Big-Number Arithmetic: HW1

Ji, Yonghyeon

July 8, 2025

Contents

1 Preliminaries

Definition 1.1 (Hexadecimal Expansion)**.** A **hexadecimal integer** of length n is any element

$$A \;=\; \sum_{i=0}^{n-1} a_i\, 16^i, \quad a_i \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \dots, \mathtt{9}, \mathtt{A}, \dots, \mathtt{F}\}.$$

We denote A in radix–16 by $(a_{n-1} \dots a_1 a_0)_{16}$.

2 Addition

2.1 Algorithmic Formulation

Definition 2.1 (Hexadecimal Addition)**.** Let

$$A = \sum_{i=0}^{n-1} a_i\, 16^i, \quad B = \sum_{i=0}^{n-1} b_i\, 16^i,$$

with $a_i, b_i \in \{\mathtt{0}, \dots, \mathtt{F}\}$. Define the recurrence:

$$r_i = a_i + b_i + c_{i-1}, \quad c_{-1} = 0,$$

and set

$$s_i = r_i \bmod 16, \qquad c_i = \left\lfloor \frac{r_i}{16} \right\rfloor,$$

for $i = 0, 1, \dots, n-1$. Then

$$A + B = \sum_{i=0}^{n-1} s_i\, 16^i + c_n\, 16^n,$$

expressed as the $(n+1)$–digit hexadecimal $(c_n\, s_{n-1} \dots s_1 s_0)_{16}$.

2.2 Examples

Example 2.1 (Addition of $\mathtt{F4C3}_{16}$ and $\mathtt{2B7D}_{16}$)**.** Let

$$A = \mathtt{F4C3}_{16}, \quad B = \mathtt{2B7D}_{16}, \quad n = 3.$$

Compute:

$$\begin{array}{cccccc}
 & \mathtt{1} & \mathtt{1} & \mathtt{1} & \mathtt{1} & \mathtt{0} \\
 & & \mathtt{F} & \mathtt{4} & \mathtt{C} & \mathtt{3} \\
+ & & \mathtt{2} & \mathtt{B} & \mathtt{7} & \mathtt{D} \\
\hline\hline
 & \mathtt{1} & \mathtt{2} & \mathtt{0} & \mathtt{4} & \mathtt{0}
\end{array}$$

i	0	1	2	3
a_i	3	C	4	F
b_i	D	7	B	2
c_{i-1}	0	1	1	1
$r_i = a_i + b_i + c_{i-1}$	16	20	16	18
$s_i = r_i \bmod 16$	0	4	0	2
$c_i = \lfloor r_i/16 \rfloor$	1	1	1	1

Since $c_3 = 1$, one obtains

$$\mathtt{F4C3}_{16} + \mathtt{2B7D}_{16} = \mathtt{12040}_{16}.$$

Example 2.2 (Addition of $\texttt{C0FFEE}_{16}$ and $\texttt{1BADF00D}_{16}$)**.** Pad to $n = 7$ by writing

$$A = \texttt{00\,C0\,FF\,EE}_{16}, \quad B = \texttt{1B\,AD\,F0\,0D}_{16}.$$

Compute for $i = 0, \ldots, 7$:

	0	1	1	1	0	0	1	0
			C	0	F	F	E	E
+	1	B	A	D	F	0	0	D
	1	C	6	E	E	F	F	B

i	0	1	2	3	4	5	6	7
a_i	E	E	F	F	0	C	0	0
b_i	D	0	0	F	D	A	B	1
c_{i-1}	0	1	0	0	1	0	1	0
r_i	27	15	15	30	14	22	12	1
$s_i = r_i \bmod 16$	B	F	F	E	E	6	C	1
$c_i = \lfloor r_i/16 \rfloor$	1	0	0	1	0	1	0	0

