Why We Fix a Lattice with $\Im(\omega_2/\omega_1) > 0$

0. Goal

We explain, in full detail, the standard normalization

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}, \qquad \Im\left(\frac{\omega_2}{\omega_1}\right) > 0,$$

used when defining a complex torus $X = \mathbb{C}/\Lambda$. This single inequality enforces:

- *Nondegeneracy:* ω_1, ω_2 are $\mathbb{R}$-linearly independent.
- *Orientation:* the ordered basis (ω_1, ω_2) has positive signed area.
- *Normalization / moduli:* every torus is represented by a unique parameter $\tau = \omega_2/\omega_1$ *in the upper half-plane* $\mathfrak{H} = \{\tau \in \mathbb{C} : \Im\tau > 0\}$ up to the natural $\mathrm{SL}_2(\mathbb{Z})$-symmetry.

1. Lattices and nondegeneracy

Definition 1 (Lattice in $\mathbb{C}$)**.** A *lattice* is a discrete, rank-2 additive subgroup $\Lambda \subset \mathbb{C}$, i.e.

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

with $\omega_1, \omega_2 \in \mathbb{C}$ that are linearly independent over $\mathbb{R}$.

Proposition 1 (Nondegeneracy via the ratio)**.** *Let $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$. Then ω_1, ω_2 are $\mathbb{R}$-linearly independent (i.e. span a rank-2 lattice) if and only if*

$$\tau := \frac{\omega_2}{\omega_1} \notin \mathbb{R} \iff \Im(\tau) \neq 0.$$

Proof. Write $\omega_j = a_j + ib_j$ with $a_j, b_j \in \mathbb{R}$. R-linear dependence means $\omega_2 = t\,\omega_1$ for some $t \in \mathbb{R}$, equivalently $\tau \in \mathbb{R}$. If $\tau \notin \mathbb{R}$, then writing

$$\omega_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \in \mathbb{R}^2,$$

the matrix $M = [\,\omega_1 \;\; \omega_2\,] = \left(\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right)$ has nonzero determinant (see Lemma 1 below), hence the vectors are independent. $\square$

2. Orientation and the signed area

Given ordered vectors $\omega_1, \omega_2 \in \mathbb{C} \cong \mathbb{R}^2$, the *signed area* of the fundamental parallelogram

$$P = \{s\,\omega_1 + t\,\omega_2 : s, t \in [0,1]\}$$

is the determinant

$$\mathrm{Area}_{\pm}(\omega_1, \omega_2) = \det \begin{pmatrix} \Re\omega_1 & \Re\omega_2 \\ \Im\omega_1 & \Im\omega_2 \end{pmatrix}.$$

Its absolute value is the geometric area; its sign records orientation.

Lemma 1 (Determinant identity)**.** *For $\omega_1 \neq 0$ and $\tau = \omega_2/\omega_1$,*

$$\mathrm{Area}_{\pm}(\omega_1, \omega_2) = |\omega_1|^2\, \Im(\tau).$$

In particular, $\mathrm{sign}\big(\mathrm{Area}_{\pm}\big) = \mathrm{sign}\big(\Im(\tau)\big)$*.*

Proof. Write $\omega_1 = re^{i\theta}$ with $r > 0$. Then

$$\omega_2 = \tau\omega_1 = re^{i\theta}\tau.$$

Rotating by $e^{-i\theta}$ (an area-preserving real linear isomorphism) sends the pair (ω_1, ω_2) to $(r, r\tau)$. The determinant scales by r^2 and becomes

$$\det \begin{pmatrix} r & r\Re\tau \\ 0 & r\Im\tau \end{pmatrix} = r^2\, \Im\tau = |\omega_1|^2\, \Im(\tau).$$

□

Proposition 2 (Orientation normalization)**.** *Replacing (ω_1, ω_2) by (ω_2, ω_1) sends $\tau \mapsto 1/\tau$, which flips the sign of $\Im(\tau)$. Thus, by swapping the two generators if needed, we may assume*

$$\Im(\omega_2/\omega_1) > 0,$$

i.e. the ordered basis has positive orientation and positive signed area.

