Riemann; Complex Analysis

- HW1 -

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We cover the following topics in this note.

- Vector Fields
- Line Integrals for Vector Fields
- Surface Integrals for Vector Fields
- TBA

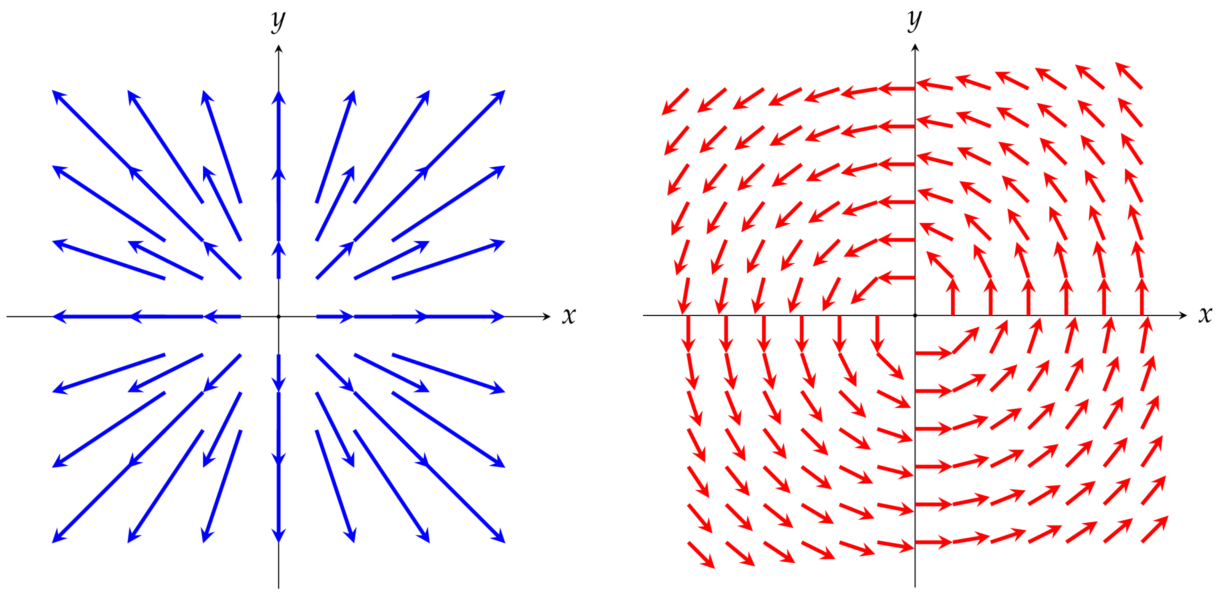
Contents

Scalar Function and Vector Fields

A **scalar function** on $\mathbb{R}^n$ is a real-valued function of an n-tuple; that is,

$$f : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n).$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $f(\mathbf{x}) \in \mathbb{R}$.



The radial field $\mathbf{F} = \langle x, y\rangle = x\mathbf{i} + y\mathbf{j}$ The spin field $\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)^{1/2}$

A **vector field** on $\mathbb{R}^n$ is a function

$$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n, \quad \mathbf{x} \mapsto \mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}), \ldots, F_n(\mathbf{x})),$$

where each component $F_i : \mathbb{R}^n \to \mathbb{R}$ is itself a scalar function.

Line Integrals

Line Integral of Scalar Function over Arc Length

Let

$$\mathbf{r}\colon [a,b] \to \mathbb{R}^m, \quad \mathbf{r}(t) = (x_1(t), \dots, x_m(t))$$

be a C^1 parametrized curve. We give two equivalent definitions of its length.

1. Polygonal Approximation. Take a partition $a = t_0 < t_1 < \cdots < t_n = b$, and set

$$\Delta S_k = \left\|\mathbf{r}(t_k) - \mathbf{r}(t_{k-1})\right\| = \sqrt{\sum_{i=1}^{m} \left[x_i(t_k) - x_i(t_{k-1})\right]^2}.$$

Then

$$L(C) = \lim_{\max|t_k - t_{k-1}| \to 0} \sum_{k=1}^{n} \Delta S_k = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta S_k.$$

For example, if $m = 2$ and $\mathbf{r}(t) = (\cos t, \sin t)$, $t \in [0, \pi/2]$, one finds $\Delta S_k \approx |t_k - t_{k-1}|$ and hence $L = \int_0^{\pi/2} 1\,dt = \frac{\pi}{2}$.

