Advanced Calculus II

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We cover the following topics in this note.

- Convergence of Sequences
- Inequality Rule for Reals
- ~~Limit Theorem (Algebraic Property of Limit of Sequence)~~

Sequence

Definition. Let $A \subseteq \mathbb{N}$ and $X \subseteq \mathbb{R}$. A **sequence** is a function

$$a : A \to X$$

with domain A and range in X.

Remark. A function a is a real sequence if

$$\begin{array}{rcccl} a & : & \mathbb{N} & \longrightarrow & \mathbb{R} \\ & & n & \longmapsto & a(n) =: a_n \end{array}$$

for $n = 1, 2, \cdots$. We write

$$\{a_n\}_{n=1}^{\infty}, \quad \{a_n\}_{n\in\mathbb{N}}, \quad \{a_n\}, \quad (a_n)_{n\in\mathbb{N}}, \quad \text{or} \quad \langle a_n \rangle_{n\in\mathbb{N}}.$$

Convergence of Sequence

Definition. A real sequence $\{a_n\}_{n=1}^{\infty}$ $(\subseteq \mathbb{R})$ is said to **converge** to $L \in \mathbb{R}$ if and only if

$$\forall \varepsilon > 0,\ \exists N_\epsilon \in \mathbb{N} \text{ such that } \left[n \geq N_\epsilon \implies |a_n - L| < \varepsilon \right].$$

Remark. A real number $L \in \mathbb{R}$ is called **the limit**. When a sequence $\{a_n\}_{n=1}^{\infty}$ has the limit L, we will use the notation

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty.$$

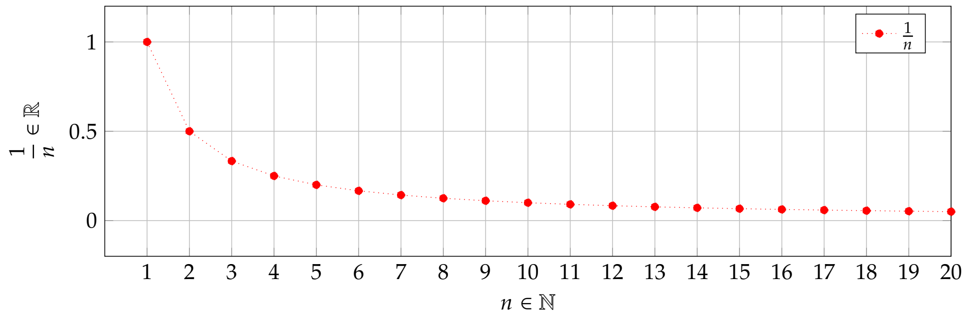
That is,

$$\lim_{n\to\infty} a_n = L \iff \forall \varepsilon > 0 : \exists N \in \mathbb{N} : \left[n \geq N \implies |a_n - L| < \varepsilon \right].$$

Note. If a sequence has a limit, we say that the sequence is **convergent**; if it has no limit, we say that the sequence is **divergent**.

Example. Consider the sequence defined by $a_n = 1/n$ for each $n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$



Proof. Let $\varepsilon > 0$. By the Archimedean property, we obtain

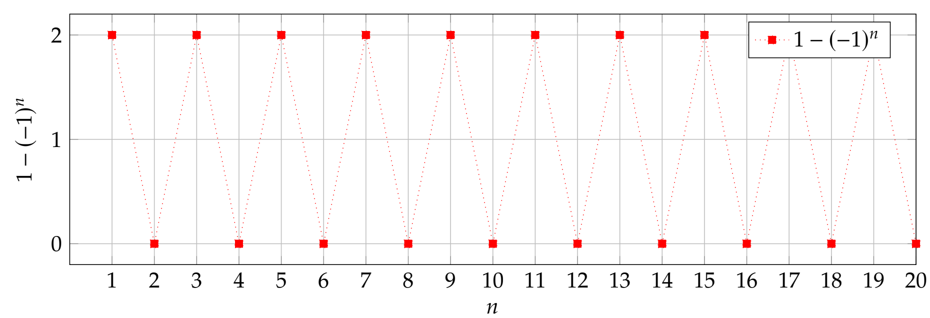
$$\exists N_\epsilon \in \mathbb{N} \quad \text{s.t.} \quad 1 < \varepsilon \cdot N_\varepsilon, \text{ i.e., } \frac{1}{N_\epsilon} < \varepsilon.$$

Assume that $n \geq N_\epsilon$ then

$$|a_n - 0| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N_\epsilon} < \varepsilon.$$

Hence $\lim_{n\to\infty} \frac{1}{n} = 0$. ∎

Example. Consider the sequence defined by $b_n = 1 - (-1)^n$ for all $n \in \mathbb{N}$. Prove that b_n does not converge.



Proof. Suppose that $\{b_n\}_{n=1}^{\infty}$ converges to $\beta \in \mathbb{R}$. Let $\varepsilon \in (0,2)$. Then if $n \geq N_\varepsilon$,

$$\begin{aligned} |b_n - \beta| &= |b_n - b_{n+1} + b_{n+1} - \beta| \\ &\leq |b_n - b_{n+1}| + |b_{n+1} - \beta| \\ &= 2 + |b_{n+1} - \beta| \end{aligned}$$

□

Absolute Value in Reals

Definition. Let $x \in \mathbb{R}$. A **absolute value** $|x|$ of x is defined by

$$|x| := \begin{cases} x & : x \geq 0 \\ -x & : x < 0 \end{cases}$$

Proposition. *Let* $x, y \in \mathbb{R}$.

