# Advanced Calculus III

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We cover the following topics in this note.

- Limit of a Function ($\varepsilon - \delta$)
- Continuity of a Function
- Monotone Convergent Theorem (MCT)
- Nested Interval Property (NIP)
- Limit Superior and Limit Inferior

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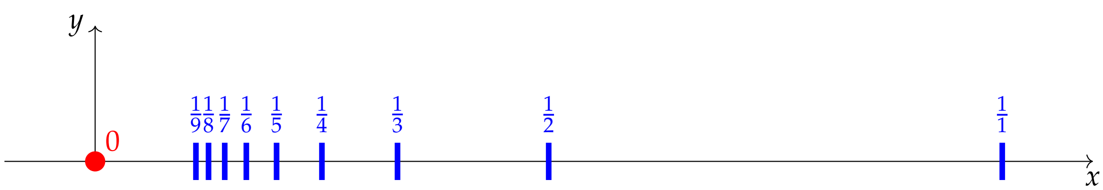
What is $0$ for the set $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$?




**Note** (Open $\varepsilon$-ball). The open $\varepsilon$-ball of $x$ in $S$ is $B_\varepsilon(x) := \left\{y \in S : d(x, y) < \varepsilon\right\}$.

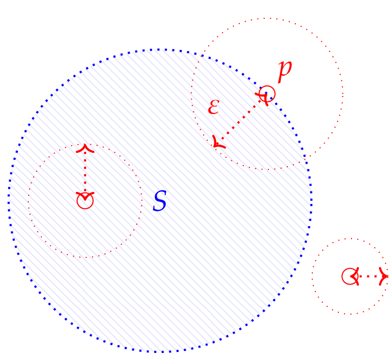
**Limit Point (Metric Space)**

**Definition.** Let $(X, d)$ be a metric space. Let $S \subseteq X$. A point $p \in X$ is a **limit point** of $S$ if and only if

$$\forall \varepsilon > 0,\ B_\varepsilon(p) \cap (S \setminus \{p\}) \neq \varnothing.$$

That is,

$$\forall \varepsilon > 0,\ \left\{x \in S : 0 < d(x, p) < \varepsilon\right\} \neq \varnothing.$$



**Remark.** Note that a limit point $p$ may NOT belong to $S$.

**Note** (Limit Point (Topology)). Let $(X, \tau)$ be a topological space. For a subset $S \subseteq X$. A point $p \in X$ is a limit point of $S$ if and only if

$$\forall U \in \tau \text{ with } p \in U,\ U \cap (S \setminus \{p\}) \neq \varnothing.$$

**Example.** Let $S = (a, b) \subseteq \mathbb{R}$:


(i) Consider $p$ with $p < a$:

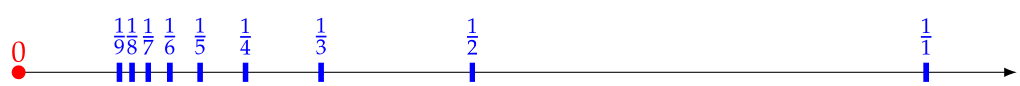

Let $\varepsilon := \frac{a-p}{2} > 0$. Then $B_\varepsilon(p) \cap (S \setminus \{p\}) = \varnothing$. Thus, $p < a$ is NOT a limit point.

(ii) Consider $p = a$:

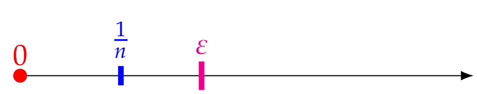

Let $\varepsilon > 0$. Then $B_\varepsilon(p) \cap (S \setminus \{p\}) \neq \varnothing$. Thus, $p = a$ is a limit point of $S = (a, b)$.

By (i) and (ii), the set of all limit points of $(a, b)$ is $[a, b]$.

**Example.** Let $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$:




- Consider $p = \frac{1}{n} \in S$. No point of $S$ is a limit point.
- Consider $p = 0$.




Let $\varepsilon > 0$. By Archimedian property, $\exists n \in \mathbb{N}$ such that $n > \frac{1}{\varepsilon}$, and so $1/n \in B_\varepsilon(0) \cap S$. Thus, $p = 0$ is a limit point of $S = \{1/n : n \in \mathbb{N}\}$.

**Example.** Let $S = \mathbb{Q}$.

- Consider $p \in \mathbb{R}$. Let $\varepsilon > 0$. By density of rationals,

$$\exists r \in \mathbb{Q} \text{ such that } p < r < p + \varepsilon.$$

Then $r \in B_\varepsilon(p) \cap S$ with $r \neq p$, i.e., $r$ is a limit points. Thus, all reals are limit points of $\mathbb{Q}$.

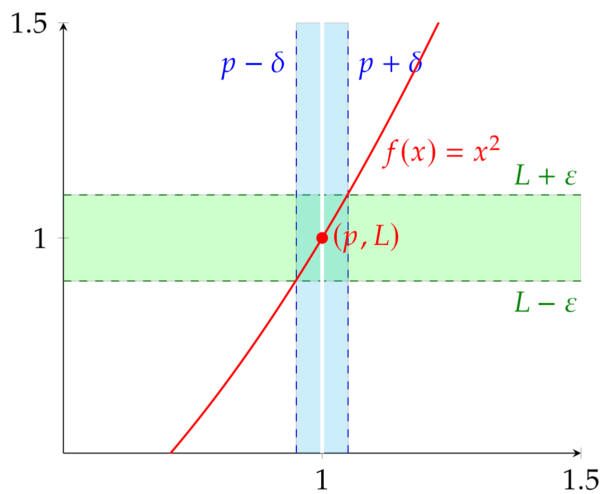
### ⋆ Limit of a Function ($\varepsilon - \delta$) ⋆

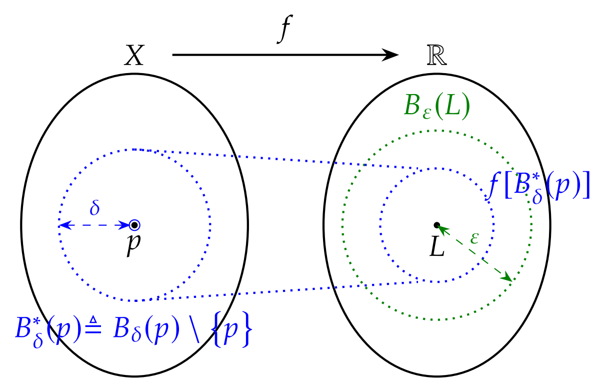
**Definition.** Let $f : X \longrightarrow \mathbb{R}$ be a function defined on a subset $X(\subseteq \mathbb{R})$ of a metric space, and let $p \in X$ be a limit point of $X$. We say that $L \in \mathbb{R}$ is the **limit of the function** $f$ **as** $x$ **approaches** $p$ if

$$\boxed{\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x \in X,\ 0 < |x - p| < \delta \implies |f(x) - L| < \varepsilon}.$$

We write

$$\boxed{\lim_{x \to p} f(x) = L}.$$




**Remark.**

$$\lim_{x \to p} f(x) \neq L \iff \exists \varepsilon > 0 : [\forall \delta > 0 : \exists x \in X : 0 < |x - p| < \delta \text{ but } |f(x) - L| > 0].$$

**Continuity of a Function**

**Definition.** Let $f : X \to \mathbb{R}$ be a function defined on a subset $X \subseteq \mathbb{R}$ of a metric space, and let $p \in X$. The function $f$ is **continuous at** $p$ if and only if

$$\lim_{x \to p} f(x) = f(p).$$

That is,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } |x-p| < \delta \implies |f(x) - f(p)| < \varepsilon.$$

**Remark** (Continuity of a Set)**.** The function $f$ is continuous on subset $S \subseteq X$ if it is continuous at every point $p \in S$.

