

Chapter 1

Closed Forms and Exact Forms

Star-convex with respect to a point

Definition 1.1. Let $U \subseteq \mathbb{R}^n$ be an open set in \mathbb{R}^n . We say that U is **star-convex** with respect to the point \mathbf{p} of U if for each $\mathbf{x} \in U$, the line segment joining \mathbf{x} and \mathbf{p} lies in U .

Remark 1.2. Let $U \subseteq \mathbb{R}^n$ be open and let $\mathbf{p} \in U$. We say that U is star-convex with respect to \mathbf{p} if

$$\forall \mathbf{x} \in U, \forall t \in [0, 1], \quad (1-t)\mathbf{p} + t\mathbf{x} \in U.$$

Equivalently, for each $\mathbf{x} \in U$, the line segment

$$\overline{\mathbf{px}} = \{(1-t)\mathbf{p} + t\mathbf{x} : t \in [0, 1]\}$$

is contained in U .

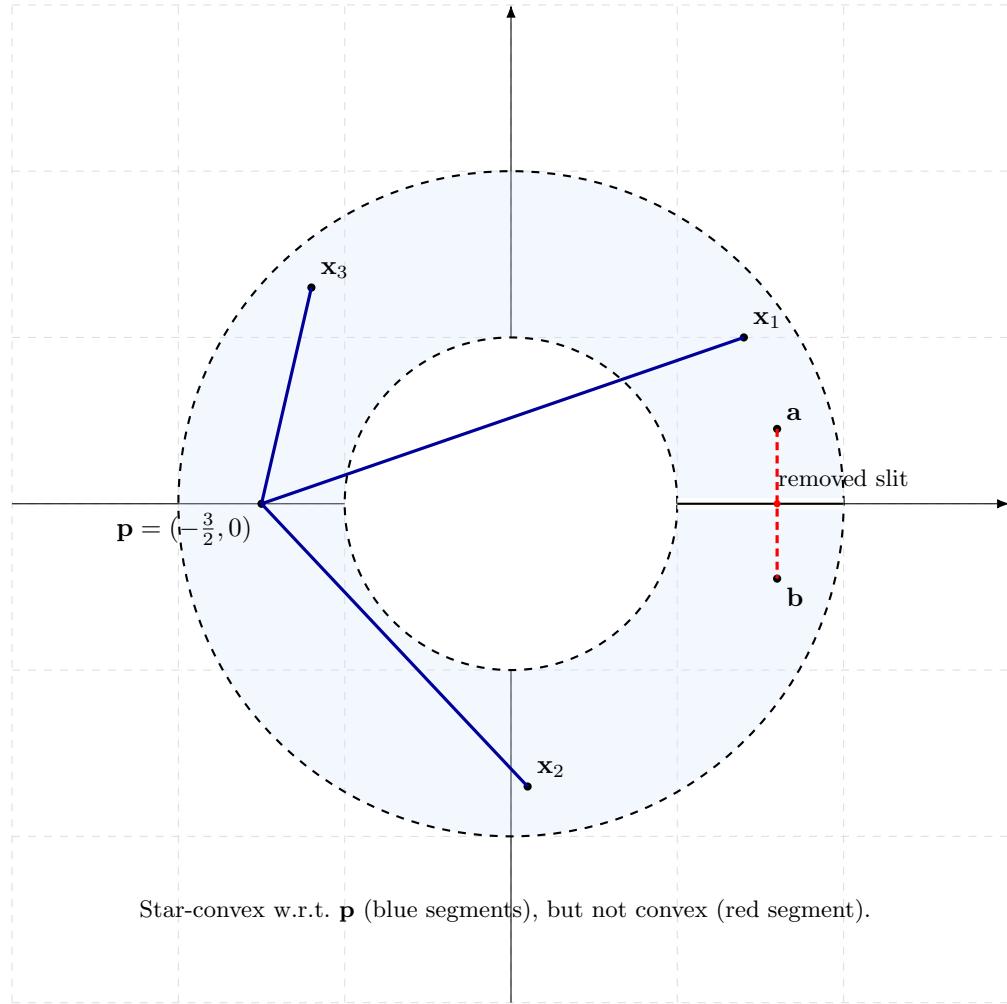
Example 1.1 (Star-convex but not convex: an annulus with a slit). Consider $n = 2$ and define the open set

$$U := \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\} \setminus \{(x, 0) \in \mathbb{R}^2 : 1 < x < 2\}.$$

Geometrically, U is the open annulus $\{1 < r < 2\}$ with the open radial segment on the positive x -axis removed. Fix the point

$$\mathbf{p} := \left(-\frac{3}{2}, 0\right) \in U.$$

Then U is star-convex with respect to \mathbf{p} , but U is not convex.



