

# Multi-variable Calculus

- HW1 -

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$$\underbrace{f}_{\Omega^0} \xrightarrow{d} \underbrace{df}_{\Omega^1} \longleftrightarrow \underbrace{\nabla f}_{\text{gradient vector field}} \longrightarrow \underbrace{\mathbf{F}}_{(\Omega^0)^m} \xrightarrow{d} \underbrace{d\mathbf{F}}_{\Omega^1 \otimes \mathbb{R}^m} \longleftrightarrow \underbrace{D\mathbf{F}}_{\text{Jacobian matrix}}$$

We cover the following topics in this note.

- Vector Fields
- Line Integrals for Vector Fields
- Surface Integrals for Vector Fields
- TBA

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## Line Integrals

### Line Integral of Scalar Function over Arc Length

For a curve  $\gamma: \mathbb{R} \rightarrow \mathbb{R}^2: t \mapsto \langle x(t), y(t) \rangle$ , the **secant vector** over  $[t, t + \Delta t]$  is

$$\frac{\Delta \gamma}{\Delta t} = \frac{\gamma(t + \Delta t) - \gamma(t)}{\Delta t} = \left\langle \frac{x(t + \Delta t) - x(t)}{\Delta t}, \frac{y(t + \Delta t) - y(t)}{\Delta t} \right\rangle.$$

As  $\Delta t \rightarrow 0$ , these secants converge (if  $\gamma$  is smooth) to

$$\begin{aligned} \gamma'(t) &= \frac{d}{dt} \gamma(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \gamma}{\Delta t} = \left\langle \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}, \lim_{\Delta t \rightarrow 0} \frac{y(t + \Delta t) - y(t)}{\Delta t} \right\rangle \\ &= \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle \\ &= \langle x'(t), y'(t) \rangle, \end{aligned}$$

which gives the **tangent vector** at  $\gamma(t)$ . The tangent vector captures how the curve is moving instantaneously at time  $t$ .

By Pythagoras' theorem, the **length moved per unit time** is  $\|\gamma'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$ , and so the small arc length traveled between  $t$  and  $t + \Delta t$  is approximately:

$$\|\gamma'(t)\| \Delta t = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot \Delta t.$$

### Arc Length of a Parametrized Curve

**Definition.** Let  $C \subset \mathbb{R}^n$  be a piecewise smooth curve, given by a smooth parameterization:

$$\gamma : [a, b] \rightarrow \mathbb{R}^n, \quad t \mapsto \gamma(t) = \langle x_1(t), x_2(t), \dots, x_n(t) \rangle.$$

Then the **arc length**  $s$  of the curve  $C$  from  $t = a$  to  $t = b$  is defined by

$$s := \int_a^b \|\gamma'(t)\| dt, \quad \text{where } \|\gamma'(t)\| = \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \dots + \left(\frac{dx_n}{dt}\right)^2}.$$

**Remark.** Let  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  be a piecewise- $C^1$  curve,  $\gamma(t) = (x_1(t), \dots, x_n(t))$ . A arc length function is defined by

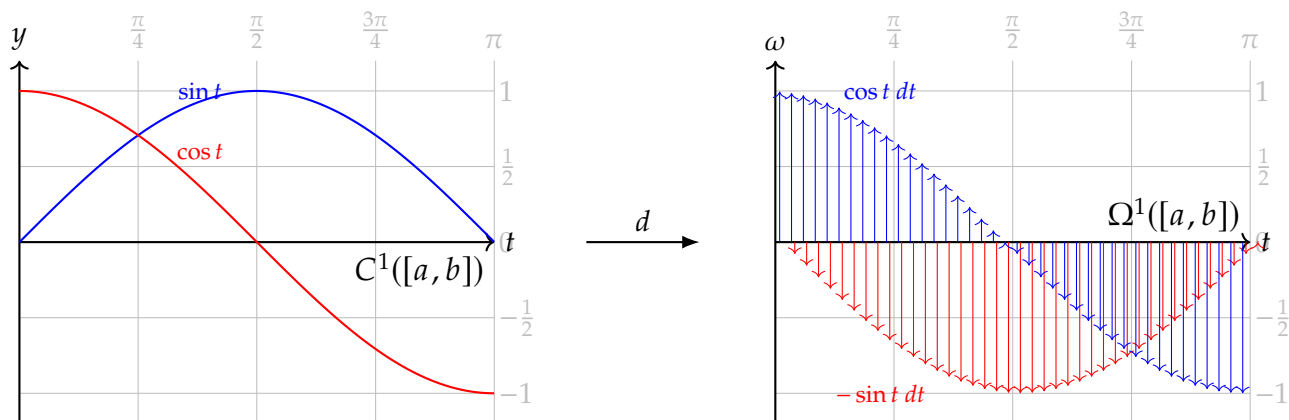
$$s : [a, b] \rightarrow \mathbb{R}, \quad t \mapsto s(t) = \int_a^t \|\gamma'(u)\| du,$$

where  $\|\gamma'(u)\| = \sqrt{\sum_{i=1}^n (x'_i(u))^2}$ . Define two sets:

$$C^1([a, b]) = \left\{ f \in \mathbb{R}^{[a, b]} : f \text{ is continuously differentiable on } [a, b] \right\}$$

$$\Omega^1([a, b]) = \left\{ \delta(t) dt : \delta \in \mathbb{R}^{[a, b]} \text{ is continuous and } t \in [a, b] \right\} = \left\{ \delta(t) dt : \delta \in C^0([a, b]) \right\}.$$

Here  $s \in C^1([a, b])$  with  $s'(t) = \frac{d}{dt} \left( \int_a^t \|\gamma'(u)\| du \right) \stackrel{\text{FTC}}{=} \|\gamma'(t)\|$ .

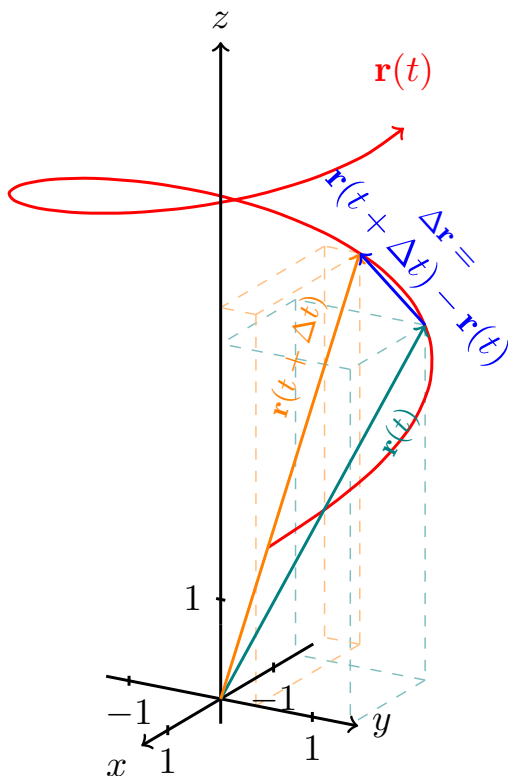


The map

$$\begin{aligned} d : C^1([a, b]) &\longrightarrow \Omega^1([a, b]) \\ f(t) &\longmapsto d(f(t)) = df \end{aligned}$$