**Remark** (Continuity in a Topological Space)**.** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ are topological spaces. $f : X \to Y$ is **continuous** if and only if

$$U_Y \in \tau_Y \implies f^{-1}[U_Y] \in \tau_X,$$

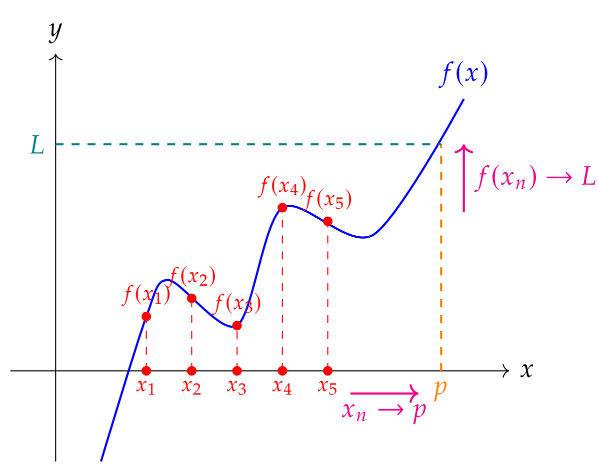
where $f^{-1}[U_Y] = \left\{x \in X : f(x) \in U_Y\right\}$ is the preimage of $U_Y$ under $f$.

**Note.** $[p \Rightarrow (q \Rightarrow r)] \equiv [p \Rightarrow (\neg q \vee r)] \equiv [\neg p \vee (\neg q \vee r)] \equiv [\neg(p \wedge q) \vee r] \equiv [(p \wedge q) \Rightarrow r]$.

**Limit of Function by Convergent Sequences**

**Theorem.** *Let $f : X \to \mathbb{R}$ be a function defined on a subset $\varnothing \neq X \subseteq \mathbb{R}$ of a metric space, and let $p$ is a limit point of $X$. Then*

$$\lim_{x \to p} f(x) = L \iff \left[\forall \{x_n\} \subseteq X \setminus \{p\}, \left(\lim_{n \to \infty} x_n = p \implies \lim_{n \to \infty} f(x_n) = L\right)\right].$$



*Proof.* ($\Rightarrow$) Let $\lim_{x\to p} f(x) = L$. Let $\{x_n\} \subseteq X \setminus \{p\}$ be a sequence, and let $\lim_{n\to\infty} x_n = p$. We NTS that

$$\lim_{n\to\infty} f(x_n) = L, \quad \text{i.e.,} \quad \forall\varepsilon > 0 : \exists N \in \mathbb{N} : n \geq N \Rightarrow |f(x_n) - L| < \varepsilon.$$

Let $\varepsilon > 0$. Since $\lim_{x\to p} f(x) = L$, we know

$$\exists\delta > 0 \text{ such that } 0 < |x - p| < \delta \implies |f(x) - L| < \varepsilon. \tag{*}$$

Since $\lim_{n\to\infty} x_n = p$, we obtain $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |x_n - p| < \delta$. Thus, if $n \geq N$ then,

$$\begin{aligned} |x_n - p| < \delta &\implies 0 < |x_n - p| < \delta \quad \because x_n \neq p \\ &\implies |f(x_n) - L| < \varepsilon \quad \text{by (*)} \end{aligned}$$

Thus, $\lim_{n\to\infty} f(x_n) = L$.

($\Leftarrow$) Let the RHS holds. Assume, for the contradiction, that $\lim_{x\to p} f(x) \neq L$, i.e.,

$$\exists\varepsilon > 0 : \forall\delta > 0 : \exists x_\delta \in X : 0 < |x_\delta - p| < \delta \text{ but } |f(x_\delta) - L| \geq \varepsilon.$$

Take $\delta = 1/n$ for $n \in \mathbb{N}$. Then

$$\exists x_n \in X \text{ such that } 0 < |x_n - p| < \frac{1}{n} \text{ but } |f(x_n) - L| \geq \varepsilon.$$

~~(Axiom of Countable Choice)~~ This means that

$$\forall n \in \mathbb{N} : \exists \{x_n\} \subseteq X \setminus \{p\} \text{ such that } 0 < |x_n - p| < \frac{1}{n} \text{ but } |f(x_n) - L| \geq \varepsilon.$$

By Squeeze Theorem, we have $\lim_{n\to\infty} x_n = p$ since $0 < |x_n - p| < 1/n$. Since the RHS holds, we obtain $\lim_{n\to\infty} f(x_n) = L$. Then, for some $\varepsilon > 0$,

$$\exists N \in \mathbb{N} \text{ such that } n \geq N \implies |f(x_n) - L| < \varepsilon \;\lightning.$$

$\square$

**Continuity of Function by Convergent Sequences**

**Corollary.** *Let $f : X \to \mathbb{R}$ be a function defined on a subset $\varnothing \neq X \subseteq \mathbb{R}$ of a metric space, and let $p$ is a limit point of $X$. Then*

$$\lim_{x\to p} f(x) = f(p) \iff \left[\forall \{x_n\} \subseteq X, \left(\lim_{n\to\infty} x_n = p \implies \lim_{n\to\infty} f(x_n) = f(p)\right)\right].$$

