Advanced Calculus I

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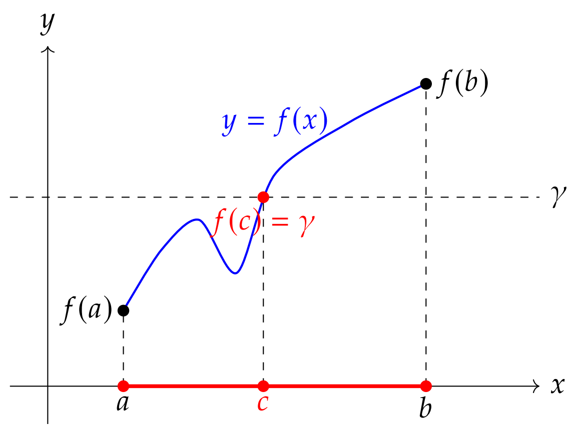
We cover the following topics in this note.

- Least Upper Bound Property (Completeness Axiom)
- Equivalence Relations
- Equivalence Classes

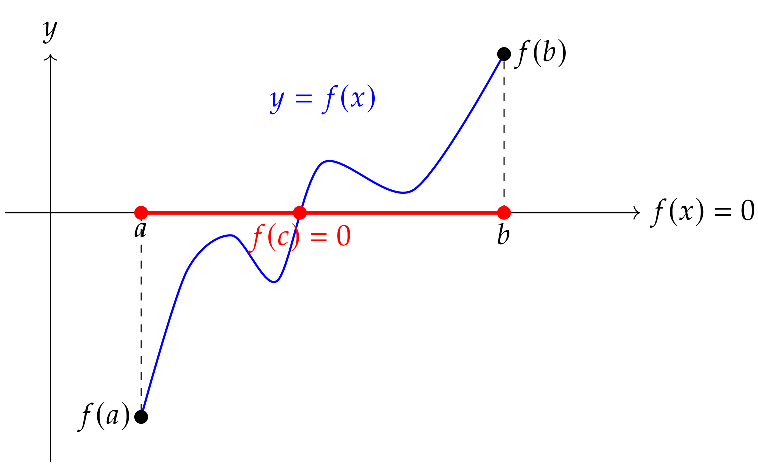
Intermediate Value Theorem

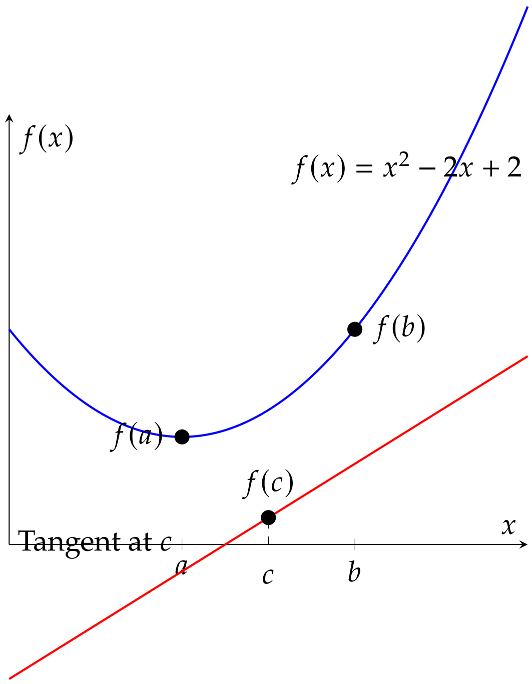
Theorem *Let $[a,b] \subseteq \mathbb{R}$ be a real interval, and let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Let $f(a) < f(b)$. If $\gamma \in \mathbb{R}$ satisfies $f(a) < \gamma < f(b)$, then*

$$\exists c \in (a,b) \text{ such that } f(c) = \gamma.$$



Remark.





Least Upper Bound Property

Bounded Above and Below

Definition. Let $S \subseteq \mathbb{R}$. We say S is bounded above (below) if

$$\exists \beta \in \mathbb{R} \text{ such that } x \leq \beta (x \geq \alpha) \text{ for each } x \in S.$$

Remark 1.

- $S = \varnothing$ is possible.
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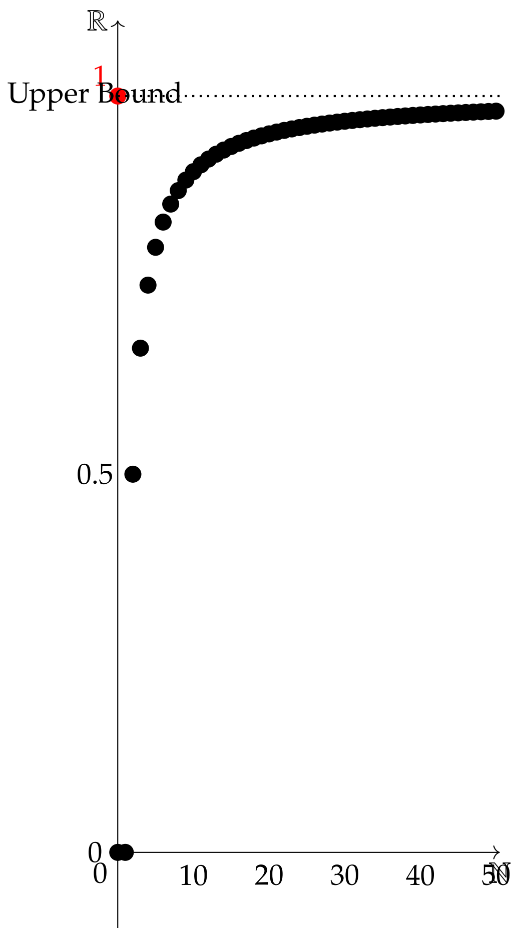
Exercise. Show that $A = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$ has an upper bound and a lower bound.

Exercise. Show that $\mathbb{N}$ has a lower bound, but do not have the upper bound.



Exercise. $B = \{r \in \mathbb{Q} : r > 0 \wedge r^2 < 2\}$. Then B has a lower bound $\alpha = 0$ but B does not have the maximum element. To show it, it is enough to show if $p \in B$, then $\exists q \in B$ s.t. $p < q$. Take any $p \in B$. Take

$$q = p + \frac{2 - p^2}{p + 2} > P.$$

Then $q \in Q$ because Q is a field.

Least Upper Bound Property (Completeness Axiom)

Axiom.

References

[1] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 4. 해석학 개론 (a) 완비성 공리." YouTube Video, 32:20. Published September 09, 2019. URL: `https://www.youtube.com/watch?v=pHIImTBdBRs`.