Weierstrass Elliptic Function from dz and $\dfrac{dz}{z}$

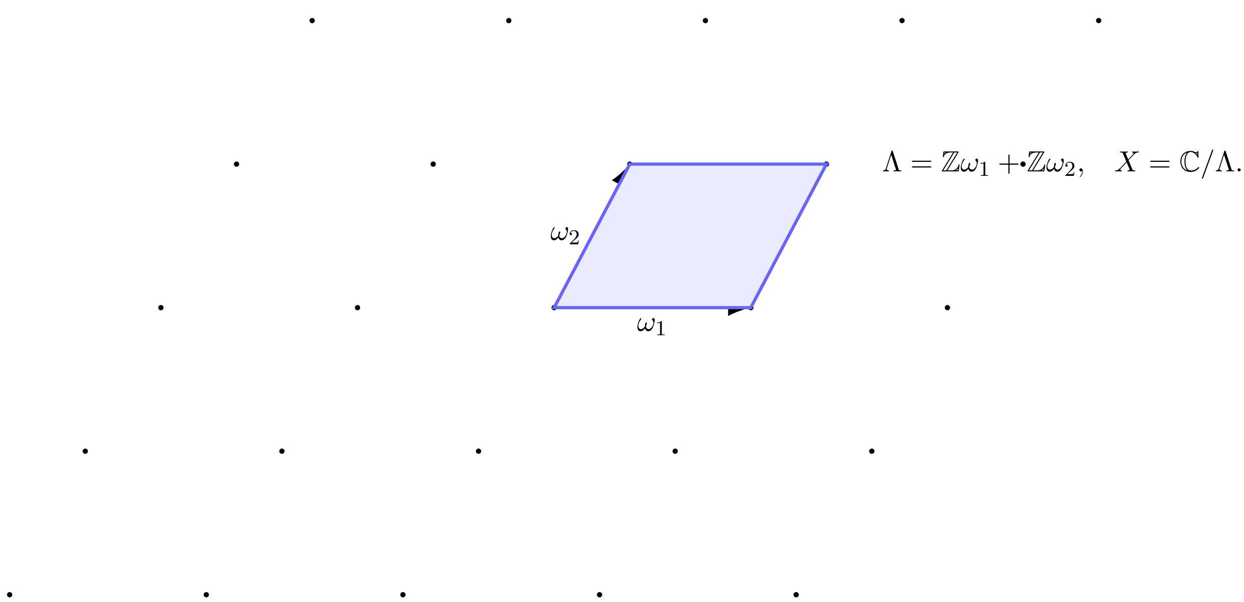
Contents

1 From dz on $\mathbb{C}$ to a torus $X = \mathbb{C}/\Lambda$

Fix two non-collinear complex numbers ω_1, ω_2 and the lattice

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

Identify $z \sim z + \lambda$ for $\lambda \in \Lambda$. The quotient $X = \mathbb{C}/\Lambda$ is a complex torus. The 1-form $\omega = dz$ is holomorphic on $\mathbb{C}$ and Λ-invariant, hence descends to a holomorphic 1-form on X. By Liouville, it is unique up to a constant; crucially, it has *no zeros*.



2 Elliptic = periodic meromorphic; why no simple poles

Definition 1. *A function $f : \mathbb{C} \to \mathbb{C}$ is* elliptic *(w.r.t. Λ) if it is meromorphic and Λ-periodic: $f(z+\lambda) = f(z)$ for all $\lambda \in \Lambda$.*

There are no nonconstant *holomorphic* elliptic functions (Liouville on a fundamental parallelogram), so any nontrivial elliptic f must have poles. Use the winding form $\frac{dz}{z}$ to control residues:

Fact 1 (Residues in a fundamental domain)**.** *Let P be a fundamental parallelogram for Λ. For an elliptic f,*

$$\iint_{\partial P} f(z)\,dz \;=\; 2\pi i \sum_{\text{poles } a \in P} \operatorname{Res}_a(f) \quad \textit{but} \quad \iint_{\partial P} f(z)\,dz = 0$$

because opposite edges cancel by periodicity. Hence $\sum \operatorname{Res}_a(f) = 0$.

If an elliptic f had a *single* simple pole in P, its residue would have to be zero, forcing the principal part to vanish—contradiction. Therefore the "smallest" possible principal part is a *double* pole with zero residue. That points us to a canonical choice.

