Advanced Calculus I

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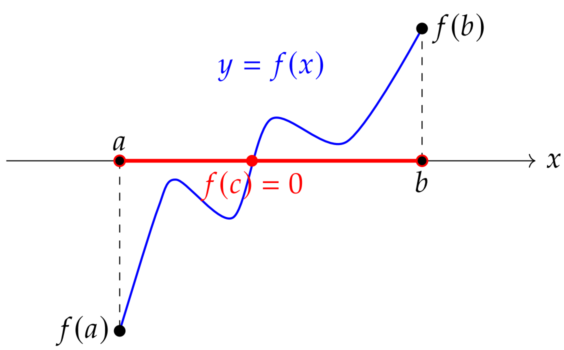
We cover the following topics in this note.

- Boundedness, Supremum and Infimum
- Least Upper Bound Property (Completeness Axiom)
- Well-Ordering Principle and Mathematical Induction
- Archimedean Property

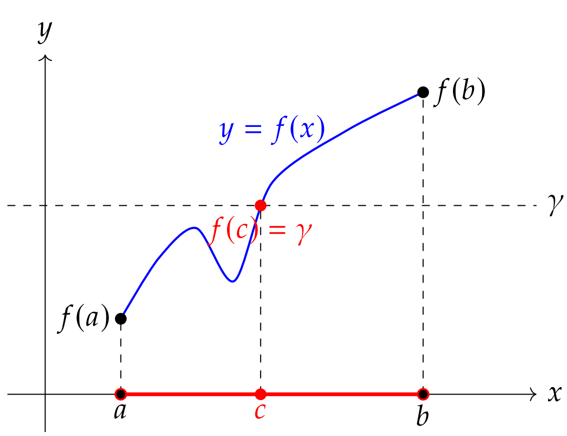
Observation. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Suppose that $f(a)$ and $f(b)$ have opposite signs, i.e., $f(a) \cdot f(b) < 0$. Then, there exists a point $c \in (a,b)$ such that $f(c) = 0$.



Intermediate Value Theorem

Theorem. *Let $[a,b] \subseteq \mathbb{R}$ be a real interval, and let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Let $f(a) < f(b)$. If $\gamma \in \mathbb{R}$ satisfies $f(a) < \gamma < f(b)$, then*

$$\exists c \in (a,b) \text{ such that } f(c) = \gamma.$$



Contents

1 Numbers

$\mathbb{N} := \{1, 2, 3, 4, \dots\}$	**N**atural Numbers
$\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \pm 4, \dots\}$	Integers (**Z**ahlen[1])
$\mathbb{Q} := \left\{\frac{q}{p} : p, q \in \mathbb{Z}, p \neq 0\right\}$	Rationals (**Q**uotient[2])
$\mathbb{R} := \left\{\text{Limit of sequences of rational numbers}\right\}$	**R**eal Numbers
$\mathbb{C} := \left\{p + q\sqrt{-1} : p, q \in \mathbb{R}\right\}$	**C**omplex numbers

Remark. The set $\mathbb{Z}_{\geq 0} := \{0\} \cup \mathbb{N} = \{0, 1, 2, \dots\}$ is called *non-negative integers*.

Remark. Let $n_0 \in \mathbb{Z}$ is given. Then

$$\mathbb{Z}_{\geq n_0} := \{n \in \mathbb{Z} : n \geq n_0\}.$$

[1]The integer set is denoted by $\mathbb{Z}$ because it comes from the German word "Zahlen", meaning "numbers".

[2]The rational set is denoted by $\mathbb{Q}$ because it stands for "Quotient", representing numbers that can be expressed as the quotient of two integers.

2 Least Upper Bound Property of $\mathbb{R}$

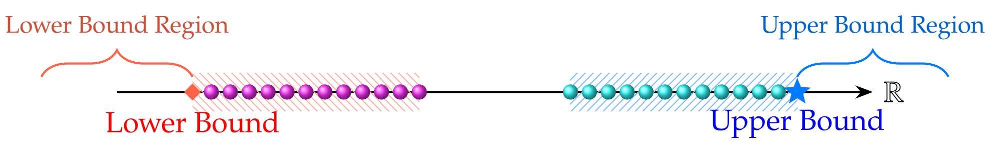
Boundedness

Definition. Let S be a non-empty subset of $\mathbb{R}$.

(1) A set S is said to be **bounded above** if $\boxed{\exists \beta \in \mathbb{R} \text{ such that for all } x \in S,\ x \le \beta}$.
A number β is called an **upper bound** of S.

(2) A set S is said to be **bounded below** if $\boxed{\exists \alpha \in \mathbb{R} \text{ such that for all } x \in S,\ \alpha \le x}$.
A number α is called an **lower bound** of S.

(3) A set S is **bounded** if it is bounded above and below.



Remark (Caution!). It is not guaranteed that $\beta \in S$ and $\alpha \in S$.

Remark. Let $\varnothing \neq S \subseteq \mathbb{R}$.

$$\begin{aligned} S \text{ is bounded above (by } \beta) &\iff S \text{ has an upper bound } \beta \\ \beta \in \mathbb{R} \text{ is an upper bound of } S &\iff \forall x \in S,\ x \le \beta \end{aligned}$$

$$\begin{aligned} S \text{ is bounded below (by } \alpha) &\iff S \text{ has an lower bound } \alpha \\ \alpha \in \mathbb{R} \text{ is an lower bound of } S &\iff \forall x \in S,\ \alpha \le x \end{aligned}$$

Remark.

1. The empty $S = \varnothing$ is bounded.

   (i) ($\varnothing$ is bounded above) We need to find a real number $\beta \in \mathbb{R}$ s.t. for all $x \in \varnothing,\ x \le \beta$. Since "$\forall x \in \varnothing, x \le \beta$" is vacuously true, we can choose any real number as β.

   (ii) ($\varnothing$ is bounded below) Similarly, we can choose any $\alpha \in \mathbb{R}$ s.t. for all $x \in \varnothing$, $\alpha \le x$.

2. Upper bound and lower bound are not unique. A set $S(\neq \varnothing) \subseteq \mathbb{R}$ may have multiple upper bounds and multiple lower bounds.

Exercise. Show that $A = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$ has an upper bound and a lower bound.

Sol. The elements of A are: $A = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\right\}$. Let $x \in A$. Then $x = 1 - 1/n$ for some $n \in \mathbb{N}$.