(i) Star-convexity with respect to \mathbf{p} . For each $\mathbf{x} \in U$, define the line segment parameterization

$$\gamma_{\mathbf{x}}(t) := (1 - t)\mathbf{p} + t\mathbf{x}, \quad t \in [0, 1].$$

The only obstruction to $\gamma_{\mathbf{x}}([0, 1]) \subseteq U$ (besides staying between the circles $r = 1$ and $r = 2$) is crossing the removed slit

$$S := \{(x, 0) : 1 < x < 2\}.$$

But S lies on the positive x -axis, whereas \mathbf{p} lies on the negative x -axis. Any segment from \mathbf{p} to a point $\mathbf{x} \in U$ either:

- has $y \neq 0$ somewhere along the segment, hence cannot meet $S \subset \{y = 0\}$; or
- stays on the x -axis, in which case \mathbf{x} must lie on the *negative* x -axis (since the positive portion $(1, 2)$ has been removed), and again the segment avoids S .

Thus $\gamma_{\mathbf{x}}(t) \notin S$ for all t , and since both endpoints lie in the annulus $1 < r < 2$, the entire segment remains in U . Hence U is star-convex with respect to \mathbf{p} .

(ii) **Failure of convexity.** Choose two points on opposite sides of the slit, for instance

$$\mathbf{a} := (1.6, 0.45) \in U, \quad \mathbf{b} := (1.6, -0.45) \in U.$$

Then the segment $\overline{\mathbf{ab}}$ is the vertical line at $x = 1.6$ and it intersects the removed slit S at $(1.6, 0) \notin U$. Hence $\overline{\mathbf{ab}} \not\subseteq U$, so U is not convex.

The Poincaré lemma

Theorem 1.3. Let $U \subseteq \mathbb{R}^3$ be a star-convex (equivalently, contractible) open set in \mathbb{R}^3 . If ω is a closed k -form on U , then ω is exact on U .

Exercise 1.1. Translate the Poincaré lemma for k -forms into theorem about scalar and vector fields in R^3 . Consider cases $k = 0, 1, 2, 3$.

Sol. Let $U \subseteq \mathbb{R}^3$ be a star-convex (equivalently, contractible) open set. Identify differential forms on U with scalar and vector fields via

$$\Omega^0(U) \cong C^\infty(U), \quad \Omega^1(U) \cong C^\infty(U; \mathbb{R}^3), \quad \Omega^2(U) \cong C^\infty(U; \mathbb{R}^3), \quad \Omega^3(U) \cong C^\infty(U),$$

by the explicit correspondences

$$\begin{aligned} f \in C^\infty(U) &\longleftrightarrow \omega^{(0)} := f, \\ \mathbf{F} = (P, Q, R) \in C^\infty(U; \mathbb{R}^3) &\longleftrightarrow \omega^{(1)} := P dx + Q dy + R dz, \\ \mathbf{G} = (U, V, W) \in C^\infty(U; \mathbb{R}^3) &\longleftrightarrow \omega^{(2)} := U dy \wedge dz + V dz \wedge dx + W dx \wedge dy, \\ g \in C^\infty(U) &\longleftrightarrow \omega^{(3)} := g dx \wedge dy \wedge dz. \end{aligned}$$

Under these identifications, the exterior derivative d corresponds to the grad–curl–div operators:

$$df \longleftrightarrow \nabla f, \quad d\omega^{(1)} \longleftrightarrow \nabla \times \mathbf{F}, \quad d\omega^{(2)} \longleftrightarrow \nabla \cdot \mathbf{G}, \quad d\omega^{(3)} = 0.$$

Then the Poincaré lemma statement

$$\omega \in \Omega^k(U), \quad d\omega = 0 \implies \exists \eta \in \Omega^{k-1}(U) : \omega = d\eta \quad (k \geq 1)$$

is equivalent (degree-by-degree) to the following assertions.

