Cartier Divisors, Invertible Sheaves, and Holomorphic Line Bundles

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We cover the following topics in this note.

- Cartier Divisors
- Invertible Sheaves
- Holomorphic Line Bundles

Contents

1 Cartier Divisors $D = \{(U_\alpha, f_\alpha)\}$

Slogan. A Cartier divisor is a coherent way to package local winding-form data (i.e. local orders) so it is independent of choices of local equations.

Why Riemann surfaces and meromorphic functions? Let X be a Riemann surface. A holomorphic coordinate change $w = w(z)$ satisfies $w'(p) \neq 0$, so vanishing multiplicity is coordinate-invariant. Restricting to **meromorphic** functions excludes essential singularities (e.g. $e^{1/z}$), ensuring the order at a point is a **finite integer** (zeros and poles have finite order).

Order of a meromorphic function. Let $f \in \mathcal{M}_X(U)$ be nonzero and $p \in U$. Choose a local coordinate z with $z(p) = 0$. Then there exists a unique $k \in \mathbb{Z}$ and a holomorphic g on a neighborhood of p with $g(p) \neq 0$ such that

$$f(z) = z^k g(z).$$

Define

$$\mathrm{ord}_p(f) := k.$$

Equivalently (winding form / logarithmic derivative), for a small positively oriented loop γ around p,

$$\mathrm{ord}_p(f) = \frac{1}{2\pi i} \int_\gamma \frac{df}{f}.$$

Units do not change order. If $h \in \mathcal{O}_X^*(V)$ is holomorphic and nowhere vanishing near p, then $\mathrm{ord}_p(h) = 0$ and hence

$$\mathrm{ord}_p(hf) = \mathrm{ord}_p(h) + \mathrm{ord}_p(f) = \mathrm{ord}_p(f).$$

Analytically: on a sufficiently small simply connected neighborhood $W \ni p$, there exists a holomorphic logarithm H with $e^H = h$, so

$$\frac{dh}{h} = dH \quad \Rightarrow \quad \int_\gamma \frac{dh}{h} = 0$$

for every closed loop $\gamma \subset W$ ("FTC for holomorphic primitives").

Weil divisors (recording integer data at points). A (Weil) divisor on X is a finite formal sum

$$D = \sum_{p \in X} n_p\, p, \qquad n_p \in \mathbb{Z}, \ \ n_p = 0 \text{ for all but finitely many } p.$$

For a nonzero meromorphic function f, its **principal divisor** is

$$(f) := \sum_{p \in X} \operatorname{ord}_p(f)\, p.$$

Cartier divisors (local equations with overlap coherence). A Cartier divisor is given by an open cover $\{U_\alpha\}$ of X and meromorphic functions $f_\alpha \in \mathcal{M}_X^*(U_\alpha)$ such that on overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta$ one has

$$\frac{f_\alpha}{f_\beta} \in \mathcal{O}_X^*(U_{\alpha\beta}).$$

This condition **is exactly** what guarantees that the local orders glue: for any $p \in U_{\alpha\beta}$,

$$\operatorname{ord}_p(f_\alpha) = \operatorname{ord}_p\Big(\frac{f_\alpha}{f_\beta} \cdot f_\beta\Big) = \operatorname{ord}_p\Big(\frac{f_\alpha}{f_\beta}\Big) + \operatorname{ord}_p(f_\beta) = \operatorname{ord}_p(f_\beta),$$

since $\frac{f_\alpha}{f_\beta}$ is a unit.

Winding-form viewpoint on the overlap condition. On $U_{\alpha\beta}$,

$$\frac{df_\alpha}{f_\alpha} - \frac{df_\beta}{f_\beta} = \frac{d(f_\alpha/f_\beta)}{(f_\alpha/f_\beta)}.$$

If f_α/f_β is a holomorphic unit, then locally it has a holomorphic logarithm, so the right-hand side is exact; therefore its integral around a small loop is 0. Hence the winding integrals (and thus the orders) computed from df_α/f_α and df_β/f_β agree.

