Multi-variable Calculus

- HW1 -

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$$\underbrace{f}_{\Omega^0} \xrightarrow{d} \underbrace{df}_{\Omega^1} \longleftrightarrow \underbrace{\nabla f}_{\substack{\text{gradient}\\\text{vector field}}} \longrightarrow \underbrace{\mathbf{F}}_{(\Omega^0)^m} \xrightarrow{d} \underbrace{d\mathbf{F}}_{\Omega^1 \otimes \mathbb{R}^m} \longleftrightarrow \underbrace{D\mathbf{F}}_{\substack{\text{Jacobian}\\\text{matrix}}}$$

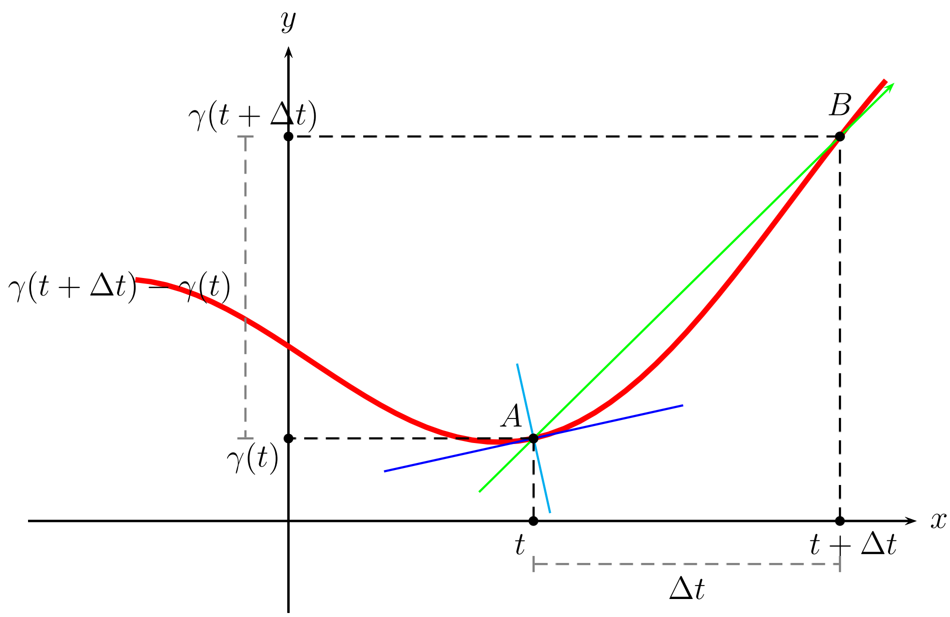
We cover the following topics in this note.

- Vector Fields
- Line Integrals for Vector Fields
- Surface Integrals for Vector Fields
- TBA

Contents

Line Integrals

Line Integral of Scalar Function over Arc Length



For a curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2\colon t \mapsto \langle x(t), y(t)\rangle$, the **secant vector** over $[t, t+\Delta t]$ is

$$\frac{\Delta\gamma}{\Delta t} = \frac{\gamma(t+\Delta t)-\gamma(t)}{\Delta t} = \left\langle \frac{x(t+\Delta t)-x(t)}{\Delta t}, \frac{y(t+\Delta t)-y(t)}{\Delta t} \right\rangle.$$

As $\Delta t \to 0$, these secants converge (if γ is smooth) to

$$\begin{aligned} \gamma'(t) = \frac{d}{dt}\gamma(t) = \lim_{\Delta t\to 0}\frac{\Delta\gamma}{\Delta t} &= \left\langle \lim_{\Delta t\to 0}\frac{x(t+\Delta t)-x(t)}{\Delta t}, \lim_{\Delta t\to 0}\frac{y(t+\Delta t)-y(t)}{\Delta t} \right\rangle \\ &= \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle \\ &= \langle x'(t), y'(t)\rangle, \end{aligned}$$

which gives the **tangent vector** at $\gamma(t)$. The tangent vector captures how the curve is moving instantaneously at time t.

By Pythagoras' theorem, the **length moved per unit time** is $\|\gamma'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$, and so the small arc length traveled between t and $t+\Delta t$ is approximately:

$$\|\gamma'(t)\|\Delta t = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot \Delta t.$$

Arc Length of a Parametrized Curve

Definition. Let $C \subset \mathbb{R}^n$ be a piecewise smooth curve, given by a smooth parameterization:

$$\gamma : [a, b] \to \mathbb{R}^n, \quad t \mapsto \gamma(t) = \langle x_1(t), x_2(t), \ldots, x_n(t)\rangle.$$

Then the **arc length** s of the curve C from $t = a$ to $t = b$ is defined by

$$\boxed{s := \int_a^b \|\gamma'(t)\|\ dt}, \quad \text{where } \|\gamma'(t)\| = \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \cdots + \left(\frac{dx_n}{dt}\right)^2}.$$

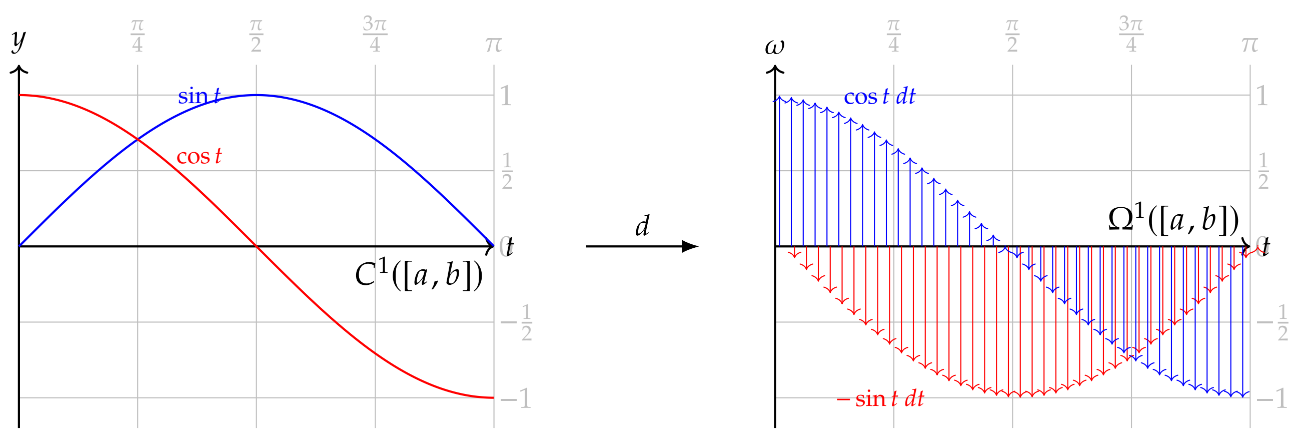
Remark. Let $\gamma : [a, b] \to \mathbb{R}^n$ be a piecewise-C^1 curve, $\gamma(t) = (x_1(t), \ldots x_n(t))$. A arc length function is defined by

$$s : [a, b] \to \mathbb{R}, \quad t \mapsto s(t) = \int_a^t \|\gamma'(u)\|\ du,$$

where $\|\gamma'(u)\| = \sqrt{\sum_{i=1}^n (x_i'(u))^2}$. Define two sets:

$$C^1([a, b]) = \left\{f \in \mathbb{R}^{[a,b]} : f \text{ is continuously differentiable on } [a, b]\right\}$$
$$\Omega^1([a, b]) = \left\{\delta(t)\ dt : \delta \in \mathbb{R}^{[a,b]} \text{ is continuous and } t \in [a, b]\right\} = \left\{\delta(t)\ dt : \delta \in C^0([a, b])\right\}.$$

Here $s \in C^1([a, b])$ with $s'(t) = \dfrac{d}{dt}\left(\int_a^t \|\gamma'(u)\|\ du\right) \overset{\text{FTC}}{=} \|\gamma'(t)\|$.

