Lecture Notes

Differential 1-Forms

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1. Introduction

In differential geometry, a [**differential 1-form**] is an object that can be integrated along curves. Intuitively, it is a field of covectors that assigns to each tangent vector a real number.

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2. Definition

Let M be a smooth manifold. A [**differential 1-form**] ω on M is a smooth section of the cotangent bundle $T^* M$.

Equivalently, at each point $p \in M$, the value ω_p is a linear map from the tangent space $T_p M$ to $\mathbb{R}$.

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3. Local Representation

In local coordinates $(x^1, x^2, ..., x^n)$, a 1-form can be written as:

$\omega = \sum_{\{i=1\}}^n f_{i(x)} d x^i$

where f_i are smooth functions on M.

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4. Examples

Example 1: Euclidean Plane

On $\mathbb{R}^2$ with coordinates (x, y), consider

$\omega = y d x + x d y$

At (x, y) and a tangent vector $v = (a, b)$,

$\omega_{\{(x,y)\}}(v) = y \cdot a + x \cdot b$

Example 2: Gradient of a Function

If $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, then its differential $d f$ is a 1-form:

$d f = \sum_{\{i=1\}}^n \left(\partial \frac{f}{\partial} x^i\right) d x^i$

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5. Integration of 1-Forms

Given a smooth curve $\gamma : [a, b] \to M$, we can integrate a 1-form ω along γ as

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6. Exact and Closed Forms

- A 1-form ω is [**exact**] if there exists f such that $\omega = df$.
- A 1-form is [**closed**] if $d\omega = 0$.

On $\mathbb{R}^n$, every closed 1-form is locally exact (Poincaré Lemma).

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7. Summary

- 1-forms are dual to vector fields.
- They can be written in local coordinates as linear combinations of dx^i.
- They can be integrated along curves.
- They are central in multivariable calculus, differential geometry, and physics.

End of Lecture