# Concrete, calculus-style examples of $\mathcal{M}(X)$ for $X = \mathbb{CP}^1$ and $X = \mathbb{C}/\Lambda$

## 1 Case 1: $X = \mathbb{CP}^1$ (Riemann sphere)

### 1.1 Setup: coordinate and function field

View $\mathbb{CP}^1$ as the Riemann sphere:

$$\mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}.$$

Use the standard affine coordinate

$$z = \frac{z_0}{z_1}$$

on the chart $U_1 = \{[z_0 : z_1] \mid z_1 \neq 0\} \cong \mathbb{C}$. The point at infinity is

$$\infty = [1 : 0].$$

A meromorphic function on $\mathbb{CP}^1$ is the same as a rational function in $z$, i.e.

$$\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z).$$

We will *explicitly* analyze one such function and its differential.

### 1.2 A concrete meromorphic function $f$ on $\mathbb{CP}^1$

Let

$$f(z) = \frac{(z-1)^2}{z(z-2)}.$$

**Step 1: Zeros and poles on $\mathbb{C}$**

- Zeros (where numerator vanishes):

$$(z-1)^2 = 0 \quad \Rightarrow \quad z = 1 \quad \text{(double zero)}.$$

  So $\mathrm{ord}_{z=1}(f) = +2$.

- Poles (where denominator vanishes):

$$z(z-2) = 0 \quad \Rightarrow \quad z = 0,\ z = 2.$$

  We check they are simple poles:

  Near $z = 0$: write $z$ as local coordinate. Then

$$f(z) = \frac{(z-1)^2}{z(z-2)} = \frac{z^2 - 2z + 1}{z(z-2)} \sim \frac{1}{z(-2)} = -\frac{1}{2z} \quad \text{as } z \to 0.$$

  So

$$\mathrm{ord}_{z=0}(f) = -1.$$

  Near $z = 2$: set $u = z - 2$, so $z = 2 + u$. Then

$$f(z) = f(2+u) = \frac{(2+u-1)^2}{(2+u)(2+u-2)} = \frac{(1+u)^2}{(2+u)u} \sim \frac{1}{2u} \quad \text{as } u \to 0.$$

  So

$$\mathrm{ord}_{z=2}(f) = -1.$$

So on the finite plane:

$$\mathrm{ord}_1(f) = +2, \quad \mathrm{ord}_0(f) = -1, \quad \mathrm{ord}_2(f) = -1.$$

**Step 2: Behavior at infinity via differential-calculus change of variable**

To study $z = \infty$, use $w = 1/z$ as local coordinate near $\infty$.

Then $z = 1/w$, and

$$f(z) = f\Big(\frac{1}{w}\Big).$$

Compute explicitly:

$$f\Big(\frac{1}{w}\Big) = \frac{(1/w - 1)^2}{(1/w)(1/w - 2)} = \frac{(1-w)^2}{\frac{1}{w^2}(1-2w)} = (1-w)^2 \cdot \frac{w^2}{1-2w}.$$

Now expand near $w = 0$. First,

$$(1 - w)^2 = 1 - 2w + w^2,$$

and

$$\frac{1}{1 - 2w} = 1 + 2w + 4w^2 + O(w^3).$$

So

$$f\left(\frac{1}{w}\right) = w^2\,(1 - 2w + w^2)\,(1 + 2w + 4w^2 + O(w^3)).$$

Multiply out just enough terms to see the leading behavior:

$$(1 - 2w + w^2)(1 + 2w + 4w^2) = 1 + (2w - 2w) + (\cdots) = 1 + O(w^2),$$

so overall

$$f\left(\frac{1}{w}\right) = w^2\,(1 + O(w^2)) = w^2 + O(w^4).$$

Thus near $w = 0$, $f(1/w)$ has a *zero of order 2* as a function of $w$. But remember: the order at $\infty$ as a point of $\mathbb{CP}^1$ is the order of this function viewed on the sphere. More directly: in terms of the original coordinate $z$, we see that as $|z| \to \infty$,