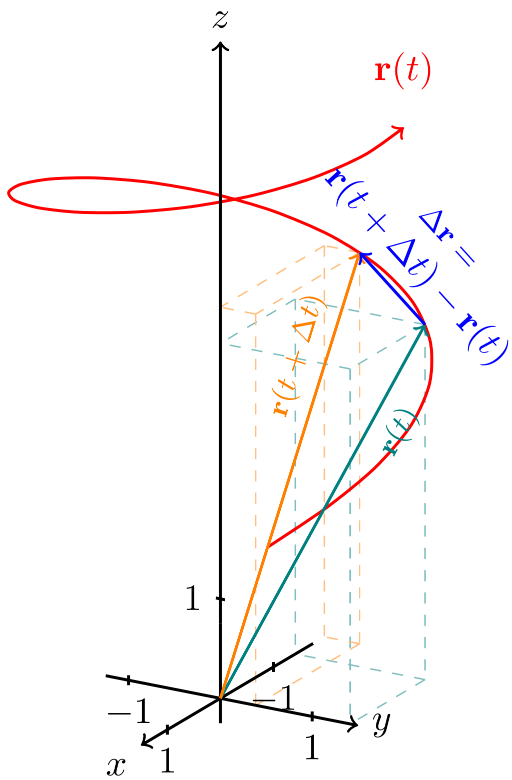


The map

$$\begin{array}{rccc} d & : & C^1([a, b]) & \longrightarrow & \Omega^1([a, b]) \\ & & f(t) & \longmapsto & d(f(t)) = df \end{array}$$

is defined by $df = f'(t)dt$, where f' is the derivative of f. Thus

$$ds := d(s(t)) = s'(t)\ dt = \|\gamma'(t)\|\ dt.$$



$$
\begin{array}{rcccl}
\mathbf{r} & : & \mathbb{R} & \longrightarrow & \mathbb{R}^3 \\
& & t & \longmapsto & \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle
\end{array}
$$

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \langle x'(t), y'(t), z'(t) \rangle$$

$$s(t) = \int_a^t \|\mathbf{r}'(t)\| \; dt$$

$$s'(t) = \|\mathbf{r}'(t)\| = \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2 + \big(z'(t)\big)^2}$$

$$ds = d(s(t)) = s'(t)\; dt = \|\mathbf{r}'(t)\| \; dt$$

Line Integral of Scalar Function over Arc Length

Definition. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar function, and let C be a piecewise smooth curve in $\mathbb{R}^n$ given by a smooth parameterization:

$$\gamma : [a, b] \to \mathbb{R}^n, \quad t \mapsto \gamma(t) = \langle x_1(t), x_2(t), \ldots, x_n(t) \rangle \in \mathbb{R}^n = \mathrm{dom}(f).$$

The **line integral of the scalar function** f along the curve C with respect to arc length is defined by

$$\int_C f \; ds := \int_a^b f(\gamma(t)) \, \|\gamma'(t)\| \; dt.$$

Line Integral of Vector Fields

Line Integral of a Vector Field in $\mathbb{R}^2$

Definition. Let C be a smooth curve parametrized by

$$\gamma : [a, b] \to \mathbb{R}^2, \quad t \mapsto \gamma(t) = \langle x(t), y(t) \rangle.$$

Let $\mathbf{F} = \langle F_1, F_2 \rangle$ be a smooth vector field on $\mathbb{R}^2$. The **line integral of the vector field F** along the curve γ is defined by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\, dt.$$

Alternatively,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (F_1, F_2) \cdot (dx, dy) = \int_C F_1 dx + F_2 dy.$$

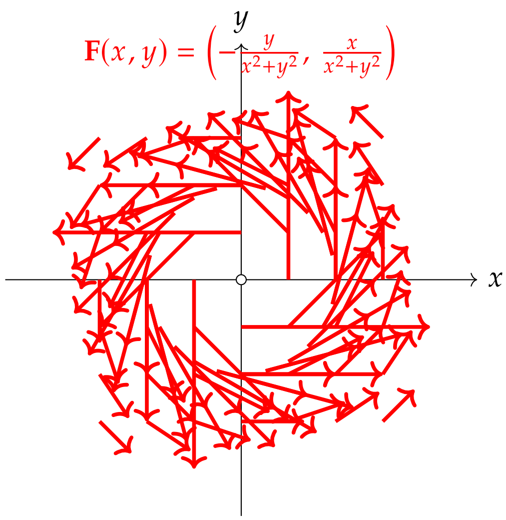
Problem #1 **(Line Integral around Unit Circle).** Let $C \subset \mathbb{R}^2$ be the unit circle defined by $C : x^2 + y^2 = 1$, traversed in the **counterclockwise direction**. Let the vector field $\mathbf{F} : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}^2$ be defined by

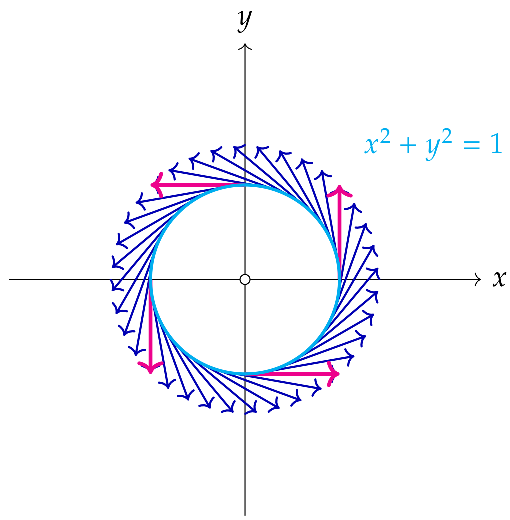
$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

Evaluate the **line integral** of **F** along C:

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

Sol.



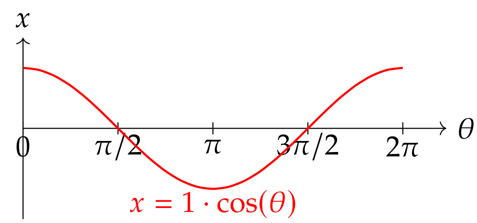


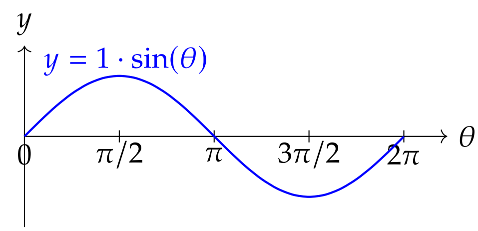
Consider the vector field $\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$, and the curve C is the unit circle $x^2 + y^2 = 1$, traversed counterclockwise.

(Parametrization) Define a function

$$\begin{array}{rcccl} \gamma & : & [0, 2\pi] & \longrightarrow & \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\} \\ & & \theta & \longmapsto & \gamma(\theta) = (\cos\theta, \sin\theta) \end{array}.$$

Here, $\frac{d\gamma}{d\theta} = (-\sin\theta, \cos\theta)$.





