Torus and Algebra

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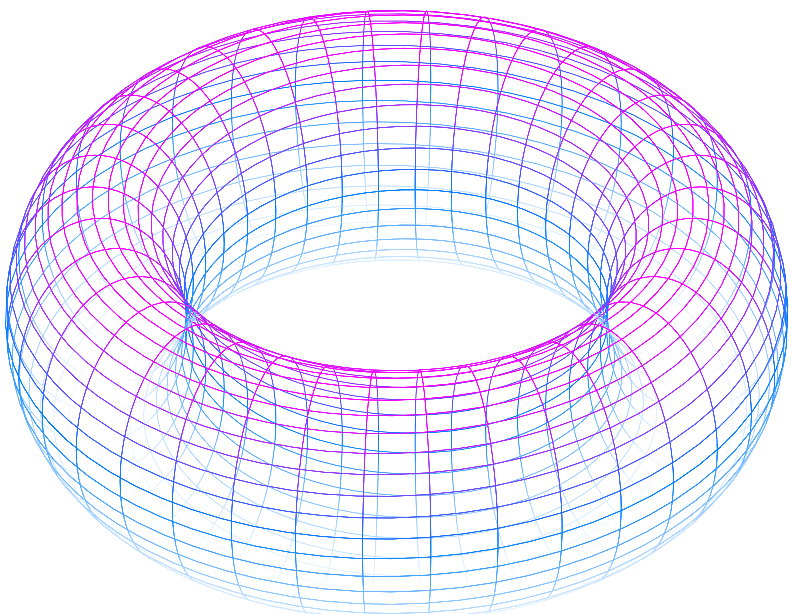
April 7, 2025

We cover the following topics in this note.

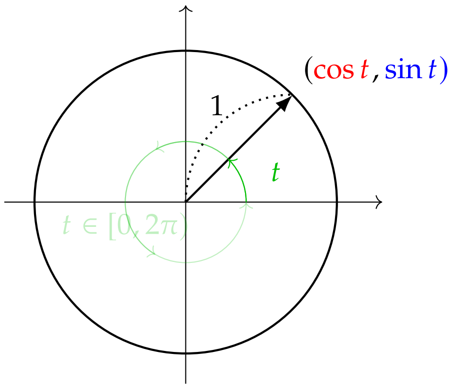
- Unit Circle
- Torus
- TBA

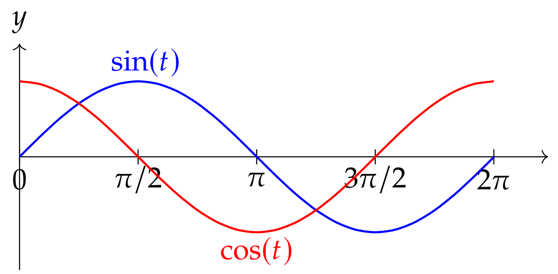


Contents

1 Unit Circle

The set $\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is called the **unit circle**.





The standard parametrization of $\mathbb{S}^1$ is given by

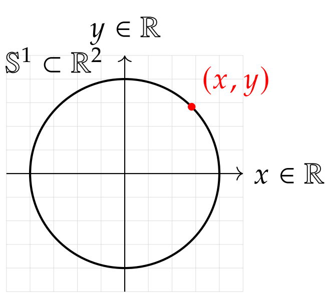
$$t \mapsto (\cos t, \sin t), \quad t \in [0, 2\pi),$$

which implies the *trigonometric identity* $\cos^2 t + \sin^2 t = 1$. The mapping

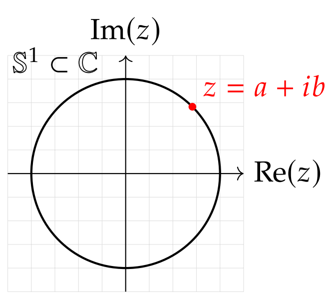
$$\begin{array}{rccc} \varphi : & [0, 2\pi) & \longrightarrow & \mathbb{S}^1 \\ & t & \longmapsto & (\cos t, \sin t) \end{array}.$$

provides a bijection between the half-open interval $[0, 2\pi)$ and the unit circle $\mathbb{S}^1$.

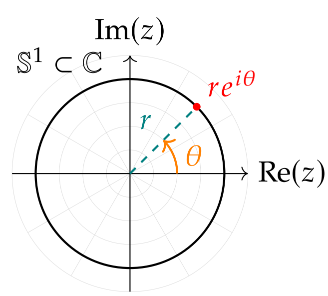
Geometrically, it represents the set of points at a fixed distance 1 from the origin in $\mathbb{R}^2$, while algebraically it can be seen as a group under complex multiplication.



$x^2 + y^2 = 1$



$|z| = 1$



$|r| = 1$ and $\theta \in \mathbb{R}$

The unit circle can be described in several equivalent ways. In $\mathbb{R}^2$, it is given by:

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

In the complex plane, we write:

$$\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\} = \{re^{i\theta} : |r| = 1 \text{ and } \theta \in \mathbb{R}\}.$$

We now show that $\mathbb{S}^1$ forms a group under complex multiplication:

(G0) **(Closure)** Let $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2} \in \mathbb{S}^1$. Then $z_1 z_2 = e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)} \in \mathbb{S}^1$.