$$f(z) = 1 + O\left(\frac{1}{z^2}\right),$$

so $f$ is holomorphic and nonzero at $\infty$. Hence

$$\operatorname{ord}_\infty(f) = 0.$$

**Step 3: The divisor of $f$**

The divisor of $f$ on $\mathbb{CP}^1$ is

$$\operatorname{Div}(f) = 2 \cdot (1) - (0) - (2) + 0 \cdot (\infty).$$

Sum of coefficients:

$$2 - 1 - 1 + 0 = 0,$$

which matches the general fact that the sum of orders of a meromorphic function on a compact Riemann surface is zero.

## 1.3 Differential: computing $df = f'(z)\,dz$ in detail

Now do calculus: compute the derivative.

**Step 1: Simplify $f$ via partial fractions**

We try to write

$$f(z) = A + \frac{B}{z} + \frac{C}{z-2}.$$

Compute directly:

$$\frac{(z-1)^2}{z(z-2)} = \frac{z^2 - 2z + 1}{z(z-2)}.$$

We look for $A, B, C$ such that

$$\frac{z^2 - 2z + 1}{z(z-2)} = A + \frac{B}{z} + \frac{C}{z-2}.$$

Multiply both sides by $z(z-2)$:

$$z^2 - 2z + 1 = Az(z-2) + B(z-2) + Cz.$$

Expand the right-hand side:

$$Az(z-2) = A(z^2 - 2z) = Az^2 - 2Az,$$

$$B(z-2) = Bz - 2B,$$

$$Cz = Cz.$$

So

$$Az^2 - 2Az + Bz - 2B + Cz = Az^2 + (-2A + B + C)z - 2B.$$

Equate coefficients with the left side $z^2 - 2z + 1$:

$$\begin{cases} A = 1, \\ -2A + B + C = -2, \\ -2B = 1. \end{cases}$$

From $-2B = 1$, we get $B = -\frac{1}{2}$. Then

$$-2A+B+C = -2 \quad \Rightarrow \quad -2(1) - \frac{1}{2} + C = -2 \quad \Rightarrow \quad -2.5 + C = -2 \quad \Rightarrow \quad C = \frac{1}{2}.$$

So

$$f(z) = 1 - \frac{1}{2z} + \frac{1}{2(z-2)}.$$

**Step 2: Differentiate term by term**

Now
$$f(z) = 1 - \frac{1}{2z} + \frac{1}{2(z-2)}.$$
Differentiate:
$$f'(z) = 0 + \frac{1}{2z^2} - \frac{1}{2(z-2)^2}.$$
So the meromorphic 1-form
$$df = f'(z)\,dz = \left(\frac{1}{2z^2} - \frac{1}{2(z-2)^2}\right) dz.$$

**Poles of $df$.** Clearly:
$$df \text{ has poles of order 2 at } z = 0 \text{ and } z = 2.$$
No simple pole terms appear in the Laurent expansions; hence all residues are 0.

**Check residues explicitly via Laurent series.** Near $z = 0$:
$$\frac{1}{2z^2} - \frac{1}{2(z-2)^2} = \frac{1}{2z^2} - \frac{1}{2(4-4z+z^2)} = \frac{1}{2z^2} - \frac{1}{8} \cdot \frac{1}{1-z+z^2/4}.$$
Expand $\frac{1}{1-z+z^2/4}$ as a power series in $z$ (no negative powers), so near $z = 0$ this second term has no negative-power part. Thus the only negative-power part is $\frac{1}{2z^2}$, which has no $(z-0)^{-1}$ term. So $\mathrm{Res}_{z=0}(df) = 0$.

Similarly near $z = 2$, set $u = z - 2$. Then
$$df = \left(\frac{1}{2(2+u)^2} - \frac{1}{2u^2}\right) d(2+u) = \left(\frac{1}{8} \cdot \frac{1}{(1+u/2)^2} - \frac{1}{2u^2}\right) du.$$
Again the first term has only nonnegative powers in $u$, the second has $u^{-2}$ but no $u^{-1}$. So $\mathrm{Res}_{z=2}(df) = 0$.

