# Advanced Calculus II

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We cover the following topics in this note.

- Convergence of Sequences
- Inequality Rule for Absolute Values
- Limit Theorem (Algebraic Property of Limit of Sequence)

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**Sequence**

**Definition.** Let $X \subseteq \mathbb{R}$. A **sequence** is a function

$$f : \mathbb{N} \to X(\subseteq \mathbb{R}), \quad n \mapsto f(n) := a_n.$$

Instead of using function notation $f(n)$, the values of the sequence are denoted by $\{a_n\}_{n=1}^{\infty}$, where $a_n = f(n)$ is called $n$-th term of the sequence.

**Remark.** A sequence in $X \subseteq \mathbb{R}$ is a function

$$a : \mathbb{N} \to X, \quad n \mapsto a_n,$$

where $a_n \in X$ for all $n \in \mathbb{N}$. We sometimes write

$$\{a_n\}, \quad \{a_n\}_{n=1}^{\infty}, \quad \{a_n\}_{n\in\mathbb{N}}, \quad (a_n)_{n\in\mathbb{N}}, \quad \text{or} \quad \langle a_n\rangle_{n\in\mathbb{N}}.$$

**Convergence of Sequence**

**Definition.** A real sequence $\{a_n\}_{n=1}^{\infty}(\subseteq \mathbb{R})$ is said to **converge** to $L \in \mathbb{R}$ if and only if

$$\forall \varepsilon > 0,\ \exists N_\epsilon \in \mathbb{N} \text{ such that } \left[n \geq N_\epsilon \implies |a_n - L| < \varepsilon\right].$$

**Remark.** The real number $L \in \mathbb{R}$ is called **the limit**[1]. When a sequence $\{a_n\}_{n=1}^{\infty}$ has the limit $L$, we will use the notation

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty.$$

That is,

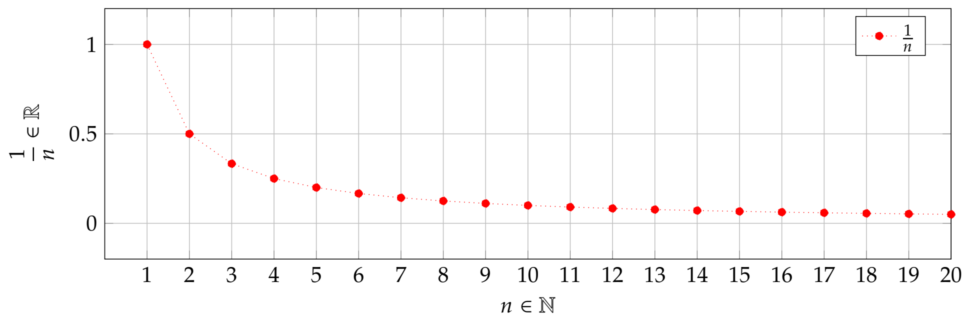
$$\lim_{n\to\infty} a_n = L \iff \forall \varepsilon > 0 : \exists N \in \mathbb{N} : \left[n \geq N \implies |a_n - L| < \varepsilon\right].$$

[1] The limit of a sequence is unique. See **Theorem 4**

**Note.** If a sequence has a limit, we say that the sequence is **convergent**; if it has no limit, we say that the sequence is **divergent**.

**Example.** Consider the sequence defined by $a_n = 1/n$ for each $n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$




*Proof.* Let $\varepsilon > 0$. By the Archimedean property, we obtain

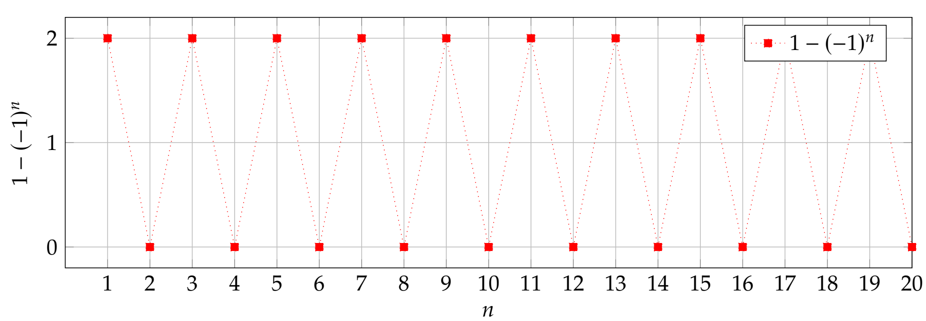
$$\exists N_\epsilon \in \mathbb{N} \quad \text{s.t.} \quad 1 < \varepsilon \cdot N_\epsilon, \text{ i.e., } \frac{1}{N_\epsilon} < \varepsilon.$$

