Set Theory II

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March 14, 2025

We cover the following topics in this note.

- Relations
- Equivalence Relations
- Equivalence Classes
- Partitions

Relation

Definition. Let $A \times B$ be the cartesian product of two sets A and B. A **(binary) relation** on $A \times B$ is a subset $\mathcal{R}$ of $A \times B$. That is,

$$\mathcal{R} \text{ is a relation on } A \times B \iff \mathcal{R} \subseteq A \times B.$$

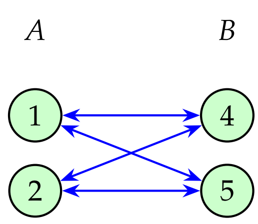
Remark. $\mathcal{R}$ is a relation on $A \iff \mathcal{R} \subseteq A \times A$.

Note (Notation). Let $(s,t) \in \mathcal{R}$. We use the notation $s\ \mathcal{R}\ t$ and we can say "s **is related to** t **by** R". If $(s,t) \notin \mathcal{R}$, we denote as: $s \not{\mathcal{R}}\ t$.

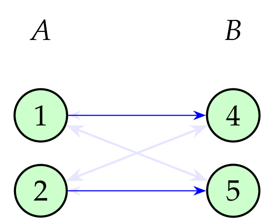
Example. Let $A = \{1,2\}$ and $B = \{4,5\}$. Then

$$A \times B = \left\{(1,4),(1,5),(2,4),(2,5)\right\}.$$

Here, $\mathcal{R} = \left\{(1,4),(2,5)\right\} \subseteq A \times B$ be a relation.



$A \times B$



$\mathcal{R} \subseteq A \times B$

Example. Let A and B are sets, and let $f : A \to B$ be a function from A to B. Then

$$(a,b) \in f \iff a\ f\ b \iff b = f(a).$$

⋆ Equivalence Relation ⋆

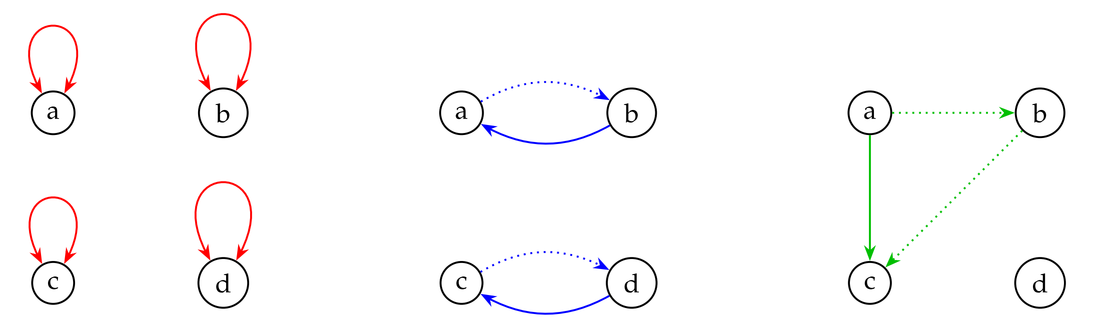
Definition. A binary relation $\mathcal{R}$ on a set S is called an **equivalence relation** if it satisfies the following three properties: for all $a, b, c \in S$,

(i) (Reflexivity) $(a, a) \in \mathcal{R}$;

(ii) (Symmetry) $(a, b) \in \mathcal{R} \implies (b, a) \in \mathcal{R}$;

(iii) (Transitivity) $(a, b) \in \mathcal{R} \wedge (b, c) \in \mathcal{R} \implies (a, c) \in \mathcal{R}$.

Remark.



Reflexivity (each element is related to itself)

Symmetry (if a is related to b, then b is related to a)

Transitivity (if a is related to b and b is related to c, then a is related to c)

Example. Let $A = \{1, 2, 3, 4\}$. Then

$$\mathcal{R} = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1)\}$$

is an equivalence relation on A.

Note. Let A, B, C are sets, and let $f : A \to B$ and $g : B \to C$ are functions.

