Topology I

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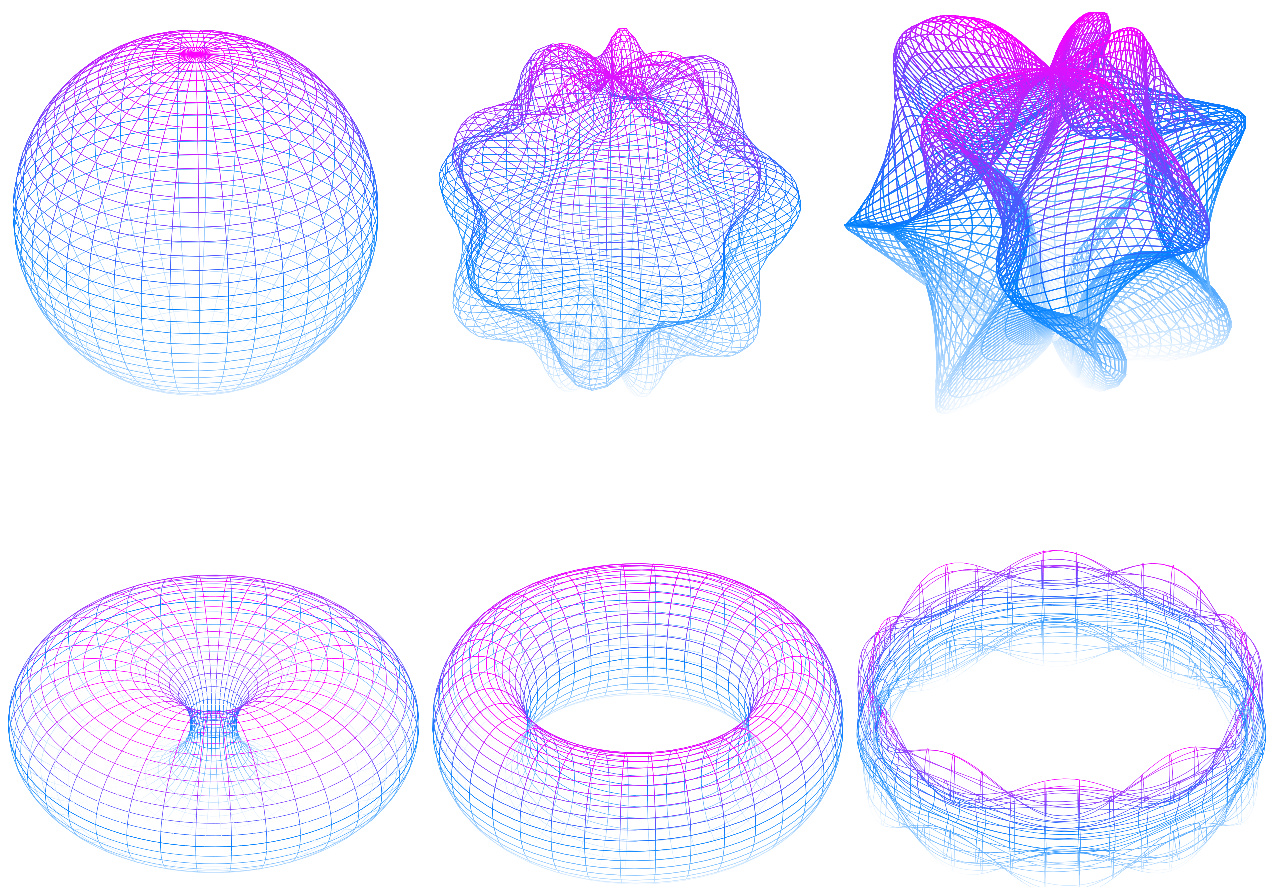
We cover the following topics in this note.

- Topology and Topological Space
- Open Set
- Continuous Mapping
- Distance Function and Metric Space
- Convergence of Sequences; Continuity of Functions
- TBA



Topology; Topological Space

Definition. Let S be a non-empty set. A **topology**[a] on S is a subset $\mathscr{T} \subseteq 2^S$, where 2^S denotes the power set of S, that satisfies the following axioms:

(O1)[b] The empty set and the entire set S belong to $\mathscr{T}$: $\boxed{S \in \mathscr{T} \text{ and } \varnothing \in \mathscr{T}}$.

(O2)[c] The union of any collection of elements in $\mathscr{T}$ is also an element of $\mathscr{T}$:

$$\boxed{\{U_i\}_{i\in I} \subseteq \mathscr{T} \implies \bigcup_{i\in I} U_i \in \mathscr{T}}.$$

(O3)[d] The intersection of any finite number of elements in $\mathscr{T}$ is also an element of $\mathscr{T}$:

$$\boxed{\{U_i\}_{i=1}^n \subseteq \mathscr{T} \implies \bigcap_{i=1}^n U_i \in \mathscr{T}}.$$

The pair $(S, \mathscr{T})$ is called a **topological space**.

[a] The word "topology" comes from the Greek roots "topos" meaning "place" and "logos" meaning "study".
[b] Empty set and Whole space
[c] Closure under *arbitrary* unions
[d] Closure under *finite* intersections

Remark. By mathematical induction, we have

$$\text{O3} \iff [\{U_1, U_2\} \subseteq \mathscr{T} \Rightarrow U_1 \cap U_2 \in \mathscr{T}].$$

Open Set (Topology)

Definition. Let $(S, \mathscr{T})$ be a topological space. $U \subseteq S$ is an **open set**, or **open** (in S) iff $U \in \mathscr{T}$.

Remark. A subset $\mathscr{T}$ of power set 2^S is a topology on S if and only if

(i) $\varnothing$ and S are open;

(ii) Let $U_1, U_2, \cdots \in \mathscr{T}$, i.e., $\{U_i\}_{i\in I} \subseteq \mathscr{T}$. Then $\bigcup_{i\in I} U_i$ is open.

