Quantum Computing

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We cover the following topics in this note.

- Vector calculus (conservative fields, irrotational field)

- Differential forms (exact forms, closed forms)

Contents

1 Spin, Qubits, and Entanglement: $\mathbb{C}^2$, $\mathbb{CP}^1$, and Two-Qubit Structure

1.1 Kinematics of a Single Qubit

1.1.1 Hilbert-Space Model

Definition 1.1 (Qubit as a ray). *Let $\mathcal{H} \cong \mathbb{C}^2$ be a two-dimensional complex Hilbert space with the standard Hermitian inner product $\langle \psi, \phi \rangle = \psi^\dagger \phi$. A **(pure) qubit state** is a ray $[\psi]$ in the complex projective line $\mathbb{CP}^1$, where $\psi \in \mathbb{C}^2 \setminus \{0\}$ and $[\psi] = \{\lambda \psi : \lambda \in \mathbb{C} \setminus \{0\}\}$. Two vectors describe the same physical state iff they differ by a nonzero complex scalar.*

Remark 1.2 (Normalization and global phase). Every state admits a representative of unit norm, $|\psi\rangle \in \mathbb{S}^3 \subset \mathbb{C}^2$, unique up to a global phase $e^{i\theta}$. Probabilities depend only on the ray $[\psi]$.

1.2 Computational Bases and Pauli Observables

Fix the computational basis $\{|0\rangle, |1\rangle\}$ with $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Define the Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Each σ_α ($\alpha \in \{x, y, z\}$) is Hermitian with eigenvalues ± 1 and eigenbases corresponding, respectively, to "horizontal" ($\{|\rightarrow\rangle, |\leftarrow\rangle\}$), "diagonal" ($\{|\nearrow\rangle, |\swarrow\rangle\}$), and "vertical" ($\{|0\rangle, |1\rangle\}$) spin/polarization directions. Up to global phase, one may take

$$|\rightarrow\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad\qquad |\leftarrow\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle),$$
$$|\nearrow\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \qquad\qquad |\swarrow\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - i|1\rangle).$$

1.3 Born Rule and Projective Measurement

Definition 1.3 (Projective measurement in basis $\mathcal{B}$). *Let $\mathcal{B} = (|b_0\rangle, |b_1\rangle)$ be an ordered orthonormal basis of $\mathbb{C}^2$. The measurement associated with $\mathcal{B}$ is the PVM $\{\Pi_0, \Pi_1\}$ where $\Pi_j = |b_j\rangle\langle b_j|$. For a normalized $|\psi\rangle$, the outcome $j \in \{0, 1\}$ is obtained with probability $p_j = \|\Pi_j \psi\|^2 = |\langle b_j | \psi \rangle|^2$, and the post-measurement state is $|b_j\rangle$.*

Remark 1.4 (Equivalence under sign and phase). If $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ (in particular $-|\psi\rangle$) differ by a global phase, they yield identical outcome probabilities in any measurement; hence they encode the same physical state (ray).

1.4   Bloch-Sphere / $\mathbb{CP}^1$ Identification

Proposition 1.5 (Hopf fibration and Bloch map)**.** *Define* $r : \mathbb{CP}^1 \to \mathbb{S}^2 \subset \mathbb{R}^3$ *by*

$$r([\psi]) = \left(\left\langle \psi \middle| \sigma_x \psi \right\rangle, \left\langle \psi \middle| \sigma_y \psi \right\rangle, \left\langle \psi \middle| \sigma_z \psi \right\rangle \right).$$

Then $\left\| r([\psi]) \right\| = 1$ *for any pure state, providing a bijection between qubit rays and points of the Bloch sphere* $\mathbb{S}^2$. *Rotations* $R \in \mathrm{SO}(3)$ *correspond to conjugations by* $U \in \mathrm{SU}(2)$ *via the double cover* $\mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$.

Remark 1.6 (Polarization and spin directions)*.* Choosing a measurement axis given by a unit vector $\hat{n} \in \mathbb{S}^2$ corresponds to measuring the observable $\sigma_{\hat{n}} = \hat{n} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. The eigenstates of $\sigma_{\hat{n}}$ are points $\pm \hat{n}$ on the Bloch sphere.

2   Kinematics of a Single Qubit

2.1   Hilbert-Space Model

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$$\left| \rightarrow \right\rangle = \tfrac{1}{\sqrt{2}}(\left| 0 \right\rangle + \left| 1 \right\rangle), \qquad\qquad \left| \leftarrow \right\rangle = \tfrac{1}{\sqrt{2}}(\left| 0 \right\rangle - \left| 1 \right\rangle),$$

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3 From Stern–Gerlach to the Postulates

3.1 Idealized Stern–Gerlach

Definition 3.1 (Stern–Gerlach splitting (idealized)). *Given a spin-$\frac{1}{2}$ particle with magnetic moment* $\boldsymbol{\mu} = \gamma \boldsymbol{S}$ *and a static inhomogeneous field* $\boldsymbol{B}(r)$, *the force is* $\boldsymbol{F}(r) = \nabla(\boldsymbol{\mu} \cdot \boldsymbol{B})$. *In the usual z-gradient setting one has* $F_z \approx \mu_z \partial_z B_z$, *producing two spatially separated beams corresponding to eigenvalues* $\pm\frac{\hbar}{2}$ *of* S_z.