3 Definition of the Weierstrass $\wp$

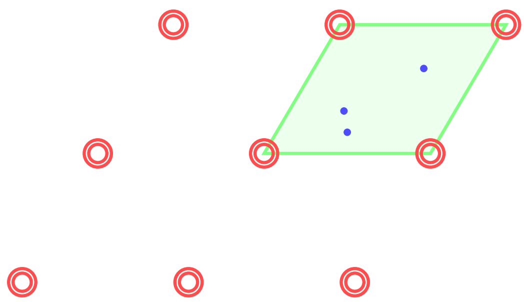
We choose the unique even elliptic function with a double pole at the lattice points and no constant term in its Laurent expansion at 0:

$$\boxed{\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)}$$

The subtraction $1/\omega^2$ kills the constant term and ensures normal convergence.

Immediate properties.

- **Double periodic:** $\wp(z+\omega_j) = \wp(z)$ for $j = 1, 2$.
- **Even:** $\wp(-z) = \wp(z)$ (the summand is even).
- **Poles:** Only at Λ, all *double* with principal part $1/z^2$.
- $\wp'$ is *odd* and elliptic, with *triple* poles at Λ.



$\wp$: double poles at Λ; $\wp'$ has three simple zeros per cell.

4 Laurent series and Eisenstein series

Expanding at $z = 0$ gives

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20} z^2 + \frac{g_3}{28} z^4 + \cdots, \qquad \wp'(z) = -\frac{2}{z^3} + \frac{g_2}{10} z + \frac{g_3}{7} z^3 + \cdots$$

where the lattice invariants are

$$g_2 = 60 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^4}, \qquad g_3 = 140 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^6}.$$

(These are built from the holomorphic data of the lattice; note how only dz and the lattice enter.)

5 The cubic and the differential equation

Consider the cubic curve

$$E: \quad y^2 = 4x^3 - g_2 x - g_3, \qquad \omega_E = \frac{dx}{y}.$$

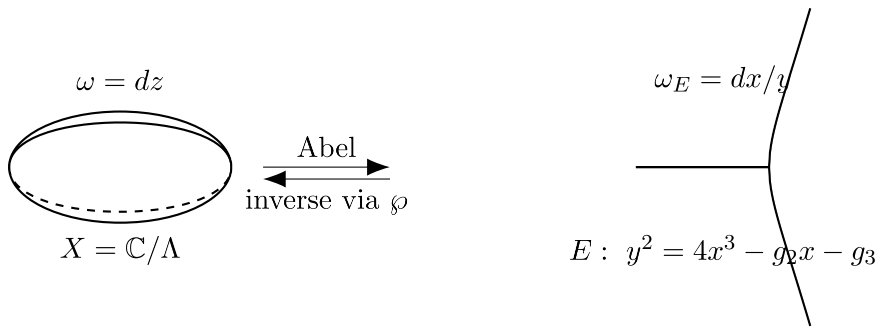
Define the *Abel map* $z = \int_\infty^{(x,y)} \omega_E$. Then the inverse map is

$$\boxed{x = \wp(z), \qquad y = \wp'(z)}$$

and the pair $(\wp(z), \wp'(z))$ automatically satisfies the cubic equation. Equivalently, one shows

$$\boxed{\big(\wp'(z)\big)^2 = 4\big(\wp(z)\big)^3 - g_2\,\wp(z) - g_3}$$

by observing that the left-hand side is an elliptic function with no poles (hence constant), and matching its Laurent expansion at 0 to force the constant to be 0.



6 Periods, zeros, and the zeta primitive

Let $2\omega_1, 2\omega_2$ be a *period basis* ($\Lambda = 2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}$). Then $\wp$ has those same periods; $\wp'$ has three simple zeros in each fundamental domain (at half-periods when the lattice is generic). Define the Weierstrass zeta function by

$$\zeta'(z) = -\wp(z), \qquad \zeta(z) = \frac{1}{z} + \sum_{\omega \neq 0} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

It is *not* elliptic (it has quasi-periods), reflecting the fact that a global primitive of $\wp$ cannot be doubly periodic—compare with $\log z$ for dz/z.

7 Everything came from dz and $\frac{dz}{z}$

- dz gives the unique holomorphic 1-form on the torus and defines periods.
- Elliptic = doubly periodic meromorphic; Stokes + periodicity + dz/z (residue calculus) force double poles and zero total residue.
- Killing the constant term canonically produces $\wp$.
- The cubic relation arises because the pole-free combination must be constant.

Quick exercises

Exercise 1. *Show that if f is elliptic then $\sum \operatorname{Res}(f) = 0$ in a fundamental domain; deduce there is no elliptic function with exactly one simple pole.*

Exercise 2. *Expand $\wp(z)$ from the definition and read off the coefficients $g_2/20$ and $g_3/28$.*

Exercise 3. *Prove $(\wp')^2 - 4\wp^3 + g_2\wp + g_3$ is elliptic with no poles and use the Laurent expansions to conclude it vanishes identically.*