Topology I

Ji, Yong-hyeon

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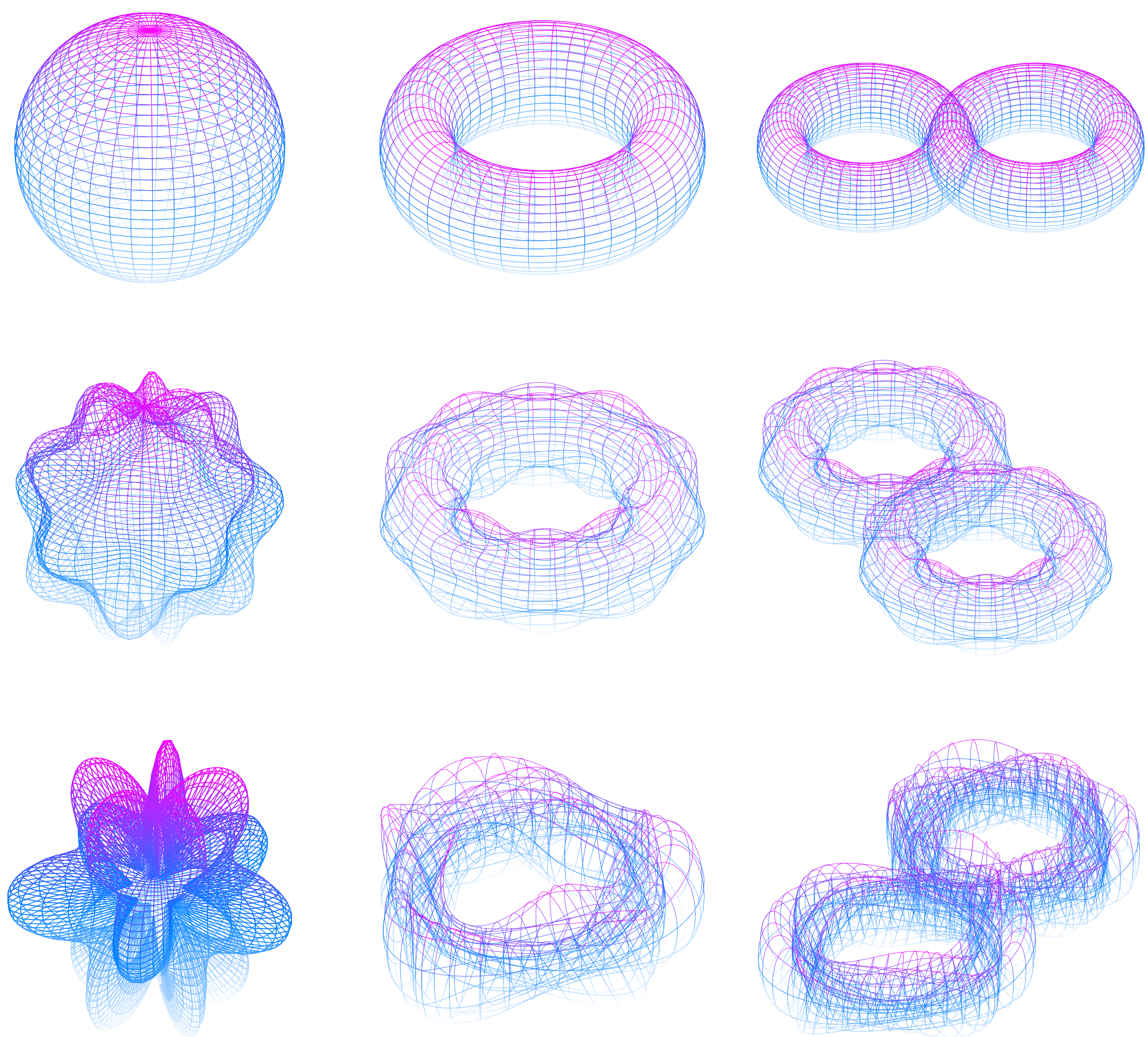
We cover the following topics in this note.