(G1) **(Associativity)** Let $z_1 = e^{i\theta_1}, z_2 = e^{i\theta_2}, z_3 = e^{i\theta_3} \in \mathbb{S}^1$ then

$$(z_1z_2)z_3 = (e^{i\theta_1}e^{i\theta_2})e^{i\theta_3} = e^{i(\theta_1+\theta_2)}e^{i\theta_3} = e^{i(\theta_1+\theta_2+\theta_3)} = e^{i\theta_1}e^{i(\theta_2+\theta_3)} = e^{i\theta_1}(e^{i\theta_2}e^{i\theta_3}) = z_1(z_2z_3).$$

(G2) **(Identity Element)** For each $z = e^{i\theta} \in \mathbb{S}^1$,

$$1 \cdot z = e^{i0}e^{i\theta} = e^{i(0+\theta)} = e^{i\theta} = z,$$

and similarly $z \cdot 1 = z$.

(G3) **(Inverses)** For any $z = e^{i\theta} \in \mathbb{S}^1$, its inverse is given by $z^{-1} = e^{-i\theta}$, since

$$z \cdot z^{-1} = e^{i\theta}e^{-i\theta} = e^{i(\theta-\theta)} = e^{i\cdot 0} = 1.$$

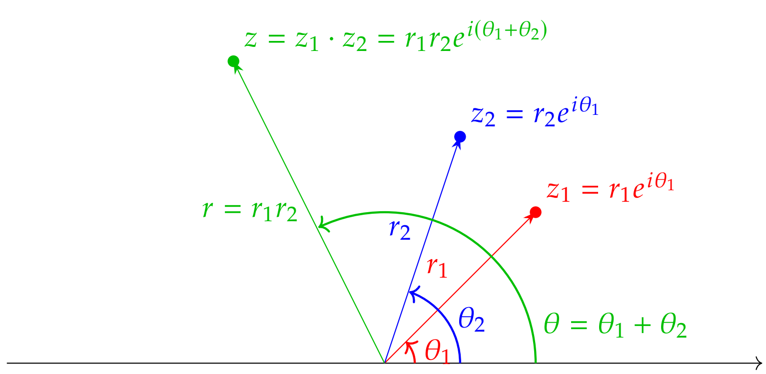
Notice that $e^{-i\theta} \in \mathbb{S}^1$ as well.

We show that **multiplication on the circle group is equivalent to addition of angles**: let

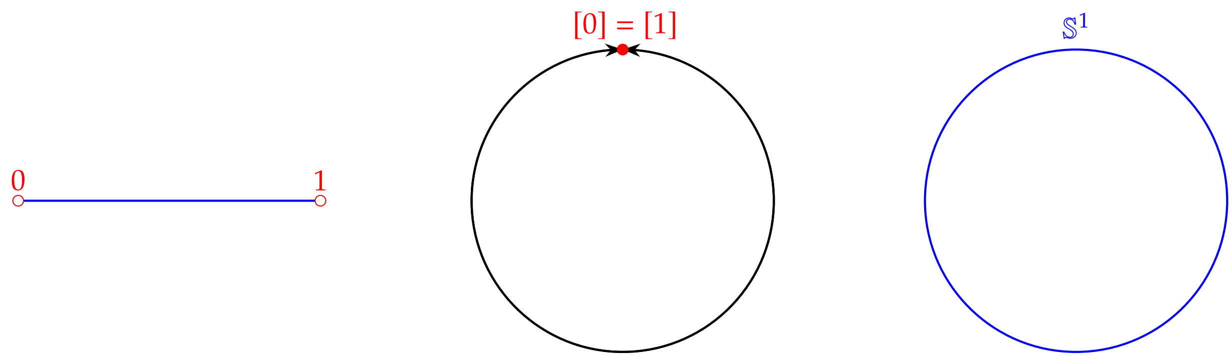
$$\begin{aligned} z_1 &= r_1e^{i\theta_1} = r_1\left(\cos\theta_1 + i\sin\theta_1\right) \in \mathbb{C} \text{ and} \\ z_2 &= r_2e^{i\theta_2} = r_2\left(\cos\theta_2 + i\sin\theta_2\right) \in \mathbb{C}. \end{aligned}$$

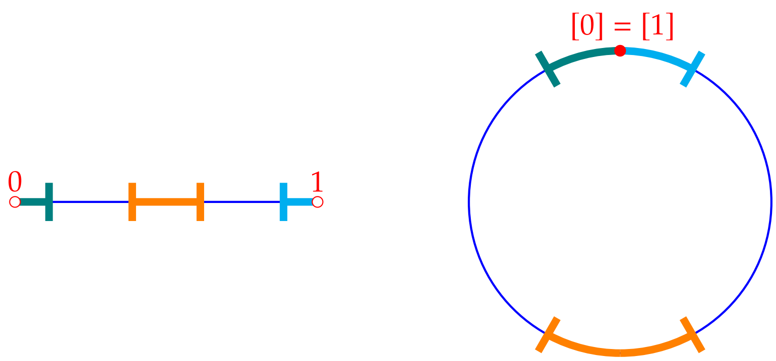
Then

$$\begin{aligned} z_1 \cdot z_2 = r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} = &= r_1r_2\left(\cos\theta_1 + i\sin\theta_1\right)\left(\cos\theta_2 + i\sin\theta_2\right) \\ &= r_1r_2\left[\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right) + i\left(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2\right)\right] \\ &= r_1r_2\left[\cos\left(\theta_1+\theta_2\right) + i\sin\left(\theta_1+\theta_2\right)\right] \\ &= r\left(\cos\theta + \sin\theta\right) \text{ with } \begin{cases} r = r_1r_2 \\ \theta = \theta_1 + \theta_2. \end{cases} \end{aligned}$$