is defined by  $df = f'(t)dt$ , where  $f'$  is the derivative of  $f$ . Thus

$$ds := d(s(t)) = s'(t) dt = \|\gamma'(t)\| dt.$$



$$\mathbf{r} : \mathbb{R} \longrightarrow \mathbb{R}^3$$

$$t \longmapsto \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \langle x'(t), y'(t), z'(t) \rangle$$

$$s(t) = \int_a^t \|\mathbf{r}'(t)\| \, dt$$

$$s'(t) = \|\mathbf{r}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

$$ds = d(s(t)) = s'(t) \, dt = \|\mathbf{r}'(t)\| \, dt$$

### Line Integral of Scalar Function over Arc Length

**Definition.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a scalar function, and let  $C$  be a piecewise smooth curve in  $\mathbb{R}^n$  given by a smooth parameterization:

$$\gamma : [a, b] \rightarrow \mathbb{R}^n, \quad t \mapsto \gamma(t) = \langle x_1(t), x_2(t), \dots, x_n(t) \rangle \in \mathbb{R}^n = \text{dom}(f).$$

The **line integral of the scalar function**  $f$  along the curve  $C$  with respect to arc length is defined by

$$\int_C f \, ds := \int_a^b f(\gamma(t)) \|\gamma'(t)\| \, dt.$$

## Line Integral of Vector Fields

### Line Integral of a Vector Field in $\mathbb{R}^2$

**Definition.** Let  $C$  be a smooth curve parametrized by

$$\gamma : [a, b] \rightarrow \mathbb{R}^2, \quad t \mapsto \gamma(t) = \langle x(t), y(t) \rangle.$$

Let  $\mathbf{F} = \langle F_1, F_2 \rangle$  be a smooth vector field on  $\mathbb{R}^2$ . The **line integral of the vector field  $\mathbf{F}$**  along the curve  $\gamma$  is defined by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t) dt.$$

Alternatively,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (F_1, F_2) \cdot (dx, dy) = \int_C F_1 dx + F_2 dy.$$

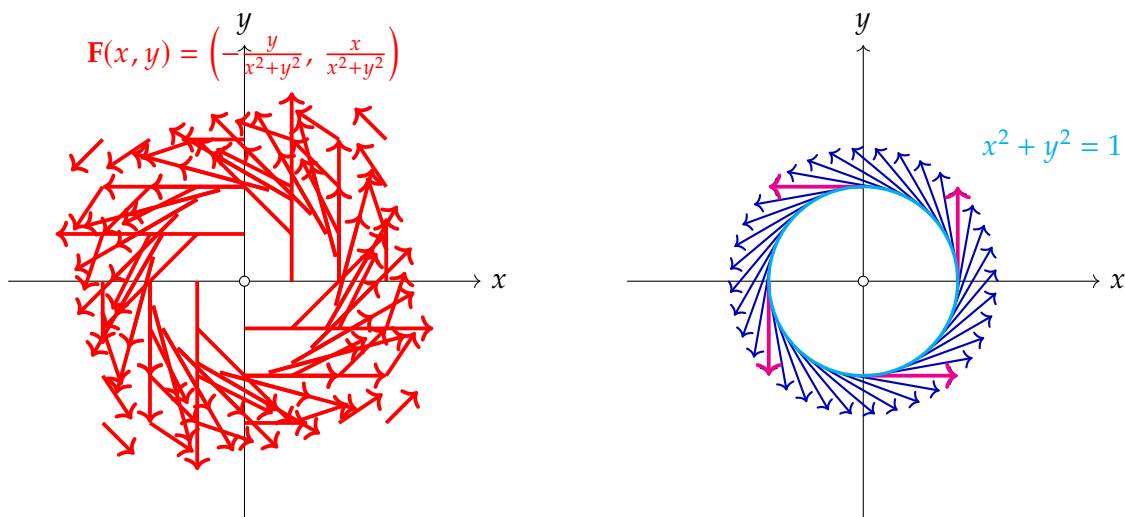
**Problem #1 (Line Integral around Unit Circle).** Let  $C \subset \mathbb{R}^2$  be the unit circle defined by  $C : x^2 + y^2 = 1$ , traversed in the **counterclockwise direction**. Let the vector field  $\mathbf{F} : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}^2$  be defined by

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

Evaluate the **line integral** of  $\mathbf{F}$  along  $C$ :

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

**Sol.**

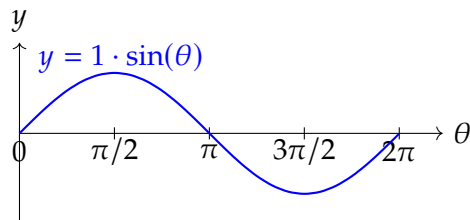
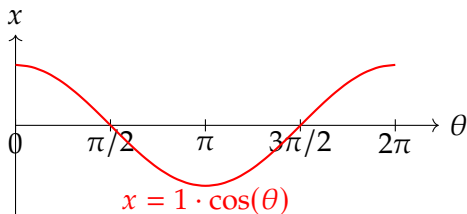


Consider the vector field  $\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ , and the curve  $C$  is the unit circle  $x^2 + y^2 = 1$ , traversed counterclockwise.

**(Parametrization)** Define a function

$$\begin{aligned} \gamma &: [0, 2\pi] \longrightarrow \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \\ \theta &\longmapsto \gamma(\theta) = (\cos \theta, \sin \theta) \end{aligned}.$$

Here,  $\frac{d\gamma}{d\theta} = (-\sin \theta, \cos \theta)$ .



**(Evaluate  $\mathbf{F}(\gamma(\theta))$  and the dot product)** We have

$$\mathbf{F}(\gamma(\theta)) = \mathbf{F}(\cos \theta, \sin \theta) \stackrel{\sin^2 \theta + \cos^2 \theta = 1}{=} \left\langle \frac{-\sin \theta}{1}, \frac{\cos \theta}{1} \right\rangle = (-\sin \theta, \cos \theta).$$

and

$$\mathbf{F}(\gamma(\theta)) \cdot \frac{d\gamma}{d\theta} = (-\sin \theta)(-\sin \theta) + (\cos \theta)(\cos \theta) = \sin^2 \theta + \cos^2 \theta = 1.$$

**(Integral)**

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\gamma(\theta)) \cdot \frac{d\gamma}{d\theta} d\theta = \int_0^{2\pi} 1 d\theta = 2\pi.$$

□

## Surface Integral for Vector Fields

**Problem #2 (Surface-Flux).** Compute the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where  $\mathbf{F}(x, y, z) = \langle x, y, -z \rangle$  and the surface  $S$  is parametrized by

$$\mathbf{r}(u, v) = \langle u + 2v, 2u + v, 3uv \rangle, \quad (u, v) \in [0, 1] \times [0, 1].$$

