Linear Algebra III

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We cover the following topics in this note.

Part I

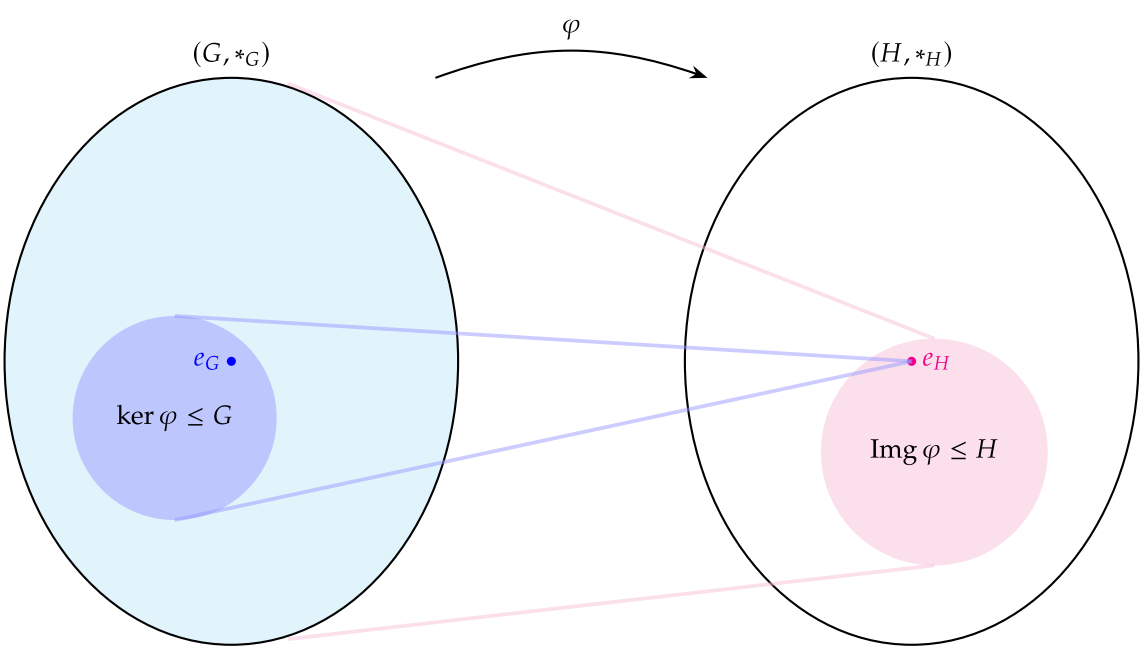
- Subspace; Span
- Subgroup

Part 2

- Homomorphism; Monomorphism; Epimorphism
- Isomorphism

Part 3

- Kernel and Images



1 Part I

Note (span). Let V be a vector space over a field $\mathbb{F}$, and let $S \subseteq V$. Recall that, for $n \in \mathbb{N}$,

$$\begin{aligned}\operatorname{span}(S) :&= \{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n \mid \lambda_i \in \mathbb{F},\ \mathbf{v}_i \in S \text{ for all } i = 1, 2, \ldots, n\} \\ &= \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \,\middle|\, \lambda_i \in \mathbb{F},\ \mathbf{v}_i \in S \text{ for all } 1 \le i \le n \right\}.\end{aligned}$$

(Vector) Subspace

Definition. Let V be a vector space over a field $\mathbb{F}$, and let $U \subseteq V$. We write $\boxed{U \le V}$ if V is a **(vector) subspace** of V. That is, $U \le V$ if and only if U satisfy the following conditions:

(i) $\mathbf{0}_V \in U$;

(ii) $\forall \mathbf{u}, \tilde{\mathbf{u}} \in U,\ \mathbf{u} + \tilde{\mathbf{u}} \in U$;

(iii) $\forall \mathbf{u} \in U,\ \forall \lambda \in \mathbb{F},\ \lambda \mathbf{u} \in U$.

Remark. If $S \subseteq V$, then $\operatorname{span}(S) \le V$.

Proof. We must verify that $\operatorname{span}(S)$ satisfies the three defining properties of a subspace of V:

(i) If $S = \varnothing$, by convention we define $\operatorname{span}(\varnothing) := \{\mathbf{0}_V\}$. Let $S \neq \varnothing$. Choose any $\mathbf{v} \in S (\subseteq V)$ and take $n = 1$ with the scalar $\lambda_1 = 0 \in \mathbb{F}$. Then $\mathbf{0}_V = 0 \cdot \mathbf{v} \in \operatorname{span}(S)$.