(Evaluate $\mathbf{F}(\gamma(\theta))$ and the dot product) We have

$$\mathbf{F}(\gamma(\theta)) = \mathbf{F}(\cos\theta, \sin\theta) \overset{\sin^2\theta+\cos^2\theta=1}{=} \left\langle \frac{-\sin\theta}{1}, \frac{\cos\theta}{1} \right\rangle = (-\sin\theta, \cos\theta).$$

and

$$\mathbf{F}(\gamma(\theta)) \cdot \frac{d\gamma}{d\theta} = (-\sin\theta)(-\sin\theta) + (\cos\theta)(\cos\theta) = \sin^2\theta + \cos^2\theta = 1.$$

(Integral)

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\gamma(\theta)) \cdot \frac{d\gamma}{d\theta}\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi.$$

□

Surface Integral for Vector Fields

Problem #2 (Surface-Flux). Compute the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = \langle x,\ y,\ -z \rangle$ and the surface S is parametrized by

$$\mathbf{r}(u, v) = \langle u + 2v,\ 2u + v,\ 3uv \rangle, \quad (u, v) \in [0, 1] \times [0, 1].$$

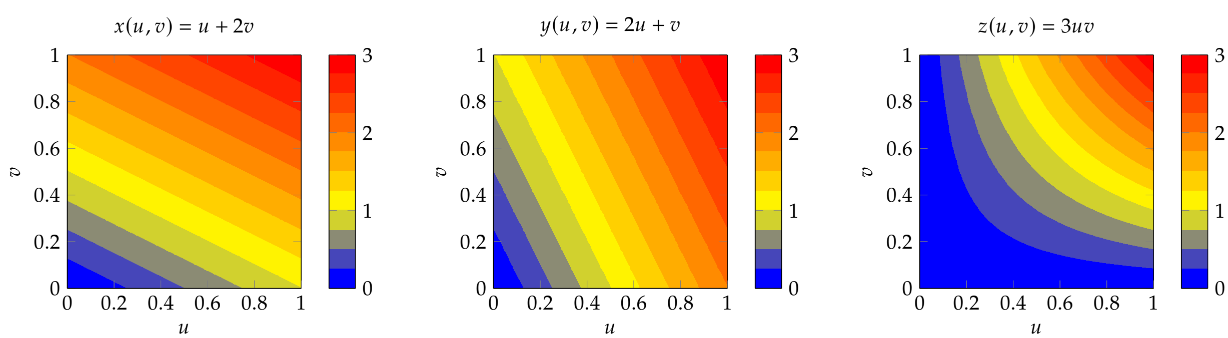
Sol.

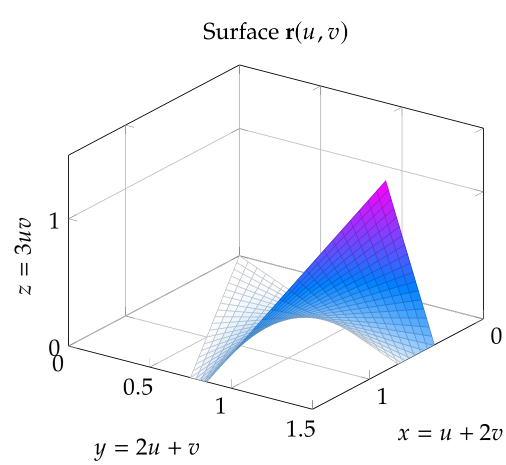
1. Parametrization and partials. The surface is

$$S = \mathbf{r}\big([0, 1]^2\big), \qquad \mathbf{r}(u, v) = (\, u + 2v,\ 2u + v,\ 3uv \,),$$

and then

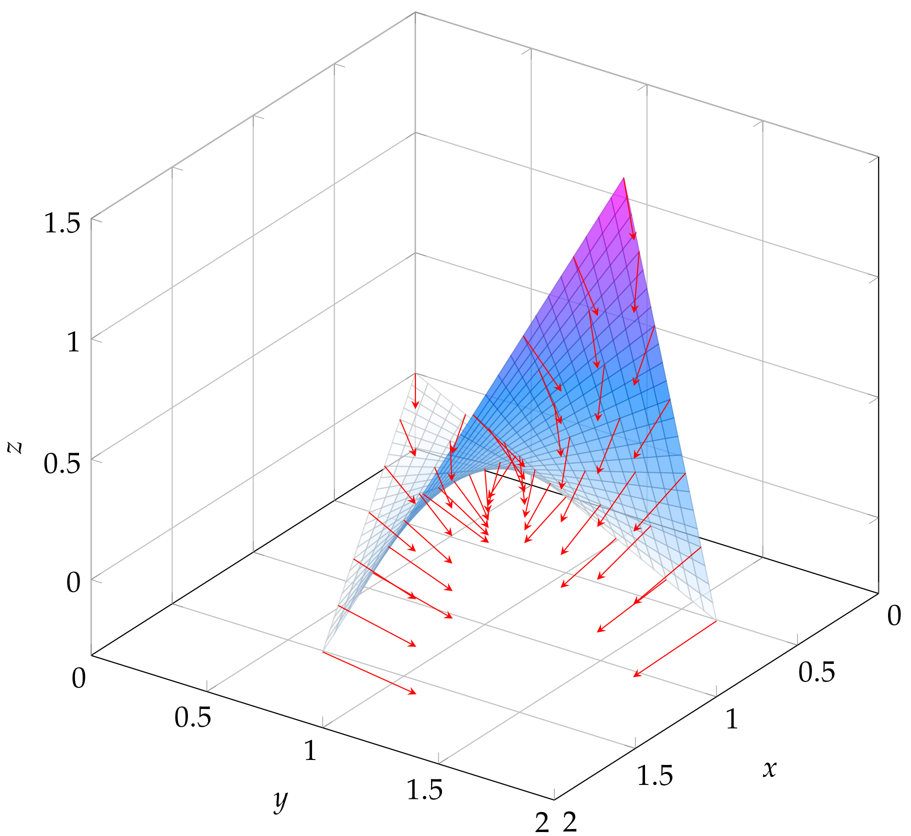
$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \langle 1,\ 2,\ 3v \rangle, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \langle 2,\ 1,\ 3u \rangle.$$





2. Oriented normal. The induced normal vector is the cross-product

$$
\begin{aligned}
\mathbf{r}_u \times \mathbf{r}_v &= \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3v \\ 2 & 1 & 3u \end{vmatrix} \\
&= \det \begin{vmatrix} 2 & 3v \\ 1 & 3u \end{vmatrix} \mathbf{i} - \det \begin{vmatrix} 1 & 3v \\ 2 & 3u \end{vmatrix} \mathbf{j} + \det \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \mathbf{k} \\
&= \langle 6u - 3v,\ -3u + 6v,\ -3 \rangle.
\end{aligned}
$$



Detailed Justification for $\mathbf{r}_u \times \mathbf{r}_v$ as Surface Normal

Let

$$\mathbf{r}(u,v) = (\,x(u,v),\ y(u,v),\ z(u,v)\,)$$

be a smooth parametrization of a surface patch $S \subset \mathbb{R}^3$. Then at each point (u,v), the two tangent vectors

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} \quad \text{and} \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$$

span the tangent plane to S. We list the logical steps that show $\mathbf{r}_u \times \mathbf{r}_v$ is the correct (non-unit) normal vector:

1. **Tangent plane.** By definition of partial derivatives,

$$\mathbf{r}_u \text{ is the velocity of the curve } v = \text{const}, \quad \mathbf{r}_v \text{ is the velocity of the curve } u = \text{const}.$$

   Both lie tangent to the surface.

2. **Cross product properties.** In $\mathbb{R}^3$, the cross product $\mathbf{a} \times \mathbf{b}$ satisfies:

$$\mathbf{a} \times \mathbf{b} \perp \mathbf{a}, \quad \mathbf{a} \times \mathbf{b} \perp \mathbf{b},$$

   and its direction is given by the right-hand rule (orientation of $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$). Thus $\mathbf{r}_u \times \mathbf{r}_v$ is perpendicular to the tangent plane.