(iii) Let $U_1, U_2, \ldots, U_n \in \mathscr{T}$, i.e., $\{U_i\}_{i=1}^n \subseteq \mathscr{T}$. Then $\bigcap_{i=1}^n U_i$ is open.

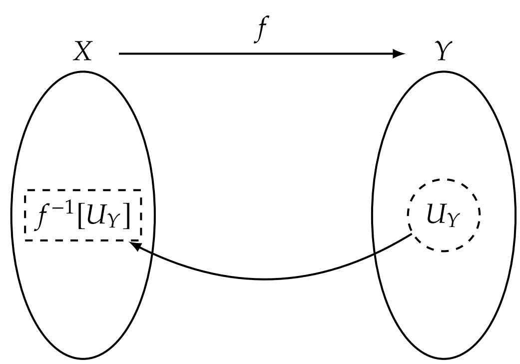
Continuous Mapping by Open Sets

Definition. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are topological spaces. Let $f : X \to Y$ be a mapping from X to Y.

(1) (Continuous Everywhere) The mapping f is **continuous on** $\boldsymbol{X}$ if and only if

$$U_Y \in \mathscr{T}_Y \implies f^{-1}[U_Y] \in \mathscr{T}_X,$$

where $f^{-1}[U_Y] = \left\{x \in X : f(x) \in U_Y\right\}$ is the preimage of U_Y under f.



Note (Preparation for **Example 1**). Let $S \neq \varnothing$ be a set, and let $\{A_\alpha\}_{\alpha \in \Lambda} \subseteq S$. Then

$$\begin{aligned} S \setminus \bigcup_{\alpha \in \Lambda} A_\alpha = S \setminus \{x \in S : \exists \alpha \in \Lambda \text{ s.t. } x \in A_\alpha\} &= \left\{x \in S : \neg[\exists \alpha \in \Lambda \text{ s.t. } x \in A_\alpha]\right\} \\ &= \{x \in S : \forall \alpha \in \Lambda,\ x \notin A_\alpha\} \\ &= \left\{x \in S : \forall \alpha \in \Lambda,\ x \in S \setminus A_\alpha\right\} \\ &= \bigcap_{\alpha \in \Lambda} (S \setminus A_\alpha). \end{aligned}$$

$$\begin{aligned} S \setminus \bigcap_{\alpha \in \Lambda} A_\alpha = S \setminus \{x \in S : \forall \alpha \in \Lambda,\ x \in A_\alpha\} &= \left\{x \in S : \neg[\forall \alpha \in \Lambda,\ x \in A_\alpha]\right\} \\ &= \{x \in S : \exists \alpha \in \Lambda \text{ s.t. } x \notin A_\alpha\} \\ &= \left\{x \in S : \exists \alpha \in \Lambda \text{ s.t. } x \in S \setminus A_\alpha\right\} \\ &= \bigcup_{\alpha \in \Lambda} (S \setminus A_\alpha). \end{aligned}$$

Note (Preparation for **Example 1**).

(1) A Subset of a Finite Set is Finite.

(2) The Intersection of Finite Sets is Finite.

Example 1 (Cofinite Topology). Let $S \neq \varnothing$ be a set. Define the cofinite topology $\mathscr{T}_C \subseteq 2^S$ by

$$\begin{aligned}\mathscr{T}_C &:= \left\{U \subseteq S : S \setminus U \text{ is finite}\right\} \cup \{\varnothing\} \\ &= \left\{U \subseteq S : U = \varnothing \text{ or } S \setminus U \text{ is finite}\right\}.\end{aligned}$$

In other words, U is open in the cofinite topology if U is the empty, or if the complement $S \setminus U$ is a finite set. We claim that $\mathscr{T}_C$ be a topology on S:

(O1) By definition, $\varnothing \in \mathscr{T}_C$. For $U = S$, the complement $S \setminus S = \varnothing$, which is finite, so $S \in \mathscr{T}_C$. Hence, both $\varnothing$ and S are elements of $\mathscr{T}_C$.

(O2) Let $\{U_i\}_{i \in I} \subseteq \mathscr{T}_C$.

(Case 1) If $U_i = \varnothing$ for all $i \in I$, then $\bigcup_{i \in I} U_i = \varnothing \in \mathscr{T}_C$.

(Case 2) Suppose that there exists $i_0 \in I$ such that $U_{i_0} \neq \varnothing$. Then

$$S \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (S \setminus U_i) \subseteq (S \setminus U_{i_0}).$$

Since $S \setminus U_{i_0}$ is finite, $S \setminus \bigcup_{i \in I} U_i$ if finite, so $\bigcup_{i \in I} U_i \in \mathscr{T}_C$.

(O3) Let $U_1 \in \mathscr{T}_C$ and $U_2 \in \mathscr{T}_C$.

(Case 1) If $U_1 = \varnothing$ or $U_2 = \varnothing$, then $U_1 \cap U_2 = \varnothing \in \mathscr{T}_C$.

(Case 2) Suppose that $U_1 \neq \varnothing$ and $U_2 \neq \varnothing$. Then $S \setminus U_1$ and $S \setminus U_2$ are finite. By the De Morgan law, we have

$$S \setminus (U_1 \cap U_2) = (S \setminus U_1) \cup (S \setminus U_2),$$

which is a finite set. Thus, $U_1 \cap U_2 \in \mathscr{T}_C$.

Example 2 (Discrete Topology)**.** Let $S \neq \varnothing$ be a set, and let $\mathscr{T} = 2^S$ be the power set of S. Then $\mathscr{T}$ is called the **discrete topology** on S and $(S, \mathscr{T}) = (S, 2^S)$ the **discrete (topological) space** on S.