Case $k = 0$ (scalar fields). If $f \in C^\infty(U)$ satisfies $df = 0$ (equivalently $\nabla f = \mathbf{0}$), then f is constant on each connected component of U .

Case $k = 1$ (curl-free vector fields are gradients). If $\mathbf{F} \in C^\infty(U; \mathbb{R}^3)$ satisfies

$$\nabla \times \mathbf{F} = \mathbf{0},$$

then there exists a scalar potential $\phi \in C^\infty(U)$ such that

$$\mathbf{F} = \nabla \phi.$$

Moreover, ϕ is unique up to addition of a constant on each connected component of U .

Case $k = 2$ (divergence-free vector fields are curls). If $\mathbf{G} \in C^\infty(U; \mathbb{R}^3)$ satisfies

$$\nabla \cdot \mathbf{G} = 0,$$

then there exists a vector potential $\mathbf{A} \in C^\infty(U; \mathbb{R}^3)$ such that

$$\mathbf{G} = \nabla \times \mathbf{A}.$$

Moreover, \mathbf{A} is not unique: if $\psi \in C^\infty(U)$ then $\nabla \times (\mathbf{A} + \nabla\psi) = \nabla \times \mathbf{A}$.

Case $k = 3$ (every 3-form is exact; every scalar is a divergence). Every 3-form on U is closed (since $\Omega^4(U) = 0$). Hence for every $g \in C^\infty(U)$ there exists $\mathbf{G} \in C^\infty(U; \mathbb{R}^3)$ such that

$$\nabla \cdot \mathbf{G} = g.$$

Equivalently, every density $g dx \wedge dy \wedge dz$ is the exterior derivative of some 2-form.

□

Uniqueness of primitives on star-convex

Theorem 1.4. Let $U \subseteq \mathbb{R}^n$ be a star-convex open set in \mathbb{R}^n . Let ω be a closed k -form on U . If $k > 1$, and if η and η_0 are two $k - 1$ forms on U with $d\eta = \omega = d\eta_0$, then

$$\eta = \eta_0 + d\theta$$

for some $(k - 2)$ -form θ on U . If $k = 1$, and if f and f_0 are two 0-forms on U with $df = \omega = df_0$, then $f = f_0 + c$ for some constant c .

Exercise 1.2. Let $U \subseteq \mathbb{R}^3$ be a star-convex open set. Use the standard identifications

$$\Omega^0(U) \cong C^\infty(U), \quad \Omega^1(U) \cong C^\infty(U; \mathbb{R}^3), \quad \Omega^2(U) \cong C^\infty(U; \mathbb{R}^3), \quad \Omega^3(U) \cong C^\infty(U),$$

given concretely by

$$\begin{aligned} (P, Q, R) &\longleftrightarrow P dx + Q dy + R dz, \\ (U, V, W) &\longleftrightarrow U dy \wedge dz + V dz \wedge dx + W dx \wedge dy, \\ g &\longleftrightarrow g dx \wedge dy \wedge dz, \end{aligned}$$

so that the exterior derivative corresponds to the operators

$$\begin{aligned} (d : \Omega^0 \rightarrow \Omega^1) &\longleftrightarrow (\nabla), \\ (d : \Omega^1 \rightarrow \Omega^2) &\longleftrightarrow (\nabla \times), \\ (d : \Omega^2 \rightarrow \Omega^3) &\longleftrightarrow (\nabla \cdot). \end{aligned}$$

Let ω be a closed k -form on U (so $d\omega = 0$). Suppose η, η_0 are two $(k - 1)$ -forms satisfying $d\eta = \omega = d\eta_0$. Then, in \mathbb{R}^3 , this statement becomes the following degree-by-degree uniqueness assertions.