Conclusion. A Cartier divisor can be viewed as a collection of **local winding/ordering data** presented by meromorphic "local equations" $\{f_\alpha\}$, with the overlap rule $f_\alpha/f_\beta \in \mathcal{O}_X^*$ ensuring the induced integer ord_p is well-defined (choice-independent) at every point.

2 Invertible Sheaf $\mathcal{O}_X(D)$

Let X be a Riemann surface. A Cartier divisor D is given by an open cover $\{U_\alpha\}$ and $f_\alpha \in \mathcal{M}_X^*(U_\alpha)$ such that on overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta$ one has

$$\frac{f_\alpha}{f_\beta} \in \mathcal{O}_X^*(U_{\alpha\beta}).$$

Definition. Define a subsheaf $\mathcal{O}_X(D) \subset \mathcal{M}_X$ by declaring

$$\mathcal{O}_X(D)\big|_{U_\alpha} := f_\alpha^{-1}\,\mathcal{O}_X\big|_{U_\alpha}.$$

Equivalently, for $V \subset U_\alpha$,

$$\mathcal{O}_X(D)(V) = \{\, s \in \mathcal{M}_X(V) \;:\; f_\alpha s \in \mathcal{O}_X(V) \,\}.$$

Gluing on overlaps. On $U_{\alpha\beta}$,

$$f_\alpha^{-1}\mathcal{O}_X = f_\beta^{-1}\left(\frac{f_\beta}{f_\alpha}\right)\mathcal{O}_X = f_\beta^{-1}\mathcal{O}_X,$$

since $\frac{f_\beta}{f_\alpha} \in \mathcal{O}_X^*(U_{\alpha\beta})$. Hence $\mathcal{O}_X(D)$ is well-defined globally.

Invertibility. Multiplication by f_α yields an isomorphism

$$\mathcal{O}_X(D)\big|_{U_\alpha} \xrightarrow{\cdot f_\alpha} \mathcal{O}_X\big|_{U_\alpha},$$

so $\mathcal{O}_X(D)$ is locally free of rank 1, i.e. an invertible sheaf.

Transition functions. If $e_\alpha := f_\alpha^{-1}$ is a local generator of $\mathcal{O}_X(D)$ on U_α, then on $U_{\alpha\beta}$,

$$e_\alpha = \left(\frac{f_\beta}{f_\alpha}\right) e_\beta,$$

so the transition functions are $g_{\alpha\beta} := \frac{f_\beta}{f_\alpha} \in \mathcal{O}_X^*(U_{\alpha\beta})$.

Logarithmic differentials (winding-form view). On $U_{\alpha\beta}$,

$$\frac{df_\alpha}{f_\alpha} - \frac{df_\beta}{f_\beta} = \frac{d(f_\alpha/f_\beta)}{(f_\alpha/f_\beta)}.$$

Since $f_\alpha/f_\beta \in \mathcal{O}_X^*$, locally it admits a holomorphic logarithm, so the right-hand side is exact and its integral over small loops is 0.

3 Cartier divisors and invertible sheaves: the precise dictionary

Let X be a Riemann surface. Write $\mathcal{O}_X$ for the sheaf of holomorphic functions, $\mathcal{O}_X^*$ for nowhere-vanishing holomorphic functions (units), $\mathcal{M}_X$ for meromorphic functions, and $\mathcal{M}_X^*$ for nonzero meromorphic functions.

1) Cartier divisors as a quotient sheaf

A **Cartier divisor** can be defined as a global section of the quotient sheaf

$$\mathcal{M}_X^*/\mathcal{O}_X^*.$$

Concretely, it is given by a cover $\{U_\alpha\}$ and $f_\alpha \in \mathcal{M}_X^*(U_\alpha)$ such that

$$\frac{f_\alpha}{f_\beta} \in \mathcal{O}_X^*(U_{\alpha\beta}) \quad \text{on } U_{\alpha\beta} := U_\alpha \cap U_\beta,$$

and two such collections $\{f_\alpha\}$ and $\{f'_\alpha\}$ define the same Cartier divisor iff $f'_\alpha/f_\alpha \in \mathcal{O}_X^*(U_\alpha)$ for all α.

