Set Theory I

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Terminology.

- Set; Collection; Family.
- Tabular (or Roster) Form
$$A = \{0, 2, 4, 8\}.$$
- Set-builder Form
$$A = \{x : x \text{ is even and } x < 10\}.$$

Example.

- $\mathbb{N} = \{1, 2, \dots\}$
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- $\mathbb{Q} = \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\}, \gcd(p, q) = 1\right\}$
- $\mathbb{R} = \{x : x \text{ is a real number}\}$
- $\mathbb{C} = \left\{z : z \text{ is a complex number}\right\}$

Exercise. Show that $\sqrt{2}$ is irrational.

Sol. Assume $\sqrt{2} \in \mathbb{Q}$, i.e., $\exists p, q \in \mathbb{Z}$ such that $\sqrt{2}q = p$, $\quad g \neq 0$ and $\gcd(p, q) = 1$. Then $2q^2 = p^2$. Since p^2 is even $\Rightarrow$ p is even,

$$p = 2k \quad \text{for some } k \in \mathbb{Z}$$

By substituting $p = 2k$ into $2q^2 = p^2$, we have

$$2q^2 = (2k)^2 \implies 2q^2 = 4k^2 \implies q^2 = 2k^2$$

Since q^2 is even $\Rightarrow$ q is even,

$$q = 2m \quad \text{for some } m \in \mathbb{Z}$$

Thus, p and q are both even $\implies$ $\gcd(p, q) \geq 2$, which contradicts the assumption $\gcd(p, q) = 1$. □

Subset and Set Equality

Definition. Let A and B are sets.

- Subset: $B \subseteq A \overset{\text{def}}{\iff} (x \in B \Rightarrow x \in A)$.
- Set Equality:

$$\begin{aligned} A = B &\overset{\text{def}}{\iff} A \subseteq B \wedge B \subseteq A \\ &\iff (x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A). \end{aligned}$$

Power Set

Definition. The **power set** of a set X is the set of all subsets of X.

$$\mathcal{P}(X) = 2^X := \{S : S \subseteq X\}.$$

Cartesian Product

Definition. Let A and B are sets. The **cartesian product** of A and B is the set

$$A \times B = \big\{(a, b) : a \in A \wedge b \in B\big\}.$$

Union, Intersection and Complement

Definition. Let U is a universal set, and let $A, B \subseteq U$.

- The **union** of A and B is the set

$$\boxed{A \cup B := \{x \in U : x \in A \vee x \in B\}}.$$

  Note that $x \in A \cup B \iff x \in A \vee x \in B$.

- The **intersection** of A and B is the set

$$\boxed{A \cap B := \{x \in U : x \in A \wedge x \in B\}}.$$

  Note that $x \in A \cap B \iff x \in A \wedge x \in B$.

- The **complement** of A is the set

$$\boxed{A^C := \big\{x \in U : \neg(x \in A)\big\} = \{x : x \notin A\}}.$$

  Note that $x \in A^C \iff x \notin A$.

Proposition 1 *Let $A, B, C \subseteq U$.*

(1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(3) $(A \cup B)^C = A^C \cap B^C$.

(4) $(A \cap B)^C = A^C \cup B^C$.

Proof. (1) Refer to the Video[1].

(2) Refer to the Video[1].

(3) $(A \cup B)^C = \{x : \neg[x \in A \vee x \in B]\} = \{x : x \notin A \wedge x \notin B\} = A^C \cap B^C$.

(4) $(A \cap B)^C = \{x : \neg[x \in A \wedge x \in B]\} = \{x : x \notin A \vee x \notin B\} = A^C \cup B^C$.

□

Exercise. Let A has n elements. Show that $\mathcal{P}(A)$ has 2^n elements.

Sol.

