Abstract Algebra II

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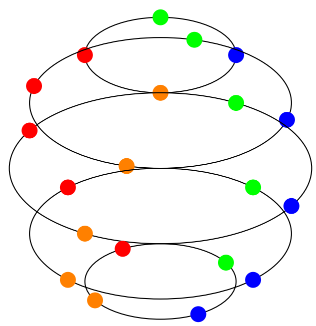
We cover the following topics in this note.

- Group Action
- Cayley Theorem
- Normal Subgroups
- Normality of the Kernel



Group Action

Definition. Let $(G, *)$ be a group and let $X \neq \emptyset$. A **(left) group action** of G on X is a function

$$\cdot : G \times X \to X, \quad (g, x) \mapsto g \cdot x$$

satisfying the followings: for all $g, h \in G$ and all $x \in X$,

(i) (Identity) $e \cdot x = x$, where $e \in G$ is the identity element of G;

(ii) (Compatibility) $(g * h) \cdot x = g \cdot (h \cdot x)$.

The pair $(X, \cdot)$ (or simply X) is then called a G-set.

Note (Notation). If a group G acts on a set X, one commonly writes: $G \curvearrowright X$.

Remark. A right group action of G on X is a function $\cdot : X \times G \to X, \quad (x, g) \mapsto x \cdot g$ satisfying:

(i) $x \cdot e = x$ for all $x \in X$;

(ii) $(x \cdot g) \cdot h = x \cdot (gh)$ for all $g, h \in G, x \in X$.

Example (Scalar Multiplication on a Vector Space)**.** Let $\mathbb{F}$ be a field, and let $X = \mathbb{F}^n$ be the n-dimensional vector space over $\mathbb{F}$. Consider the multiplicative group of nonzero scalars in $\mathbb{F}$:

$$G = (\mathbb{F}^\times, \times), \quad \text{where } \mathbb{F}^\times = \mathbb{F} \setminus \{0\}.$$

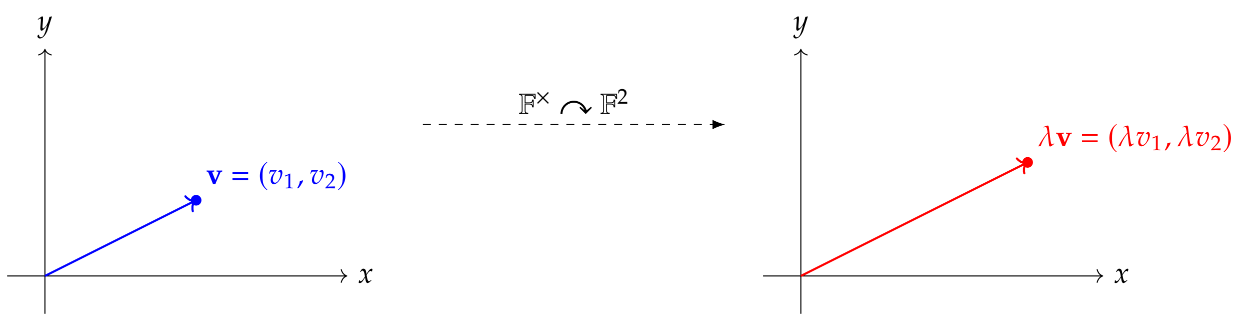
We define an action $G \curvearrowright X$ by scalar multiplication:

$$\begin{array}{rccc} \cdot & : & \mathbb{F}^\times \times \mathbb{F}^n & \longrightarrow & \mathbb{F}^n \\ & & (\lambda, \mathbf{v}) & \longmapsto & \lambda \cdot \mathbf{v} \end{array},$$

where the product $\lambda \cdot \mathbf{v}$ is defined componentwise. Then

(i) $1 \cdot \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{F}^n$.

(ii) $(\lambda\mu) \cdot \mathbf{v} = \lambda \cdot (\mu \cdot \mathbf{v})$ for all $\lambda, \mu \in \mathbb{F}^\times$, $\mathbf{v} \in \mathbb{F}^n$.



Example (Conjugation Action on the Group Itself)**.** Let G be any group, and consider $X = G$. Define an action of G on itself by conjugation:

$$G \curvearrowright G, \quad (g, x) \mapsto g \cdot x := g * x * g^{-1}.$$

Then

(i) $e \cdot x = e * x * e^{-1} = x$ for all $x \in G$.

(ii) Note that

$$\begin{aligned} (g * h) \cdot x &= (g * h) * x * (g * h)^{-1} \\ &= (g * h) * x * (h^{-1} * g^{-1}) \\ &= g * (h * x * h^{-1}) * g^{-1} \\ &= g * (h \cdot x) * g^{-1} \\ &= g \cdot (h \cdot x). \end{aligned}$$

Thus, this is a left group action.

Example (Trivial G-Set)**.** Let G be any group and define the set $X = \{x\}$, a singleton. Define the action

$$G \curvearrowright X, \quad (g, x) \mapsto g \cdot x := x \quad \text{for all } g \in G.$$

This is the **trivial action**, where every group element acts as the identity on X:

(i) $e \cdot x = x$.

