Lecture Notes: From Differentials to Jacobians

Overview

We shall trace the passage from the simplest differential of a function of one variable all the way to the Jacobian matrix of a vector field, using the unifying language of differential forms and matrix notation:

$$\underbrace{f}_{\Omega^0} \xrightarrow{d} \underbrace{df}_{\Omega^1} \longleftrightarrow \underbrace{\nabla f}_{\substack{\text{gradient}\\ \text{vector field}}} \longrightarrow \underbrace{\mathbf{F}}_{(\Omega^0)^m} \xrightarrow{d} \underbrace{d\mathbf{F}}_{\Omega^1 \otimes \mathbb{R}^m} \longleftrightarrow \underbrace{D\mathbf{F}}_{\substack{\text{Jacobian}\\ \text{matrix}}} .$$

Each step is an instance of the exterior derivative d or its reinterpretation under the standard Euclidean metric.

1 Differentials of Scalar Functions

1.1 0-Forms and 1-Forms

Definition 1 (Spaces of Forms).

- *$\Omega^0(\mathbb{R}^n)$ is the space of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ (also called 0-forms).*
- *$\Omega^1(\mathbb{R}^n)$ is the space of smooth 1-forms $\omega = \sum_{i=1}^{n} g_i(x)\, dx^i$.*

1.2 The Exterior Derivative d

Definition 2 (Exterior Derivative on Functions). *For $f \in \Omega^0(\mathbb{R}^n)$, the differential df is the 1-form*

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\, dx^i \ \in \ \Omega^1(\mathbb{R}^n).$$

Example 1 (One-Variable Case). *If $n = 1$ and $f(x) \in \Omega^0(\mathbb{R})$, then*

$$df = f'(x)\, dx,$$

and the Fundamental Theorem of Calculus *is* $\int_a^b df = f(b) - f(a)$.

1.3 Gradient as Metric Dual

Endow $\mathbb{R}^n$ with the standard dot-product. Then each 1-form $\omega = \sum_i g_i\, dx^i$ corresponds uniquely to a vector field $\displaystyle\sum_i g_i \frac{\partial}{\partial x_i}$. In particular, the 1-form df corresponds to the *gradient* vector field

$$\nabla f(x) = \begin{pmatrix} \dfrac{\partial f}{\partial x_1}(x) \\ \dfrac{\partial f}{\partial x_2}(x) \\ \vdots \\ \dfrac{\partial f}{\partial x_n}(x) \end{pmatrix} \;\in\; \mathbb{R}^n.$$

Thus we have the identification

$$df \;\longleftrightarrow\; \nabla f.$$

2 Differentials of Vector-Valued Functions

2.1 Vector Fields as $(\Omega^0)^m$

A smooth vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ is an m-tuple of scalar fields,

$$\mathbf{F}(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_m(x) \end{pmatrix} \;\in\; (\Omega^0(\mathbb{R}^n))^m.$$

2.2 Applying d Componentwise

Definition 3 (Differential of a Vector Field)**.** *The exterior derivative d acts on each component F_i, producing a column of 1-forms:*

$$d\mathbf{F} = \begin{pmatrix} dF_1 \\ dF_2 \\ \vdots \\ dF_m \end{pmatrix} \;\in\; \Omega^1(\mathbb{R}^n) \otimes \mathbb{R}^m.$$

Explicitly,

$$dF_i = \sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j}(x)\, dx^j.$$

2.3 Jacobian Matrix

Choosing the basis $\{dx^1, \ldots, dx^n\}$ for $\Omega^1(\mathbb{R}^n)$ identifies each dF_i with the row vector $\big(\partial_1 F_i, \ldots, \partial_n F_i\big)$. Stacking these rows yields the *Jacobian matrix*:

$$D\mathbf{F}(x) = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1}(x) & \dfrac{\partial F_1}{\partial x_2}(x) & \cdots & \dfrac{\partial F_1}{\partial x_n}(x) \\ \dfrac{\partial F_2}{\partial x_1}(x) & \dfrac{\partial F_2}{\partial x_2}(x) & \cdots & \dfrac{\partial F_2}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial F_m}{\partial x_1}(x) & \dfrac{\partial F_m}{\partial x_2}(x) & \cdots & \dfrac{\partial F_m}{\partial x_n}(x) \end{pmatrix}.$$

Remark 1 (Linear Approximation)**.** *For a small increment $h \in \mathbb{R}^n$,*

$$\mathbf{F}(x+h) = \mathbf{F}(x) \; + \; D\mathbf{F}(x)\, h \; + \; o(\|h\|),$$

so $D\mathbf{F}(x)$ is the total derivative (the best linear approximation) of $\mathbf{F}$ at x.

Putting It All Together

We summarize the full chain of ideas:

$$\underbrace{f}_{\Omega^0} \xrightarrow{d} \underbrace{df}_{\Omega^1} \longleftrightarrow \underbrace{\nabla f}_{\substack{\text{gradient} \\ \text{vector field}}} \quad \longrightarrow \quad \underbrace{\mathbf{F}}_{(\Omega^0)^m} \xrightarrow{d} \underbrace{d\mathbf{F}}_{\Omega^1 \otimes \mathbb{R}^m} \longleftrightarrow \underbrace{D\mathbf{F}}_{\substack{\text{Jacobian} \\ \text{matrix}}}.$$

Key Takeaway: The single operator d applied to 0-forms produces 1-forms. Under the Euclidean metric we identify those 1-forms with gradients (vector fields). When d is applied to each component of a vector field, it produces exactly the rows of the Jacobian matrix, which encodes the full linearization of the vector-valued function.

When Is a Vector Field a Gradient?

Let $U \subset \mathbb{R}^n$ be a region (ideally simply-connected), and let

$$\mathbf{F} : U \longrightarrow \mathbb{R}^n, \qquad \mathbf{F}(x) = \big(F_1(x), \dots, F_n(x)\big).$$

Define the associated 1-form

$$\alpha = F_1\, dx^1 + \cdots + F_n\, dx^n \ \in \ \Omega^1(U).$$

We seek a scalar potential f with

$$\mathbf{F} = \nabla f \quad \Longleftrightarrow \quad \alpha = df.$$

Closed vs. Exact Forms

Definition 4. *A 1-form α is* closed *if $d\alpha = 0$, and* exact *if there exists f with $\alpha = df$.*

On any domain $U \subset \mathbb{R}^n$,

$$d\alpha = 0 \iff \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \quad \text{for all } i, j.$$

In vector-calculus language (for $n = 3$), $d\alpha = 0$ is exactly $\nabla \times \mathbf{F} = 0$.

Poincaré Lemma (Simply–Connected Case)

If U is simply connected, then

$$\big(d\alpha = 0\big) \implies \big(\alpha \text{ is exact}\big).$$

Hence on such a U, the condition

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \quad \forall\, i, j$$

is *necessary and sufficient* for the existence of a scalar potential f. In that case one recovers

$$f(x) = \int_{x_0}^{x} \alpha = \int_{x_0}^{x} \mathbf{F} \cdot d\mathbf{r},$$

and indeed $\nabla f = \mathbf{F}$.

Summary

$$\mathbf{F} = \nabla f \quad \Longleftrightarrow \quad d\alpha = 0 \quad \text{(on a simply-connected domain)}.$$

Equivalently, in coordinates,

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \quad \forall\, i, j.$$

This integrability condition ensures the tight connection between the differential df and the vector field $\mathbf{F}$.