(pf 1) For each element of A, there are two choices:

1. Include the element in the subset.
2. Exclude the element from the subset.

Since we have two independent choices (include or exclude), the total number of subsets is:

$$\underbrace{2 \times 2 \times \cdots 2}_{n \text{ times}} = 2^n.$$

(pf 2) We use mathematical induction.
(Basic Step) Let $A = \varnothing$ (so $|A| = 0$). Then $\mathcal{P}(A) = \{\varnothing\}$ and so $\left|\mathcal{P}(A)\right| = \left|\{\varnothing\}\right| = 1$.
(Inductive Step) Assume that $\left|\mathcal{P}(A)\right| = 2^k$ where $|A| = k$ for some $k \in \mathbb{Z}_{\geq 0}$. Let $A' = A \cup \{x\}$ where $|A| = k$ and $x \notin A$. That is, $|A'| = k + 1$. Then

$$\mathcal{P}(A') = \mathcal{P}(A) \cup \left\{S \cup \{x\} : S \in \mathcal{P}(A)\right\}.$$

This implies $\left|\mathcal{P}(A')\right| = \left|\mathcal{P}(A)\right| + \left|\mathcal{P}(A)\right|$. Therefore, by assumption, $\left|\mathcal{P}(A')\right| = 2^k + 2^k = 2^{k+1}$.

□

Function

Definition. Let A and B are sets. A relation $f \subseteq A \times B$ is a **function from** A **to** B if

$$\boxed{\forall a \in A,\ \exists! b \in B \text{ such that } (a,b) \in f.}$$

That is, every element of A relates to exactly one element of B.

Remark.

- The **domain** of f is $\mathrm{Dom}(f) = A$.
- The **codomain** of f is $\mathrm{Cdm}(f) = B$.
- The **image** of f is the set

$$\begin{aligned} \mathrm{Img}(f) = f[A] :&= \left\{ b \in B : \exists a \in A \text{ s.t. } (a,b) \in f \right\} \\ &= \left\{ b \in B : \exists a \in A \text{ s.t. } f(a) = b \right\} \\ &= \left\{ b \in B : b = f(a) \text{ for at least one } a \in A \right\}. \end{aligned}$$

Simply we can express it as $f[A] = \left\{ f(a) \in B : a \in A \right\}$. Note that $f[A] \subseteq B = \mathrm{Cdm}(f)$.

Note that

$$b \in f[A] \iff b = f(a) \text{ for some } a \in A.$$

- The **preimage** of f is the set

$$\begin{aligned} \mathrm{Img}^{-1}(f) = f^{-1}[B] :&= \left\{ a \in A : \exists b \in B \text{ s.t. } (a,b) \in f \right\} \\ &= \left\{ a \in A : \exists! b \in B \text{ s.t. } b = f(a) \right\} \text{ by def. of a function} \\ &= \left\{ a \in A : f(a) = b \text{ for exactly one } b \in B \right\}. \end{aligned}$$

"Exactly one" ensures a unique assignment for every element of A, while "at most one" allows no assignment. Simply we can express it as $f^{-1}[B] = \left\{ a \in A : f(a) \in B \right\}$. Note that $f^{-1}[B] \subseteq A = \mathrm{Dom}(f)$.

Note that

$$a \in f^{-1}[B] \iff f(a) \in B.$$

Proposition 2 *Let $f : A \longrightarrow B$ be a function from A to B, and let $A_1, A_2 \subseteq A$.*

(1) $f[A_1 \cup A_2] = f[A_1] \cup f[A_2]$.

(2) $f[A_1 \cap A_2] \subseteq f[A_1] \cap f[A_2]$.

Proof. Recall that

$$b \in f[A] \iff b = f(a) \text{ for some } a \in A.$$

(1) ($\subseteq$) Let $b \in f[A_1 \cup A_2]$. By the definition of the image, $b = f(a)$ for some $a \in A_1 \cup A_2$. Then, either $a \in A_1$ or $a \in A_2$.

(Case 1) $a \in A_1 \Rightarrow f(a) \in f[A_1]$.
(Case 2) $a \in A_2 \Rightarrow f(a) \in f[A_2]$.

Thus, $b = f(a) \in f[A_1] \cup f[A_2]$, and so $f[A_1 \cup A_2] \subseteq f[A_1] \cup f[A_2]$.

