Complex Analysis & Vector Calculus Homework Notes

Contents

1 Homework 1: Vector Calculus and Differential Forms

Line Integrals for Vector Fields

Definition 1. Given a curve $\gamma : [a,b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$), let F be a vector field defined on a neighborhood of γ. It makes sense to talk about $F(\gamma(t)) \cdot \gamma'(t)$ for each $t \in (a,b)$. Define the **line integral** of a vector field F along a curve γ as:

$$\int_\gamma F \cdot d\gamma := \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt$$

Concretely, if $F(x,y) = (F_1(x,y), F_2(x,y)) \in \mathbb{R}^2$ and $dr = (dx, dy)$:

$$\int_\gamma F \cdot d\gamma = \int (F_1, F_2) \cdot (dx, dy) = \int F_1 dx + F_2 dy$$

Exercise 1. Let C be the unit circle traversed in a counterclockwise direction. Let $F(x,y) = \left(-\frac{y}{r^2}, \frac{x}{r^2}\right)$, where $r^2 = x^2 + y^2$. Compute $\int_C F \cdot dr$.

Surface Integrals for Vector Fields

Let S be a surface in $\mathbb{R}^3$. Let F be a vector field on S. Let $T : D \subseteq \mathbb{R}^2 \to S$ be a parametrization, $(u,v) \mapsto T(u,v)$. Define the outward normal N.

$$\iint_S F \cdot dS := \iint_D F(T(u,v)) \cdot \left(\frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v}\right) dA$$

Exercise 2. Compute $\iint_S F \cdot dS$. Here, $F(x,y,z) = (x, y, -z)$. Parametrization: $x = u + v$, $y = v - u$, $z = 3u$ for $0 \le u, v \le 1$.

Exercise 3. Let $F(x,y,z) = (y, xz, 1)$. Let C be the unit circle in the xy-plane, i.e., $x^2+y^2 = 1$, $z = 0$, oriented counterclockwise. Compute $\int_C F \cdot dr$.

Exercise 4. Same F as Exercise 3. Surface S is the disk in the xy-plane: $x^2 + y^2 \le 1, z = 0$. Normal $N = (0,0,1)$. Compute $\iint_S \mathrm{curl} F \cdot dS$.

Exercise 5. Same F as Exercise 3. S is the hemisphere: $x^2 + y^2 + z^2 = 1, z \ge 0$. Compute $\iint_S \mathrm{curl} F \cdot dS$.

Exercise 6. Same F as Exercise 3. S is the paraboloid: $z = 1 - x^2 - y^2$, $z \ge 0$. Compute $\iint_S \mathrm{curl} F \cdot dS$.

Exercise 7. Exercise 3부터 Exercise 6까지 결과값이 전부다 같음을 주목하고 왜 그런지 눈치채시오. 특히, 스토크스 정리의 특수한 경우로서 그린의 정리를 매우 직관적으로 이해하다. (Translation: Notice that the results from Exercise 3 to Exercise 6 are all the same and realize why. In particular, understand Green's Theorem intuitively as a special case of Stokes' Theorem.)

Exterior Derivative and Differential Forms

Definition 2 (Exterior Derivative)**.** For each smooth function f, we say f is a **0-form**. For coordinates $x_1, \ldots, x_n$, define $dx_1, \ldots, dx_n$. A **1-form** is a linear combination $a_1 dx_1 + \cdots + a_n dx_n$, where coefficients are smooth functions. For a 0-form $f(x_1, \ldots, x_n)$, define:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

Define the **wedge product** $\wedge$ between forms:

- $\alpha \wedge \beta = -\beta \wedge \alpha$ (Anticommutativity)
- $\alpha \wedge \alpha = 0$ (e.g., $dx_1 \wedge dx_1 = 0$)
- Associativity: $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$

A **k-form** ω can be written as:

$$\omega = \sum_I a_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Define the $(k+1)$-form $d\omega$ by:

$$d\omega = \sum_I da_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Fact: $d(d\omega) = 0$ for any form ω.

Complex Case: $z = x + iy$, $dz = dx + idy$. For $f(z, \bar{z})$, $df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$.

Exercise 8. Compute $d\left(\frac{f(w,\bar{w})}{w-z} dw\right)$. Show that it equals $\frac{1}{w-z} \frac{\partial f}{\partial \bar{w}} d\bar{w} \wedge dw$ for $z \in \Omega$ (bounded domain with smooth boundary).

Theorems in Differential Forms

- **Fundamental Theorem of Calculus:** $\int_a^b f'(x)dx = f(b) - f(a) \iff \int_{[a,b]} df = \int_{\partial[a,b]} f$.
- **Fundamental Theorem of Line Integrals (FTLI):** $\int_C \nabla f \cdot dr = f(q) - f(p) \iff \int_C df = \int_{\partial C} f$.
- **Stokes' Theorem:** $\iint_S \text{curl} F \cdot dS = \int_C F \cdot dr \iff \int_\Omega d\eta = \int_{\partial\Omega} \eta$.

Exercise 9. Let $\eta = Pdx + Qdy + Rdz$. Compute $d\eta$. Conclude that $\int_{\partial\Omega} \eta = \int_C F \cdot dr$ relates to $\int_\Omega d\eta = \iint_S \text{curl} F \cdot dS$.