2) From a Cartier divisor D to the invertible sheaf $\mathcal{O}_X(D)$

Given Cartier data $\{(U_\alpha, f_\alpha)\}$, define

$$\mathcal{O}_X(D)\big|_{U_\alpha} := f_\alpha^{-1}\,\mathcal{O}_X\big|_{U_\alpha} \subset \mathcal{M}_X\big|_{U_\alpha}.$$

Equivalently, for $V \subset U_\alpha$,

$$\mathcal{O}_X(D)(V) = \{\, s \in \mathcal{M}_X(V) \;:\; f_\alpha s \in \mathcal{O}_X(V)\,\}.$$

On overlaps,

$$f_\alpha^{-1}\mathcal{O}_X = f_\beta^{-1}\left(\frac{f_\beta}{f_\alpha}\right)\mathcal{O}_X = f_\beta^{-1}\mathcal{O}_X \quad \text{since } \frac{f_\beta}{f_\alpha} \in \mathcal{O}_X^*,$$

so the local definitions glue. Moreover multiplication by f_α gives an isomorphism

$$\mathcal{O}_X(D)\big|_{U_\alpha} \xrightarrow{\cdot f_\alpha} \mathcal{O}_X\big|_{U_\alpha},$$

hence $\mathcal{O}_X(D)$ is locally free of rank 1, i.e. an invertible sheaf.

3) From an invertible sheaf to a Cartier divisor

Let $\mathcal{L}$ be an invertible sheaf on X. Choose a cover $\{U_\alpha\}$ and local frames e_α identifying

$$\mathcal{L}|_{U_\alpha} \cong \mathcal{O}_X|_{U_\alpha}.$$

On overlaps $U_{\alpha\beta}$ there are transition functions

$$e_\alpha = g_{\alpha\beta}\, e_\beta, \qquad g_{\alpha\beta} \in \mathcal{O}_X^*(U_{\alpha\beta}),$$

and $\{g_{\alpha\beta}\}$ is a Čech 1-cocycle with values in $\mathcal{O}_X^*$. Now pick meromorphic trivializations s_α of $\mathcal{L}$ on U_α (nonzero meromorphic sections); write $s_\alpha = f_\alpha e_\alpha$ with $f_\alpha \in \mathcal{M}_X^*(U_\alpha)$. Then on overlaps,

$$\frac{f_\alpha}{f_\beta} = g_{\alpha\beta} \in \mathcal{O}_X^*(U_{\alpha\beta}),$$

so $\{f_\alpha\}$ defines a Cartier divisor. (Intuitively: $\mathcal{L}$ becomes meromorphically trivial, and the failure of holomorphic triviality is measured by the divisor.)

4) Principal divisors correspond to trivial line bundles

If $D = (f)$ is principal (coming from a global $f \in \mathcal{M}_X^*(X)$), take $f_\alpha = f|_{U_\alpha}$. Then

$$\mathcal{O}_X((f)) = f^{-1}\mathcal{O}_X \cong \mathcal{O}_X$$

via multiplication by f. Thus principal divisors map to the trivial bundle.

5) Picard group and the exact-sequence explanation

The **Picard group** is the group of isomorphism classes of invertible sheaves:

$$\mathrm{Pic}(X) := \{\text{invertible sheaves on } X\}/\cong,$$

with tensor product as the group law.

There is a short exact sequence of sheaves of abelian groups

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{M}_X^* \longrightarrow \mathcal{M}_X^*/\mathcal{O}_X^* \longrightarrow 1.$$

Taking cohomology yields a connecting homomorphism

$$\delta:\ H^0\Big(X, \mathcal{M}_X^*/\mathcal{O}_X^*\Big) \longrightarrow H^1(X, \mathcal{O}_X^*).$$

Here

$$H^0\Big(X, \mathcal{M}_X^*/\mathcal{O}_X^*\Big) \cong \{\text{Cartier divisors on } X\}, \qquad H^1(X, \mathcal{O}_X^*) \cong \mathrm{Pic}(X).$$

Under these identifications, $\delta(D)$ is exactly the isomorphism class of the invertible sheaf $\mathcal{O}_X(D)$.