At $\infty$, we could check via $w = 1/z$. A general fact: for any meromorphic function $f$ on a compact Riemann surface,
$$\sum_p \mathrm{Res}_p(df) = 0.$$
Since the residues at 0 and 2 are zero, the residue at $\infty$ is also zero.

### 1.4 Moral for $\mathbb{CP}^1$: function field via differential forms

In general, if $f$ is *any* meromorphic function on $\mathbb{CP}^1$, then

- $df$ is a meromorphic 1-form whose residues all vanish.
- By analyzing the principal parts and using Liouville, one shows that $f$ must be a rational function in the coordinate $z$.

Thus the function field is

$$\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z).$$

Our explicit $f$ shows concretely how poles/zeros and the differential look, and how partial fractions naturally appear from calculus.

## 2 Case 2: $X = \mathbb{C}/\Lambda$ (complex torus)

Now we move to a more subtle example and use differential forms very concretely.

### 2.1 The torus and the basic holomorphic 1-form

Let $\Lambda \subset \mathbb{C}$ be a lattice:

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2, \quad \Im\left(\frac{\omega_2}{\omega_1}\right) > 0.$$

Define the complex torus

$$X = \mathbb{C}/\Lambda.$$

The projection $\pi : \mathbb{C} \to X$ is holomorphic and locally biholomorphic. The 1-form $dz$ on $\mathbb{C}$ is invariant under translations by $\Lambda$, so it descends to a global holomorphic 1-form on $X$. In fact,

$$H^0(X, \Omega^1_X) = \mathbb{C} \cdot dz,$$

i.e. there is a one-dimensional space of holomorphic 1-forms.

## 2.2 The Weierstrass $\wp$ and its derivative $\wp'$

Define the Weierstrass $\wp$-function:

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left( \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

Facts:

- $\wp(z+\lambda) = \wp(z)$ for all $\lambda \in \Lambda$ ($\Lambda$-periodic).
- $\wp$ is even: $\wp(-z) = \wp(z)$.
- $\wp$ is meromorphic on $\mathbb{C}$ with poles of order 2 at each lattice point.

Differentiate term by term (justified by uniform convergence on compact sets away from poles):

$$\wp'(z) = -\frac{2}{z^3} - 2 \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{(z-\lambda)^3}.$$

Then:

- $\wp'$ is odd: $\wp'(-z) = -\wp'(z)$.
- $\wp'(z)$ has poles of order 3 at each lattice point.

Both $\wp$ and $\wp'$ are $\Lambda$-periodic, so they descend to meromorphic functions on $X = \mathbb{C}/\Lambda$:

$$\wp_X([z]) = \wp(z), \qquad (\wp_X)'([z]) = \wp'(z).$$

## 2.3 The meromorphic 1-form $\wp'(z)\,dz = d\wp(z)$

Consider the 1-form

$$\omega = \wp'(z)\,dz.$$

This is clearly

$$\omega = d\wp(z),$$

as in calculus: derivative times $dz$.

**Local expansion near a pole**

Focus on the point $[0] \in X$ (the image of $0 \in \mathbb{C}$).

Near $z = 0$,

$$\wp(z) = \frac{1}{z^2} + c_2 z^2 + c_4 z^4 + \cdots$$

for some complex coefficients $c_k$ (coming from the lattice). Then

$$\wp'(z) = -\frac{2}{z^3} + 2c_2 z + 4c_4 z^3 + \cdots .$$

So near $z = 0$,

$$\omega = \left( -\frac{2}{z^3} + 2c_2 z + 4c_4 z^3 + \cdots \right) dz.$$

In Laurent series form:

$$\omega = -2z^{-3} dz + 2c_2 z\, dz + 4c_4 z^3 dz + \cdots .$$

Note there is *no* term of the form $c_{-1} z^{-1} dz$, so

$$\mathrm{Res}_{z=0}(\omega) = 0.$$

Similarly, at any other lattice point $\lambda \in \Lambda$, shifting $z \mapsto z - \lambda$ gives the same type of expansion: pole of order 3, no $1/(z - \lambda)$ term. Thus

$$\mathrm{Res}_{[0]}(\omega) = 0, \quad \mathrm{Res}_{[\lambda]}(\omega) = 0 \quad \text{in the quotient } X.$$