2. Differential-Form Definition. In coordinates, let

$$dx_i = x_i'(t)\,dt, \quad i = 1, \dots, m.$$

Define the line-element (a 1-form)

$$ds = \sqrt{(dx_1)^2 + (dx_2)^2 + \cdots + (dx_m)^2} = \sqrt{\sum_{i=1}^{m} (dx_i)^2}.$$

Then the arc-length is

$$L(C) = \int_C ds = \int_a^b \sqrt{\sum_{i=1}^{m} \left(x_i'(t)\right)^2}\,dt = \int_a^b \left\|\mathbf{r}'(t)\right\| dt.$$

In particular:

- *2-D:* $ds = \sqrt{dx^2 + dy^2}$, so $L = \int_C \sqrt{dx^2 + dy^2}$.
- *3-D:* $ds = \sqrt{dx^2 + dy^2 + dz^2}$, so $L = \int_C \sqrt{dx^2 + dy^2 + dz^2}$.
- *n-D:* $ds = \sqrt{\sum_{i=1}^{n} dx_i^2}$, so $L = \int_C ds$ as above.

Example (Quarter-Circle). On $C: x^2 + y^2 = 1$, $\mathbf{r}(t) = (\cos t, \sin t)$, $t \in [0, \pi/2]$, we have

$$ds = \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = 1 \cdot dt,$$

so

$$L = \int_0^{\pi/2} ds = \int_0^{\pi/2} 1\, dt = \frac{\pi}{2}.$$

On $\mathbb{R}^n$ let $(x^1, \ldots, x^n)$ be Cartesian coordinates. Define the *line element* (a differential 1-form)

$$\boxed{ds \;=\; \sqrt{(dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2} \;=\; \sqrt{\sum_{i=1}^{n} (dx^i)^2}\,.}$$

Then for any smooth curve

$$\mathbf{r}\colon [a,b] \longrightarrow \mathbb{R}^n, \quad t \mapsto \big(x^1(t), \ldots, x^n(t)\big),$$

its *arc-length* is the pullback of ds integrated over $[a,b]$:

$$\boxed{L(C) \;=\; \int_C ds \;=\; \int_a^b \big\|\mathbf{r}'(t)\big\|\,dt \;=\; \int_a^b \sqrt{\sum_{i=1}^{n} \big(x^{i\prime}(t)\big)^2}\;dt.}$$

In particular:

- *2-D:* $ds = \sqrt{dx^2 + dy^2}$, so $L = \int_C \sqrt{dx^2 + dy^2}$.
- *3-D:* $ds = \sqrt{dx^2 + dy^2 + dz^2}$, so $L = \int_C \sqrt{dx^2 + dy^2 + dz^2}$.
- *n-D:* as above with n differentials.

Justification. By definition of the pullback,

$$\mathbf{r}^*(dx^i) = x^{i\prime}(t)\,dt,$$

so

$$\mathbf{r}^*(ds) = \sqrt{\sum_i \big(x^{i\prime}(t)\,dt\big)^2} = \sqrt{\sum_i \big(x^{i\prime}(t)\big)^2}\;dt \;=\; \|\mathbf{r}'(t)\|\,dt.$$

Hence $L = \int_C ds \;=\; \int_a^b \|\mathbf{r}'(t)\|\,dt$, recovering the usual formula.