- Topology; Topological Space
- Open Set
- Continuous Mapping
- ~~Distance Function; Metric Topology~~



Topology

Definition. Let S be a non-empty set. A **topology** on S is a subset

$$\mathscr{T} = \{U : U \subseteq S\} \subseteq 2^S$$

that satisfies the axioms:

(O1) S and $\varnothing$ are elements of $\mathscr{T}$: $S \in \mathscr{T}$ and $\varnothing \in \mathscr{T}$.

(O2)[a] The union of an arbitrary subset of $\mathscr{T}$ is an element of $\mathscr{T}$:

$$\{U_\alpha\}_{\alpha\in\Lambda} \subseteq \mathscr{T} \implies \bigcup_{\alpha\in\Lambda} U_\alpha \in \mathscr{T}.$$

(O3)[b] The intersection of any finite subset of $\mathscr{T}$ is an element of $\mathscr{T}$:

$$\{U_i\}_{i=1}^n \subseteq \mathscr{T} \implies \bigcap_{i=1}^n U_i \in \mathscr{T}.$$

[a] $\mathscr{T}$ is closed under *arbitrary* unions
[b] $\mathscr{T}$ is closed under *finite* intersection

Remark. By mathematical induction, we have

$$\text{O3} \iff [\{U_1, U_2\} \subseteq \mathscr{T} \Rightarrow U_1 \cap U_2 \in \mathscr{T}].$$

Topological Space

Definition. Let S be a set. Let $\mathscr{T}$ be a topology on S. Then the ordered pair $(S, \mathscr{T})$ is called a **topological space**.

Open Set

Definition. Let $(S, \mathscr{T})$ be a topological space. $E \subseteq S$ is an **open set**, or **open** (in S) iff $E \in \mathscr{T}$.

Remark. A subset $\mathscr{T} \subseteq 2^S$ is a topology on S if and only if

(i) $\varnothing$ and S are open;

(ii) Let $\{E_\alpha\}_{\alpha\in\Lambda} \subseteq \mathscr{T}$. Then $\bigcup_{\alpha\in\Lambda} E_\alpha$ is open.

(iii) Let $\{E_i\}_{i=1}^n \subseteq \mathscr{T}$. Then $\bigcap_{i=1}^n E_i$ is open.

Continuous Mapping

Definition. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are topological spaces. Let $f : X \to Y$ be a mapping from X to Y.

(1) (Continuous at a Point) Let $x \in X$. The mapping f is **continuous at x** if and only if

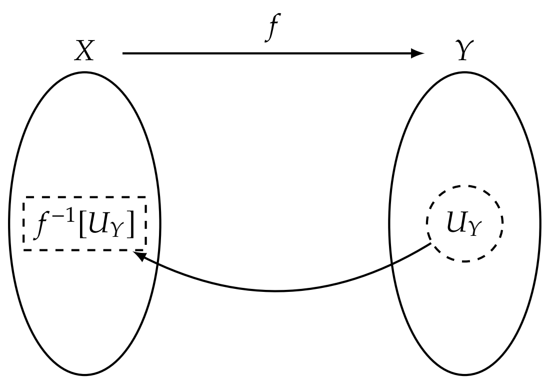
$$\forall U_Y \in \mathscr{T}_Y, \ (f(x) \in U_Y \implies \exists U_X \in \mathscr{T}_X \text{ such that } x \in U_X \wedge f[U_X] \subseteq U_Y.)$$

(2) (Continuous on a Set) Let $S \subseteq X$. The mapping f is **continuous on S** if and only if f is continuous at every point $x \in S$.

(3) (Continuous Everywhere) The mapping f is **continuous on X** if and only if

$$U_Y \in \mathscr{T}_Y \implies f^{-1}[U_Y] \in \mathscr{T}_X,$$

where $f^{-1}[U_Y] = \{x \in X : f(x) \in U_Y\}$ is the preimage of U_Y under f.



