

Let

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle R t - R \sin t, R - R \cos t \rangle, \quad t \in [0, 2\pi].$$

Then

$$\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle = \langle R(1 - \cos t), R \sin t \rangle.$$

The differential arc-length is

$$\left\|\mathbf{r}'(t)\right\|dt = \sqrt{\left[R(1-\cos t)\right]^2 + \left[R\sin t\right]^2} dt.$$

Inside the square-root:

$$R^2(1-\cos t)^2 + R^2\sin^2 t = R^2\left[1-2\cos t + \cos^2 t + \sin^2 t\right] = R^2\left[2-2\cos t\right] = 2R^2\left(1-\cos t\right).$$

Hence

$$\|\mathbf{r}'(t)\| = \sqrt{2R^2(1-\cos t)} = R\sqrt{2(1-\cos t)} = 2R|\sin\frac{t}{2}|.$$

On $0 \le t \le 2\pi$, $\sin(t/2) \ge 0$, so

$$\|\mathbf{r}'(t)\| = 2R\sin\frac{t}{2}.$$

The total length of one arch is

$$L = \int_0^{2\pi} \|\mathbf{r}'(t)\| dt = \int_0^{2\pi} 2R \sin \frac{t}{2} dt.$$

Make the substitution $u = \frac{t}{2}$, so dt = 2 du and when t runs from 0 to 2π , u runs from 0 to π :

$$L = 2R \int_0^{2\pi} \sin \frac{t}{2} dt = 2R \int_0^{\pi} \sin u \, (2 \, du) = 4R \int_0^{\pi} \sin u \, du = 4R \left[-\cos u \right]_0^{\pi} = 4R \left[-(-1) - (-1) \right] = 8R.$$

$$L_{\text{one arch}} = 8R.$$