

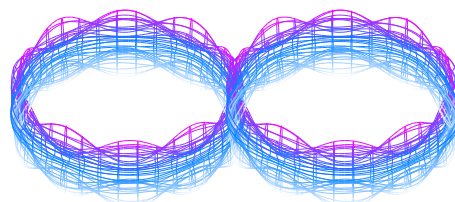
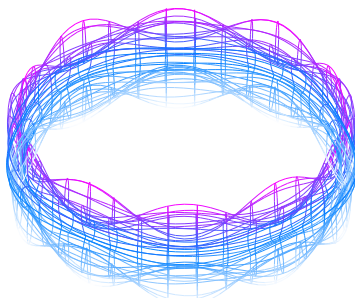
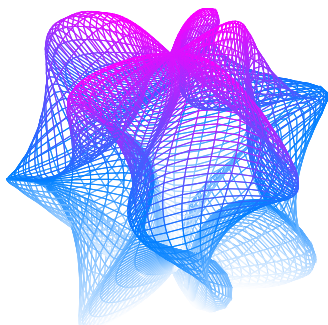
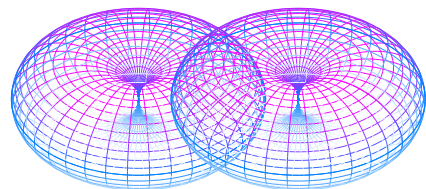
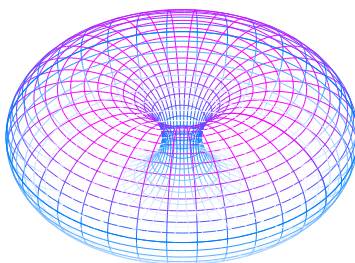
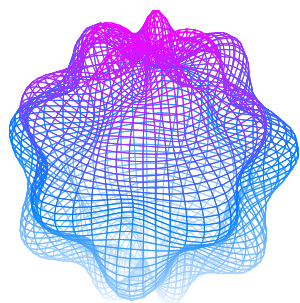
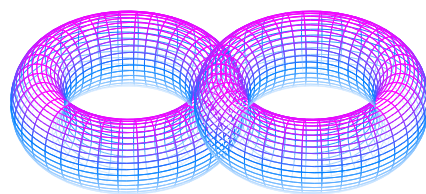
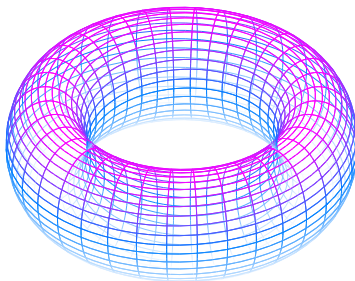
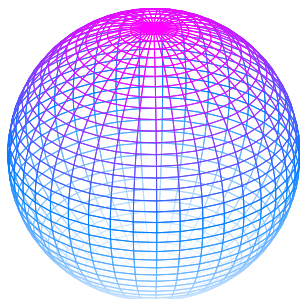
Topology I

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We cover the following topics in this note.

- Topology; Topological Space
 - Open Set
 - Continuous Mapping
 - Distance Function; Metric Topology
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Topology

Definition. Let S be a non-empty set. A **topology**^a on S is a subset

$$\mathcal{T} \subseteq 2^S = \{U : U \subseteq S\}$$

that satisfies the axioms:

(O1) S and \emptyset are elements of \mathcal{T} : $S \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.

(O2)^b The union of an arbitrary collection in \mathcal{T} is an element of \mathcal{T} :

$$\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \mathcal{T} \implies \bigcup_{\alpha \in \Lambda} U_\alpha \in \mathcal{T}.$$

(O3)^c The intersection of any finite collection in \mathcal{T} is an element of \mathcal{T} :

$$\{U_i\}_{i=1}^n \subseteq \mathcal{T} \implies \bigcap_{i=1}^n U_i \in \mathcal{T}.$$

^aThe word “topology” comes from the Greek roots “topos” meaning “place” and “logos” meaning “study”.

^b \mathcal{T} is closed under *arbitrary* unions

^c \mathcal{T} is closed under *finite* intersection

Remark. By mathematical induction, we have

$$O3 \iff [\{U_1, U_2\} \subseteq \mathcal{T} \Rightarrow U_1 \cap U_2 \in \mathcal{T}].$$

Topological Space

Definition. Let $S \neq \emptyset$ be a set. Let \mathcal{T} be a topology on S . Then the ordered pair (S, \mathcal{T}) is called a **topological space**.

