Notes on Complex Analysis and Riemann Surface Theory toward Algebraic Geometry

Ji, Yonghyeon

A document presented for
the Algebraic Geometry

December 7, 2025

Contents

Chapter 1

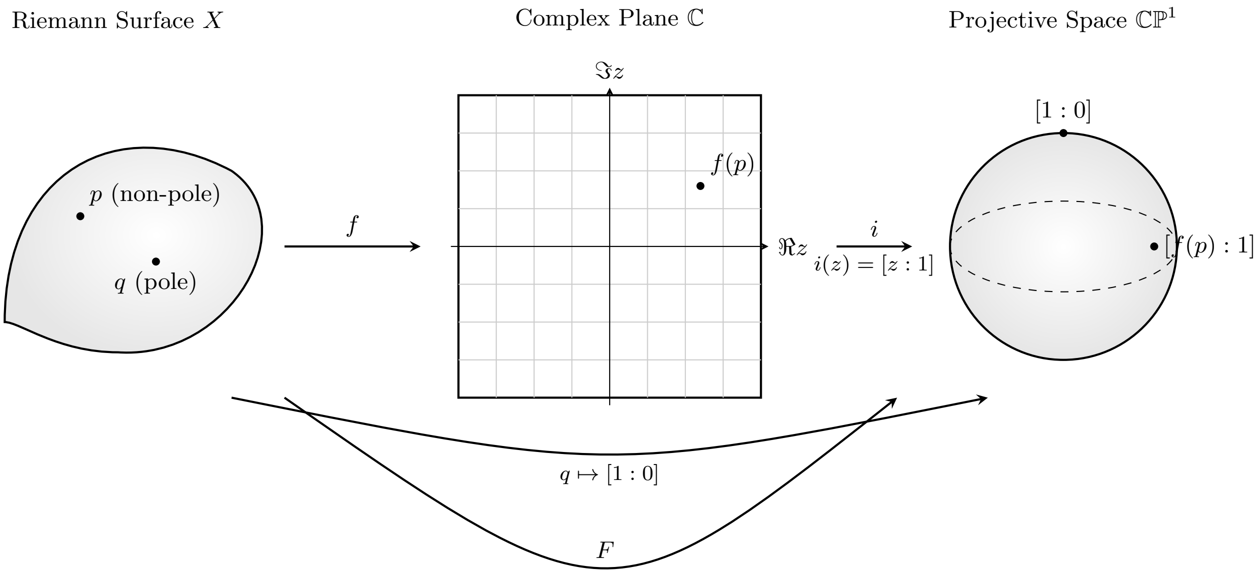
Elliptic Curve and Torus

1.1 Note 1: Meromorphic Function and Order

1.2 Note 2: Meromorphic $f \in \mathbb{C}^X$ and Holomorphic $F \in (\mathbb{CP}^1)^X$

Given a meromorphic $f : X \to \mathbb{C}$ on a Riemann surface X, we define

$$\begin{aligned} F \quad : \quad X &\longrightarrow \mathbb{CP}^1 \\ p &\longmapsto F(p) = \begin{cases} [f(p) : 1] & \text{if } p \text{ is not a pole} \\ [1 : 0] & \text{if } p \text{ is a pole} \end{cases} \end{aligned}$$



In other word,

$$\begin{array}{ccccc} X & \xrightarrow{\quad f \quad} & \mathbb{C} & \xrightarrow{\quad i \quad} & \mathbb{CP}^1 \\ p_{\text{non-pole}} & \longmapsto & f(p) & \longmapsto & [f(p) : 1] \\ q_{\text{pole}} & & \longmapsto & & [1 : 0] \end{array}$$

1.2.1 Example 1: $X = \mathbb{CP}^1$ (Riemann sphere)

We view $\mathbb{CP}^1$ as the Riemann sphere. On the affine chart

$$U_1 = \{[z_0 : z_1] \in \mathbb{CP}^1 \mid z_1 \neq 0\},$$

we use the coordinate $z = z_0/z_1$. The point at infinity is $\infty = [1 : 0]$.