- We claim that $\boxed{(g \circ f)[A] = g[f[A]]}$:

$$(g \circ f)[A] = \big\{(g \circ f)(a) : a \in A\big\} = \big\{g(f(a)) : a \in A\big\} = \big\{g(b) : b = f(a) \in f[A]\big\} = g[f[A]].$$

- We claim that $\boxed{f \text{ is surjective} \iff \operatorname{Img}(f) = f[A] = B}$:

$$f : A \twoheadrightarrow B \iff \forall b \in B,\ \exists a \in A \text{ s.t. } f(a) = b \iff f[A] = \big\{f(a) \in B : a \in A\big\} = B.$$

Lemma 1. *Let A, B and C are sets, and let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are functions.*

(1) If f and g are both one-to-one, then $(g \circ f) : A \longrightarrow C$ is one-to-one.

(2) If f and g are both onto, then $(g \circ f) : A \longrightarrow C$ is onto.

Proof.

□

Lemma 2. *Let A, B and C are sets, and let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are functions.*

(1) If $(g \circ f) : A \longrightarrow C$ is one-to-one, then f is one-to-one.

(2) If $(g \circ f) : A \longrightarrow C$ is onto, then g is onto.

Proof.

□

Equivalence Relation on 2^A Based on Bijection

Proposition 3. *Let A be a set, and 2^A be its power set. Define a relation $\mathcal{R}$ on 2^A as follows:*

$$X \sim_{\mathcal{R}} Y \iff \exists f \in Y^X \text{ such that } f \text{ is bijective},$$

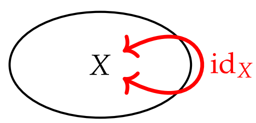
for $X, Y \in 2^A$. In other words,

$$\mathcal{R} := \left\{(X, Y) \in 2^A \times 2^A : \exists \text{ a bijection } f \in Y^X\right\}.$$

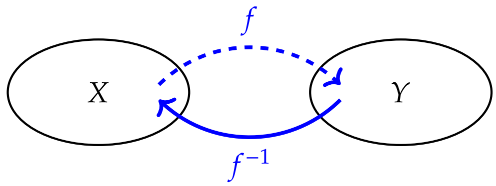
Then $\mathcal{R}$ is an equivalence relation on 2^A.

Proof. Let $X, Y, Z \in 2^A$. We must show that $\mathcal{R}$ is reflexive, symmetric and transitive:

(i) (Reflexivity)



(ii) (Symmetry)



(iii) (Transitivity)

$$
\begin{tikzcd}
X \arrow[r, dashed, "f"] \arrow[rr, bend left, "g \circ f"] & Y \arrow[r, dashed, "g"] & Z
\end{tikzcd}
$$

$\square$

Indexed Family

Definition. Let I and S are sets. Consider a function $A : I \to S$ defined by $i \mapsto A(i) =: A_i$. The image $\mathrm{Img}(A)$ is called an **indexed family of elements in S indexed by I**. We write this indexed family as: $\langle A_i \rangle_{i \in I}$. Note that

$$\mathrm{Img}(A) = \left\{A(i) : i \in I\right\} = \{A_i : i \in I\} = \langle A_i \rangle_{i \in I} .$$

Example (Sequence)**.** Let $I = \mathbb{N}$ be an indexing set. Then

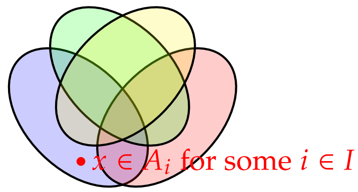
$$S := \{A_1, A_2, A_3, A_4, \cdots\} = \{A_i : i \in \mathbb{N}\} = \langle A_i \rangle_{i \in \mathbb{N}}$$

is an indexed family of elements in S indexed by $\mathbb{N}$.

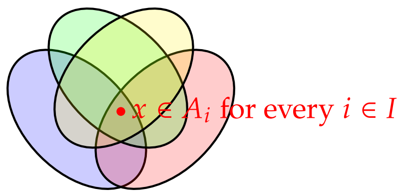
Union and Intersection of an Indexed Family

Definition. Let I and S are sets, and let $\langle A_i \rangle_{i \in I}$ be an indexed family in S.