Open Set

Definition. Let (S, \mathcal{T}) be a topological space. $U \subseteq S$ is an **open set**, or **open** (in S) iff $U \in \mathcal{T}$.

Remark. A subset $\mathcal{T} \subseteq 2^S$ is a topology on S if and only if

- (i) \emptyset and S are open;
- (ii) Let $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \mathcal{T}$. Then $\bigcup_{\alpha \in \Lambda} U_\alpha$ is open.
- (iii) Let $\{U_i\}_{i=1}^n \subseteq \mathcal{T}$. Then $\bigcap_{i=1}^n U_i$ is open.

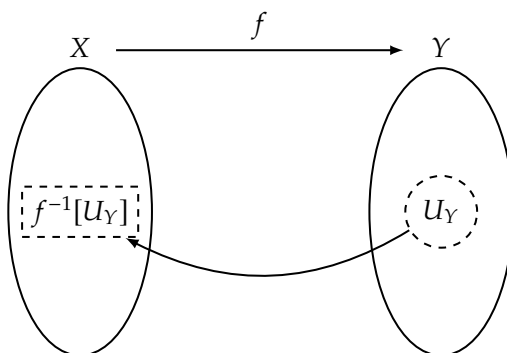
Continuous Mapping by Open Sets

Definition. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are topological spaces. Let $f : X \rightarrow Y$ be a mapping from X to Y .

(1) (Continuous Everywhere) The mapping f is **continuous on X** if and only if

$$U_Y \in \mathcal{T}_Y \implies f^{-1}[U_Y] \in \mathcal{T}_X,$$

where $f^{-1}[U_Y] = \{x \in X : f(x) \in U_Y\}$ is the preimage of U_Y under f .



Note (Preparation for **Example 1**). Let $S \neq \emptyset$ be a set, and let $\{A_\alpha\}_{\alpha \in \Lambda} \subseteq S$. Then

$$\begin{aligned} S \setminus \bigcup_{\alpha \in \Lambda} A_\alpha &= S \setminus \{x \in S : \exists \alpha \in \Lambda \text{ s.t. } x \in A_\alpha\} = \{x \in S : \neg[\exists \alpha \in \Lambda \text{ s.t. } x \in A_\alpha]\} \\ &= \{x \in S : \forall \alpha \in \Lambda, x \notin A_\alpha\} \\ &= \{x \in S : \forall \alpha \in \Lambda, x \in S \setminus A_\alpha\} \\ &= \bigcap_{\alpha \in \Lambda} (S \setminus A_\alpha). \end{aligned}$$

$$\begin{aligned} S \setminus \bigcap_{\alpha \in \Lambda} A_\alpha &= S \setminus \{x \in S : \forall \alpha \in \Lambda, x \in A_\alpha\} = \{x \in S : \neg[\forall \alpha \in \Lambda, x \in A_\alpha]\} \\ &= \{x \in S : \exists \alpha \in \Lambda \text{ s.t. } x \notin A_\alpha\} \\ &= \{x \in S : \exists \alpha \in \Lambda \text{ s.t. } x \in S \setminus A_\alpha\} \\ &= \bigcup_{\alpha \in \Lambda} (S \setminus A_\alpha). \end{aligned}$$

Note (Preparation for **Example 1**).

- (1) A Subset of a Finite Set is Finite.
- (2) The Intersection of Finite Sets is Finite.

Example 1 (Cofinite Topology). Let $S \neq \emptyset$ be a set. Define the cofinite topology $\mathcal{T}_C \subseteq 2^S$ by

$$\begin{aligned}\mathcal{T}_C &:= \{U \subseteq S : S \setminus U \text{ is finite}\} \cup \{\emptyset\} \\ &= \{U \subseteq S : U = \emptyset \text{ or } S \setminus U \text{ is finite}\}.\end{aligned}$$

In other words, U is open in the cofinite topology if U is the empty, or if the complement $S \setminus U$ is a finite set. We claim that \mathcal{T}_C be a topology on S :

(O1) By definition, $\emptyset \in \mathcal{T}_C$. For $U = S$, the complement $S \setminus S = \emptyset$, which is finite, so $S \in \mathcal{T}_C$. Hence, both \emptyset and S are elements of \mathcal{T}_C .

(O2) Let $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \mathcal{T}_C$.

(Case 1) If $U_\alpha = \emptyset$ for all $\alpha \in \Lambda$, then $\bigcup_{\alpha \in \Lambda} U_\alpha = \emptyset \in \mathcal{T}_C$.