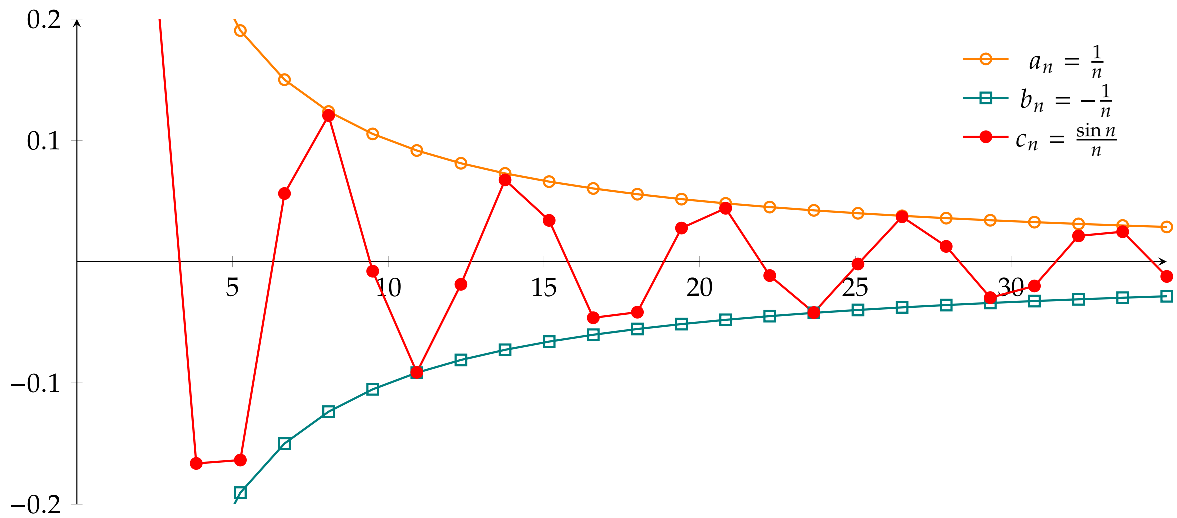
**Squeeze Theorem; Sandwich Theorem**

**Theorem.** *Let*

*(i)* $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$*;*

*(ii)* $\exists n_0 \in \mathbb{N}$ *such that* $a_n \leq c_n \leq b_n$ *for all* $n \geq n_0$*.*

*Then* $\lim_{n\to\infty} c_n = L$*.*




*Proof.* Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = L$, we have

$$\exists n_1 \in \mathbb{N} \text{ such that } n \geq n_1 \implies L - \varepsilon < a_n < L + \varepsilon,$$
$$\exists n_2 \in \mathbb{N} \text{ such that } n \geq n_2 \implies L - \varepsilon < b_n < L + \varepsilon.$$

Let $N := \max\{n_0, n_1, n_2\}$. If $n \geq N$ then

$$L - \varepsilon < a_n \leq c_n \leq b_n < L_+\varepsilon,$$

and so $|c_n - L| < \varepsilon$. □

**Note.** Recall that

"A convergent sequence is bounded."

Formally,

$$\exists A \in \mathbb{R} \text{ s.t. } A = \lim_{n\to\infty} a_n \implies \exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, |a_n| \le M.$$

However, the converse is not necessarily true:

$$\exists A \in \mathbb{R} \text{ s.t. } A = \lim_{n\to\infty} a_n \;\not\Longleftarrow\; \exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, |a_n| \le M.$$

To illustrate, consider the sequence $\{a_n\} = 1 - (-1)^n$ that is bounded, yet it does not converge.

## Monotone Sequence

**Definition.** A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be **monotone** if it is either **monotone increasing** or **monotone decreasing**.

(1) A sequence $\{a_n\}_{n=1}^{\infty}$ is **monotone increasing** if $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$.

Alternatively, it is **strictly increasing** if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.

(2) A sequence $\{a_n\}_{n=1}^{\infty}$ is **monotone decreasing** if $a_{n+1} \le a_n$ for all $n \in \mathbb{N}$.

Alternatively, it is **strictly decreasing** if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$.

**Remark.** A sequence $\{a_n\}$ is monotone if $\begin{cases} a_n \le a_{n+1} & \text{(monotone increasing)} \\ a_{n+1} \le a_n & \text{(monotone decreasing)} \end{cases}$.

**Example.**

- $\{n\}_{n=1}^{\infty}$ is monotone increasing.
- $\left\{1/n\right\}_{n=1}^{\infty}$ is monotone decreasing.

**Monotone Convergence Theorem (MCT)**

**Theorem.** *A monotone sequence of real numbers $\{a_n\}$ is convergent if and only if it is bounded.*

*(1) Let $\{a_n\}$ be an monotone increasing sequence of real numbers that is bounded above. Then*

$$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

*(2) Let $\{b_n\}$ be an monotone decreasing sequence of real numbers that is bounded below. Then*

$$\lim_{n\to\infty} b_n = \inf\{b_n : n \in \mathbb{N}\}.$$

*Proof.*

(1) Suppose that a sequence $\{a_n\}$ is monotone increasing and bounded above. Consider the set $\{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$, which is non-empty and bounded above by assumption. By **Least Upper Bound Property**[1],

$$\exists \alpha \in \mathbb{R} \text{ such that } \alpha = \sup\{a_n : n \in \mathbb{N}\}.$$

We claim that:

$$\lim_{n\to\infty} a_n = \alpha = \sup\{a_n : n \in \mathbb{N}\}.$$

Let $\varepsilon > 0$. Since $\alpha$ is the supremum (*least* upper bound) of $\{a_n : n \in \mathbb{N}\}$, it follows that $\alpha - \varepsilon$ is not an upper bound of $\{a_n : n \in \mathbb{N}\}$. Thus, $\neg[\forall N \in \mathbb{N},\ a_N \le \alpha - \varepsilon]$, i.e.,

$$\exists N \in \mathbb{N} \text{ such that } \alpha - \varepsilon < a_N.$$

Since $\{a_n\}$ is monotone increasing,

$$\alpha - \varepsilon < a_N \le a_n$$

for all $n \ge N$. Therefore,

$$\alpha - \varepsilon \underset{\varepsilon>0}{\overset{\alpha=\sup\{a_n\}}{<}} a_N \underset{n\ge N}{\overset{\{a_n\}\text{ is monotone increasing}}{\le}} a_n \overset{\alpha\text{ is an upper bound}}{\le} \alpha \overset{\varepsilon>0}{<} \alpha + \varepsilon.$$

This implies that $|a_n - \alpha| < \varepsilon$ for all $n \ge N$.

[1]Every non-empty subset of $\mathbb{R}$ that is bounded above has the supremum in $\mathbb{R}$.

(2) Suppose that a sequence $\{b_n\}$ is monotone decreasing and bounded below. Consider the set $\{b_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$, which is non-empty and bounded below by assumption. By **Greatest Lower Bound Property**[2],

$$\exists \beta \in \mathbb{R} \text{ such that } \beta = \inf\{b_n : n \in \mathbb{N}\}.$$

We claim that:

$$\lim_{n\to\infty} b_n = \beta = \inf\{b_n : n \in \mathbb{N}\}.$$

Let $\varepsilon > 0$. Since $\beta$ is the infimum (*greatest* lower bound) of $\{b_n : n \in \mathbb{N}\}$, it follows that $\beta + \varepsilon$ is not a lower bound of $\{b_n : n \in \mathbb{N}\}$. Thus, $\neg[\forall N \in \mathbb{N},\ \beta + \varepsilon \le b_N]$, i.e.,

$$\exists N \in \mathbb{N} \text{ such that } b_N < \beta + \varepsilon.$$

Since $\{b_n\}$ is monotone decreasing,

$$b_n \le b_N < \beta + \varepsilon$$

for all $n \ge N$. Therefore,

$$\beta - \varepsilon \overset{\varepsilon>0}{<} \beta \overset{\beta\text{ is a lower bound}}{\le} b_n \underset{n\ge N}{\overset{\{b_n\}\text{ is monotone decreasing}}{\le}} b_N \underset{\varepsilon>0}{\overset{\beta=\inf\{b_n\}}{<}} \beta + \varepsilon$$

This implies that $|b_n - \beta| < \varepsilon$ for all $n \ge N$.