(i) Since $n \in \mathbb{N}$, we have $1/n > 0$. Therefore

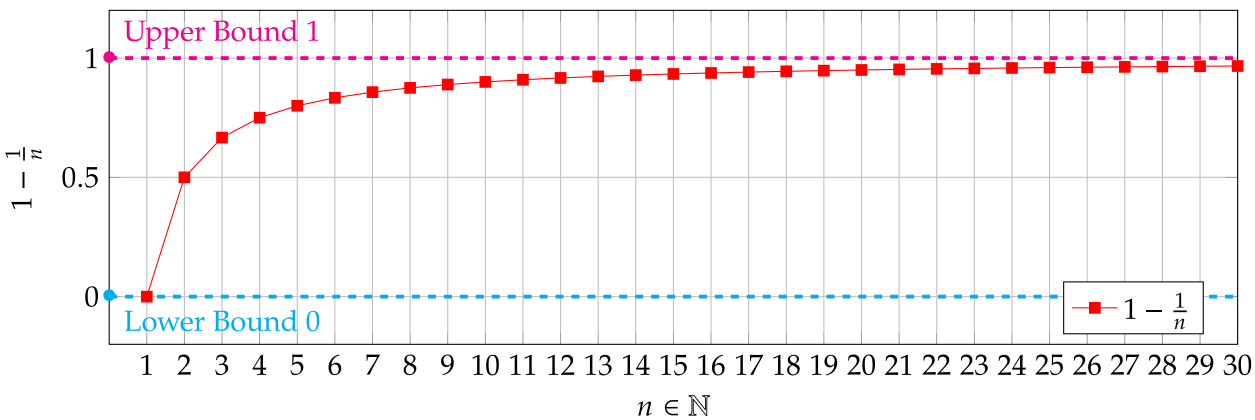
$$1 - \frac{1}{n} < 1.$$

Thus, for all $x = 1 - 1/n \in A$, we have $x \leq 1$. Hence 1 be an upper bound of A.

(ii) Since $n \in \mathbb{N}$, we have $n \geq 1$, so $1/n \leq 1$. Therefore,

$$1 - \frac{1}{n} \geq 1 - 1 = 0.$$

Thus, for all $x = 1 - 1/n \in A$, we have $x \geq 0$. Hence 0 is a lower bound of A.



□

Exercise. Show that $\mathbb{N}$ has a lower bound but does not have an upper bound.

Sol. Let $\mathbb{N}$ is the set of natural numbers.

(i) For each $n \in \mathbb{N}$, we have $n \geq 1$, since the smallest element of $\mathbb{N}$ is 1. Therefore, 1 is a lower bound of $\mathbb{N}$.

(ii) Assume that

$$\exists \beta \in \mathbb{R} \text{ such that } \forall n \in \mathbb{N},\ n \leq \beta.$$

However, for any $M \in \mathbb{R}$, we can always find a natural number $n \in \mathbb{N}$ such that $\beta \leq n$, by choosing $n = \lfloor \beta \rfloor + 1 > \beta$. It is a contradiction. Thus $\mathbb{N}$ does not have an upper bound.

□

Exercise ($\star$). Consider a set

$$A := \left\{ r \in \mathbb{Q} : r > 0,\ r^2 < 2 \right\}$$

of positive rational numbers whose squares are less than 2. Then A has a lower bound 0. Prove that A does not have the maximum element.

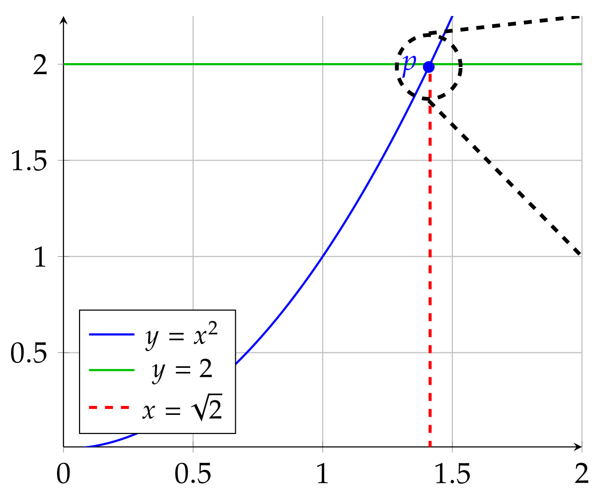
Sol. Note that

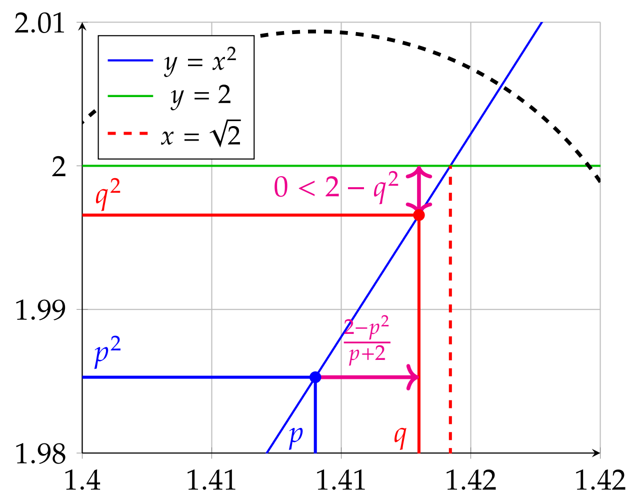
$$A := \left\{ r \in \mathbb{Q} : r > 0,\ r^2 < 2 \right\} = \{\cdots, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.4141213, 1.4142135, \cdots\}.$$

Let $a \in A$. Then

$$a^2 < 2 < 4 \implies a^2 < 4 \implies |a| < 2 \implies a < 2.$$

That is, A is bounded above by 2. Suppose that p is a maximum of A. Since $p \in A$, $p > 0$ and $p^2 < 2$.





Define a rational number $q \in \mathbb{Q}$, for contradiction, by

$$q := p + \frac{2 - p^2}{p + 2} = \frac{2p + 2}{p + 2},$$

where $p < q$. We must show that $q \in A$.

