Set Theory II

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We cover the following topics in this note.

- Relations
- Equivalence Relations
- Partitions

Relation

Definition. $\mathcal{R} \subseteq A \times B$

Example. Let $A = \{1, 2\}$ and $B = \{4, 5\}$. Then

$$A \times B = \{(1,4),(1,5),(2,4),(2,5)\}.$$

Here, $R = \{(1,4),(2,5)\} \subseteq A \times B$ be a relation.

Example.

$$(a, b) \in f \iff a\ f\ b \iff b = f(a)$$

Equivalence Relation

Definition. Let R be a relation on A.

(i) (Reflexivity)

(ii) (Symmetry)

(iii) (Transitivity)

Example. Let $A = \{1, 2, 3, 4\}$.

$$\begin{aligned} R = \{&(1,1),(2,2),(3,3),(4,4),\\ &(1,2),(2,1)\} \end{aligned}$$

Proposition 1 *Let X and Y are sets. Let $f : X \to Y$ be*

(1) If $g \circ f$ is one-to-one, then f is one-to-one.

(2) If $g \circ f$ is onto, then g is onto.

Proposition 2 *Let X and Y are sets. Let $f : X \to Y$ be*

(1) If f and g are both one-to-one, then $g \circ f$ is one-to-one.

(2) If f and g are both one-to-one, then $g \circ f$ is onto.

Example. $F = \mathcal{P}(A)$

$$R := \left\{(X \times Y) \in F \times F : \exists \text{a bijection } f : X \to Y\right\}$$

We claim that R is an equivalence relation on $\mathcal{P}(A)$:

(i) (Reflexivity)

(ii) (Symmetry)

(iii) (Transitivity)

Family (Indexed Set)

Definition.

Partitions

Definition.

Fundamental Theorem of Equivalence Relation

Theorem 1

Example. Let $\mathbb{Z}$ be a set of integers. Define

$$[k]_3 := \{n \in \mathbb{Z} : x = 3a + k,\ a \in \mathbb{Z}\} = \left\{n \in \mathbb{Z} : 3a \mid n,\ a \in \mathbb{Z}\right\}$$

References

[1] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 3. 집합론 기초 (c)." YouTube Video, 35:04. Published September 07, 2019. URL: https://www.youtube.com/watch?v=2gM-Vh8CY8I&list=PL4m4z_pFWq2pLwFsWf0KJX_uMNo-jktN5&index=136.