**Exactness and periods**

Since $\omega = d\wp$, it is an exact differential. On the torus, this implies for any closed loop $\gamma$ in $X$,

$$\oint_\gamma \omega = \oint_\gamma d\wp = 0.$$

This is the global version of the fact that an exact differential has zero integral over closed paths.

## 2.4 A very concrete elliptic function built from $\wp$

Fix some $a \in \mathbb{C}$ with $[a] \neq [0]$ in $X$, and also $[a] \neq [-a]$ (i.e. $2a \notin \Lambda$). Consider the function

$$g(z) = \wp(z) - \wp(a),$$

which is $\Lambda$-periodic and meromorphic.

Define:

$$f(z) = \frac{\wp'(z)}{\wp(z) - \wp(a)}.$$

This is a classical elliptic function: it is the derivative of the logarithm of $g(z)$:

$$f(z) = \frac{d}{dz} \log(\wp(z) - \wp(a)).$$

**Poles of $f(z)$ on the torus $X$**

**At $z = a$.** Near $z = a$, write $\zeta = z - a$. Then expand:

$$\wp(z) = \wp(a) + \wp'(a)\zeta + \frac{1}{2}\wp''(a)\zeta^2 + \cdots .$$

So

$$\wp(z) - \wp(a) = \wp'(a)\zeta + \frac{1}{2}\wp''(a)\zeta^2 + \cdots .$$

Similarly

$$\wp'(z) = \wp'(a) + \wp''(a)\zeta + \cdots .$$

Thus

$$f(z) = \frac{\wp'(z)}{\wp(z) - \wp(a)} = \frac{\wp'(a) + \wp''(a)\zeta + \cdots}{\wp'(a)\zeta + \frac{1}{2}\wp''(a)\zeta^2 + \cdots}.$$

Pull out $\wp'(a)\zeta$ from the denominator:

$$f(z) = \frac{\wp'(a) + \wp''(a)\zeta + \cdots}{\wp'(a)\zeta \left(1 + \frac{\wp''(a)}{2\wp'(a)}\zeta + \cdots\right)}.$$

Now expand:

$$\frac{1}{1 + \alpha\zeta + \cdots} = 1 - \alpha\zeta + \cdots ,$$

so near $\zeta = 0$,

$$f(z) \sim \frac{1}{\zeta} \cdot \frac{\wp'(a)}{\wp'(a)} = \frac{1}{\zeta} = \frac{1}{z - a}.$$

Thus $f$ has a *simple pole* at $z = a$ with residue

$$\mathrm{Res}_{z=a}\big(f(z)\, dz\big) = 1.$$

**At $z = -a$.** Because $\wp$ is even and $\wp'$ is odd,

$$\wp(-z) = \wp(z), \quad \wp'(-z) = -\wp'(z).$$

The equation $\wp(z) = \wp(a)$ has the two solutions $z = \pm a$ modulo $\Lambda$. So $\wp(z) - \wp(a)$ also vanishes at $z = -a$. Repeating the same expansion, or just using the symmetry, we find

$$\operatorname{Res}_{z=-a}\big(f(z)\,dz\big) = 1.$$

**At $z = 0$ (and other lattice points).** Near $z = 0$,

$$\wp(z) \sim \frac{1}{z^2}, \quad \wp'(z) \sim -\frac{2}{z^3}.$$

Then

$$f(z) = \frac{\wp'(z)}{\wp(z) - \wp(a)} \sim \frac{-2z^{-3}}{z^{-2} - \wp(a)} = \frac{-2z^{-3}}{z^{-2}(1 - \wp(a)z^2)} = \frac{-2}{z} \cdot \frac{1}{1 - \wp(a)z^2}.$$

