Extremely Detailed Explanation of $\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z)$ and Why $\mathcal{M}(X) \cong \mathbb{C}(z) \iff X \cong \mathbb{CP}^1$

Contents

1 Basic objects and philosophy

We work over $\mathbb{C}$. There are three levels of structure:

- **Complex-analytic / calculus**: holomorphic and meromorphic functions, Laurent series, residues, contour integrals.
- **Algebraic**: rational functions $p(z)/q(z)$, projective coordinates, divisors, function fields.
- **Riemann surface theory**: compact Riemann surfaces, meromorphic maps $X \to \mathbb{CP}^1$, degree, genus.

Our main goals:

1. Show in detail:
$$\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z),$$
i.e. every meromorphic function on the Riemann sphere is a rational function in one variable.
2. Show:
$$\mathcal{M}(X) \cong \mathbb{C}(z) \quad \Longleftrightarrow \quad X \cong \mathbb{CP}^1$$
for a compact Riemann surface X.

2 Analytic / calculus side: $\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z)$

2.1 Identifying $\mathbb{CP}^1$ with the Riemann sphere

As a set,
$$\mathbb{CP}^1 = \{[z_0 : z_1] \neq [0 : 0]\}/\sim, \qquad [z_0 : z_1] \sim [\lambda z_0 : \lambda z_1],\ \lambda \neq 0.$$

Analytically, we identify
$$\mathbb{CP}^1 \cong \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$$
as follows:

- Affine chart $U_1 = \{[z_0 : z_1] \mid z_1 \neq 0\}$ with coordinate
$$z = \frac{z_0}{z_1} : U_1 \to \mathbb{C}.$$
- The point at infinity $[1 : 0]$ corresponds to the symbol ∞.

So we think of $\mathbb{CP}^1$ as the complex plane plus one extra point at infinity.

2.2 Meromorphic functions and meromorphic 1-forms

A *meromorphic function* on $\mathbb{CP}^1$ is a function

$$f : \mathbb{CP}^1 \to \mathbb{CP}^1$$

that is holomorphic except possibly at isolated points where it can have poles (but NO essential singularities).

On $\mathbb{C} \subset \mathbb{CP}^1$ we can write f as an ordinary meromorphic function $f(z)$ (holomorphic except for isolated poles). At ∞ we use coordinate $w = 1/z$; then

$$F(w) := f\left(\frac{1}{w}\right)$$

is meromorphic on a deleted neighborhood of $w = 0$ and has a pole or removable singularity there.

We attach to f the meromorphic 1-form

$$\omega = f(z)\, dz.$$

Integrals of ω around closed curves encode information about f (residues, principal parts).

2.3 Simple explicit example to see everything

Consider

$$f(z) = \frac{1}{z^2(z-1)}$$

as a meromorphic function on $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

2.3.1 Poles and orders (calculus style)

At $z = 0$. Clearly $z = 0$ is a pole because the denominator has z^2. Write

$$f(z) = \frac{1}{z^2(z-1)}.$$

Near $z = 0$, $(z-1) \approx -1$, so

$$f(z) \sim \frac{1}{z^2 \cdot (-1)} = -\frac{1}{z^2}.$$

Thus f has a pole of order 2 at $z = 0$.

At $z = 1$. Set $u = z - 1$. Then $z = u + 1$, so

$$f(z) = \frac{1}{(u+1)^2 u}.$$

Near $u = 0$, $(u+1)^2 \approx 1$, so

$$f(z) \sim \frac{1}{u} = \frac{1}{z-1}.$$

So $z = 1$ is a simple pole (order 1).

At ∞. We analyze $z \to \infty$. As $|z| \to \infty$,

$$f(z) = \frac{1}{z^2(z-1)} = \frac{1}{z^3(1-1/z)}.$$

Expand $\frac{1}{1-1/z} = 1 + \frac{1}{z} + \frac{1}{z^2} + \dots$, so

$$f(z) = \frac{1}{z^3}\Big(1 + \frac{1}{z} + \cdots\Big) = \frac{1}{z^3} + O\Big(\frac{1}{z^4}\Big).$$

Thus $f(z) \to 0$ as $z \to \infty$, and in fact has a zero of order 3 at ∞; equivalently, it's holomorphic at ∞ and $f(\infty) = 0$.