1.1 Quotient Space





Let

$$\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}, \quad x \mapsto x + \mathbb{Z},$$

be the canonical projection onto the quotient group, where the equivalence relation is given by

$$x \sim y \quad \Longleftrightarrow \quad x - y \in \mathbb{Z}.$$

Denote by

$$[x] = \{\, y \in \mathbb{R} \mid y \sim x \,\} = x + \mathbb{Z}$$

the equivalence class of x. Then $\mathbb{R}/\mathbb{Z} = \{[x] : x \in \mathbb{R}\}$. Thus, each element of $\mathbb{R}/\mathbb{Z}$ is a coset of the subgroup $\mathbb{Z}$ in $\mathbb{R}$. Consider the canonical projection map

$$\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}, \quad \pi(x) = [x].$$

The topology on $\mathbb{R}/\mathbb{Z}$ is defined as the **quotient topology** induced by π. That is, a subset $U \subseteq \mathbb{R}/\mathbb{Z}$ is declared open if and only if its preimage under π is open in $\mathbb{R}$; symbolically,

$$U \in \tau \quad \Longleftrightarrow \quad \pi^{-1}(U) \in \tau_{\mathbb{R}},$$

where $\tau_{\mathbb{R}}$ is the standard topology on $\mathbb{R}$ (i.e., the topology generated by the open intervals).

3. Formal Definition Summary

- **Set:**

$$\mathbb{R}/\mathbb{Z} = \{\, [x] : x \in \mathbb{R} \,\},$$

where $[x] = \{x + n : n \in \mathbb{Z}\}$.

- **Topology:** The topology τ on $\mathbb{R}/\mathbb{Z}$ is given by

$$\tau = \{\, U \subseteq \mathbb{R}/\mathbb{Z} \mid \pi^{-1}(U) \in \tau_{\mathbb{R}} \,\}.$$

This makes π a continuous, surjective map, and by definition, $\mathbb{R}/\mathbb{Z}$ becomes a topological space known as the quotient space of $\mathbb{R}$ by $\mathbb{Z}$.

4. Additional Comments

It is a classical result that $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the circle S^1. One explicit homeomorphism is given by

$$\varphi : \mathbb{R}/\mathbb{Z} \to S^1, \quad \varphi([x]) = e^{2\pi i x},$$

which is continuous, bijective, and has a continuous inverse.

This completes the formal construction of the topological space $\mathbb{R}/\mathbb{Z}$ by specifying both its underlying set and the quotient topology derived from the standard topology on $\mathbb{R}$.

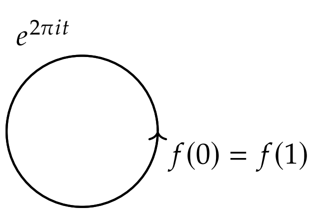
It is well known that the topological space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the circle S^1. For instance, define

$$\varphi : \mathbb{R}/\mathbb{Z} \to S^1, \quad \varphi(x + \mathbb{Z}) = e^{2\pi i x}.$$

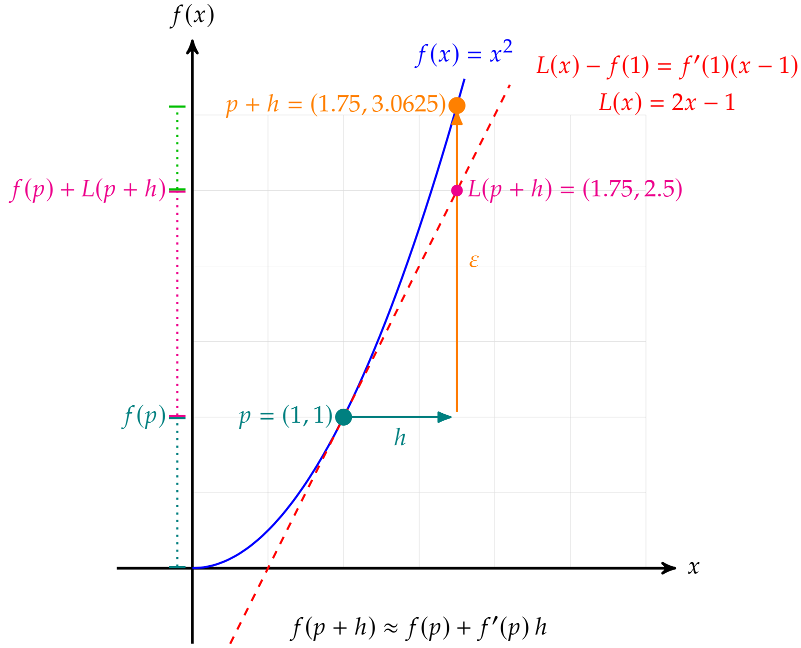
Then φ is a continuous bijection with continuous inverse, hence a homeomorphism.

