Theorem 1 (Harmonic $\Rightarrow$ constant on a compact Riemann surface, no Hodge star / no $\partial, \bar{\partial}$ / no "conformal"). *Let X be a compact, connected Riemann surface. If $u \in C^\infty(X, \mathbb{R})$ is harmonic, then u is constant.*

Definition of harmonic (chart-based). A Riemann surface has an atlas of holomorphic charts $\phi : U \to V \subset \mathbb{C}$, $\phi(p) = z = x + iy$. We say u is *harmonic* if in every holomorphic chart,

$$\frac{\partial^2 (u \circ \phi^{-1})}{\partial x^2} + \frac{\partial^2 (u \circ \phi^{-1})}{\partial y^2} = 0 \quad \text{on } V.$$

This is independent of the holomorphic chart: if $w = h(z)$ is a holomorphic coordinate change with $h'(z_0) = a + ib \neq 0$, then at the corresponding point

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = (a^2 + b^2) \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right),$$

so vanishing of the x, y-Laplacian is equivalent to vanishing of the u, v-Laplacian.

Key local identity (plain calculus). Fix one holomorphic chart (x, y) on $U \subset X$. Set

$$P := -u\, u_y, \qquad Q := u\, u_x.$$

A direct computation (product rule) gives

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = u_x^2 + u_y^2 + u\, (u_{xx} + u_{yy}) = |\nabla u|^2 + u\, \Delta u, \qquad (*)$$

where here $\Delta u := u_{xx} + u_{yy}$ in this chart.

Green's theorem with a bump function. Let $\rho \in C_c^\infty(U)$ (compactly supported in U) and define

$$P_\rho := -\rho\, u\, u_y, \qquad Q_\rho := \rho\, u\, u_x.$$

Then

$$\frac{\partial Q_\rho}{\partial x} - \frac{\partial P_\rho}{\partial y} = \rho\big(|\nabla u|^2 + u\, \Delta u\big) + u\, u_x\, \rho_x + u\, u_y\, \rho_y. \qquad (**)$$

Take $U' \Subset U$ with smooth boundary and $\operatorname{supp} \rho \subset U'$. Green's theorem gives

$$\int_{U'} \left(\frac{\partial Q_\rho}{\partial x} - \frac{\partial P_\rho}{\partial y} \right) dx\, dy = \int_{\partial U'} P_\rho\, dx + Q_\rho\, dy = 0$$

because $\rho = 0$ on $\partial U'$.

Globalization via a partition of unity. Cover X by finitely many holomorphic coordinate discs $U_1, \ldots, U_N$. Choose a smooth partition of unity $\{\rho_j\}_{j=1}^N$ with $\operatorname{supp} \rho_j \Subset U_j$ and $\sum_{j=1}^N \rho_j \equiv 1$ on X. Apply the previous step in each U_j and sum:

$$0 = \sum_{j=1}^N \int_{U_j} \left(\frac{\partial Q_{\rho_j}}{\partial x} - \frac{\partial P_{\rho_j}}{\partial y} \right) dx\, dy = \int_X \sum_{j=1}^N \left[\rho_j \big(|\nabla u|^2 + u\, \Delta u \big) + u\, u_x\, \rho_{j,x} + u\, u_y\, \rho_{j,y} \right] dx\, dy.$$

Since $\sum_j \rho_j \equiv 1$ and hence $\sum_j \rho_{j,x} \equiv 0 \equiv \sum_j \rho_{j,y}$ (in overlapping coordinates), this becomes

$$0 \;=\; \int_X \big(|\nabla u|^2 + u\, \Delta u \big)\, dx\, dy.$$

Finish. If u is harmonic, then $\Delta u = 0$ in each chart, so

$$0 = \int_X |\nabla u|^2\, dx\, dy.$$

The integrand is pointwise nonnegative; thus $|\nabla u| \equiv 0$ everywhere, so u is locally constant. By connectedness of X, u is constant. $\qquad\square$

What we used (and nothing more):

- Holomorphic coordinate charts on a Riemann surface.

- Chain rule to note that the Euclidean Laplacian scales by a positive factor under holomorphic coordinate changes, so "$\Delta u = 0$" is chart-independent.

- Plain product rule identity $(*)$.

- Green's theorem in the plane (applied in charts) $+$ a partition of unity to glue the local integrals.