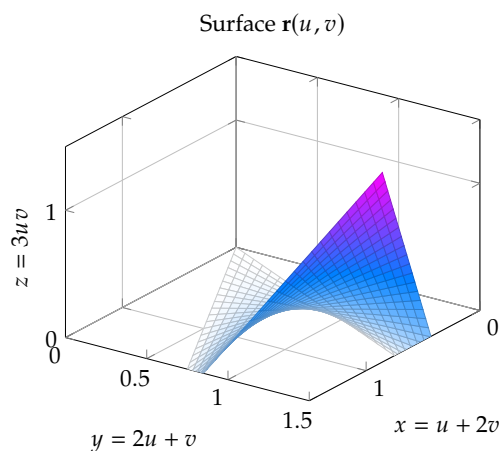
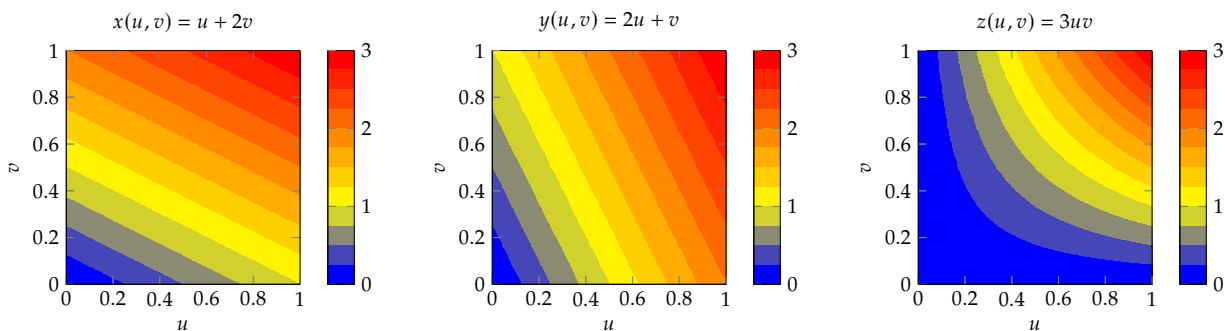
**Sol.**

**1. Parametrization and partials.** The surface is

$$S = \mathbf{r}([0, 1]^2), \quad \mathbf{r}(u, v) = (u + 2v, 2u + v, 3uv),$$

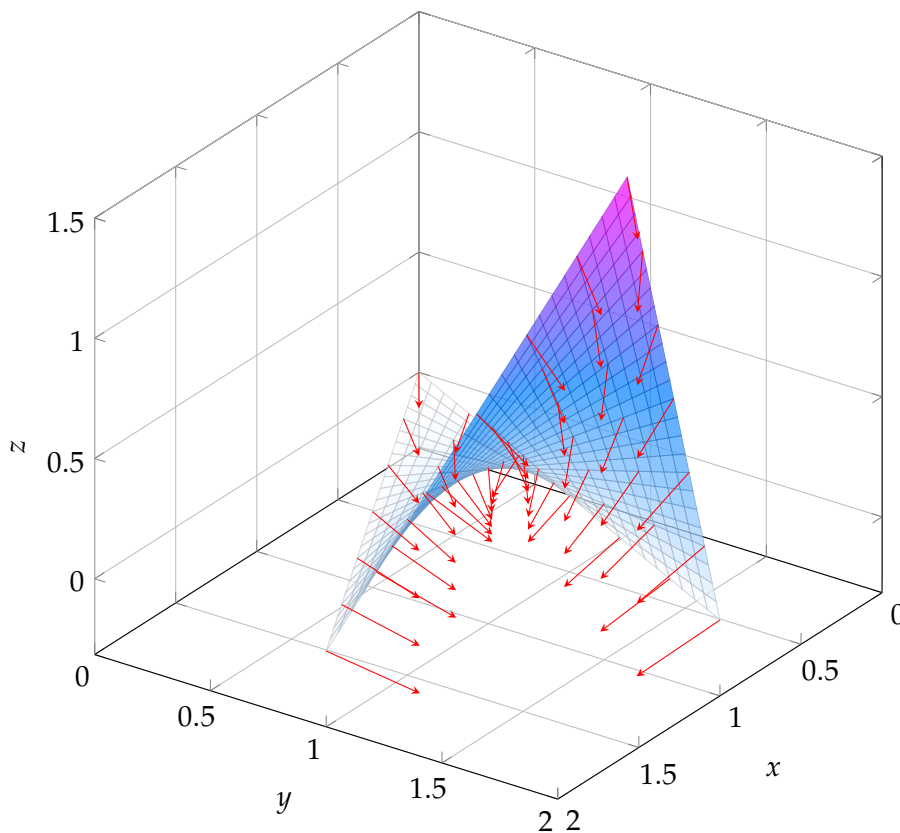
and then

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \langle 1, 2, 3v \rangle, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \langle 2, 1, 3u \rangle.$$



**2. Oriented normal.** The induced normal vector is the cross-product

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3v \\ 2 & 1 & 3u \end{vmatrix} \\ &= \det \begin{vmatrix} 2 & 3v \\ 1 & 3u \end{vmatrix} \mathbf{i} - \det \begin{vmatrix} 1 & 3v \\ 2 & 3u \end{vmatrix} \mathbf{j} + \det \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \mathbf{k} \\ &= \langle 6u - 3v, -3u + 6v, -3 \rangle.\end{aligned}$$



### Detailed Justification for $\mathbf{r}_u \times \mathbf{r}_v$ as Surface Normal

Let

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

be a smooth parametrization of a surface patch  $S \subset \mathbb{R}^3$ . Then at each point  $(u, v)$ , the two tangent vectors

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} \quad \text{and} \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$$

span the tangent plane to  $S$ . We list the logical steps that show  $\mathbf{r}_u \times \mathbf{r}_v$  is the correct (non-unit) normal vector:

1. **Tangent plane.** By definition of partial derivatives,

$\mathbf{r}_u$  is the velocity of the curve  $v = \text{const}$ ,  $\mathbf{r}_v$  is the velocity of the curve  $u = \text{const}$ .

Both lie tangent to the surface.

2. **Cross product properties.** In  $\mathbb{R}^3$ , the cross product  $\mathbf{a} \times \mathbf{b}$  satisfies:

$$\mathbf{a} \times \mathbf{b} \perp \mathbf{a}, \quad \mathbf{a} \times \mathbf{b} \perp \mathbf{b},$$

and its direction is given by the right-hand rule (orientation of  $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ ). Thus  $\mathbf{r}_u \times \mathbf{r}_v$  is perpendicular to the tangent plane.

3. **Determinant formula.** One defines

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix},$$

which expands by minors to give the familiar component formula  $(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$ .

