

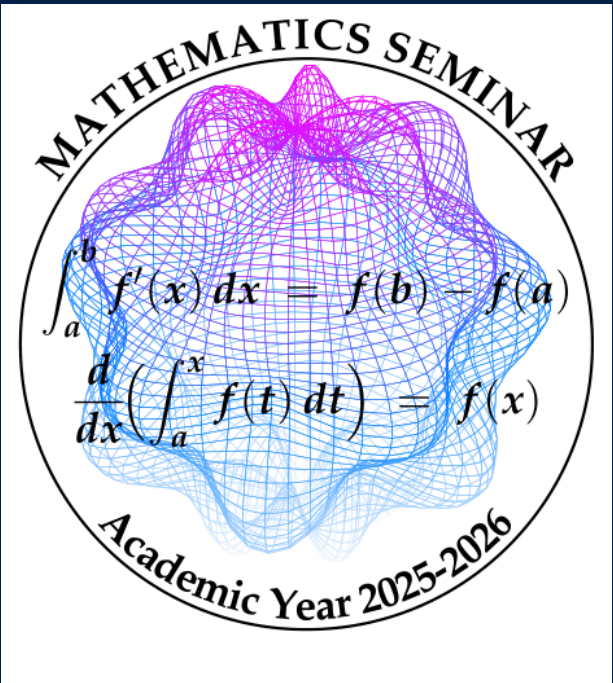
# Riemann-Roch Theorem and its Application

## - From de Rham to sheaf cohomology, Serre duality -

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### Preliminaries I: Vector calculus as a cochain complex

On  $\mathbb{R}^3$ , the familiar operators form a complex:

$$C^\infty \xrightarrow{\nabla} \Gamma(T^*) \xrightarrow{\nabla \times} \Gamma(\Lambda^2 T^*) \xrightarrow{\nabla \cdot} C^\infty,$$

with  $(\nabla \times) \circ \nabla = 0$ ,  $(\nabla \cdot) \circ (\nabla \times) = 0$ . The de Rham complex is  $(\Omega^\bullet(M), d)$ :

$$0 \rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots$$

Cohomology measures global obstructions to solving  $d\eta = \omega$ .

### Preliminaries II: Local-to-global glueing: Mayer–Vietoris

For  $M = U \cup V$ , we uses a partition of unity to build the short exact sequence

$$0 \rightarrow \Omega^k(M) \rightarrow \Omega^k(U) \oplus \Omega^k(V) \rightarrow \Omega^k(U \cap V) \rightarrow 0,$$

and then the long exact Mayer–Vietoris sequence in cohomology.

This means that “curl-free locally” does not imply “gradient globally” when topology obstructs global potentials.

Given  $M = U \cup V$ , we uses the short exact sequence of de Rham complexes

$$0 \rightarrow \Omega^k(M) \xrightarrow{\alpha^k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\beta^k} \Omega^k(U \cap V) \rightarrow 0$$

with  $\beta^k(\omega_U, \omega_V) = \omega_U|_{U \cap V} - \omega_V|_{U \cap V}$ , and surjectivity is built using a partition of unity  $\{\rho_U, \rho_V\}$ .

**Why this belongs on a Riemann–Roch:**

- Fine sheaves (built from partitions of unity) let we compute sheaf cohomology via resolutions.
- This is the conceptual bridge from `grad/curl/div` (de Rham) to  $\bar{\partial}$ - and sheaf-cohomological formulas that culminate in Riemann–Roch.

### Preliminaries III: From de Rham to Complex Geometry

On a compact Riemann surface  $M$ , complex structure refines  $d = \partial + \bar{\partial}$ . Holomorphic data live in sheaves:

$$\mathcal{O} \text{ (holomorphic functions),} \quad \Omega \text{ (holomorphic 1-forms).}$$

Divisors  $D = \sum_p n_p p$  encode zeros/poles, and determine a line bundle  $\mathcal{O}(D)$ :

$$\mathcal{O}(D)(V) = \{TBA\}$$

### Core I: Riemann–Roch (Curves)

Let  $M$  be a compact Riemann surface of genus  $g$ , and  $D$  a divisor. Define

$$\ell(D) = \dim H^0(M, \mathcal{O}(D)), \quad i(D) = \dim H^0(M, \Omega(-D)).$$

**Riemann–Roch:**

$$\boxed{\ell(D) - i(D) = 1 - g + \deg D} \quad \text{equivalently} \quad i(D) = \ell(K - D).$$

### Core II: Cohomological mechanism: Euler characteristic + Serre duality

We derives Riemann–Roch from the Euler characteristic of  $\mathcal{O}(D)$ :

$$\chi(\mathcal{O}(D)) := \dim H^0(M, \mathcal{O}(D)) - \dim H^1(M, \mathcal{O}(D)) = \ell(D) - i(D),$$

using Serre duality to identify  $H^1(M, \mathcal{O}(D))^* \simeq I(D)$ .

**Key principle:**

$$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}_M) + \deg D, \quad \chi(\mathcal{O}_M) = 1 - g.$$

So  $\chi$  is “topology + degree,” while  $\ell(K - D)$  is the global obstruction.

### Examples

**Sphere  $\mathbb{P}^1$**  ( $g = 0$ ,  $K = -2[\infty]$ ). For  $D = n[\infty]$ ,  $\ell(D) = n+1$  for  $n \geq -1$  (polynomials of degree  $\leq n$ ).

