Set Theory I

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Terminology.

- Set; Collection; Family.
- Tabular (or Roster) Form
$$A = \{0, 2, 4, 8\}.$$
- Set-builder Form
$$A = \{x : x \text{ is even and } x < 10\}.$$

Example.

- $\mathbb{N} = \{1, 2, \ldots\}$
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{Q} = \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\}, \gcd(p, q) = 1\right\}$
- $\mathbb{R} = \{x : x \text{ is a real number}\}$
- $\mathbb{C} = \left\{a + bi : a, b \in \mathbb{R}, i^2 = -1\right\}$

Exercise. Show that $\sqrt{2}$ is irrational.

Sol.

$\square$

Subset and Set Equality

Definition. Let A and B are sets.

- Subset: $B \subseteq A \overset{\text{def}}{\iff} (x \in B \Rightarrow x \in A)$.
- Set Equality:

$$\begin{aligned} A = B &\overset{\text{def}}{\iff} A \subseteq B \wedge B \subseteq A \\ &\iff (x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A). \end{aligned}$$

Power Set

Definition. The **power set** of a set X is the set of all subsets of X.

$$\mathcal{P}(X) = 2^X := \{S : S \subseteq X\}.$$

Cartesian Product

Definition. Let A and B are sets. The **cartesian product** of A and B is the set

$$A \times B = \big\{(a, b) : a \in A \wedge b \in B\big\}.$$

Union, Intersection and Complement

Definition. Let U is a universal set, and let $A, B \subseteq U$.

- The **union** of A and B is the set

$$\boxed{A \cup B := \{x \in U : x \in A \vee x \in B\}}.$$

Note that $x \in A \cup B \iff x \in A \vee x \in B$.

- The **intersection** of A and B is the set

$$\boxed{A \cap B := \{x \in U : x \in A \wedge x \in B\}}.$$

Note that $x \in A \cap B \iff x \in A \wedge x \in B$.

- The **complement** of A is the set

$$\boxed{A^C := \big\{x \in U : \neg(x \in A)\big\} = \{x : x \notin A\}}.$$

Note that $x \in A^C \iff x \notin A$.

Proposition 1. *Let $A, B, C \subseteq U$.*

(1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(3) $(A \cup B)^C = A^C \cap B^C$.

(4) $(A \cap B)^C = A^C \cup B^C$.

Proof.

□

Exercise. Let A has n elements. Show that $\mathcal{P}(A)$ has 2^n elements.

Sol.

$\square$

Function

Definition. Let A and B are sets. A **function** $f \subseteq A \times B$ **from** A **to** B is a relation on $A \times B$ satisfying as follows:

(i) Every element of A relates to some element of B.

$$\boxed{\forall a \in A : \exists b \in B \text{ such that } (a, b) \in f.}$$

(ii) Every element of A relates to no more than one element of its B.

$$\boxed{\forall a \in A : \forall b_1, b_2 \in B : (a, b_1), (a, b_2) \in f \implies b_1 = b_2.}$$

Remark. A **function** f from a set A to a set B, written as

$$f : A \longrightarrow B,$$

is defined as a relation that assigns to each element $a \in A$ *exactly one* element $b \in B$. More formally, f is a subset of the Cartesian product $A \times B$ with the property that for every $a \in A$, there exists a unique $b \in B$ such that

$$(a, b) \in f.$$

This uniqueness is essential: it means that an element in the domain cannot be assigned more than one output in the codomain.

Remark. A relation $f \subseteq A \times B$ is a function if $\boxed{\forall a \in A : \exists! b \in B : (a, b) \in f.}$

- The **domain** of f is $\mathrm{Dom}(f) = A$.
- The **codomain** of f is $\mathrm{Cdm}(f) = B$.

- The **image of A under f** is the set

$$\begin{aligned}\mathrm{Img}(f) = f[A] &:= \{b \in B : \exists a \in A \text{ s.t. } (a,b) \in f\} \\ &= \{b \in B : \exists a \in A \text{ s.t. } f(a) = b\} \\ &= \{b \in B : b = f(a) \text{ for at least one } a \in A\}.\end{aligned}$$

Simply we can express it as $f[A] = \{f(a) \in B : a \in A\}$.

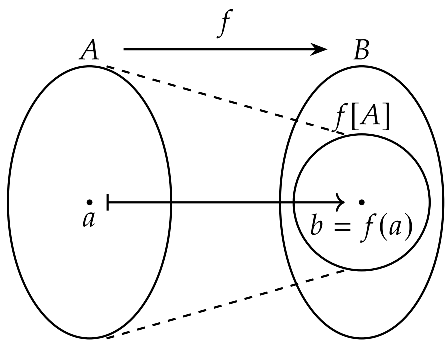


Figure 1: Image of A under f.