Expand $\frac{1}{1-\wp(a)z^2} = 1 + \wp(a)z^2 + \cdots$, so

$$f(z) \sim \frac{-2}{z} + (\text{holomorphic terms}).$$

Thus at $z = 0$ we have a simple pole with residue

$$\operatorname{Res}_{z=0}\big(f(z)\,dz\big) = -2.$$

Similarly, at other lattice points $\lambda \neq 0$, the local behavior is like near 0 shifted by $\lambda$, giving the same residue pattern, but on the torus $X = \mathbb{C}/\Lambda$ all lattice points project to a finite set of points, and you can group residues within one fundamental domain.

### Residue theorem on the torus

On the compact Riemann surface $X$,

$$\sum_{p \in X} \operatorname{Res}_p\big(f(z)\,dz\big) = 0.$$

From the computations:

- $\operatorname{Res}_{[a]} = 1$,

- $\mathrm{Res}_{[-a]} = 1$,
- $\mathrm{Res}_{[0]} = -2$

(and no other poles in a fundamental parallelogram), so

$$1 + 1 - 2 = 0.$$

This is a very concrete check of the residue theorem on a torus for this particular elliptic function.

### 2.5 Function field $\mathcal{M}(\mathbb{C}/\Lambda)$ via $\wp$ and $\wp'$

A fundamental theorem: $\wp$ and $\wp'$ satisfy

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

where $g_2, g_3$ are complex constants depending on $\Lambda$.

If we set

$$X := \wp(z), \quad Y := \wp'(z),$$

then

$$Y^2 = 4X^3 - g_2X - g_3.$$

From the differential-form viewpoint:

$$dX = d\wp(z) = \wp'(z)\, dz = Y\, dz \quad \Rightarrow \quad dz = \frac{dX}{Y}.$$

So the basic holomorphic 1-form $dz$ on the torus becomes

$$dz = \frac{dX}{\sqrt{4X^3 - g_2X - g_3}},$$

when we view the torus as the algebraic curve $Y^2 = 4X^3 - g_2X - g_3$.

The function field is then

$$\mathcal{M}(\mathbb{C}/\Lambda) = \mathbb{C}(\wp, \wp') \cong \mathbb{C}(X, Y)/\big(Y^2 - 4X^3 + g_2X + g_3\big).$$

Concretely, any meromorphic function $F$ on $X$ is of the form

$$F([z]) = R\big(\wp(z), \wp'(z)\big)$$

for some rational function $R(X, Y)$.

Our explicit example

$$f(z) = \frac{\wp'(z)}{\wp(z) - \wp(a)} = \frac{Y}{X - \wp(a)}$$

is exactly such a rational function in $(X, Y) = (\wp, \wp')$, and we computed its poles and residues using differential-form techniques in detail.

## Conclusion

- For $X = \mathbb{CP}^1$, we picked a very concrete rational function $f(z) = (z-1)^2/(z(z-2))$, computed its divisor, wrote it via partial fractions, and computed $df$ as a meromorphic 1-form, verifying the residue behavior. This illustrates $\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z)$.
- For $X = \mathbb{C}/\Lambda$, we used $\wp, \wp'$ and a very explicit elliptic function $f(z) = \wp'(z)/(\wp(z)-\wp(a))$, expanded near poles, and computed residues of the meromorphic 1-form $f(z)\,dz$. This illustrates concretely how $\mathcal{M}(\mathbb{C}/\Lambda)$ is generated by $\wp, \wp'$, and how differential forms (like $\wp'(z)\,dz = d\wp(z)$) control the structure of the function field.