We begin with the intuitive idea: approximate a smooth curve by joining successive points with straight chords, sum their lengths, and let the mesh of the partition tend to zero. Concretely:

1. Planar Curves (2-D)

Let C be a smooth curve in the plane parametrized by

$$\mathbf{r}(t) = (x(t), y(t)), \quad t \in [a, b],$$

with $x, y \in C^1$. Choose a partition

$$a = t_0 < t_1 < \cdots < t_n = b,$$

and for each k let

$$\Delta S_k = \left\|\mathbf{r}(t_k) - \mathbf{r}(t_{k-1})\right\| = \sqrt{\left[x(t_k) - x(t_{k-1})\right]^2 + \left[y(t_k) - y(t_{k-1})\right]^2}.$$

Then the length of C is

$$L(C) = \lim_{\max|t_k - t_{k-1}| \to 0} \sum_{k=1}^{n} \Delta S_k = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta S_k.$$

Example. For the quarter-circle of radius 1, $\mathbf{r}(t) = (\cos t, \sin t)$, $t \in [0, \pi/2]$, one computes

$$\Delta S_k \approx \sqrt{2 - 2\cos(t_k - t_{k-1})} \;\approx\; |t_k - t_{k-1}| \quad \text{(for small } \Delta t\text{)},$$

so summing and taking limits yields $L = \displaystyle\int_0^{\pi/2} 1\, dt = \frac{\pi}{2}$.

2. Space Curves (3-D)

Let C be a smooth curve in space, $\mathbf{r}(t) = (x(t), y(t), z(t))$, $t \in [a, b]$. With the same partition,

$$\Delta S_k = \sqrt{\left[x(t_k) - x(t_{k-1})\right]^2 + \left[y(t_k) - y(t_{k-1})\right]^2 + \left[z(t_k) - z(t_{k-1})\right]^2},$$

and

$$L(C) = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta S_k = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

Example. For the helix $\mathbf{r}(t) = (\cos t, \sin t, t)$, $t \in [0, 2\pi]$, one finds $\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$, so $L = \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2}$.

3. Curves in $\mathbb{R}^n$

In $\mathbb{R}^n$, a smooth curve $\mathbf{r}(t) = (x_1(t), \ldots, x_n(t))$ admits

$$\Delta S_k = \sqrt{\sum_{i=1}^{n} \big[x_i(t_k) - x_i(t_{k-1})\big]^2},$$

and

$$L(C) = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta S_k = \int_a^b \sqrt{\sum_{i=1}^{n} [x_i'(t)]^2}\, dt.$$

Why this works: each chord $\mathbf{r}(t_k) - \mathbf{r}(t_{k-1})$ approximates $\mathbf{r}'(t)\,\Delta t$, so $\|\Delta \mathbf{r}\| \approx \|\mathbf{r}'(t)\|\Delta t$, and summing gives the Riemann integral of $\|\mathbf{r}'(t)\|$.

Consider the quarter-circle

$$C : x^2 + y^2 = 1, \quad x \geq 0,\ y \geq 0.$$

A natural parametrization is

$$\mathbf{r} : [0, \tfrac{\pi}{2}] \longrightarrow \mathbb{R}^2, \qquad \mathbf{r}(t) = \big(\cos t,\ \sin t\big).$$

Then

$$\mathbf{r}'(t) = \big(-\sin t,\ \cos t\big),$$

and its Euclidean norm is

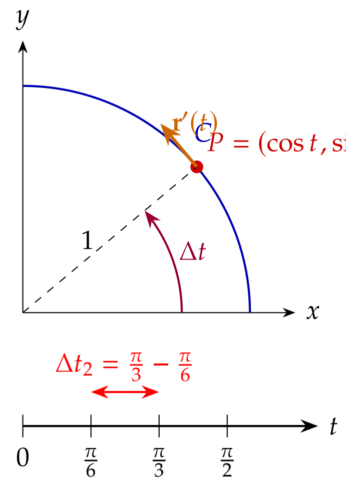
$$\big\|\mathbf{r}'(t)\big\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = \sqrt{\sin^2 t + \cos^2 t} = 1.$$

By the definition of arc length,

$$\text{Length}(C) = \int_0^{\pi/2} \big\|\mathbf{r}'(t)\big\|\, dt = \int_0^{\pi/2} 1\, dt = [t]_0^{\pi/2} = \frac{\pi}{2}.$$

$$\boxed{\text{Hence the quarter-circle has length } \frac{\pi}{2}.}$$

□



Let $I = [a, b] \subset \mathbb{R}$ and let

$$\mathbf{r}\colon I \longrightarrow \mathbb{R}^n, \quad t \mapsto \mathbf{r}(t)$$

be a continuously differentiable (C^1) curve.