Clearly $q > 0$. We claim that $q^2 < 2$:

$$\begin{aligned}
2 - q^2 = 2 - \left(p + \frac{2-p^2}{p+2}\right)^2 &= 2 - \left(p^2 + \frac{2p(2-p^2)}{p+2} + \frac{(2-p^2)^2}{(p+2)^2}\right) \\
&= 2 - p^2 - \frac{2p(2-p^2)}{p+2} - \frac{(2-p^2)^2}{(p+2)^2} \\
&= \frac{(2-p^2)(p+2)^2 - 2p(2-p^2)(p+2) - (2-p^2)^2}{(p+2)^2} \\
&= \frac{1}{(p+2)^2}\left[(2-p^2)(p^2+4p+4) + (2p^2+4p)(p^2-2) + (-p^4+4p^2-4)\right] \\
&= \frac{1}{(p+2)^2}\left[\cancelto{0}{(-1+2-1)p^4} + \cancelto{0}{(-4+4)p^3} + (2-4-\cancel{4}+\cancel{4})p^2 + \cancelto{0}{(8-8)p} + (8-4)\right] \\
&= \frac{1}{(p+2)^2}\left[-2p^2+4\right] = \frac{2(2-p^2)}{(p+2)^2} > 0.
\end{aligned}$$

Thus, $q \in A$ with $p < q$. It contradicts to the assumption that $p = \max A$. Hence it is proved. $\square$

Note (Existence of $\sqrt{2}$). There exists $x \in \mathbb{R}$ such that $x^2 = 2$. We write $x = \sqrt{2} > 0$.

Proof. Consider a set

$$A = \left\{x \in \mathbb{R} : x > 0,\ x^2 < 2\right\} \subseteq \mathbb{R}.$$

Since $1 \in A$, we know $A \neq \varnothing$. Clearly, A is bounded above (by 2). By the completeness axiom, A has the supremum $\lambda = \sup A \in \mathbb{R}$. Define a number $\mu \in \mathbb{R}$ by

$$\mu := \lambda + \frac{2-\lambda^2}{\lambda+2}, \text{ and then } \mu^2 - 2 = \frac{2(\lambda^2-2)}{(\lambda+2)^2}.$$

(Case I) Let $\lambda^2 < 2$. Then $\lambda < \mu$ and $\mu^2 < 2$. That is, $\mu \in A$ with $\lambda < \mu$. It contradicts to the fact that λ is an upper bound of A.

(Case II) Let $\lambda^2 > 2$. Then

$$\begin{aligned}
\mu < \lambda \text{ and } 2 < \mu^2 \implies \forall x \in A,\ x^2 < 2 < \mu^2 &\implies \forall x \in A,\ x < \mu \\
&\implies \mu \text{ is an upper bound of } A \text{ with } \mu < \lambda.
\end{aligned}$$

It contradicts to the fact that λ is an upper bound of A.

By trichotomy, we obtain $\lambda^2 = 2$ with $\lambda \in \mathbb{R}$. $\square$

Supremum and Infimum

Definition. Let $\varnothing \neq S \subseteq X$.

(1) Let S is bounded above. The number $\beta \in \mathbb{R}$ is the **supremum** (or the **least upper bound**) of S if and only if

(i) β is an upper bound of S, i.e., $\forall x \in S,\ x \leq \beta$;

(ii) u is any upper bound of $S \implies \beta \leq u$.

We write $\boxed{\beta = \sup S} \in \mathbb{R}$.

(2) Let S is bounded below. The number $\alpha \in \mathbb{R}$ is the **infimum** (or the **greatest lower bound**) of S if and only if

(i) α is an lower bound of S, i.e., $\forall x \in S,\ \alpha \leq x$;

(ii) if ℓ is any lower bound of S then $\ell \leq \alpha$.

We write $\boxed{\alpha = \inf S} \in \mathbb{R}$.

Remark (Caution!). It is not guaranteed that $\sup S \in S$ and that $\inf S \in S$.

Remark. Let $\varnothing \neq S \subseteq \mathbb{R}$.

(1) Suppose S is bounded above. Then

$$\begin{aligned}\beta = \sup S \iff & \text{(i) } \forall x \in S, x \leq \beta; \\ & \text{(ii) } \forall \text{upper bound } u \in S,\ \beta \leq u.\end{aligned}$$

(2) Suppose S is bounded below. Then

$$\begin{aligned}\alpha = \inf S \iff & \text{(i) } \forall x \in S, \alpha \leq x; \\ & \text{(ii) } \forall \text{lower bound } \ell \in S,\ \ell \leq \alpha.\end{aligned}$$

Remark.

$$\begin{aligned}[u \text{ is any upper bound of } S \implies \beta \leq u] &\iff [u < \beta \implies \beta \text{ is NOT an upper bound of } S] \\ &\iff \beta \leq u \text{ for all upper bound } u \text{ of } S.\end{aligned}$$

$$\begin{aligned}[\ell \text{ is any lower bound of } S \implies \ell \leq \alpha] &\iff [\alpha < \ell \implies \alpha \text{ is NOT an lower bound of } S] \\ &\iff \ell \leq \alpha \text{ for all lower bound } \ell \text{ of } S.\end{aligned}$$

Remark (Uniqueness of Supremum and Infimum).

(*Proof by Trichotomy*) Let $\varnothing \neq T \subseteq \mathbb{R}$ and T is bounded above. Suppose that $\sup T = a$ and $\sup T = b$ also. By trichotomy, exactly one of the following holds:

$$a = b,\ a < b,\ \text{or } b < a.$$

However, $a < b$ and $b < a$ are impossible, as a and b are upper bounds, respectively. Hence $a = b$. Similarly, infimum also is unique. □

(*Proof by Anti-symmetry*[3] *of* $\leq$) Let $\varnothing \neq S \subseteq \mathbb{R}$ and S is bounded. Suppose that $\sup S = a$ and that $\sup S = b$ also. Then

(a) a is an upper bound of S in $\mathbb{R}$; (b) a is a supremum of S in $\mathbb{R}$;

(c) b is an upper bound of S in $\mathbb{R}$; (d) b is a supremum of S in $\mathbb{R}$.

By (a) and (d), we have $b \leq a$, and by (b) and (c), we have $a \leq b$. Thus by the anti-symmetry of $\leq$: $a = b$. Similarly, infimum also is unique. □

Unbounded Sets

Definition. Let $\varnothing \neq S \subseteq \mathbb{R}$.

(1) If S is NOT bounded above, then we write $\boxed{\sup S = \infty}$.

(2) If S is NOT bounded below, then we write $\boxed{\inf S = -\infty}$.