Assume that $n \geq N_\varepsilon$ then

$$|a_n - 0| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N_\varepsilon} < \varepsilon.$$

Hence $\lim_{n\to\infty} \frac{1}{n} = 0.$ □

**Example.** Consider the sequence defined by $b_n = 1 - (-1)^n$ for all $n \in \mathbb{N}$. Prove that $b_n$ does not converge.



*Proof.* The sequence $\{b_n\}$ alternates between 0 and 2:

$$b_n = \begin{cases} 0 & : n = 2k \\ 2 & : n = 2k+1 \end{cases}$$

with $k \in \mathbb{N}$. Suppose that $\{b_n\}_{n=1}^{\infty}$ converges to some limit $B \in \mathbb{R}$ and set $\varepsilon = 1$. Then, by the definition of convergence:

$$\exists N_\varepsilon \in \mathbb{N} \text{ s.t. } n \geq N_\varepsilon \implies |b_n - B| < 1.$$

(Case 1) For all even $n \geq N$, we have $b_n = 0$. Then the inequality $|b_n - B| < 1$ becomes

$$|0 - B| = |B| < 1, \text{ i.e., } B \in (-1, 1). \tag{1}$$

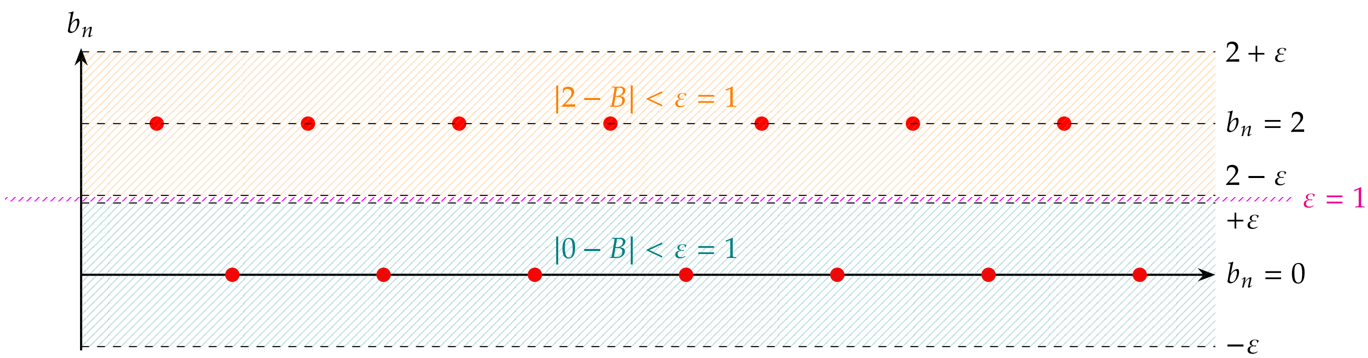
(Case 2) For all odd $n \geq N$, we have $b_n = 2$. Then the inequality $|b_n - B| < 1$ becomes

$$|2 - B| < 1, \text{ i.e., } B \in (1, 3) \tag{2}$$

By (1) and (2), there is no intersection between these ranges;

$$B \in (-1, 1) \cap (1, 3) = \varnothing$$

which proves that $b_n$ does not converge.




□

**Absolute Value in Reals**

**Definition.** Let $x \in \mathbb{R}$. A **absolute value** $|x|$ of $x$ is defined by

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

**Remark.** For $x \in \mathbb{R}$,

$$|x| = \begin{cases} x & : x > 0 \\ 0 & : x = 0 \\ -x & : x < 0 \end{cases}$$

**Proposition 1.** *Let* $x, y \in \mathbb{R}$.

(1) $|x| = +\sqrt{x^2}$

(2) $|x| \geq 0$

(3) $|x| = 0 \Leftrightarrow x = 0$

(4) $|x| = |-x|$

(5) $|xy| = |x||y|$

(6) *(Fundamental Theorem of Absolute Values) For* $c \geq 0$, *we have*

$$|x| \leq c \iff -c \leq x \leq c$$

(7) $-|x| \leq x \leq |x|$

*Proof.* (1) If $(x \geq 0)$ then $|x| = x = \sqrt{x^2}$. Similarly if $x < 0$ then $|x| = -x = \sqrt{x^2}$.