Definition 1 (Arc Length)**.** The *arc length* of the curve $\mathbf{r}(t)$ over the interval $[a, b]$ is

$$L[\mathbf{r}] = \sup_{\mathcal{P}} \sum_{i=1}^{N} \left\| \mathbf{r}(t_i) - \mathbf{r}(t_{i-1}) \right\|,$$

where the supremum is taken over all partitions $\mathcal{P}\colon a = t_0 < t_1 < \cdots < t_N = b$ of the interval $[a, b]$, and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.

Remark 1 (Motivation)**.**

- We approximate the smooth curve by a polygonal path joining the points $\mathbf{r}(t_0), \mathbf{r}(t_1), \ldots, \mathbf{r}(t_N)$.
- Each segment has length $\|\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})\|$.
- Taking the supremum over all finer and finer partitions captures the intuitive "length of the curve."
- This definition reduces to the usual distance when $\mathbf{r}(t)$ is a straight line.

Theorem 1. *If* $\mathbf{r} \in C^1([a, b], \mathbb{R}^n)$*, then the above supremum equals the integral*

$$L[\mathbf{r}] = \int_a^b \left\| \mathbf{r}'(t) \right\| dt.$$

Sketch of proof.

1. On each subinterval $[t_{i-1}, t_i]$, the Mean Value Theorem gives $\mathbf{r}(t_i) - \mathbf{r}(t_{i-1}) = \mathbf{r}'(\xi_i)\,(t_i - t_{i-1})$ for some ξ_i.

2. Hence $\|\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})\| = \|\mathbf{r}'(\xi_i)\| (t_i - t_{i-1})$.

3. Summing and passing to the limit as the mesh of the partition goes to zero yields the Riemann integral $\int_a^b \|\mathbf{r}'(t)\| \, dt$.

□

Why this definition?

- It agrees with our geometric intuition of "length" via polygonal approximation.
- It is *reparametrization invariant*: if one changes t–coordinate without reversing direction, the integral of $\|\mathbf{r}'(t)\| \, dt$ remains the same.
- It generalizes the classical arclength formula for graphs $y = y(x)$:

$$L = \int_a^b \sqrt{1 + (y'(x))^2} \, dx.$$

Definition (Line Integral of Scalar Function over Arc Length)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar function, and let C be a piecewise smooth curve in $\mathbb{R}^n$ given by a smooth parameterization:

$$\gamma : [a, b] \to \mathbb{R}^n, \quad t \mapsto \gamma(t) = (x_1(t), x_2(t), \ldots, x_n(t)).$$

Then the *arc length* of the curve C from $t = a$ to $t = b$ is defined by

$$L(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt} \right\| \, dt, \quad \text{where } \left\| \frac{d\gamma}{dt} \right\| = \sqrt{\left(\frac{dx_1}{dt} \right)^2 + \left(\frac{dx_1}{dt} \right)^2 + \cdots + \left(\frac{dx_1}{dt} \right)^2}.$$

The **line integral of the scalar function** f along the curve C with respect to arc length is defined by

$$\int_C f \; ds := \int_a^b f(\gamma(t)) \, \|\gamma'(t)\| \; dt.$$

Here, $ds = \|\gamma'(t)\| \; dt$ is the *infinitesimal arc length*.

Line Integral of Vector Fields

Excellent — this is a key question for understanding how **vector calculus** connects with **differential forms** and notation. Let's walk through how to **rigorously deduce**:

$$\int_C F_1 \, dx + F_2 \, dy \quad \text{from} \quad \int_C \mathbf{F} \cdot d\mathbf{r}$$

for a vector field $\mathbf{F}(x, y) = (F_1(x, y), F_2(x, y))$.