Note that $f[A] \subseteq B = \mathrm{Cdm}(f)$ and that $b \in f[A] \iff b = f(a)$ for some $a \in A$.

- The **preimage of $B_1 \subseteq B$ under f** is the set

$$\begin{aligned}f^{-1}[B_1] &:= \{a \in A : \exists b \in B_1 \text{ s.t. } (a,b) \in f\} \\ &= \{a \in A : \exists! b \in B_1 \text{ s.t. } b = f(a)\} \text{ by def. of a function} \\ &= \{a \in A : f(a) = b \text{ for exactly one } b \in B_1\}.\end{aligned}$$

"Exactly one" ensures a unique assignment for every element of A, while "at most one" allows no assignment. Simply we can express it as $f^{-1}[B_1] = \{a \in A : f(a) \in B_1\}$.

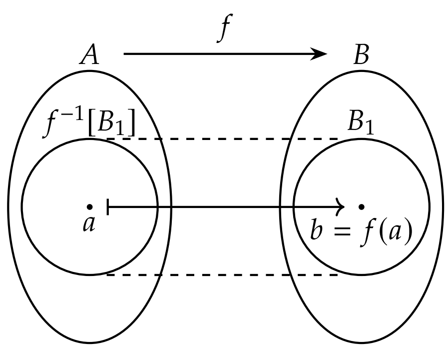


Figure 2: Preimage of $B_1 \subseteq B$ under f.

Note that $f^{-1}[B_1] \subseteq A = \mathrm{Dom}(f)$ and that $a \in f^{-1}[B_1] \iff f(a) \in B_1$.

Proposition 2. *Let $f : A \longrightarrow B$ be a function from A to B, and let $A_1, A_2 \subseteq A$.*

(1) $f[A_1 \cup A_2] = f[A_1] \cup f[A_2]$.

(2) $f[A_1 \cap A_2] \subseteq f[A_1] \cap f[A_2]$.

Proof.

□

Proposition 3. *Let $f : A \longrightarrow B$ be a function from A to B, and let $B_1, B_2 \subseteq B$.*

(1) $f^{-1}[B_1 \cup B_2] = f^{-1}[B_1] \cup f^{-1}[B_2]$.

(2) $f^{-1}[B_1 \cap B_2] = f^{-1}[B_1] \cap f^{-1}[B_2]$.

(3) $f^{-1}[B_1^C] = (f^{-1}[B_1])^C$.

Proof.

$\square$

Proposition 4. *Let $f : A \longrightarrow B$ be a function from A to B. Let $A_1 \subseteq A$ and $B_1 \subseteq B$.*

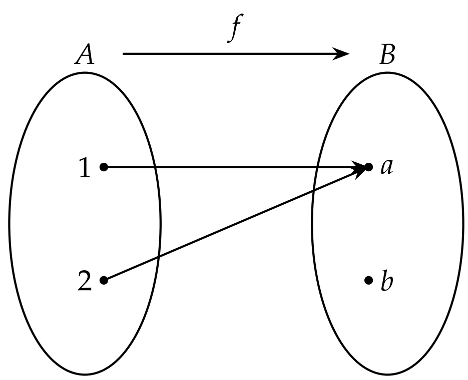
(1) $f[f^{-1}[B_1]] \subseteq B_1$.

(2) $A_1 \subseteq f^{-1}[f[A_1]]$.

Proof.

$\square$

Example (Counterexample). Consider a function $f : A \longrightarrow B$, where $A = \{1, 2\}$ and $B = \{a, b\}$.



(1) Let $B_1 = \{b\} \subseteq B$. Then $f^{-1}[B_1] = \varnothing$ and so

$$f[f^{-1}[B]] = f[\varnothing] = \varnothing \neq \{b\} = B_1.$$

(2) Let $A_1 = \{1\} \subseteq A$. Then $f[A_1] = f[\{1\}] = \{a\}$ and so

$$f^{-1}[f[A_1]] = f^{-1}[\{a\}] = \{1, 2\} \neq \{1\} = A_1.$$

Injection and Surjection

Definition. Let $f : A \longrightarrow B$ is a function from A to B.

- A function f is **an injection** or **injective** (or **one-to-one**) if and only if

  $$\boxed{\forall a_1, a_2 \in A,\ f(a_1) = f(a_2) \implies a_1 = a_2.}$$

  An **injection** is a function s.t. the output uniquely determines its input.

- A function f is **a surjection** or **surjective** (or **onto**) if and only if

  $$\boxed{\forall b \in B,\ \exists a \in A \text{ such that } f(a) = b.}$$

  A **surjection** is a function s.t. every element of B is related to by some element of A.