4. **Application to our  $\mathbf{r}_u, \mathbf{r}_v$ .** In our case,

$$\mathbf{r}_u = \langle 1, 2, 3v \rangle, \quad \mathbf{r}_v = \langle 2, 1, 3u \rangle.$$

Plugging into the determinant formula yields

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3v \\ 2 & 1 & 3u \end{vmatrix} = \langle 2 \cdot 3u - 3v \cdot 1, -(1 \cdot 3u - 3v \cdot 2), 1 \cdot 1 - 2 \cdot 2 \rangle = \langle 6u - 3v, -3u + 6v, -3 \rangle.$$

5. **Verification of orthogonality.** One can check directly

$$(\mathbf{r}_u \times \mathbf{r}_v) \cdot \mathbf{r}_u = 0, \quad (\mathbf{r}_u \times \mathbf{r}_v) \cdot \mathbf{r}_v = 0,$$

confirming it is indeed normal.

6. **Geometric interpretation.** The magnitude  $\|\mathbf{r}_u \times \mathbf{r}_v\|$  equals the area of the parallelogram spanned by  $\mathbf{r}_u, \mathbf{r}_v$ . Thus in surface integrals  $\iint_S \mathbf{F} \cdot d\mathbf{S}$ , one uses  $\mathbf{r}_u \times \mathbf{r}_v du dv$  as the oriented area element.

In summary, the cross-product of the partial derivatives is the unique algebraic construction in

$\mathbb{R}^3$  that (i) is bilinear and alternating, (ii) yields a vector orthogonal to both inputs, and (iii) has magnitude equal to the parallelogram area—exactly capturing the normal and area element needed for surface integrals.

## The Cross Product in $\mathbb{R}^3$

### Definition

For two vectors

$$\mathbf{u} = (u_1, u_2, u_3), \quad \mathbf{v} = (v_1, v_2, v_3)$$

in  $\mathbb{R}^3$ , their **cross product**  $\mathbf{u} \times \mathbf{v}$  is defined to be the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

Equivalently, if  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  is the standard basis, one writes formally

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

meaning “take the determinant by expanding along the first row.”

### Key Properties

1. **Bilinearity and Alternation:**  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ , and  $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ .
2. **Perpendicularity:**  $\mathbf{u} \times \mathbf{v}$  is orthogonal to both  $\mathbf{u}$  and  $\mathbf{v}$ :  $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$ ,  $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$ .
3. **Magnitude = Area:**

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| |\sin \theta| = \text{area of the parallelogram spanned by } \mathbf{u}, \mathbf{v},$$

where  $\theta$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ .

4. **Right-Hand Rule (Orientation):** The triple  $(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v})$  has the same “right-hand” orientation as the standard basis  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ .

## Why the Determinant Formula?

The determinant expression

$$\mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

is not a literal  $3 \times 3$  determinant of numbers, but a **mnemonic** encoding exactly those three component-wise minors which satisfy:

- **Alternation:** Swapping the two rows ( $u_i$ ) and ( $v_i$ ) changes the sign of each minor, matching  $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$ .
- **Bilinearity:** Expanding a determinant along the top row is linear in each column.
- **Compatibility with basis:** For the standard basis vectors,

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2,$$

and all others follow by bilinearity and alternation.

Concretely, expanding along the first row gives

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3v \\ 2 & 1 & 3u \end{pmatrix} \\ &= \mathbf{i}(2 \cdot 3u - 1 \cdot 3v) - \mathbf{j}(1 \cdot 3u - 2 \cdot 3v) + \mathbf{k}(1 \cdot 1 - 2 \cdot 2) \\ &= \langle 6u - 3v, -3u + 6v, -3 \rangle. \end{aligned}$$

which reproduces the component-wise definition above.

## Geometric Interpretation via Volumes

Another way to see the determinant is to note that in  $\mathbb{R}^3$  the scalar triple product  $\det[\mathbf{u}, \mathbf{v}, \mathbf{w}]$  gives the signed volume of the parallelepiped spanned by  $\mathbf{u}, \mathbf{v}, \mathbf{w}$ . If one fixes  $\mathbf{u}, \mathbf{v}$ , then the unique vector  $\mathbf{n}$  satisfying

$$\det[\mathbf{u}, \mathbf{v}, \mathbf{n}] = \|\mathbf{u} \times \mathbf{v}\|^2 \quad \text{and} \quad \mathbf{n} \perp \mathbf{u}, \mathbf{v}$$

is precisely  $\mathbf{u} \times \mathbf{v}$ . The determinant-of-a-matrix formula encodes that same volume-and-orientation condition intrinsically.

**Conclusion:** The cross product is the unique bilinear, alternating, oriented map  $\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$  whose length measures area; the  $3 \times 3$  “determinant” notation succinctly packages its component formula and all of its key properties in one place.

2. \*\*

3. \*\*Pullback of the field.\*\* The given field is  $\mathbf{F}(x, y, z) = (x, y, -z)$ . Along the patch,

$$\mathbf{F}(\mathbf{r}(u, v)) = (u + 2v, 2u + v, -3uv).$$

4. \*\*Integrand.\*\* Taking the dot-product,

$$\begin{aligned}\mathbf{F}(\mathbf{r}) \cdot (\mathbf{r}_u \times \mathbf{r}_v) &= (u + 2v)(6u - 3v) + (2u + v)(-3u + 6v) + (-3uv)(-3) \\ &= 6u^2 - 3uv + 12uv - 6v^2 - 6u^2 + 12uv - 3uv + 6v^2 + 9uv \\ &= (-3uv + 12uv + 12uv - 3uv + 9uv) = 27uv.\end{aligned}$$

5. \*\*Double integral.\*\* Thus the flux is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{[0,1]^2} 27uv \, du \, dv = 27 \int_0^1 \int_0^1 uv \, du \, dv = 27 \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) = \frac{27}{4}.$$

Hence

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{27}{4}}.$$

1. Compute the partial derivatives:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \langle 1, 2, 3v \rangle, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \langle 2, 1, 3u \rangle.$$

2. Form the cross-product to get the oriented area element:

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3v \\ 2 & 1 & 3u \end{vmatrix} = \langle 6u - 3v, -3u + 6v, -3 \rangle.$$

Hence

$$d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v) du dv.$$

3. Evaluate  $\mathbf{F}$  on the parametrization:

$$\mathbf{F}(\mathbf{r}(u, v)) = \langle u + 2v, 2u + v, -3uv \rangle.$$

4. Compute the integrand

$$\mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = (u + 2v)(6u - 3v) + (2u + v)(-3u + 6v) + (-3uv)(-3).$$

Expand term by term:

$$(u + 2v)(6u - 3v) = 6u^2 + 9uv - 6v^2, \quad (2u + v)(-3u + 6v) = -6u^2 + 9uv + 6v^2, \quad (-3uv)(-3) = 9uv.$$

Summing gives

$$6u^2 + 9uv - 6v^2 + (-6u^2 + 9uv + 6v^2) + 9uv = 27uv.$$

5. Finally integrate over the unit square:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 27uv du dv = 27 \left( \int_0^1 u du \right) \left( \int_0^1 v dv \right) = 27 \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) = \frac{27}{4}.$$

**Answer:**  $\frac{27}{4}$ .

□

## Appendices

### A Function and Derivative

**1. Single-Variable Function and Derivative.** Consider a single-variable function

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto f(x).$$

Its derivative at  $x$  is the linear map

$$\frac{df}{dx}(x) = f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \in \mathbb{R},$$

characterized by

$$f(x+h) = f(x) + f'(x)h + o(h).$$

#### 1. Setup

Let

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto f(x)$$

be a real-valued function. We wish to define its derivative at a point  $x \in \mathbb{R}$  as the best linear approximation to the increment  $f(x+h) - f(x)$ .