Example 3 (Indiscrete Topology)**.** Let $S \neq \varnothing$ be a set, and let $\mathscr{T} = \{S, \varnothing\}$. Then $\mathscr{T}$ is called the **indiscrete topology** on S and $(S, \mathscr{T}) = (S, \{S, \varnothing\})$ the **indiscrete (topological) space** on S.

Note.

(1) Discrete Topology is Finest Topology.

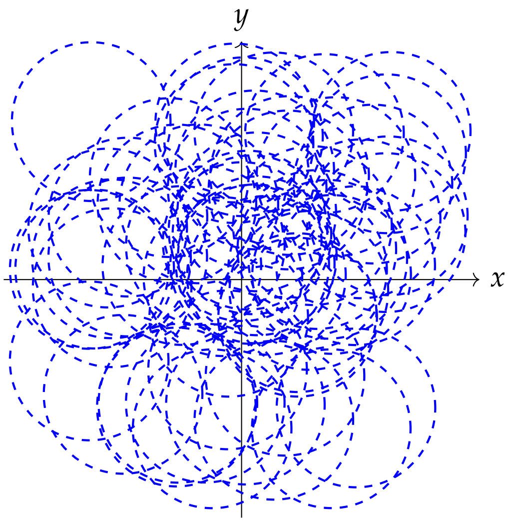
(2) Indiscrete Topology is Coarsest Topology.

Coarser Topology and Finer Topology

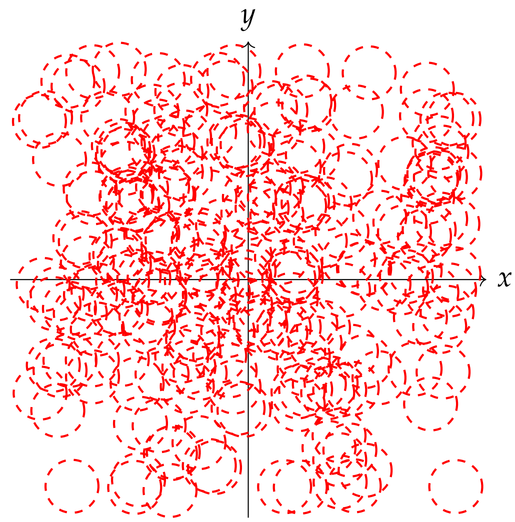
Definition. Let $S \neq \varnothing$ be a set. Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be topologies on S.

(1) $\mathscr{T}_1$ is said to be **coarser** than $\mathscr{T}_2$ if $\mathscr{T}_1 \subseteq \mathscr{T}_2$.

(2) $\mathscr{T}_1$ is said to be **finer** than $\mathscr{T}_2$ if $\mathscr{T}_2 \subseteq \mathscr{T}_1$.



Coarser Topology



Finer Topology

Distance Function

Definition. Let S be a set. The real-valued function of two variable

$$d : S \times S \to \mathbb{R}$$

is called a **distance function** (or **metric**) if it satisfies the following properties:

(i)[a] $\forall x, y \in S,\ d(x,y) \geq 0 \quad \text{and} \quad d(x,y) = 0 \Leftrightarrow x = y.$

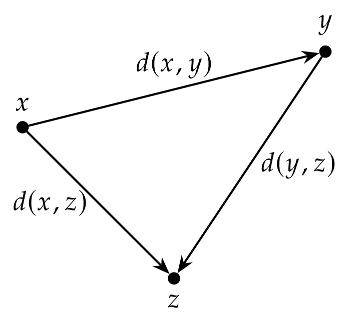
(ii)[b] $\forall x, y \in S,\ d(x,y) = d(y,x).$

(iii)[c] $\forall x, y, z \in S,\ d(x,z) \leq d(x,y) + d(y,z).$

The pair (S,d) is called a **metric space**.

[a] Non-negativity and Zero only for identical points
[b] Symmetry
[c] Triangle inequality

Remark.



Example 4.

- Let $S = \mathbb{R}$, the set of real numbers. Define the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

  $$d(x,y) = \left| x - y \right|$$

  for $x, y \in \mathbb{R}$.

- Let $S = \mathbb{R}^n$, the n-dimensional Euclidean space. Define the function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

  $$d(\mathbf{x}, \mathbf{y}) = \left\| \mathbf{x} - \mathbf{y} \right\| = \sqrt{\sum_{i=0}^{n-1} \left| x_i - y_i \right|^2},$$

  where $\mathbf{x} = (x_0, x_1, \ldots x_{n-1})$ and $\mathbf{y} = (y_0, \ldots, y_{n-1})$ are vectors in $\mathbb{R}^n$.

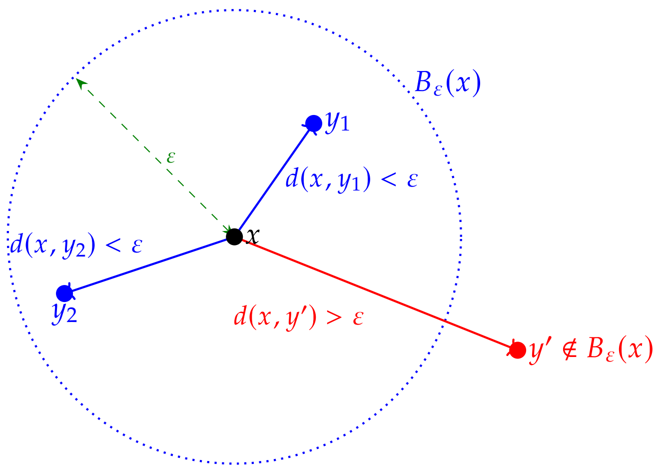
Open Epsilon-Ball

Definition. Let (S,d) be a metric space, where S is a set and $d : S \times S \to \mathbb{R}$ is a metric. For $x \in S$ and $\varepsilon \in \mathbb{R}_{>0}$, the **open ε-ball**[a] **of** x **in** S, denoted by $B_\varepsilon(x)$, is defined as

$$B_\varepsilon(x) := \left\{ y \in S : d(x,y) < \varepsilon \right\}.$$

[a] Open ball with center x and radius ε

Remark.



