Advanced Calculus III

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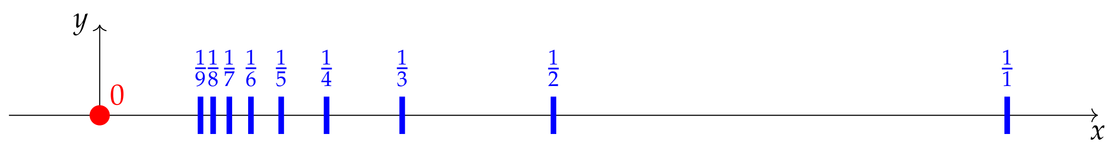
We cover the following topics in this note.

- Limit of a Function
- Continuity of a Function
- TBA

What is 0 for the set $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$?



Note (Open ε-ball)**.** The open ε-ball of x in S is $B_\varepsilon(x) := \{y \in S : d(x, y) < \varepsilon\}$.

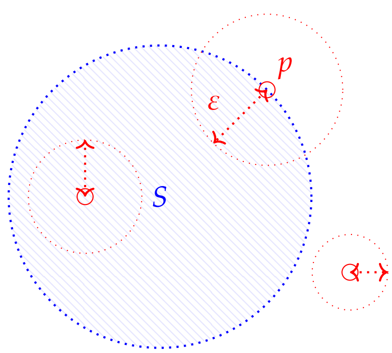
Limit Point (Metric Space)

Definition. Let (X, d) be a metric space. Let $S \subseteq X$. A point $p \in X$ is a **limit point** of S if and only if

$$\forall \varepsilon > 0,\ B_\varepsilon(p) \cap (S \setminus \{p\}) \neq \varnothing.$$

That is,

$$\forall \varepsilon > 0,\ \{x \in S : 0 < d(x, p) < \varepsilon\} \neq \varnothing.$$



Remark. Note that a limit point p may NOT belong to S.

Note (Limit Point (Topology))**.** Let (X, τ) be a topological space. For a subset $S \subseteq X$. A point $p \in X$ is a limit point of S if and only if

$$\forall U \in \tau \text{ with } p \in U,\ U \cap (S \setminus \{p\}) \neq \varnothing.$$

Example. Let $S = (a, b) \subseteq \mathbb{R}$:

- Consider p with $p < a$:

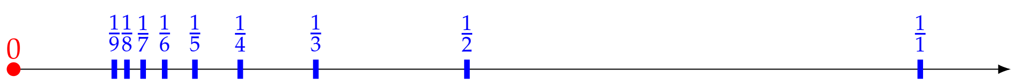
Let $\varepsilon := \frac{a-p}{2} > 0$. Then $B_\varepsilon(p) \cap (S \setminus \{p\}) = \varnothing$. Thus, $p < a$ is NOT a limit point.

- Consider $p = a$:

Let $\varepsilon > 0$. Then $B_\varepsilon(p) \cap (S \setminus \{p\}) \neq \varnothing$. Thus, $p = a$ is a limit point of $S = (a, b)$.

Hence the set of all limit points of (a, b) is $[a, b]$.

Example. Let $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$:



- Consider $p = \frac{1}{n} \in S$. No point of S is a limit point.
- Consider $p = 0$.

Let $\varepsilon > 0$. By Archimedian property,

$$\exists n \in \mathbb{N} \text{ such that } n > \frac{1}{\varepsilon},$$

and so $1/n \in B_\varepsilon(0) \cap S$. Thus, $p = 0$ is a limit point of $S = \{1/n : n \in \mathbb{N}\}$.

Example. Let $S = \mathbb{Q}$.

- Consider $p \in \mathbb{R}$. Let $\varepsilon > 0$. By density of rationals,

$$\exists r \in \mathbb{Q} \text{ such that } p < r < p + \varepsilon.$$

Then $r \in B_\varepsilon(p) \cap S$ with $r \neq p$, i.e., r is a limit points. Thus, all reals are limit points of $\mathbb{Q}$.