(3) $\sup \varnothing := -\infty$ and $\inf \varnothing := \infty$.

Example. $\sup \mathbb{N} = \infty$ and $\inf \mathbb{Z} = -\infty$.

Remark. Suppose that $\varnothing \neq S \subseteq \mathbb{R}$ is not bounded above. Then

$$\neg[\exists \beta \in \mathbb{R} \text{ s.t. } \forall x \in S,\ x \leq \beta] \equiv [\forall \beta \in \mathbb{R},\ \exists x_\beta \in S \text{ s.t. } \beta < x_\beta].$$

Suppose that $\varnothing \neq T \subseteq \mathbb{R}$ is not bounded below. Then

$$\neg[\exists \alpha \in \mathbb{R} \text{ s.t. } \forall x \in S,\ \alpha \leq x] \equiv [\forall \alpha \in \mathbb{R},\ \exists x_\alpha \in S \text{ s.t. } x_\alpha < \alpha].$$

[3] A relation $\mathcal{R}$ on a set S is anti-symmetric if, for $a, b \in \mathcal{R}$, $a\,\mathcal{R}\,b \wedge b\,\mathcal{R}\,a \implies a = b$.

Approximation Property for Supremum and Infinum I

Proposition 1.

(1) Let $\varnothing \neq S \subseteq \mathbb{R}$ which is bounded above, and let λ be an upper bound of S in $\mathbb{R}$.

$$\lambda = \sup S \iff \forall \varepsilon > 0,\ \exists x_\varepsilon \in S \text{ s.t. } \lambda - \varepsilon < x_\varepsilon \leq \lambda.$$

(2) Let $\varnothing \neq T \subseteq \mathbb{R}$ which is bounded below, and let γ be a lower bound of T in $\mathbb{R}$.

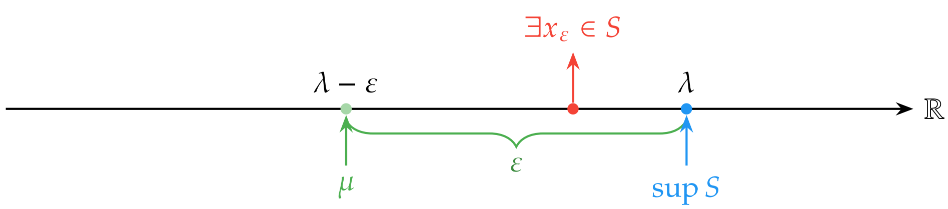
$$\gamma = \inf T \iff \forall \varepsilon > 0,\ \exists x_\varepsilon \in T \text{ s.t. } \gamma \leq x_\varepsilon < \gamma + \varepsilon.$$

Proof. (1) ($\Rightarrow$) Let $\lambda = \sup S$. Since λ is an upper bound, $\forall x \in S,\ x \leq \lambda$. That is, $x_\varepsilon \leq \lambda$ holds.
Let $\varepsilon > 0$. Suppose that

$$\neg[\exists x_\varepsilon \in S \text{ s.t. } \lambda - \varepsilon < x_\varepsilon] \equiv [\forall x_\varepsilon \in S,\ x_\varepsilon \leq \lambda - \varepsilon].$$

Then $\lambda - \varepsilon$ be an upper bound. Then $\lambda - \varepsilon < \lambda$ but λ is the *least* upper bound. It is a contradiction. Therefore $\exists x_\varepsilon \in S$ s.t. $\lambda - \varepsilon < x_\varepsilon$.

($\Leftarrow$) Let the RHS holds. We claim that λ is the *least* upper bound of S:



Assume, for contradiction, that there exists a smaller upper bound $\mu \in \mathbb{R}$ s.t. $\mu < \lambda$.
Let $\varepsilon = \lambda - \mu > 0$. Then

$$\exists x_\varepsilon \in S \text{ such that } \lambda - \varepsilon < x_\epsilon \leq \lambda,$$

and so

$$\mu = \lambda - \varepsilon < x_\varepsilon \leq \lambda \implies \mu < x_\varepsilon \leq \lambda.$$

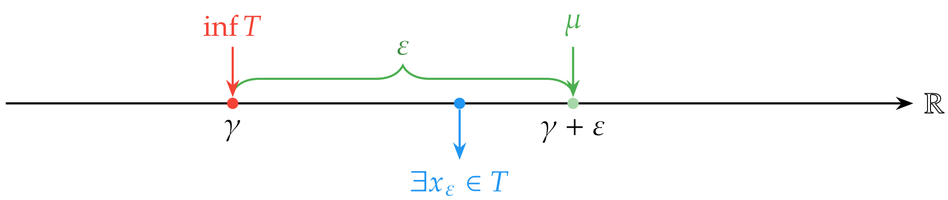
It contradicts to the assumption that μ is an upper bound of S. Thus, λ be the least upper bound of S.

(2) (⇒) Let $\gamma = \inf T$. Since γ is a lower bound, $\forall x \in S,\ \gamma \leq x$. That is, $\gamma \leq x_\varepsilon$ holds.
Let $\varepsilon > 0$. Suppose that

$$\neg[\exists x \in T \text{ s.t. } x < \gamma + \varepsilon] \equiv [\forall x \in T,\ \gamma + \varepsilon \leq x].$$

Then $\gamma + \varepsilon$ be a lower bound. Then $\gamma < \gamma + \varepsilon$ but γ is the *greatest* lower bound. It is a contradiction. Therefore $\exists x_\varepsilon \in T$ s.t. $x_\varepsilon < \gamma + \varepsilon$.

(⇐) Let the RHS holds. We claim that γ is the *greatest* lower bound of T:



Assume, for contradiction, that there exists a greater lower bound $\gamma < \mu$.
Let $\varepsilon = \mu - \gamma > 0$. Then

$$\exists x_\varepsilon \in T \text{ such that } \gamma \leq x_\varepsilon < \gamma + \varepsilon,$$

and so $\gamma \leq x_\varepsilon < \mu$. It contradicts to the assumption that γ is an lower bound of T. Thus, γ be the greatest lower bound of T.

□

Remark. See **Approximation Property for Supremum and Infinum II**.

Least Upper Bound Property (Completeness Axiom) of Real Number

Axiom. Every non-empty subset of $\mathbb{R}$ that is bounded above has the <u>supremum</u> in $\mathbb{R}$.