On $\mathbb{CP}^1$, a meromorphic function is the same as a rational function. Take for instance

$$f(z) = \frac{z^2 - 1}{z - 2}.$$

This is meromorphic on $\mathbb{CP}^1$, with a simple pole at $z = 2$, and (possibly) a pole at ∞.

Define

$$F : \mathbb{CP}^1 \to \mathbb{CP}^1, \qquad F(p) = \begin{cases} [f(p) : 1], & p \text{ not a pole of } f, \\ [1 : 0], & p \text{ a pole of } f. \end{cases}$$

Concretely, for $p = [z : 1]$ with $z \neq 2$,

$$F([z : 1]) = [f(z) : 1] = \left[\frac{z^2 - 1}{z - 2} : 1\right],$$

and at the pole $p = [2 : 1]$,

$$F([2 : 1]) = [1 : 0].$$

Similarly one checks the value at $\infty = [1 : 0]$ using the behavior of $f(z)$ as $|z| \to \infty$.

To see that F is holomorphic, we use the usual charts on $\mathbb{CP}^1$:

- **At a non-pole point** p**.** Suppose p is not a pole of f. Then f is holomorphic near p and finite there, so $F(p) = [f(p) : 1] \in U_1$. Let

  $$w = \frac{z_0}{z_1} : U_1 \to \mathbb{C}$$

  be the affine coordinate on U_1. In this chart,

  $$(w \circ F)(q) = \left.\frac{z_0}{z_1}\right|_{F(q)} = f(q),$$

  which is holomorphic in any local coordinate around p. Hence F is holomorphic at non-poles.

- **At a pole** p**.** Let p be a pole of order $m > 0$. Choose a local coordinate z on $\mathbb{CP}^1$ with $z(p) = 0$. Then

  $$f(z) = z^{-m} g(z), \qquad g \text{ holomorphic, } g(0) \neq 0.$$

  Here $F(p) = [1 : 0]$. Use the chart

  $$U_0 = \{[z_0 : z_1] \in \mathbb{CP}^1 \mid z_0 \neq 0\},$$

  with coordinate

  $$u = \frac{z_1}{z_0} : U_0 \to \mathbb{C}.$$

For $z \neq 0$ near p,
$$F(z) = [f(z) : 1] = [z^{-m} g(z) : 1].$$
Multiplying homogeneous coordinates by z^m (which does not change the point in projective space), we get
$$[z^{-m} g(z) : 1] = [g(z) : z^m].$$
Thus, in the chart U_0,
$$(u \circ F)(z) = \frac{z^m}{g(z)}.$$
Since $g(z)$ is holomorphic with $g(0) \neq 0$, the function $\frac{1}{g(z)}$ is holomorphic near 0, and hence
$$\frac{z^m}{g(z)}$$
is holomorphic near 0 (and vanishes to order m). Therefore F is holomorphic at the pole p.

Since we have holomorphicity in local charts at every point of $\mathbb{CP}^1$, $F : \mathbb{CP}^1 \to \mathbb{CP}^1$ is a holomorphic map.

1.2.2 Example 2: $X = \mathbb{C}/\Lambda$ (complex torus)

Let $\Lambda \subset \mathbb{C}$ be a lattice and consider the complex torus
$$X = \mathbb{C}/\Lambda.$$
The quotient map is
$$\pi : \mathbb{C} \to X, \qquad \pi(z) = [z].$$
A meromorphic function $f : X \to \mathbb{C}$ corresponds to a Λ-periodic meromorphic function $\tilde{f} : \mathbb{C} \to \mathbb{C}$ satisfying
$$\tilde{f}(z + \lambda) = \tilde{f}(z), \quad \forall \lambda \in \Lambda,$$
and
$$f([z]) = \tilde{f}(z).$$
A standard example is the Weierstrass $\wp$-function $\wp : \mathbb{C} \to \mathbb{C}$, which is Λ-periodic and meromorphic with double poles at lattice points. Thus it descends to a meromorphic
$$f : X \to \mathbb{C}, \qquad f([z]) = \wp(z).$$
We define
$$F : X \to \mathbb{CP}^1, \qquad F(p) = \begin{cases} [f(p) : 1], & p \text{ not a pole of } f, \\ [1 : 0], & p \text{ a pole of } f. \end{cases}$$
For our example $f([z]) = \wp(z)$:

- $\wp(z)$ has poles precisely at lattice points $z \in \Lambda$, which all represent the same point on the torus, usually denoted $[0]$.
- For $[z] \neq [0]$, we set $F([z]) = [\wp(z) : 1]$.
- At $[0]$, we set $F([0]) = [1 : 0]$.

