M}(\mathbb{CP}^1)$.

2c. Define the map Φ. We have a function

$$\Phi : \mathbb{C}(x) \longrightarrow \mathcal{M}(\mathbb{CP}^1), \qquad \Phi(R) = F_R.$$

One checks directly:

$$\Phi(R_1 + R_2) = \Phi(R_1) + \Phi(R_2), \qquad \Phi(R_1 R_2) = \Phi(R_1)\,\Phi(R_2),$$

so Φ is a field homomorphism.

Step 3: Every meromorphic function on $\mathbb{CP}^1$ is rational

Now we go in the other direction.

Let $F \in \mathcal{M}(\mathbb{CP}^1)$, so $F : \mathbb{CP}^1 \to \mathbb{CP}^1$ is holomorphic.

3a. Restrict to the affine chart $\mathbb{CP}^1 \setminus \{\infty\} \cong \mathbb{C}$. On the chart U_1 with coordinate $x = \phi_1$, consider

$$f := \phi_1 \circ F \circ \phi_1^{-1} : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}.$$

In words: write F in the coordinate x. Then f is a meromorphic function on the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

3b. Use the theorem from complex analysis. A standard result: a meromorphic function on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ is a rational function. Therefore there exists some $R(x) \in \mathbb{C}(x)$ such that

$$f(x) = R(x) \quad \text{for all } x \in \mathbb{C} \cup \{\infty\}.$$

Translating back through ϕ_1, this means

$$\phi_1 \circ F \circ \phi_1^{-1} = R \quad \Longleftrightarrow \quad F = \phi_1^{-1} \circ R \circ \phi_1 = F_R.$$

Thus every $F \in \mathcal{M}(\mathbb{CP}^1)$ is of the form F_R for a unique $R \in \mathbb{C}(x)$.

Step 4: Φ is an isomorphism

We already defined

$$\Phi : \mathbb{C}(x) \to \mathcal{M}(\mathbb{CP}^1), \quad R \mapsto F_R.$$

- **Injective:** If $\Phi(R_1) = \Phi(R_2)$, then $F_{R_1} = F_{R_2}$. On $U_1 \simeq \mathbb{C}$,

  $$\phi_1 \circ F_{R_1} = R_1 \circ \phi_1, \qquad \phi_1 \circ F_{R_2} = R_2 \circ \phi_1.$$

  So $R_1 = R_2$ as functions on $\mathbb{C}$, hence as elements of $\mathbb{C}(x)$. Therefore $R_1 = R_2$.

- **Surjective:** Given any $F \in \mathcal{M}(\mathbb{CP}^1)$, Step 3 produced a unique $R \in \mathbb{C}(x)$ with $F = F_R$. So every F is in the image of Φ.

Hence Φ is a bijective field homomorphism:

$$\boxed{\mathbb{C}(x) \;\cong\; \mathcal{M}(\mathbb{CP}^1)}.$$

Step 5: Intuitive summary

- The chart ϕ_1 identifies $\mathbb{CP}^1 \setminus \{\infty\}$ with $\mathbb{C}$.
- A rational function $R(x)$ is just an ordinary meromorphic function on $\mathbb{C} \cup \{\infty\}$.

- Using ϕ_1, we can transport R to a map $F_R : \mathbb{CP}^1 \to \mathbb{CP}^1$, which is meromorphic on the sphere.
- Conversely, any meromorphic map $F : \mathbb{CP}^1 \to \mathbb{CP}^1$ becomes a meromorphic f on $\mathbb{C} \cup \{\infty\}$, hence a rational function $R(x)$.

So "rational functions in the variable x" and "meromorphic functions on the Riemann sphere" are two different *descriptions* of the same field.