Case $k = 1$ (scalar potentials for a curl-free vector field). Let $\mathbf{F} \in C^\infty(U; \mathbb{R}^3)$ satisfy

$$\nabla \times \mathbf{F} = \mathbf{0}.$$

If $\phi, \phi_0 \in C^\infty(U)$ satisfy

$$\nabla \phi = \mathbf{F} = \nabla \phi_0,$$

then there exists a constant $c \in \mathbb{R}$ (on each connected component of U) such that

$$\phi = \phi_0 + c.$$

Equivalently, scalar potentials of the same vector field are unique up to addition of constants.

Case $k = 2$ (vector potentials for a divergence-free vector field). Let $\mathbf{G} \in C^\infty(U; \mathbb{R}^3)$ satisfy

$$\nabla \cdot \mathbf{G} = 0.$$

If $\mathbf{A}, \mathbf{A}_0 \in C^\infty(U; \mathbb{R}^3)$ satisfy

$$\nabla \times \mathbf{A} = \mathbf{G} = \nabla \times \mathbf{A}_0,$$

then there exists a scalar field $\psi \in C^\infty(U)$ such that

$$\mathbf{A} = \mathbf{A}_0 + \nabla\psi.$$

Equivalently, vector potentials of the same field are unique up to “gauge” transformations $\mathbf{A} \mapsto \mathbf{A} + \nabla\psi$.

Case $k = 3$ (primitives of a 3-form / solutions to a divergence equation). Let $g \in C^\infty(U)$ and suppose $\mathbf{H}, \mathbf{H}_0 \in C^\infty(U; \mathbb{R}^3)$ satisfy

$$\nabla \cdot \mathbf{H} = g = \nabla \cdot \mathbf{H}_0.$$

Then there exists a vector field $\mathbf{B} \in C^\infty(U; \mathbb{R}^3)$ such that

$$\mathbf{H} = \mathbf{H}_0 + \nabla \times \mathbf{B}.$$

Equivalently, solutions to $\nabla \cdot \mathbf{H} = g$ are unique up to addition of a curl field, because $\nabla \cdot (\nabla \times \mathbf{B}) = 0$ and (on star-convex U) every divergence-free field is a curl.

Chapter 2

Differentiable Manifolds and Riemannian Manifolds

Exercise 2.1. Show that if $\mathbf{v} \in T_p(M)$, then \mathbf{v} is the velocity vector of some C^∞ curve γ in M passing through p .

Sol. Let M be a smooth manifold, let $p \in M$, and let $\mathbf{v} \in T_p M$. Then there exist an open interval $I \subset \mathbb{R}$ with $0 \in I$ and a C^∞ curve $\gamma : I \rightarrow M$ such that

$$\gamma(0) = p, \quad \gamma'(0) = \mathbf{v}.$$

Moreover, one may take $I = \mathbb{R}$.

Choose a smooth chart (U, φ) about p , with $p \in U$ and $\varphi : U \rightarrow \varphi(U) \subset \mathbb{R}^n$ a diffeomorphism onto an open set. Define

$$a := (d\varphi)_p(\mathbf{v}) \in T_{\varphi(p)}\mathbb{R}^n \cong \mathbb{R}^n.$$

Since $\varphi(U)$ is open, there exists $\varepsilon > 0$ such that

$$\varphi(p) + ta \in \varphi(U) \quad \text{for all } |t| < \varepsilon.$$

Define a smooth curve $\tilde{\gamma} : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$ by

$$\tilde{\gamma}(t) := \varphi(p) + ta,$$

and define $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ by

$$\gamma(t) := \varphi^{-1}(\tilde{\gamma}(t)) = \varphi^{-1}(\varphi(p) + ta).$$

Then γ is C^∞ (being the composition of smooth maps) and

$$\gamma(0) = \varphi^{-1}(\varphi(p)) = p.$$

To prove $\gamma'(0) = \mathbf{v}$, let $f \in C^\infty(M)$ be arbitrary. By the chain rule,

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} (f \circ \gamma)(t) &= \frac{d}{dt}\Big|_{t=0} ((f \circ \varphi^{-1})(\varphi(p) + ta)) \\ &= d(f \circ \varphi^{-1})_{\varphi(p)}(a). \end{aligned}$$