Remark 1 (What if $\Im(\tau) = 0$?)**.** Then $\tau \in \mathbb{R}$ and Lemma 1 gives zero area; the two vectors lie on one real line, so $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is rank 1, not a lattice in the sense we need.

3. Scaling and the single complex parameter τ

Proposition 3 (Scaling does not change the torus up to isomorphism)**.** *For any $\lambda \in \mathbb{C}^{\times}$, the map $z \mapsto \lambda z$ induces a biholomorphism*

$$\mathbb{C}/\Lambda \xrightarrow{\cong} \mathbb{C}/(\lambda\Lambda), \qquad \Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

Proof. The map is $\mathbb{C} \to \mathbb{C}$, holomorphic, surjective, and $\lambda\Lambda$-periodic; its kernel is exactly Λ. Hence it descends to a holomorphic bijection with holomorphic inverse $z \mapsto \lambda^{-1}z$. □

Normalization by scaling. Taking $\lambda = \omega_1^{-1}$ we may assume $\omega_1 = 1$. Then the lattice is

$$\Lambda = \mathbb{Z} + \mathbb{Z}\tau, \qquad \tau = \omega_2/\omega_1.$$

By the orientation choice above, we further assume $\tau \in \mathfrak{H} = \{\Im\tau > 0\}$.

4. Change of basis and the $\mathrm{SL}_2(\mathbb{Z})$-action

Different *bases* can generate the *same* lattice.

Proposition 4 (Same lattice, new basis)**.** *Let* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$*. Then*

$$(\omega_1', \omega_2') = (a\omega_1 + b\omega_2,\ c\omega_1 + d\omega_2)$$

is a new ordered $\mathbb{Z}$*-basis of the same lattice* Λ*.*

Proof. The matrix is unimodular, so the change of basis is invertible over $\mathbb{Z}$, and $\mathbb{Z}\omega_1' + \mathbb{Z}\omega_2' = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. ☐

Proposition 5 (Möbius action on τ)**.** *If* $\tau = \omega_2/\omega_1$ *and* $(\omega_1', \omega_2') = (a\omega_1 + b\omega_2,\ c\omega_1 + d\omega_2)$ *with* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$*, then*

$$\tau' = \frac{\omega_2'}{\omega_1'} = \frac{c\omega_1 + d\omega_2}{a\omega_1 + b\omega_2} = \frac{c + d\tau}{a + b\tau}.$$

Equivalently, writing the basis as $(1, \tau)$*, the action on* τ *is the Möbius map*

$$\tau \longmapsto \frac{a\tau + b}{c\tau + d}.$$

Proof. Factor out $\omega_1 \neq 0$ to reduce to the pair $(1, \tau)$ and compute directly. ☐

Lemma 2 (Upper half-plane is preserved)**.** *For* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ *and* $\tau \in \mathfrak{H}$*,*

$$\Im\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{\Im(\tau)}{|c\tau + d|^2} > 0.$$

Proof. Compute $\frac{a\tau+b}{c\tau+d} = \frac{(a\tau+b)\overline{(c\tau+d)}}{|c\tau+d|^2}$ and take imaginary parts. The numerator simplifies to $\Im(\tau)$ using $ad - bc = 1$. ☐

Conclusion. Two bases related by $\mathrm{SL}_2(\mathbb{Z})$ define the same lattice; they differ by the Möbius action on τ, which preserves $\mathfrak{H}$. Thus the *moduli* of complex tori is the quotient

$$\mathrm{Tori}(\mathbb{C}) \ \cong\ \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H},$$

and the condition $\Im(\omega_2/\omega_1) > 0$ is exactly "choose the representative τ in the upper half-plane."

5. Putting it together (one line)

After scaling so that $\omega_1 = 1$ and choosing the ordered basis to make the signed area positive, we *fix*

$$\Lambda = \mathbb{Z} + \mathbb{Z}\tau \quad \text{with} \quad \tau \in \mathfrak{H} \ \ (\Im\tau > 0).$$

This simultaneously encodes nondegeneracy, orientation, and normalization modulo the $\mathrm{SL}_2(\mathbb{Z})$-symmetry.