Local coordinate on the torus near a pole

To get a local coordinate near $[0] \in X$, choose a small disc $D \subset \mathbb{C}$ around 0 such that $\pi|_D : D \to \pi(D)$ is a biholomorphism. Then

$$\varphi : \pi(D) \to \mathbb{C}, \qquad \varphi([z]) = z,$$

is a local coordinate on X near $[0]$.

The local behavior of $\wp(z)$ at $z = 0$ is

$$\wp(z) = \frac{1}{z^2} + \text{holomorphic terms},$$

so more precisely,

$$\wp(z) = z^{-2} g(z), \qquad g(z) \text{ holomorphic, } g(0) \neq 0.$$

Thus, for the induced f,

$$f([z]) = \wp(z) = z^{-2} g(z),$$

so f has a pole of order $m = 2$ at $[0]$.

Holomorphicity of F at the pole $[0]$

As before, we use the chart around $[1 : 0] \in \mathbb{CP}^1$:

$$U_0 = \{[z_0 : z_1] \mid z_0 \neq 0\}, \qquad u = \frac{z_1}{z_0} : U_0 \to \mathbb{C}.$$

For $z \neq 0$ small, we have $p = [z] \neq [0]$ and

$$F([z]) = [f([z]) : 1] = [\wp(z) : 1] = [z^{-2} g(z) : 1].$$

Multiplying the homogeneous coordinates by z^2 gives

$$[z^{-2} g(z) : 1] = [g(z) : z^2].$$

So in the chart U_0,

$$(u \circ F)([z]) = \frac{z^2}{g(z)}.$$

Since $g(z)$ is holomorphic with $g(0) \neq 0$, the function $\dfrac{1}{g(z)}$ is holomorphic near 0, and hence $\dfrac{z^2}{g(z)}$ is holomorphic near 0 and vanishes at $z = 0$. In the local coordinate $\varphi([z]) = z$ on X, the expression

$$u \circ F \circ \varphi^{-1}(z) = \frac{z^2}{g(z)}$$

is holomorphic, so F is holomorphic at the pole $[0]$.

At a non-pole point $[z_0] \in X$, the same argument as in Example 1 applies: f is holomorphic and finite, and in the affine chart

$$U_1 = \{[z_0 : z_1] \mid z_1 \neq 0\}, \qquad w = \frac{z_0}{z_1},$$

we have

$$(w \circ F)([z]) = f([z]) = \wp(z),$$

which is holomorphic in the local coordinate on X.

Conclusion

For both examples $X = \mathbb{CP}^1$ and $X = \mathbb{C}/\Lambda$, the construction

$$f : X \to \mathbb{C} \text{ meromorphic} \quad \longmapsto \quad F : X \to \mathbb{CP}^1, \quad F(p) = \begin{cases} [f(p) : 1], & p \text{ not a pole,} \\ [1 : 0], & p \text{ a pole,} \end{cases}$$

produces a holomorphic map $F : X \to \mathbb{CP}^1$. This concretely illustrates the general principle that a meromorphic function on a Riemann surface is the same as a holomorphic map to $\mathbb{CP}^1$.