(ii) Let $\mathbf{u}, \tilde{\mathbf{u}} \in \operatorname{span}(S)$, say, $\mathbf{u} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i$ and $\tilde{\mathbf{u}} = \sum_{j=1}^{m} \mu_j \tilde{\mathbf{v}}_j$, where $n, m \in \mathbb{N}$, $\lambda_i, \mu_j \in \mathbb{F}$, and $\mathbf{v}_i, \tilde{\mathbf{v}}_j \in S$ for all indices i, j. Then

$$\mathbf{u} + \tilde{\mathbf{u}} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i + \sum_{j=1}^{m} \mu_j \tilde{\mathbf{v}}_j = \overbrace{\underbrace{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n}_{n \text{ terms}} + \underbrace{\mu_1 \tilde{\mathbf{v}}_1 + \mu_2 \tilde{\mathbf{v}}_2 + \cdots + \mu_m \tilde{\mathbf{v}}_m}_{m \text{ terms}}}^{n+m \text{ terms}} \in \operatorname{span}(S).$$

(iii) Let $\alpha \in \mathbb{F}$. Let $\mathbf{u} \in \operatorname{span}(S)$, say, $\mathbf{u} = \displaystyle\sum_{i=1}^{n} \lambda_i \mathbf{v}_i$, where $n \in \mathbb{N}$, $\lambda_i, \in \mathbb{F}$, and $\mathbf{v}_i \in S$ for each $1 \le i \le n$. Then

$$\alpha \mathbf{u} = \alpha \left(\sum_{i=1}^{n} \lambda_i \mathbf{v}_i \right) = \sum_{i=1}^{n} (\alpha \lambda_i) \mathbf{v}_i \in \operatorname{span}(S).$$

since $\alpha \lambda_i \in \mathbb{F}$ for all $i = 1, 2, \ldots, n$.

$\square$

Proposition. *Let V be a vector space over a field $\mathbb{F}$, and let $S \subseteq V$. Then*

(1) $S \subseteq \operatorname{span}(S) \subseteq V$.

(2) If $U \leq V$ is any subspace of V such that $S \subseteq U$, then $\operatorname{span}(S) \subseteq U$.

Proof.

(1) Let $\mathbf{s} \in S$. Then, choosing $n = 1$ and $\lambda_1 = 1 \in \mathbb{F}$, we have $\mathbf{s} = 1 \cdot \mathbf{s} \in \operatorname{span}(S)$. Each element $\mathbf{s} \in \operatorname{span}(S)$ is of the form

$$\mathbf{s} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i,$$

where $\mathbf{v}_i \in S \subseteq V$ and $\lambda_i \in \mathbb{F}$. Since V is a vector space and is closed under finite linear combinations, it follows that $\mathbf{s} \in V$.

(2) Let $U \leq V$ and $S \subseteq U$. Let $\mathbf{s} \in \operatorname{span}(S)$. Then, there exist $n \in \mathbb{N}$, scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F}$, and vectors $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n \in S \subseteq V$ such that

$$\mathbf{s} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \in \operatorname{span}(S).$$

Since

- $S \subseteq U$, i.e., $\mathbf{v}_i \in S \subseteq U$ for each $i = 1, 2, \ldots, n$, and
- $U \leq V$, i.e., $\mathbf{u} + \tilde{\mathbf{u}} \in U$ and $\alpha \mathbf{u} \in U$ for any $\mathbf{u}, \tilde{\mathbf{u}} \in U,\ \alpha \in \mathbb{F}$,

it follows that

$$\forall i \in \{1, 2, \ldots, n\},\ \lambda_i \mathbf{v}_i \in U \quad \text{and} \quad \mathbf{s} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \in U.$$

$\square$

Proposition. *Let V be a vector space over a field $\mathbb{F}$, and let $S \subseteq V$. Let $\mathcal{U} := \{U \leq V : S \subseteq U\}$. Then*

$$\operatorname{span}(S) = \bigcap_{U \in \mathcal{U}} U.$$

In other words, $\operatorname{span}(S)$ *is the smallest subspace of V containing S.*

Proof. We want to show that $\operatorname{span}(S) = \bigcap_{U \in \mathcal{U}} U$.