Moreover, $\ker(\delta)$ consists of divisors that come from global meromorphic functions (principal divisors), so one gets the fundamental identification

$$\mathrm{CaCl}(X) := \frac{\{\text{Cartier divisors}\}}{\{\text{principal Cartier divisors}\}} \cong \mathrm{Pic}(X), \qquad [D] \longmapsto [\mathcal{O}_X(D)].$$

6) How this matches your winding-form intuition

On overlaps $U_{\alpha\beta}$, put $u_{\alpha\beta} := f_\alpha / f_\beta \in \mathcal{O}_X^*$. Then

$$\frac{df_\alpha}{f_\alpha} - \frac{df_\beta}{f_\beta} = \frac{du_{\alpha\beta}}{u_{\alpha\beta}}.$$

Locally (on a small simply connected set) $u_{\alpha\beta} = e^{h_{\alpha\beta}}$ for a holomorphic $h_{\alpha\beta}$, so

$$\frac{du_{\alpha\beta}}{u_{\alpha\beta}} = dh_{\alpha\beta}$$

is exact. Hence small-loop integrals of $\frac{df_\alpha}{f_\alpha}$ agree across overlaps, so the local winding/order data glue. The same units $u_{\alpha\beta}$ are also exactly the transition functions of the line bundle $\mathcal{O}_X(D)$:

$$e_\alpha = u_{\beta\alpha}\, e_\beta \quad \text{with} \quad u_{\beta\alpha} = f_\beta / f_\alpha \in \mathcal{O}_X^*.$$

Sheaves and the definition of $\mathcal{O}_X(D)$

Definition (presheaf and sheaf). Let X be a topological space. A **presheaf** $\mathcal{F}$ (of sets / abelian groups / rings / modules) assigns to each open set $U \subset X$ an object $\mathcal{F}(U)$ and to each inclusion $V \subset U$ a restriction map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$, $s \mapsto s|_V$, such that

$$s|_U = s, \qquad (s|_V)|_W = s|_W \quad (W \subset V \subset U).$$

A presheaf is a **sheaf** if for every open cover $U = \bigcup_i U_i$ the following hold:

(S1) Locality. If $s, t \in \mathcal{F}(U)$ and $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$.
(S2) Gluing. If $s_i \in \mathcal{F}(U_i)$ satisfy $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

Setup. Let X be a Riemann surface. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions and $\mathcal{M}_X$ the sheaf of meromorphic functions. A Cartier divisor D is given by an open cover $\{U_\alpha\}$ and $f_\alpha \in \mathcal{M}_X^*(U_\alpha)$ such that on overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta$,

$$\frac{f_\alpha}{f_\beta} \in \mathcal{O}_X^*(U_{\alpha\beta}).$$

Definition (the sheaf $\mathcal{O}_X(D)$). Define a presheaf $\mathcal{O}_X(D) \subset \mathcal{M}_X$ by

$$\mathcal{O}_X(D)(V) := \left\{ s \in \mathcal{M}_X(V) \ : \ (f_\alpha s)\Big|_{V \cap U_\alpha} \in \mathcal{O}_X(V \cap U_\alpha) \ \forall \alpha \right\}$$

for every open $V \subset X$. Restriction maps are the restrictions in $\mathcal{M}_X$: $s \mapsto s|_W$ for $W \subset V$.

Lemma (well-defined on overlaps). On $V \cap U_{\alpha\beta}$, the condition $(f_\alpha s) \in \mathcal{O}_X$ is equivalent to $(f_\beta s) \in \mathcal{O}_X$.