3. **Determinant formula.** One defines

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix},$$

   which expands by minors to give the familiar component formula $\big(a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1\big)$.

4. **Application to our $\mathbf{r}_u, \mathbf{r}_v$.** In our case,

$$\mathbf{r}_u = \langle 1,\ 2,\ 3v \rangle, \quad \mathbf{r}_v = \langle 2,\ 1,\ 3u \rangle.$$

   Plugging into the determinant formula yields

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3v \\ 2 & 1 & 3u \end{vmatrix} = \big\langle 2 \cdot 3u - 3v \cdot 1,\ -(1 \cdot 3u - 3v \cdot 2),\ 1 \cdot 1 - 2 \cdot 2 \big\rangle = \langle 6u - 3v,\ -3u + 6v,\ -3 \rangle.$$

5. **Verification of orthogonality.** One can check directly

$$(\mathbf{r}_u \times \mathbf{r}_v) \cdot \mathbf{r}_u = 0, \quad (\mathbf{r}_u \times \mathbf{r}_v) \cdot \mathbf{r}_v = 0,$$

   confirming it is indeed normal.

6. **Geometric interpretation.** The magnitude $\|\mathbf{r}_u \times \mathbf{r}_v\|$ equals the area of the parallelogram spanned by $\mathbf{r}_u, \mathbf{r}_v$. Thus in surface integrals $\iint_S \mathbf{F} \cdot d\mathbf{S}$, one uses $\mathbf{r}_u \times \mathbf{r}_v \, du \, dv$ as the oriented area element.

In summary, the cross-product of the partial derivatives is the unique algebraic construction in

$\mathbb{R}^3$ that (i) is bilinear and alternating, (ii) yields a vector orthogonal to both inputs, and (iii) has magnitude equal to the parallelogram area—exactly capturing the normal and area element needed for surface integrals.

The Cross Product in $\mathbb{R}^3$

Definition

For two vectors

$$\mathbf{u} = (u_1, u_2, u_3), \qquad \mathbf{v} = (v_1, v_2, v_3)$$

in $\mathbb{R}^3$, their **cross product** $\mathbf{u} \times \mathbf{v}$ is defined to be the vector

$$\mathbf{u} \times \mathbf{v} = \big(u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1\big).$$

Equivalently, if $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the standard basis, one writes formally

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

meaning "take the determinant by expanding along the first row."

Key Properties

1. **Bilinearity and Alternation:** $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$, and $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.

2. **Perpendicularity:** $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$: $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$.

3. **Magnitude = Area:**

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\big|\sin\theta\big| = \text{area of the parallelogram spanned by } \mathbf{u}, \mathbf{v},$$

   where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

4. **Right-Hand Rule (Orientation):** The triple $(\mathbf{u}, \mathbf{v}, \mathbf{u}\times\mathbf{v})$ has the same "right-hand" orientation as the standard basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

Why the Determinant Formula?

The determinant expression

$$\mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

is not a literal 3×3 determinant of numbers, but a **mnemonic** encoding exactly those three component-wise minors which satisfy:

- **Alternation:** Swapping the two rows (u_i) and (v_i) changes the sign of each minor, matching $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$.
- **Bilinearity:** Expanding a determinant along the top row is linear in each column.
- **Compatibility with basis:** For the standard basis vectors,

  $$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2,$$

  and all others follow by bilinearity and alternation.

Concretely, expanding along the first row gives

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3v \\ 2 & 1 & 3u \end{pmatrix} \\ &= \mathbf{i}\,(2 \cdot 3u - 1 \cdot 3v) - \mathbf{j}\,(1 \cdot 3u - 2 \cdot 3v) + \mathbf{k}\,(1 \cdot 1 - 2 \cdot 2) \\ &= \langle 6u - 3v,\ -3u + 6v,\ -3 \rangle. \end{aligned}$$

which reproduces the component-wise definition above.

Geometric Interpretation via Volumes

Another way to see the determinant is to note that in $\mathbb{R}^3$ the scalar triple product $\det[\mathbf{u}, \mathbf{v}, \mathbf{w}]$ gives the signed volume of the parallelepiped spanned by $\mathbf{u}, \mathbf{v}, \mathbf{w}$. If one fixes $\mathbf{u}, \mathbf{v}$, then the unique vector $\mathbf{n}$ satisfying

$$\det[\mathbf{u}, \mathbf{v}, \mathbf{n}] = \|\mathbf{u} \times \mathbf{v}\|^2 \quad \text{and} \quad \mathbf{n} \perp \mathbf{u}, \mathbf{v}$$

is precisely $\mathbf{u} \times \mathbf{v}$. The determinant-of-a-matrix formula encodes that same volume-and-orientation condition intrinsically.

Conclusion: The cross product is the unique bilinear, alternating, oriented map $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ whose length measures area; the 3×3 "determinant" notation succinctly packages its component formula and all of its key properties in one place.

2. **
3. **Pullback of the field.** The given field is $\mathbf{F}(x, y, z) = (x, y, -z)$. Along the patch,

$$\mathbf{F}\big(\mathbf{r}(u, v)\big) = \big(u + 2v,\ 2u + v,\ -3uv\big).$$

4. **Integrand.** Taking the dot-product,

$$\begin{aligned}\mathbf{F}(\mathbf{r}) \cdot (\mathbf{r}_u \times \mathbf{r}_v) &= (u + 2v)(6u - 3v) \ + \ (2u + v)(-3u + 6v) \ + \ (-3uv)(-3)\\ &= 6u^2 - 3uv + 12uv - 6v^2 \ - \ 6u^2 + 12uv - 3uv + 6v^2 \ + \ 9uv\\ &= (-3uv + 12uv + 12uv - 3uv + 9uv) \ = \ 27\, u\, v.\end{aligned}$$

5. **Double integral.** Thus the flux is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \ = \ \iint_{[0,1]^2} 27\, u\, v\, du\, dv \ = \ 27 \int_0^1 \int_0^1 u\, v\, du\, dv \ = \ 27\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) \ = \ \frac{27}{4}.$$

Hence

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} \ = \ \frac{27}{4}}.$$

1. Compute the partial derivatives:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \langle 1,\, 2,\, 3v \rangle, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \langle 2,\, 1,\, 3u \rangle.$$

2. Form the cross-product to get the oriented area element:

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3v \\ 2 & 1 & 3u \end{vmatrix} = \left\langle 6u - 3v,\; -3u + 6v,\; -3 \right\rangle.$$

Hence

$$d\mathbf{S} = \left(\mathbf{r}_u \times \mathbf{r}_v\right) du\, dv.$$

3. Evaluate **F** on the parametrization:

$$\mathbf{F}\big(\mathbf{r}(u,v)\big) = \left\langle u + 2v,\; 2u + v,\; -3uv \right\rangle.$$

4. Compute the integrand

$$\mathbf{F}(\mathbf{r}(u,v)) \;\cdot\; (\mathbf{r}_u \times \mathbf{r}_v) = (u + 2v)(6u - 3v) + (2u + v)(-3u + 6v) + (-3uv)(-3).$$

Expand term by term:

$$(u+2v)(6u-3v) = 6u^2+9uv-6v^2, \quad (2u+v)(-3u+6v) = -6u^2+9uv+6v^2, \quad (-3uv)(-3) = 9uv.$$

Summing gives

$$6u^2 + 9uv - 6v^2 \;+\; (-6u^2 + 9uv + 6v^2) \;+\; 9uv = 27\,uv.$$

5. Finally integrate over the unit square:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^1\!\!\int_0^1 27\,uv\,du\,dv = 27\left(\int_0^1 u\,du\right)\!\left(\int_0^1 v\,dv\right) = 27\left(\tfrac{1}{2}\right)\!\left(\tfrac{1}{2}\right) = \frac{27}{4}.$$

Answer: $\dfrac{27}{4}$. □

Appendices

A Function and Derivative

1. Single–Variable Function and Derivative. Consider a single-variable function

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto f(x).$$

Its derivative at x is the linear map

$$\frac{df}{dx}(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \in \mathbb{R},$$

characterized by

$$f(x+h) = f(x) + f'(x)\,h + o(h).$$

1. Setup

Let

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \qquad x \mapsto f(x)$$

be a real–valued function. We wish to define its derivative at a point $x \in \mathbb{R}$ as the best linear approximation to the increment $f(x+h) - f(x)$.