($\supseteq$) Let $b \in f[A_1] \cup f[A_2]$. Then either $b \in f[A_1]$ or $b \in f[A_2]$.

(Case 1) $b \in f[A_1] \Rightarrow b = f(a_1)$ for some $a_1 \in A_1$.
(Case 2) $b \in f[A_1] \Rightarrow b = f(a_2)$ for some $a_2 \in A_2$.

That is, $\exists a \in A_1 \cup A_2$ such that $f(a) = b$ and $a \in \{a_1, a_2\}$. Thus, $b \in f[A_1 \cup A_2]$.

(2) Let $b \in f[A_1 \cap A_2]$. By the definition of the image, $b = f(a)$ for some $a \in A_1 \cap A_2$. Since $a \in A_1 \cap A_2$, we have $a \in A_1$ and $a \in A_2$. Then *both* of the following hold:

(i) $a \in A_1 \implies f(a) \in f[A_1]$

(ii) $a \in A_2 \implies f(a) \in f[A_2]$

Therefore, $b = f(a) \in f[A_1] \cap f[A_2]$.

□

Proposition 3 *Let $f : A \longrightarrow B$ be a function from A to B, and let $B_1, B_2 \subseteq B$.*

(1) $f^{-1}[B_1 \cup B_2] = f^{-1}[B_1] \cup f^{-1}[B_2]$.

(2) $f^{-1}[B_1 \cap B_2] = f^{-1}[B_1] \cap f^{-1}[B_2]$.

(3) $f^{-1}[B_1^C] = (f^{-1}[B_1])^C$.

Proof. Recall that

$$a \in f^{-1}[B] \iff f(a) \in B.$$

(1) ($\subseteq$) Let $a \in f^{-1}[B_1 \cup B_2]$. By the definition of the preimage, we have $f(a) \in B_1 \cup B_2$. That is, either $f(a) \in B_1$ or $f(a) \in B_2$.

(Case 1) $f(a) \in B_1 \implies a \in f^{-1}[B_1]$.
(Case 2) $f(a) \in B_2 \implies a \in f^{-1}[B_2]$.

Thus, $a \in f^{-1}[B_1] \cup f[B_2]$.

($\supseteq$) Let $a \in f^{-1}[B_1] \cup f^{-1}[B_2]$. Then either $a \in f^{-1}[B_1]$ or $a \in f^{-1}[B_2]$.

(Case 1) $a \in f^{-1}[B_1] \implies f(a) \in B_1$.
(Case 2) $a \in f^{-1}[B_2] \implies f(a) \in B_2$.

That is, $f(a) \in B_1 \cup B_2$. Thus, $a \in f^{-1}[B_1 \cup B_2]$.

(2) ($\subseteq$) Let $a \in f^{-1}[B_1 \cap B_2]$. By the definition of the preimage, $f(a) \in B_1 \cap B_2$ and so $f(a) \in B_1$ and $f(a) \in B_2$. Then *both* of the following hold:

(i) $f(a) \in B_1 \implies a \in f^{-1}[B_1]$.
(ii) $f(a) \in B_2 \implies a \in f^{-1}[B_2]$.

Thus, $a \in f^{-1}[B_1] \cap f[B_2]$.

($\supseteq$) Let $a \in f^{-1}[B_1] \cap f^{-1}[B_2]$. Then $a \in f^{-1}[B_1]$ and $a \in f^{-1}[B_2]$. Then *both* of the following hold:

(i) $a \in f^{-1}[B_1] \implies f(a) \in B_1$.
(ii) $a \in f^{-1}[B_2] \implies f(a) \in B_2$.

That is, $f(a) \in B_1 \cap B_2$. Thus, $a \in f^{-1}[B_1 \cap B_2]$.