Remark 3.2. At the level of state vectors, a beam in state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ impinging on a z-analyzer is **projectively** resolved into $|0\rangle$ with probability $|\alpha|^2$ and $|1\rangle$ with probability $|\beta|^2$; the spatial separation is a macroscopic record of the projective outcome.

3.2 Rotation of the Analyzer

Proposition 3.3 (Spinor half-angle). *If the analyzer is rotated by a physical angle* θ *about some axis, the corresponding basis in* $\mathbb{C}^2$ *is obtained by an* $\mathrm{SU}(2)$ *action with* **half-angle**: *the eigenvectors of* $\sigma_{\hat{n}(\theta)}$ *correspond to* $U(\theta/2) \in \mathrm{SU}(2)$. *Concretely, for rotations in the x–z plane,*

$$|b_0(\theta)\rangle = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}, \qquad |b_1(\theta)\rangle = \begin{bmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix}.$$

Consequently, if a particle is prepared in $|0\rangle$ *and measured along* θ, *the +1 outcome occurs with probability* $\cos^2(\theta/2)$ *and the −1 outcome with probability* $\sin^2(\theta/2)$.

4 Linear Algebra Prerequisites in $\mathbb{C}^2$

4.1 Bras, Kets, and Inner Products

Vectors $|v\rangle \in \mathbb{C}^2$ are columns, bras are conjugate-transposes $\langle v| = |v\rangle^\dagger$, inner products are $\langle u|v\rangle = u^\dagger v$, and orthonormality means $\langle b_i|b_j\rangle = \delta_{ij}$. For any orthonormal basis $\{|b_0\rangle, |b_1\rangle\}$, every $|v\rangle$ admits the expansion $|v\rangle = \sum_j \langle b_j|v\rangle |b_j\rangle$ with $\|v\|^2 = \sum_j |\langle b_j|v\rangle|^2$.

4.2 Unitary Transformations and Gates

Definition 4.1 (Unitary). *$U \in \mathbb{C}^{2\times 2}$ is **unitary** if $U^\dagger U = I$. Unitaries preserve inner products and implement reversible dynamics on $\mathbb{CP}^1$ via $[\psi] \mapsto [U\psi]$. Examples: the Hadamard $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ diagonalizes σ_x; phase gates and general rotations $R_{\hat{n}}(\theta) = e^{-i\theta\,\hat{n}\cdot\boldsymbol{\sigma}/2}$ generate SU(2).*

5 Photons: Linear Polarization in $\mathbb{C}^2$

Remark 5.1. Classically, a linear polarizer transmits the field component along its axis and absorbs the orthogonal component. Quantum mechanically, a single-photon polarization qubit $|\psi\rangle$ is resolved by the PVM aligned with the polarizer axis; Malus's law $\cos^2(\theta)$ appears as $\cos^2(\theta/2)$ on the Bloch sphere because the physical angle between axes equals 2 times the geodesic angle between the corresponding rays in $\mathbb{CP}^1$.

Example 5.2 (Three polarizers). Let Z be vertical, X be horizontal, and D be $45°$. A vertically polarized photon $|0\rangle$ has $\frac{1}{2}$ transmission probability through D (state becomes $|\rightarrow\rangle$ or $|\leftarrow\rangle$), and then again $\frac{1}{2}$ through X, yielding an overall $\frac{1}{4}$ transmission, whereas without D the transmission through X is zero. The intermediate measurement changes the state and thereby the statistics.

6 Two Qubits and Tensor Products

6.1 Tensor Products and Bases

For $\mathcal{H}_A \cong \mathcal{H}_B \cong \mathbb{C}^2$, the composite space is $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \cong \mathbb{C}^4$ with computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. A pure product state has the form $|\psi_A\rangle \otimes |\phi_B\rangle$.

Definition 6.1 (Product vs. entangled state). *A unit vector $|\Psi\rangle \in \mathcal{H}_{AB}$ is a **product** state if $|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle$ for some $|\psi\rangle, |\phi\rangle \in \mathbb{C}^2$. Otherwise it is **entangled**.*

6.2 Schmidt Decomposition in $\mathbb{C}^2 \otimes \mathbb{C}^2$

Theorem 6.2 (Schmidt decomposition). *Every $|\Psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ admits orthonormal bases $\{|a_0\rangle, |a_1\rangle\}$ of A and $\{|b_0\rangle, |b_1\rangle\}$ of B such that*

$$|\Psi\rangle = \sqrt{\lambda}\,|a_0\rangle \otimes |b_0\rangle + \sqrt{1-\lambda}\,|a_1\rangle \otimes |b_1\rangle, \qquad 0 \leq \lambda \leq 1.$$

Entanglement occurs iff $\lambda \in (0, 1)$.