2. Linear Approximation with Remainder

We say f is **differentiable** at x if there exists a real number A and a function $r(h)$ such that

$$f(x+h) = f(x) + A\,h + r(h), \tag{1}$$

where the remainder $r(h)$ satisfies

$$\lim_{h \to 0} \frac{r(h)}{h} = 0.$$

In Landau notation, $r(h) = o(h)$ as $h \to 0$.

3. Definition of the Derivative

Definition 1. If (1) holds for some real number A and $r(h) = o(h)$, then f is differentiable at x, and the **derivative** of f at x is

$$f'(x) := A.$$

Equivalently, the linear map

$$T_x\mathbb{R} \cong \mathbb{R} \longrightarrow T_{f(x)}\mathbb{R} \cong \mathbb{R}, \qquad h \mapsto f'(x)\,h$$

is the unique linear approximation to the increment $f(x+h) - f(x)$.

4. Equivalence with the Limit Formulation

One also defines

$$f'(x) \;=\; \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

when this limit exists. To see that this agrees with the linear–plus–remainder definition:

- If $f'(x) = A$ and $r(h) = o(h)$ satisfy (1), then

$$\frac{f(x+h) - f(x)}{h} = A + \frac{r(h)}{h} \;\longrightarrow\; A,$$

  since $\lim\limits_{h\to 0} \frac{r(h)}{h} = 0$.

- Conversely, if $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = A$, set $r(h) = f(x+h) - f(x) - A\,h$. Then

$$\frac{r(h)}{h} = \frac{f(x+h) - f(x)}{h} - A \;\longrightarrow\; 0,$$

  so $r(h) = o(h)$ and (1) holds.

5. Notation $o(h)$

The notation $r(h) = o(h)$ means

$$\forall \varepsilon > 0,\ \exists \delta > 0: \quad 0 < |h| < \delta \implies \big|r(h)/h\big| < \varepsilon.$$

6. Summary

Thus the derivative $f'(x)$ is the unique scalar A making

$$f(x+h) = f(x) + A\,h + o(h),$$

and equivalently the limit $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

1. Definition Recap

A function $f : \mathbb{R} \to \mathbb{R}$ is **differentiable** at x if there exists a real number A and a remainder $r(h)$ such that

$$f(x+h) = f(x) + A\,h + r(h), \qquad \lim_{h\to 0} \frac{r(h)}{h} = 0.$$

In that case $f'(x) = A$, and $r(h) = o(h)$.

2. Example: $f(t) = \sin t$

We check

$$\sin(x+h) \;=\; \sin x + \cos x\, h + r(h).$$

Using the addition formula,

$$\begin{aligned}\sin(x+h) &= \sin x \cos h + \cos x \sin h \\ &= \sin x\Big(1 - \tfrac{h^2}{2} + o(h^2)\Big) + \cos x\Big(h - \tfrac{h^3}{6} + o(h^3)\Big) \\ &= \sin x + \cos x\, h \;+\; \Big(-\tfrac{\sin x}{2} h^2 + o(h^2)\Big).\end{aligned}$$

Thus

$$r(h) = -\tfrac{\sin x}{2}\, h^2 + o(h^2),$$

and

$$\frac{r(h)}{h} = -\tfrac{\sin x}{2}\, h + o(h) \xrightarrow[h\to 0]{} 0.$$

Hence $\sin t$ is differentiable with

$$f'(x) = \cos x,$$

and indeed $d(\sin t) = \cos t\, dt$.

3. Example: $f(t) = \cos t$

Similarly,

$$\cos(x+h) \;=\; \cos x - \sin x\, h + r(h).$$

By the addition formula,

$$\begin{aligned}\cos(x+h) &= \cos x \cos h - \sin x \sin h \\ &= \cos x\Big(1 - \tfrac{h^2}{2} + o(h^2)\Big) - \sin x\Big(h - \tfrac{h^3}{6} + o(h^3)\Big) \\ &= \cos x - \sin x\, h \;+\; \Big(-\tfrac{\cos x}{2} h^2 + o(h^2)\Big).\end{aligned}$$

Thus

$$r(h) = -\tfrac{\cos x}{2}\, h^2 + o(h^2),$$

and

$$\frac{r(h)}{h} = -\tfrac{\cos x}{2}\, h + o(h) \xrightarrow[h\to 0]{} 0.$$

Therefore $\cos t$ is differentiable with

$$f'(x) = -\sin x,$$

and indeed $d(\cos t) = -\sin t\, dt$.

4. Summary

In both cases we have exhibited the decomposition

$$f(x+h) = f(x) + f'(x)\, h + o(h),$$

with the remainder $r(h)$ vanishing faster than h. This formalizes that $f'(x)$ is the unique scalar making

$$h \;\mapsto\; f'(x)\, h$$

the best linear approximation to the increment $f(x+h) - f(x)$.

2. Scalar Function and Gradient

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}, \quad \mathbf{x} \mapsto f(\mathbf{x}).$$

Its **gradient** at $\mathbf{x}$ is the vector

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial f}{\partial x_1}(\mathbf{x}) \\ \dfrac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \dfrac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^n,$$

characterized by the first-order Taylor expansion

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) \; + \; \nabla f(\mathbf{x}) \cdot \mathbf{h} \; + \; o(\|\mathbf{h}\|).$$

3. Vector Field and Jacobian (Total Derivative)

$$\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad \mathbf{x} \mapsto \mathbf{F}(\mathbf{x}) = \begin{pmatrix} F_1(\mathbf{x}) \\ \vdots \\ F_m(\mathbf{x}) \end{pmatrix}.$$

Its **Jacobian matrix** (total derivative) at $\mathbf{x}$ is

$$D\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial F_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial F_m}{\partial x_n}(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^{m\times n},$$

characterized by

$$\mathbf{F}(\mathbf{x}+\mathbf{h}) = \mathbf{F}(\mathbf{x}) \; + \; D\mathbf{F}(\mathbf{x})\,\mathbf{h} \; + \; o(\|\mathbf{h}\|).$$

Special Case: $n = m = 3$**.** For a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$: - The **divergence** is the trace of the Jacobian: $\nabla\cdot\mathbf{F} = \partial_x F_1 + \partial_y F_2 + \partial_z F_3$. - The **curl** is a new vector field: $\nabla\times\mathbf{F} = \begin{pmatrix} \partial_y F_3 - \partial_z F_2 \\ \partial_z F_1 - \partial_x F_3 \\ \partial_x F_2 - \partial_y F_1 \end{pmatrix}$.

Summary of the Hierarchy:

$$\underbrace{f'(x)}_{\text{single-variable derivative}} \quad \longleftrightarrow \quad \underbrace{\nabla f(\mathbf{x})}_{\substack{\text{gradient of} \\ \text{scalar } f}} \quad \longleftrightarrow \quad \underbrace{D\mathbf{F}(\mathbf{x})}_{\substack{\text{Jacobian of} \\ \text{vector field } \mathbf{F}}} .$$

We summarize the familiar hierarchy

$$\text{single-variable derivative} \;\longleftrightarrow\; \text{gradient of a scalar field} \;\longleftrightarrow\; \text{Jacobian of a vector field}$$

by using the exterior derivative d on differential forms.

