Calculus / differential-form viewpoint on $\mathbb{C}(x) \simeq \mathcal{M}(\mathbb{CP}^1)$

We want to show:

$$\mathcal{M}(\mathbb{CP}^1) \;\cong\; \mathbb{C}(x),$$

using mainly complex analysis (Laurent series, Liouville, change of variables) and a bit of differential-form language.

1. Charts on $\mathbb{CP}^1$ and the coordinate x

Think of $\mathbb{CP}^1$ as the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Use two standard charts:

- $U_1 = \mathbb{CP}^1 \setminus \{\infty\}$ with coordinate

  $$z = \frac{z_0}{z_1} : U_1 \to \mathbb{C}.$$

  We call this coordinate x. So $x = z$ on $\mathbb{C}$.

- $U_0 = \mathbb{CP}^1 \setminus \{0\}$ with coordinate

  $$w = \frac{z_1}{z_0} : U_0 \to \mathbb{C},$$

  so that on the overlap $U_0 \cap U_1$ we have

  $$w = \frac{1}{z}, \qquad z = \frac{1}{w}.$$

  Here w is a coordinate near ∞ (since $\infty = [1 : 0]$ corresponds to $w = 0$).

2. Meromorphic functions and meromorphic 1-forms

A *meromorphic function* f on $\mathbb{CP}^1$ is the same as a holomorphic map

$$F : \mathbb{CP}^1 \to \mathbb{CP}^1,$$

whose restriction to $\mathbb{C} \subset \mathbb{CP}^1$ is an ordinary meromorphic function $f(z)$ on $\mathbb{C}$, and with some allowed behavior at ∞.

From the viewpoint of calculus/differential forms, you often look at the 1-form

$$\omega = f(z)\, dz.$$

On $\mathbb{C}$, this is a meromorphic 1-form. To extend f (or ω) to the sphere, we must check what happens near ∞. That means rewriting everything in the coordinate $w = 1/z$.

On the overlap:

$$z = \frac{1}{w}, \qquad dz = -\frac{1}{w^2}\, dw.$$

Thus

$$\omega = f(z)\, dz = f\left(\frac{1}{w}\right) \cdot \left(-\frac{1}{w^2}\, dw\right) = -\, f\left(\frac{1}{w}\right)\, w^{-2}\, dw.$$

So the expression of ω near ∞ (in the coordinate w) is

$$\omega = G(w)\, dw, \qquad G(w) = -\, f\left(\frac{1}{w}\right)\, w^{-2}.$$

Meromorphicity at ∞ means: $G(w)$ has a Laurent series at $w = 0$ with only finitely many negative powers.

3. Global calculus proof that meromorphic on $\mathbb{CP}^1$ is rational

Let $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be meromorphic. Restricting to $\mathbb{C} \subset \mathbb{CP}^1$, we view f as a meromorphic function $f(z)$ on $\mathbb{C}$.

- On a compact Riemann surface like $\mathbb{CP}^1$, a meromorphic function has only finitely many poles. So there exist points $a_1, \ldots, a_k \in \mathbb{C} \cup \{\infty\}$ and positive integers m_j such that all poles of f lie at the a_j.

- For each finite pole $a_j \in \mathbb{C}$ (with local coordinate $z - a_j$), f has a Laurent expansion

$$f(z) = \sum_{n=-m_j}^{\infty} c_{j,n}\, (z - a_j)^n$$

  near $z = a_j$. Its *principal part* at a_j is

$$P_j(z) := \sum_{n=-m_j}^{-1} c_{j,n}\, (z - a_j)^n.$$

- Near ∞, change variables to $w = 1/z$. Then

$$f(z) = f\left(\tfrac{1}{w}\right)$$

has a Laurent expansion in w:

$$f\left(\tfrac{1}{w}\right) = \sum_{n=-M}^{\infty} b_n\, w^n,$$

with finitely many negative powers w^n (this is the definition of "meromorphic at ∞"). Equivalently,

$$f(z) = \sum_{n=-M}^{\infty} b_n\, z^{-n}$$

for large $|z|$. Its principal part at ∞ is

$$P_\infty(z) := \sum_{n=-M}^{-1} b_n\, z^{-n}.$$

Now define a *rational function* $R(z)$ by summing all principal parts:

$$R(z) = \sum_{j=1}^{k} P_j(z) + P_\infty(z).$$

This is a rational function because it is a finite sum of expressions of the form $(z - a_j)^{-n}$ and powers of z (which are all rational).

Then consider

$$g(z) := f(z) - R(z).$$

By construction:

- At each finite pole a_j, g has no negative powers in its Laurent expansion, hence is holomorphic at a_j.
- At ∞, subtracting the principal part $P_\infty(z)$ removes all negative powers in the expansion in $w = 1/z$, so g is holomorphic at ∞ as well.

Thus g is entire on $\mathbb{C}$ and *holomorphic at* ∞. Being holomorphic at ∞ means exactly that g is *bounded* for large $|z|$. By Liouville's theorem:

$$g \text{ entire and bounded } \implies g \text{ is constant.}$$

So there exists $C \in \mathbb{C}$ such that

$$f(z) = R(z) + C.$$

But $R(z) + C$ is still rational, so f is a rational function:

$$f(z) \in \mathbb{C}(z).$$

Therefore, any meromorphic function f on $\mathbb{CP}^1$ is a rational function in the coordinate $x = z$. This shows $\mathcal{M}(\mathbb{CP}^1) \subset \mathbb{C}(x)$, and the converse inclusion (every rational function defines a meromorphic map $\mathbb{CP}^1 \to \mathbb{CP}^1$) is straightforward. Hence

$$\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(x).$$

4. Differential-form language (optional viewpoint)

Instead of looking directly at f, one may look at the meromorphic 1-form

$$\omega = f(z)\,dz.$$

- At each finite pole a_j, ω has a Laurent expansion

$$\omega = \left(\sum_{n=-m_j}^{\infty} c_{j,n}(z - a_j)^n \right) dz.$$

  Its residue at a_j is $c_{j,-1}$.

- Near ∞, write $z = 1/w$, $dz = -w^{-2}dw$. Then

$$\omega = f(1/w)\,dz = -\,f(1/w)\,w^{-2}dw.$$

  Meromorphicity at ∞ is the condition that the coefficient of dw has only finitely many negative powers of w.

- On a compact Riemann surface (like $\mathbb{CP}^1$), the sum of the residues of any meromorphic 1-form is zero:

$$\sum_{p \in \mathbb{CP}^1} \mathrm{Res}_p(\omega) = 0.$$

  This is a global version of the Cauchy integral theorem.

The partial fractions decomposition of f can be seen as reconstructing ω from its local principal parts (residues, higher-order terms) at each pole, then subtracting this from the given ω to get a holomorphic 1-form on $\mathbb{CP}^1$. But any global holomorphic 1-form on $\mathbb{CP}^1$ must be zero, so what remains is just a constant multiple of dz, which corresponds to the polynomial part of f. This reproduces the same "$f =$ rational $+$ constant" conclusion.

In summary, from a calculus/differential-forms viewpoint:

- Meromorphic f on $\mathbb{CP}^1$ has finitely many poles (including possibly ∞).
- Using Laurent expansions and the change of variables $w = 1/z$, we build a rational function $R(z)$ with the same principal parts as f.
- Their difference $g = f - R$ is entire and bounded at ∞, hence constant by Liouville.
- Therefore f is rational: $\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(x)$.