($\subseteq$) Let $\mathbf{u} \in \operatorname{span}(S)$. By definition, there exists $n \in \mathbb{N}$, scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F}$, and vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in S$ such that

$$\mathbf{u} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i.$$

Let $U \in \mathcal{U}$ be arbitrary. Since $S \subseteq U$ and $U \leq V$, it is closed under finite linear combinations:

$$\sum_{i=1}^{n} \lambda_i \mathbf{v}_i \in U.$$

Since $\forall U \in \mathcal{U},\ \mathbf{u} \in U \Leftrightarrow \mathbf{u} \in \bigcap_{U \in \mathcal{U}} U$, we obtain

$$\mathbf{u} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \in \bigcap_{U \in \mathcal{U}} U.$$

($\supseteq$) Since $S \subseteq \operatorname{span}(S)$ and $\operatorname{span}(S) \leq V$, we know $\operatorname{span}(S) \in \mathcal{U}$. Let $\mathbf{u} \in \bigcap_{U \in \mathcal{U}} U$. Then

$$\mathbf{u} \in \bigcap_{U \in \mathcal{U}} U \iff \forall U \in \mathcal{U},\ \mathbf{u} \in U \implies \mathbf{u} \in \operatorname{span}(S).$$

Hence, we conclude that $\operatorname{span}(S) = \bigcap_{U \in \mathcal{U}} U$. $\square$

Subgroup

Definition. Let G be a group. Let $H \subseteq G$. We say that H is a **subgroup** of G, denoted by $H \leq G$, if and only if H is itself a group (with the operation inherited from G).

Example.

- $(\mathbb{Q}, +) \leq (\mathbb{R}, +)$.
- $(\mathbb{Q}^\times, \times) \leq (\mathbb{R}^\times, \times)$.

Subgroup Test

Proposition. *Let G be a group, and let $H \subseteq G$ with $H \neq \varnothing$.*

(1) (2-step Test)

$$H \leq G \iff \left(x, y \in H \implies xy \in H,\ x^{-1} \in H\right)$$

(2) (1-step Test)

$$H \leq G \iff \left(x, y \in H \implies xy^{-1} \in H\right)$$

Proof. We want to show that

$$\underbrace{H \leq G}_{\text{(a)}} \iff \underbrace{\left(x, y \in H \implies xy \in H,\ x^{-1} \in H\right)}_{\text{(b)}} \iff \underbrace{\left(x, y \in H \implies xy^{-1} \in H\right)}_{\text{(c)}}$$

((a) $\Rightarrow$ (b)) Let $H \leq G$. Let $x, y \in H$. Since every subgroup is closed under the group operation and taking inverses, we have

$$xy \in H \quad \text{and} \quad x^{-1} \in H.$$

((b) $\Rightarrow$ (c)) Let $x, y \in H$. Suppose that $xy \in H$ and $x^{-1} \in H$. Clearly, $xy^{-1} \in H$.

((c) $\Rightarrow$ (a)) Let $x, y \in H$. Suppose that

$$xy^{-1} \in H.$$

Since $H \neq \varnothing$, $\exists a \in H$, and so

$$aa^{-1} \in H \implies e \in H.$$

Since $x \in H$ and $e \in H$, we have

$$ex^{-1} \in H \implies x^{-1} \in H.$$

Then, since $x, y \in H$ and $y^{-1} \in H$, we obtain

$$x(y^{-1})^{-1} \in H \implies xy \in H,$$

i.e., H is closed under binary operation on G.

$\square$

Subgroup Generated by S

Definition. Let G be a group, and let $S \subseteq G$. The **subgroup of G generated by S**, denoted by $\langle S \rangle$, is defined as:

$$\langle S \rangle := \bigcap \{H \leq G : S \subseteq H\} = \bigcap_{S \subseteq H \leq G} H.$$

Exercise. Let G be a group, and let $S \subseteq G$. Show that $\langle S \rangle$ is the unique smallest subgroup of G containing S.

Sol. TBA □

Exercise. Let G be a group, and let $S \subseteq G$. Let $H_i \leq G$ for each $i \in I$. Show that

$$\bigcap_{i \in I} H_i \leq G.$$

Sol. TBA □

Proposition. *Let $(G, +)$ be an abelian group with identity 0_G, and let $x, y \in G$. Then*