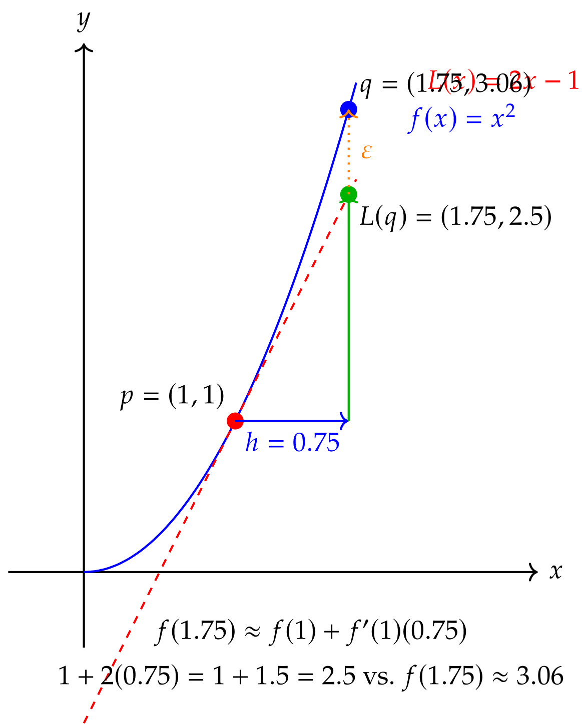
$$\begin{tikzcd} \mathbb{R} \arrow[r, "\pi"] \arrow[rr, bend left, "E(t) = e^{2\pi i t}"] & \mathbb{R}/\mathbb{Z} \arrow[r, "F"] & S^1 \end{tikzcd}$$

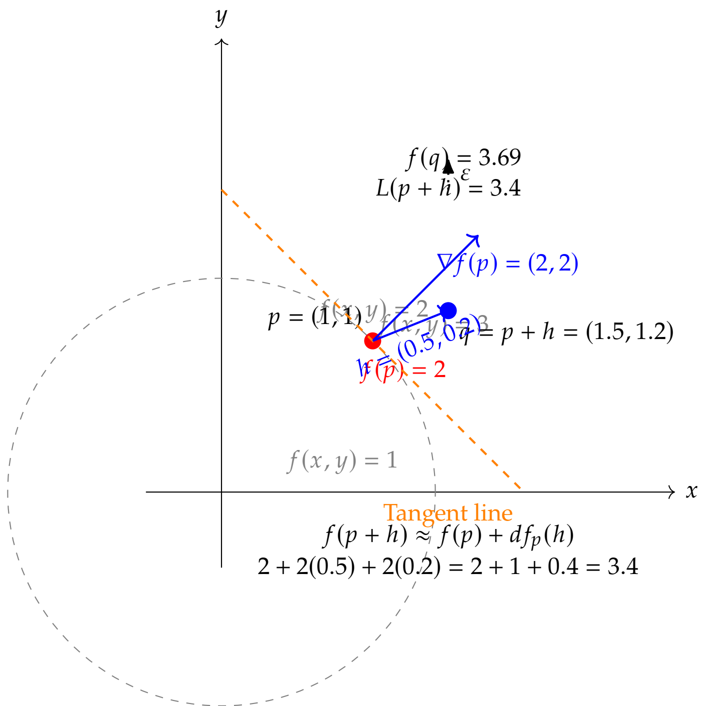
Visualization of $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \to S^1$



1.2 Linear Approximation







Below is a detailed explanation, along with a concrete calculation, that defines what a differential is in the context of smooth manifolds.

1. Differential as the Best Linear Approximation

Let $f\colon M \to \mathbb{R}$ be a smooth function defined on a smooth manifold M. The **differential of f at a point $p \in M$**, denoted by df_p, is defined as the best linear approximation to f near p. Concretely, df_p is a linear map

$$df_p\colon T_pM \to \mathbb{R},$$

where T_pM is the tangent space at p.

1.1. Definition via Curves

One way to define df_p is by using smooth curves. Let $v \in T_pM$ be a tangent vector, and let $\gamma\colon (-\varepsilon, \varepsilon) \to M$ be any smooth curve with

$$\gamma(0) = p \quad \text{and} \quad \gamma'(0) = v.$$

Then, the differential df_p applied to v is given by:

$$df_p(v) = \left.\frac{d}{dt}\right|_{t=0} f\big(\gamma(t)\big).$$

This derivative is independent of the particular choice of curve γ as long as it has the correct initial velocity v.

1.2. Linear Approximation

In elementary calculus for functions $f\colon \mathbb{R} \to \mathbb{R}$, the derivative $f'(x)$ gives the linear (first-order) approximation:

$$f(x+h) \approx f(x) + f'(x)h \quad \text{for small } h.$$

In higher dimensions, df_p generalizes this idea. It is the linear map that approximates the change in f near p. If $p \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, then

$$f(p+v) \approx f(p) + df_p(v).$$

2. Differential in Local Coordinates

If (U, φ) is a coordinate chart around p with coordinates $(x^1, \ldots, x^n)$, then f can be written locally as a function $f(x^1, \ldots, x^n)$. In these coordinates, the differential is given by:

$$df = \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2 + \cdots + \frac{\partial f}{\partial x^n} dx^n.$$

Here, $dx^1, dx^2, \ldots, dx^n$ are the canonical 1–forms associated with the coordinate functions. For a tangent vector $v = (v^1, \ldots, v^n)$ (which in these coordinates is written as $v = v^i \partial/\partial x^i$), we have:

$$df_p(v) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)\, v^i.$$

3. Differential as a Section of the Cotangent Bundle

The assignment $p \mapsto df_p$ defines a smooth section of the cotangent bundle T^*M. This means that df is a 1–form on M. In other words, it assigns to each point p a covector (a linear functional on T_pM) in a smooth (i.e. differentiable) manner.

