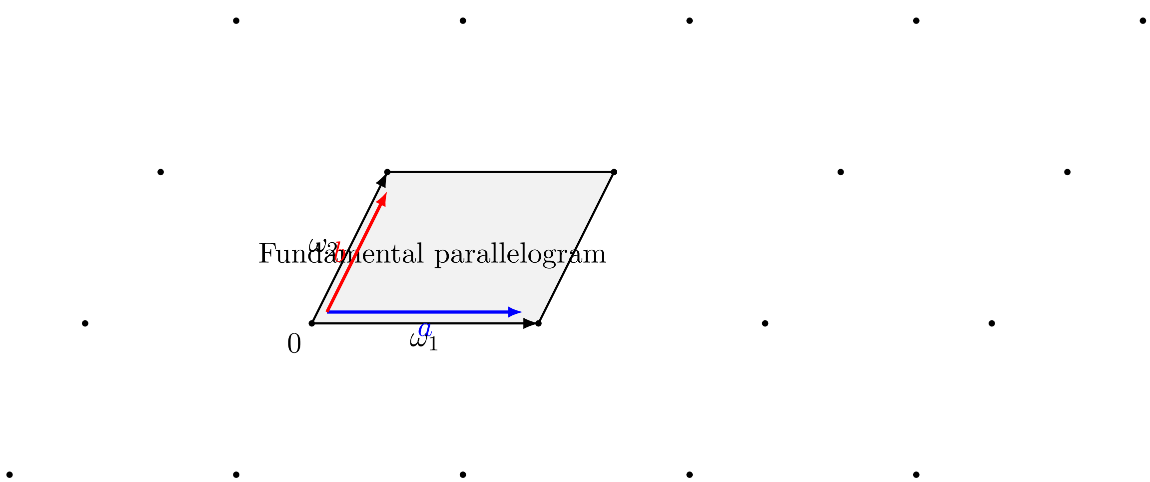
From Complex Tori to Elliptic Curves via the Weierstrass $\wp$-Function

0. Setup

Fix a lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ with $\Im(\omega_2/\omega_1) > 0$. The complex torus is $X = \mathbb{C}/\Lambda$; write the class of $z \in \mathbb{C}$ as $[z] \in X$.

Figure 1. Lattice, periods, and a fundamental parallelogram.



1. Why nonconstant holomorphic functions don't exist on compact Riemann surfaces

Proposition 1 (Maximum principle)**.** *If X is compact and $f : X \to \mathbb{C}$ is holomorphic, then f is constant.*

Sketch. On compact X, $|f|$ attains a maximum. A holomorphic function with an interior maximum is constant. Equivalently, $\Re f$ and $\Im f$ are harmonic and attain maxima/minima, hence are constant. □

Moral. To get nontrivial functions on a compact surface (e.g. a torus), we must allow poles: **meromorphic** functions.

2. Function fields: the warm-up $\mathbb{CP}^1$

On $\mathbb{CP}^1$, the function field is $\mathbb{C}(x)$: every meromorphic function is a rational expression in a single generator x (with a pole at ∞).

3. The basic elliptic function $\wp$

Definition 1 (Weierstrass $\wp$)**.** For $z \in \mathbb{C}$,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

Key properties:

- **Doubly periodic:** $\wp(z+\omega) = \wp(z)$ for all $\omega \in \Lambda$.
- **Parity:** $\wp$ is even, $\wp'$ is odd.
- **Poles:** a double pole at each Λ-point and no other singularities.
- Descends to a meromorphic function $X \to \mathbb{CP}^1$.