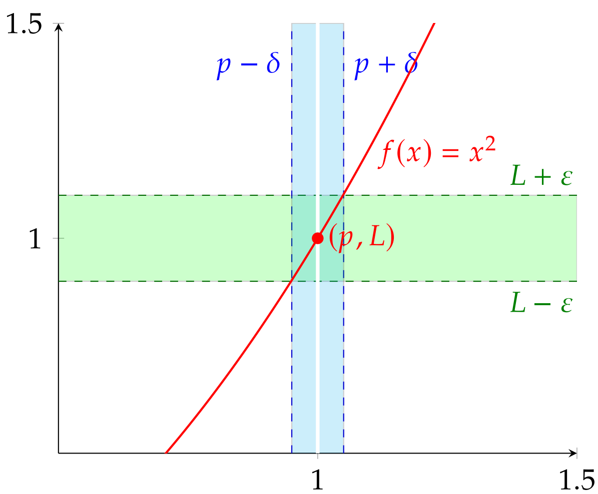
⋆ Limit of a Function ($\varepsilon - \delta$) ⋆

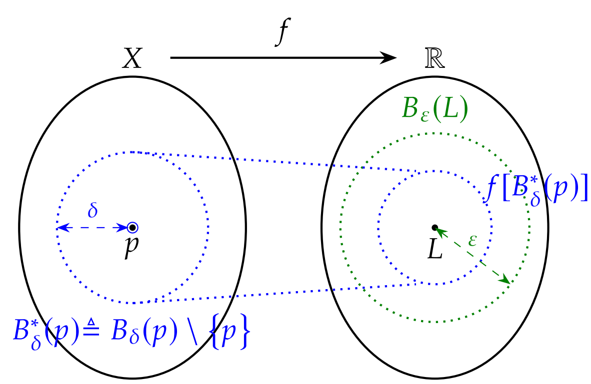
Definition. Let $f : X \longrightarrow \mathbb{R}$ be a function defined on a subset $X (\subseteq \mathbb{R})$ of a metric space, and let $p \in X$ be a limit point of X. We say that $L \in \mathbb{R}$ is the **limit of the function** f **as** x **approaches** p if

$$\boxed{\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x \in X,\ 0 < |x - p| < \delta \implies |f(x) - L| < \varepsilon}.$$

We write

$$\boxed{\lim_{x \to p} f(x) = L}.$$





Remark.

$$\lim_{x \to p} f(x) \neq L \iff \exists \varepsilon > 0 : [\forall \delta > 0 : \exists x \in X : 0 < |x - p| < \delta \text{ but } |f(x) - L| > 0].$$

Continuity of a Function

Definition. Let $f : X \longrightarrow \mathbb{R}$ be a function defined on a subset $X \subseteq \mathbb{R}$ of a metric space, and let $p \in X$. The function f is **continuous at** p if and only if

$$\lim_{x \to p} f(x) = f(p).$$

That is,

$$\forall \varepsilon > 0, \ \exists \delta > 0 \text{ such that } |x - p| < \delta \implies |f(x) - f(p)| < \varepsilon.$$

Remark (Continuity of a Set)**.** The function f is continuous on subset $S \subseteq X$ if it it continuous at every point $p \in S$.

Remark (Continuity in a Topological Space)**.** Let (X, τ_X) and (Y, τ_Y) are topological spaces. $f : X \longrightarrow Y$ is **continuous** if and only if

$$U_Y \in \tau_Y \implies f^{-1}[U_Y] \in \tau_X,$$

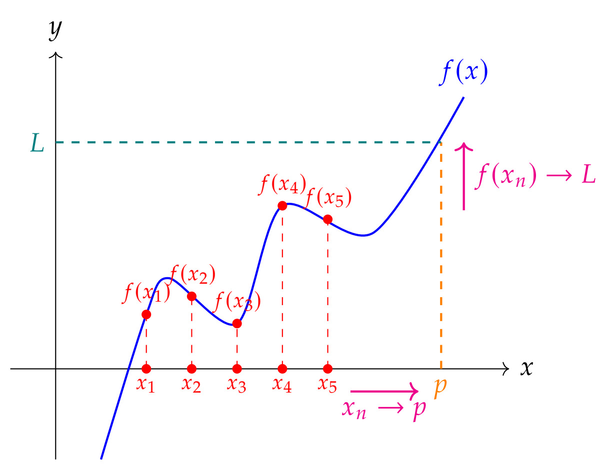
where $f^{-1}[U_Y] = \left\{x \in X : f(x) \in U_Y\right\}$ is the preimage of U_Y under f.