□

**Divergence of Sequence**

**Definition.** Let $\{a_n\}$ be a sequence of real numbers.

(1) We say that the sequence $\{a_n\}$ **diverges to infinity** (or **tends to infinity**) if

$$\forall M \in \mathbb{R},\ \exists N \in \mathbb{N} \text{ such that } n \ge N \implies M < a_n,$$

and write $\lim_{n\to\infty} a_n = +\infty$.

(2) We say that the sequence $\{a_n\}$ **diverges to minus infinity** (or **tends to infinity**) if

$$\forall M \in \mathbb{R},\ \exists N \in \mathbb{N} \text{ such that } n \ge N \implies a_n < M,$$

and write $\lim_{n\to\infty} a_n = -\infty$.

(3) We say that $\{a_n\}$ is properly divergent in case we have either $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} = -\infty$.

[2] Every non-empty subset of $\mathbb{R}$ that is bounded below has the infimum in $\mathbb{R}$.

**Note.** Recall that

**(Monotonicity)** A sequence $\{a_n\}$ is monotone increasing if $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$;

**(Not Bounded Above)** The sequence $\{a_n\}$ is not bounded above if

$$\neg[\exists M \in \mathbb{R},\ \forall n \in \mathbb{N},\ a_n \le M] \equiv [\forall M \in \mathbb{R},\ \exists n \in \mathbb{N} \text{ such that } a_n > M].$$

We claim that a sequence $\{a_n\}$ that is monotone increasing and not bounded above diverges to infinity:

*Proof.* Let $M \in \mathbb{R}$. Since $\{a_n\}$ is not bounded above,

$$\exists n_0 \in \mathbb{N} \text{ such that } a_{n_0} > M.$$

Since $\{a_n\}$ is monotonic increasing, it fllows that

$$a_{n_0} \le a_n,\ \forall n \ge n_0.$$

Thus

$$n \ge n_0 \overset{\text{monotone increasing}}{\Longrightarrow} a_{n_0} \le a_n \overset{\text{Not Bounded Above}}{\Longrightarrow} M < a_{n_0} < a_n.$$

Hence it is proved. □

Note that

**Lemma.** *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Then*

$$[\forall n \in \mathbb{N},\ a_n \le b_n] \implies \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n.$$

*Proof.* Let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Suppose that $a > b$. Let $\varepsilon = a - b > 0$. Then

$$\exists N_1 \in \mathbb{N} \text{ such that } n \ge N_1 \implies |a_n - a| < \varepsilon,$$
$$\exists N_2 \in \mathbb{N} \text{ such that } n \ge N_2 \implies |b_n - b| < \varepsilon.$$

Let $N := \max\{N_1, N_2\}$. Then $b_N < b + \varepsilon < a + \varepsilon < a_N$ ↯. Hence $a \le b$, i.e., $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$. □

**Note.** Let $I_n = \left(0, \frac{1}{n}\right) \subseteq \mathbb{R}$ for all $n \in \mathbb{N}$.


Suppose that $x \in \bigcap_{n=1}^{\infty} I_n$ then $x \in I_n$ for all $n \ge 1$. That is,

$$0 < x < \frac{1}{n} \quad \text{for all} \quad n \ge 1.$$

By Archimedian property, $\exists n_0 \in \mathbb{N}$ s.t. $n_0 x > 1$ ↯. Hence $\bigcap_{n=1}^{\infty} I_n = \varnothing$.

**Note.** Let $I_n = [n, \infty) \subseteq \mathbb{R}$ for all $n \in \mathbb{N}$.


Suppose that $x \in \bigcap_{n=1}^{\infty} I_n$ then $x \in I_n$ for all $n \ge 1$. That is,

$$n \le x \quad \text{for all} \quad n \ge 1.$$

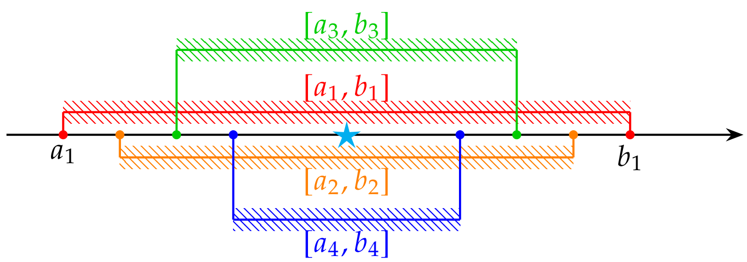
By Archimedian property, $\exists n_0 \in \mathbb{N}$ s.t. $x < n_0$ ↯. Hence $\bigcap_{n=1}^{\infty} I_n = \varnothing$.

**Nested Interval Property (NIP)**

**Theorem.** *Let $a_n \le b_n$ for all $n \in \mathbb{N}$, and let $\left\{[a_n, b_n]\right\}_{i=1}^{\infty} \subseteq \mathbb{R}$ be a sequence of bounded and closed intervals satisfying $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for all $n \in \mathbb{N}$. Then*

$$\bigcap_{n=1}^{\infty} [a_n, b_n] := \left\{x \in \mathbb{R} : x \in [a_n, b_n] \text{ for all } n \in \mathbb{N}\right\} \neq \varnothing.$$

If $\lim_{n\to\infty}(b_n - a_n) = 0$, then $\left|\bigcap_{n=1}^{\infty}[a_n, b_n]\right| = 1$.




*Proof.* Since $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ for all $n \in \mathbb{N}$, we know the sequence $\{a_n\}$ is monotone increasing, and the sequence $\{b_n\}$ is monotone decreasing. In other words,

$$a_1 \le a_2 \le \cdots \le a_n \le \cdots \le b_n \le \cdots b_2 \le b_1.$$

By Monotone Convergence Theorem, we obtain

$$\lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} a_n \quad \text{and} \quad \lim_{n\to\infty} b_n = \inf_{n\in\mathbb{N}} b_n$$

Thus,

$$[\forall n \in \mathbb{N},\ a_n \le b_n] \implies \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n \implies \sup_{n\in\mathbb{N}} a_n \le \inf_{n\in\mathbb{N}} b_n \tag{*}$$

Then

$$x \in \bigcap_{n=1}^{\infty}[a_n, b_n] \iff \forall n \in \mathbb{N},\ a_n \le x \le b_n \overset{\text{by (*) for} \Rightarrow}{\iff} \sup_{n\in\mathbb{N}} a_n \le x \le \inf_{n\in\mathbb{N}} b_n$$
$$\iff x \in \left[\sup_{n\in\mathbb{N}} a_n, \inf_{n\in\mathbb{N}} b_n\right].$$

By Set Equality, we have

$$\bigcap_{n=1}^{\infty}[a_n, b_n] = \left[\sup_{n\in\mathbb{N}} a_n, \inf_{n\in\mathbb{N}} b_n\right],$$

and so $\left[\sup_{n\in\mathbb{N}} a_n, \inf_{n\in\mathbb{N}} b_n\right] \neq \varnothing$ by Least Upper Bound Property. □

**Monotonicity of Supremum and Infimum**

**Proposition.** *Let $\{a_n\}, \{b_n\} \subseteq \mathbb{R}$ be sequences of real numbers. Let $\{b_n\}$ is a subsequence of $\{a_n\}$,* i.e., $\{b_n\} \subseteq \{a_n\}$. *Then*

(1) $\sup\{b_n\} \leq \sup\{a_n\}$;

(2) $\inf\{a_n\} \leq \inf\{b_n\}$.