## Integrating $\omega = \dfrac{\wp'(z)}{\wp(z) - \wp(a)}\,dz$ around the torus

We continue with the torus

$$X = \mathbb{C}/\Lambda, \qquad \Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2, \quad \Im\left(\frac{\omega_2}{\omega_1}\right) > 0.$$

Recall the elliptic function

$$f(z) = \frac{\wp'(z)}{\wp(z) - \wp(a)},$$

for some fixed $a \in \mathbb{C}$ with $2a \notin \Lambda$, and the meromorphic 1-form

$$\omega = f(z)\,dz.$$

We already computed the residues:

$$\mathrm{Res}_{z=a}(\omega) = 1, \qquad \mathrm{Res}_{z=-a}(\omega) = 1, \qquad \mathrm{Res}_{z=0}(\omega) = -2$$

(modulo the lattice). Now we will:

- Fix a fundamental parallelogram in $\mathbb{C}$.
- Integrate $\omega$ around its boundary.
- Use periodicity of $f$ to show the boundary integral is 0.
- Use the residue theorem to show that this equals $2\pi i$ times the sum of residues inside.

## 1. Fundamental parallelogram and its boundary

Pick a fundamental parallelogram (fundamental domain) for $\Lambda$ in $\mathbb{C}$:

$$\mathcal{P} := \{s\omega_1 + t\omega_2 \mid 0 \le s \le 1,\ 0 \le t \le 1\}.$$

Its vertices are

$$0, \quad \omega_1, \quad \omega_1 + \omega_2, \quad \omega_2.$$

Define the oriented boundary $\partial\mathcal{P}$ as the piecewise smooth path:

$$\partial\mathcal{P} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4,$$

where

$$\begin{aligned}
&\gamma_1 : 0 \to \omega_1,\\
&\gamma_2 : \omega_1 \to \omega_1 + \omega_2,\\
&\gamma_3 : \omega_1 + \omega_2 \to \omega_2,\\
&\gamma_4 : \omega_2 \to 0.
\end{aligned}$$

More concretely, parametrize each side:

- $\gamma_1(t) = t\,\omega_1, \quad 0 \le t \le 1.$
- $\gamma_2(t) = \omega_1 + t\,\omega_2, \quad 0 \le t \le 1.$
- $\gamma_3(t) = \omega_1 + \omega_2 - t\,\omega_1, \quad 0 \le t \le 1.$
- $\gamma_4(t) = \omega_2 - t\,\omega_2, \quad 0 \le t \le 1.$

We will compute

$$\oint_{\partial\mathcal{P}} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega + \int_{\gamma_3} \omega + \int_{\gamma_4} \omega.$$

## 2. Periodicity of $f$ and cancellation of integrals on opposite sides

Since $f$ is $\Lambda$-periodic, we have

$$f(z + \omega_1) = f(z), \quad f(z + \omega_2) = f(z), \quad \forall z \in \mathbb{C}.$$

We use this to relate the integrals on opposite sides of $\mathcal{P}$.

### 2.1. Compare $\int_{\gamma_1} \omega$ and $\int_{\gamma_3} \omega$

Write

$$I_1 := \int_{\gamma_1} \omega, \qquad I_3 := \int_{\gamma_3} \omega.$$

First compute $I_1$ explicitly:

$$\gamma_1(t) = t\,\omega_1, \quad \gamma_1'(t) = \omega_1, \quad 0 \le t \le 1.$$

Then

$$I_1 = \int_{\gamma_1} f(z)\,dz = \int_0^1 f(\gamma_1(t))\,\gamma_1'(t)\,dt = \int_0^1 f(t\omega_1)\,\omega_1\,dt = \omega_1 \int_0^1 f(t\omega_1)\,dt.$$

Now compute $I_3$. Parametrization:

$$\gamma_3(t) = \omega_1 + \omega_2 - t\,\omega_1, \quad \gamma_3'(t) = -\omega_1, \quad 0 \le t \le 1.$$

So

$$I_3 = \int_{\gamma_3} f(z)\,dz = \int_0^1 f(\gamma_3(t))\,\gamma_3'(t)\,dt = \int_0^1 f(\omega_1 + \omega_2 - t\omega_1) \cdot (-\omega_1)\,dt.$$