(2) $|x| = \begin{cases} x \geq 0 & : x \geq 0 \\ -x > 0 & : x < 0 \end{cases} \geq 0.$

(3) $(\Leftarrow)$ If $x = 0$ then $|x| = x = 0$.

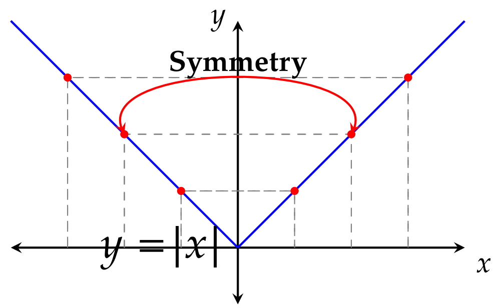
$(\Rightarrow)$ Let $|x| = 0$. Suppose that $x \neq 0$.

(i) $x > 0 \implies |x| = x > 0$ ↯

(ii) $x < 0 \implies |x| = -x > 0$ ↯

Thus $x$ must be zero.

(4) $|-x| = \begin{cases} -x & : -x \geq 0 \text{ (i.e., } x \leq 0) \\ -(-x) = x & : -x < 0 \text{ (i.e., } x > 0) \end{cases} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} = |x|.$




(5) $|xy| = \begin{cases} xy = |x||y| & : x \geq 0, y \geq 0 \\ -xy = x(-y) = |x||y| & : x \geq 0, y < 0 \\ -xy = (-x)y = |x||y| & : x < 0, y \geq 0 \\ xy = (-x)(-y) = |x||y| & : x < 0, y < 0 \end{cases}$

(6) ($\Rightarrow$) Let $|x| \leq c$.

(i) $x \geq 0 \implies x = |x| \leq c$, i.e., $-c \leq 0 \leq x \leq c$.

(ii) $x < 0 \implies -x = |x| \leq c$, i.e., $-c \leq x < 0 \leq c$.

Thus, $-c \leq x \leq c$.

($\Leftarrow$) Let $-c \leq x \leq c$.

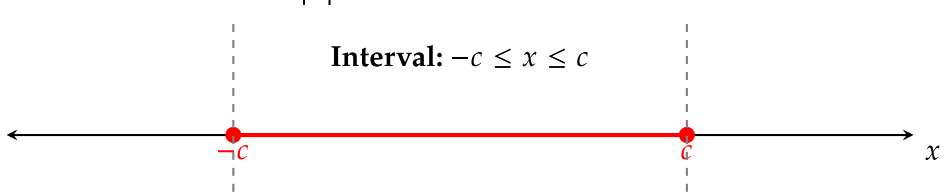
(i) $x \geq 0 \implies |x| = x \leq c$.

(ii) $x < 0 \implies |x| = -x \leq c$.

Thus, $|x| \leq c$.

**Key equivalence:**

$|x| \leq c \iff -c \leq x \leq c$




(7) Let $c = |x|$, where $c \geq 0$. By (6), thus, the result follows.

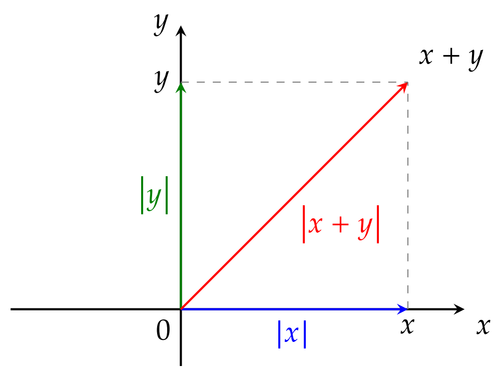
$\square$

**Triangle Inequality**

**Proposition 2.** *Let* $x, y \in \mathbb{R}$.

(1) $|x+y| \leq |x| + |y|$

(2) $\big||x| - |y|\big| \leq |x-y|$.