Note. $[p \Rightarrow (q \Rightarrow r)] \equiv [p \Rightarrow (\neg q \vee r)] \equiv [\neg p \vee (\neg q \vee r)] \equiv [\neg(p \wedge q) \vee r] \equiv [(p \wedge q) \Rightarrow r]$.

Limit of Function by Convergent Sequences

Theorem. *Let $f : X \longrightarrow \mathbb{R}$ be a function defined on a subset $\varnothing \neq X \subseteq \mathbb{R}$ of a topological space, and let p is a limit point of X. Then*

$$\lim_{x \to p} f(x) = L \iff \left[\forall \left\{x_n\right\} \subseteq X \setminus \left\{p\right\}, \left(\lim_{n \to \infty} x_n = p \implies \lim_{n \to \infty} f(x_n) = L\right)\right].$$



Proof. ($\Rightarrow$) Suppose that $\lim_{x\to p} f(x) = L$. Let $\{x_n\} \subseteq X \setminus \{p\}$ be a sequence, and let $\lim_{n\to\infty} x_n = p$. We NTS that

$$\lim_{n\to\infty} f(x_n) = L, \quad \text{i.e.,} \quad \forall \varepsilon > 0 : \exists N \in \mathbb{N} : n \geq N \Rightarrow |f(x_n) - L| < \varepsilon.$$

Let $\varepsilon > 0$. Since $\lim_{x\to p} f(x) = L$, we know

$$\exists \delta > 0 \text{ such that } 0 < |x - p| < \delta \implies |f(x) - L| < \varepsilon. \tag{*}$$

Since $\lim_{n\to\infty} x_n = p$, we obtain

$$\exists N \in \mathbb{N} \text{ such that } n \geq N \implies |x_n - p| < \delta.$$

Thus, if $n \geq N$ then,

$$\begin{aligned} |x_n - p| < \delta &\implies 0 < |x_n - p| < \delta \quad \because x_n \neq p \\ &\implies |f(x_n) - L| < \varepsilon \quad \text{by (*)} \end{aligned}$$

Thus, $\lim_{n\to\infty} f(x_n) = L$.

($\Leftarrow$) Let the RHS holds. Assume, for the contradiction, that $\lim_{x\to p} f(x) \neq L$, i.e.,

$$\exists \varepsilon > 0 : \forall \delta > 0 : \exists x_\delta \in X : 0 < |x_\delta - p| < \delta \text{ but } |f(x_\delta) - L| \geq \varepsilon.$$

Take $\delta = 1/n$ for $n \in \mathbb{N}$. Then

$$\exists x_n \in X \text{ such that } 0 < |x_n - p| < \delta \text{ but } |f(x_n) - L| \geq \varepsilon.$$

~~(Axiom of Countable Choice)~~ This means that

$$\forall n \in \mathbb{N} : \exists \{x_n\} \subseteq X \setminus \{p\} \text{ such that } 0 < |x_n - p| < \frac{1}{n} \text{ but } |f(x_n) - L| \geq \varepsilon.$$

By Squeeze Theorem, we have $\lim_{n\to\infty} x_n = p$ since $0 < |x_n - p| < 1/n$. Since the RHS holds, we obtain $\lim_{n\to\infty} f(x_n) = L$. Then, for some $\varepsilon > 0$,

$$\exists N \in \mathbb{N} \text{ such that } n \geq N \implies \left|f(x_n) - L\right| < \varepsilon \lightning.$$

Hence it is proved.