Example. $\mathbb{Q}$ does NOT hold completeness axiom. We already showed that $\{x \in \mathbb{Q} : x > 0,\ x^2 < 2\}$ has NO supremum in $\mathbb{Q}$.

Infimum Property

Axiom. Every non-empty subset of $\mathbb{R}$ that is bounded below has the <u>infimum</u> in $\mathbb{R}$.

3 Well-Ordering Principle and Mathematical Induction

Well-Ordering Principle (Principle of the Least Element)

Axiom. Every non-empty subset S of $\mathbb{N}$ has a least element, i.e.,

$$\varnothing \neq S \subseteq \mathbb{N} \implies \exists n \in S \text{ s.t. } \forall k \in S,\ n \leq k.$$

In other words, $[\varnothing \neq S \subseteq \mathbb{N} \Rightarrow \exists n \in S \text{ s.t. } n = \min(S)]$.

Remark (general version). $\varnothing \neq S \subseteq \mathbb{Z}_{\geq n_0} \implies \exists n \in S \text{ s.t. } n = \min S \geq n_0$.

Principle of Mathematical Induction

Axiom. Suppose that $S \subseteq \mathbb{N}$ satisfies the following two conditions:

1. (Basic Step) $1 \in S$, and
2. (Inductive Step) $n \in S \implies n+1 \in S$.

Then $S = \mathbb{N}$.

Remark (general version). Let $n_0 \in \mathbb{Z}$ be given, and let $S \subseteq \mathbb{Z}_{\geq n_0}$. Suppose that S satisfies the following two conditions:

1. (Basic Step) $n_0 \in S$, and
2. (Inductive Step) $\forall n \in \mathbb{Z}_{\geq n_0} : [n \in S \implies n+1 \in S]$.

Then $\forall n \in \mathbb{Z}_{\geq n_0} : n \in S$, i.e., $S = \mathbb{Z}_{\geq n_0}$.

Remark. To show that a mathematical statement $P(n)$ (property for n) holds for $n \in \mathbb{N}$, simply verify that the set

$$S := \{n \in \mathbb{N} : P(n) \text{ holds}\}$$

satisfies the following conditions:

(Step 1) Show that $P(1)$ holds.

(Step 2) Show that $P(n+1)$ holds with the assumption $P(n)$ holds.

Equivalence of Well-Ordering Principle and Induction

Theorem.

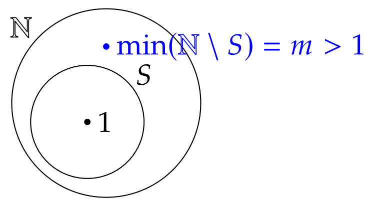
The Well-Ordering Principle and Principle of Mathematical Induction are equivalent.

Proof. **(WOP $\Rightarrow$ MI)** Let $S \subseteq \mathbb{N}$ satisfy the followings: (i) $1 \in S$ and (ii) $k \in S \Rightarrow k+1 \in S$. We want to establish that $S = \mathbb{N}$ by the Well-Ordering Principle (WOP).

Assume for contradiction that $S \neq \mathbb{N}$. Then $S \subsetneq \mathbb{N}$, which means $\mathbb{N} \setminus S \neq \emptyset$. By the WOP,

$$\exists m \in \mathbb{N} \setminus S \text{ s.t. } m = \min(\mathbb{N} \setminus S).$$

Since $1 \in S$, we have $1 \notin \mathbb{N} \setminus S$, so $m \neq 1$ and thus $m > 1$ (or $m \geq 2$).



Then

$$m = \min(\mathbb{N} \setminus S) \overset{\text{by minimality of } m}{\Longrightarrow} m - 1 \notin \mathbb{N} \setminus S \implies m - 1 \in S \overset{\text{by (ii)}}{\Longrightarrow} m \in S \ ↯.$$

Hence $S = \mathbb{N}$.

(MI $\Rightarrow$ WOP) Suppose that $\varnothing \neq S \subseteq \mathbb{N}$ has no least element. Define the set $T \subseteq \mathbb{N}$ by

$$T := \{n \in \mathbb{N} : 1, 2, 3, \ldots, n \notin S\}.$$

For example, if $3 \in T$ then $1, 2, 3 \notin S$; conversely, if $1, 2, 3 \notin S$ then $3 \in T$. We claim that T satisfies the condition of MI:

(i) (Basic Step) Since S has no least element, $1 \notin S$. Therefore, $1 \in T$.

(ii) (Inductive Step) Suppose that $k \in T$. This means that $1, 2, \ldots, k \notin S$. Since S has no least element, $k + 1 \notin S$ (otherwise $k + 1$ would be a least element of S). Therefore

$$1, 2, \ldots, k, k+1 \notin S, \text{ i.e., } k + 1 \in T.$$

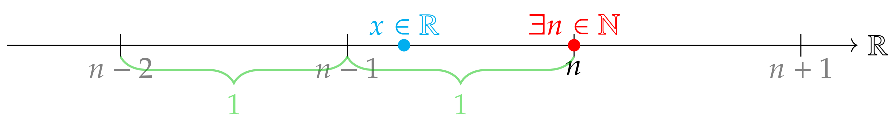
By the Principle of Mathematical Induction, we have $T = \mathbb{N}$. It follows that no natural number is in S, which contradicts $S \neq \varnothing$. Hence it is proved. $\square$

4 Archimedean Principle

Archimedean Property (The Unboundedness of Natural Numbers)

Theorem. *Let $x \in \mathbb{R}$. Then*

$$\exists n \in \mathbb{N} \text{ such that } x < n.$$



Proof. Assume, for contradiction, that

$$\forall n \in \mathbb{N},\ n \le x.$$

That is $\mathbb{N} \subseteq \mathbb{R}$ is bounded above by $x \in \mathbb{R}$. By the completeness axiom of $\mathbb{R}$,

$$\exists \sup \mathbb{N} =: s \in \mathbb{R}.$$

By **Proposition 1**, for $\varepsilon = 1 > 0$, $\exists n \in \mathbb{N}$ s.t. $s - 1 < n$. Then

$$s - 1 < n \implies s < n + 1 \overset{n+1 \in \mathbb{N}}{\implies} s < n + 1 \le \sup \mathbb{N} = s\ ↯.$$