(3) ($\subseteq$) Let $a \in f^{-1}[B_1^C]$. By the definition of the preimage,

$$f(a) \in B_1^C \implies f(a) \notin B_1 \implies a \notin f^{-1}[B_1] \implies a \in (f^{-1}[B_1])^C.$$

($\supseteq$) Let $a \in (f^{-1}[B_1])^C$. By the definition of the preimage,

$$a \notin f^{-1}[B_1] \implies f(a) \notin B_1 \implies f(a) \in B_1^C \implies a \in f^{-1}[B_1^C].$$

□

Proposition 4 *Let $f : A \to B$ be a function from A to B. Let $A_1 \subseteq A$ and $B_1 \subseteq B$.*

(1) $f[f^{-1}[B_1]] \subseteq B_1$.

(2) $A_1 \subseteq f^{-1}[f[A_1]]$.

Proof. Recall that

$$f^{-1}[B_1] := \{a \in A : f(a) \in B_1\}, \qquad f[A_1] := \{f(a) \in B : a \in A_1\},$$
$$f[f^{-1}[B_1]] := \{f(a) \in B : a \in f^{-1}[B_1]\}, \qquad f^{-1}[f[A_1]] := \{a \in A : f(a) \in f[A_1]\}.$$

(1) Let $b \in f[f^{-1}[B_1]]$. By the definition of the image,

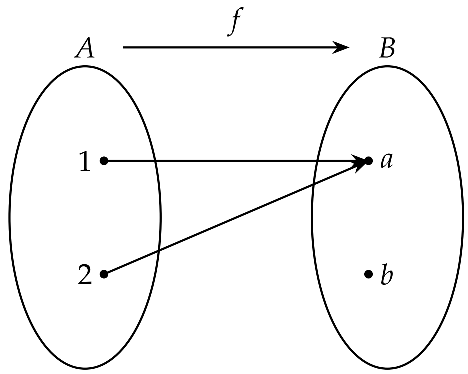
$$\exists a \in f^{-1}[B_1] \text{ such that } b = f(a).$$

From the definition of the preimage, $a \in f^{-1}[B_1] \Rightarrow f(a) \in B_1$. Thus $b = f(a) \in B_1$.

(2) Let $a \in A_1$. By the definition of the image, we know that $f(a) \in f[A_1]$. By the definition of the preimage, $f(a) \in f[A_1] \Rightarrow a \in f^{-1}[f[A_1]]$.

□

Example (Counterexample). Consider a function $f : A \to B$, where $A = \{1, 2\}$ and $B = \{a, b\}$.



(1) Let $B_1 = \{b\} \subseteq B$. Then $f^{-1}[B_1] = \varnothing$ and so

$$f[f^{-1}[B]] = f[\varnothing] = \varnothing \neq \{b\} = B_1.$$

(2) Let $A_1 = \{1\} \subseteq A$. Then $f[A_1] = f[\{1\}] = \{a\}$ and so

$$f^{-1}[f[A_1]] = f^{-1}[\{a\}] = \{1, 2\} \neq \{1\} = A_1.$$

Injection and Surjection

Definition. Let $f : A \to B$ is a function from A to B.

- A function f is **an injection** or **injective** (or **one-to-one**) if and only if

  $$\boxed{\forall a_1, a_2 \in A : [f(a_1) = f(a_2) \implies a_1 = a_2].}$$

  That is, an **injection** is a mapping such that the output uniquely determines its input.

- A function f is **a surjection** or **surjective** (or **onto**) if and only if

  $$\boxed{\forall b \in B : [\exists a \in A \text{ such that } f(a) = b].}$$

  That is, a **surjection** is a mapping such that every element of B is related to by some element of A.

Remark. A function f is **bijective** if and only if f is both injective and surjective.

- f is **a bijection** (or **bijective**).
- f is **one-to-one and onto** (or **a one-to-one correspondence**).

Composition of Functions

Definition. Let $f_1 : A \to B$ and $f_2 : B \to C$ be functions such that $\mathrm{Cdm}(f_1) = B = \mathrm{Dom}(f_2)$. The **composition** $f_2 \circ f_1$ is defined as:

$$(f_2 \circ f_1)(a) := f_2(f_1(a)).$$

for all $a \in A$.

Note (Diagram).