Substituting $a = (d\varphi)_p(\mathbf{v})$ and using the functoriality of the differential,

$$d(f \circ \varphi^{-1})_{\varphi(p)} \circ (d\varphi)_p = d((f \circ \varphi^{-1}) \circ \varphi)_p = (df)_p,$$

we obtain

$$\frac{d}{dt} \Big|_{t=0} (f \circ \gamma)(t) = (df)_p(\mathbf{v}) = \mathbf{v}(f).$$

By the defining characterization of the velocity vector $\gamma'(0) \in T_p M$ as the derivation $f \mapsto \frac{d}{dt}|_{t=0} (f \circ \gamma)(t)$, this equality for all $f \in C^\infty(M)$ implies $\gamma'(0) = \mathbf{v}$.

Finally, to obtain a curve defined on all of \mathbb{R} , choose $\beta \in C^\infty(\mathbb{R})$ such that

$$\beta(t) = t \text{ for } |t| \text{ small,} \quad |\beta(t)| < \varepsilon \text{ for all } t \in \mathbb{R}.$$

(For example, one may take $\beta(t) = \varepsilon \tanh(t/\varepsilon)$.) Define

$$\Gamma(t) := \varphi^{-1}(\varphi(p) + \beta(t)a), \quad t \in \mathbb{R}.$$

Then $\Gamma \in C^\infty(\mathbb{R}, M)$, $\Gamma(0) = p$, and $\Gamma'(0) = \mathbf{v}$ since $\beta'(0) = 1$. \square

Example 2.1 (Unit circle version of the chart-pushforward construction). Let

$$M = S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Fix the point

$$p = (1, 0) \in S^1.$$

Then

$$T_p S^1 = \{(0, \lambda) : \lambda \in \mathbb{R}\} \subset \mathbb{R}^2.$$

Choose the tangent vector

$$\mathbf{v} = (0, 1) \in T_p S^1.$$

A concrete smooth chart around p

Let

$$U := \{(x, y) \in S^1 : x > 0\},$$

an open neighborhood of p in S^1 . Define

$$\varphi : U \rightarrow (-1, 1), \quad \varphi(x, y) := y.$$

Then φ is a C^∞ diffeomorphism onto the open interval $(-1, 1)$, with inverse

$$\varphi^{-1} : (-1, 1) \rightarrow U, \quad \varphi^{-1}(u) = (\sqrt{1 - u^2}, u).$$

Note that

$$\varphi(p) = \varphi(1, 0) = 0 \in (-1, 1).$$

Computing $a = (d\varphi)_p(\mathbf{v})$

Since φ is (the restriction to S^1) of the ambient coordinate function $(x, y) \mapsto y$, the differential at p satisfies

$$(d\varphi)_p(w) = w_y \quad \text{for all } w = (w_x, w_y) \in T_p S^1.$$

Hence

$$a := (d\varphi)_p(\mathbf{v}) = (d\varphi)_p(0, 1) = 1 \in T_{\varphi(p)} \mathbb{R} \cong \mathbb{R}.$$

The straight line in coordinates and its pullback to the manifold

Because $\varphi(U) = (-1, 1)$ is open, there exists $\varepsilon > 0$ such that

$$\varphi(p) + ta = 0 + t \in (-1, 1) \quad \text{for all } |t| < \varepsilon.$$

For instance, one may take $\varepsilon = 1$.

Define

$$\tilde{\gamma} : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}, \quad \tilde{\gamma}(t) := \varphi(p) + ta = t.$$

Define a curve in S^1 by pulling back via φ^{-1} :

$$\gamma : (-\varepsilon, \varepsilon) \rightarrow S^1, \quad \gamma(t) := \varphi^{-1}(\tilde{\gamma}(t)) = \varphi^{-1}(t) = (\sqrt{1 - t^2}, t).$$

Then γ is C^∞ and

$$\gamma(0) = (\sqrt{1 - 0}, 0) = (1, 0) = p.$$

Verifying $\gamma'(0) = \mathbf{v}$

For any $f \in C^\infty(S^1)$, the chain rule gives

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} (f \circ \gamma)(t) &= \frac{d}{dt} \Big|_{t=0} ((f \circ \varphi^{-1})(\varphi(p) + ta)) \\ &= d(f \circ \varphi^{-1})_{\varphi(p)}(a). \end{aligned}$$