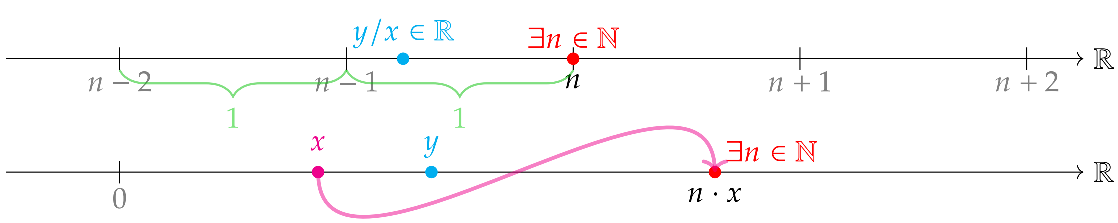
□

Corollary. *Let $x, y \in \mathbb{R}$ with $x > 0$. Then $\exists n \in \mathbb{N}$ such that $y < n \cdot x$.*

Proof. Since $\mathbb{R}$ is a field, we know $\frac{y}{x} \in \mathbb{R}$. By the Archimedean property,

$$\exists n \in \mathbb{N} \text{ such that } \frac{y}{x} < n, \text{ i.e., } y < n \cdot x.$$

□

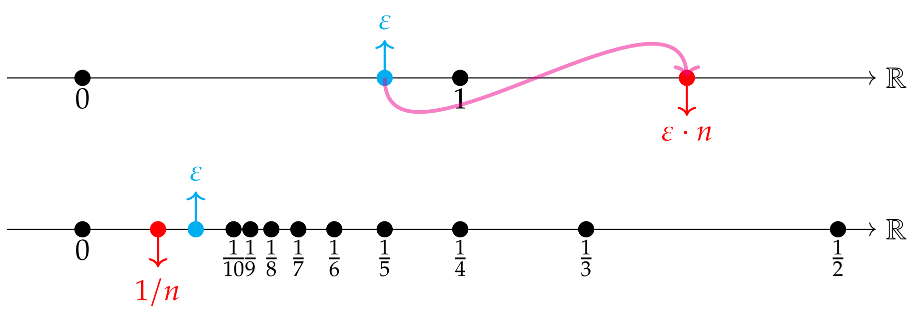


Corollary. $\forall \varepsilon > 0,\ \exists n \in \mathbb{N}$ *such that* $\frac{1}{n} < \varepsilon$.

Proof. For $\varepsilon \in \mathbb{R}^+$ and $1 \in \mathbb{R}$, by Archimedean property, we have

$$\exists n \in \mathbb{N} \text{ such that } 1 < \varepsilon \cdot n, \text{ i.e., } \frac{1}{n} < \varepsilon.$$

□



Note (Archimedean Property in Number Theory). Let $a, b \in \mathbb{N}$. Then $\exists n \in \mathbb{N}$ such that $b < na$.

Proof. Suppose that $\exists a, b \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, na \leq b.$$

Define a set S by

$$S := \{b - na \geq 0 : n \in \mathbb{N}\} \subseteq \mathbb{Z}_{\geq 0}.$$

By the well-ordering principle, $\exists s := \min S$. Since $s = \min S \in S$, we have

$$s = b - ma \text{ for some } m \in \mathbb{N}.$$

Since $m + 1 \in \mathbb{N}$ also, we have $b - (m+1)a \in S$, and so

$$\begin{aligned} b - (m+1)a &= b - ma - a \\ &< b - ma \quad \because a \in \mathbb{N}, \text{i.e., } a > 0 \\ &= \min S \ ↯. \end{aligned}$$

Hence it is proved. □

Approximation Property for Supremum and Infinum II

Proposition 2.

(1) Let $\varnothing \neq S \subseteq \mathbb{R}$ which is bounded above, and let λ be an upper bound of S in $\mathbb{R}$.

$$\begin{aligned} \lambda = \sup S &\iff \forall \varepsilon > 0,\ \exists x_\varepsilon \in S \text{ s.t. } \lambda - \varepsilon < x_\varepsilon \le \lambda \\ &\iff \forall n \in \mathbb{N},\ \exists x_n \in S \text{ s.t. } \lambda - \frac{1}{n} < x_n \le \lambda. \end{aligned}$$

(2) Let $\varnothing \neq T \subseteq \mathbb{R}$ which is bounded below, and let γ be a lower bound of T in $\mathbb{R}$.

$$\begin{aligned} \gamma = \inf T \in \mathbb{R} &\iff \forall \varepsilon > 0,\ \exists x_\varepsilon \in T \text{ s.t. } \gamma \le x_\varepsilon < \gamma + \varepsilon \\ &\iff \forall n \in \mathbb{N},\ \exists x_n \in T \text{ s.t. } \gamma \le x_n < \gamma + \frac{1}{n}. \end{aligned}$$

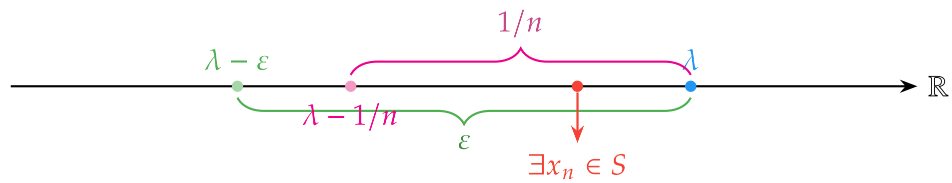
Proof. (1) We NTS that

$$[\forall \varepsilon > 0,\ \exists x_\varepsilon \in S \text{ s.t. } \lambda - \varepsilon < x_\varepsilon \le \lambda] \iff [\forall n \in \mathbb{N},\ \exists x_n \in S \text{ s.t. } \lambda - \frac{1}{n} < x_n \le \lambda].$$

($\Rightarrow$) Assume that the LHS holds. Let $n \in \mathbb{N}$. Since $1/n > 0$, we can take $\varepsilon = 1/n$ for each $n \in \mathbb{N}$. By assumption, for this choice of ε,

$$\exists x_n \in S \text{ s.t. } \lambda - \frac{1}{n} < x_n \le \lambda.$$

($\Leftarrow$) Assume that the RHS holds.



Let $\varepsilon > 0$. By Archimedean property, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$. By assumption, for this n,

$$\lambda - \frac{1}{n} < x_n \le \lambda.$$

Since $1/n < \varepsilon$, we have $\lambda - \varepsilon < \lambda - \frac{1}{n}$, and so

$$\lambda - \varepsilon < \lambda - \frac{1}{n} < x_n \le \lambda, \text{ i.e., } \lambda - \varepsilon < x_n \le \lambda.$$

Letting $x_n = x_\varepsilon$, we see that such an $x_\varepsilon \in S$ exists for any $\varepsilon > 0$.