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[dr, dashed, "f_2 \circ f_1"'] & B \arrow[d, "f_2"] \\ & C \end{tikzcd}$$

Figure 1: Diagram of $f_2 \circ f_1$

Note (Illustration).

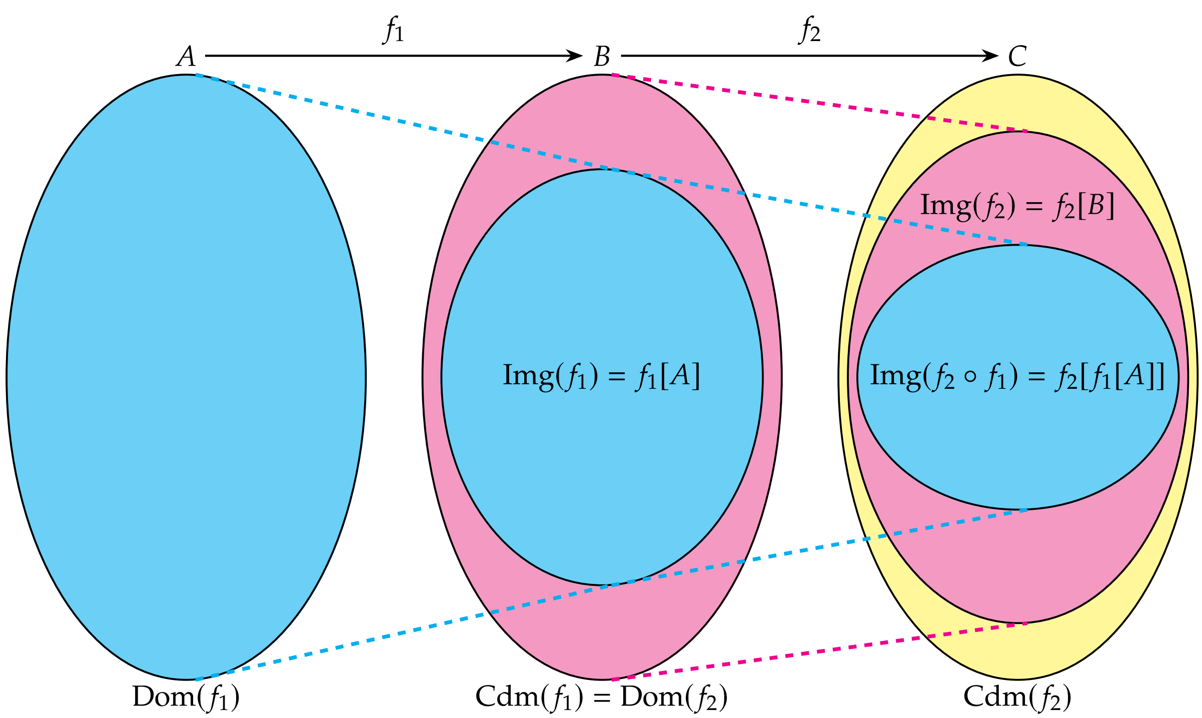


Figure 2: Illustration of $f_2 \circ f_1$

Theorem 1 *Let A and B are sets. Let $f : A \longrightarrow B$ be a function.*

(1) f is one-to-one if and only if there exists the function $g : B \longrightarrow A$ such that $g \circ f = \mathrm{id}_A$.

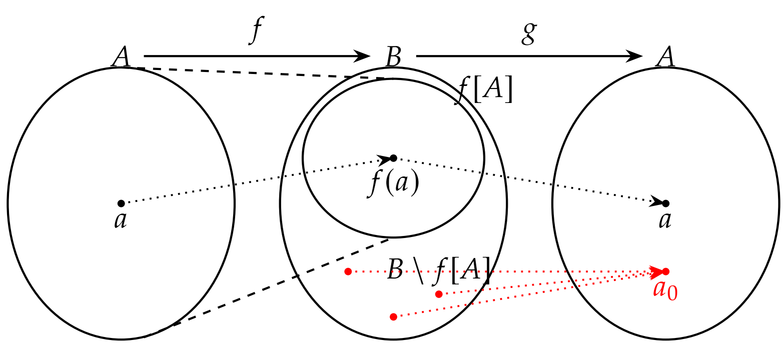
(2) f is onto if and only if there exists the function $g : B \longrightarrow A$ such that $f \circ g = \mathrm{id}_B$.