1. Single-variable case. A smooth function $f : \mathbb{R} \to \mathbb{R}$ is a 0–form, $f \in \Omega^0(\mathbb{R})$. Its derivative is the 1–form

$$df = f'(x)\,dx \;\in\; \Omega^1(\mathbb{R}),$$

and the Fundamental Theorem of Calculus reads $\displaystyle\int_a^b df = f(b) - f(a)$.

2. Scalar field in $\mathbb{R}^n$. A smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is again a 0–form, $f \in \Omega^0(\mathbb{R}^n)$. Its exterior derivative

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\,dx^i \;\in\; \Omega^1(\mathbb{R}^n)$$

is the differential 1–form whose components are the partials. Under the Euclidean metric this 1–form corresponds to the gradient vector field ∇f.

3. Vector field in $\mathbb{R}^n$. A smooth vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ can be viewed as an $\mathbb{R}^m$-valued 0–form

$$\mathbf{F} = (F_1, \ldots, F_m) \;\in\; \big(\Omega^0(\mathbb{R}^n)\big)^m .$$

Applying d to each component yields the **matrix of 1–forms**

$$d\mathbf{F} = \big(dF_1, \ldots, dF_m\big) \;\in\; \underbrace{\Omega^1(\mathbb{R}^n) \times \cdots \times \Omega^1(\mathbb{R}^n)}_{m \text{ copies}} \;\cong\; \Omega^1(\mathbb{R}^n) \otimes \mathbb{R}^m ,$$

whose ith entry is

$$dF_i = \sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j}\,dx^j .$$

Choosing the basis $\{dx^1, \dots, dx^n\}$ identifies $d\mathbf{F}$ with the Jacobian matrix

$$D\mathbf{F} = \left[\tfrac{\partial F_i}{\partial x_j}\right]_{1\le i\le m,\, 1\le j\le n}.$$

Summary in One Diagram:

$$\underbrace{f}_{\Omega^0} \xrightarrow{d} \underbrace{df}_{\Omega^1} \longleftrightarrow \underbrace{\nabla f}_{\substack{\text{gradient}\\ \text{vector field}}} \longrightarrow \underbrace{\mathbf{F}}_{(\Omega^0)^m} \xrightarrow{d} \underbrace{d\mathbf{F}}_{\Omega^1\otimes\mathbb{R}^m} \longleftrightarrow \underbrace{D\mathbf{F}}_{\substack{\text{Jacobian}\\ \text{matrix}}}.$$

Each arrow d is the same exterior derivative, producing higher-rank forms whose coefficients encode the familiar derivatives.

From Derivatives to Differentials: A Unified Matrix–Form View

We compare three levels of maps and their differentials in the language of exterior derivatives and matrices.

Case	Map	Differential / Matrix Form
Single–variable	$f:\mathbb{R}\longrightarrow\mathbb{R},\ x\mapsto f(x)$	$df = f'(x)\,dx \quad \big(\in \Omega^1(\mathbb{R})\big)$
Scalar field	$f:\mathbb{R}^n\longrightarrow\mathbb{R},\ \mathbf{x}\mapsto f(\mathbf{x})$	$df = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\mathbf{x})\,dx^i \quad \big(\in \Omega^1(\mathbb{R}^n)\big)$, $\nabla f(\mathbf{x}) = \begin{pmatrix}\partial_1 f\\ \vdots\\ \partial_n f\end{pmatrix} \in \mathbb{R}^{n\times 1}$
Vector field	$\mathbf{F}:\mathbb{R}^n\longrightarrow\mathbb{R}^n,\ \mathbf{x}\mapsto \mathbf{F}(\mathbf{x})$	$d\mathbf{F} = (dF_1,\dots,dF_n) = \Big(\sum_j \partial_j F_i\,dx^j\Big)_{i=1}^{n} \in \Omega^1(\mathbb{R}^n)\otimes\mathbb{R}^n$, $D\mathbf{F}(\mathbf{x}) = \begin{pmatrix}\partial_1 F_1 & \cdots & \partial_n F_1\\ \vdots & \ddots & \vdots\\ \partial_1 F_n & \cdots & \partial_n F_n\end{pmatrix} \in \mathbb{R}^{n\times n}.$

Key points:

- In each case, the exterior derivative d raises the form-degree by one:

$$d : \Omega^0 \to \Omega^1, \quad d(f) = df, \quad d(F_i) = dF_i.$$

- For $f : \mathbb{R}^n \to \mathbb{R}$, df is a 1–form whose coefficients are the partials $\partial_i f$. Under the Euclidean metric these correspond to the gradient vector ∇f.
- For $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, applying d to each component yields the matrix of 1–forms $d\mathbf{F}$. Choosing the basis $\{dx^j\}$ identifies $d\mathbf{F}$ with the Jacobian matrix $D\mathbf{F}$, which is the total derivative (linear approximation) of $\mathbf{F}$.
- Thus the familiar hierarchy

$$f'(x) \longleftrightarrow \nabla f(\mathbf{x}) \longleftrightarrow D\mathbf{F}(\mathbf{x})$$

  is simply the degrees–of–freedom of the single exterior derivative d applied to scalar vs. vector-valued functions, packaged in matrix form.

B Scalar Function and Vector Fields

Definition. A **scalar function** on $\mathbb{R}^n$ is a real-valued function of an n-tuple; that is,

$$f : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n).$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $f(\mathbf{x}) \in \mathbb{R}$.

Definition 2 (Scalar Function)**.** A **scalar function** on $\mathbb{R}^n$ is a mapping

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad \mathbf{x} = (x_1, \ldots, x_n) \longmapsto f(\mathbf{x}),$$

which assigns to each point $\mathbf{x} \in \mathbb{R}^n$ a real value $f(\mathbf{x})$. If f has continuous partial derivatives on an open set $U \subset \mathbb{R}^n$, we write $f \in C^1(U)$.

Definition (Gradient of a Scalar Function)**.** Let $f \in C^1(U)$ be a scalar function on an open set $U \subset \mathbb{R}^n$. Its **gradient** is the vector-valued function

$$\nabla f : U \longrightarrow \mathbb{R}^n, \qquad \nabla f(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial f}{\partial x_1}(\mathbf{x}) \\ \dfrac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \dfrac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix}.$$

Equivalently, $\nabla f(\mathbf{x})$ is the unique vector in $\mathbb{R}^n$ satisfying

$$df(\mathbf{x})[\mathbf{h}] = \nabla f(\mathbf{x}) \cdot \mathbf{h} \quad \text{for all } \mathbf{h} \in \mathbb{R}^n,$$

where $df(\mathbf{x})$ is the differential of f at $\mathbf{x}$.

Remark. In particular, its differential (or gradient) may be written in matrix (row-vector) form as

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \cdots & \dfrac{\partial}{\partial x_n} \end{pmatrix} \begin{pmatrix} \dfrac{\partial f}{\partial x_1}(x) \\ \\ \dfrac{\partial f}{\partial x_2}(x) \\ \\ \vdots \\ \\ \dfrac{\partial f}{\partial x_n}(x) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f}{\partial x_1}(\mathbf{x}) \\ \\ \dfrac{\partial f}{\partial x_2}(\mathbf{x}) \\ \\ \vdots \\ \\ \dfrac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^{1\times n}.$$

Although $\nabla f(\mathbf{x})$ is **not** the product of a fixed matrix by $\mathbf{x}$, the symbol

$$\nabla = \begin{pmatrix} \dfrac{\partial}{\partial x_1} \\ \vdots \\ \dfrac{\partial}{\partial x_n} \end{pmatrix}$$

can itself be viewed as a "column-vector" whose entries are the partial-derivative operators. Then for any scalar function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\nabla f = \begin{pmatrix} \dfrac{\partial}{\partial x_1} \\ \vdots \\ \dfrac{\partial}{\partial x_n} \end{pmatrix} f = \begin{pmatrix} \dfrac{\partial f}{\partial x_1} \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{pmatrix},$$

which is exactly the usual gradient.