2.3.2 Partial fraction decomposition and derivative

We try to write

$$f(z) = \frac{1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1}.$$

Multiply by $z^2(z-1)$:

$$1 = Az(z-1) + B(z-1) + Cz^2.$$

Compute:

$$Az(z-1) = A(z^2 - z) = Az^2 - Az,$$
$$B(z-1) = Bz - B,$$
$$Cz^2 = Cz^2.$$

So

$$Az^2 - Az + Bz - B + Cz^2 = (A+C)z^2 + (-A+B)z - B.$$

We want this equal to 1 for all z, meaning:

$$(A+C)z^2 + (-A+B)z - B = 0 \cdot z^2 + 0 \cdot z + 1.$$

So we must have

$$A + C = 0, \quad -A + B = 0, \quad -B = 1.$$

From $-B = 1$ we get $B = -1$. Then $-A + (-1) = 0 \Rightarrow A = -1$. Then $A + C = 0 \Rightarrow C = 1$.

So

$$f(z) = \frac{-1}{z} + \frac{-1}{z^2} + \frac{1}{z-1}.$$

Now compute the derivative:

$$f'(z) = \frac{d}{dz}\left(-\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z-1}\right) = \frac{1}{z^2} + \frac{2}{z^3} - \frac{1}{(z-1)^2}.$$

Then

$$df = f'(z)\,dz = \left(\frac{1}{z^2} + \frac{2}{z^3} - \frac{1}{(z-1)^2}\right) dz.$$

Poles of df:

- At $z = 0$: order 3 pole (from $2/z^3$ and $1/z^2$), residue 0 (no $1/z$ term).
- At $z = 1$: order 2 pole, residue 0.
- At ∞: since f is holomorphic at ∞, df is also holomorphic there and residue 0.

This matches the general fact: df always has *total residue* 0.

2.4 General analytic proof that $\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z)$

Now let f be *any* meromorphic function on $\widehat{\mathbb{C}}$.

2.4.1 Step 1: finitely many poles

Because $\mathbb{CP}^1$ is compact and poles are isolated, f has only finitely many poles:

$$\{a_1, \dots, a_N\} \subset \mathbb{C} \cup \{\infty\}.$$

2.4.2 Step 2: Laurent expansions at each pole

At each finite pole $a_j \in \mathbb{C}$, there is a small circle γ_j around a_j enclosing no other poles, and a Laurent expansion

$$f(z) = \sum_{n=-m_j}^{\infty} c_{j,n}(z - a_j)^n, \quad 0 < |z - a_j| < \varepsilon,$$

with $m_j \geq 1$. Coefficients are given by integrals:

$$c_{j,n} = \frac{1}{2\pi i} \oint_{\gamma_j} \frac{f(\zeta)}{(\zeta - a_j)^{n+1}} \, d\zeta.$$

The principal part at a_j is

$$\mathrm{PP}_{a_j}(f)(z) := \sum_{n=-m_j}^{-1} c_{j,n}(z - a_j)^n.$$

At ∞, in coordinate $w = 1/z$:

$$F(w) := f\left(\frac{1}{w}\right) = \sum_{n=-M}^{\infty} b_n w^n$$

for some $M \geq 0$, and

$$b_n = \frac{1}{2\pi i} \oint_{|\xi|=\rho} \frac{F(\xi)}{\xi^{n+1}} \, d\xi.$$

The principal part at ∞ is

$$\mathrm{PP}_{\infty}(f)(w) := \sum_{n=-M}^{-1} b_n w^n.$$

In terms of $z = 1/w$, this is a polynomial $P(z)$.

2.4.3 Step 3: Build a rational function $R(z)$

Define

$$R(z) := P(z) + \sum_{j=1}^{N} \mathrm{PP}_{a_j}(f)(z).$$

Concretely,

$$R(z) = \sum_{k=1}^{M} \tilde{b}_k z^k + \sum_{j=1}^{N} \sum_{k=1}^{m_j} \frac{c_{j,-k}}{(z - a_j)^k}.$$

It is clear that $R(z)$ is a rational function in z, i.e. belongs to $\mathbb{C}(z)$.
By construction:

- At each finite pole a_j, the principal part of R coincides with that of f.
- At ∞, the principal part of R coincides with that of f.