*Proof.* (1) Since

$$\beta \in \{b_n\} \overset{\{b_n\}\subseteq\{a_n\}}{\Longrightarrow} \beta \in \{a_n\} \overset{\sup\{a_n\}}{\Longrightarrow} \beta \leq \sup\{a_n\},$$

$\sup\{a_n\}$ be an upper bound of $\{b_n\}$. Since $\sup\{b_n\}$ is the *least* upper bound of $\{b_n\}$, we have $\sup\{b_n\} \leq \sup\{a_n\}$.

(2) Since

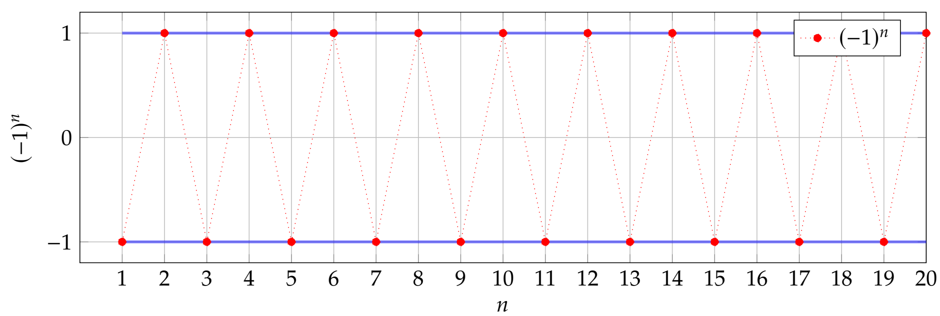
$$\beta \in \{b_n\} \overset{\{b_n\}\subseteq\{a_n\}}{\Longrightarrow} \beta \in \{a_n\} \overset{\inf\{a_n\}}{\Longrightarrow} \inf\{a_n\} \leq \beta,$$

$\inf\{a_n\}$ be a lower bound of $\{b_n\}$. Since $\inf\{b_n\}$ is the *greatest* lower bound of $\{b_n\}$, we have $\inf\{a_n\} \leq \inf\{b_n\}$.

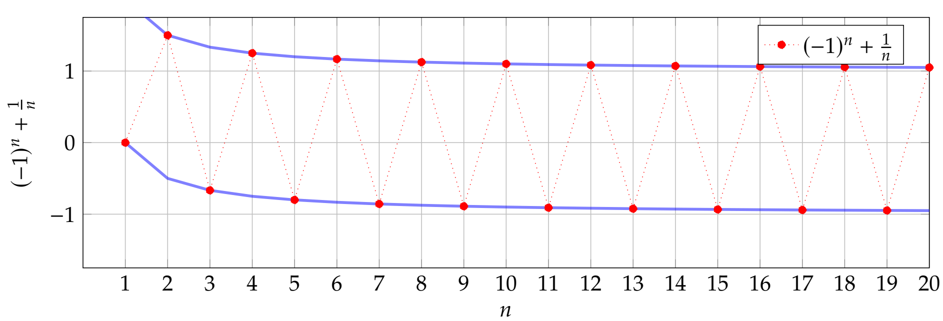
□

**Observation.**

- What is $\pm 1$ for the set $S = \left\{(-1)^n : n \in \mathbb{N}\right\}$?




- What is $\pm 1$ for the set $S = \left\{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\right\}$?



Let $\{x_n\}_{i=1}^{\infty}$ be a sequence in $\mathbb{R}$. Define

$$\begin{aligned}
s_1 &= \sup\{x_1, x_2, x_3, \ldots\} = \sup\{x_k : k \geq 1\}, \\
s_1 &= \sup\{x_2, x_3, x_4, \ldots\} = \sup\{x_k : k \geq 2\}, \\
&\vdots \\
s_n &= \sup\{x_k, x_{k+1}, \ldots\} = \sup\{x_k : k \geq n\}.
\end{aligned}$$

By monotonicity of supremum,

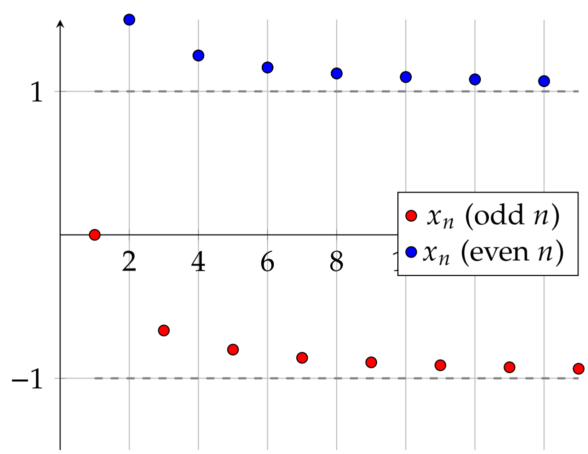
$$s_1 \geq s_2 \geq \cdots \geq s_n \geq s_{n+1} \geq \cdots.$$

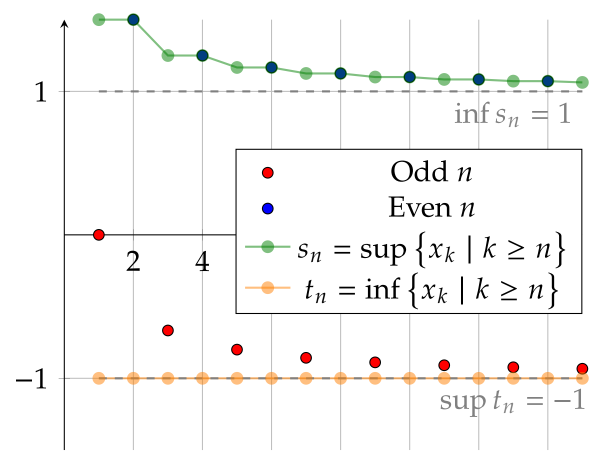
That is, $\{s_n\}_{n=1}^{\infty}$ be a monotone decreasing sequence. Similarly, for $t_n = \inf\{x_k, x_{k+1}, \ldots\} = \inf\{x_k : k \geq n\}$, we have a monotone increasing sequence $\{t_n\}_{n=1}^{\infty}$. For example,