(ii) $(g * h) \cdot x = x = g \cdot (h \cdot x)$.

Example (Action on Coset Space G/H)**.** Let $(G, *)$ be a group, and let $H \leq G$. Let $X = G/H$ be the set of left cosets of H in G, i.e.,

$$X = G/H = \{gH \mid g \in G\}.$$

Define an action

$$G \curvearrowright G/H, \quad (g, aH) \mapsto (ga)H.$$

This is well-defined because if $a_1H = a_2H$, then $a_1^{-1}a_2 \in H$, so: $ga_1H = ga_2H$.. Since

(i) $e \cdot aH = aH$;

(ii) $(gh) \cdot aH = g \cdot (h \cdot aH)$.

Group Elements Act as Permutations

Proposition. *Let G be a group action on a set X via a left action $G \curvearrowright X$, given by $(g, x) \mapsto g \cdot x$. Then for each $g \in G$, the map*

$$\sigma_g : X \to X, \quad x \mapsto g \cdot x$$

is one-to-one and onto. That is, $\sigma_g \in \mathrm{Sym}(X)$, the group of all permutations of X.

Proof. TBA □

Group Actions Induce Permutation Representations

Theorem. *Let G be a group action on a set X via a left group action $G \curvearrowright X$, $(g, x) \mapsto g \cdot x$. For each $g \in G$, define the bijection $\sigma_g : X \to X$ by $\sigma_g(x) := g \cdot x$. Then the map*

$$\phi : G \to \mathrm{Sym}(X), \quad g \mapsto \sigma_g,$$

*is a **group homomorphism** from G to the symmetric group $\mathrm{Sym}(X)$. In other words, for all $g, h \in G$,*

$$\phi(g * h) = \sigma_{g*h} = \sigma_g \circ \sigma_h = \phi(g) \circ \phi(h).$$

Remark. A group action $G \curvearrowright X$ is equivalent to a group homomorphism $G \to \mathrm{Sym}(X)$, i.e., a **permutation representation** of G.

Proof. TBA □

Cayley Theorem

Theorem. *Let G be a group. Consider the action of G on itself by left multiplication. For each $g \in G$, define*

$$\sigma_g : G \longrightarrow G, \qquad x \mapsto g \cdot x.$$

Then the map

$$\phi : G \longrightarrow \mathrm{Sym}(G), \qquad g \mapsto \sigma_g$$

is an injective group homomorphism (group monomorphism). In particular,

$$\phi(G) \simeq G \quad \textit{and} \quad \varphi(G) \leq \mathrm{Sym}(G).$$

Proof. TBA □

Normal Subgroups

Observation. Consider $4\mathbb{Z} \leq \mathbb{Z}$. Then

$$\mathbb{Z}/4\mathbb{Z} = \{0 + 4\mathbb{Z}, 1 + 4\mathbb{Z}, 2 + 4\mathbb{Z}, 3 + 4\mathbb{Z}\} = \{[0], [1], [2], [3]\}.$$

- $[0] + [1] = (0 + 4\mathbb{Z}) + (1 + 4\mathbb{Z}) \overset{\text{def.}}{=} (0 + 1) + 4\mathbb{Z} = 1 + 4\mathbb{Z} = [1]$.
- $[1] + [2] = (1 + 4\mathbb{Z}) + (2 + 4\mathbb{Z}) \overset{\text{def.}}{=} (1 + 2) + 4\mathbb{Z} = 3 + 4\mathbb{Z} = [3]$.
- $[1] + [3] = (2 + 4\mathbb{Z}) + (3 + 4\mathbb{Z}) \overset{\text{def.}}{=} (1 + 3) + 4\mathbb{Z} = 4 + 4\mathbb{Z} = 0 + 4\mathbb{Z} = [0]$.

Existence of the Quotient Group

Proposition. *Let $(G, *)$ be a group and let $H \leq G$ be a subgroup. Define a binary operation $\boxtimes$ on the set of left cosets G/H by*

$$(g * H) \boxtimes (g' * H) \;=\; (g * g') * H$$

where $g, g' \in G$. Then this operation is well-defined if and only if

$$g * h * g^{-1} \;\in\; H.$$

for all $g \in G$, $h \in H$.

Proof. TBA $\square$

Normal Subgroup

Definition. Let $(G, *)$ be a group and let $H \leq G$. We say that H is **normal** in G, written

$$H \trianglelefteq G,$$

if $g * h * g^{-1} \in H$ for any $g \in G$ and $h \in H$.

Remark. The set of (left) cosets G/H be a well-defined group structure via

$$(g * H) \boxtimes (k * H) = (g * k) * H,$$

making G/H the quotient group of G by H.