$\square$

Continuity of Function by Convergent Sequences

Corollary. *Let $f : X \to \mathbb{R}$ be a function defined on a subset $\varnothing \neq X \subseteq \mathbb{R}$ of a topological space, and let p is a limit point of X. Then*

$$\lim_{x \to p} f(x) = f(p) \iff \left[\forall \{x_n\} \subseteq X, \left(\lim_{n\to\infty} x_n = p \implies \lim_{n\to\infty} f(x_n) = f(p) \right) \right].$$

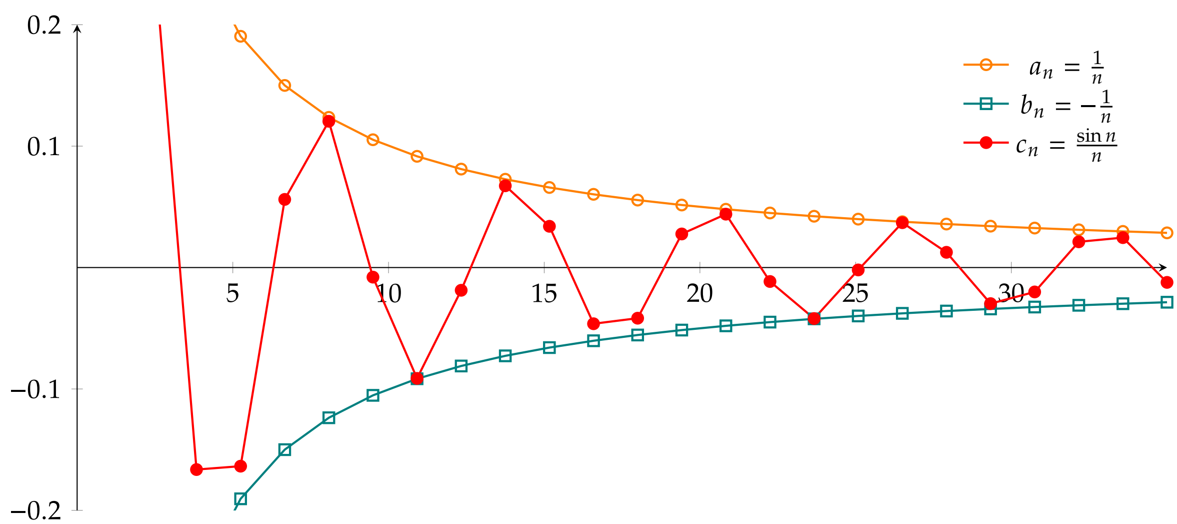
Squeeze Theorem; Sandwich Theorem

Theorem. *Let*

(i) $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$;

(ii) $\exists n_0 \in \mathbb{N}$ *such that* $a_n \leq c_n \leq b_n$ *for all* $n \geq n_0$.

Then $\lim_{n\to\infty} c_n = L$.



Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = L$, we have

$$\exists n_1 \in \mathbb{N} \text{ such that } n \geq n_1 \implies L - \varepsilon < a_n < L + \varepsilon,$$
$$\exists n_2 \in \mathbb{N} \text{ such that } n \geq n_2 \implies L - \varepsilon < b_n < L + \varepsilon.$$

Let $N := \max\{n_0, n_1, n_2\}$. If $n \geq N$ then

$$L - \varepsilon < a_n \leq c_n \leq b_n < L_+\varepsilon,$$

and so $|c_n - L| < \varepsilon$. □

Monotone Convergence Theorem (MCT)

Theorem. *TBA*

Proof. TBA □

Nested Interval Property (NIP)

Theorem. *TBA*

Proof. TBA □

References

[1] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 10. 해석학 개론 (e) 엡실론-델타와 수열의 수렴성” YouTube Video, 25:57. Published September 29, 2019. URL: `https://youtu.be/2Ml3G_Duffk?si=qo-CVgW3Ukd4ADRL`.

[2] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 11. 해석학 개론 (f) MCT and NIP” YouTube Video, 20:17. Published October 1, 2019. URL: `https://youtu.be/YdnBQaY5eDk?si=BNe0Ue4iq2P9Fxsd`.