#### 2. Linear Approximation with Remainder

We say  $f$  is **differentiable** at  $x$  if there exists a real number  $A$  and a function  $r(h)$  such that

$$f(x+h) = f(x) + Ah + r(h), \tag{1}$$

where the remainder  $r(h)$  satisfies

$$\lim_{h \rightarrow 0} \frac{r(h)}{h} = 0.$$

In Landau notation,  $r(h) = o(h)$  as  $h \rightarrow 0$ .

#### 3. Definition of the Derivative

**Definition 1.** If (1) holds for some real number  $A$  and  $r(h) = o(h)$ , then  $f$  is differentiable at  $x$ , and the **derivative** of  $f$  at  $x$  is

$$f'(x) := A.$$

Equivalently, the linear map

$$T_x \mathbb{R} \cong \mathbb{R} \longrightarrow T_{f(x)} \mathbb{R} \cong \mathbb{R}, \quad h \mapsto f'(x) h$$

is the unique linear approximation to the increment  $f(x+h) - f(x)$ .

## 4. Equivalence with the Limit Formulation

One also defines

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

when this limit exists. To see that this agrees with the linear-plus-remainder definition:

- If  $f'(x) = A$  and  $r(h) = o(h)$  satisfy (1), then

$$\frac{f(x+h) - f(x)}{h} = A + \frac{r(h)}{h} \longrightarrow A,$$

since  $\lim_{h \rightarrow 0} \frac{r(h)}{h} = 0$ .

- Conversely, if  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = A$ , set  $r(h) = f(x+h) - f(x) - A h$ . Then

$$\frac{r(h)}{h} = \frac{f(x+h) - f(x)}{h} - A \longrightarrow 0,$$

so  $r(h) = o(h)$  and (1) holds.

## 5. Notation $o(h)$

The notation  $r(h) = o(h)$  means

$$\forall \varepsilon > 0, \exists \delta > 0: \quad 0 < |h| < \delta \implies |r(h)/h| < \varepsilon.$$

## 6. Summary

Thus the derivative  $f'(x)$  is the unique scalar  $A$  making

$$f(x+h) = f(x) + A h + o(h),$$

and equivalently the limit  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ .

## 1. Definition Recap

A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is **differentiable** at  $x$  if there exists a real number  $A$  and a remainder  $r(h)$  such that

$$f(x+h) = f(x) + A h + r(h), \quad \lim_{h \rightarrow 0} \frac{r(h)}{h} = 0.$$

In that case  $f'(x) = A$ , and  $r(h) = o(h)$ .

## 2. Example: $f(t) = \sin t$

We check

$$\sin(x+h) = \sin x + \cos x h + r(h).$$

Using the addition formula,

$$\begin{aligned} \sin(x+h) &= \sin x \cos h + \cos x \sin h \\ &= \sin x \left(1 - \frac{h^2}{2} + o(h^2)\right) + \cos x \left(h - \frac{h^3}{6} + o(h^3)\right) \\ &= \sin x + \cos x h + \left(-\frac{\sin x}{2} h^2 + o(h^2)\right). \end{aligned}$$

Thus

$$r(h) = -\frac{\sin x}{2} h^2 + o(h^2),$$

and

$$\frac{r(h)}{h} = -\frac{\sin x}{2} h + o(h) \xrightarrow{h \rightarrow 0} 0.$$

Hence  $\sin t$  is differentiable with

$$f'(x) = \cos x,$$

and indeed  $d(\sin t) = \cos t dt$ .

## 3. Example: $f(t) = \cos t$

Similarly,

$$\cos(x+h) = \cos x - \sin x h + r(h).$$

By the addition formula,

$$\begin{aligned} \cos(x+h) &= \cos x \cos h - \sin x \sin h \\ &= \cos x \left(1 - \frac{h^2}{2} + o(h^2)\right) - \sin x \left(h - \frac{h^3}{6} + o(h^3)\right) \\ &= \cos x - \sin x h + \left(-\frac{\cos x}{2} h^2 + o(h^2)\right). \end{aligned}$$

Thus

$$r(h) = -\frac{\cos x}{2} h^2 + o(h^2),$$

and

$$\frac{r(h)}{h} = -\frac{\cos x}{2} h + o(h) \xrightarrow{h \rightarrow 0} 0.$$

Therefore  $\cos t$  is differentiable with

$$f'(x) = -\sin x,$$

and indeed  $d(\cos t) = -\sin t \, dt$ .

## 4. Summary

In both cases we have exhibited the decomposition

$$f(x+h) = f(x) + f'(x)h + o(h),$$

with the remainder  $r(h)$  vanishing faster than  $h$ . This formalizes that  $f'(x)$  is the unique scalar making

$$h \mapsto f'(x)h$$

the best linear approximation to the increment  $f(x+h) - f(x)$ .

## 2. Scalar Function and Gradient

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}, \quad \mathbf{x} \mapsto f(\mathbf{x}).$$

Its **gradient** at  $\mathbf{x}$  is the vector

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^n,$$

characterized by the first-order Taylor expansion

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{h} + o(\|\mathbf{h}\|).$$

## 3. Vector Field and Jacobian (Total Derivative)

$$\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad \mathbf{x} \mapsto \mathbf{F}(\mathbf{x}) = \begin{pmatrix} F_1(\mathbf{x}) \\ \vdots \\ F_m(\mathbf{x}) \end{pmatrix}.$$

Its **Jacobian matrix** (total derivative) at  $\mathbf{x}$  is

$$D\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial F_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial F_m}{\partial x_n}(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^{m \times n},$$

characterized by

$$\mathbf{F}(\mathbf{x} + \mathbf{h}) = \mathbf{F}(\mathbf{x}) + D\mathbf{F}(\mathbf{x}) \mathbf{h} + o(\|\mathbf{h}\|).$$

**Special Case:**  $n = m = 3$ . For a vector field  $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ : - The **divergence** is the trace of the

Jacobian:  $\nabla \cdot \mathbf{F} = \partial_x F_1 + \partial_y F_2 + \partial_z F_3$ . - The **curl** is a new vector field:  $\nabla \times \mathbf{F} = \begin{pmatrix} \partial_y F_3 - \partial_z F_2 \\ \partial_z F_1 - \partial_x F_3 \\ \partial_x F_2 - \partial_y F_1 \end{pmatrix}.$

### Summary of the Hierarchy:

$$\underbrace{f'(x)}_{\text{single-variable derivative}} \longleftrightarrow \underbrace{\nabla f(\mathbf{x})}_{\text{gradient of scalar } f} \longleftrightarrow \underbrace{D\mathbf{F}(\mathbf{x})}_{\text{Jacobian of vector field } \mathbf{F}}.$$

We summarize the familiar hierarchy

$$\text{single-variable derivative} \longleftrightarrow \text{gradient of a scalar field} \longleftrightarrow \text{Jacobian of a vector field}$$

by using the exterior derivative  $d$  on differential forms.