Exercise 10. Let $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$. Compute $d\omega$. Conclude this case corresponds to the Divergence Theorem.

Exercise 11 (Cauchy-Green Formula)**.** Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with smooth boundary $\partial\Omega$. For any $f \in C^1(\bar{\Omega})$:

$$f(z) = \frac{1}{2\pi i} \left[\int_{\partial\Omega} \frac{f(w)}{w-z} dw - \iint_\Omega \frac{\frac{\partial f}{\partial \bar{w}}}{w-z} d\bar{w} \wedge dw \right]$$

Prove the formula above.

2 Homework 2: Potential Functions

Exercise 12. Given a vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $F(x, y) := (3x^2 + 6xy, 3x^2 + 6y)$. Find a potential function $f : \mathbb{R}^2 \to \mathbb{R}$ of F, i.e., f satisfies $F = \nabla f$.

Exercise 13. Exercise 1을 다른 방법으로 푸시오. (Solve Exercise 1 in a different way.)

Definition 3. Given a vector field F, $\int_C F \cdot dr$ is **path-independent** for any two points p, q, if the integral yields the same value for any path connecting p and q.

Example 1 (FTLI)**.** If $F = \nabla f$, then $\int_C F \cdot dr = f(q) - f(p)$, which is path-independent.

Exercise 14. 다음을 보여라: (Show the following:) $\int_C F \cdot dr$ is path-independent for any two points if and only if $\oint_C F \cdot dr = 0$ for any closed loops C.

Exercise 15. Let D be a connected region in $\mathbb{R}^2$, $F : D \to \mathbb{R}^2$. Assume F satisfies Exercise 3. 그러면 $F = \nabla f$ 인 f가 존재함을 보여라. (Show that there exists an f such that $F = \nabla f$.) Such an F is called a **conservative vector field**.

Exercise 16. Check if the vector field F from HW1 Exercise 1 is a gradient vector field or not.

3 Homework 3 / Notes: Winding Numbers

Topic: Complex Analysis

Let $F(x, y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$. Consider the path integral $\int_C F \cdot dr$.

Example Calculation

Let C be a circle of radius 1 centered at $(2, 0)$. Parametrization:

$$x = \cos t + 2, \quad y = \sin t, \quad 0 \le t \le 2\pi$$

Then $x^2+y^2 = (\cos t+2)^2+\sin^2 t = \cos^2 t+4\cos t+4+\sin^2 t = 5+4\cos t$. Also $dx = -\sin t dt$, $dy = \cos t dt$.

$$\begin{aligned}
\int_C F \cdot dr &= \int_C \left(-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right) \\
&= \int_0^{2\pi} \frac{1}{5+4\cos t}\left(-\sin t(-\sin t) + (\cos t+2)(\cos t)\right) dt \\
&= \int_0^{2\pi} \frac{\sin^2 t + \cos^2 t + 2\cos t}{5+4\cos t} dt \\
&= \int_0^{2\pi} \frac{1+2\cos t}{5+4\cos t} dt
\end{aligned}$$

Method 1: Using FTLI

If $F = \nabla f$ (locally), and the loop does not enclose the singularity at $(0, 0)$, then $\int_C F \cdot dr = 0$.

Method 2: Direct Calculation

To compute $I = \int_0^{2\pi} \frac{1+2\cos t}{5+4\cos t} dt$. Use substitution $u = \tan(t/2)$.

$$\sin t = \frac{2u}{1+u^2}, \quad \cos t = \frac{1-u^2}{1+u^2}, \quad dt = \frac{2}{1+u^2}du$$

Substituting these into the integral:

$$\begin{aligned}
\frac{1+2\cos t}{5+4\cos t} &= \frac{1+2\left(\frac{1-u^2}{1+u^2}\right)}{5+4\left(\frac{1-u^2}{1+u^2}\right)} = \frac{(1+u^2)+2(1-u^2)}{5(1+u^2)+4(1-u^2)} \\
&= \frac{3-u^2}{9+u^2}
\end{aligned}$$

Thus,

$$I = \int_{-\infty}^{\infty} \frac{3-u^2}{9+u^2} \frac{2}{1+u^2} du$$

Evaluating this integral yields 0 (via Partial Fractions or Residue Calculus).

Complexification

Observe that

$$F(x,y) \cdot dr = \frac{-ydx + xdy}{x^2 + y^2}$$

Let $z = x + iy$. Then $dz = dx + idy$, $d\bar{z} = dx - idy$. Using differential forms algebra:

$$\frac{1}{z}dz = \frac{\bar{z}}{z\bar{z}}dz = \frac{x - iy}{x^2 + y^2}(dx + idy) = \frac{xdx + ydy}{x^2 + y^2} + i\frac{xdy - ydx}{x^2 + y^2}$$

Thus,

$$\int_C F \cdot dr = \mathrm{Im} \int_C \frac{1}{z}dz$$

- If C encloses the origin (e.g., unit circle centered at 0), $\int_C \frac{1}{z}dz = 2\pi i$, so imaginary part is 2π.
- If C does not enclose the origin (like the example centered at $(2, 0)$), the function is analytic inside, so by Cauchy Integral Theorem, integral is 0.

Keywords

- Cauchy Integral Theorem
- Morera's Theorem
- Meaning of Complex Analytic Functions / Cauchy-Riemann Equations / Holomorphic Functions