# Why $\partial Q/\partial x - \partial P/\partial y$ is the Curl in 2D

## Local circulation density from a small rectangle

Let $\vec{F} = \langle P(x,y), Q(x,y)\rangle$ be $C^1$. Consider a small, axis-aligned rectangle centered at $(x_0, y_0)$ with side lengths $\Delta x, \Delta y$. Its counterclockwise circulation is

$$\oint_{\partial R} \vec{F} \cdot d\vec{r} = \int_{\text{right}} Q\,dy + \int_{\text{top}} P\,dx + \int_{\text{left}} Q\,dy + \int_{\text{bottom}} P\,dx.$$

Taylor expand $P, Q$ to first order along each edge and keep first-order terms. One obtains

$$\oint_{\partial R} \vec{F} \cdot d\vec{r} \approx \big(Q_x(x_0,y_0) - P_y(x_0,y_0)\big)\,\Delta x\,\Delta y.$$

Dividing by the area $\Delta A = \Delta x\,\Delta y$ and shrinking the rectangle,

$$\lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint_{\partial R} \vec{F} \cdot d\vec{r} = Q_x(x_0,y_0) - P_y(x_0,y_0).$$

Thus the scalar $Q_x - P_y$ is the *circulation per unit area*—the 2D curl.

## Green's theorem (global circulation)

If $C = \partial D$ is a positively oriented simple closed curve enclosing a region $D$, Green's theorem states

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

So the total circulation equals the area integral of the local circulation density.

## Example 1: Rigid rotation and angular velocity

Consider the rigid rotation field with angular speed $\omega$:

$$\vec{F}(x,y) = \langle -\omega y,\ \omega x\rangle.$$

Then

$$\frac{\partial Q}{\partial x} = \omega, \qquad \frac{\partial P}{\partial y} = -\omega \quad \Rightarrow \quad \operatorname{curl} \vec{F} = Q_x - P_y = 2\omega.$$

This shows curl equals twice the angular velocity. For a circle of radius $R$, parametrize $r(t) = (R\cos t, R\sin t)$, $dr = (-R\sin t, R\cos t)\,dt$. Then

$$\oint \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \omega R^2 \, dt = 2\pi\omega R^2.$$

Meanwhile, $\iint_D (2\omega)\,dA = 2\omega \cdot \pi R^2 = 2\pi\omega R^2$, agreeing with Green's theorem.

## Example 2: Curl-free but not conservative (topology matters)

On $\mathbb{R}^2 \setminus \{(0,0)\}$, define

$$\vec{F}(x,y) = \Big\langle -\frac{y}{x^2+y^2},\ \frac{x}{x^2+y^2} \Big\rangle.$$

A direct calculation shows $Q_x - P_y = 0$ wherever defined (curl-free). However, the circulation around the unit circle is

$$\oint \vec{F} \cdot d\vec{r} = 2\pi \neq 0.$$

Hence there is no global potential function; the puncture creates a topological obstruction. This illustrates that $\operatorname{curl} \vec{F} = 0$ captures *local* rotation, while global circulation can persist in domains with holes.

## Summary checklist

- $Q_x - P_y$ is the infinitesimal (per-area) circulation density.
- Green's theorem sums local curl to give total circulation.
- Rigid rotation: curl $= 2\omega$ (twice angular velocity).
- Curl $= 0$ can still have nonzero loop integrals if the domain has holes.