**1. Single-variable case.** A smooth function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a 0-form,  $f \in \Omega^0(\mathbb{R})$ . Its derivative is the 1-form

$$df = f'(x) dx \in \Omega^1(\mathbb{R}),$$

and the Fundamental Theorem of Calculus reads  $\int_a^b df = f(b) - f(a)$ .

**2. Scalar field in  $\mathbb{R}^n$ .** A smooth function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is again a 0-form,  $f \in \Omega^0(\mathbb{R}^n)$ . Its exterior derivative

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx^i \in \Omega^1(\mathbb{R}^n)$$

is the differential 1-form whose components are the partials. Under the Euclidean metric this 1-form corresponds to the gradient vector field  $\nabla f$ .

**3. Vector field in  $\mathbb{R}^n$ .** A smooth vector field  $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  can be viewed as an  $\mathbb{R}^m$ -valued 0-form

$$\mathbf{F} = (F_1, \dots, F_m) \in (\Omega^0(\mathbb{R}^n))^m.$$

Applying  $d$  to each component yields the **matrix of 1-forms**

$$d\mathbf{F} = (dF_1, \dots, dF_m) \in \underbrace{\Omega^1(\mathbb{R}^n) \times \dots \times \Omega^1(\mathbb{R}^n)}_{m \text{ copies}} \cong \Omega^1(\mathbb{R}^n) \otimes \mathbb{R}^m,$$

whose  $i$ th entry is

$$dF_i = \sum_{j=1}^n \frac{\partial F_i}{\partial x_j} dx^j.$$

Choosing the basis  $\{dx^1, \dots, dx^n\}$  identifies  $d\mathbf{F}$  with the Jacobian matrix

$$D\mathbf{F} = \left[ \frac{\partial F_i}{\partial x_j} \right]_{1 \leq i \leq m, 1 \leq j \leq n}.$$

Summary in One Diagram:

$$\underbrace{f}_{\Omega^0} \xrightarrow{d} \underbrace{df}_{\Omega^1} \longleftrightarrow \underbrace{\nabla f}_{\text{gradient vector field}} \longrightarrow \underbrace{\mathbf{F}}_{(\Omega^0)^m} \xrightarrow{d} \underbrace{d\mathbf{F}}_{\Omega^1 \otimes \mathbb{R}^m} \longleftrightarrow \underbrace{D\mathbf{F}}_{\text{Jacobian matrix}}.$$

Each arrow  $d$  is the same exterior derivative, producing higher-rank forms whose coefficients encode the familiar derivatives.

## From Derivatives to Differentials: A Unified Matrix–Form View

We compare three levels of maps and their differentials in the language of exterior derivatives and matrices.

Case	Map	Differential / Matrix Form
Single-variable	$f : \mathbb{R} \longrightarrow \mathbb{R}, x \mapsto f(x)$	$df = f'(x) dx \quad (\in \Omega^1(\mathbb{R}))$
Scalar field	$f : \mathbb{R}^n \longrightarrow \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x})$	$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x}) dx^i \quad (\in \Omega^1(\mathbb{R}^n)),$ $\nabla f(\mathbf{x}) = \begin{pmatrix} \partial_1 f \\ \vdots \\ \partial_n f \end{pmatrix} \in \mathbb{R}^{n \times 1}$
Vector field	$\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \mathbf{x} \mapsto \mathbf{F}(\mathbf{x})$	$d\mathbf{F} = (dF_1, \dots, dF_n) = \left( \sum_j \partial_j F_i dx^j \right)_{i=1}^n \in \Omega^1(\mathbb{R}^n) \otimes \mathbb{R}^n,$ $D\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \partial_1 F_1 & \cdots & \partial_n F_1 \\ \vdots & \ddots & \vdots \\ \partial_1 F_n & \cdots & \partial_n F_n \end{pmatrix} \in \mathbb{R}^{n \times n}.$

Key points:

- In each case, the exterior derivative  $d$  raises the form-degree by one:

$$d : \Omega^0 \rightarrow \Omega^1, \quad d(f) = df, \quad d(F_i) = dF_i.$$

- For  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $df$  is a 1-form whose coefficients are the partials  $\partial_i f$ . Under the Euclidean metric these correspond to the gradient vector  $\nabla f$ .
- For  $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , applying  $d$  to each component yields the matrix of 1-forms  $d\mathbf{F}$ . Choosing the basis  $\{dx^j\}$  identifies  $d\mathbf{F}$  with the Jacobian matrix  $D\mathbf{F}$ , which is the total derivative (linear approximation) of  $\mathbf{F}$ .
- Thus the familiar hierarchy

$$f'(x) \longleftrightarrow \nabla f(\mathbf{x}) \longleftrightarrow D\mathbf{F}(\mathbf{x})$$

is simply the degrees-of-freedom of the single exterior derivative  $d$  applied to scalar vs. vector-valued functions, packaged in matrix form.

## B Scalar Function and Vector Fields

**Definition.** A **scalar function** on  $\mathbb{R}^n$  is a real-valued function of an  $n$ -tuple; that is,

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, \quad \mathbf{x} \mapsto f(\mathbf{x}) = f(x_1, x_2, \dots, x_n).$$

where  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  and  $f(\mathbf{x}) \in \mathbb{R}$ .

**Definition 2** (Scalar Function). A **scalar function** on  $\mathbb{R}^n$  is a mapping

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, \quad \mathbf{x} = (x_1, \dots, x_n) \mapsto f(\mathbf{x}),$$

which assigns to each point  $\mathbf{x} \in \mathbb{R}^n$  a real value  $f(\mathbf{x})$ . If  $f$  has continuous partial derivatives on an open set  $U \subset \mathbb{R}^n$ , we write  $f \in C^1(U)$ .

**Definition** (Gradient of a Scalar Function). Let  $f \in C^1(U)$  be a scalar function on an open set  $U \subset \mathbb{R}^n$ . Its **gradient** is the vector-valued function

$$\nabla f : U \rightarrow \mathbb{R}^n, \quad \nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix}.$$

Equivalently,  $\nabla f(\mathbf{x})$  is the unique vector in  $\mathbb{R}^n$  satisfying

$$df(\mathbf{x})[\mathbf{h}] = \nabla f(\mathbf{x}) \cdot \mathbf{h} \quad \text{for all } \mathbf{h} \in \mathbb{R}^n,$$

where  $df(\mathbf{x})$  is the differential of  $f$  at  $\mathbf{x}$ .

**Remark.** In particular, its differential (or gradient) may be written in matrix (row-vector) form as

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial}{\partial x_n} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^{1 \times n}.$$

Although  $\nabla f(\mathbf{x})$  is **not** the product of a fixed matrix by  $\mathbf{x}$ , the symbol

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

can itself be viewed as a “column-vector” whose entries are the partial-derivative operators. Then for any scalar function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$\nabla f = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix},$$

which is exactly the usual gradient.