Thus

$$\texttt{C0FFEE}_{16} + \texttt{1BADF00D}_{16} = \texttt{1C6EEFFB}_{16}.$$

3 Subtraction

3.1 Algorithmic Formulation

Definition 3.1 (Hexadecimal Subtraction)**.** Let

$$A = \sum_{i=0}^{n-1} a_i\, 16^i, \quad B = \sum_{i=0}^{n-1} b_i\, 16^i,$$

with $a_i, b_i \in \{\texttt{0}, \ldots, \texttt{F}\}$. Define the recurrence:

$$r_i = a_i - b_i - d_{i-1}, \quad d_{-1} = 0,$$

and set

$$t_i = r_i \bmod 16, \qquad d_i = \begin{cases} 1, & r_i < 0, \\ 0, & r_i \geq 0. \end{cases}$$

Then

$$A - B = \sum_{i=0}^{n-1} t_i\, 16^i \ - \ d_n\, 16^{n+1}.$$

3.2 Examples

Example 3.1 (Subtraction of $\texttt{A5B2}_{16}$ and $\texttt{3C7F}_{16}$)**.** Let

$$A = \texttt{A5B2}_{16}, \quad B = \texttt{3C7F}_{16}, \quad n = 3.$$

Compute:

	1	0	1	0
	A	5	B	2
−	3	C	7	F
+	6	9	3	3

i	0	1	2	3
a_i	2	B	5	A
b_i	F	7	C	3
d_{i-1}	0	1	0	1
$r_i = a_i - b_i - d_{i-1}$	−13	3	−7	6
$t_i = r_i \bmod 16$	3	3	9	6
$d_i = \mathbf{1}_{r_i<0}$	1	0	1	0

Hence

$$\texttt{A5B2}_{16} - \texttt{3C7F}_{16} = \texttt{6933}_{16}.$$

Example 3.2 (Subtraction of $\texttt{DEAD}_{16}$ and $\texttt{BEEF}_{16}$)**.** Let

$$A = \texttt{DEAD}_{16}, \quad B = \texttt{BEEF}_{16}, \quad n = 3.$$

Compute:

$$\begin{array}{c|c|c|c|c|}
 & \texttt{1} & \texttt{1} & \texttt{1} & \color{red}{0} \\
\cline{2-5}
 & \texttt{D} & \texttt{E} & \texttt{A} & \texttt{D} \\
\cline{2-5}
- & \texttt{B} & \texttt{E} & \texttt{E} & \texttt{F} \\
\hline\hline
+ & \texttt{1} & \texttt{F} & \texttt{B} & \texttt{E} \\
\cline{2-5}
\end{array}$$

i	0	1	2	3
a_i	D	A	E	D
b_i	F	E	E	B
d_{i-1}	0	1	1	1
$r_i = a_i - b_i - d_{i-1}$	-2	-5	-1	1
$t_i = r_i \bmod 16$	E	B	F	1
$d_i = \mathbf{1}_{r_i<0}$	1	1	1	0

Hence

$$\texttt{DEAD}_{16} - \texttt{BEEF}_{16} = \texttt{2CCD}_{16}.$$

4 Exercises

1. $\texttt{A5B2}_{16} + \texttt{C3F9}_{16}$
2. $\texttt{7D3E}_{16} + \texttt{1A4C}_{16}$
3. $\texttt{F4C3}_{16} + \texttt{2B7D}_{16}$
4. $\texttt{9AFE}_{16} + \texttt{65B1}_{16}$
5. $\texttt{BEEF}_{16} + \texttt{DEAD}_{16}$
6. $\texttt{C0FFEE}_{16} + \texttt{1BADF00D}_{16}$
7. $\texttt{F4C3}_{16} - \texttt{2A9D}_{16}$
8. $\texttt{A5B2}_{16} - \texttt{3C7F}_{16}$
9. $\texttt{DEAD}_{16} - \texttt{BEEF}_{16}$
10. $\texttt{1BADF00D}_{16} - \texttt{C0FFEE}_{16}$
11. $\texttt{7D3E}_{16} - \texttt{1A4C}_{16}$
12. $\texttt{9AFE}_{16} - \texttt{65B1}_{16}$