- The **union of** $\langle A_i \rangle_{i \in I}$ is defined by $\boxed{\displaystyle\bigcup_{i \in I} A_i := \{x \in S : \exists i \in I \text{ such that } x \in A_i\}}$.



- The **intersection of** $\langle A_i \rangle_{i \in I}$ is defined by $\boxed{\displaystyle\bigcap_{i \in I} A_i := \{x \in S : \forall i \in I,\ x \in A_i\}}$.



Remark. Let $I = \{1, \ldots, n\}$. Then

- $\displaystyle\bigcup_{i \in I} S_i = \bigcup_{i=1}^{n} S_i = S_1 \cup S_2 \cup \cdots \cup S_n.$
- $\displaystyle\bigcap_{i \in I} S_i = \bigcap_{i=1}^{n} S_i = S_1 \cap S_2 \cap \cdots \cap S_n.$

⋆ Partitions ⋆

Definition. Let S be a set, and let the function $A : I \longrightarrow 2^S$ as $i \mapsto A_i := A(i) \subseteq S$, for all $i \in I$. Consider a family of subsets $\langle A_i \rangle_{i \in I}$, where $A_i \subseteq S$ for every index $i \in I$. The family $\langle A_i \rangle_{i \in I}$ is called a **partition of** S if the following conditions are satisfied:

(i) (**Non-empty Subsets**) Each subset A_i is non-empty, i.e., $\boxed{\forall i \in I,\ A_i \neq \varnothing}$.

(ii) (**Pairwise disjoint**) For all distinct $i, j \in I$, the subsets A_i and A_j are disjoint, i.e.,

$$\boxed{\forall i, j \in I,\ i \neq j \implies A_i \cap A_j = \varnothing}.$$

(iii) (**Union covers the entire set**) The union of all subsets A_i covers the whole set S, i.e.,

$$\boxed{\bigcup_{i \in I} A_i = S}.$$

Example. Let $\mathbb{Z}$ be a set of integers. We define an indexed family $\langle A_i \rangle_{i \in \{0,1,2\}}$ of subsets of $\mathbb{Z}$ as follows:

$$\begin{aligned}
A_0 &= \left\{n \in \mathbb{Z} : n \equiv 0 \pmod 3\right\} = \{n \in \mathbb{Z} : n = 3k + 0 \text{ for some } k \in \mathbb{Z}\} =: [0],\\
A_1 &= \left\{n \in \mathbb{Z} : n \equiv 1 \pmod 3\right\} = \{n \in \mathbb{Z} : n = 3k + 1 \text{ for some } k \in \mathbb{Z}\} =: [1],\\
A_2 &= \left\{n \in \mathbb{Z} : n \equiv 2 \pmod 3\right\} = \{n \in \mathbb{Z} : n = 3k + 2 \text{ for some } k \in \mathbb{Z}\} =: [2].
\end{aligned}$$

Then

(i) $[0] \neq \varnothing$, $[1] \neq \varnothing$ and $[2] \neq \varnothing$.

(ii)

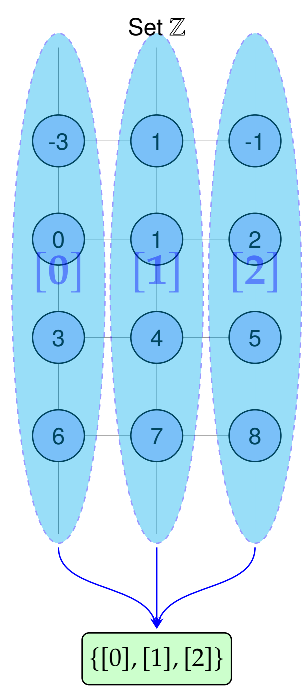
$$\begin{aligned}
[0] \cap [1] &= \varnothing,\\
[1] \cap [2] &= \varnothing,\\
[2] \cap [0] &= \varnothing.
\end{aligned}$$

(iii) $[0] \cup [1] \cup [2] = \mathbb{Z}$.