4. Concrete Calculation Example

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = x^2 + y^2.$$

Let $p = (1, 2)$. We want to compute df_p.

1. **Compute the Partial Derivatives:**

$$\frac{\partial f}{\partial x}(x, y) = 2x, \quad \frac{\partial f}{\partial y}(x, y) = 2y.$$

2. **Evaluate at $p = (1, 2)$:**

$$\frac{\partial f}{\partial x}(1, 2) = 2, \quad \frac{\partial f}{\partial y}(1, 2) = 4.$$

3. **Express df in Coordinates:**

$$df = 2x\, dx + 2y\, dy.$$

At p, this becomes:

$$df_{(1,2)} = 2\,dx + 4\,dy.$$

4. **Apply df_p to a Tangent Vector:**
Let $v = (3, -1) \in T_{(1,2)}\mathbb{R}^2$. Then:

$$df_{(1,2)}(v) = 2 \cdot 3 + 4 \cdot (-1) = 6 - 4 = 2.$$

This number 2 is the rate at which f increases in the direction of the vector v at the point $(1, 2)$.
Summary

- **Differential df_p:** For a smooth function $f : M \to \mathbb{R}$, the differential at p, df_p, is a linear map from the tangent space T_pM to $\mathbb{R}$ that gives the best linear approximation of f near p.

- **Definition via Curves:** It is defined by

$$df_p(v) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)),$$

for any curve γ with $\gamma(0) = p$ and $\gamma'(0) = v$.

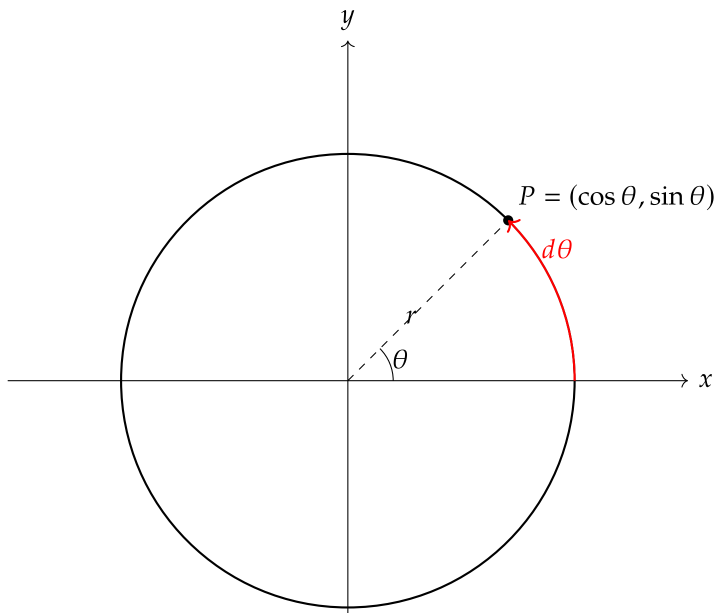
- **Local Expression:** In local coordinates,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\,dx^i,$$

which acts on tangent vectors by the dot product with the gradient of f.

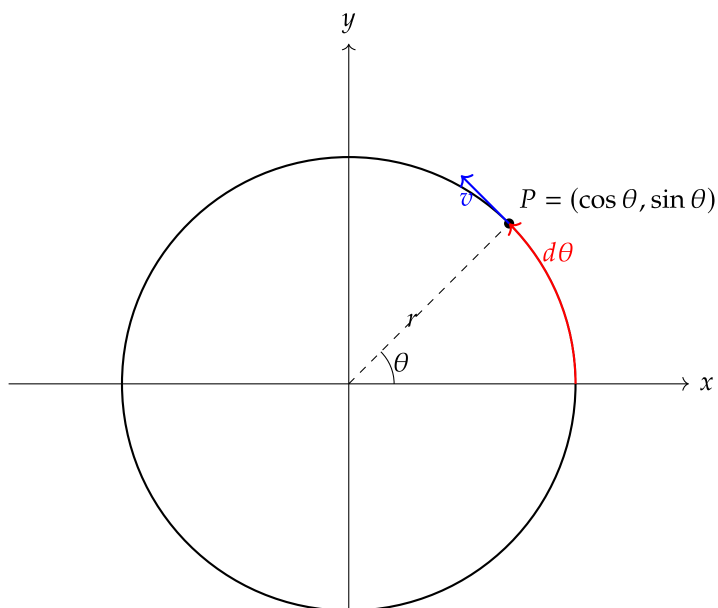
- **Global Perspective:** The map $p \mapsto df_p$ defines a smooth section of the cotangent bundle T^*M, and thus df is a 1–form on M.

This detailed explanation, along with the concrete calculation for a function on $\mathbb{R}^2$, illustrates what a differential is and why it is such a central concept in differential geometry and analysis.



$$\frac{x\,dy - y\,dx}{x^2 + y^2} = \frac{r^2\,d\theta}{r^2} = d\theta$$

$d\theta$ is globally defined, whereas θ is only local (mod 2π).



$$\frac{x\,dy - y\,dx}{x^2 + y^2} = \frac{r^2\,d\theta}{r^2} = d\theta$$

$d\theta$ is globally defined as a 1–form, unlike θ which is only local.