Epsilon-Neighborhood (Real Analysis)

Definition. Consider the Euclidean space $(\mathbb{R}^1, d)$. The **ε-neighborhood of** $\alpha \in \mathbb{R}$ is defined as the open interval:

$$\mathcal{N}_\varepsilon(\alpha) := \left\{ x \in \mathbb{R} : |x-\alpha| < \varepsilon \right\} = (\alpha-\varepsilon, \alpha+\varepsilon)$$

where $\varepsilon \in \mathbb{R}_{>0}$.

Neighborhood (Topology)

Definition. Let (S,τ) be a topological space.

(1) (Neighborhood of a Set) Let $A \subseteq S$. $\mathcal{N}_A$ is a **neighborhood of** A if

$$\exists U \in \tau \text{ such that } A \subseteq U \subseteq \mathcal{N}_A \subseteq S.$$

(2) (Neighborhood of a Point) Consider a singleton $\{a\} = A \subseteq S$, that is, $a \in S$ be a point in S. Then $\mathcal{N}_a$ is a **neighborhood** of $a \in S$ if

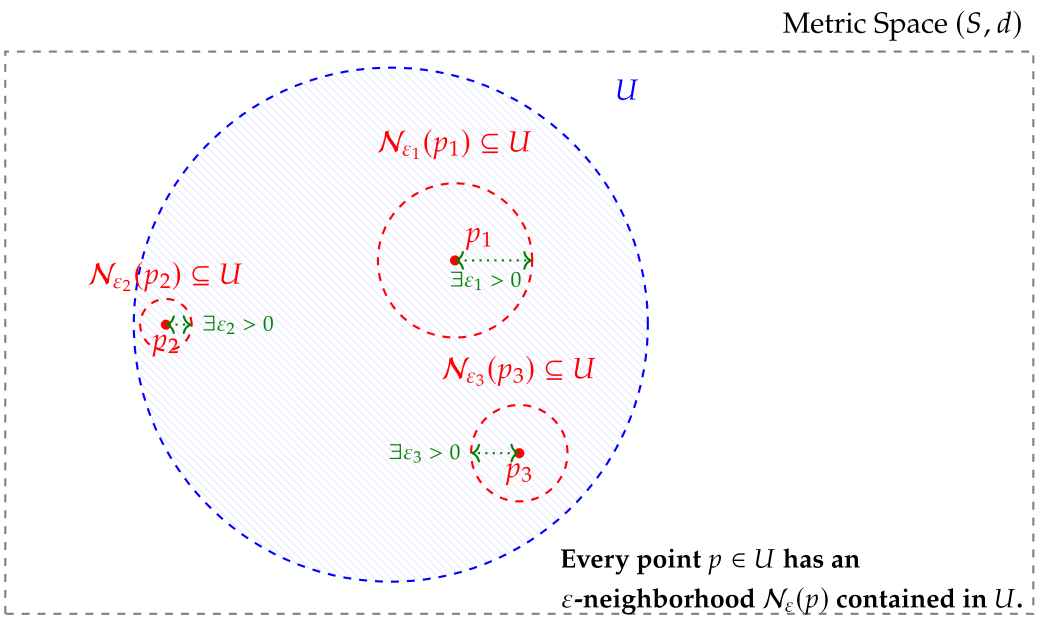
$$\exists U \in \tau \text{ such that } a \in U \subseteq \mathcal{N}_a \subseteq S.$$

Open Set (Metric Space)

Definition. Let (S, d) be a metric space, where S is a set and $d : S \times S \to \mathbb{R}$ is a metric. Then

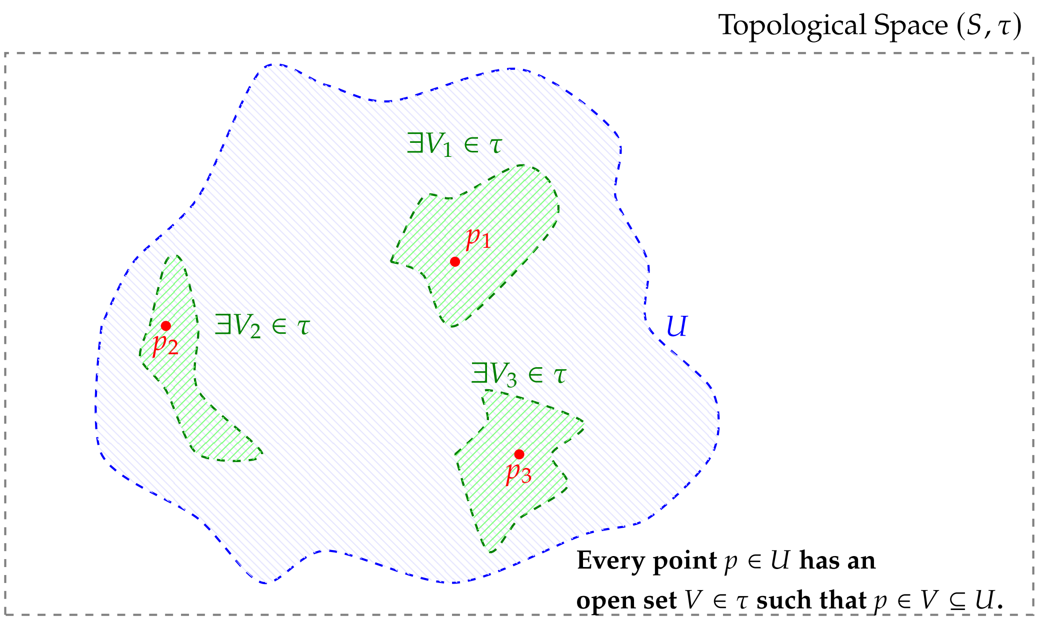
$$U \subseteq S \text{ is } \textbf{open} \text{ in } S \overset{\text{def}}{\iff} \forall p \in U,\ \exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(p) \subseteq U.$$

Remark.