*Proof.* (1) By (7) of **Proposition 1**, we have

$$-|x| \leq x \leq |x|, \quad -|y| \leq y \leq |y|.$$

Then

$$\begin{array}{rcccccc} & -|x| & \leq & x & \leq & |x| \\ + & -|y| & \leq & y & \leq & |y| \\ \hline & -(|x|+|y|) & \leq & x+y & \leq & |x|+|y| \end{array}$$

Thus, we have $|x+y| \leq |x| + |y|$.

(2)

(i) Note that

$$\begin{aligned} |x| &= |x-y+y| \\ &\leq |x-y| + |y| \quad \text{by (1) of } \textbf{Proposition 2} \end{aligned}$$

Thus $|x| - |y| \leq |x-y|$.

(ii) Note that

$$\begin{aligned}|y| &= |x-(x-y)| \\ &\leq |x| + |-(x-y)| \quad \text{by (1) of } \textbf{Proposition 2} \\ &= |x| + |x-y| \quad \text{by (4) of } \textbf{Proposition 1}\end{aligned}$$

Therefore $-|x-y| \leq |x| - |y|$.

By (i) and (ii), we know

$$-|x-y| \leq |x| - |y| \leq |x-y|, \quad \text{i.e.,} \quad \big||x| - |y|\big| \leq |x-y|.$$

□

**Boundedness of Sequence**

**Definition.** Let $\{a_n\}_{n=1}^{\infty} (\subseteq \mathbb{R})$ is a sequence. $\{a_n\}$ is said to be **bounded** if

$$\exists M \in \mathbb{R} \text{ such that } \forall n \in \mathbb{N}, |a_n| \leq M.$$

**Proposition 3.** *A convergent sequence is bounded.*

*Proof.* Let $\lim_{n\to\infty} a_n = L$. By the definition of convergence, for $\varepsilon = 1$,

$$\exists N \in \mathbb{N} \text{ such that } n \geq N \implies |a_n - L| < 1.$$

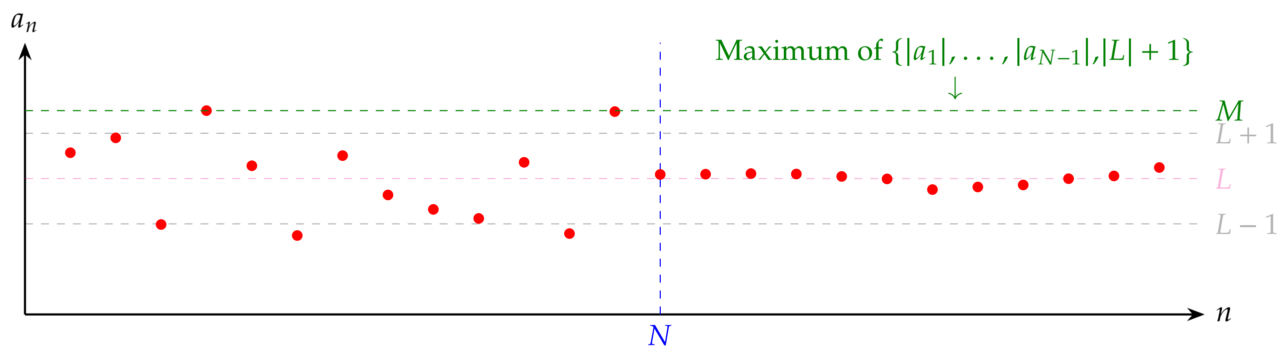
By triangle inequality, we have

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| < 1 + |L|.$$

Let $M := \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |L|\}$. Then

$$|a_n| \leq M$$

for all $n \in \mathbb{N}$. Therefore $\{a_n\}$ is bounded.




□

**Note.** We have established that if the limit of a sequence $a_n$ exists as $n$ approaches infinity, then there exists a real number $M$ such that $|a_n| \leq M$ for all $n$:

$$\exists A \in \mathbb{R} \text{ s.t. } A = \lim_{n\to\infty} a_n \implies \exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, |a_n| \leq M.$$

However, the converse is not necessarily true:

$$\exists A \in \mathbb{R} \text{ s.t. } A = \lim_{n\to\infty} a_n \not\Longleftarrow \exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, |a_n| \leq M.$$

To illustrate, consider the sequence $\{a_n\} = 1-(-1)^n$. This sequence is bounded, yet it does not converge, serving as a counterexample.