On $\mathbb{R}^2 \setminus \{0\}$, when we express the coordinates in polar form with

$$x = r\cos\theta, \quad y = r\sin\theta,$$

the 1–form
$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}$$
simplifies to
$$d\theta.$$

Verification in Polar Coordinates

1. **Express dx and dy in polar coordinates:**

$$dx = \cos\theta\,dr - r\sin\theta\,d\theta, \quad dy = \sin\theta\,dr + r\cos\theta\,d\theta.$$

2. **Compute the numerator:**

$$\begin{aligned} x\,dy - y\,dx &= r\cos\theta\,(\sin\theta\,dr + r\cos\theta\,d\theta) - r\sin\theta\,(\cos\theta\,dr - r\sin\theta\,d\theta) \\ &= r\cos\theta\sin\theta\,dr + r^2\cos^2\theta\,d\theta - r\sin\theta\cos\theta\,dr + r^2\sin^2\theta\,d\theta \\ &= r^2(\cos^2\theta + \sin^2\theta)\,d\theta \\ &= r^2\,d\theta. \end{aligned}$$

3. **Compute the denominator:**

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2.$$

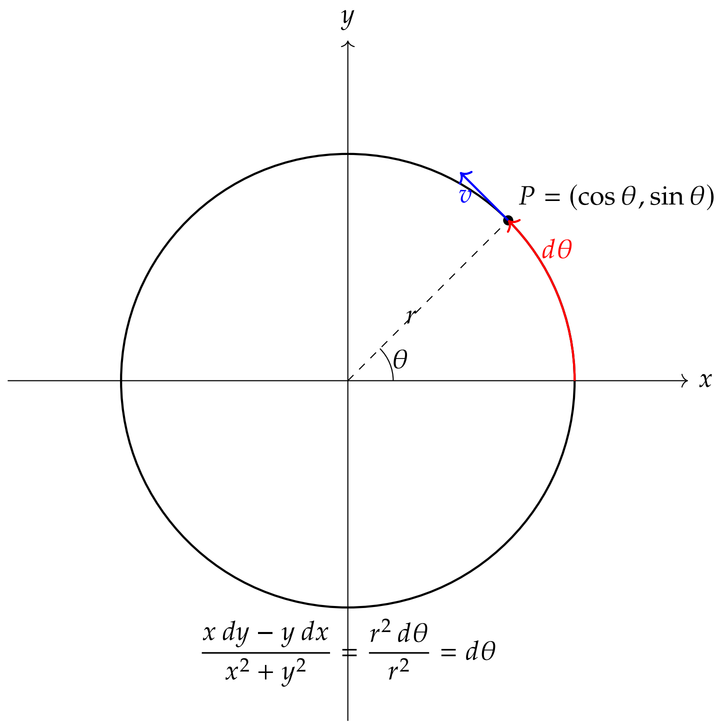
4. **Combine the results:**
$$\omega = \frac{r^2\,d\theta}{r^2} = d\theta.$$

Interpretation

- **$d\theta$:** This differential represents the infinitesimal change in the angular coordinate θ. Although the function θ is not globally well-defined on $\mathbb{R}^2 \setminus \{0\}$ (because angles are defined modulo 2π), its differential $d\theta$ is a well-defined 1–form on this space.

- **Winding Number:** When you integrate ω (or equivalently $d\theta$) over a closed curve γ in $\mathbb{R}^2 \setminus \{0\}$,
$$\int_\gamma \omega = \int_\gamma d\theta,$$
you obtain the total angular change along γ. Normalizing this by 2π gives the winding number of γ about the origin.

In summary, on $\mathbb{R}^2 \setminus \{0\}$, the form
$$\frac{x\,dy - y\,dx}{x^2 + y^2}$$
is equivalent to $d\theta$ in polar coordinates.



The 1–form

$$\omega = \frac{x\,dy - y\,dx}{x^2+y^2}$$

can seem mysterious at first, but it becomes clear when we change to polar coordinates. Here is a step-by-step explanation:

1. Domain and Invariance

- **Domain:** The form ω is defined on $\mathbb{R}^2 \setminus \{0\}$ (the plane minus the origin), because at $(0,0)$ the denominator $x^2 + y^2$ vanishes.

- **Invariance:** Notice that ω is invariant under rotations and scalings. This invariance makes it a natural candidate for measuring how much a curve "turns around" the origin.

2. Change to Polar Coordinates

Introduce polar coordinates by setting:

$$x = r\cos\theta, \quad y = r\sin\theta,$$

where $r > 0$ and $\theta \in \mathbb{R}$ (with θ defined modulo 2π).

Compute the differentials:

$$dx = \cos\theta\,dr - r\sin\theta\,d\theta, \quad dy = \sin\theta\,dr + r\cos\theta\,d\theta.$$

Now, substitute these into the numerator $x\,dy - y\,dx$:

$$\begin{aligned} x\,dy - y\,dx &= (r\cos\theta)(\sin\theta\,dr + r\cos\theta\,d\theta) - (r\sin\theta)(\cos\theta\,dr - r\sin\theta\,d\theta) \\ &= r\cos\theta\,\sin\theta\,dr + r^2\cos^2\theta\,d\theta - r\sin\theta\,\cos\theta\,dr + r^2\sin^2\theta\,d\theta \\ &= r^2(\cos^2\theta + \sin^2\theta)\,d\theta \\ &= r^2\,d\theta. \end{aligned}$$

Similarly, the denominator becomes:

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2.$$

Thus, in polar coordinates, the form simplifies to:

$$\omega = \frac{r^2\,d\theta}{r^2} = d\theta.$$

3. Interpretation as the Angular Differential

- **Angular Differential:** The expression $d\theta$ represents the infinitesimal change in the angle θ. Thus, ω measures the rate at which the angle changes as you move along a curve in $\mathbb{R}^2 \setminus \{0\}$.