Proof. On $V \cap U_{\alpha\beta}$,

$$f_\alpha s = \left(\frac{f_\alpha}{f_\beta}\right)(f_\beta s).$$

Since $\frac{f_\alpha}{f_\beta} \in \mathcal{O}_X^*(U_{\alpha\beta})$, multiplication by this factor preserves holomorphicity, hence $f_\alpha s$ is holomorphic iff $f_\beta s$ is holomorphic. □

Proposition (sheaf property). The presheaf $\mathcal{O}_X(D)$ is a sheaf (indeed, an $\mathcal{O}_X$-module subsheaf of $\mathcal{M}_X$).

Proof. Let $V = \bigcup_i V_i$ be an open cover.

Locality (S1). If $s, t \in \mathcal{O}_X(D)(V)$ and $s|_{V_i} = t|_{V_i}$ for all i, then $s = t$ in $\mathcal{M}_X(V)$ because $\mathcal{M}_X$ is a sheaf. Hence $s = t$ in $\mathcal{O}_X(D)(V)$.

Gluing (S2). Let $s_i \in \mathcal{O}_X(D)(V_i)$ satisfy $s_i = s_j$ on $V_i \cap V_j$. Since $\mathcal{M}_X$ is a sheaf, there exists a unique $s \in \mathcal{M}_X(V)$ with $s|_{V_i} = s_i$. We claim $s \in \mathcal{O}_X(D)(V)$.

Fix α. On each $V_i \cap U_\alpha$,

$$(f_\alpha s)|_{V_i \cap U_\alpha} = f_\alpha(s|_{V_i \cap U_\alpha}) = f_\alpha(s_i|_{V_i \cap U_\alpha}),$$

and the right-hand side lies in $\mathcal{O}_X(V_i \cap U_\alpha)$ because $s_i \in \mathcal{O}_X(D)(V_i)$. Moreover these holomorphic functions agree on overlaps because the s_i agree. Since $\mathcal{O}_X$ is a sheaf, they glue to a holomorphic function on $V \cap U_\alpha$. Thus $(f_\alpha s)|_{V \cap U_\alpha} \in \mathcal{O}_X(V \cap U_\alpha)$ for all α, i.e. $s \in \mathcal{O}_X(D)(V)$.

Uniqueness follows from uniqueness in $\mathcal{M}_X$. Therefore $\mathcal{O}_X(D)$ satisfies (S1) and (S2), hence is a sheaf. $\square$

Sheaves and $\mathcal{O}_{\mathbf{CP}^1}(D)$ on $\mathbf{CP}^1$

Definition (sheaf). Let X be a topological space. A **presheaf** $\mathcal{F}$ (of sets / abelian groups / rings / modules) assigns to every open set $U \subset X$ an object $\mathcal{F}(U)$ and to every inclusion $V \subset U$ a restriction map

$$\mathrm{res}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V), \qquad s \mapsto s|_V,$$

such that $\mathrm{res}_{U,U} = \mathrm{id}$ and $\mathrm{res}_{V,W} \circ \mathrm{res}_{U,V} = \mathrm{res}_{U,W}$ for $W \subset V \subset U$. A presheaf is a **sheaf** if for every open cover $U = \bigcup_i U_i$:

(S1) Locality. If $s, t \in \mathcal{F}(U)$ and $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$.

(S2) Gluing. If $s_i \in \mathcal{F}(U_i)$ satisfy $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

The standard cover of $\mathbf{CP}^1$. Let $X = \mathbf{CP}^1$ with the standard affine cover

$$U_0 = \{[z_0 : z_1] \in \mathbf{CP}^1 : \ z_0 \neq 0\}, \qquad U_\infty = \{[z_0 : z_1] \in \mathbf{CP}^1 : \ z_1 \neq 0\}.$$

On U_0 use the coordinate $z = z_1/z_0 \in \mathbf{C}$, and on U_∞ use $w = z_0/z_1 \in \mathbf{C}$. On the overlap $U_0 \cap U_\infty$ one has $w = 1/z$ and $U_0 \cap U_\infty \simeq \mathbf{C}^*$.

