Advanced Calculus II

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We cover the following topics in this note.

- Convergence of Real Sequences
- Inequality Rule for Real Sequences
- Limit Theorems for Real Sequences

Sequence

Definition. A **real sequence**(or a sequence in $\mathbb{R}$) is a function defined on the set N whose range is contained in the set $\mathbb{R}$.

Remark. A function a is a real sequence if

$$\begin{array}{rcccl} a & : & \mathbb{N} & \longrightarrow & \mathbb{R} \\ & & n & \longmapsto & a(n) =: a_n \end{array}$$

for $n = 1, 2, \cdots$. We write $\{a_n\}_{n=1}^{\infty}$ (or $\{a_n\}$).

Definition 1 (Convergence). A real sequence $\{a_n\}$ is said to converge to $L \in \mathbb{R}$ or L is said to be a limit of $\{a_n\}$, if $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}$ such that

$$n \geq N(\varepsilon) \implies |a_n - L| < \varepsilon.$$

If a sequence has a limit, we say that the sequence is convergent; if it has no limit, we say that the sequence is divergent.

Remark 1.

$$\begin{aligned} \lim_{n\to\infty} a_n = L &\iff \forall \varepsilon > 0,\ \exists N(\varepsilon) \in \mathbb{N} \text{ such that } n \geq N(\varepsilon) \implies |a_n - L| < \varepsilon \\ &\iff \forall \varepsilon > 0,\ \#\left\{n \in \mathbb{N} : a_n \notin (x - \varepsilon, x + \varepsilon)\right\} < \infty. \end{aligned}$$

Remark 2. We can abbreviate $\lim_{n\to\infty} a_n = L$ to $a_n \to L$.

Definition 2 (Divergence). Let $\{a_n\}$ be a real sequence. We say that

- $\{a_n\}$ **diverges to infinity**(or **tends to infinity**) if $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \implies a_n > M$$

and write

$$\lim_{n\to\infty} a_n = +\infty.$$

- $\{a_n\}$ **diverges to minus infinity**(or **tends to minus infinity**) if $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \implies a_n < M$$

and write

$$\lim_{n\to\infty} a_n = -\infty.$$

- $\{a_n\}$ is **properly divergent** in case we have either

$$\lim_{n\to\infty} a_n = -\infty \text{ or } \lim_{n\to\infty} a_n = -\infty.$$

Remark 3.

$$\lim_{n\to\infty} a_n = +\infty \iff \forall M \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } n \geq N \implies M < a_n$$

$$\lim_{n\to\infty} a_n = -\infty \iff \forall M \in \mathbb{R}, \exists N \in \mathbb{N} \text{ such that } n \geq N \implies a_n < M.$$

0.1 Theorems 1

Theorem 1 (Uniqueness of Limits). *The limit of a real sequence is unique.*

Proof. Let a real sequence $\{a_n\}$ has limit L_1 and L_2, and let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = L_1$,

$$\exists N_1 \in \mathbb{N} : n \geq N_1 \implies |a_n - L_1| < \frac{\varepsilon}{2}.$$

Also we have

$$\exists N_2 \in \mathbb{N} : n \geq N_2 \implies |a_n - L_2| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$, then if $n \geq N$,

$$\begin{aligned}|L_2 - L_1| &= |L_2 - L_1 + a_n - a_n| \\ &= \big|(a_n - L_1) - (a_n - L_2)\big| \\ &\leq |a_n - L_1| + |a_n - L_2| \quad (\because |a-b| \leq |a| + |b|) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

□

Remark 4. Let $\lim_{n\to\infty}$ is a function as $\lim_{n\to\infty} : F_{\text{Con}}$[1] $\to \mathbb{R}$ given by

$$\{a_n\} \mapsto \lim_{n\to\infty} a_n.$$

Then

$$\{a_n\} = \{b_n\} \implies \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

that is, $\lim_{n\to\infty}$ preserves "equality(=)".

References

[1] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 6. 해석학 개론 (c) 수열의 수렴성." YouTube Video, 26:29. Published September 20, 2019. URL: `https://www.youtube.com/watch?v=jwLfzJyIxmU`.

[2] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 7. 해석학 개론 (d) 극한 정리" YouTube Video, 26:46. Published September 26, 2019. URL: `https://www.youtube.com/watch?v=1TRD34QbIaw`.

[1] $F_{\text{Con}} := \big\{\text{Convergent sequences on } \mathbb{R}\big\}$