Corollary 6.3 (Rank criterion). *Writing $|\Psi\rangle = \sum_{i,j\in\{0,1\}} c_{ij} |ij\rangle$ and arranging coefficients as the 2×2 matrix $C = [c_{ij}]$, we have: $|\Psi\rangle$ is a product state iff $\operatorname{rank} C = 1$, equivalently $\det C = 0$; it is entangled iff $\det C \neq 0$.*

6.3 Bell States and Correlations

Define the (maximally entangled) Bell states

$$\left|\Phi^\pm\right\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \qquad\qquad \left|\Psi^\pm\right\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle).$$

Measuring both qubits in the same basis yields perfect (anti-)correlations while marginal statistics of each subsystem are maximally mixed: $\rho_A = \rho_B = \tfrac{1}{2}I$.

6.4 CNOT as an Entangler

Definition 6.4 (CNOT). *The controlled-NOT gate in the ordered basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is*

$$\mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

If the control is A and the target B, then $\mathrm{CNOT}\,(H\,|0\rangle) \otimes |0\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \left|\Phi^+\right\rangle$, producing entanglement from a product input.

7 Measurement Locality and No-Signalling

Proposition 7.1 (Local measurement update). *Let $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and let $\{\Pi_j\}$ be a PVM on A. Upon obtaining outcome j (with probability $p_j = \langle\Psi|\,(\Pi_j \otimes I)\,|\Psi\rangle$), the post-measurement state is $|\Psi_j\rangle = (\Pi_j \otimes I)\,|\Psi\rangle / \sqrt{p_j}$. The reduced state on B becomes $\rho_B^{(j)} = \operatorname{Tr}_A |\Psi_j\rangle\langle\Psi_j|$ and depends on j; however, the **unconditioned** state on B is $\sum_j p_j\, \rho_B^{(j)} = \operatorname{Tr}_A |\Psi\rangle\langle\Psi|$, independent of whether A was measured.*

Corollary 7.2 (No superluminal signalling). *Local operations and classical ignorance ensure that marginal statistics on B are unaffected by a space-like separated measurement on A; hence entanglement alone cannot be used for faster-than-light communication.*

8 Worked Calculations in $\mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^2$

8.1 Axis Rotation and Probabilities

Example 8.1. Prepare $|\psi\rangle = |0\rangle$ and measure along axis at physical angle θ from z. The outcome "+" (eigenvalue +1 of $\sigma_{\hat{n}}$) occurs with probability $\cos^2(\theta/2)$; the outcome "−" occurs with probability $\sin^2(\theta/2)$. For $\theta = 60°$, these are $3/4$ and $1/4$, respectively.

8.2 Entanglement Test via Coefficients

Example 8.2. Consider $|\Psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + 0 \cdot |11\rangle$. The coefficient matrix is $C = \begin{bmatrix} 1/2 & 1/2 \\ 1/\sqrt{2} & 0 \end{bmatrix}$ with $\det C = -\frac{1}{2\sqrt{2}} \neq 0$, hence $|\Psi\rangle$ is entangled. Measuring A in the $\{|0\rangle, |1\rangle\}$ basis yields outcome 0 or 1 with equal probabilities $1/2$; the corresponding conditional states of B are $|\rightarrow\rangle$ and $|0\rangle$, respectively.

9 From $\mathbb{CP}^1$ to Geometry of Gates

9.1 Geodesics and Two-Level Interference

On $\mathbb{CP}^1$ endowed with the Fubini–Study metric $ds^2 = \arccos^2(|\langle\psi|\phi\rangle|)$, the probability $|\langle\psi|\phi\rangle|^2$ is the squared cosine of half the geodesic distance on the Bloch sphere; interference phases shift points along great circles.

9.2 SU(2) Action and Euler Angles

Any unitary $U \in \mathrm{SU}(2)$ can be written $U = e^{-i\alpha\sigma_z/2}e^{-i\beta\sigma_y/2}e^{-i\gamma\sigma_z/2}$; on $\mathbb{S}^2$ this is the SO(3) rotation with Euler angles (α, β, γ). Thus physical rotations of analyzers/polarizers correspond to unitary conjugations on $\mathbb{C}^2$.

10 Appendix: Basic Probability for Qubits

Definition 10.1 (Discrete probability space). *An experiment with outcomes $\{E_i\}_{i=1}^n$ assigns probabilities $p_i \in [0,1]$ with $\sum_i p_i = 1$. For qubit measurements in an ONB $\{|b_0\rangle, |b_1\rangle\}$, the distribution is $p_j = |\langle b_j|\psi\rangle|^2$.*

Remark 10.2 (Law of total probability for projective measurements). Given a refinement by an intermediate measurement (e.g., three-polarizer setup), classical conditioning applies to the **quantum-updated** states, not to the unmeasured counterfactuals; hence inserting a compatible intermediate polarizer can increase transmission by altering the state.

Notation summary: $|\cdot\rangle$ (kets), $\langle\cdot|$ (bras), $\langle\phi|\psi\rangle$ (inner product), $|\psi\rangle\langle\psi|$ (rank-1 projector), Tr (trace), I (identity), $\sigma_{x,y,z}$ (Pauli), H (Hadamard), CNOT (controlled-NOT), $\mathbb{CP}^1$ (rays), Bloch sphere $\mathbb{S}^2$ (pure states).