—

**Key point:** -  $\nabla$  itself is an **operator-valued** vector, not a numeric matrix. - When you write  $\nabla f$ , you are applying each row of that “matrix” of operators to the single-valued function  $f$ . - By the same token, for a vector field  $\mathbf{F} = (F_1, \dots, F_n)^T$ , the **Jacobian** can be written symbolically as

$$D\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \nabla^T F_1(\mathbf{x}) \\ \vdots \\ \nabla^T F_n(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \end{pmatrix} \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}.$$

Here the row  $(\partial_1, \dots, \partial_n)$  acts on each component  $F_i$ .

In this sense,  $\nabla$  really is a “matrix” (vector) of operators, whose multiplication by a function or vector field produces the gradient, divergence, curl, or Jacobian, depending on how you contract it.

1.  $\nabla$  is not a “point-wise” vector in  $\mathbb{R}^n$ . Rather,

$$\nabla = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_n} \end{pmatrix}$$

is a **vector of differential operators**, each  $\partial_{x_i}$  acting on functions. It lives in the space of linear maps  $\text{Hom}(C^\infty(\mathbb{R}^n), \Omega^1(\mathbb{R}^n))$ , not in  $\mathbb{R}^n$  itself. We write it in “vector form” simply to mirror how it acts component-wise.

2.  $f$  is a scalar field, an element of the function space  $C^\infty(\mathbb{R}^n)$ . Concretely,

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}, \quad \mathbf{x} \longmapsto f(\mathbf{x}).$$

This  $f$  is **not** itself a vector in  $\mathbb{R}^n$ ; it is a real-valued function on  $\mathbb{R}^n$ .

3. The action  $\nabla f$  produces a genuine vector field. When you apply the operator-vector  $\nabla$  to the scalar  $f$ , you get

$$\nabla f = \begin{pmatrix} \partial_{x_1} f \\ \partial_{x_2} f \\ \vdots \\ \partial_{x_n} f \end{pmatrix},$$

which is a true vector-valued function on  $\mathbb{R}^n$ , i.e. an element of  $\Omega^1(\mathbb{R}^n)$  or equivalently  $(C^\infty(\mathbb{R}^n))^n$ .

**Summary:**

- $\nabla$  itself is an **operator** (a “vector” of partial-derivatives), not a point in physical space.
- $f$  is a scalar field (a function) on  $\mathbb{R}^n$ .
- $\nabla f$  is the gradient vector field, a bona fide element of  $(C^\infty(\mathbb{R}^n))^n$ .

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^{n \times 1}.$$

Equivalently, its transpose (the differential) is the  $1 \times n$  row-vector

$$df(\mathbf{x}) = \nabla f(\mathbf{x})^\top = \left( \frac{\partial f}{\partial x_1}(\mathbf{x}) \quad \frac{\partial f}{\partial x_2}(\mathbf{x}) \quad \cdots \quad \frac{\partial f}{\partial x_n}(\mathbf{x}) \right).$$

**Example.** Let  $n = 2$ . We consider two **scalar functions** on  $\mathbb{R}^2 \setminus \{(0, 0)\}$ :

$$\begin{aligned} f_1 : \mathbb{R}^2 \setminus \{(0, 0)\} &\longrightarrow \mathbb{R} & f_2 : \mathbb{R}^2 \setminus \{(0, 0)\} &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto f_1(x, y) = -\frac{y}{x^2+y^2} & (x, y) &\longmapsto f_2(x, y) = \frac{x}{x^2+y^2} . \end{aligned}$$

These functions assign to each point of the punctured plane a single real value, in accordance with the definition of a scalar function on  $\mathbb{R}^2$ .

Writing

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r > 0, \theta \in [0, 2\pi),$$

we obtain the equivalent descriptions in  $(r, \theta)$ -space:

$$\begin{aligned} f_1 : (0, \infty) \times [0, 2\pi) &\longrightarrow \mathbb{R} \\ (r, \theta) &\longmapsto f_1(r, \theta) = -\frac{r \sin \theta}{r^2} = -\frac{\sin \theta}{r} , \end{aligned}$$

$$\begin{aligned} f_2 : (0, \infty) \times [0, 2\pi) &\longrightarrow \mathbb{R} \\ (r, \theta) &\longmapsto f_2(r, \theta) = \frac{r \cos \theta}{r^2} = \frac{\cos \theta}{r} . \end{aligned}$$

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^{n \times 1}.$$

Equivalently, its transpose (the differential) is the  $1 \times n$  row-vector

$$df(\mathbf{x}) = \nabla f(\mathbf{x})^\top = \left( \frac{\partial f}{\partial x_1}(\mathbf{x}) \quad \frac{\partial f}{\partial x_2}(\mathbf{x}) \quad \cdots \quad \frac{\partial f}{\partial x_n}(\mathbf{x}) \right).$$

**Definition.** A **vector field** on  $\mathbb{R}^n$  is a function

$$\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$\mathbf{x} \longmapsto \mathbf{F}(\mathbf{x}) = \begin{pmatrix} F_1(\mathbf{x}) \\ F_2(\mathbf{x}) \\ \vdots \\ F_n(\mathbf{x}) \end{pmatrix},$$

where each component  $F_i : \mathbb{R}^n \rightarrow \mathbb{R}$  is itself a scalar function.

**Remark.** Its Jacobian matrix—which encodes the best linear approximation of  $F$  at each point—is

$$\mathbf{J}_{\mathbf{F}} = \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\mathbf{x}) & \frac{\partial F_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial F_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial F_2}{\partial x_1}(\mathbf{x}) & \frac{\partial F_2}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial F_2}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(\mathbf{x}) & \frac{\partial F_n}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial F_n}{\partial x_n}(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Let

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}, \quad \mathbf{x} = (x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n).$$

Its **Jacobian** is the  $1 \times n$  matrix whose entries are the first-order partial derivatives of  $f$ . Concretely,

$$J_f(x) = \left( \frac{\partial f}{\partial x_1}(x) \quad \frac{\partial f}{\partial x_2}(x) \quad \cdots \quad \frac{\partial f}{\partial x_n}(x) \right) \in \mathbb{R}^{1 \times n}.$$

Equivalently, one writes

$$df(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) dx^i,$$

so that in the basis  $\{dx^1, \dots, dx^n\}$ ,

$$df(x) = \begin{pmatrix} \partial_1 f(x) & \partial_2 f(x) & \cdots & \partial_n f(x) \end{pmatrix} \begin{pmatrix} dx^1 \\ dx^2 \\ \vdots \\ dx^n \end{pmatrix}.$$

**Examples:**

- If  $n = 2$  and  $f(x, y) = x^2y + e^y$ , then

$$J_f(x, y) = \left( \frac{\partial}{\partial x}(x^2y + e^y) \quad \frac{\partial}{\partial y}(x^2y + e^y) \right) = \begin{pmatrix} 2xy & x^2 + e^y \end{pmatrix}.$$

- If  $n = 3$  and  $f(x, y, z) = \sin(xy) + z^3$ , then

$$J_f(x, y, z) = (f_x, f_y, f_z) = \begin{pmatrix} y \cos(xy), & x \cos(xy), & 3z^2 \end{pmatrix}.$$

## Deriving the Jacobian Matrix from the Gradient

Let

$$\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad \mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}), \dots, F_n(\mathbf{x}))^T,$$

where each  $F_i : \mathbb{R}^n \rightarrow \mathbb{R}$  is a scalar function. We will build the  $n \times n$  Jacobian matrix

$$J_F(x) = \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)},$$

step by step, starting from the familiar gradient of a single scalar function.