| $n$ | $(-1)^n$ | $\frac{1}{n}$ | $x_n = (-1)^n + \frac{1}{n}$ | $\sup\{x_k : k \geq n\} (=: s_n)$ | $\inf\{x_k : k \geq n\} (=: t_n)$ |
|---|---|---|---|---|---|
| 1 | $-1$ | 1 | 0 | 1.5 | $-1$ |
| 2 | 1 | $\frac{1}{2} = 0.5$ | $\frac{3}{2} = 1.5$ | 1.5 | $-1$ |
| 3 | $-1$ | $\frac{1}{3} \approx 0.33$ | $-\frac{2}{3} \approx -0.67$ | 1.25 | $-1$ |
| 4 | 1 | $\frac{1}{4} = 0.25$ | $\frac{5}{4} = 1.25$ | 1.25 | $-1$ |
| 5 | $-1$ | $\frac{1}{5} = 0.2$ | $-\frac{4}{5} = -0.8$ | 1.17 | $-1$ |
| 6 | 1 | $\frac{1}{6} \approx 0.17$ | $\frac{7}{6} \approx 1.17$ | 1.17 | $-1$ |

By Monotone Convergent Theorem, $\{s_n\}$ and $\{t_n\}$ are converges, and so

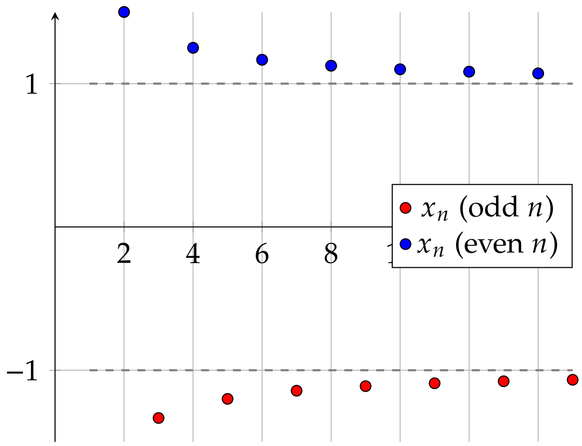
$$\lim_{n\to\infty} s_n \overset{\{s_n\}\text{ is monotone decreasing}}{\boxed{=}} \inf s_n \underset{\text{definition of } s_n}{\boxed{=}} \inf\left(\sup_{k\geq n} x_k\right), \quad \lim_{n\to\infty} t_n \overset{\{t_n\}\text{ is monotone increasing}}{\boxed{=}} \sup t_n \underset{\text{definition of } t_n}{\boxed{=}} \sup\left(\inf_{k\geq n} x_k\right).$$

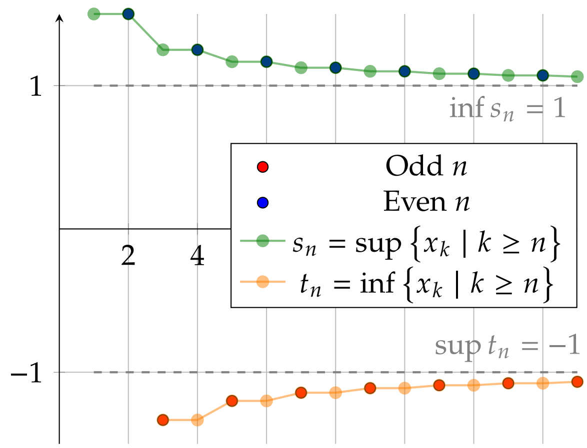


**Remark.** Consider

$$x_n := (-1)^n + (-1)^n \cdot \frac{1}{n}$$




### Limit Superior and Limit Inferior

**Definition.** Let $\{x_n\}$ be a sequence of real numbers. ~~Suppose that $\{x_n\}$ is bounded.~~

(1) The **limit superior** of $\{x_n\}$, denoted by $\limsup_{n\to\infty} x_n$ (or $\overline{\lim_{n\to\infty}} x_n$) is defined as

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right) = \inf_{n\in\mathbb{N}}\left(\sup_{k\geq n} x_k\right),$$

where $\sup_{k\geq n} x_k$ represents the supremum of the subsequence $\{x_k : k \geq n\}$.

(2) The **limit inferior** of $\{x_n\}$, denoted by $\liminf_{n\to\infty} x_n$ (or $\underline{\lim_{n\to\infty}} x_n$) is defined as

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right) = \sup_{n\in\mathbb{N}}\left(\inf_{k\geq n} x_k\right),$$

where $\inf_{k\geq n} x_k$ represents the infimum of the subsequence $\{x_k : k \geq n\}$.

**Note** (Extended Real Number Line)**.** The **extended real number line** $\overline{\mathbb{R}}$ is defined as

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}.$$

That is, the set of real numbers together with two symbols $+\infty, -\infty$ such that

$$\forall x \in \mathbb{R},\ -\infty < x < +\infty.$$

**Bolzano-Weierstrass Theorem**

**Theorem.** *A bounded sequence of real numbers has a convergent subsequence.*

*Proof.* TBA □

**Proposition.** *Let* $\{x_n\}, \{y_n\}$ *be* ~~*bounded*~~ *sequences of real numbers. Then*

*(1)* $\liminf\limits_{n\to\infty} x_n \le \limsup\limits_{n\to\infty} x_n$.

*(2)* $\limsup\limits_{n\to\infty} x_n = L = \liminf\limits_{n\to\infty} x_n \iff \exists \lim\limits_{n\to\infty} x_n = L$.

**Remark.** $\limsup x_n = \liminf x_n \iff \exists \lim\limits_{n\to\infty} x_n \in \mathbb{R} \cup \{\pm\infty\}$.

*Proof.* Let $s_n := \sup\limits_{k\ge n} x_k$ and $t_n := \inf\limits_{k \ge n} x_k$ for each $n \ge 1$. Then $\{s_n\}$ is monotone decreasing and $\{t_n\}$ is monotone increasing. And so

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\ge n} x_k\right) = \lim_{n\to\infty} s_n \quad \text{and} \quad \liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\ge n} x_k\right) = \lim_{n\to\infty} t_n.$$

(1) $[\forall n \in \mathbb{N},\ t_n \le s_n] \implies \lim\limits_{n\to\infty} t_n \le \lim\limits_{n\to\infty} s_n \implies \liminf (x_n) \le \limsup (x_n)$.

(2) ($\Rightarrow$) Note that

$$t_n = \inf_{k\ge n} x_k \le x_n \le \sup_{k\ge n} x_k = s_n.$$

By Squeeze Theorem, we have $\lim\limits_{n\to\infty} x_n = L$.