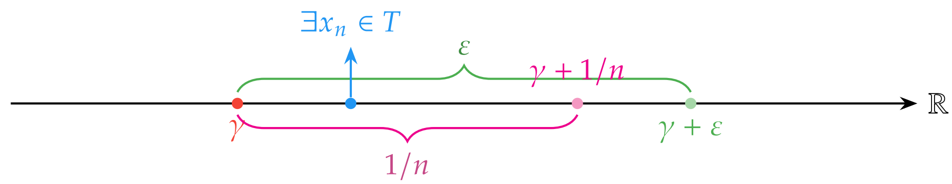
(2) We NTS that

$$[\forall \varepsilon > 0,\ \exists x_\varepsilon \in T \text{ s.t. } \gamma \leq x_\varepsilon < \gamma + \varepsilon] \iff [\forall n \in \mathbb{N},\ \exists x_n \in T \text{ s.t. } \gamma \leq x_n < \gamma + \frac{1}{n}].$$

($\Rightarrow$) Assume that the LHS holds. Let $n \in \mathbb{N}$. Since $1/n > 0$, we can take $\varepsilon = 1/n$ for each $n \in \mathbb{N}$. By assumption, for this choice of ε,

$$\exists x_n \in T \text{ s.t. } \gamma \leq x_n < \gamma + \varepsilon.$$

($\Leftarrow$) Assume that the RHS holds.



Let $\varepsilon > 0$. By Archimedean property, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$. By assumption, for this n,

$$\gamma \leq x_n < \gamma + \frac{1}{n}.$$

Since $1/n < \varepsilon$, we have $\gamma + \frac{1}{n} < \gamma + \varepsilon$, and so

$$\gamma \leq x_n < \gamma + \frac{1}{n} < \gamma + \varepsilon, \text{ i.e., } \gamma \leq x_n < \gamma + \varepsilon.$$

Letting $x_n = x_\varepsilon$, we see that such an $x_\varepsilon \in T$ exists for any $\varepsilon > 0$.

□

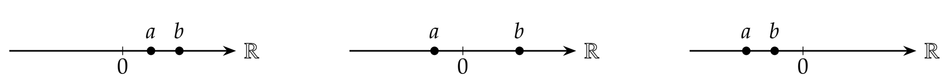
Remark. See **Approximation Property for Supremum and Infinum I**.

Density of the Rationals

Theorem. *Let $a, b \in \mathbb{R}$.*

$$a < b \implies \exists q \in \mathbb{Q} \text{ such that } a < q < b.$$

Proof. Let $a, b \in \mathbb{R}$. Suppose that $a < b$.



We consider the following two cases:

(Case I) $(0 \le a)$ Since $b - a > 0$, we have $\frac{1}{b-a} \in \mathbb{R}$. By the Archimedean property,

$$\exists n \in \mathbb{N} \quad \text{s.t.} \quad \frac{1}{b-a} < n, \text{ i.e., } na + 1 < nb.$$

Clearly $na \in \mathbb{R}$. Define a set A by

$$A := \{k \in \mathbb{N} : na < k\} \subseteq \mathbb{N}.$$

By the Archimedean property, $\exists k \in \mathbb{N}$ such that $na < k$. That is, $A \neq \emptyset$. By the well-ordering principle, $\exists m = \min A$. By the minimality of m, we know $m - 1 \notin A$, i.e., $m - 1 \le na$, and so $m \le na + 1$. Thus, we obtain

$$na \overset{m \in A}{<} m \le na + 1 \overset{\text{by A.P.}}{<} nb.$$

Therefore

$$na < m < ny \implies a < \frac{m}{n} < b.$$

Thus, $q := \frac{m}{n} \in \mathbb{Q}$ satisfies $a < q < b$.

(Case II) $(a < 0)$ Note that $-a \in \mathbb{R}^+$. By the Archimedean property,

$$\exists n \in \mathbb{N} : -a < n, \text{ i.e., } 0 < a + n.$$

By (Case I), we have

$$\exists q \in \mathbb{Q} : a + n < q < b + n$$

since $0 < a + n < b + n$. Let $q' = q - n \in \mathbb{Q}$. Then $a < q' < b$.

$\square$

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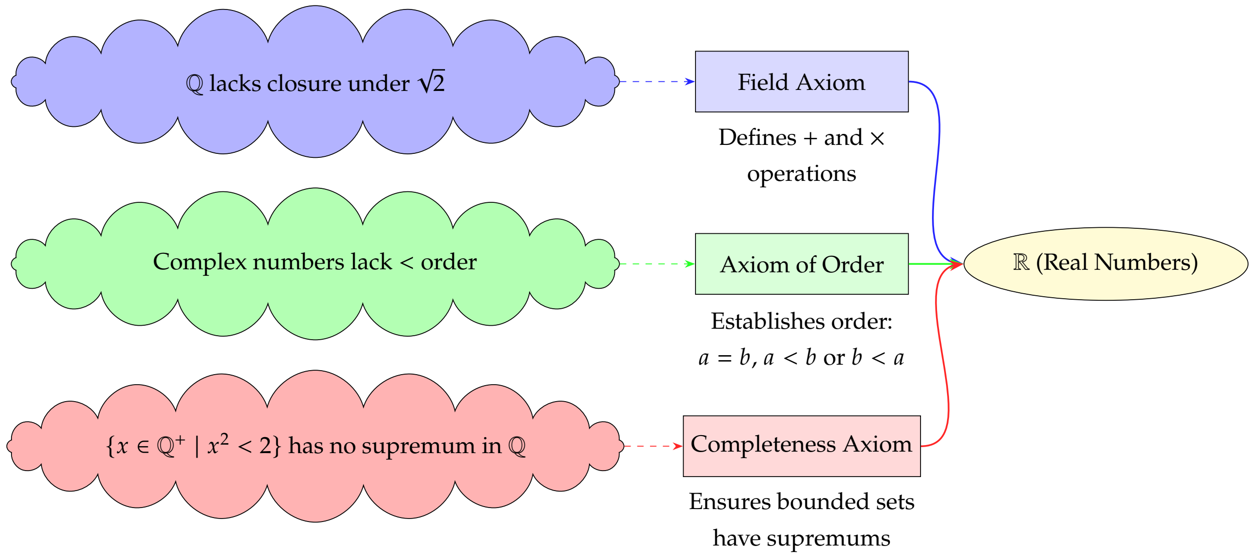
A Axioms of the Real Numbers



The following axioms define the real numbers $\mathbb{R}$ as a complete ordered field.