Setup (Cartier divisor). Let D be a Cartier divisor on X given by meromorphic functions

$$f_0 \in \mathcal{M}_X^*(U_0), \qquad f_\infty \in \mathcal{M}_X^*(U_\infty),$$

such that on the overlap

$$\frac{f_0}{f_\infty} \in \mathcal{O}_X^*(U_0 \cap U_\infty).$$

Definition (the presheaf $\mathcal{O}_X(D)$). For each open set $V \subset X$, define

$$\mathcal{O}_X(D)(V) := \left\{ s \in \mathcal{M}_X(V) : (f_0 s)\big|_{V \cap U_0} \in \mathcal{O}_X(V \cap U_0),\ (f_\infty s)\big|_{V \cap U_\infty} \in \mathcal{O}_X(V \cap U_\infty) \right\}.$$

Restriction maps are the restrictions in $\mathcal{M}_X$: for $W \subset V$, $\mathrm{res}_{V,W}(s) = s|_W$.

Lemma (overlap consistency). On $V \cap U_0 \cap U_\infty$, the condition $(f_0 s) \in \mathcal{O}_X$ is equivalent to $(f_\infty s) \in \mathcal{O}_X$.

Proof. On $U_0 \cap U_\infty$ we have $f_0 = u\, f_\infty$ where $u := f_0/f_\infty \in \mathcal{O}_X^*(U_0 \cap U_\infty)$. Hence

$$f_0 s = u\,(f_\infty s).$$

Since u is holomorphic and nowhere vanishing, multiplication by u preserves holomorphicity. Therefore $f_0 s$ is holomorphic iff $f_\infty s$ is holomorphic. □

Proposition (sheaf property on $\mathbf{CP}^1$). The presheaf $\mathcal{O}_X(D)$ is a sheaf (indeed, an $\mathcal{O}_X$-module subsheaf of $\mathcal{M}_X$).

Proof. Let $V = \bigcup_i V_i$ be an open cover.

Locality (S1). If $s, t \in \mathcal{O}_X(D)(V)$ and $s|_{V_i} = t|_{V_i}$ for all i, then $s = t$ in $\mathcal{M}_X(V)$ because $\mathcal{M}_X$ is a sheaf. Hence $s = t$ in $\mathcal{O}_X(D)(V)$.

Gluing (S2). Let $s_i \in \mathcal{O}_X(D)(V_i)$ satisfy $s_i = s_j$ on $V_i \cap V_j$. Since $\mathcal{M}_X$ is a sheaf, there exists a unique $s \in \mathcal{M}_X(V)$ with $s|_{V_i} = s_i$. We show $s \in \mathcal{O}_X(D)(V)$.

On $V \cap U_0$, the open cover $\{V_i \cap U_0\}_i$ satisfies

$$(f_0 s)|_{V_i \cap U_0} = f_0(s|_{V_i \cap U_0}) = f_0(s_i|_{V_i \cap U_0}) \in \mathcal{O}_X(V_i \cap U_0),$$

because $s_i \in \mathcal{O}_X(D)(V_i)$. These holomorphic functions agree on overlaps (since the s_i agree), hence glue (by the sheaf property of $\mathcal{O}_X$) to a holomorphic function on $V \cap U_0$. Thus $(f_0 s)|_{V \cap U_0} \in \mathcal{O}_X(V \cap U_0)$. The same argument on $V \cap U_\infty$ shows $(f_\infty s)|_{V \cap U_\infty} \in \mathcal{O}_X(V \cap U_\infty)$. Therefore $s \in \mathcal{O}_X(D)(V)$.

Uniqueness is inherited from uniqueness in $\mathcal{M}_X$. □

Example: $D = n[\infty]$ gives $\mathcal{O}(n)$. Let $D = n[\infty]$. Take $f_0 = 1$ on U_0 and $f_\infty = w^n$ on U_∞. On $U_0 \cap U_\infty$,

$$\frac{f_0}{f_\infty} = \frac{1}{w^n} = z^n \in \mathcal{O}_X^*(\mathbf{C}^*),$$

so this is Cartier data. Then $s \in \mathcal{O}(n)(X)$ iff s is holomorphic on $U_0 \simeq \mathbf{C}$ and $w^n s$ is holomorphic at $w = 0$ (i.e. at ∞), hence s is a polynomial in z of degree $\le n$. Consequently,

$$H^0(\mathbf{CP}^1, \mathcal{O}(n)) \cong \langle 1, z, \ldots, z^n \rangle.$$

Why the condition $f_\alpha s \in \mathcal{O}_X$?