($\Leftarrow$) Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} x_n = L$,

$$\exists n \in \mathbb{N} \text{ such that } n \ge N \implies |x_n - L| < \frac{\varepsilon}{2}.$$

Since $\{s_n\}$ is monotone decreasing and $\{t_n\}$ is monotone increasing, we have

Each $t_i$ is the “greatest” lower bound　　　Each $s_i$ is the “least” upper bound

$$L - \varepsilon < L - \frac{\varepsilon}{2} \boxed{\le} t_N \boxed{\le} t_n \le s_n \boxed{\le} s_N \boxed{\le} L + \frac{\varepsilon}{2} < L + \varepsilon.$$

$\{t_n\}$ is monotone increasing and $n \ge N$　　　$\{s_n\}$ is monotone decreasing and $n \ge N$

Therefore, $\limsup (x_n) = \lim\limits_{n\to\infty} s_n = L$ and $\liminf (x_n) = \lim\limits_{n\to\infty} t_n = L$.

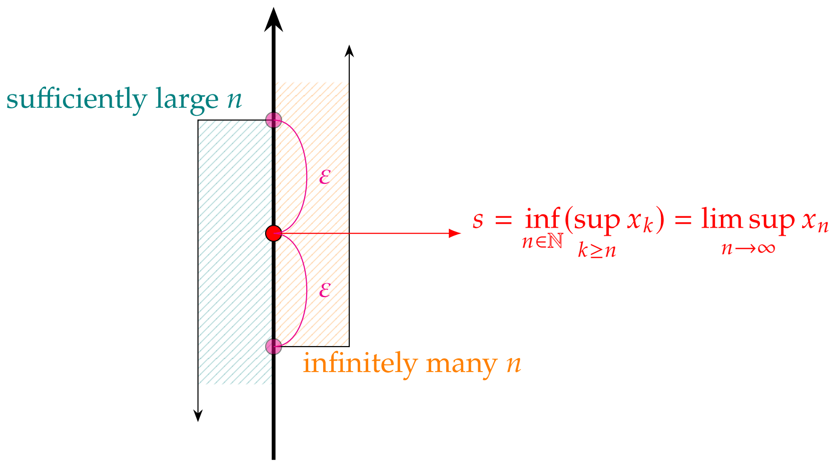
□

**Theorem.** *Let* $\limsup_{n\to\infty} x_n \in \mathbb{R}$ *and* $\liminf_{n\to\infty} x_n \in \mathbb{R}$.

*(1)* $\limsup x_n = s \iff \forall \varepsilon > 0, \begin{cases} \text{(i) } \exists n_0 \in \mathbb{N} \textit{ such that } \forall n \ge n_0,\ x_n < s+\varepsilon \\ \text{(ii) } \forall n \in \mathbb{N},\ \exists k \ge n \textit{ such that } s-\varepsilon < x_k \end{cases}.$

*(2)* $\liminf x_n = t \iff \forall \varepsilon > 0, \begin{cases} \text{(i) } \exists n_0 \in \mathbb{N} \textit{ such that } \forall n \ge n_0,\ t-\varepsilon < x_n \\ \text{(ii) } \forall n \in \mathbb{N},\ \exists k \ge n \textit{ such that } x_k < t+\varepsilon \end{cases}.$

*Proof.* (1)




($\Rightarrow$) Assume that $\limsup_{n\to\infty} x_n = s$. Let $\varepsilon > 0$.

(i) Since $s = \lim_{n\to\infty}\left(\sup_{k\ge n} x_k\right)$,

$$\begin{aligned} \exists n_0 \in \mathbb{N} \text{ such that } n \ge n_0 &\implies \left|\sup_{k\ge n} x_k - s\right| < \varepsilon \\ &\implies s-\varepsilon < \sup_{k\ge n} x_k < s+\varepsilon \\ &\implies x_n \le \sup_{k\ge n} x_k < s+\varepsilon \\ &\implies x_n < s+\varepsilon. \end{aligned}$$

Thus, there exits $n_0 \in \mathbb{N}$ such that if $n \ge n_0$ then $x_n < s+\varepsilon$.

(ii) Let $n \in \mathbb{N}$. Recall that, for $S \subseteq \mathbb{R}$,

$$\boxed{\lambda = \sup S \iff \forall \varepsilon > 0,\ \exists x_\varepsilon \in S \text{ s.t. } \lambda - \varepsilon < x_\varepsilon \leq \lambda}.$$

This guarantee the following:

$$\exists x \in \{x_k : k \geq n\} \text{ s.t. } \sup_{k \geq n} x_k - \varepsilon < x \leq \sup_{k \geq n} x_k.$$

In other words, $\exists k \geq n$ s.t. $\sup\limits_{k \geq n} x_k - \varepsilon < x_k$, and so

$$\underbrace{\inf_{n \geq 1}\left(\sup_{k \geq n} x_k\right)}_{=s} - \varepsilon \leq \sup_{k \geq n} x_k - \varepsilon < x_k.$$

($\Leftarrow$) Let $\varepsilon > 0$. Assume that $s \in \mathbb{R}$ satisfies (i) and (ii). By (i), we know that

$$\exists n_0(\varepsilon) \in \mathbb{N} \text{ s.t. } \forall n \geq n_0(\varepsilon),\ x_n < s + \varepsilon.$$

Then if $k \geq n \geq n_0$, we also have $k \geq n_0$. This means that $x_k < s + \varepsilon$. Thus,

$$\sup_{k \geq n} x_k \leq x_k < s + \varepsilon \text{ for all } n \geq n_0.$$

Form (ii), we have

$$\forall n \in \mathbb{N},\ \exists k \geq n \text{ s.t. } s - \varepsilon < x_k.$$

By the definition of supremum, $s - \varepsilon < x_k \leq \sup\limits_{k \geq n} x_k$ and so

$$s - \varepsilon < \sup_{k \geq n} x_k \text{ for all } n \in \mathbb{N}.$$

Here, we get two inequalities:

- Upper bound for large $n$: $\forall n \geq n_0(\varepsilon),\ \sup\limits_{k \geq n} x_k < s + \varepsilon$.
- Lower bound for all $n$: $\forall n \in \mathbb{N},\ s - \varepsilon < \sup\limits_{k \geq n} x_k$.

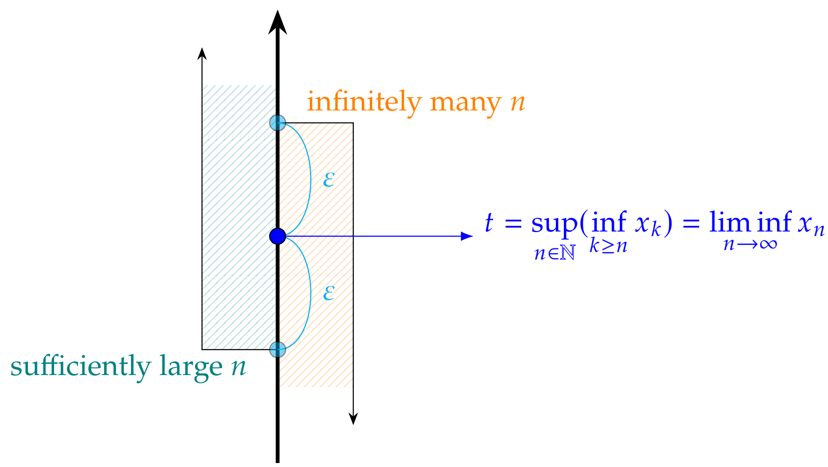
Then, for all $n \geq n_0(\varepsilon)$,

$$s - \varepsilon < \sup_{k \geq n} x_k < s + \varepsilon.$$

Hence,

$$\lim_{n \to \infty}\left(\sup_{k \geq n} x_k\right) = s, \quad \text{i.e.} \quad , \limsup_{n \to \infty} x_n = s.$$

(2)




($\Longrightarrow$) Assume that $\liminf_{n\to\infty} x_n = t$. Let $\varepsilon > 0$.

