Lecture Note: Coordinates and Differentials on a Plane Curve

1 Setup

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a C^1–function and define the embedded curve

$$C \;=\; \big\{(x, y) \in \mathbb{R}^2 \mid y = f(x)\big\}.$$

Fix $a \in \mathbb{R}$ and set

$$p \;=\; (a,\; f(a)) \;\in\; C.$$

2 Coordinate System on C

1. Define the global parametrization

$$\Phi\colon \mathbb{R} \;\longrightarrow\; C, \quad \Phi(t) = \big(t, f(t)\big).$$

2. Its inverse is

$$\Phi^{-1}\colon C \;\longrightarrow\; \mathbb{R}, \quad (x, y) \mapsto x.$$

3. Thus t is a *coordinate on* C, and every $p \in C$ is uniquely $p = \Phi(t)$.

3 Coordinate System on T_pC

1. Differentiate Φ at $t = a$:

$$d\Phi_a(1) = \frac{d}{dt}(t, f(t))\bigg|_{t=a} = (1,\; f'(a)) =: \vec{v} \in T_pC.$$

2. By definition,

$$T_pC = \operatorname{span}\{\, (1, f'(a))\} \subset T_p\mathbb{R}^2.$$

3. Every $v \in T_pC$ writes $v = \tau\,(1, f'(a))$ for a unique $\tau \in \mathbb{R}$. Hence τ is a *fiber coordinate* on T_pC.

4 The Functions $x, y\colon C \to \mathbb{R}$

$$\begin{aligned} x &= \pi_1\big|_C\colon C \to \mathbb{R}, \quad (x,y) \mapsto x, \\ y &= \pi_2\big|_C\colon C \to \mathbb{R}, \quad (x,y) \mapsto y. \end{aligned}$$

Restricted to $\Phi(t)$, these give

$$x\big(\Phi(t)\big) = t, \quad y\big(\Phi(t)\big) = f(t).$$

5 The Differentials $dx, dy\colon T_pC \to \mathbb{R}$

1. In the ambient $\mathbb{R}^2$, the differentials act by

$$dx(v_1, v_2) = v_1, \quad dy(v_1, v_2) = v_2.$$

2. On the generator $\vec{v} = (1, f'(a)) \in T_pC$,

$$dx(\vec{v}) = 1, \quad dy(\vec{v}) = f'(a).$$

3. Hence dx, dy extract the x– and y–components of any vector in T_pC.

Abstract Graduate–Level Synthesis

Let $M = C$ be the 1–dimensional submanifold of $\mathbb{R}^2$ defined by $y = f(x)$. The chart $\Phi\colon \mathbb{R} \to M$ endows M with the coordinate t, and its differential $d\Phi$ trivializes the tangent bundle,

$$d\Phi\colon T\mathbb{R} \cong \mathbb{R} \longrightarrow TM, \quad \tau \mapsto \tau\,\Phi'(t).$$

Dually, the ambient projections restrict to

$$dx,\, dy\colon TM \longrightarrow \mathbb{R},$$

providing a coframe. Thus (t) on M and (dx, dy) on TM constitute local frames that distinguish base points from tangent vectors, a paradigm that generalizes to all smooth manifolds.