**Elliptic curve** ( $g = 1$ ,  $\deg K = 0$ ). Riemann–Roch becomes  $\ell(D) - i(D) = \deg D$ , and for  $\deg D > 0$  one has  $\ell(D) = \deg D$ .

### Topology link: Gauss–Bonnet parallel

We emphasizes the analogy with Gauss–Bonnet:

$$\int_{C_{\text{mes}}} \omega = 2\pi \chi(M) = 2\pi(2 - 2g).$$

### Applications I: Algebraic–geometric (AG) codes from Riemann–Roch

Let  $X/\mathbb{F}_q$  be a smooth projective curve of genus  $g$ . Choose:

$$D = P_1 + \dots + P_n \text{ (distinct rational points),}$$

$$G \text{ (divisor, } \text{supp}(G) \cap \text{supp}(D) = \emptyset \text{).}$$

Define the evaluation code

$$C_L(D, G) = \{ (f(P_1), \dots, f(P_n)) : f \in L(G) \} \subseteq \mathbb{F}_q^n,$$

$$L(G) = H^0(X, \mathcal{O}(G)).$$

**Dimension:**  $k = \dim C_L(D, G) = \ell(G) - \ell(G - D)$ , and for  $\deg G$  sufficiently large,

$$\ell(G) = \deg G + 1 - g,$$

as large-degree/embedding discussion.

**Designed distance:**  $d \geq n - \deg G$ . (Standard AG-code bound.)

### Example for Applications I: Classic McEliece and FALOMA

### Applications II: “Surface direction” via Riemann–Roch for surfaces

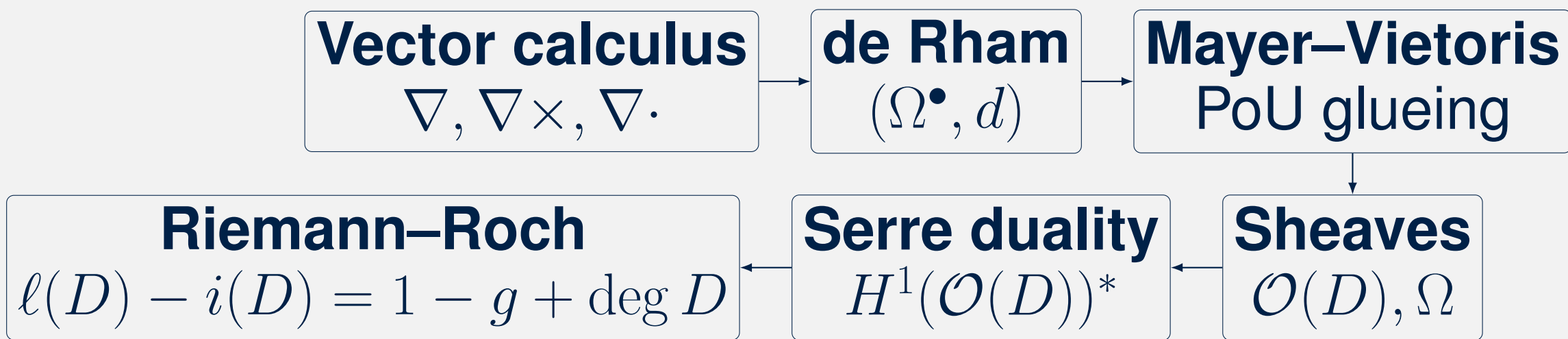
We states the surface analogue (for a smooth projective surface  $X$  and divisor  $D$ ):

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2}(D \cdot D - D \cdot K_X),$$

where  $\cdot$  is the intersection pairing and  $K_X$  is the canonical divisor.

- AG surface codes:** evaluate  $H^0(X, \mathcal{O}_X(D))$  on rational points/curves; use the above  $\chi$  plus vanishing theorems to estimate  $\dim H^0$ .
- Topological/surface codes (quantum):** homological dimension depends on  $H_1$  (genus); the same genus parameter appears in curve Riemann–Roch and in the obstruction term.

### The pipeline: Calculus (grad/curl/div) $\rightarrow$ Riemann-Roch



### Vector calculus as a cochain complex

On an oriented surface  $M$ , smooth differential forms package familiar operators:

$$0 \rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \rightarrow 0$$

- $d : \Omega^0 \rightarrow \Omega^1$  is the *gradient* operator in disguise.
- $d : \Omega^1 \rightarrow \Omega^2$  is the *curl* (scalar curl on a surface).
- The *divergence* is the codifferential  $d^*$  (adjoint of  $d$  via a metric).
- The Laplacian  $\Delta = dd^* + d^*d$  yields **Hodge decomposition**:

$$\Omega^1(M) = \underbrace{d\Omega^0(M)}_{\text{grad part}} \oplus \underbrace{d^*\Omega^2(M)}_{\text{div part}} \oplus \underbrace{\mathcal{H}^1(M)}_{\text{harmonic}}$$

and  $\mathcal{H}^1(M) \cong H_{\text{dR}}^1(M)$  (closed mod exact).

**Index-counting slogan:** “#(global solutions) – #(global obstructions)” is an Euler characteristic.

Riemann–Roch is precisely such an index formula for  $\mathcal{O}(D)$ .