Equivalent Definitions of Normal Subgroup

Proposition. *Let $(G, *)$ be a group and let $H \leq G$. The Following Are Equivalent:*

(1)[a] *H is normal in G,* i.e., *$H \trianglelefteq G$;*

(2)[b] *$g * h * g^{-1} \in H$ for all $g \in G$, $h \in H$;*

(3)[c] *$g * H * g^{-1} = H$ for all $g \in G$;*

(4)[d] *$g * H = H * g$ for all $g \in G$.*

[a] Terminology and Notation
[b] (Elementwise Conjugation)
[c] (Conjugation Invariance)
[d] (Coset Equality)

Proof. ((2)⟹(3)) TBA

((3)⟹(4)) TBA

((4)⟹(2)) TBA

□

Normality of Kernel

Theorem. *Let* $\phi : (G, *) \longrightarrow (H, *')$ *be a group homomorphism, and define its kernel by*

$$\ker\phi \;=\; \left\{ g \in G : \varphi(g) = e_H \right\}.$$

Then $\ker\phi$ *is a normal subgroup of* G*; that is,* $\ker\phi \;\trianglelefteq\; G$*.*

Proof. Since ϕ is a homomorphism, for every $g \in G$ and every $k \in \ker\phi$ we have

$$\phi\left(g * k * g^{-1}\right) = \phi(g) *' \varphi(k) *' \phi(g)^{-1} = \phi(g) *' e_H *' \phi(g)^{-1} = e_H,$$

so $g * k * g^{-1} \in \ker\varphi$. Thus,

$$g * (\ker\varphi) * g^{-1} \;=\; \ker\varphi \quad \forall\, g \in G,$$

i.e. $\ker\varphi$ is invariant under conjugation and hence normal in G. □

References

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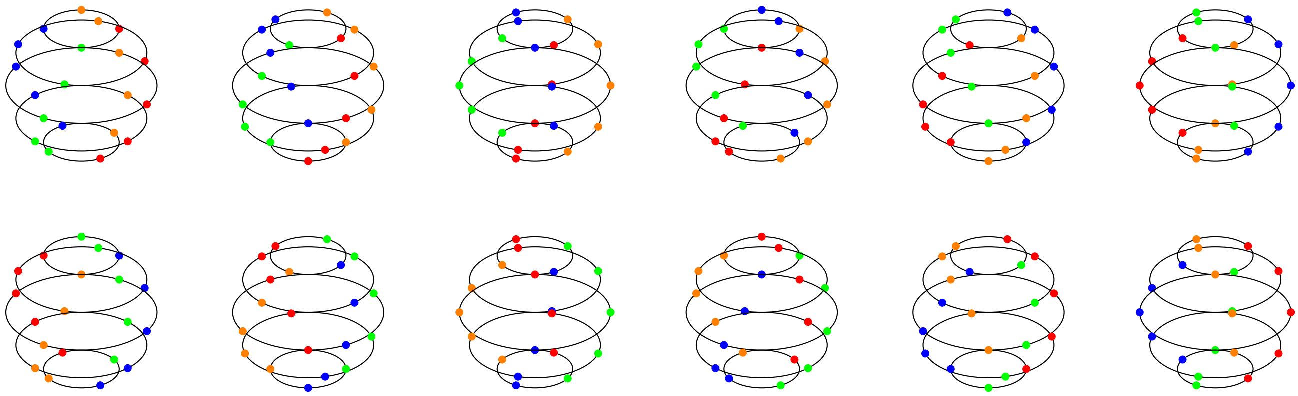
[2] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 24. 추상대수학 (e) 정규부분군의 정의 def of normal subgroups" YouTube Video, 23:00. Published October 25, 2019. URL: `https://www.youtube.com/watch?v=3UJILZr4CNo`.

A Appendices

A.1 The Rotation Action of $\mathbb{S}^1$ on $\mathbb{S}^2$



Consider

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} = \{e^{i\theta} : \theta \in \mathbb{R}\},$$
$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Define the map

$$\Phi \colon \mathbb{S}^1 \times \mathbb{S}^2 \longrightarrow \mathbb{S}^2, \qquad (e^{i\theta}, P) \mapsto \mathsf{Rot}_\theta(P),$$

where, for each $e^{i\theta} \in S^1$, define the rotation

$$\mathsf{Rot}_\theta \colon \mathbb{S}^2 \to \mathbb{S}^2, \qquad \mathsf{Rot}_\theta(x, y, z) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \\ z \end{pmatrix}.$$

Here, $\Phi(e^{i\theta}, P) = \mathsf{Rot}_\theta(P)$. Then

(i) (Identity) The identity in $\mathbb{S}^1$ is $1 = e^{i\cdot 0}$. Since $\cos 0 = 1$, $\sin 0 = 0$, we have

$$\Phi(1, P) = \mathsf{Rot}_0(P) = (x, y, z) = P,$$

for every $P \in \mathbb{S}^2$.

(ii) (Compatibility) For any $e^{i\theta}, e^{i\phi} \in \mathbb{S}^1$ and $P \in \mathbb{S}^2$,

$$\Phi\big(e^{i\theta}e^{i\phi}, P\big) = \Phi\big(e^{i(\theta+\phi)}, P\big) = \mathsf{Rot}_{\theta+\phi}(P) = \mathsf{Rot}_\theta\big(\mathsf{Rot}_\phi(P)\big) = \Phi\big(e^{i\theta}, \Phi(e^{i\phi}, P)\big).$$

Hence Φ be a left group action. To be continue $\cdots$.