(i) Since $t = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right)$,

$$\begin{aligned} \exists n_0 \in \mathbb{N} \text{ such that } n \geq n_0 &\Longrightarrow \left|\inf_{k\geq n} x_k - t\right| < \varepsilon \\ &\Longrightarrow t - \varepsilon < \inf_{k\geq n} x_k < t + \varepsilon \\ &\Longrightarrow t - \varepsilon < \inf_{k\geq n} x_k \leq x_n \\ &\Longrightarrow t - \varepsilon < x_n. \end{aligned}$$

Thus, there exits $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $s - \varepsilon < x_n$.

(ii) Let $n \in \mathbb{N}$. Recall that, for $T \subseteq \mathbb{R}$,

$$\boxed{\gamma = \inf T \iff \forall\varepsilon > 0,\ \exists x_\varepsilon \in T \text{ s.t. } \gamma \leq x_\varepsilon < \gamma + \varepsilon}.$$

This guarantee the following:

$$\exists x \in \{x_k : k \geq n\} \text{ s.t. } \inf_{k\geq n} x_k \leq x < \inf_{k\geq n} x_k + \varepsilon.$$

In other words, $\exists k \geq n$ s.t. $x_k < \inf_{k\geq n} x_k + \varepsilon$, and so

$$x_k < \inf_{k\geq n} x_k + \varepsilon \leq \underbrace{\sup_{n\geq 1}\left(\inf_{k\geq n} x_k\right)}_{=t} + \varepsilon.$$

($\Leftarrow$) Let $\varepsilon > 0$. Assume that $t \in \mathbb{R}$ satisfies (i) and (ii). By (i), we know that

$$\exists n_0(\varepsilon) \in \mathbb{N} \text{ s.t. } \forall n \geq n_0(\varepsilon),\ t - \varepsilon < x_n.$$

Then if $k \geq n \geq n_0$, we also have $k \geq n_0$. This means that $t - \varepsilon < x_k$. Thus,

$$t - \varepsilon < x_k \leq \inf_{k \geq n} x_k \text{ for all } n \geq n_0.$$

Form (ii), we have

$$\forall n \in \mathbb{N},\ \exists k \geq n \text{ s.t. } x_k < t + \varepsilon.$$

By the definition of infimum, $\inf_{k \geq n} x_k \leq x_k < t + \varepsilon$ and so

$$\inf_{k \geq n} x_k < t + \varepsilon \text{ for all } n \in \mathbb{N}.$$

Here, we get two inequalities:

- Upper bound for large $n$: $\forall n \geq n_0(\varepsilon),\ t - \varepsilon < \inf_{k \geq n} x_k$.
- Lower bound for all $n$: $\forall n \in \mathbb{N},\ \inf_{k \geq n} x_k < t + \varepsilon$.

Then, for all $n \geq n_0(\varepsilon)$,

$$t - \varepsilon < \inf_{k \geq n} x_k < t + \varepsilon.$$

Hence,

$$\lim_{n \to \infty} \left( \inf_{k \geq n} x_k \right) = t, \quad \text{i.e.} \quad , \liminf_{n \to \infty} x_n = t.$$

□

**Proposition.** *Let $\{x_n\}, \{y_n\}$ be bounded sequences of real numbers. Then*

*(1)* $\liminf (x_n) + \liminf (y_n) \leq \liminf (x_n + y_n)$.

*(2)* $\limsup (x_n + y_n) \leq \limsup (x_n) + \limsup (y_n)$.

*Proof.* (1) Since

$$\inf_{k \geq n} x_k \leq x_k \text{ and } \inf_{k \geq n} y_k \leq y_k \implies \inf_{k \geq n} x_k + \inf_{k \geq n} y_k \leq x_k + y_k$$

for each $k \geq n$, we have

$$\forall n \in \mathbb{N},\ \inf_{k \geq n} x_k + \inf_{k \geq n} y_k \ \leq\ \inf_{k \geq n}(x_k + y_k).$$

This implies that

$$\begin{aligned}
\lim_{n\to\infty}\left(\inf_{k\geq n} x_k + \inf_{k\geq n} y_k\right) &\leq \lim_{n\to\infty}\left(\inf_{k\geq n}(x_k + y_k)\right), \\
\lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right) + \lim_{n\to\infty}\left(\inf_{k\geq n} y_k\right) &\leq \lim_{n\to\infty}\left(\inf_{k\geq n}(x_k + y_k)\right), \\
\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n &\leq \liminf_{n\to\infty}\,(x_n + y_n).
\end{aligned}$$

(2) Since

$$x_k \leq \sup_{k\geq n} x_k \text{ and } y_k \leq \sup_{k\geq n} y_k \implies x_k + y_k \leq \sup_{k\geq n} x_k + \sup_{k\geq n} y_k$$

for each $k \geq n$, we have

$$\forall n \in \mathbb{N},\ \sup_{k\geq n}(x_k + y_k) \leq \sup_{k\geq n} x_k + \sup_{k\geq n} y_k.$$

This implies that

$$\begin{aligned}
\lim_{n\to\infty}\left(\sup_{k\geq n}(x_k + y_k)\right) &\leq \lim_{n\to\infty}\left(\sup_{k\geq n} x_k + \sup_{k\geq n} y_k\right), \\
\lim_{n\to\infty}\left(\sup_{k\geq n}(x_k + y_k)\right) &\leq \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right) + \lim_{n\to\infty}\left(\sup_{k\geq n} y_k\right), \\
\limsup_{n\to\infty}\,(x_n + y_n) &\leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.
\end{aligned}$$

□

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# A Equivalent Statements of the Least Upper Bound Property

$$\begin{aligned}\text{Least Upper Bound Property} &\iff \text{Monotone Convergence Theorem}\\ &\iff \text{Nested Interval Property}\end{aligned}$$

**Theorem.** *Monotone Convergence Theorem* $\iff$ *Nested Interval Property*

*Proof.* ($\Rightarrow$) See **Nested Interval Property**.

($\Leftarrow$) TBA

□

**Theorem.** *Least Upper Bound Property* $\iff$ *Monotone Convergence Theorem*

*Proof.* ($\Rightarrow$) See **Monotone Convergence Theorem**.

($\Leftarrow$) TBA

□