6. Figures

Figure A: Orientation and signed area

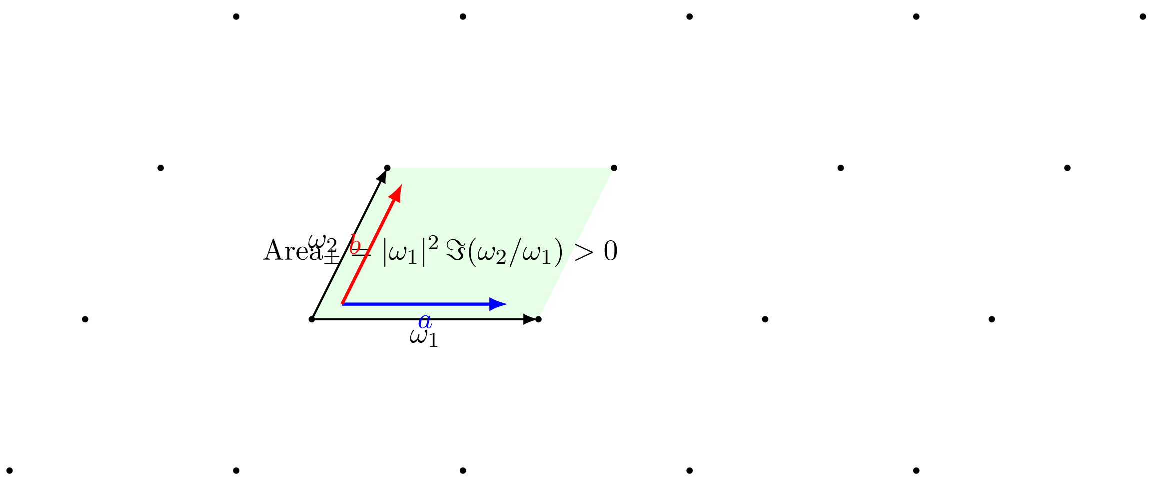
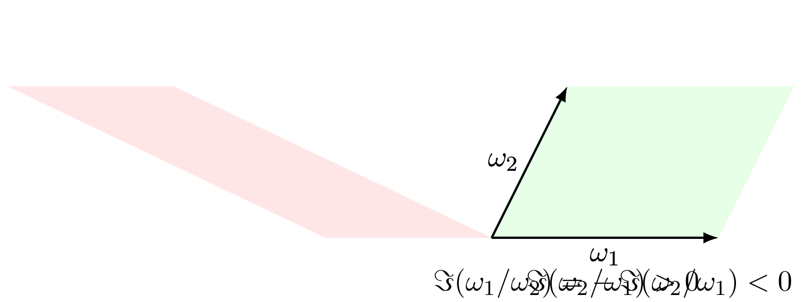


Figure B: Swapping the basis flips the sign of $\Im(\tau)$



7. Quick checks and exercises

- **Check 1 (Nondegeneracy).** Show $\Im(\tau) \neq 0$ iff ω_1, ω_2 are $\mathbb{R}$-independent.
- **Check 2 (Area formula).** Prove Lemma 1 directly by expanding $\det\begin{pmatrix} \Re\omega_1 & \Re\omega_2 \\ \Im\omega_1 & \Im\omega_2 \end{pmatrix}$ and comparing with $|\omega_1|^2 \Im(\omega_2/\omega_1)$.
- **Exercise (Modular action).** Verify $\Im\left(\dfrac{a\tau + b}{c\tau + d}\right) = \dfrac{\Im\tau}{|c\tau + d|^2}$ and conclude that $\mathfrak{H}$ is stable under $\mathrm{SL}_2(\mathbb{Z})$.
- **Exercise (Normalization).** Using scaling, put $\omega_1 = 1$. Using orientation, arrange $\Im(\tau) > 0$. Explain why this is a canonical choice up to $\mathrm{SL}_2(\mathbb{Z})$.

8. One-sentence summary

We require $\Im(\omega_2/\omega_1) > 0$ to ensure the pair (ω_1, ω_2) generates a genuine rank-2 lattice (nondegenerate), gives it a consistent orientation (positive signed area), and yields a unique parameter $\tau \in \mathfrak{H}$ modulo $\mathrm{SL}_2(\mathbb{Z})$, which cleanly parametrizes isomorphism classes of complex tori.