Remark. A function f is **bijective** if and only if f is both injective and surjective.

- f is **a bijection** (or **bijective**).
- f is **one-to-one and onto** (or **a one-to-one correspondence**).

Remark. Let $f : A \longrightarrow B$ be a function from a set A to a set B.

- (Distinct Images) f is injective $\iff \forall a \in A,\ \mathrm{Img}\left(\{a\}\right) \cap \mathrm{Img}\left(A \setminus \{a\}\right) = \varnothing$.
- (Every Codomain Element has a Preimage) f is surjective $\iff \forall b \in B,\ \mathrm{Img}^{-1}\left(\{b\}\right) \neq \varnothing$.

Composition of Functions

Definition. Let $f_1 : A \longrightarrow B$ and $f_2 : B \longrightarrow C$ be functions such that $\mathrm{Cdm}(f_1) = B = \mathrm{Dom}(f_2)$. The **composition** $f_2 \circ f_1$ is defined as:

$$(f_2 \circ f_1)(a) := f_2(f_1(a)) \quad \text{for all } a \in A.$$

Note (Diagram)**.**

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[dr, dashed, "f_2\circ f_1"'] & B \arrow[d, "f_2"] \\ & C \end{tikzcd}$$

Figure 3: Diagram of $f_2 \circ f_1$.

Note (Illustration)**.**

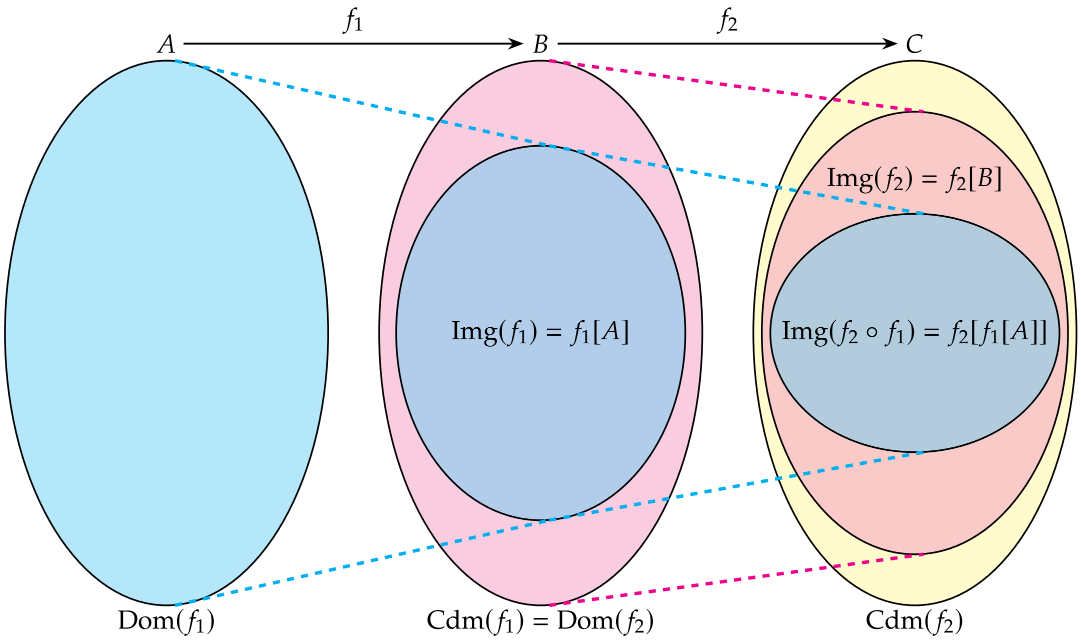


Figure 4: Illustration of $f_2 \circ f_1$

Remark. The composition is associative. For any $f, g, h \in G$, $(f \circ g) \circ h = f \circ (g \circ h)$.