Remark.

(1) $$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f=\mathrm{id}_A"'] & B \arrow[r, "\exists g"] & A \end{tikzcd}$$

(2) $$\begin{tikzcd} B \arrow[r, "\exists g"] \arrow[rr, bend right, "f\circ g=\mathrm{id}_B"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Proof. (1) ($\Longrightarrow$) Assume that $f : A \longrightarrow B$ is injective. We need to construct a function $g : B \longrightarrow A$ such that $g \circ f = \mathrm{id}_A$.



We define a function $g : B \longrightarrow A$ given by

$$g(b) = \begin{cases} a & \text{if } \exists! a \in A \text{ such that } f(a) = b \\ a_0 & \text{if } b \notin f[A] \end{cases}$$

for all $b \in B$, where $a_0 \in A$ is an arbitrary element of A. Since f is one-to-one, g is well-defined. For any $a \in A$, we have $f(a) \in B$. By the definition of g, we obtain $g(f(a)) = a$. Thus,

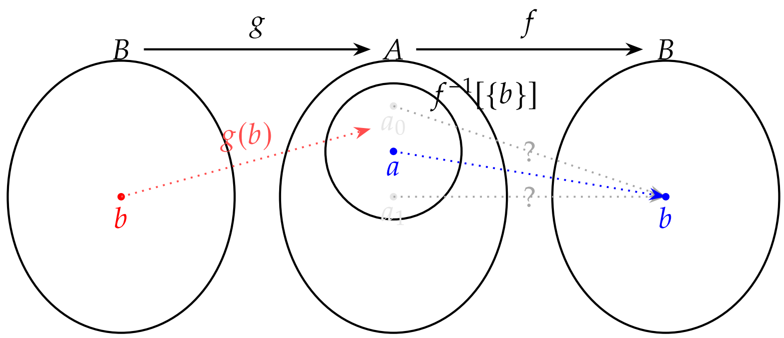
$$(g \circ f)(a) = g(f(a)) = a = \mathrm{id}_A(a)$$

for all $a \in A$.

($\Longleftarrow$) Assume that there exists $g : B \longrightarrow A$ such that $g \circ f = \mathrm{id}_A$. Suppose that $f(a_1) = f(a_2)$ for any $a_1, a_2 \in A$. Then

$$\begin{aligned} f(a_1) = f(a_2) &\implies g(f(a_1)) = g(f(a_2)) \quad \text{by def. of a function} \\ &\implies a_1 = a_2 \quad \text{by assumption } g \circ f = \mathrm{id}_A. \end{aligned}$$

(2) ($\Rightarrow$) Assume $f : A \to B$ is surjective. Then, for every $b \in B$, there exists at least one $a \in A$ such that $f(a) = b$. We need to construct a function $g : B \to A$ such that $f \circ g = \mathrm{id}_B$, i.e., $f(g(b)) = b$ for every $b \in B$.



The **Axiom of Choice**[1] allows us to define $g : B \to A$ given by

$$g(b) = a \in f^{-1}[\{b\}]$$

for each $b \in B$. Thus,

$$\begin{aligned}
(f \circ g)(b) &= f(g(b)) && \text{by def. of composition} \\
&= f(a) && \text{by def. of } g \\
&= b && \text{by assumption} \\
&= \mathrm{id}_B(b)
\end{aligned}$$

for all $b \in B$. That is, $f \circ g = \mathrm{id}_B$. Without the Axiom of Choice, we cannot always guarantee the existence of such a selection function, especially when the sets $f^{-1}[\{b\}]$ are uncountable.

