Lecture Note: 1-Form as Scalar Projection onto a Line in $\mathbb{R}^2$

1 Ambient Space and Tangent Spaces

Let

$$\mathbb{R}^2 \;=\; \{\,(x,y) : x,y \in \mathbb{R}\}$$

be equipped with the standard basis

$$\mathbf{e}_1 = (1,0), \quad \mathbf{e}_2 = (0,1).$$

At any point $p \in \mathbb{R}^2$, the tangent space is canonically

$$T_p\mathbb{R}^2 \;=\; \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\} \;\cong\; \mathbb{R}^2.$$

We write a general tangent vector at p as

$$v \;=\; v^1\,\mathbf{e}_1 \;+\; v^2\,\mathbf{e}_2, \qquad v^1, v^2 \in \mathbb{R}.$$

2 Definition of the Line and Its Unit Direction

Fix a nonzero vector

$$\mathbf{w} \;=\; (w_1, w_2) \;\in\; \mathbb{R}^2, \qquad \mathbf{w} \neq (0,0).$$

This determines the line

$$L \;=\; \operatorname{span}\{\mathbf{w}\} \;\subset\; \mathbb{R}^2.$$

Define the unit–direction vector

$$\hat{\mathbf{w}} \;=\; \frac{\mathbf{w}}{\|\mathbf{w}\|} \;=\; \frac{1}{\sqrt{w_1^2 + w_2^2}}\,(w_1, w_2),$$

so that $\|\hat{\mathbf{w}}\| = 1$.

3 Scalar Projection Functional

Definition 1. *The* scalar projection onto the line L *is the map*

$$\alpha\colon T_p\mathbb{R}^2 \longrightarrow \mathbb{R}$$

given by

$$\alpha(v) \;=\; \big\langle \hat{\mathbf{w}},\, v \big\rangle \;=\; \hat{w}_1\, v^1 \;+\; \hat{w}_2\, v^2.$$

Lemma 1. *α is a linear functional on $T_p\mathbb{R}^2$. Hence it defines a differential 1–form on $\mathbb{R}^2$.*

Proof. For $v, v' \in T_p\mathbb{R}^2$ and $\lambda \in \mathbb{R}$,

$$\alpha(v + v') = \big\langle \hat{\mathbf{w}},\, v + v' \big\rangle = \big\langle \hat{\mathbf{w}},\, v \big\rangle + \big\langle \hat{\mathbf{w}},\, v' \big\rangle = \alpha(v) + \alpha(v'),$$

$$\alpha(\lambda\, v) = \big\langle \hat{\mathbf{w}},\, \lambda\, v \big\rangle = \lambda \,\big\langle \hat{\mathbf{w}},\, v \big\rangle = \lambda\, \alpha(v).$$

Thus $\alpha \in (T_p\mathbb{R}^2)^*$. □

4 Expression in the Dual Basis

Recall that the dual basis $\{dx, dy\}$ of $\{\mathbf{e}_1, \mathbf{e}_2\}$ is defined by

$$dx(\mathbf{e}_1) = 1, \quad dx(\mathbf{e}_2) = 0, \qquad dy(\mathbf{e}_1) = 0, \quad dy(\mathbf{e}_2) = 1.$$

Writing $v = v^1\mathbf{e}_1 + v^2\mathbf{e}_2$, one has

$$dx(v) = v^1, \quad dy(v) = v^2.$$

Therefore

$$\alpha(v) = \hat{w}_1\, v^1 + \hat{w}_2\, v^2 = \hat{w}_1\, dx(v) + \hat{w}_2\, dy(v),$$

and as a 1–form,

$$\boxed{\alpha = \hat{w}_1\, dx \;+\; \hat{w}_2\, dy \;=\; \frac{w_1}{\sqrt{w_1^2 + w_2^2}}\, dx \;+\; \frac{w_2}{\sqrt{w_1^2 + w_2^2}}\, dy.}$$

Interpretation

- $x, y\colon \mathbb{R}^2 \to \mathbb{R}$ are the coordinate functions associated to the ambient basis $(\mathbf{e}_1, \mathbf{e}_2)$.
- $dx, dy\colon T_p\mathbb{R}^2 \to \mathbb{R}$ are the dual coordinate functions on each tangent space.
- The 1-form α is precisely the functional which takes any tangent vector v and returns its signed scalar projection onto the fixed direction $\hat{\mathbf{w}}$.