We start with a meromorphic function

$$f : X \to \mathbb{C}$$

on a Riemann surface X, and define a map

$$F : X \to \mathbb{CP}^1$$

by

$$F(p) = \begin{cases} [f(p) : 1], & p \text{ not a pole of } f, \\ [1 : 0], & p \text{ a pole of } f. \end{cases}$$

You're asking: **why is this F holomorphic as a map of Riemann surfaces?**

1. Definition to remember

A map $F : X \to Y$ between Riemann surfaces is **holomorphic** if, for every point $p \in X$, you can choose local coordinates

- φ: neighborhood of $p \to \mathbb{C}$,
- ψ: neighborhood of $F(p) \to \mathbb{C}$,

such that the coordinate expression

$$\psi \circ F \circ \varphi^{-1} : (\text{open in } \mathbb{C}) \to \mathbb{C}$$

is an ordinary holomorphic function.

So we need to check this around:

1. a point where f is holomorphic (no pole),
2. a point where f has a pole.

2. Case 1: p is not a pole (easy)

If p is not a pole, then f is holomorphic near p and finite there.

- On X: choose any local coordinate z with $z(p) = 0$.
- On $\mathbb{CP}^1$: since $F(p) = [f(p) : 1]$ has second coordinate $\neq 0$, it lies in the chart

  $$U_1 = \{[z_0 : z_1] \mid z_1 \neq 0\}$$

  with coordinate

  $$w = \frac{z_0}{z_1} : U_1 \to \mathbb{C}.$$

Then on some neighborhood of p,

$$(w \circ F)(q) = \frac{z_0}{z_1}\Big|_{F(q)} = \frac{f(q)}{1} = f(q),$$

which is holomorphic in z.

So $\psi \circ F \circ \varphi^{-1} = f$ is holomorphic $\Rightarrow F$ is holomorphic at non-pole points.

3. Case 2: p is a pole of order $m > 0$

This is the interesting part.

Let p be a pole of f of order m. Choose a local coordinate z on X with $z(p) = 0$. By the definition of meromorphic:

$$f(z) = z^{-m} g(z),$$

where g is holomorphic and $g(0) \neq 0$.

By definition,

$$F(p) = [1 : 0] \in \mathbb{CP}^1.$$

Now we must look at a chart of $\mathbb{CP}^1$ that contains $[1 : 0]$. That is:

$$U_0 = \{[z_0 : z_1] \mid z_0 \neq 0\},$$

with coordinate

$$u = \frac{z_1}{z_0} : U_0 \to \mathbb{C},$$

and in this chart $[1 : 0]$ corresponds to $u = 0$.

For $z \neq 0$ near p,

$$F(z) = [f(z) : 1] = [z^{-m} g(z) : 1].$$

Multiply homogeneous coordinates by z^m (allowed in projective space):

$$[z^{-m} g(z) : 1] = [g(z) : z^m].$$

So in the chart U_0 we have:

$$u(F(z)) = \frac{z^m}{g(z)}.$$

Now, check holomorphicity:

- $g(z)$ is holomorphic with $g(0) \neq 0 \Rightarrow 1/g(z)$ is holomorphic near 0.
- z^m is holomorphic.
- The product $z^m \cdot \dfrac{1}{g(z)}$ is holomorphic near 0.

So

$$u \circ F(z) = \frac{z^m}{g(z)}$$

is an ordinary holomorphic function of z on a neighborhood of 0, and it extends to $z = 0$ with value 0.

Thus, in local coordinates,

$$\psi \circ F \circ \varphi^{-1} = u \circ F$$

is holomorphic at $z = 0$. Therefore, F **is holomorphic at the pole** p.

4. Conclusion

We have checked:

- At non-poles: in the chart U_1, $w \circ F = f$ is holomorphic.
- At poles: in the chart U_0, $u \circ F = z^m/g(z)$ is holomorphic.

So at **every** point $p \in X$, we can choose charts making the coordinate expression of F holomorphic. That's exactly the definition:

$$\boxed{F : X \longrightarrow \mathbb{CP}^1 \text{ is holomorphic.}}$$

This is why we can safely say:

1.3 Note 3: The Isomorphism $\mathcal{M}(\mathbb{CP}^1) \simeq \mathbb{C}(x)$

We explain that the field of meromorphic functions on $\mathbb{CP}^1$ is isomorphic to the field $\mathbb{C}(x)$ of rational functions in one variable.