Furthermore, we note the following important theorems:

1. **Monotone Convergence Theorem**:
   (i) If a sequence $\{a_n\}$ is bounded above and monotone increasing, then it converges.
   (ii) If a sequence $\{a_n\}$ is bounded below and monotone decreasing, then it converges.
2. **Bolzano-Weierstrass Theorem**: Every bounded sequence of real numbers has a convergent subsequence. That is, if there exists a real number $M$ such that $|a_n| < M$ for all $n$, then there exists a convergent subsequence $\{a_{n_k}\}$ of $\{a_n\}$.

**Limit Theorem (Algebraic Property of Limit of Sequence)**

**Theorem.** *Let* $\lim_{n\to\infty} a_n = \alpha$, $\lim_{n\to\infty} b_n = \beta$, *and* $k \in \mathbb{R}$. *Then*

*(1)* $\lim_{n\to\infty} ka_n = k\alpha = k \lim_{n\to\infty} a_n$.

*(2)* $\lim_{n\to\infty} a_n \pm b_n = \alpha \pm \beta = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$.

*(3)* $\lim_{n\to\infty} a_nb_n = \alpha\beta = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$.

*(4)* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\alpha}{\beta} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$. (Here, $\beta \neq 0$ and $b_n \neq 0$)

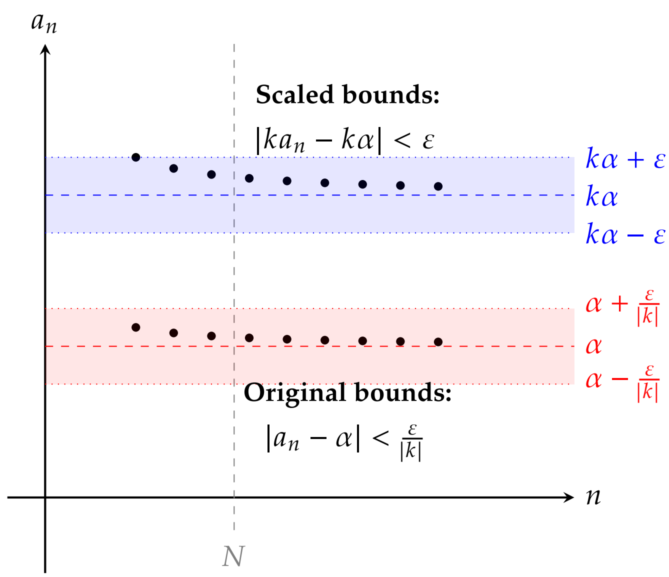
*Proof.* Let $\varepsilon > 0$.

(1) If $k = 0$, it is trivial. Let $k \neq 0$. Since $\lim_{n\to\infty} a_n = \alpha$, we know

$$\exists N \in \mathbb{N} \text{ such that } \left[ n \geq N \implies |a_n - \alpha| < \frac{\varepsilon}{|k|} \right] \qquad (*)$$

Thus, if $n \geq N$ then

$$\begin{aligned} |ka_n - k\alpha| &= \left|k(a_n - \alpha)\right| \\ &= |k||a_n - \alpha| \qquad \because \left|xy\right| = |x|\left|y\right| \\ &< |k| \cdot \frac{\varepsilon}{|k|} \qquad \text{by } (*) \\ &= \varepsilon. \end{aligned}$$



(2) Since $\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} b_n = \beta$, we know that

$$\exists N_1 \in \mathbb{N} \text{ such that } \left[n \geq N_1 \implies |a_n - \alpha| < \frac{\varepsilon}{2}\right] \quad (**)$$
$$\exists N_2 \in \mathbb{N} \text{ such that } \left[n \geq N_2 \implies |b_n - \beta| < \frac{\varepsilon}{2}\right] \quad (***)$$