—

Key point: - ∇ itself is an **operator-valued** vector, not a numeric matrix. - When you write ∇f, you are applying each row of that "matrix" of operators to the single-valued function f. - By the same token, for a vector field $\mathbf{F} = (F_1, \ldots, F_n)^T$, the **Jacobian** can be written symbolically as

$$D\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \nabla^\mathsf{T} F_1(\mathbf{x}) \\ \vdots \\ \nabla^\mathsf{T} F_n(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial}{\partial x_1} & \cdots & \dfrac{\partial}{\partial x_n} \end{pmatrix} \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}.$$

Here the row $\left(\partial_1, \ldots, \partial_n\right)$ acts on each component F_i.

In this sense, ∇ really **is** a "matrix" (vector) of operators, whose multiplication by a function or vector field produces the gradient, divergence, curl, or Jacobian, depending on how you contract it.

1. ∇ is not a "point-wise" vector in $\mathbb{R}^n$. Rather,

$$\nabla = \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_n} \end{pmatrix}$$

is a **vector of differential operators**, each ∂_{x_i} acting on functions. It lives in the space of linear maps $\mathrm{Hom}\big(C^\infty(\mathbb{R}^n),\,\Omega^1(\mathbb{R}^n)\big)$, not in $\mathbb{R}^n$ itself. We write it in "vector form" simply to mirror how it acts component-wise.

2. f is a scalar field, an element of the function space $C^\infty(\mathbb{R}^n)$. Concretely,

$$f:\mathbb{R}^n \longrightarrow \mathbb{R}, \quad \mathbf{x} \longmapsto f(\mathbf{x}).$$

This f is **not** itself a vector in $\mathbb{R}^n$; it is a real-valued function on $\mathbb{R}^n$.

3. The action ∇f produces a genuine vector field. When you apply the operator-vector ∇ to the scalar f, you get

$$\nabla f = \begin{pmatrix} \partial_{x_1} f \\ \partial_{x_2} f \\ \vdots \\ \partial_{x_n} f \end{pmatrix},$$

which **is** a true vector-valued function on $\mathbb{R}^n$, i.e. an element of $\Omega^1(\mathbb{R}^n)$ or equivalently $(C^\infty(\mathbb{R}^n))^n$.

Summary:

- ∇ itself is an **operator** (a "vector" of partial-derivatives), not a point in physical space.
- f is a scalar field (a function) on $\mathbb{R}^n$.
- ∇f is the gradient vector field, a bona fide element of $(C^\infty(\mathbb{R}^n))^n$.

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial f}{\partial x_1}(\mathbf{x}) \\ \dfrac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \dfrac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^{n\times 1}.$$

Equivalently, its transpose (the differential) is the $1\times n$ row-vector

$$df(\mathbf{x}) = \nabla f(\mathbf{x})^{\mathsf{T}} = \begin{pmatrix} \dfrac{\partial f}{\partial x_1}(\mathbf{x}) & \dfrac{\partial f}{\partial x_2}(\mathbf{x}) & \cdots & \dfrac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix}.$$

Example. Let $n = 2$. We consider two **scalar functions** on $\mathbb{R}^2 \setminus \{(0,0)\}$:

$$\begin{array}{rcrcl} f_1 & : & \mathbb{R}^2 \setminus \{(0,0)\} & \longrightarrow & \mathbb{R} \\ & & (x,y) & \longmapsto & f_1(x,y) = -\frac{y}{x^2+y^2} \end{array}, \qquad \begin{array}{rcrcl} f_2 & : & \mathbb{R}^2 \setminus \{(0,0)\} & \longrightarrow & \mathbb{R} \\ & & (x,y) & \longmapsto & f_2(x,y) = \frac{x}{x^2+y^2} \end{array}.$$

These functions assign to each point of the punctured plane a single real value, in accordance with the definition of a scalar function on $\mathbb{R}^2$.

Writing

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r > 0, \ \theta \in [0, 2\pi),$$

we obtain the equivalent descriptions in (r,θ)-space:

$$\begin{array}{rcrcl} f_1 & : & (0,\infty) \times [0,2\pi) & \longrightarrow & \mathbb{R} \\ & & (r,\theta) & \longmapsto & f_1(r,\theta) = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r} \end{array},$$

$$\begin{array}{rcrcl} f_2 & : & (0,\infty) \times [0,2\pi) & \longrightarrow & \mathbb{R} \\ & & (r,\theta) & \longmapsto & f_2(r,\theta) = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r} \end{array}.$$

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial f}{\partial x_1}(\mathbf{x}) \\ \dfrac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \dfrac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^{n\times 1}.$$

Equivalently, its transpose (the differential) is the $1 \times n$ row-vector

$$df(\mathbf{x}) = \nabla f(\mathbf{x})^{\mathsf{T}} = \begin{pmatrix} \dfrac{\partial f}{\partial x_1}(\mathbf{x}) & \dfrac{\partial f}{\partial x_2}(\mathbf{x}) & \cdots & \dfrac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix}.$$

Definition. A **vector field** on $\mathbb{R}^n$ is a function

$$\begin{aligned} \mathbf{F} \;:\; \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ \mathbf{x} &\longmapsto \mathbf{F}(\mathbf{x}) = \begin{pmatrix} F_1(\mathbf{x}) \\ F_2(\mathbf{x}) \\ \vdots \\ F_n(\mathbf{x}) \end{pmatrix}, \end{aligned}$$

where each component $F_i : \mathbb{R}^n \to \mathbb{R}$ is itself a scalar function.

Remark. Its Jacobian matrix—which encodes the best linear approximation of F at each point—is

$$\mathbf{J}_{\mathbf{F}} = \frac{\partial(F_1,\ldots,F_n)}{\partial(x_1,\ldots,x_n)} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\mathbf{x}) & \frac{\partial F_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial F_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial F_2}{\partial x_1}(\mathbf{x}) & \frac{\partial F_2}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial F_2}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(\mathbf{x}) & \frac{\partial F_n}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial F_n}{\partial x_n}(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Let

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad x = (x_1,\ldots,x_n) \mapsto f(x_1,\ldots,x_n).$$

Its **Jacobian** is the $1\times n$ matrix whose entries are the first-order partial derivatives of f. Concretely,

$$J_f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) & \frac{\partial f}{\partial x_2}(x) & \cdots & \frac{\partial f}{\partial x_n}(x) \end{pmatrix} \in \mathbb{R}^{1\times n}.$$

Equivalently, one writes

$$df(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\, dx^i,$$

so that in the basis $\{dx^1,\ldots,dx^n\}$,

$$df(x) = \begin{pmatrix} \partial_1 f(x) & \partial_2 f(x) & \cdots & \partial_n f(x) \end{pmatrix} \begin{pmatrix} dx^1 \\ dx^2 \\ \vdots \\ dx^n \end{pmatrix}.$$

Examples:

- If $n = 2$ and $f(x, y) = x^2y + e^y$, then

$$J_f(x, y) = \begin{pmatrix} \frac{\partial}{\partial x}(x^2y + e^y) & \frac{\partial}{\partial y}(x^2y + e^y) \end{pmatrix} = \begin{pmatrix} 2xy & x^2 + e^y \end{pmatrix}.$$

- If $n = 3$ and $f(x, y, z) = \sin(xy) + z^3$, then

$$J_f(x, y, z) = (f_x, f_y, f_z) = \left(y\cos(xy),\ x\cos(xy),\ 3z^2\right).$$

Deriving the Jacobian Matrix from the Gradient

Let

$$\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \qquad \mathbf{F}(\mathbf{x}) = \left(F_1(\mathbf{x}),\ F_2(\mathbf{x}),\ \ldots,\ F_n(\mathbf{x})\right)^T,$$

where each $F_i : \mathbb{R}^n \to \mathbb{R}$ is a scalar function. We will build the $n \times n$ Jacobian matrix

$$J_F(x) = \frac{\partial(F_1, \ldots, F_n)}{\partial(x_1, \ldots, x_n)},$$

step by step, starting from the familiar gradient of a single scalar function.