- **Integration and Winding Number:** If $\gamma : [a,b] \to \mathbb{R}^2 \setminus \{0\}$ is a smooth closed curve parametrized by t, the integral

$$\int_\gamma \omega = \int_a^b d\theta$$

gives the total change in the angle as you traverse γ. Dividing by 2π normalizes this change to count the number of complete rotations (or "windings") around the origin:

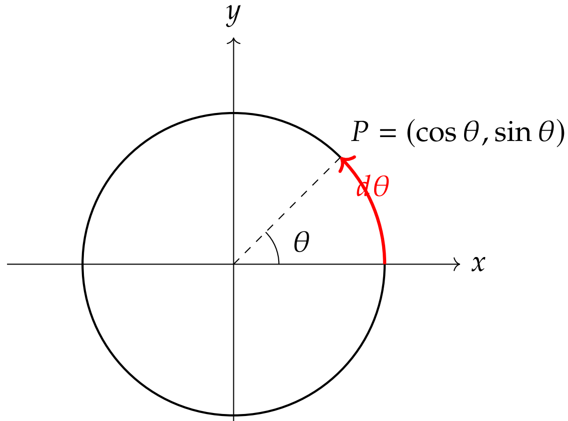
$$\mathrm{wind}(\gamma, 0) = \frac{1}{2\pi}\int_\gamma \omega \in \mathbb{Z}.$$

- **Global Issue of θ:** Although the function θ is not globally well-defined (because it is multivalued modulo 2π), the differential $d\theta$ is a well-defined 1–form on $\mathbb{R}^2 \setminus \{0\}$. This is why ω is a powerful tool in defining the winding number.

4. Summary

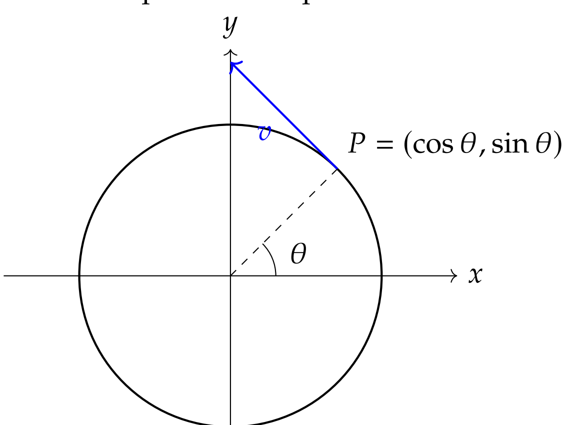
- The form $\omega = \frac{x\,dy - y\,dx}{x^2+y^2}$ is equivalent to $d\theta$ when expressed in polar coordinates. - It captures the infinitesimal angular change along a path. - Integrating ω along a closed curve yields 2π times the winding number of the curve around the origin.

This formalism is central in several areas of mathematics, including complex analysis, differential geometry, and algebraic topology, where it provides a rigorous way to measure how many



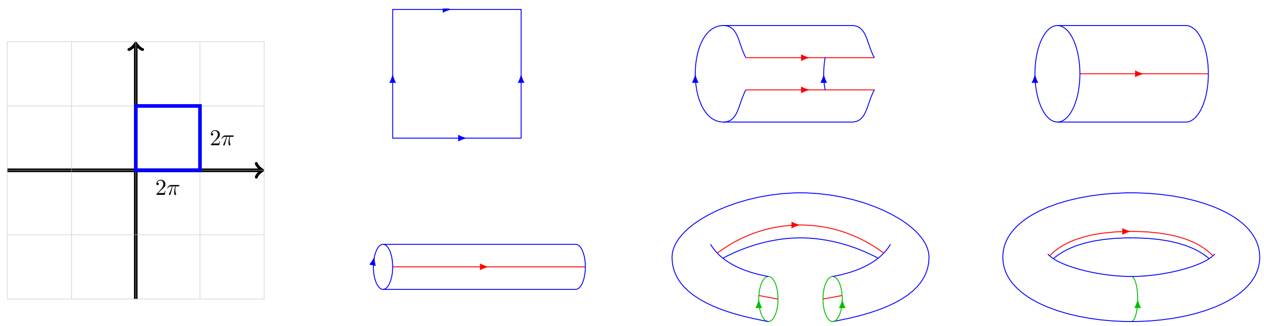
In polar coordinates,
$$\frac{x\,dy - y\,dx}{x^2 + y^2} = d\theta$$

times a curve wraps around a point.



The 1-form
$\omega = \frac{x\,dy - y\,dx}{x^2+y^2}$
assigns a real number to each tangent vector,
here giving the angular change $d\theta$.