Setup. Let X be a Riemann surface and let D be a Cartier divisor represented by $\{(U_\alpha, f_\alpha)\}$ with $f_\alpha \in \mathcal{M}_X^*(U_\alpha)$ and $f_\alpha/f_\beta \in \mathcal{O}_X^*(U_{\alpha\beta})$.

1) Local-frame (line bundle) viewpoint. On U_α define the local generator (frame)

$$e_\alpha := f_\alpha^{-1}.$$

The intended meaning of $\mathcal{O}_X(D)$ is: "sections are holomorphic multiples of the local frame." Thus a section s on $V \subset U_\alpha$ should be of the form

$$s = h \cdot e_\alpha = h \cdot f_\alpha^{-1} \quad \text{for some } h \in \mathcal{O}_X(V).$$

Rearranging gives

$$f_\alpha s = h \in \mathcal{O}_X(V).$$

Hence the defining condition

$$s \in \mathcal{O}_X(D)(V) \iff f_\alpha s \in \mathcal{O}_X(V)$$

simply says: "s has holomorphic coefficient in the chosen local frame."

2) Order / winding-form viewpoint. For a nonzero meromorphic function g and a point p, let γ_p be a small positively oriented loop. Then

$$\mathrm{ord}_p(g) = \frac{1}{2\pi i}\int_{\gamma_p} \frac{dg}{g}.$$

Using $\dfrac{d(f_\alpha s)}{f_\alpha s} = \dfrac{df_\alpha}{f_\alpha} + \dfrac{ds}{s}$, we get additivity:

$$\mathrm{ord}_p(f_\alpha s) = \mathrm{ord}_p(f_\alpha) + \mathrm{ord}_p(s).$$

Now $f_\alpha s$ is holomorphic near p iff $\mathrm{ord}_p(f_\alpha s) \geq 0$, so

$$f_\alpha s \text{ holomorphic near } p \quad \iff \quad \mathrm{ord}_p(s) \geq -\mathrm{ord}_p(f_\alpha).$$

Thus the condition $f_\alpha s \in \mathcal{O}_X$ is exactly the statement that the pole order of s is bounded by the divisor D.

3) Independence of chart (overlap consistency). On $U_{\alpha\beta}$, write $f_\alpha = u_{\alpha\beta} f_\beta$ with $u_{\alpha\beta} \in \mathcal{O}_X^*(U_{\alpha\beta})$. Then

$$f_\alpha s \in \mathcal{O}_X \iff u_{\alpha\beta}(f_\beta s) \in \mathcal{O}_X \iff f_\beta s \in \mathcal{O}_X,$$

since multiplication by a holomorphic unit preserves holomorphicity (and contributes zero winding/order).

Example on $\mathbf{CP}^1$: $D = n[\infty]$ and $\mathcal{O}(n)$. Let $X = \mathbf{CP}^1$ with charts $U_0 \simeq \mathbf{C}$ (coordinate z) and $U_\infty \simeq \mathbf{C}$ (coordinate $w = 1/z$). For $D = n[\infty]$, take

$$f_0 = 1 \text{ on } U_0, \qquad f_\infty = w^n \text{ on } U_\infty.$$

Then $s \in \mathcal{O}(n)(X)$ iff

$$s \text{ is holomorphic on } U_0 \simeq \mathbf{C}, \qquad w^n s \text{ is holomorphic at } w = 0 \text{ (i.e. at } \infty).$$

Equivalently, $\mathrm{ord}_\infty(s) \geq -n$, so s has a pole at ∞ of order at most n. This forces s to be a polynomial in z of degree $\leq n$, hence

$$H^0(\mathbf{CP}^1, \mathcal{O}(n)) \cong \langle 1, z, \ldots, z^n \rangle.$$