Substituting $a = (d\varphi)_p(\mathbf{v})$ and using functoriality of differentials,

$$d(f \circ \varphi^{-1})_{\varphi(p)} \circ (d\varphi)_p = d((f \circ \varphi^{-1}) \circ \varphi)_p = (df)_p,$$

hence

$$\frac{d}{dt} \Big|_{t=0} (f \circ \gamma)(t) = (df)_p(\mathbf{v}) = \mathbf{v}(f).$$

By the defining characterization of the velocity vector $\gamma'(0) \in T_p S^1$ as the derivation $f \mapsto \frac{d}{dt} \Big|_{t=0} (f \circ \gamma)(t)$, it follows that

$$\gamma'(0) = \mathbf{v}.$$

(Indeed, differentiating $\gamma(t) = (\sqrt{1 - t^2}, t)$ gives $\gamma'(0) = (0, 1)$.)

Extending to a globally defined smooth curve

Choose $\beta \in C^\infty(\mathbb{R})$ such that

$$\beta(t) = t \text{ for } |t| \text{ small,} \quad |\beta(t)| < \varepsilon \text{ for all } t \in \mathbb{R}.$$

For example, if $\varepsilon = 1$, take $\beta(t) = \tanh(t)$. Define

$$\Gamma : \mathbb{R} \rightarrow S^1, \quad \Gamma(t) := \varphi^{-1}(\varphi(p) + \beta(t)a) = \varphi^{-1}(\beta(t)) = (\sqrt{1 - \beta(t)^2}, \beta(t)).$$

Then $\Gamma \in C^\infty(\mathbb{R}, S^1)$, $\Gamma(0) = p$, and $\Gamma'(0) = \mathbf{v}$, since $\beta'(0) = 1$.

The Poincaré lemma

The derivation associated to a tangent vector

Let M be a C^∞ manifold and let $p \in M$. Assume the tangent space $T_p M$ is defined as equivalence classes of C^∞ curves through p : two curves $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \rightarrow M$ with $\gamma_i(0) = p$ are equivalent if, for some (hence any) chart (U, φ) with $p \in U$,

$$\frac{d}{dt} \Big|_{t=0} (\varphi \circ \gamma_1)(t) = \frac{d}{dt} \Big|_{t=0} (\varphi \circ \gamma_2)(t) \in \mathbb{R}^n.$$

Let $\mathbf{v} \in T_p M$, and choose a representative curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ with $\gamma(0) = p$ and $[\gamma] = \mathbf{v}$.

Define an operator

$$X_{\mathbf{v}} : C^\infty(M) \longrightarrow \mathbb{R}$$

by

$$X_{\mathbf{v}}(f) := \frac{d}{dt} \Big|_{t=0} (f \circ \gamma)(t), \quad f \in C^\infty(M). \quad (2.1)$$

Proposition 2.1 (Well-definedness and basic properties of $X_{\mathbf{v}}$). *With notation as above:*

- (a) *The value $X_{\mathbf{v}}(f)$ is independent of the choice of representative curve γ for \mathbf{v} . Hence $X_{\mathbf{v}}$ is well-defined.*
- (b) *The operator $X_{\mathbf{v}}$ satisfies:*
 - (1) **Locality:** If $f, g \in C^\infty(M)$ agree on some neighborhood of p , then $X_{\mathbf{v}}(f) = X_{\mathbf{v}}(g)$.
 - (2) **Linearity:** For all $a, b \in \mathbb{R}$ and $f, g \in C^\infty(M)$,

$$X_{\mathbf{v}}(af + bg) = aX_{\mathbf{v}}(f) + bX_{\mathbf{v}}(g).$$

- (3) **Product rule:** For all $f, g \in C^\infty(M)$,

$$X_{\mathbf{v}}(f \cdot g) = X_{\mathbf{v}}(f)g(p) + f(p)X_{\mathbf{v}}(g).$$

Proof. (a) **Well-definedness.** Let $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \rightarrow M$ be two C^∞ curves with $\gamma_1(0) = \gamma_2(0) = p$ representing the same tangent vector \mathbf{v} . By definition of the equivalence relation, there exists (equivalently, for every) chart (U, φ) about p such that

$$\frac{d}{dt} \Big|_{t=0} (\varphi \circ \gamma_1)(t) = \frac{d}{dt} \Big|_{t=0} (\varphi \circ \gamma_2)(t) \in \mathbb{R}^n. \quad (2.2)$$