—

Step 1: Parametrize the Curve C
Let $\gamma : [a,b] \to \mathbb{R}^2$ be a smooth parameterization of the curve C, given by:

$$\gamma(t) = (x(t), y(t)), \quad t \in [a,b].$$

Then the **derivative of the curve** is:

$$\gamma'(t) = (x'(t), y'(t)),$$

and the **differential displacement** is:

$$d\mathbf{r} = \gamma'(t)\,dt = (dx, dy) = (x'(t)\,dt, y'(t)\,dt).$$

—

Step 2: Write the Line Integral of $\mathbf{F}$
The **vector line integral** is defined as:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\,dt = \int_a^b \left[F_1(x(t), y(t))\,x'(t) + F_2(x(t), y(t))\,y'(t)\right] dt.$$

—

Step 3: Recognize the Total Differentials
From calculus, we know:

$$dx = x'(t)\,dt, \quad dy = y'(t)\,dt.$$

Substitute into the integral:

$$\int_a^b F_1(x(t), y(t))\,dx + F_2(x(t), y(t))\,dy.$$

So we have:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b F_1(x(t), y(t))\,dx + F_2(x(t), y(t))\,dy = \int_C F_1\,dx + F_2\,dy.$$

—

Final Deduction
Thus:

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C F_1\,dx + F_2\,dy}$$

where: - $\mathbf{F} = (F_1, F_2)$, - $d\mathbf{r} = (dx, dy)$, - The dot product $\mathbf{F} \cdot d\mathbf{r} = F_1\,dx + F_2\,dy$.

—

Interpretation
- The vector line integral becomes an integral of a **1-form**:

$$\mathbf{F} \cdot d\mathbf{r} \quad \leftrightarrow \quad F_1\,dx + F_2\,dy.$$

- The dot product turns into a sum of **components times differentials**.

—

Would you like to go one step further and express this in terms of pullbacks or show how it generalizes to $\mathbb{R}^3$?

Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{F} : U \to \mathbb{R}^n$ a continuous vector field. Suppose $C \subset U$ is a smooth curve parametrized by

$$\mathbf{r} \colon [a,b] \longrightarrow \mathbb{R}^n, \quad t \mapsto \mathbf{r}(t),$$

with nonzero velocity $\mathbf{r}'(t)$. Then the *line integral* of $\mathbf{F}$ along C is defined by

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\, dt = \int_a^b \sum_{i=1}^n F_i\big(\mathbf{r}(t)\big)\, x_i'(t)\, dt,}$$

where $\mathbf{r}(t) = (x_1(t), \ldots, x_n(t))$ and $\mathbf{F} = (F_1, \ldots, F_n)$.

This integral "accumulates" at each infinitesimal step dt the projection of $\mathbf{F}$ onto the tangent vector $\mathbf{r}'(t)$, yielding a single real number that captures the *circulation* or *work* of $\mathbf{F}$ along C.

Example. Take $n = 2$ and $\mathbf{F}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ on $U = \mathbb{R}^2 \setminus \{(0,0)\}$. Let C be the unit circle $x^2 + y^2 = 1$, counterclockwise. Parametrize $\mathbf{r}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Then

$$\mathbf{r}'(t) = (-\sin t, \cos t), \qquad \mathbf{F}\big(\mathbf{r}(t)\big) = \big(-\sin t, \cos t\big),$$

so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \big(-\sin t, \cos t\big) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} \big(\sin^2 t + \cos^2 t\big)\, dt = 2\pi.$$

Thus the total circulation (or "work") of $\mathbf{F}$ around the unit circle is 2π.

Problem #1 (Line Integral around Unit Circle). Let $C \subset \mathbb{R}^2$ be the unit circle defined by

$$C : x^2 + y^2 = 1,$$

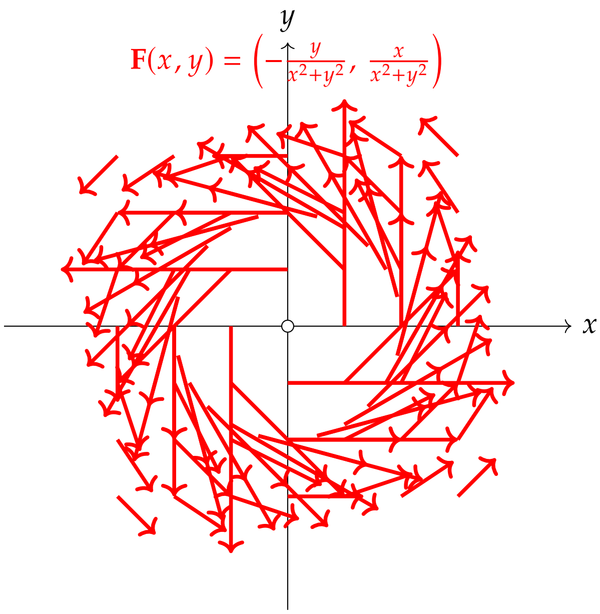
traversed in the *counterclockwise direction*. Let the vector field $\mathbf{F} : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ be defined by