Thus,

$$\{A_1, A_2, A_3\} = \big\{[0], [1], [2]\big\}$$

is a partition of $\mathbb{Z}$.

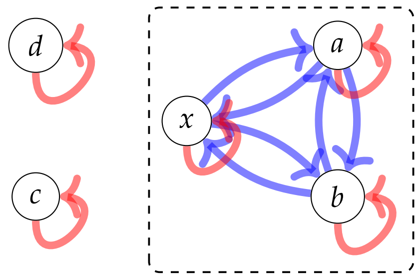


⋆ Equivalence Class ⋆

Definition. Let $\mathcal{R} \subseteq S \times S$ be an equivalence relation on S. The **equivalence class of** $x \in S$ **under** $\mathcal{R}$ is the set

$$[x]_{\mathcal{R}} = \left\{ y \in S : x \mathrel{\mathcal{R}} y \right\}.$$

Note. Note that $\alpha \mathrel{\mathcal{R}} x \iff \alpha \in [x]_{\mathcal{R}} \iff x \mathrel{\mathcal{R}} \alpha$.



Equivalence class of x

Lemma 4. *Let $\mathcal{R}$ be an equivalence relation on a set S. For any $x, y \in S$, let $[x]$ and $[y]$ represent the equivalence classes of x and y, respectively, under $\mathcal{R}$.*

(1) $\forall x \in S,\ x \in [x]$.

(2) $x \mathcal{R} y \iff [x] = [y]$.

(3) $x \not\mathcal{R} y \iff [x] \cap [y] = \varnothing$.

Proof.

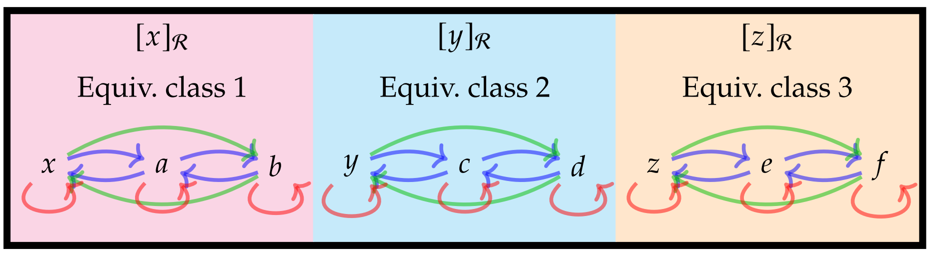
$\square$

⋆ ⋆ ⋆ Fundamental Theorem on Equivalence Relations ⋆ ⋆ ⋆

Theorem 5. *Let S be a set and let $\mathcal{R}$ be an equivalence relation on S. Define the set of equivalence classes*

$$\mathcal{P} := \left\{[x]_{\mathcal{R}} : x \in S\right\}, \text{ where } [x]_{\mathcal{R}} = \left\{y \in S : x \,\mathcal{R}\, y\right\}.$$

Then $\mathcal{P}$ forms a partition of S.



$S = \left\{x, y, z, a, b, c, d, e, f\right\}$

$\mathcal{P} = \{[x]_{\mathcal{R}}, [y]_{\mathcal{R}}, [z]_{\mathcal{R}}\}$

Proof.

□

⋆ Relation Induced by Partition is Equivalence ⋆

Theorem 6. *Let S be a set and $\mathcal{P} = \langle P_i \rangle_{i \in I}$ be a partition of S. We define a relation $\mathcal{R}$ on S:*

$$x \sim_{\mathcal{R}} y \iff \exists i \in I \text{ such that } x, y \in P_i$$

for all $x, y \in S$. That is, x is related to y under $\mathcal{R}$ if and only if x and y belong to the same subset P_i in the partition. Then $\mathcal{R}$ is the equivalence relation induced by a partition $\mathcal{P}$.

Proof.

□