2 Torus



Consider the Cartesian product $\mathbb{R}^2$ and the subgroup $\mathbb{Z}^2 \subset \mathbb{R}^2$, where

$$\mathbb{Z}^2 = \{(m,n) \mid m,n \in \mathbb{Z}\}.$$

The quotient space is defined by the equivalence relation on $\mathbb{R}^2$

$$(x,y) \sim (x',y') \quad \Longleftrightarrow \quad (x-x', y-y') \in \mathbb{Z}^2.$$

Denote the quotient by

$$T^2 := \mathbb{R}^2/\mathbb{Z}^2.$$

We claim that there exists a natural homeomorphism

$$\Phi : \mathbb{R}^2/\mathbb{Z}^2 \to (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}).$$

Define the map

$$\Phi\big((x,y) + \mathbb{Z}^2\big) = \big(x + \mathbb{Z},\ y + \mathbb{Z}\big).$$

This mapping is well-defined because if (x,y) and (x',y') represent the same equivalence class, then $(x-x', y-y') \in \mathbb{Z}^2$ and hence $x + \mathbb{Z} = x' + \mathbb{Z}$ and $y + \mathbb{Z} = y' + \mathbb{Z}$. Moreover, Φ is bijective and continuous with continuous inverse.

Since we have already established that $\mathbb{R}/\mathbb{Z} \cong S^1$ (via the homeomorphism φ), it follows that

$$(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) \cong S^1 \times S^1.$$

By definition, the torus is the topological space

$$T^2 \equiv S^1 \times S^1.$$

Thus, we conclude that

$$\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1,$$

which shows that $\mathbb{R}^2/\mathbb{Z}^2$ is, indeed, a torus.

Summary in Symbolic Notation

1. Let

$$\mathbb{R}/\mathbb{Z} \overset{\varphi}{\cong} S^1, \quad \varphi(x+\mathbb{Z}) = e^{2\pi i x}.$$

2. Define the quotient

$$T^2 = \mathbb{R}^2/\mathbb{Z}^2, \quad (x,y) \sim (x',y') \iff (x-x', y-y') \in \mathbb{Z}^2.$$

3. Then the canonical isomorphism

$$\Phi : \mathbb{R}^2/\mathbb{Z}^2 \to (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}), \quad \Phi((x,y)+\mathbb{Z}^2) = (x+\mathbb{Z},\ y+\mathbb{Z})$$

implies

$$\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1,$$

and hence, T^2 is topologically a torus.

This completes the formal demonstration that $\mathbb{R}^2/\mathbb{Z}^2$ is a torus.

From the perspective of algebraic geometry—what you refer to as "zariskitology"—it is indeed more natural and fruitful to work with complex tori rather than real tori. Here's why:

1. Enhanced Structure of $\mathbb{C}$ vs. $\mathbb{R}^2$

- **Complex Field $\mathbb{C}$:** The field $\mathbb{C}$ carries a richer structure than $\mathbb{R}^2$ when viewed as a real vector space. In particular, $\mathbb{C}$ is not only a two-dimensional real vector space but also a one-dimensional complex vector space. This extra algebraic structure permits the definition of holomorphic functions and endows the quotient $\mathbb{C}/\Lambda$ with the structure of a complex manifold.

- **Real Quotient $\mathbb{R}^2/\mathbb{Z}^2$:** Although $\mathbb{R}^2/\mathbb{Z}^2$ is homeomorphic to the torus $S^1 \times S^1$ in the Euclidean (or metric) topology, it does not naturally carry a complex structure. Thus, from an algebraic-geometric point of view, the analytic and algebraic tools available over $\mathbb{C}$ are not directly applicable.

2. Complex Tori and Elliptic Curves

- **Definition of a Complex Torus:** Let $\Lambda \subset \mathbb{C}$ be a lattice, i.e.,

$$\Lambda = \{\omega_1 m + \omega_2 n \mid m, n \in \mathbb{Z}\},$$

where $\omega_1, \omega_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent. Then the quotient

$$E = \mathbb{C}/\Lambda$$

is a complex torus. It naturally inherits the structure of a compact Riemann surface of genus one.

- **Algebraic Structure:** A fundamental result in complex algebraic geometry states that every complex torus of dimension one is isomorphic (as a complex analytic manifold) to an elliptic curve, i.e., a smooth projective algebraic curve of genus one. This identification allows one to apply tools from the theory of elliptic functions, modular forms, and scheme theory.

- **Zariski Topology:** In algebraic geometry, the Zariski topology on an algebraic variety is defined in terms of polynomial equations. The coordinate rings and the associated structure sheaves are far more natural when working over $\mathbb{C}$ than over $\mathbb{R}$, because algebraically closed fields like $\mathbb{C}$ yield a richer theory (e.g., every non-constant polynomial over $\mathbb{C}$ has a root).

3. Summary and Conclusion

Replacing $\mathbb{R}^2$ with $\mathbb{C}$ in the quotient construction shifts the setting from a purely topological or metric one (as in $\mathbb{R}^2/\mathbb{Z}^2$) to one with both complex analytic and algebraic structure. Concretely, the quotient

$$\mathbb{C}/\Lambda,$$

where Λ is a lattice in $\mathbb{C}$, is not only a topological torus but also a complex manifold and, by the theory of elliptic curves, an algebraic curve. This richer structure is essential in algebraic geometry and offers deeper insights and more powerful tools than what the real quotient $\mathbb{R}^2/\mathbb{Z}^2$ provides.

Thus, from a "zariskitology" (algebraic geometry) perspective, it is indeed better and more natural to consider complex tori $\mathbb{C}/\Lambda$ rather than real tori $\mathbb{R}^2/\mathbb{Z}^2$.