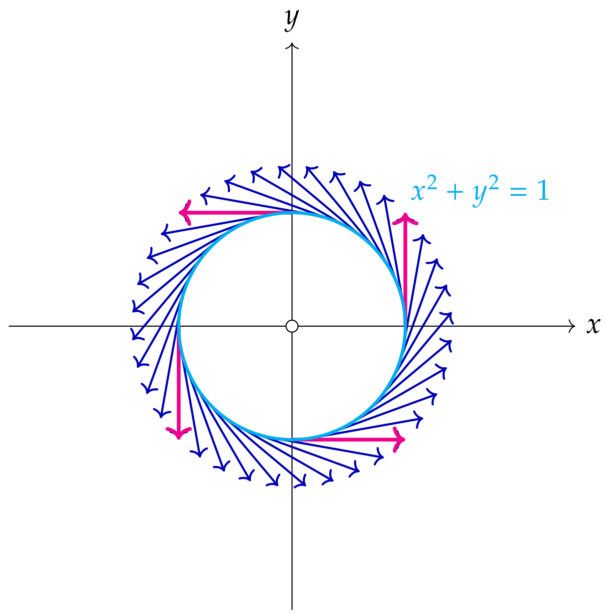
$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right).$$

Evaluate the *line integral* of $\mathbf{F}$ along C:

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

Sol.





Consider the vector field:

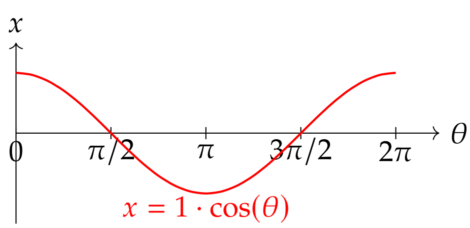
$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right),$$

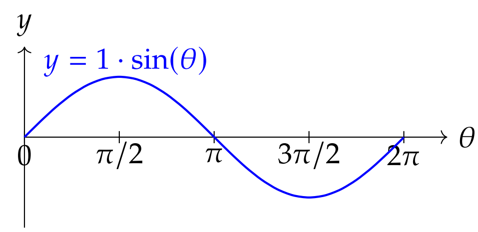
and the curve C is the unit circle $x^2+y^2=1$, traversed counterclockwise.

Step 1. (Parametrization) Define a function

$$\begin{array}{rcccl} \gamma & : & [0,2\pi] & \longrightarrow & \left\{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\right\} \\ & & \theta & \longmapsto & \gamma(\theta) = (\cos\theta, \sin\theta) \end{array}.$$

Here, $\frac{d\gamma}{d\theta} = (-\sin\theta, \cos\theta)$.





Step 2. (Evaluate $\mathbf{F}(\gamma(\theta))$ and the dot product) We have

$$\mathbf{F}(\gamma(\theta)) = \mathbf{F}(\cos\theta, \sin\theta) \overset{\sin^2\theta+\cos^2\theta=1}{=} \left(\frac{-\sin\theta}{1}, \frac{\cos\theta}{1}\right) = (-\sin\theta, \cos\theta).$$

and

$$\mathbf{F}(\gamma(\theta))\cdot\frac{d\gamma}{d\theta} = (-\sin\theta)(-\sin\theta) + (\cos\theta)(\cos\theta) = \sin^2\theta + \cos^2\theta = 1.$$

Step 3. (Integral)

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\gamma(\theta)) \cdot \frac{d\gamma}{d\theta}\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi.$$

□

A Differential Geometry