**1. Gradient of a single scalar function.** If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , then its **gradient** is the column-vector of partials:

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \frac{\partial f}{\partial x_2}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix} \in \mathbb{R}^n.$$

Equivalently, the differential  $df(x)$  is the row-vector  $(\partial_1 f(x), \dots, \partial_n f(x)) \in \mathbb{R}^{1 \times n}$ .

**2. View each component  $F_i$  as a scalar function.** Since  $F(x) = (F_1(x), \dots, F_n(x))^T$ , we can take the gradient of each  $F_i$ :

$$\nabla F_i(x) = \begin{pmatrix} \partial_1 F_i(x) \\ \partial_2 F_i(x) \\ \vdots \\ \partial_n F_i(x) \end{pmatrix},$$

and the corresponding differential (row) is

$$dF_i(x) = (\partial_1 F_i(x), \partial_2 F_i(x), \dots, \partial_n F_i(x)) \in \mathbb{R}^{1 \times n}.$$

**3. Assemble the Jacobian by stacking these row-vectors.** By definition, the Jacobian  $J_F(x)$  is the matrix whose  $i$ th row is  $dF_i(x)$ . Thus

$$J_F(x) = \begin{pmatrix} dF_1(x) \\ dF_2(x) \\ \vdots \\ dF_n(x) \end{pmatrix} = \begin{pmatrix} \partial_1 F_1(x) & \partial_2 F_1(x) & \cdots & \partial_n F_1(x) \\ \partial_1 F_2(x) & \partial_2 F_2(x) & \cdots & \partial_n F_2(x) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 F_n(x) & \partial_2 F_n(x) & \cdots & \partial_n F_n(x) \end{pmatrix}.$$

**4. Interpretation as the linear approximation.** For a small increment  $h \in \mathbb{R}^n$ ,

$$F(x + h) = F(x) + J_F(x) h + o(\|h\|),$$

so  $J_F(x)$  is exactly the matrix representing the derivative (total differential) of the vector field  $F$  at  $x$ .

**Summary:** - The gradient  $\nabla f$  of a scalar  $f$  is a single column of partial derivatives. - For a vector field  $F = (F_1, \dots, F_n)$ , we take gradients of each component  $F_i$ . - Stacking these gradients (as row-vectors) produces the Jacobian matrix  $J_F$ .

This construction ensures that  $J_F(x)$  captures all first-order variations of the vector field in every coordinate direction.

$$\underbrace{\begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix}}_{\text{"row" of partials of the vector field}} = \underbrace{\begin{bmatrix} \nabla^T f_1 \\ \vdots \\ \nabla^T f_m \end{bmatrix}}_{\substack{\text{stacked transposed} \\ \text{gradients of each component}}} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}}_{\text{the usual Jacobian matrix of } \mathbf{f}=(f_1, \dots, f_m)^T}.$$

$$\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad \mathbf{F}(\mathbf{x}) = \begin{pmatrix} F_1(\mathbf{x}) \\ F_2(\mathbf{x}) \\ \vdots \\ F_n(\mathbf{x}) \end{pmatrix}, \quad F_i : \mathbb{R}^n \rightarrow \mathbb{R} \text{ (scalar functions).}$$

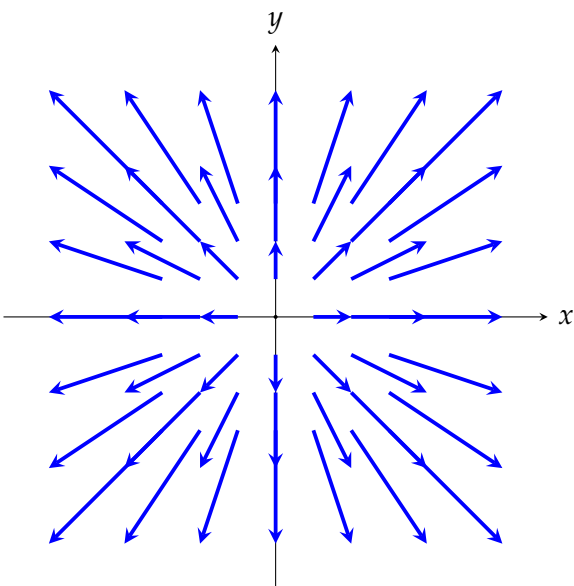
Then the **Jacobian matrix**  $D\mathbf{F}(\mathbf{x})$  is obtained by taking the gradient of each component  $F_i$  and

stacking them as rows:

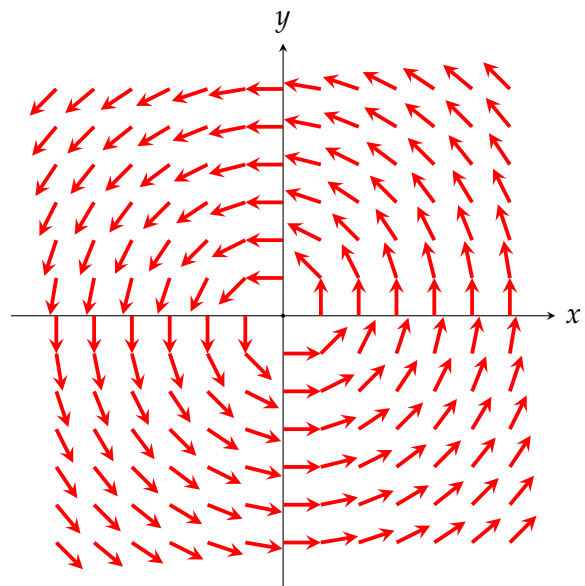
$$D\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \nabla^T F_1(\mathbf{x}) \\ \nabla^T F_2(\mathbf{x}) \\ \vdots \\ \nabla^T F_n(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\mathbf{x}) & \frac{\partial F_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial F_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial F_2}{\partial x_1}(\mathbf{x}) & \frac{\partial F_2}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial F_2}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(\mathbf{x}) & \frac{\partial F_n}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial F_n}{\partial x_n}(\mathbf{x}) \end{pmatrix}.$$

Each row  $\nabla^T F_i(\mathbf{x})$  is the transpose of the gradient of the scalar function  $F_i$ .

**Example.**



The radial field  $\mathbf{F} = \langle x, y \rangle = x\mathbf{i} + y\mathbf{j}$



The spin field  $\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)^{1/2}$