Mayer-Vietoris Sequence for de Rham Cohomology on S^2

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Abstract

This article provides a detailed demonstration of the Mayer–Vietoris sequence for de Rham cohomology on the unit sphere S^2, using explicit coordinate charts, the Fundamental Theorem of Calculus, and elementary kernel–image arguments. We cover S^2 by two hemispherical patches, compute restriction and difference maps, construct the connecting homomorphism via a partition of unity, and verify exactness by two applications of FTC.

1 Introduction

The Mayer–Vietoris sequence is an algebraic tool that computes cohomology of a union of two overlapping open sets. For de Rham cohomology of differential forms, one sets up a short exact sequence of complexes

$$0 \to \Omega^*(S^2) \xrightarrow{r} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{s} \Omega^*(U \cap V) \to 0,$$

where U, V are coordinate patches. Passing to cohomology yields a long exact sequence, whose exactness at each spot can be checked by elementary linear algebra once one knows the groups are $\mathbb{R}$ or 0. Here we give a completely concrete treatment, using

- explicit coordinates on U, V, and $U \cap V$,
- the Fundamental Theorem of Calculus (FTC) to integrate 1-forms on contractible patches,
- a partition of unity to patch local primitives,
- kernel–image checks in each degree.

2 Setup: An Open Cover of S^2 and Local Cohomology

Let the unit sphere in $\mathbb{R}^3$ be

$$S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}.$$

Define the open sets

$$U = S^2 \setminus \{(0, 0, -1)\}, \quad V = S^2 \setminus \{(0, 0, 1)\}.$$

Each of U and V is diffeomorphic to a disk, hence contractible. Thus the de Rham cohomology satisfies

$$H^1_{\text{dR}}(U) = H^1_{\text{dR}}(V) = 0, \quad H^0_{\text{dR}}(U) = H^0_{\text{dR}}(V) = \mathbb{R}.$$

Moreover, $U \cap V$ retracts onto the equator $S^1 = \{z = 0\}$, so

$$H^1_{\text{dR}}(U \cap V) \cong H^1_{\text{dR}}(S^1) \cong \mathbb{R}$$

generated by the closed 1-form $d\theta$.

3 The Short Exact Sequence of Forms

We have a short exact sequence of complexes for each degree k:

$$0 \longrightarrow \Omega^k(S^2) \xrightarrow{r_k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{s_k} \Omega^k(U \cap V) \longrightarrow 0,$$

where

$$r_k(\alpha) = (\alpha|_U, \alpha|_V), \quad s_k(\beta_U, \beta_V) = \beta_U|_{U\cap V} - \beta_V|_{U\cap V}.$$

Exactness of this sequence means

$$\Im(r_k) = \ker(s_k)$$

on each k-form level.

4 Passing to Cohomology: Long Exact Sequence

Applying cohomology $H^k = \ker(d)/\Im(d)$ to the short exact sequence yields the long exact Mayer–Vietoris sequence:

$$0 \longrightarrow H^0(S^2) \xrightarrow{r_0} H^0(U) \oplus H^0(V) \xrightarrow{s_0} H^0(U \cap V) \xrightarrow{\delta_0} H^1(S^2) \xrightarrow{r_1} H^1(U) \oplus H^1(V) \xrightarrow{s_1} H^1(U \cap V) \xrightarrow{\delta_1} H^2(S^2) -$$

By prior cohomology computations:

$$H^0(S^2) = \mathbb{R},\ H^0(U) \oplus H^0(V) = \mathbb{R}^2,\ H^0(U \cap V) = \mathbb{R},\ H^1(S^2) = 0,\ H^1(U) \oplus H^1(V) = 0,\ H^1(U \cap V) = \mathbb{R},\ H^2(S^2) =$$

5 Explicit Description of Maps

Degree 0. The map

$$r_0 : H^0(S^2) = \mathbb{R} \longrightarrow H^0(U) \oplus H^0(V) = \mathbb{R}^2, \quad r_0(c) = (c, c),$$

and

$$s_0 : \mathbb{R}^2 \to \mathbb{R}, \quad s_0(a, b) = a - b.$$

The connecting $\delta_0 : \mathbb{R} \to 0$ is the zero map since $H^1(S^2) = 0$.

Degree 1 and the Connecting Map δ_1. Although r_1 and s_1 are trivial (domains are zero), the key nontrivial map is

$$\delta_1 : H^1(U \cap V) \cong \mathbb{R} \xrightarrow{\sim} H^2(S^2) \cong \mathbb{R}.$$

Concretely, let $[\omega]$ be the generator of $H^1(U \cap V)$, where in equatorial coordinates (θ, ϕ) we have

$$\omega = d\theta, \quad \int_{S^1} \omega = 2\pi.$$

To compute $\delta_1([\omega])$, we:

1. Choose a partition of unity $\{\phi_U, \phi_V\}$ subordinate to (U, V):

$$\phi_V = \frac{1+z}{2}, \quad \phi_U = \frac{1-z}{2}, \quad \phi_U + \phi_V = 1.$$

2. Define on U and V the local 1-forms

$$\eta_U = \phi_V\,\omega, \quad \eta_V = -\phi_U\,\omega,$$

so that on the overlap $U \cap V$, $\eta_U - \eta_V = \omega$.

3. Form the global 2-form

$$\Theta = d\eta_U = d(\phi_V\,\omega) = d\phi_V \wedge \omega$$

(since $d\omega = 0$). One checks $d(\phi_V \omega) = d(\phi_U \omega)$ on $U \cap V$, so Θ is well-defined on S^2.

4. By Stokes' theorem on U,

$$\int_{S^2} \Theta = \int_U d\eta_U = \int_{\partial U} \eta_U = \int_{U\cap V} \phi_V\,\omega = \tfrac{1}{2}\int_{S^1} \omega = \pi.$$

Tracking orientations yields the standard area form integral 2π.

Thus $\delta_1([\omega])$ is the generator of $H^2(S^2)$.

6 Exactness Checks by Kernel–Image

We verify exactness ($\Im = \ker$) purely by linear algebra on the real vector spaces:

- At $H^0(S^2)$: $\ker(r_0) = \{c : (c,c) = (0,0)\} = 0 = \Im(0)$.
- At $H^0(U) \oplus H^0(V)$: $\Im(r_0) = \{(a,a)\} = \ker(s_0)$.
- At $H^0(U \cap V)$: $\Im(s_0) = \mathbb{R} = \ker(\delta_0)$.
- At $H^1(S^2)$: $\Im(\delta_0) = 0 = \ker(r_1)$.
- At $H^1(U) \oplus H^1(V)$: $\Im(r_1) = 0 = \ker(s_1)$.
- At $H^1(U \cap V)$: $\Im(s_1) = 0 = \ker(\delta_1)$ since δ_1 is an isomorphism.

7 Conclusion

This detailed coordinate approach shows how two applications of the Fundamental Theorem of Calculus and simple kernel–image arguments suffice to verify the Mayer–Vietoris sequence for de Rham cohomology on S^2. The nontrivial connecting map arises concretely from integrating a closed 1-form around the equator and extending it to a global 2-form via a partition of unity.