Remark. Let (S, τ) be a topological space. Then

$$U \subseteq S \text{ is } \textbf{open} \text{ in } S \overset{\text{def}}{\iff} \forall p \in U, \exists V \in \tau \text{ such that } p \in V \subseteq U.$$



Exercise (Metric Topology)**.** Let (S,d) be a metric space, where S is a set and $d : S \times S \to \mathbb{R}$ is a metric. Consider the set τ of all open sets of S:

$$\begin{aligned}\tau :&= \left\{U \subseteq S : U \text{ is open in } S\right\} \\ &= \left\{U \subseteq S : \forall p \in U,\ \exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(p) \subseteq U\right\}.\end{aligned}$$

We claim that τ is the topology induced by the metric d on the space S:

(O1) $S \in \tau$ and $\varnothing \in \tau$:

($\varnothing \in \tau$) The condition

$$\text{“}\forall p \in U,\ \exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(p) \subseteq U\text{”}$$

is vacuously true for $U = \varnothing$. Therefore $\varnothing \in \tau$.

($S \in \tau$) For $p \in S$, the ε-neighborhhod of p is defined as

$$\mathcal{N}_\varepsilon(p) = \left\{q \in S : d(p,q) < \varepsilon\right\} \subseteq S.$$

Since S is the entire space, $\mathcal{N}_\varepsilon(p) \subseteq S$ for any $\varepsilon > 0$.

(O2) τ is closed under arbitrary unions:

Let $\{U_i\}_{i\in I}$ be an arbitrary collection of sets in τ. Let $p \in \bigcup_{i\in I} U_i$. Then

$$\exists i_0 \in I \quad \text{such that} \quad p \in U_{i_0}.$$

Since $U_{i_0} \in \tau$, there exists $\varepsilon > 0$ such that $\mathcal{N}_\epsilon(p) \subseteq U_{i_0}$. Then

$$\mathcal{N}_\epsilon(p) \subseteq U_{i_0} \subseteq \bigcup_{i\in I} U_i.$$

Thus, $\bigcup_{i\in I} U_i \in \tau$.

(O3) τ is closed under finite intersections:

Let $U_1, U_2 \in \tau$, and let $p \in (U_1 \cap U_2)$. Then

$$\begin{aligned}\exists \varepsilon_1 > 0 \quad &\text{such that} \quad \mathcal{N}_{\varepsilon_1}(p) \subseteq U_1, \\ \exists \varepsilon_2 > 0 \quad &\text{such that} \quad \mathcal{N}_{\varepsilon_2}(p) \subseteq U_2.\end{aligned}$$

Define $\varepsilon := \min(\varepsilon_1, \varepsilon_2)$. Then

$$\mathcal{N}_\varepsilon(p) \subseteq \mathcal{N}_{\varepsilon_i}(p) \subseteq U_i \quad \text{for} \quad i = 1,2.$$

Thus $\mathcal{N}_\varepsilon(p) \subset U_1 \cap U_2$, and so $U_1 \cap U_2 \in \tau$.

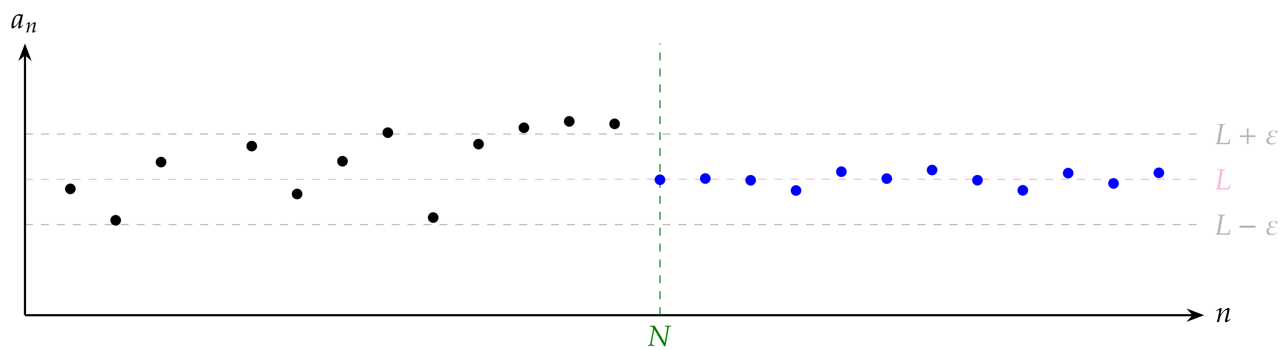
Note (Convergence of Sequences). We consider the topological space $(\mathbb{R}, \tau)$ where

$$\tau = \left\{ U \subseteq \mathbb{R} : U = \bigcup_{i \in I} (a_i, b_i) \right\}$$