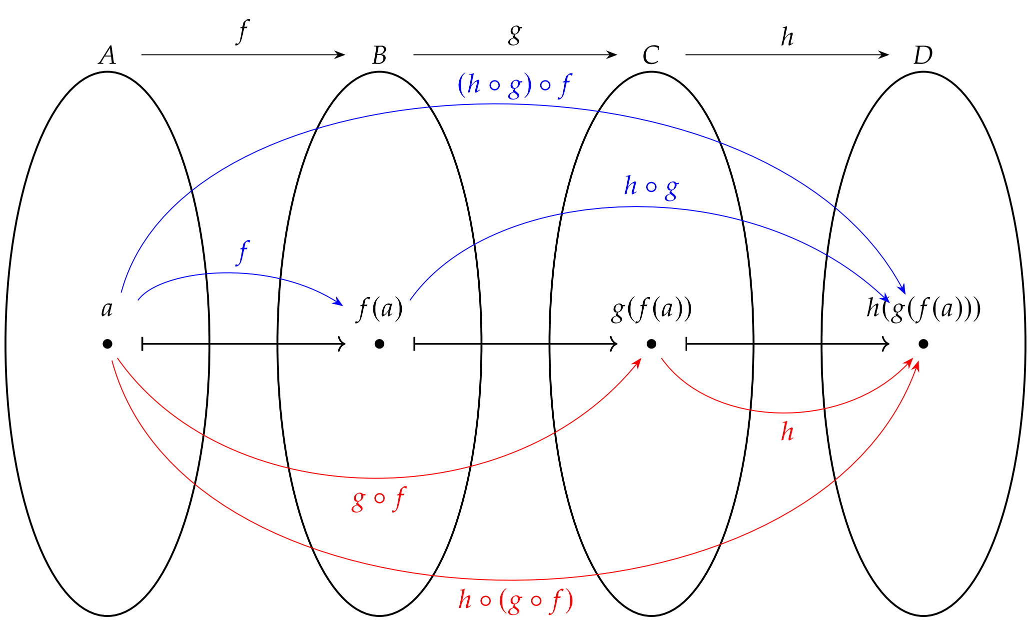


Figure 5: Associativity of Composition.

Theorem 5. *Let A and B are sets. Let $f : A \longrightarrow B$ be a function.*

(1) f is one-to-one if and only if there exists the function $g : B \longrightarrow A$ such that $g \circ f = \mathrm{id}_A$.

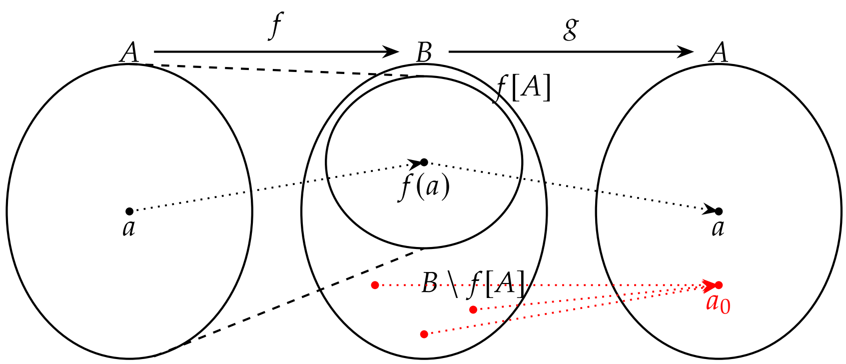
(2) f is onto if and only if there exists the function $g : B \longrightarrow A$ such that $f \circ g = \mathrm{id}_B$.

Remark.

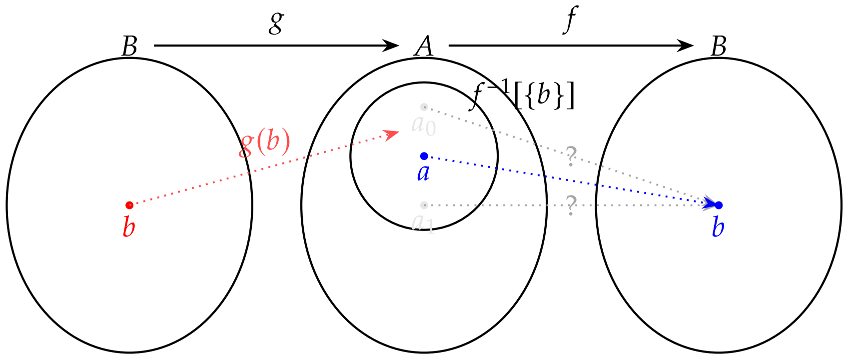
(1)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f=\mathrm{id}_A"'] & B \arrow[r, "\exists g"] & A \end{tikzcd}$$

(2)
$$\begin{tikzcd} B \arrow[r, "\exists g"] \arrow[rr, bend right, "f\circ g=\mathrm{id}_B"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Proof. (1)



(2)



$\square$

Note (Axiom of Choice). Let $\{X_i\}_{i\in I}$ be a family of non-empty sets.

"It is always possible to assert the existence of a choice function
that selects one element from each member of the set."

Formally,

$$\forall \{X_i\}_{i\in I} : \left[\varnothing \notin \{X_i\}_{i\in I} \implies \exists \left(f : \{X_i\}_{i\in I} \to \bigcup_{i\in I} X_i \right) \text{ s.t. } \forall A \in \{X_i\}_{i\in I} \, , \; f[A] \in A \right].$$

For example, let $\{X_i\}_{i\in I} = \{A, B, C, \dots\}$ and $\bigcup_{i\in I} X_i = A \cup B \cup C \cup \cdots$.