Homework1: Matrix Representations of Linear Transformations

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Let V and W be vector spaces over a field $\mathbb{F}$ with bases

$$\mathcal{B}_V = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \quad \text{and} \quad \mathcal{B}_W = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$$

and let $T : V \to W$ be a linear transformation whose matrix with respect to $\mathcal{B}_V$ and $\mathcal{B}_W$ is

$$[T]_{\mathcal{B}_V}^{\mathcal{B}_W} = \begin{bmatrix} \vdots & & \vdots \\ T(\mathbf{v}_1) & \cdots & T(\mathbf{v}_n) \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

$$V \xrightarrow{\quad T \quad} W$$

$$\begin{aligned}
\mathbf{v}_1 &\longmapsto T(\mathbf{v}_1) = a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{m1}\mathbf{w}_m = \textstyle\sum_{i=1}^m a_{i1}\mathbf{w}_i \\
\mathbf{v}_2 &\longmapsto T(\mathbf{v}_2) = a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{m2}\mathbf{w}_m = \textstyle\sum_{i=1}^m a_{i2}\mathbf{w}_i \\
&\vdots \qquad\qquad \vdots \\
\mathbf{v}_n &\longmapsto T(\mathbf{v}_n) = a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \cdots + a_{mn}\mathbf{w}_m = \textstyle\sum_{i=1}^m a_{in}\mathbf{w}_i
\end{aligned}$$

Problem 1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

1. Compute $T\left(\begin{bmatrix} 5 & 6 \end{bmatrix}^T\right)$.
2. Verify that A represents T by checking $T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 & 0 \end{bmatrix}^T\right)$ and $T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 & 1 \end{bmatrix}^T\right)$.

Solution

$\square$

Problem 2. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T(x, y, z) = \big(x + 2y - z,\ 3x - y + 4z\big).$$

Find the matrix $[T]_{\mathcal{E}_3}^{\mathcal{E}_2}$ with respect to the standard bases $\mathcal{E}_3$ of $\mathbb{R}^3$ and $\mathcal{E}_2$ of $\mathbb{R}^2$.

Solution

$\square$

Problem 3. Let $V = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$ with basis

$$\mathcal{B}_V = \{1 + x,\ x - 1,\ x^2 + 2\},$$

and $W = \mathbb{R}^2$ with basis

$$\mathcal{B}_W = \{\mathbf{e}_1 = (1, 0),\ \mathbf{e}_2 = (1, 1)\}.$$

Suppose the matrix of $T : V \to W$ with respect to these bases is

$$[T]_{\mathcal{B}_V}^{\mathcal{B}_W} = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \end{bmatrix}.$$

Compute $T\big(-x^2 + 3x + 2\big)$ in standard coordinates of $\mathbb{R}^2$.

Solution

$\square$

Problem 4. Let $V = \mathbb{R}^{2\times 2}$ with basis

$$\mathcal{B}_V = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

and $W = \mathbb{R}^2$ with its standard basis. Define

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \big(a + b,\ c - d\big).$$

Find the matrix $[T]^{\mathcal{E}_2}_{\mathcal{B}_V}$.

Solution

□

Problem 5. Let $V = \left\{ax^3 + bx^2 + cx + d : a, b, c, d \in \mathbb{R}\right\}$ with standard basis $\mathcal{B}_V = \{1, x, x^2, x^3\}$ and $W = \left\{ax^2 + bx + c : a, b, c \in \mathbb{R}\right\}$ with basis $\mathcal{B}_W = \{1 + x,\ x^2,\ 1\}$. Define

$$T(p(x)) = p'(x) + p(1).$$

Find the matrix $[T]_{\mathcal{E}_4}^{\mathcal{B}_W}$.

Solution

$\square$

Problem 6. Choose any nonzero matrix

$$A \in \mathbb{F}^{m \times n}$$

and choose bases

- $\mathcal{B}_V$ for an n-dimensional vector space V over a field $\mathbb{F}$ and
- $\mathcal{B}_W$ for an m-dimensional vector space W over a field $\mathbb{F}$.

Define the linear map $T : V \to W$ by

$$[T]_{\mathcal{B}_V}^{\mathcal{B}_W} = A.$$

1. Pick a vector $\mathbf{v} \in V$ (in coordinates relative to $\mathcal{B}_V$) and compute $T(\mathbf{v})$ in the coordinates of W.
2. Determine $\ker T$, $\operatorname{im} T$, and $\operatorname{rank} T$.

Solution

$\square$

$$
\begin{tikzcd}
V \arrow[r, "\phi_n = \mathrm{Id}_n"] \arrow[d, "T"'] \arrow[d, shift left=2] & \mathbb{F}^n \arrow[l, shift left=2, "\phi_n^{-1} = \mathrm{Id}_n"] \arrow[d, "A = \phi_m \circ T \circ \phi_n^{-1}"'] \\
W \arrow[r, "\phi_m = \mathrm{Id}_m"] \arrow[r, shift left=2] & \mathbb{F}^m
\end{tikzcd}
$$