Fix such a chart (U, φ) and let $f \in C^\infty(M)$ be arbitrary. Set $F := f \circ \varphi^{-1} \in C^\infty(\varphi(U))$. For $i = 1, 2$ and $|t|$ small we have $\gamma_i(t) \in U$ and hence

$$(f \circ \gamma_i)(t) = (F \circ \varphi \circ \gamma_i)(t).$$

Applying the chain rule in \mathbb{R}^n yields

$$\frac{d}{dt} \Big|_{t=0} (f \circ \gamma_i)(t) = dF_{\varphi(p)} \left(\frac{d}{dt} \Big|_{t=0} (\varphi \circ \gamma_i)(t) \right).$$

Using (2.2) for the velocities, we conclude

$$\frac{d}{dt} \Big|_{t=0} (f \circ \gamma_1)(t) = \frac{d}{dt} \Big|_{t=0} (f \circ \gamma_2)(t).$$

Thus the right-hand side of (2.1) depends only on \mathbf{v} , not on the representative curve. Therefore $X_{\mathbf{v}}$ is well-defined.

(b1) Locality. Assume $f = g$ on a neighborhood W of p . Since $\gamma(0) = p$ and γ is continuous, there exists $\delta > 0$ such that $\gamma((-\delta, \delta)) \subset W$. Hence for $|t| < \delta$ we have $(f \circ \gamma)(t) = (g \circ \gamma)(t)$, and therefore

$$X_{\mathbf{v}}(f) = \frac{d}{dt} \Big|_{t=0} (f \circ \gamma)(t) = \frac{d}{dt} \Big|_{t=0} (g \circ \gamma)(t) = X_{\mathbf{v}}(g).$$

(b2) Linearity. Let $a, b \in \mathbb{R}$ and $f, g \in C^\infty(M)$. Then for all t ,

$$(af + bg) \circ \gamma = a(f \circ \gamma) + b(g \circ \gamma).$$

Differentiate at $t = 0$ and use linearity of the derivative in \mathbb{R} :

$$X_{\mathbf{v}}(af + bg) = \frac{d}{dt} \Big|_{t=0} (a(f \circ \gamma)(t) + b(g \circ \gamma)(t)) = aX_{\mathbf{v}}(f) + bX_{\mathbf{v}}(g).$$

(b3) Product rule. For all t ,

$$(fg) \circ \gamma = (f \circ \gamma)(g \circ \gamma),$$

so by the ordinary product rule for functions of one real variable,

$$\begin{aligned} X_{\mathbf{v}}(fg) &= \frac{d}{dt} \Big|_{t=0} ((f \circ \gamma)(t)(g \circ \gamma)(t)) \\ &= \frac{d}{dt} \Big|_{t=0} (f \circ \gamma)(t) \cdot (g \circ \gamma)(0) + (f \circ \gamma)(0) \cdot \frac{d}{dt} \Big|_{t=0} (g \circ \gamma)(t). \end{aligned}$$

Since $(g \circ \gamma)(0) = g(\gamma(0)) = g(p)$ and $(f \circ \gamma)(0) = f(p)$, this becomes

$$X_{\mathbf{v}}(fg) = X_{\mathbf{v}}(f)g(p) + f(p)X_{\mathbf{v}}(g).$$

This proves (1)–(3). □

A concrete example of $X_{\mathbf{v}}$ on the unit circle

Let

$$M = S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Fix the point

$$p = (1, 0) \in S^1,$$

and the tangent vector

$$\mathbf{v} = (0, 1) \in T_p S^1 = \{(0, \lambda) : \lambda \in \mathbb{R}\}.$$

A natural smooth curve representing \mathbf{v} is the counterclockwise parametrization

$$\gamma(t) = (\cos t, \sin t), \quad t \in \mathbb{R}.$$

Indeed,

$$\gamma(0) = (1, 0) = p, \quad \gamma'(0) = (-\sin 0, \cos 0) = (0, 1) = \mathbf{v}.$$