(1) $\langle x \rangle = \{nx : n \in \mathbb{Z}\}$

(2) $\langle x, y \rangle = \left\{nx + my : n, m \in \mathbb{Z}\right\}$

Proof. TBA □

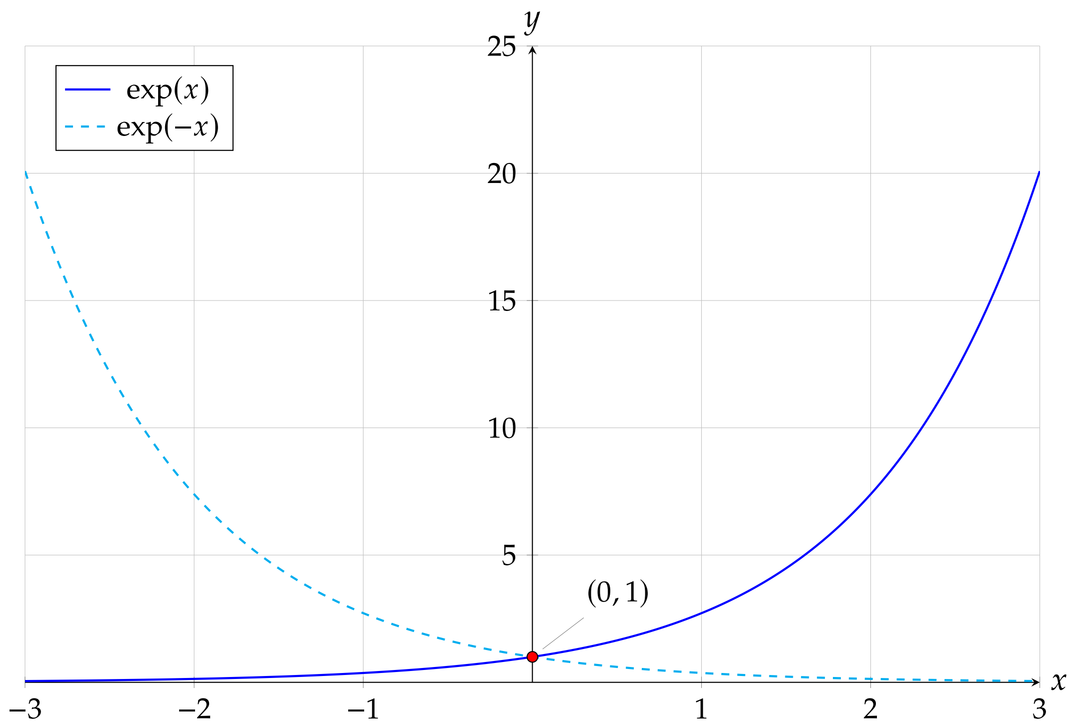
2 Part II

Observation. Let

- $(\mathbb{R}, +)$ is the additive group of real numbers, and
- $(\mathbb{R}_{>0}, \cdot)$ is the multiplicative group of positive real numbers.

The **exponential function** is defined by

$$\begin{array}{rccc} \exp & : & (\mathbb{R}, +) & \longrightarrow & (\mathbb{R}_{>0}, \cdot) \\ & & x & \longmapsto & e^x \end{array}.$$



Then,

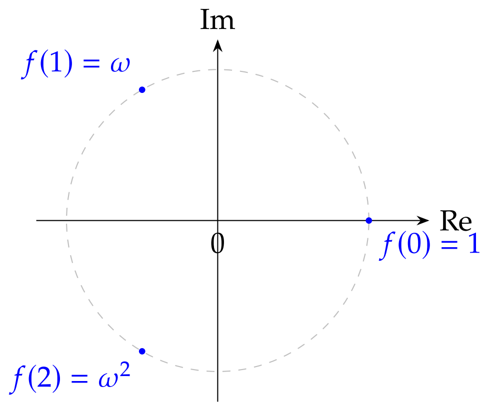
(i) $\exp(x+y) = e^{x+y} = e^x \cdot e^y = \exp(x) \cdot \exp(y)$;

(ii) $\exp(0) = e^0 = 1$;

(iii) $\exp(-x) = e^{-x} = (e^x)^{-1} = (\exp(x))^{-1}$.

Observation. Consider the exponential map

$$f : \mathbb{Z}_3 \to U_3, \quad f(x) = \exp\left(\frac{2\pi i}{3} x\right).$$



Then f is a group homomorphism from the additive group $(\mathbb{Z}_3, +)$ to the multiplicative group $(U_3, \cdot)$ of the third roots of unity. Here,

- $\mathbb{Z}_3 = \{0, 1, 2\}$ with addition modulo 3
- $U_3 = \{1,\ \omega,\ \omega^2\}$, with $\omega = \exp\left(\frac{2\pi i}{3}\right)$ which satisfies $\omega^3 = 1$.