($\Leftarrow$) Assume that there exists $g : B \to A$ such that $f \circ g = \mathrm{id}_B$. Let $b \in B$. Since $f \circ g = \mathrm{id}_B$, we have $f(g(b)) = \mathrm{id}_B(b) = b$. Thus, for every $b \in B$,

$$\exists a = g(b) \in A \quad \text{such that} \quad f(a) = f(g(b)) = b.$$

□

[1] Here, $\mathbb{S} = \left\{ f^{-1}[\{b\}] \subseteq A : b \in B \right\}$ and $\bigcup \mathbb{S} = \bigcup_{b \in B} f^{-1}[\{b\}] = A$. That is, there is a choice function $F : \mathcal{P}(A) \setminus \{\emptyset\} \to A$.

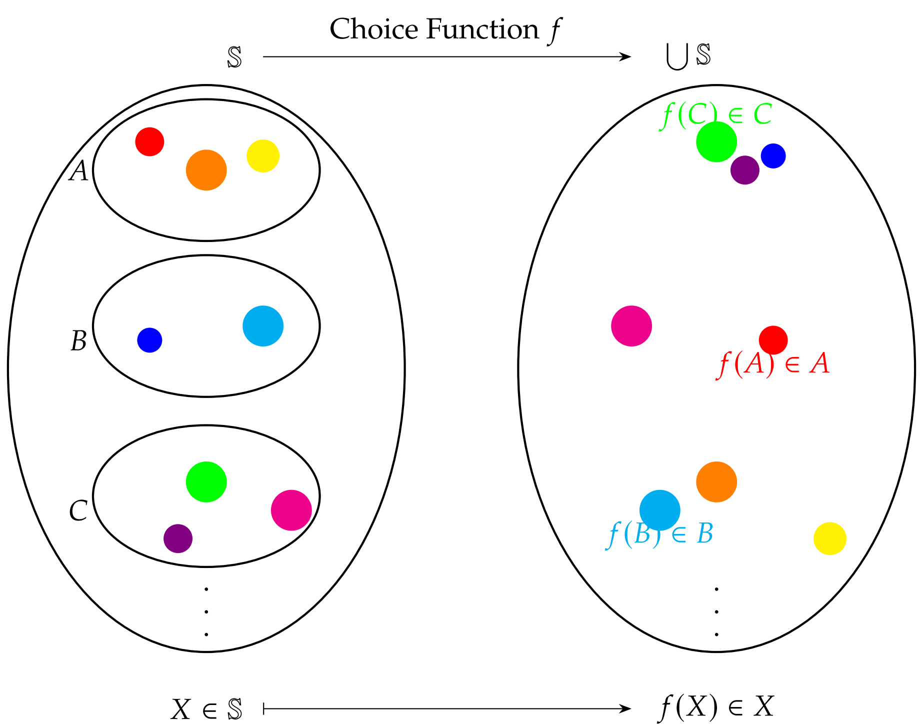
Note (Axiom of Choice). Let $\mathbb{S}$ be a set of non-empty sets.

"It is always possible to construct a choice function
that selects a one element from each member of the set."

Formally,

$$\forall \mathbb{S} : \left[\varnothing \notin \mathbb{S} \implies \exists \left(f : \mathbb{S} \to \bigcup \mathbb{S} \right) \text{ s.t. } \forall X \in \mathbb{S} : [f(X) \in X] \right].$$

For example, let $\mathbb{S} = \{A, B, C, \ldots\}$ and $\bigcup \mathbb{S} = A \cup B \cup C \cup \cdots$.



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[1] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 1. 집합론 기초 (a)." YouTube Video, 30:55. Published September 05, 2019. URL: `https://www.youtube.com/watch?v=9HUk8zays2E&list=PL4m4z_pFWq2pLwFsWf0KJX_uMNo-jktN5&index=132`.

[2] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 2. 집합론 기초 (b)." YouTube Video, 29:06. Published September 05, 2019. URL: `https://www.youtube.com/watch?v=k53Sr9Q9NR8&list=PL4m4z_pFWq2pLwFsWf0KJX_uMNo-jktN5&index=133`.