Definition of X_v in this example

For any $f \in C^\infty(S^1)$ define

$$X_v(f) := \frac{d}{dt} \Big|_{t=0} (f \circ \gamma)(t).$$

Thus, in this concrete situation, $X_v(f)$ is literally the directional derivative of f along S^1 at p in the counterclockwise direction.

Concrete computations

Let $x, y : S^1 \rightarrow \mathbb{R}$ be the coordinate restriction functions $x(x, y) = x$, $y(x, y) = y$. Then

$$(x \circ \gamma)(t) = \cos t, \quad (y \circ \gamma)(t) = \sin t,$$

so

$$X_v(x) = \frac{d}{dt} \Big|_0 \cos t = 0, \quad X_v(y) = \frac{d}{dt} \Big|_0 \sin t = 1.$$

Now define two smooth functions on S^1 by

$$f(x, y) := y, \quad g(x, y) := x y.$$

Then

$$(f \circ \gamma)(t) = \sin t, \quad (g \circ \gamma)(t) = (\cos t)(\sin t),$$

so

$$X_v(f) = \frac{d}{dt} \Big|_0 \sin t = 1, \quad X_v(g) = \frac{d}{dt} \Big|_0 (\cos t \sin t) = \frac{d}{dt} \Big|_0 \frac{1}{2} \sin(2t) = 1.$$

Also

$$f(p) = f(1, 0) = 0, \quad g(p) = g(1, 0) = 0.$$

Verification of the three properties in this example

(1) Locality. Let $f_1, f_2 \in C^\infty(S^1)$ and suppose there exists an open arc $W \subset S^1$ containing p such that $f_1|_W = f_2|_W$. Since $\gamma(0) = p$ and γ is continuous, there exists $\delta > 0$ such that $\gamma((-\delta, \delta)) \subset W$. Hence $(f_1 \circ \gamma)(t) = (f_2 \circ \gamma)(t)$ for $|t| < \delta$, and therefore

$$X_v(f_1) = \frac{d}{dt} \Big|_0 (f_1 \circ \gamma)(t) = \frac{d}{dt} \Big|_0 (f_2 \circ \gamma)(t) = X_v(f_2).$$

(2) Linearity. For $a, b \in \mathbb{R}$ and $f_1, f_2 \in C^\infty(S^1)$,

$$((af_1 + bf_2) \circ \gamma)(t) = a(f_1 \circ \gamma)(t) + b(f_2 \circ \gamma)(t),$$

so differentiating at 0 yields

$$X_v(af_1 + bf_2) = aX_v(f_1) + bX_v(f_2).$$

(3) Product rule (verified by explicit functions). Take $f(x, y) = y$ and $g(x, y) = xy$ as above. Then

$$X_{\mathbf{v}}(fg) = X_{\mathbf{v}}(y \cdot xy) = X_{\mathbf{v}}(xy^2).$$

Compute:

$$(fg) \circ \gamma(t) = (\sin t) \cdot (\cos t \sin t) = \cos t \sin^2 t,$$

hence

$$X_{\mathbf{v}}(fg) = \frac{d}{dt} \Big|_0 (\cos t \sin^2 t) = 0.$$

On the other hand,

$$X_{\mathbf{v}}(f) g(p) + f(p) X_{\mathbf{v}}(g) = (1) \cdot 0 + 0 \cdot (1) = 0,$$

so indeed

$$X_{\mathbf{v}}(fg) = X_{\mathbf{v}}(f) g(p) + f(p) X_{\mathbf{v}}(g).$$

(The general product rule follows from the one-variable product rule for $(f \circ \gamma)(t)$ and $(g \circ \gamma)(t)$.)

The Poincaré lemma

The Poincaré lemma

The Poincaré lemma

The Poincaré lemma

The Poincaré lemma