Lemma 1. *Let X be a set.*

(1) (***The Intersection of Finite Sets is Finite or Empty***) *Let $\{A_i\}_{i\in I}$ be a collection of finite sets. Then*

$$\bigcap_{i\in I} A_i$$

is finite if I is finite, or empty otherwise.

(2) (***The Union of Finitely Many Finite Sets is Finite***) *Let $\{A_i\}_{i=1}^{n}$ be a finite collection of finite sets. Then*

$$\bigcup_{i=1}^{n} A_i$$

is finite.

Proof. Let $|A_i| < \infty$ for all $i \in I$. The intersection is defined as:

$$\bigcap_{i\in I} A_i = \{x \in X \mid x \in A_i \text{ for all } i \in I\}.$$

Part 1 (Case 1: I is finite)**.** Suppose $I = \{i_1, i_2, \ldots, i_n\}$ is finite. Then:

$$\bigcap_{i\in I} A_i = A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_n}.$$

Since each A_{i_k} is finite, the intersection cannot introduce new elements:

$$\bigcap_{k=1}^{n} A_{i_k} \subseteq A_{i_k} \quad \text{for all } k.$$

Therefore, $\bigcap_{i\in I} A_i$ is a subset of the smallest A_{i_k} and is finite.

Part 2 (Case 2: I is infinite)**.** If I is infinite, the intersection may be empty. For any $x \in \bigcap_{i\in I} A_i$, x must belong to all A_i. If the A_i's shrink (e.g., $A_i = \{1, 2, \ldots, i\}$), then for large i, $A_i \cap A_j = \emptyset$. Hence:

$$\bigcap_{i\in I} A_i = \emptyset.$$

Thus, the intersection of finite sets is finite if I is finite, or empty otherwise. □

Proof. Each A_i is finite, meaning $|A_i| < \infty$ for $i = 1, 2, \ldots, n$. The union satisfies:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n.$$

To compute the size of the union, we use the inclusion-exclusion principle:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \cdots + (-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|.$$

- Each term in this expansion represents the size of overlaps between the A_i, all of which are finite because the A_i's are finite.

Since a finite sum of finite numbers is finite, we conclude that:

$$\bigcup_{i=1}^{n} A_i$$

is finite. □

Example 1 (Cofinite Topology). Let S be a set. Define a subset $\mathscr{T}_C \subseteq 2^S$ by

$$\mathscr{T}_C := \left\{T \subseteq S : T^C \text{ is a finite set}\right\} \cup \{\varnothing\}$$

We claim that $\mathscr{T}_C$ be a topology on S:

(i) Clearly $\subseteq\in \mathscr{T}_C$. Since $S^C = \varnothing$ and $\varnothing$ is finite, $S \in \mathscr{T}$.

(ii) Let $\{E_\alpha\}_{\alpha\in\Lambda} \subseteq \mathscr{T}_C$. Then

$$\left(\bigcup_{\alpha\in\Lambda} E_\alpha\right)^C = \bigcap_{\alpha\in\Lambda} E_\alpha^C$$

and so

(iii)

Example 2 (Discrete Topology).

Example 3 (Indiscrete Topology).

Finer and Coarser

Definition.

Distance Function

Definition. TBA

Metric Topology

Definition. TBA

References

[1] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 8. 위상수학 (a) 위상공간의 정의." YouTube Video, 41:25. Published September 27, 2019. URL: https://www.youtube.com/watch?v=q8BtXIFzo2Q.

[2] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 9. 위상수학 (b) 해석학개론과 거리위상" YouTube Video, 33:43. Published September 29, 2019. URL: https://www.youtube.com/watch?v=uJOGw7Yxk7c&t=242s.

A Complement of Family

Note.

$$\left(\bigcup_{i\in\Lambda} E_i\right)^C = \bigcap_{i\in\Lambda} (E_i)^C$$

Proof. content... □