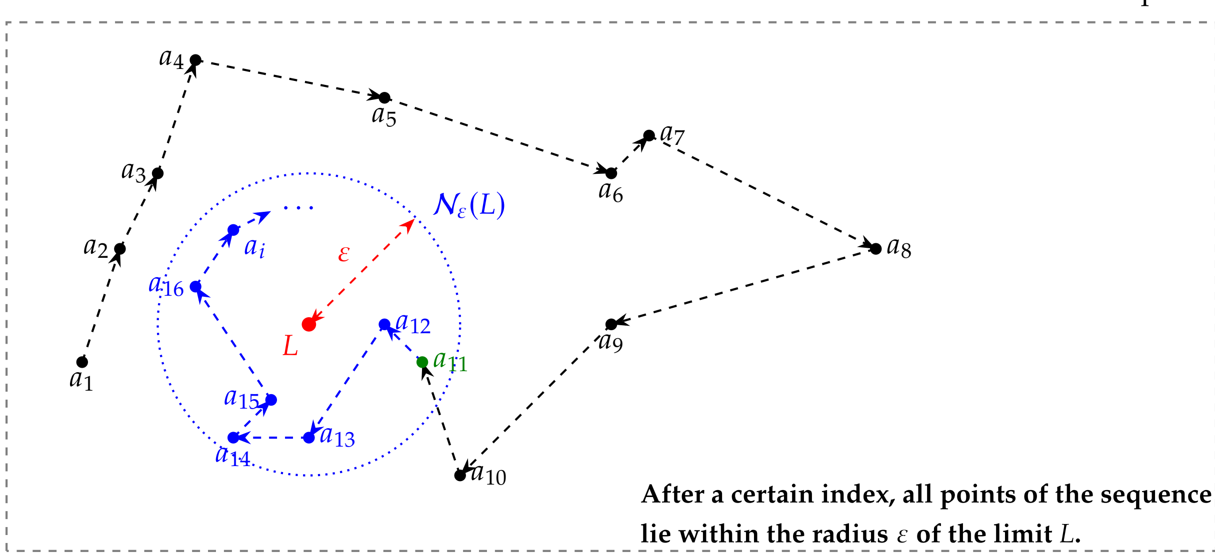
where each (a_i, b_i) is an open interval with $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$, that is, τ consists of all open intervals (and unions of such intervals).

A sequence $\{a_n\}_{n=1}^{\infty} (\subseteq \mathbb{R})$ is **converge** to $L \in \mathbb{R}$ if and only if

$$\begin{aligned}
&\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \left[n \geq N \implies |a_n - L| < \varepsilon \right] \\
\iff &\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \left[n \geq N \implies d(a_n, L) < \varepsilon \right] \\
\iff &\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \left[n \geq N \implies a_n \in \mathcal{N}_\varepsilon(L) \right] \\
\iff &\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \left[n \geq N \implies a_n \in (L - \varepsilon, L + \varepsilon) \right] \\
\iff &\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } [n \geq N \implies a_n \in U_\varepsilon] \\
\iff &\forall U \in \tau \text{ with } L \in U,\ \exists N \in \mathbb{N} \text{ such that } [n \geq N \implies a_n \in U]
\end{aligned}$$



Metric Space



Continuity of Functions

Definition. Let $S \subseteq$ be a non-empty subset of $\mathbb{R}$. Let $f : S \to \mathbb{R}$ be a real-valued function, and let $a \in S$. We say that f is **continuous at** a if and only if

$$\lim_{x \to a} f(x) = f(a).$$

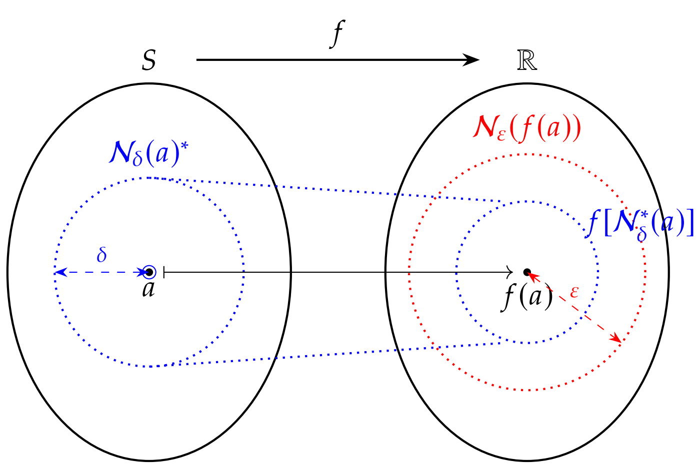
That is,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad 0 < |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

If f is continuous on every point of S, then f is called a **continuous function on** S.

Remark.

$$\begin{aligned}
&\forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad 0 < |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon \\
\iff &\forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad x \in (a - \delta, a) \cup (a, a + \delta) \implies f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon) \\
\iff &\forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad x \in \mathcal{N}_\delta(a) \setminus \{a\} \implies f(x) \in \mathcal{N}_\varepsilon(f(a)) \\
\iff &\forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad f(x) \in f[\mathcal{N}_\delta^*(a)] \implies f(x) \in \mathcal{N}_\varepsilon(f(a)) \qquad \because f[\mathcal{N}_\delta^*(a)] = \left\{f(x) : x \in \mathcal{N}_\delta^*(a)\right\} \\
\iff &\forall \varepsilon > 0,\ \exists \delta > 0 \quad \text{such that} \quad f[\mathcal{N}_\delta^*(a)] \subseteq \mathcal{N}_\varepsilon\left(f(a)\right).
\end{aligned}$$



Remark. f is discontinuous at a if and only if

$$\begin{aligned}
&\exists \varepsilon > 0 \quad \text{such that} \quad \forall \delta > 0,\ |x - a| < \delta \text{ but } |f(x) - f(a)| \geq \varepsilon \\
\iff &\exists \varepsilon > 0 \quad \text{such that} \quad \forall \delta > 0,\ \mathcal{N}_\varepsilon\left(f(a)\right) \subset f\left[\mathcal{N}_\delta^*(a)\right].
\end{aligned}$$

Note. Consider a topological space $(\mathbb{R}, \tau_d)$, where

$$\tau_d := \left\{U \subseteq \mathbb{R} : \forall p \in U,\ \exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(p) \subseteq U\right\}.$$