(Case 2) Suppose that there exists $\alpha_0 \in \Lambda$ such that $U_{\alpha_0} \neq \emptyset$. Then

$$S \setminus \bigcup_{\alpha \in \Lambda} U_\alpha = \bigcap_{\alpha \in \Lambda} (S \setminus U_\alpha) \subseteq S \setminus U_{\alpha_0}.$$

Since $S \setminus U_{\alpha_0}$ is finite, $S \setminus \bigcup_{\alpha \in \Lambda} U_\alpha$ is finite, so $\bigcup_{\alpha \in \Lambda} U_\alpha \in \mathcal{T}_C$.

(O3) Let $U_1 \in \mathcal{T}_C$ and $U_2 \in \mathcal{T}_C$.

(Case 1) If $U_1 = \emptyset$ or $U_2 = \emptyset$, then $U_1 \cap U_2 = \emptyset \in \mathcal{T}_C$.

(Case 2) Suppose that $U_1 \neq \emptyset$ and $U_2 \neq \emptyset$. Then $S \setminus U_1$ and $S \setminus U_2$ are finite. By the De Morgan law, we have

$$S \setminus (U_1 \cap U_2) = (S \setminus U_1) \cup (S \setminus U_2),$$

which is a finite set. Thus, $U_1 \cap U_2 \in \mathcal{T}_C$.

Example 2 (Discrete Topology). Let $S \neq \emptyset$ be a set, and let $\mathcal{T} = 2^S$ be the power set of S . Then \mathcal{T} is called the **discrete topology** on S and $(S, \mathcal{T}) = (S, 2^S)$ the **discrete (topological) space** on S .

Example 3 (Indiscrete Topology). Let $S \neq \emptyset$ be a set, and let $\mathcal{T} = \{S, \emptyset\}$. Then \mathcal{T} is called the **indiscrete topology** on S and $(S, \mathcal{T}) = (S, \{S, \emptyset\})$ the **indiscrete (topological) space** on S .

Note.

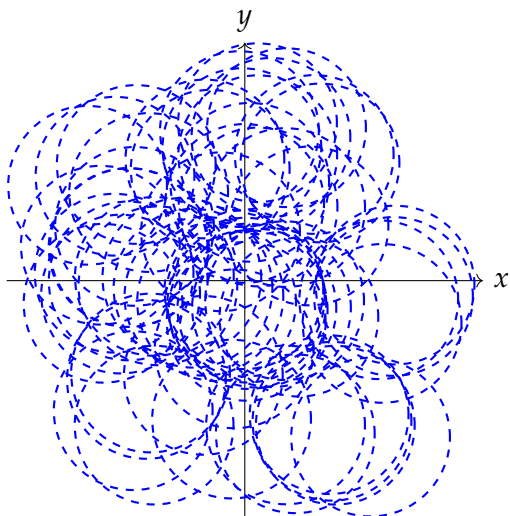
(1) Discrete Topology is Finest Topology.

(2) Indiscrete Topology is Coarsest Topology.

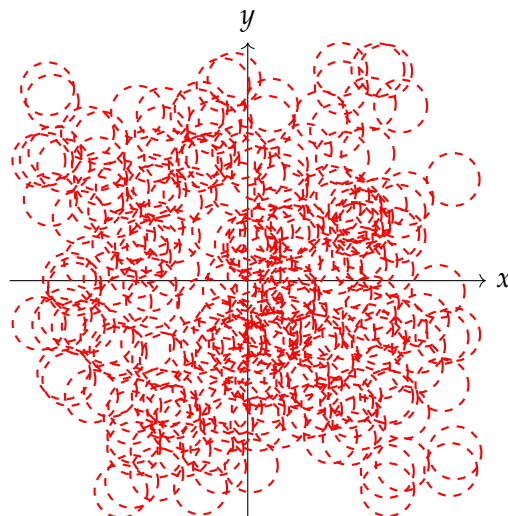
Coarser Topology and Finer Topology

Definition. Let $S \neq \emptyset$ be a set. Let \mathcal{T}_1 and \mathcal{T}_2 be topologies on S .

- (1) \mathcal{T}_1 is said to be **coarser** than \mathcal{T}_2 if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.
- (2) \mathcal{T}_1 is said to be **finer** than \mathcal{T}_2 if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.



Coarser Topology



Finer Topology

Distance Function

Definition. TBA

Metric Topology

Definition. TBA

References

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- [2] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 9. 위상수학 (b) 해석학개론과 거리위상” YouTube Video, 33:43. Published September 29, 2019. URL: <https://www.youtube.com/watch?v=uJ0Gw7Yxk7c&t=242s>.