2.4.4 Step 4: The difference $g = f - R$ is holomorphic everywhere

Set

$$g(z) := f(z) - R(z).$$

At each finite pole a_j, the negative-power terms in the Laurent expansion cancel, so g has no pole there (holomorphic at a_j).

At ∞, using $w = 1/z$, f and R have the same principal part in w, so g has no negative powers in w and is holomorphic at $w = 0$ (i.e. at ∞).

Therefore g is holomorphic on all of $\mathbb{CP}^1 = \widehat{\mathbb{C}}$. A holomorphic function on a compact Riemann surface is constant (by the maximum modulus principle or Liouville), so $g(z) \equiv C$ for some $C \in \mathbb{C}$.

Thus

$$f(z) = R(z) + C.$$

Since $R(z) \in \mathbb{C}(z)$, we conclude

$$f(z) \in \mathbb{C}(z).$$

$$\boxed{\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z).}$$

This is the *calculus* argument: it uses Laurent series, integrals for coefficients, residues, and Liouville.

3 Algebraic / projective viewpoint on $\mathcal{M}(\mathbb{CP}^1)$

Now we give a more algebraic description.

3.1 Affine chart and coordinate function z

On the chart

$$U_1 = \{[z_0 : z_1] \in \mathbb{CP}^1 \mid z_1 \neq 0\},$$

we define

$$z = \frac{z_0}{z_1} : U_1 \to \mathbb{C}.$$

This z is a holomorphic function on U_1. On $\mathbb{CP}^1$ it extends *meromorphically* with a single simple pole at the point $[1 : 0]$ (i.e. ∞).

The function z generates the field $\mathbb{C}(z)$ of rational functions.

3.2 Rational functions as homogeneous maps

Let $R(z) = p(z)/q(z)$ with $p, q \in \mathbb{C}[z]$, $q \not\equiv 0$. Set

$$m = \max\{\deg p, \deg q\}.$$

Define homogeneous polynomials:

$$P(z_0, z_1) = z_1^m p\left(\frac{z_0}{z_1}\right), \quad Q(z_0, z_1) = z_1^m q\left(\frac{z_0}{z_1}\right).$$

Then P, Q are homogeneous of degree m, and we define a map

$$F_R : \mathbb{CP}^1 \to \mathbb{CP}^1, \qquad F_R([z_0 : z_1]) = \begin{cases} [P(z_0, z_1) : Q(z_0, z_1)], & Q(z_0, z_1) \neq 0, \\ [1 : 0], & Q(z_0, z_1) = 0. \end{cases}$$

This is well-defined: scaling (z_0, z_1) by $\lambda \neq 0$ scales both P and Q by λ^m, so the projective point $[P : Q]$ is the same.

On the affine chart U_1, with $z = z_0/z_1$, we have

$$F_R([z : 1]) = [p(z) : q(z)],$$

and in the chart where $q(z) \neq 0$, this corresponds to

$$\frac{p(z)}{q(z)} = R(z).$$

So any rational function $R(z)$ yields a meromorphic map $F_R : \mathbb{CP}^1 \to \mathbb{CP}^1$.

3.3 Conversely: maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ are rational

Conversely, any holomorphic map $F : \mathbb{CP}^1 \to \mathbb{CP}^1$ between projective lines is given by homogeneous polynomials of the same degree:

$$F([z_0 : z_1]) = [P(z_0, z_1) : Q(z_0, z_1)],$$

where P, Q are homogeneous of the same degree and have no common factor. Restricting to U_1 with $z = z_0/z_1$,

$$F([z : 1]) = [P(z, 1) : Q(z, 1)];$$

if $Q(z, 1) \neq 0$ then in the affine chart we get

$$F([z : 1]) \mapsto \frac{P(z, 1)}{Q(z, 1)} \in \mathbb{C},$$

which is a rational function of z.