The homomorphism property means that for all $x, y \in \mathbb{Z}_3$ we have:

$$f(x + y) = f(x)f(y).$$

(Addition Table in $\mathbb{Z}_3$)

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

(Multiplicative Table in $\mathbb{Z}_3$)

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

After applying the exponential map, the corresponding elements are:

$$f(0) = 1, \quad f(1) = \omega, \quad f(2) = \omega^2.$$

Thus, the multiplication table is:

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$\cdot$	$f(0)$	$f(1)$	$f(2)$
$f(0)$	$f(0)$	$f(1)$	$f(2)$
$f(1)$	$f(1)$	$f(2)$	$f(0)$
$f(2)$	$f(2)$	$f(0)$	$f(1)$

Homomorphism, Monomorphism, Epicmorphism, and Isomorphism

Definition. Let $(G, *_G)$ and $(H, *_H)$ be groups with identity elements e_G and e_H, respectively.

(1) A function $\varphi : G \longrightarrow H$ is said to be a **group homomorphism** if and only if

$$\varphi(x *_G y) = \varphi(x) *_H \varphi(y) \quad \text{for all } x, y \in G.$$

(2) A group homomorphism $\varphi : G \longrightarrow H$ is called a **group monomorphism** iff it is injective.

(3) A group homomorphism $\varphi : G \longrightarrow H$ is called an **group epimorphism** iff it is surjective.

(4) A group homomorphism $\varphi : G \longrightarrow H$ is called an **group isomorphism** iff it is bijective.

Ring Homomorphism

Definition. Let $(R, +_R, \cdot_R)$ and $(S, +_S, \cdot_S)$ be rings (with unity). A function

$$\varphi : (R, +_R, \cdot_R) \longrightarrow (S, +_S, \cdot_S)$$

is called a **ring homomorphism** if

(i) $\varphi(a +_R b) = \varphi(a) +_S \varphi(b)$ for all $a, b \in R$

(ii) $\varphi(a \cdot_R b) = \varphi(a) \cdot_S \varphi(b)$.

and, if the rings are unital, one additionally requires $\varphi(1_R) = 1_S$. It is immediate that this definition implies $\varphi(0_R) = 0_S$ since

$$\varphi(0_R) = \varphi(0_R +_R 0_R) = \varphi(0_R) +_S \varphi(0_R).$$

Module Homomorphism

Definition. Let R be a ring and let $(M, +_M, \cdot_M)$ and $(N, +_N, \cdot_N)$ be R-modules. A function

$$f : (M, +_M, \cdot_M) \longrightarrow (N, +_N, \cdot_N)$$

is an R-**module homomorphism** if the following hold: for all $m_1, m_2 \in M$ and for all $r \in R$

(i) $f(m_1 +_M m_2) = f(m_1) +_N f(m_2)$

(ii) $f(r \cdot_M m_1) = r \cdot_N f(m_1)$.

Linear Transformation (revised via Module Homomorphism)

Definition. Let F be a field and let V and W be vector spaces over $\mathbb{F}$; that is, V and W are F-modules. A function

$$T : V \longrightarrow W$$

is called a **linear transformation** if the followings are satisfied: for every $\mathbf{v}_1, \mathbf{v}_2 \in V$ and every scalar $\lambda \in F$

(i) $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$;

(ii) $T(\lambda\,\mathbf{v}_1) = \lambda\,T(\mathbf{v}_1)$.

Thus, a linear transformation is precisely an $\mathbb{F}$-module homomorphism.

Preservation of Identity and Inverses

Proposition. *Let $(G, \cdot_G)$ and $(H, \cdot_H)$ be groups with respective identity elements e_G and e_H, and let $\varphi : G \longrightarrow H$ be a group homomorphism, that is,*

$$\varphi(a \cdot_G b) = \varphi(a) \cdot_H \varphi(b) \quad \textit{for all } a, b \in G.$$

Then the following hold:

(1) ***Preservation of Identity:*** $\varphi(e_G) = e_H$.

(2) ***Preservation of Inverse:*** $\varphi\left(a^{-1}\right) = \left(\varphi(a)\right)^{-1}$ *for all* $a \in G$.

Proof. TBA □

3 Part III

Kernel

Definition. Let $\varphi : G \to H$ be a group homomorphism. The **kernel of** φ is the subset of G defined by

$$\ker(\varphi) := \{g \in G : \varphi(g) = e_H\}.$$

Remark. The set $\ker(\varphi)$ is a ~~normal~~ subgroup of G.

Proof. TBA □

Image

Definition. Let $\varphi : G \to H$ be a group homomorphism. The **image** of φ is the subset of H given by

$$\mathrm{Img}(\varphi) := \{h \in H : \exists\, g \in G \text{ such that } \varphi(g) = h\} = \left\{\varphi(g) : g \in G\right\}.$$

Remark. The set $\mathrm{Img}(\varphi)$ forms a subgroup of H.

Proof. TBA □