We now use periodicity to simplify $f(\omega_1 + \omega_2 - t\omega_1)$:

$$\omega_1 + \omega_2 - t\omega_1 = \omega_2 + (1-t)\omega_1.$$

Since $f$ is periodic with period $\omega_2$, we can subtract $\omega_2$:

$$f(\omega_2 + (1-t)\omega_1) = f((1-t)\omega_1).$$

Thus

$$I_3 = \int_0^1 f((1-t)\omega_1) \cdot (-\omega_1)\,dt.$$

Make the change of variable

$$s = 1 - t \quad \Rightarrow \quad t = 1 - s, \quad dt = -ds,$$

and when $t = 0$, $s = 1$; when $t = 1$, $s = 0$. Then

$$I_3 = \int_{s=1}^{s=0} f(s\omega_1) \cdot (-\omega_1) \cdot (-ds) = \int_{s=1}^{s=0} f(s\omega_1)\,\omega_1\,ds.$$

Swap limits:

$$I_3 = -\int_{s=0}^{s=1} f(s\omega_1)\,\omega_1\,ds = -\omega_1 \int_0^1 f(s\omega_1)\,ds.$$

Comparing with

$$I_1 = \omega_1 \int_0^1 f(t\omega_1)\, dt,$$

we see

$$I_3 = -I_1.$$

So the integrals along the bottom and top sides cancel:

$$\int_{\gamma_1} \omega + \int_{\gamma_3} \omega = I_1 + I_3 = 0.$$

**2.2. Compare $\int_{\gamma_2} \omega$ and $\int_{\gamma_4} \omega$**

Similarly define

$$I_2 := \int_{\gamma_2} \omega, \qquad I_4 := \int_{\gamma_4} \omega.$$

Parametrize $\gamma_2$:

$$\gamma_2(t) = \omega_1 + t\omega_2, \quad \gamma_2'(t) = \omega_2, \quad 0 \le t \le 1.$$

Then

$$I_2 = \int_{\gamma_2} f(z)\, dz = \int_0^1 f(\omega_1 + t\omega_2)\, \omega_2\, dt.$$

Parametrize $\gamma_4$:

$$\gamma_4(t) = \omega_2 - t\omega_2, \quad \gamma_4'(t) = -\omega_2, \quad 0 \le t \le 1.$$

Then

$$I_4 = \int_{\gamma_4} f(z)\, dz = \int_0^1 f(\omega_2 - t\omega_2) \cdot (-\omega_2)\, dt.$$

Simplify $f(\omega_2 - t\omega_2)$:

$$\omega_2 - t\omega_2 = (1-t)\omega_2.$$

Because $f(z + \omega_1) = f(z)$, we can add $\omega_1$ if we want:

$$f((1-t)\omega_2) = f(\omega_1 + (1-t)\omega_2)$$

as well. But we do not strictly need that here. Do the change of variable

$$s = 1 - t \quad \Rightarrow \quad dt = -ds,$$

and when $t = 0 \to s = 1$, $t = 1 \to s = 0$. Then

$$I_4 = \int_{s=1}^{s=0} f(s\omega_2)\cdot(-\omega_2)\cdot(-ds) = \int_{s=1}^{s=0} f(s\omega_2)\,\omega_2\,ds = -\int_{s=0}^{s=1} f(s\omega_2)\,\omega_2\,ds.$$

But

$$I_2 = \int_0^1 f(\omega_1 + t\omega_2)\,\omega_2\,dt.$$

Use periodicity with period $\omega_1$:

$$f(\omega_1 + t\omega_2) = f(t\omega_2).$$

So

$$I_2 = \int_0^1 f(t\omega_2)\,\omega_2\,dt = \omega_2 \int_0^1 f(t\omega_2)\,dt.$$

Using $s$ instead of $t$ as dummy variable,

$$I_2 = \omega_2 \int_0^1 f(s\omega_2)\,ds.$$

Thus

$$I_4 = -\omega_2 \int_0^1 f(s\omega_2)\,ds = -I_2.$$

So the integrals along the left and right sides cancel:

$$\int_{\gamma_2} \omega + \int_{\gamma_4} \omega = I_2 + I_4 = 0.$$

### 2.3. Total boundary integral

Putting together:

$$\oint_{\partial\mathcal{P}} \omega = I_1 + I_2 + I_3 + I_4 = (I_1 + I_3) + (I_2 + I_4) = 0 + 0 = 0.$$

So we have shown purely from periodicity and calculus parametrizations:

$$\oint_{\partial\mathcal{P}} f(z)\,dz = 0.$$

## 3. Residue theorem inside the fundamental parallelogram

Now apply the residue theorem from complex analysis.