$$\mathcal{M}(X) = \left\{ \overline{i \circ f} \in (\mathbb{CP}^1)^X \;\middle|\; f \text{ meromorphic on } X \right\},$$

$$\mathcal{M}(X) = \{ F : X \to \mathbb{CP}^1 \mid F \text{ holomorphic} \}.$$

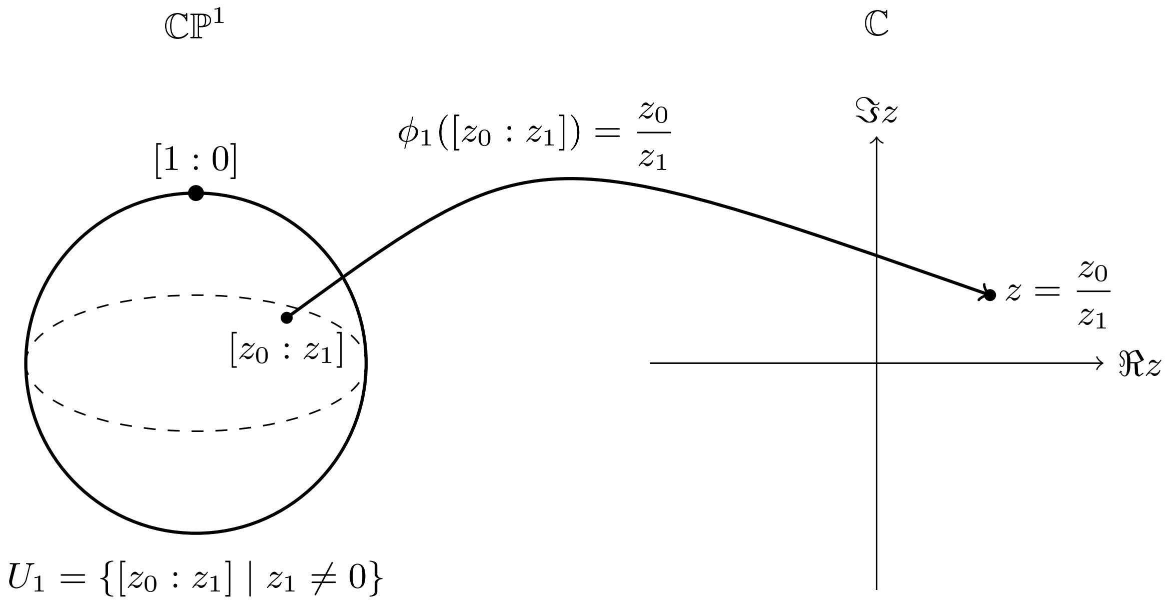
1.3.1 Charts on $\mathbb{CP}^1$ and Field of Meromorphic Functions

View $\mathbb{CP}^1$ as the Riemann sphere. Consider the standard affine chart

$$U_1 = \{[z_0 : z_1] \in \mathbb{CP}^1 \mid z_1 \neq 0\}$$

with coordinate map

$$\begin{array}{cccc} \phi_1 & : & U_1 & \longrightarrow & \mathbb{C} \\ & & [z_0 : z_1] & \longmapsto & \dfrac{z_0}{z_1} \end{array}.$$



We write

$$x := \phi_1,$$

and think of x as the *coordinate function* on U_1. This function extends meromorphically to all of $\mathbb{CP}^1$, with a simple pole at $\infty = [1 : 0]$.

We define the field of meromorphic functions on $\mathbb{CP}^1$ as

$$\mathcal{M}(\mathbb{CP}^1) = \{F : \mathbb{CP}^1 \to \mathbb{CP}^1 \mid F \text{ holomorphic}\},$$

viewing a meromorphic function as a holomorphic map into $\mathbb{CP}^1$ (via the usual convention "finite value $\mapsto [f(p) : 1]$, pole $\mapsto [1 : 0]$").

On the other hand, the field $\mathbb{C}(x)$ is

$$\mathbb{C}(x) = \left\{ \frac{p(x)}{q(x)} \;\middle|\; p, q \in \mathbb{C}[x],\ q \not\equiv 0 \right\} \Big/ \sim,$$

where $\frac{p}{q} \sim \frac{p'}{q'}$ if $p(x)q'(x) = p'(x)q(x)$.