A sequence $\{a_n\}_{i=1}^\infty \subseteq \mathbb{R}$ converges to $L \in \mathbb{R}$ if and only if

$$\begin{aligned}
&\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \left[n \geq N \implies |a_n - L| < \varepsilon\right] \\
\iff &\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \left[n \geq N \implies a_n \in (L-\varepsilon, L+\varepsilon)\right] \\
\iff &\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \left[n \geq N \implies a_n \in \mathcal{N}_\varepsilon(L)\right] \\
\iff &\forall U \in \tau_d \text{ with } L \in U,\ \exists N \in \mathbb{N} \text{ such that } [n \geq N \implies a_n \in U].
\end{aligned}$$

Limit Theorem (Topology)

Theorem. *Consider a topological space* $(\mathbb{R}, \tau_d)$, *where*

$$\tau_d := \left\{U \subseteq \mathbb{R} : \forall p \in U,\ \exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(p) \subseteq U\right\}.$$

Let $\{a_n\} \subseteq \mathbb{R}$ *and* $\{b_n\} \subseteq \mathbb{R}$. *Let* $\lim_{n\to\infty} a_n = \alpha \in \mathbb{R}$, $\lim_{n\to\infty} b_n = \beta \in \mathbb{R}$, *and* $k \in \mathbb{R}$. *Then*

(1) $\lim_{n\to\infty} ka_n = k\alpha = k \lim_{n\to\infty} a_n$.

(2) $\lim_{n\to\infty} a_n \pm b_n = \alpha \pm \beta = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$.

(3) $\lim_{n\to\infty} a_n b_n = \alpha\beta = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$.

(4) $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\alpha}{\beta} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$. (Here, $\beta \neq 0$ and $b_n \neq 0$)

Proof.

(1) Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = \alpha$, we know

$$\exists N \in \mathbb{N} \text{ s.t. } n \geq N \implies |a_n - \alpha| < \frac{\varepsilon}{|k|+1}$$

Thus, if $n \geq N$ then

$$\begin{aligned} |ka_n - k\alpha| &= \left|k(a_n - \alpha)\right| \\ &= |k||a_n - \alpha| \qquad \because \left|xy\right| = |x|\left|y\right| \\ &< |k| \cdot \frac{\varepsilon}{|k|+1} \\ &< \varepsilon. \end{aligned}$$

(1) Let $U \in \tau_d$ with $k\alpha \in U$. Since $k\alpha \in U$, by definition of τ_d, we have

$$\exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(k\alpha) \subseteq U.$$

Since $\lim_{n\to\infty} a_n = \alpha$, we know

$$\exists N \in \mathbb{N} \text{ s.t. } n \geq N \Rightarrow a_n \in \left(\alpha - \frac{\varepsilon}{|k|+1}, \alpha + \frac{\varepsilon}{|k|+1}\right).$$

Thus, if $n \geq N$ then

$$\begin{aligned} k\alpha_n &\in \left(k\alpha - k \cdot \frac{\varepsilon}{|k|+1}, k\alpha + k \cdot \frac{\varepsilon}{|k|+1}\right) \\ &\subseteq (k\alpha - \varepsilon, k\alpha + \varepsilon) \\ &= \mathcal{N}_\varepsilon(k\alpha) \subseteq U. \end{aligned}$$

(2) Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} b_n = \beta$, we know

$$\exists N_1 \in \mathbb{N} \text{ s.t. } n \geq N_1 \Rightarrow |a_n - \alpha| < \frac{\varepsilon}{2},$$
$$\exists N_2 \in \mathbb{N} \text{ s.t. } n \geq N_2 \Rightarrow |b_n - \beta| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. If $n \geq N$ then

$$\begin{aligned}\left|(a_n + b_n) - (\alpha + \beta)\right| &= \left|(a_n - \alpha) + (b_n - \beta)\right| \\ &\leq |a_n - \alpha| + |b_n - \beta| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

and

$$\begin{aligned}\left|(a_n - b_n) - (\alpha - \beta)\right| &= \left|(a_n - \alpha) + (-b_n + \beta)\right| \\ &\leq |a_n - \alpha| + |b_n - \beta| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

(2) Let $U \in \tau_d$ with $\alpha \pm \beta \in U$. Since $\alpha \pm \beta \in U$, by definition of τ_d, we have

$$\exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(\alpha \pm \beta) \subseteq U.$$

Since $\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} b_n = \beta$,

$$\exists N_1 \in \mathbb{N} \text{ s.t. } n \geq N_1 \Rightarrow a_n \in \left(\alpha - \frac{\varepsilon}{2}, \alpha + \frac{\varepsilon}{2}\right),$$
$$\exists N_2 \in \mathbb{N} \text{ s.t. } n \geq N_2 \Rightarrow b_n \in \left(\beta - \frac{\varepsilon}{2}, \beta + \frac{\varepsilon}{2}\right).$$

Let $N = \max\{N_1, N_2\}$. if $n \geq N$ then

$$\begin{aligned}a_n + b_n &\in \left(\alpha - \frac{\varepsilon}{2} + \beta - \frac{\varepsilon}{2}, \alpha + \frac{\varepsilon}{2} + \beta + \frac{\varepsilon}{2}\right) \\ &= (\alpha + \beta - \varepsilon, \alpha + \beta + \varepsilon) \\ &= \mathcal{N}_\varepsilon(\alpha + \beta) \subseteq U\end{aligned}$$

and

$$\begin{aligned}a_n + (-b_n) &\in \left(\alpha - \frac{\varepsilon}{2} - \beta - \frac{\varepsilon}{2}, \alpha + \frac{\varepsilon}{2} - \beta + \frac{\varepsilon}{2}\right) \\ &= (\alpha - \beta - \varepsilon, \alpha - \beta + \varepsilon) \\ &= \mathcal{N}_\varepsilon(\alpha - \beta) \subseteq U.\end{aligned}$$