(1) $|x| = |-x| = \sqrt{x^2}$

(2) $|xy| = |x||y|$

(3) *For each* $r > 0$,

$$|x| < r \iff -r < x < r$$

(4)

$$\delta < |x| \iff \delta < x \text{ or } x < -\delta$$

(5) $-|x| \leq x \leq |x|$

(6) *(Triangle Inequality)*

$$|x + y| \leq |x| + |y|$$

Proof. (a)

□

Boundedness of Sequence

Definition. Let $\{a_n\}_{n=1}^{\infty} (\subseteq \mathbb{R})$ is a sequence. $\{a_n\}$ is said to be **bounded** if

$$\exists M \in \mathbb{R} \text{ such that } \forall n \in \mathbb{N}, |a_n| \leq M.$$

Proposition. *A convergent sequence is bounded.*

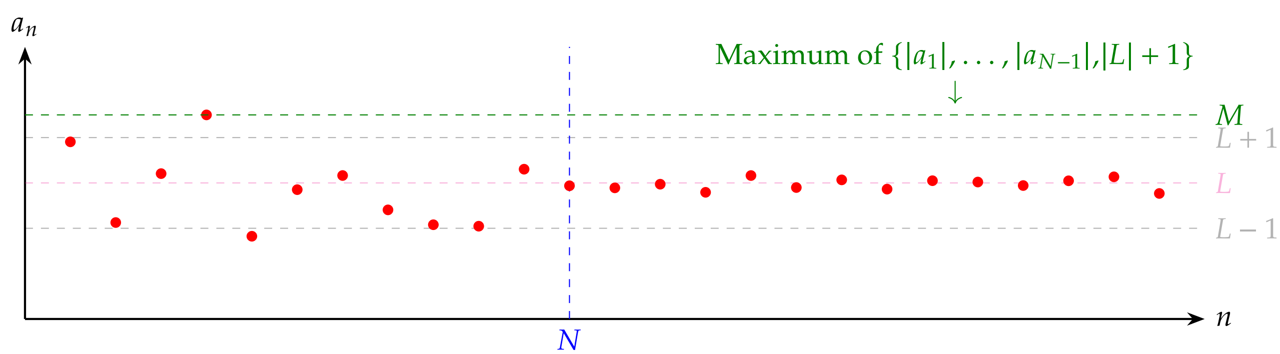
Proof. Let $\lim_{n\to\infty} a_n = L$. For $\varepsilon = 1$, $\exists N \in \mathbb{N}$ such that $n \geq N \implies |a_n - L| < 1$. Then we see that

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| < 1 + |L|.$$

Let $M := \max\left\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |L|\right\}$. Then

$$|a_n| \leq M$$

for all $n \in \mathbb{N}$. That is, $\{a_n\}$ is bounded. □



Note. We have established that if the limit of a sequence a_n exists as n approaches infinity, then there exists a real number M such that $|a_n| \leq M$ for all n. However, the converse is not necessarily true. To illustrate, consider the sequence $\{a_n\} = 1 - (-1)^n$. This sequence is bounded, yet it does not converge, serving as a counterexample.

Furthermore, we note the following important theorems:

1. **Monotone Convergence Theorem**:
   (i) If a sequence $\{a_n\}$ is bounded above and monotone increasing, then it converges.
   (ii) If a sequence $\{a_n\}$ is bounded below and monotone decreasing, then it converges.
2. **Bolzano-Weierstrass Theorem**: If there exists a real number M such that $|a_n| < M$ for all n, then there exists a convergent subsequence $\{a_{n_k}\}$ of $\{a_n\}$.

We confirmed that

$$\exists \lim_{n\to\infty} a_n \implies \exists M \in \mathbb{R} : |a_n| \leq M.$$

However,

$$\exists \lim_{n\to\infty} a_n \;\not\Longleftarrow\; \exists M \in \mathbb{R} : |a_n| \leq M$$

because there is a counterexample: $\{a_n\} = 1-(-1)^n$ is bounded sequence but not convergent. Note that

1. (Monotone Convergent Theorem)
   (i) if $\{a_n\}$ is bounded above and monotone increasing, then is convergent.
   (ii) if $\{a_n\}$ is bounded below and monotone decreasing, then is convergent.
2. (Bolzano-Weierstrass Theorem) $\exists M \in \mathbb{R} : |a_n| < M \implies \exists \lim_{k\to\infty} \left\{a_{n_k}\right\}$.

Limit Theorem (Algebraic Property of Limit of Sequence)

Theorem. *Let* $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$, *and* $\alpha \in \mathbb{R}$. *Then*

(1) $\lim_{n\to\infty} \alpha a_n = \alpha \lim_{n\to\infty} a_n = \alpha A.$

(2) $\lim_{n\to\infty} a_n \pm b_n = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B.$

(3) $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n = AB.$

(4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$. (Here, $B \neq 0$ and $b_n \neq 0$)

(5) $\lim_{n\to\infty} |a_n| = \left|\lim_{n\to\infty} a_n\right| = |A|.$

(6) $\lim_{n\to\infty} (a_n)^p = A^p$. (Here, $p \in \mathbb{N}$).

(7) $[\exists K : n \geq K \implies a_n \leq b_n] \implies A \leq B.$

Proof. To be continue... □

References

[1] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 6. 해석학 개론 (c) 수열의 수렴성." YouTube Video, 26:29. Published September 20, 2019. URL: `https://www.youtube.com/watch?v=jwLfzJyIxmU`.

[2] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 7. 해석학 개론 (d) 극한 정리" YouTube Video, 26:46. Published September 26, 2019. URL: `https://www.youtube.com/watch?v=1TRD34QbIaw`.