A.1 Field Axioms

Addition:

1. **Closure under addition**: $\forall a, b \in \mathbb{R},\ a + b \in \mathbb{R}$
2. **Associativity of addition**:$\forall a, b, c \in \mathbb{R},\ (a + b) + c = a + (b + c)$
3. **Commutativity of addition**:$\forall a, b \in \mathbb{R},\ a + b = b + a$
4. **Existence of additive identity**: $\exists 0 \in \mathbb{R}$ such that $\forall a \in \mathbb{R},\ a + 0 = a$
5. **Existence of additive inverses**: $\forall a \in \mathbb{R},\ \exists - a \in \mathbb{R}$ such that $a + (-a) = 0$

Multiplication:

1. **Closure under multiplication**: $\forall a, b \in \mathbb{R},\ a \cdot b \in \mathbb{R}$
2. **Associativity of multiplication**: $\forall a, b, c \in \mathbb{R},\ (a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. **Commutativity of multiplication**: $\forall a, b \in \mathbb{R},\ a \cdot b = b \cdot a$
4. **Existence of multiplicative identity**: $\exists 1 \in \mathbb{R}, 1 \neq 0,$ such that $\forall a \in \mathbb{R},\ a \cdot 1 = a$
5. **Existence of multiplicative inverses**: $\forall a \in \mathbb{R}, a \neq 0,\ \exists a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = 1$

Distributive law:

1. **Distributivity of multiplication over addition**: $\forall a, b, c \in \mathbb{R},\ a \cdot (b + c) = a \cdot b + a \cdot c$

A.2 Axiom of Order

A relation $<$ defined on $\mathbb{R}$ satisfy the followings:

1. **Trichotomy**: For $a, b \in \mathbb{R}$, exactly one of the following holds:

$$a = b, \quad a < b, \quad \text{or} \quad b < a.$$

2. **Transitivity**: For $a, b, c \in \mathbb{R}$,

$$a < b \text{ and } b < c \implies a < c$$

3. **Additive compatibility**: For $a, b, c \in \mathbb{R}$,

$$a < b \implies a + c < b + c$$

4. **Multiplicative compatibility**: For $a, b \in \mathbb{R}$ and $c \in \mathbb{R}^+$,

$$a < b \implies a \cdot c < b \cdot c$$

A.3 Completeness Axiom

The least upper bound property (or **supremum property**):

$$\forall S \subseteq \mathbb{R},\ S \neq \emptyset,\ \text{if } S \text{ is bounded above then } \exists \sup(S) \in \mathbb{R}$$

B Application of Well-Ordering Principle

Theorem. $\sqrt{2}$ *is irrational*, i.e., $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

Proof. Suppose $\sqrt{2} \in \mathbb{Q}$. That is, $\exists p, q \in \mathbb{N}$ s.t. $p\sqrt{2} = q$. Define a set S by

$$S := \left\{ k\sqrt{2} \in \mathbb{N} : k \in \mathbb{N} \right\} \subseteq \mathbb{N}.$$

Since $p\sqrt{2} = q \in \mathbb{N}$, we have $S \neq \varnothing$. By the Well-Ordering Principle,

$$\exists s = \min(S) \in S.$$

Then $s = t\sqrt{2}$ for some $t \in \mathbb{N}$. Define a number

$$r := s\sqrt{2} - s.$$

(Claim 1) $r \in S$:

$$\begin{aligned} r &= s\sqrt{2} - s \\ &= s\sqrt{2} - t\sqrt{2} \\ &= (s-t)\sqrt{2} \\ &\in S & \because s = t\sqrt{2} > t \Rightarrow s - t > 0 \Rightarrow s - t \in \mathbb{N}. \end{aligned}$$

(Claim 2) $r < s = \min(S)$:

$$\begin{aligned} r &= s\sqrt{2} - s \\ &= s(\sqrt{2} - 1) \\ &< s = \min S & \because s \in \mathbb{N} \text{ and } 1 < \sqrt{2} < 2 \Rightarrow 0 < \sqrt{2} - 1 < 1. \end{aligned}$$

It is a contradiction. Hence $\sqrt{2} \notin \mathbb{Q}$. □

C The 2nd Principle of Mathematical Induction

The 2nd Principle of Mathematical Induction

Theorem. *Suppose that $T \subseteq \mathbb{N}$ satisfies the following two conditions:*

1. *(Basic Step)* $1 \in T$, *and*
2. *(Inductive Step)* $1, 2, \ldots n \in T \implies n+1 \in T$.

Then $T = \mathbb{N}$.

Proof. We use the first principle of mathematical induction. Define the set T' by

$$T' := \{n \in \mathbb{N} : 1, 2, \ldots, n \in T\} \subseteq \mathbb{N}.$$

For example, if $1, 2, 3 \in T$ then $3 \in T'$; conversely, if $3 \in T'$ then $1, 2, 3 \in T$. Since $n \in T' \Rightarrow n \in T$, we have $T' \subseteq T \subseteq \mathbb{N}$. We claim that T' satisfies the condition of MI:

(i) (Basic Step) Clearly $1 \in T'$.

(ii) (Inductive Step) Suppose that $k \in T'$. This means that $1, 2, \ldots k \in T$. By condition 2,

$$1, 2, \ldots, k, k+1 \in T, \text{ i.e., } k+1 \in T'.$$

Therefore by the first principle of mathematical induction, $T' = \mathbb{N}$. That is,

$$\mathbb{N} = T' \subseteq T \subseteq \mathbb{N} \implies T = \mathbb{N}.$$

Hence it is proved. □

Remark. To show that a mathematical statement $P(n)$ (property for n) holds for $n \in \mathbb{N}$, verify that the set

$$S := \{n \in \mathbb{N} : P(n) \text{ holds}\}$$

satisfies the following conditions:

(Step 1) Show that $P(1)$ holds.

(Step 2) Show that $P(n+1)$ holds assuming $P(k)$ holds for all $k \leq n$.