(3) Let $\varepsilon > 0$. Since $\{a_n\}$ is bounded,

$$\exists M > 0 \text{ such that } \forall n \in N,\ |a_n| \leq M.$$

Since $\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} b_n = \beta$,

$$\exists N_1 \in \mathbb{N} \text{ s.t. } n \geq N_1 \Rightarrow |a_n - \alpha| < \frac{\varepsilon}{2|\beta|+1},$$
$$\exists N_2 \in \mathbb{N} \text{ s.t. } n \geq N_2 \Rightarrow |b_n - \beta| < \frac{\varepsilon}{2M}.$$

Let $N = \max\{N_1, N_2\}$. If $n \geq N$ then

$$\begin{aligned} |a_nb_n - \alpha\beta| &= |a_nb_n - \alpha\beta + a_n\beta - a_n\beta| \\ &= |a_n(b_n - \beta) + \beta(a_n - \alpha)| \\ &\leq |a_n||b_n - \beta| + |\beta||a_n - \alpha| \\ &< M \cdot \frac{\varepsilon}{2M} + \frac{|\beta| \cdot \varepsilon}{2|\beta|+1} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Note that $2|\beta| < 2|\beta| + 1 \Leftrightarrow \frac{|\beta|}{2|\beta|+1} < \frac{1}{2}$.

(3) Let $U \in \tau_d$ with $\alpha\beta \in U$. Since $\alpha\beta \in U$, by definition of τ_d, we have

$$\exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(\alpha\beta) \subseteq U.$$

Since $\{a_n\}$ is bounded,

$$\exists M > 0 \text{ such that } \forall n \in N,\ |a_n| \leq M.$$

Since $\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} b_n = \beta$,

$$\exists N_1 \in \mathbb{N} \text{ s.t. } n \geq N_1 \Rightarrow a_n \in \left(\alpha - \frac{\varepsilon}{2|\beta|+1}, \alpha + \frac{\varepsilon}{2|\beta|+1}\right),$$
$$\exists N_2 \in \mathbb{N} \text{ s.t. } n \geq N_2 \Rightarrow b_n \in \left(\beta - \frac{\varepsilon}{2M}, \beta + \frac{\varepsilon}{2M}\right).$$

Let $N = \max\{N_1, N_2\}$, and let $n \geq N$. Then

$$a_nb_n - \alpha\beta = a_nb_n - a_n\beta + a_n\beta = \underbrace{a_n(b_n - \beta)}_{\text{(i)}} + \underbrace{\beta(a_n - \alpha)}_{\text{(ii)}}.$$

(i) $b_n \in \left(\beta - \frac{\varepsilon}{2M}, \beta + \frac{\varepsilon}{2M}\right)$

$$\Longrightarrow b_n - \beta \in \left(-\frac{\varepsilon}{2M}, \frac{\varepsilon}{2M}\right)$$
$$\Longrightarrow a_n(b_n - \beta) \in \left(-M \cdot \frac{\varepsilon}{2M}, M \cdot \frac{\varepsilon}{2M}\right) = \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right).$$

(ii) $a_n \in \left(\alpha - \frac{\varepsilon}{2|\beta|+1}, \alpha + \frac{\varepsilon}{2|\beta|+1}\right)$

$$\Longrightarrow a_n - \alpha \in \left(-\frac{\varepsilon}{2|\beta|+1}, \frac{\varepsilon}{2|\beta|+1}\right)$$
$$\Longrightarrow \beta(a_n - \alpha) \in \left(-\frac{|\beta| \cdot \varepsilon}{2|\beta|+1}, \frac{|\beta| \cdot \varepsilon}{2|\beta|+1}\right) \subseteq \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right).$$

Thus,

$$a_nb_n - \alpha\beta = a_n(b_n - \beta) + \beta(a_n - \alpha) \in (-\varepsilon, \varepsilon)$$
$$\Longrightarrow a_nb_n \in (\alpha\beta - \varepsilon, \alpha\beta + \varepsilon) = \mathcal{N}_\varepsilon(\alpha\beta) \subseteq U.$$

Let $\varepsilon > 0$.

(1) It is enough to prove that $\lim\limits_{n\to\infty} \frac{1}{b_n} = \frac{1}{\beta}$ with $b_n \neq 0$ and $\beta \neq 0$. Note that Triangle Inequality implies that

$$|y| = |y - x + x| \leq |y - x| + |x| \iff |y| - |x| \leq |y - x|$$

for any $x, y \in \mathbb{R}$. Since $\lim\limits_{n\to\infty} b_n = \beta$, for $\frac{1}{2}|\beta| > 0$, $\exists N_1 \in \mathbb{N}$ such that if $n \geq N_1$

$$|\beta| - |b_n| \leq |\beta - b_n| = |b_n - \beta| < \frac{1}{2}|\beta|.$$

Thus, we obtain that

$$|\beta| - |b_n| < \frac{1}{2}|\beta| \implies \frac{1}{2}|\beta| < |b_n| \implies \frac{1}{b_n} < \frac{2}{|\beta|}$$

And

$$\exists N_2 \in \mathbb{N} : \left[n \geq N_2 \implies |b_n - \beta| < \frac{\beta^2}{2}\varepsilon \right].$$

Let $N = \max\{N_1, N_2\}$. If $n \geq N$ then

$$\left|\frac{1}{b_n} - \frac{1}{\beta}\right| = \left|\frac{\beta - b_n}{\beta b_n}\right| = \frac{|b_n - \beta|}{|\beta||b_n|} < \varepsilon \cdot \frac{\beta^2}{2} \cdot \frac{1}{|\beta|} \cdot \frac{2}{|\beta|} = \varepsilon.$$

□

Note. TBA

References

[1] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 8. 위상수학 (a) 위상공간의 정의." YouTube Video, 41:25. Published September 27, 2019. URL: `https://www.youtube.com/watch?v=q8BtXIFzo2Q`.

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