4. Invariants and the cubic relation

Define Eisenstein invariants

$$g_2 = 60 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^4}, \qquad g_3 = 140 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^6},$$

and discriminant $\Delta = g_2^3 - 27g_3^2 \neq 0$. Then

$$\big(\wp'(z)\big)^2 = 4\big(\wp(z)\big)^3 - g_2\,\wp(z) - g_3.$$

Set $x = \wp(z)$, $y = \wp'(z)$; we get the affine cubic

$$y^2 = 4x^3 - g_2 x - g_3.$$

5. Function field of the torus

Theorem 1. *Every meromorphic function on X is rational in $\wp$ and $\wp'$:*

$$\mathbb{C}(X) = \mathbb{C}(\wp, \wp') \quad \textit{with} \quad (\wp')^2 = 4\wp^3 - g_2\wp - g_3.$$

Analogy: $\mathbb{CP}^1$ needs one generator x; a genus 1 curve needs two generators x, y with one relation $y^2 = 4x^3 - g_2 x - g_3$.

6. Embedding X as a projective cubic (elliptic curve)

Define

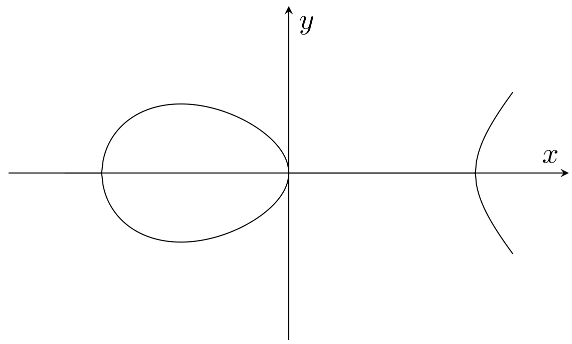
$$\Phi : X \longrightarrow \mathbb{P}^2, \qquad [z] \longmapsto [x : y : 1] = [\wp(z) : \wp'(z) : 1].$$

Its image is the smooth cubic

$$E : \quad y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3 \subset \mathbb{P}^2,$$

and Φ extends over the poles so that $[0 : 1 : 0]$ is the point at infinity. The group law on E (via chord-and-tangent) matches addition on X modulo Λ.

Figure 2. A sample nonsingular cubic $y^2 = 4x^3 - g_2 x - g_3$ with $g_2 = 4$, $g_3 = 0$ (so $\Delta = 64 > 0$).



7. Periods, winding, and motivation

Let $\omega = dz$ on $\mathbb{C}$. Periods over a fundamental parallelogram's boundary generate Λ:

$$\int_a \omega = \omega_1, \qquad \int_b \omega = \omega_2.$$

The logarithmic *winding form* $\frac{dz}{z}$ on $\mathbb{C}^\times$ satisfies $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z} \in \mathbb{Z}$ for loops γ, recording winding number. On the torus, taking suitable derivatives of the (doubly periodic) Green/Zeta potentials leads naturally to $\wp$ with minimal pole order 2.

8. Riemann–Roch viewpoint ("why $\wp$ is minimal")

On a genus 1 curve, $K \sim 0$. For the divisor $2[0]$, $\ell(2[0]) = 2$ with basis $\{1, \wp\}$: $\wp$ is the unique (up to affine change) elliptic function with a *double* pole at 0 and no other poles. For $3[0]$, $\ell(3[0]) = 3$ with basis $\{1, \wp, \wp'\}$; eliminating gives the cubic relation.

9. Minimal formula sheet

$$\wp(-z) = \wp(z), \quad \wp'(z) = -\wp'(-z), \qquad \wp(z+\omega) = \wp(z)\ (\omega \in \Lambda),$$
$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \qquad \Delta = g_2^3 - 27g_3^2 \neq 0,$$
$$\mathbb{C}(X) = \mathbb{C}(\wp, \wp'), \quad [z] \longmapsto [\wp(z) : \wp'(z) : 1] \in E :\ y^2z = 4x^3 - g_2xz^2 - g_3z^3.$$

10. Exercises

Exercise 1. Show any elliptic function has as many zeros as poles (with multiplicity) in a fundamental parallelogram.

Exercise 2. Using the Laurent series of $\wp$ at 0, verify it has a double pole with zero residue.

Exercise 3. Prove that $\wp$ is even and $\wp'$ is odd.

Exercise 4. Derive the cubic relation by comparing Laurent expansions of $\wp$ and $\wp'$.

Exercise 5. Check that the point at infinity $[0 : 1 : 0]$ is the identity for the group law on E.