Let $N = \max\{N_1, N_2\}$. If $n \geq N$ then

$$\begin{aligned}\left|(a_n + b_n) - (\alpha + \beta)\right| &= \left|(a_n - \alpha) + (b_n - \beta)\right| \\ &\leq |a_n - \alpha| + \left|b_n - \beta\right| \quad \text{by Triangle Inequality} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{by (**) and (***)} \\ &= \varepsilon.\end{aligned}$$

and

$$\left|(a_n - b_n) - (\alpha - \beta)\right| = \left|(a_n - \alpha) + (-b_n + \beta)\right| \leq |a_n - \alpha| + \left|b_n - \beta\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(3) By **Proposition 3**, $\{a_n\}$ is bounded and so

$$\exists M > 0 \text{ such that } \forall n \in N, |a_n| \leq M.$$

Note that

$$\lim_{n\to\infty} a_n = \alpha \implies \exists N_1 \in \mathbb{N} : \left[n \geq N_1 \implies |a_n - \alpha| < \frac{\varepsilon}{2|\beta| + 1}\right],$$
$$\lim_{n\to\infty} b_n = \beta \implies \exists N_2 \in \mathbb{N} : \left[n \geq N_2 \implies |b_n - \beta| < \frac{\varepsilon}{2M}\right].$$

Let $N = \max\{N_1, N_2\}$. If $n \geq N$ then

$$\begin{aligned}\left|a_n b_n - \alpha\beta\right| = \left|a_n b_n - \alpha\beta + a_n\beta - a_n\beta\right| &= \left|a_n(b_n - \beta) + \beta(a_n - \alpha)\right| \\ &\leq |a_n|\left|b_n - \beta\right| + \left|\beta\right||a_n - \alpha| \\ &< M \cdot \frac{\varepsilon}{2M} + \frac{|\beta|}{2|\beta| + 1} \cdot \varepsilon \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Note that $2|\beta| < 2|\beta| + 1 \Leftrightarrow \frac{|\beta|}{2|\beta|+1} < \frac{1}{2}$.

(4) It is enough to prove that $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{\beta}$ with $b_n \neq 0$ and $\beta \neq 0$. Note that Triangle Inequality implies that

$$|y| = |y - x + x| \leq |y - x| + |x| \iff |y| - |x| \leq |y - x|$$

for any $x, y \in \mathbb{R}$. Since $\lim_{n\to\infty} b_n = \beta$, for $\frac{1}{2}|\beta| > 0$, $\exists N_1 \in \mathbb{N}$ such that if $n \geq N_1$

$$|\beta| - |b_n| \leq |\beta - b_n| = |b_n - \beta| < \frac{1}{2}|\beta|.$$

Thus, we obtain that

$$|\beta| - |b_n| < \frac{1}{2}|\beta| \implies \frac{1}{2}|\beta| < |b_n| \implies \frac{1}{b_n} < \frac{2}{|\beta|}$$

And

$$\exists N_2 \in \mathbb{N} : \left[ n \geq N_2 \implies |b_n - \beta| < \frac{\beta^2}{2}\varepsilon \right].$$

Let $N = \max\{N_1, N_2\}$. If $n \geq N$ then

$$\left| \frac{1}{b_n} - \frac{1}{\beta} \right| = \left| \frac{\beta - b_n}{\beta b_n} \right| = \frac{|b_n - \beta|}{|\beta||b_n|} < \varepsilon \cdot \frac{\beta^2}{2} \cdot \frac{1}{|\beta|} \cdot \frac{2}{|\beta|} = \varepsilon.$$

□

**Uniqueness of Limits**

**Theorem 4.** *The limit of a sequence is unique.*

*Proof.* We want to show that

$$\lim_{n\to\infty} a_n = \alpha \text{ and } \lim_{n\to\infty} a_n = \beta \implies \alpha = \beta.$$

Let a sequence $\{a_n\}$ has limit $\alpha$ and $\beta$, and let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} a_n = \beta$, we have

$$\exists N_1 \in \mathbb{N} \text{ such that } n \geq N_1 \implies |a_n - \alpha| < \frac{\varepsilon}{2}.$$
$$\exists N_2 \in \mathbb{N} \text{ such that } n \geq N_2 \implies |a_n - \beta| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. If $n \geq N$, then

$$|\beta - \alpha| = |\beta - \alpha + a_n - a_n| = |(a_n - \alpha) + (-a_n + \beta)| \leq |a_n - \alpha| + |a_n - \beta| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

## References

[1] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 6. 해석학 개론 (c) 수열의 수렴성." YouTube Video, 26:29. Published September 20, 2019. URL: https://www.youtube.com/watch?v=jwLfzJyIxmU.

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