Here ϕ_1 is a biholomorphism between U_1 and $\mathbb{C}$, its inverse is

$$\begin{array}{cccc} \phi_1^{-1} & : & \mathbb{C} & \longrightarrow & U_1 \\ & & z & \longmapsto & [z:1] \end{array}.$$

We'll write

$$x := \phi_1$$

and think of x as the *coordinate function* on U_1. It extends meromorphically to all of $\mathbb{CP}^1$ with a simple pole at $[1:0]$ (the point at infinity).

1. Describe both sides with ϕ_1

Side 1: $\mathcal{M}(\mathbb{CP}^1)$

We use the "holomorphic map to $\mathbb{CP}^1$" definition:

$$\mathcal{M}(\mathbb{CP}^1) = \left\{ F : \mathbb{CP}^1 \to \mathbb{CP}^1 \;\middle|\; F \text{ holomorphic} \right\}.$$

We want to use ϕ_1, so whenever the image of F lies in U_1, we can look at

$$\phi_1 \circ F : \text{ (some open set)} \to \mathbb{C}.$$

That's just the "affine coordinate" of the value of F.

Side 2: $\mathbb{C}(x)$

$$\mathbb{C}(x) = \left\{ \frac{p(x)}{q(x)} \;\middle|\; p(x), q(x) \in \mathbb{C}[x],\ q(x) \not\equiv 0 \right\} \Big/ \sim,$$

where $\frac{p}{q} \sim \frac{p'}{q'}$ iff $p(x)q'(x) = p'(x)q(x)$.

Here the symbol x is exactly your coordinate function

$$x = \phi_1 : U_1 \to \mathbb{C}.$$

2. Map $\mathbb{C}(x) \to \mathcal{M}(\mathbb{CP}^1)$ using ϕ_1

Take a rational function

$$R(x) = \frac{p(x)}{q(x)} \in \mathbb{C}(x).$$

On the affine chart U_1:

Given a point $[z_0 : z_1] \in U_1$, write

$$x([z_0 : z_1]) = \phi_1([z_0 : z_1]) = z_0/z_1 =: z.$$

We *define* a map $F_R : \mathbb{CP}^1 \to \mathbb{CP}^1$ by saying on U_1,

$$\phi_1\big(F_R([z_0 : z_1])\big) = R\big(\phi_1([z_0 : z_1])\big) = R(z).$$

In other words,

$$F_R|_{U_1} = \phi_1^{-1} \circ R \circ \phi_1.$$

Concretely:

$$F_R([z_0 : z_1]) = [R(z_0/z_1) : 1] \quad (\text{for } z_1 \neq 0,\ R(z) \neq \infty).$$

At points where $R(z) = \infty$ (i.e. $q(z) = 0$), we set

$$F_R([z_0 : z_1]) = [1 : 0].$$

This defines F_R on $U_1 \cup \{\infty\}$, but one must check it is *holomorphic at* ∞. Using homogeneous polynomials is a cleaner way:

- Let $\deg p \leq m,\ \deg q \leq m$. Define

  $$P(z_0, z_1) = z_1^m p(z_0/z_1), \quad Q(z_0, z_1) = z_1^m q(z_0/z_1),$$

  homogeneous of degree m.

- Then set

  $$F_R([z_0 : z_1]) = \begin{cases} [P(z_0, z_1) : Q(z_0, z_1)], & Q(z_0, z_1) \neq 0, \\ [1 : 0], & Q(z_0, z_1) = 0. \end{cases}$$

This is well-defined and holomorphic on all of $\mathbb{CP}^1$. In the chart U_1, this is exactly $\phi_1^{-1} \circ R \circ \phi_1$. So we get a map

$$\Phi : \mathbb{C}(x) \to \mathcal{M}(\mathbb{CP}^1), \quad R \mapsto F_R.$$

3. Use ϕ_1 to go backwards: from F to $R(x)$

Now take any

$$F \in \mathcal{M}(\mathbb{CP}^1), \quad F : \mathbb{CP}^1 \to \mathbb{CP}^1 \text{ holomorphic.}$$

We want to show: *there exists a unique rational function* $R(x) \in \mathbb{C}(x)$ *such that*

$$F = F_R.$$

Using ϕ_1:

1. Consider the open set where the image of F stays inside U_1:

   $$V := F^{-1}(U_1) \subset \mathbb{CP}^1.$$

2. On V, define

   $$f := \phi_1 \circ F : V \to \mathbb{C}.$$

   In local coordinates, f is holomorphic. So f is a holomorphic function on the Riemann surface V.