### 3.1. Poles inside $\mathcal{P}$

We consider the poles of $\omega$ inside the parallelogram $\mathcal{P}$. We chose $a$ such that $0, a, -a$ represent three distinct points mod $\Lambda$, and (for a generic choice of $a$) we can assume that within the chosen fundamental domain $\mathcal{P}$, the only poles of $f(z)$ are at

$$z = 0, \quad z = a, \quad z = -a.$$

(Any other poles are at lattice translates of these, lying in other translates of the parallelogram.)

We computed the residues:

$$\mathrm{Res}_{z=a}(\omega) = 1, \quad \mathrm{Res}_{z=-a}(\omega) = 1, \quad \mathrm{Res}_{z=0}(\omega) = -2.$$

### 3.2. Residue theorem

The residue theorem applied to $\omega = f(z)\,dz$ on $\mathcal{P}$ states:

$$\oint_{\partial\mathcal{P}} \omega = 2\pi i \sum_{p\in\mathcal{P}} \mathrm{Res}_{z=p}(\omega),$$

where the sum is over all poles of $\omega$ inside $\mathcal{P}$.

From the previous subsection:

$$\oint_{\partial\mathcal{P}} \omega = 0.$$

Thus

$$0 = 2\pi i\left(\mathrm{Res}_{z=a}(\omega) + \mathrm{Res}_{z=-a}(\omega) + \mathrm{Res}_{z=0}(\omega)\right) = 2\pi i(1 + 1 - 2).$$

So indeed

$$1 + 1 - 2 = 0.$$

## 4. Interpretation on the torus $X = \mathbb{C}/\Lambda$

The fundamental parallelogram $\mathcal{P}$ is a fundamental domain for the projection map $\pi : \mathbb{C} \to X = \mathbb{C}/\Lambda$. Its boundary edges are identified in pairs:

$$\gamma_1 \sim \gamma_3, \quad \gamma_2 \sim \gamma_4.$$

The fact that

$$\int_{\gamma_1} \omega + \int_{\gamma_3} \omega = 0, \quad \int_{\gamma_2} \omega + \int_{\gamma_4} \omega = 0$$

is exactly the statement that, when you go to the quotient torus $X$, the integral of $\omega$ over the resulting closed loop is well-defined and "wraps around" consistently.

On the compact Riemann surface $X$, the residue theorem becomes

$$\sum_{p \in X} \operatorname{Res}_p(\omega) = 0.$$

In our explicit example,

$$\operatorname{Res}_{[a]}(\omega) + \operatorname{Res}_{[-a]}(\omega) + \operatorname{Res}_{[0]}(\omega) = 1 + 1 - 2 = 0,$$

in perfect agreement with the global theory.

**Summary.** For the elliptic function

$$f(z) = \frac{\wp'(z)}{\wp(z) - \wp(a)}$$

on the torus $X = \mathbb{C}/\Lambda$, we have:

- an explicit meromorphic 1-form $\omega = f(z)\,dz$;
- an explicit fundamental parallelogram $\mathcal{P}$ in $\mathbb{C}$;
- a direct computation using periodicity shows $\displaystyle\oint_{\partial\mathcal{P}} \omega = 0$;
- the residue theorem computes the same integral as $2\pi i \sum \operatorname{Res}_p(\omega)$, giving $2\pi i(1 + 1 - 2) = 0$.

This is a fully concrete, calculus-level demonstration of how residues and periodicity interact on the torus, and how global facts about the function field $\mathcal{M}(\mathbb{C}/\Lambda)$ arise from local Laurent expansions and contour integrals.