1. Gradient of a single scalar function. If $f : \mathbb{R}^n \to \mathbb{R}$, then its **gradient** is the column-vector of partials:

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \frac{\partial f}{\partial x_2}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix} \in \mathbb{R}^n.$$

Equivalently, the differential $df(x)$ is the row-vector $\left(\partial_1 f(x), \ldots, \partial_n f(x)\right) \in \mathbb{R}^{1\times n}$.

2. View each component F_i as a scalar function. Since $F(x) = (F_1(x), \ldots, F_n(x))^T$, we can take the gradient of each F_i:

$$\nabla F_i(x) = \begin{pmatrix} \partial_1 F_i(x) \\ \partial_2 F_i(x) \\ \vdots \\ \partial_n F_i(x) \end{pmatrix},$$

and the corresponding differential (row) is

$$dF_i(x) = \left(\partial_1 F_i(x),\ \partial_2 F_i(x),\ \ldots,\ \partial_n F_i(x)\right) \in \mathbb{R}^{1\times n}.$$

3. Assemble the Jacobian by stacking these row-vectors. By definition, the Jacobian $J_F(x)$ is the matrix whose ith row is $dF_i(x)$. Thus

$$J_F(x) = \begin{pmatrix} dF_1(x) \\ dF_2(x) \\ \vdots \\ dF_n(x) \end{pmatrix} = \begin{pmatrix} \partial_1 F_1(x) & \partial_2 F_1(x) & \cdots & \partial_n F_1(x) \\ \partial_1 F_2(x) & \partial_2 F_2(x) & \cdots & \partial_n F_2(x) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 F_n(x) & \partial_2 F_n(x) & \cdots & \partial_n F_n(x) \end{pmatrix}.$$

4. Interpretation as the linear approximation. For a small increment $h \in \mathbb{R}^n$,

$$F(x+h) = F(x) + J_F(x)\,h + o(\|h\|),$$

so $J_F(x)$ is exactly the matrix representing the derivative (total differential) of the vector field F at x.

Summary: - The gradient ∇f of a scalar f is a single column of partial derivatives. - For a vector field $F = (F_1, \ldots, F_n)$, we take gradients of each component F_i. - Stacking these gradients (as row-vectors) produces the Jacobian matrix J_F.

This construction ensures that $J_F(x)$ captures all first-order variations of the vector field in every coordinate direction.

$$\underbrace{\begin{bmatrix} \dfrac{\partial \mathbf{f}}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix}}_{\text{“row” of partials of the vector field}} = \underbrace{\begin{bmatrix} \nabla^{\mathsf{T}} f_1 \\ \vdots \\ \nabla^{\mathsf{T}} f_m \end{bmatrix}}_{\substack{\text{stacked transposed} \\ \text{gradients of each component}}} = \underbrace{\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}}_{\text{the usual Jacobian matrix of } \mathbf{f}=(f_1,\ldots,f_m)^{\mathsf{T}}.}$$

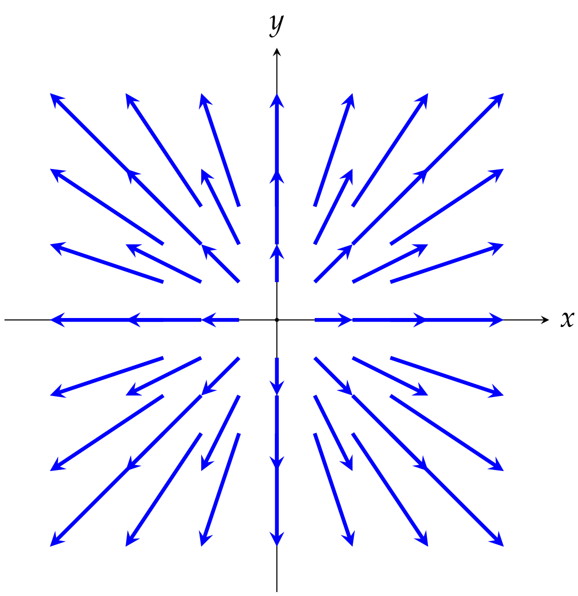
$$\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \qquad \mathbf{F}(\mathbf{x}) = \begin{pmatrix} F_1(\mathbf{x}) \\ F_2(\mathbf{x}) \\ \vdots \\ F_n(\mathbf{x}) \end{pmatrix}, \quad F_i : \mathbb{R}^n \to \mathbb{R} \text{ (scalar functions)}.$$

Then the **Jacobian matrix** $D\mathbf{F}(\mathbf{x})$ is obtained by taking the gradient of each component F_i and stacking them as rows:

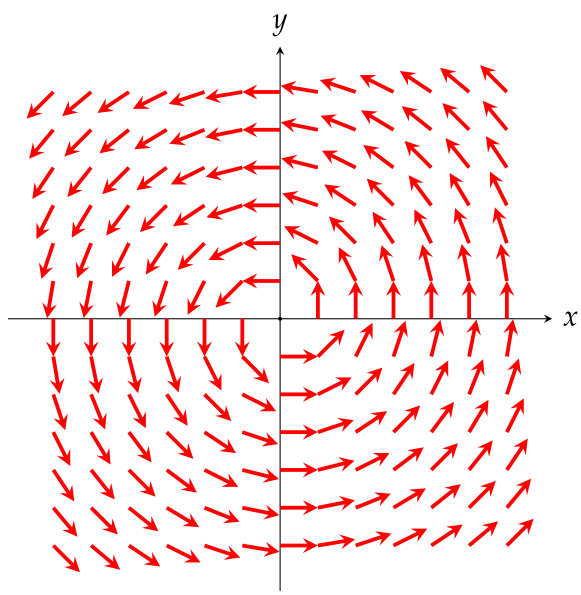
$$D\mathbf{F}(\mathbf{x}) = \begin{pmatrix} \nabla^{\mathsf{T}} F_1(\mathbf{x}) \\ \nabla^{\mathsf{T}} F_2(\mathbf{x}) \\ \vdots \\ \nabla^{\mathsf{T}} F_n(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(\mathbf{x}) & \frac{\partial F_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial F_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial F_2}{\partial x_1}(\mathbf{x}) & \frac{\partial F_2}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial F_2}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(\mathbf{x}) & \frac{\partial F_n}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial F_n}{\partial x_n}(\mathbf{x}) \end{pmatrix}.$$

Each row $\nabla^{\mathsf{T}} F_i(\mathbf{x})$ is the transpose of the gradient of the scalar function F_i.

Example.



The radial field $\mathbf{F} = \langle x, y \rangle = x\mathbf{i} + y\mathbf{j}$



The spin field $\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)^{1/2}$