3. The complement $\mathbb{CP}^1 \setminus V = F^{-1}(\infty)$ is a *finite set* (preimages of the point $[1 : 0]$ under a holomorphic map from a compact Riemann surface). At those points, we'll see f has poles. So in the chart ϕ_1, f is a *meromorphic function on* $\mathbb{C}$ with finitely many poles.

Now, via ϕ_1, we can identify $\mathbb{CP}^1 \setminus \{\infty\}$ with $\mathbb{C}$. Under this, F becomes a meromorphic function

$$\tilde{f} : \mathbb{C} \to \mathbb{C} \cup \{\infty\},$$

which has only finitely many poles (coming from $F^{-1}(\infty)$) and maybe a pole at ∞.

From standard complex analysis:

> A meromorphic function on $\mathbb{CP}^1$ (i.e. on $\mathbb{C} \cup \{\infty\}$) is *rational*.

Concretely, we do the principal-part argument *in the coordinate* ϕ_1:

- In the x-coordinate (i.e. using ϕ_1 as your chart), $f(x)$ has Laurent expansions at each finite pole $x = a_j$.
- You build a rational function $R(x)$ whose principal parts match those of f at all finite poles and at ∞.
- Then $f(x) - R(x)$ is entire and holomorphic at ∞, so it's constant. So $f(x) = R(x) + C$, still rational.

Thus there exists some $R(x) \in \mathbb{C}(x)$ such that

$$f(x) = R(x) \quad \text{as meromorphic functions on } \mathbb{C} \cup \{\infty\}.$$

But $f = \phi_1 \circ F$ and $R \circ \phi_1$ have the same values on U_1, so

$$\phi_1 \circ F = R \circ \phi_1 \quad \text{on } U_1,$$

hence

$$F|_{U_1} = \phi_1^{-1} \circ R \circ \phi_1 = F_R|_{U_1}.$$

Both F and F_R are holomorphic maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ that agree on the nonempty open set U_1, so by the identity theorem they agree everywhere:

$$F = F_R.$$

So every $F \in \mathcal{M}(\mathbb{CP}^1)$ comes from a unique $R \in \mathbb{C}(x)$. That's surjectivity and injectivity of Φ.

4. Summary in your language

Using your chart

$$\phi_1 : U_1 \to \mathbb{C}, \quad [z_0 : z_1] \mapsto z_0/z_1,$$

we have:

- Define $x := \phi_1$. This is a meromorphic function on $\mathbb{CP}^1$ with one pole at $[1 : 0]$.
- Given $R(x) \in \mathbb{C}(x)$, we define a holomorphic map $F_R : \mathbb{CP}^1 \to \mathbb{CP}^1$ by

  $$F_R = \phi_1^{-1} \circ R \circ \phi_1 \quad \text{on } U_1,$$

  extended holomorphically to ∞.

- Given a holomorphic $F : \mathbb{CP}^1 \to \mathbb{CP}^1$, its coordinate expression

  $$f = \phi_1 \circ F \circ \phi_1^{-1} : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$$

  is a meromorphic function on the sphere, hence a rational function $R(x)$. Then $F = F_R$.

So precisely:

$$\boxed{\mathcal{M}(\mathbb{CP}^1) = \{F : \mathbb{CP}^1 \to \mathbb{CP}^1 \text{ holomorphic}\} \;\cong\; \{R(x) \in \mathbb{C}(x)\}}$$

and the chart ϕ_1 is the bridge that makes this identification explicit.