Therefore, algebraically,

$$\mathcal{M}(\mathbb{CP}^1) = \{\text{meromorphic maps } \mathbb{CP}^1 \to \mathbb{CP}^1\} \cong \mathbb{C}(z).$$

4 General compact Riemann surface X and $\mathcal{M}(X)$

Now consider an arbitrary compact Riemann surface X.

4.1 Non-constant meromorphic functions and maps $X \to \mathbb{CP}^1$

A non-constant meromorphic function f on X corresponds to a holomorphic map

$$f : X \to \mathbb{CP}^1$$

defined by

$$f(p) = \begin{cases} [f(p) : 1], & f(p) \text{ finite}, \\ [1 : 0], & f(p) = \infty. \end{cases}$$

This map is *finite-to-one*: for a generic point $w \in \mathbb{CP}^1$, the preimage $f^{-1}(w)$ consists of finitely many points, counted with multiplicity. The number of points in a generic fiber is the *degree* of f, denoted $\deg(f)$.

Locally, in coordinates, near a point $p \in X$, we can choose z as a local coordinate on X at p and ζ as a local coordinate on $\mathbb{CP}^1$ at $f(p)$ so that

$$\zeta = f(z) = z^k + (\text{higher order terms}),$$

with $k \geq 1$. The integer k is the local degree (ramification index) at p. Summing these local degrees over $f^{-1}(w)$ for generic w gives $\deg(f)$.

4.2 Function field extension viewpoint

The map $f : X \to \mathbb{CP}^1$ induces a field embedding

$$f^* : \mathcal{M}(\mathbb{CP}^1) \hookrightarrow \mathcal{M}(X),$$

by pullback: if $R(z) \in \mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z)$, then

$$f^*(R) := R \circ f \in \mathcal{M}(X).$$

Thus we get an inclusion

$$\mathbb{C}(z) \hookrightarrow \mathcal{M}(X).$$

A deep theorem (from the theory of compact Riemann surfaces / algebraic curves) says:

$$[\mathcal{M}(X) : \mathbb{C}(z)] = \deg(f),$$

i.e. the degree of the field extension equals the topological degree of the map f. Intuitively, each branch contributes one "copy" of $\mathbb{C}(z)$ in the extension.

4.3 If $\mathcal{M}(X) \cong \mathbb{C}(z)$, then $X \cong \mathbb{CP}^1$

Suppose we have *as fields*:

$$\mathcal{M}(X) \cong \mathbb{C}(z).$$

This means that $\mathcal{M}(X)$ is a purely transcendental extension of $\mathbb{C}$ of transcendence degree 1, with no nontrivial algebraic relations besides those already in $\mathbb{C}(z)$.

Pick any non-constant meromorphic function $f \in \mathcal{M}(X)$. The map

$$f : X \to \mathbb{CP}^1$$

is non-constant, hence of some degree $d \geq 1$. The induced extension

$$\mathbb{C}(z) \hookrightarrow \mathcal{M}(X)$$

has degree d. But by assumption $\mathcal{M}(X) \cong \mathbb{C}(z)$, so as a field extension, $[\mathcal{M}(X) : \mathbb{C}(z)] = 1$. Therefore, $\deg(f) = 1$.

So we have a non-constant map $f : X \to \mathbb{CP}^1$ of degree 1.

Degree 1 implies biholomorphism

We claim a holomorphic map $f : X \to \mathbb{CP}^1$ of degree 1 between compact Riemann surfaces must be an isomorphism of Riemann surfaces.

- Since f is non-constant holomorphic, it is open and its image is an open connected subset of $\mathbb{CP}^1$.
- Compactness of X plus continuity of f implies $f(X)$ is compact, hence closed in $\mathbb{CP}^1$.
- $\mathbb{CP}^1$ is connected, so the only nonempty closed and open subset is all of $\mathbb{CP}^1$. Thus f is surjective.
- $\deg(f) = 1$ means that for a generic point $w \in \mathbb{CP}^1$, the fiber $f^{-1}(w)$ consists of exactly one point (counted with multiplicity). Roughly speaking, this means f is "one-to-one almost everywhere".
- One can show (using local behavior and that there is no branching if the total degree is 1) that f is globally one-to-one.
- A bijective holomorphic map between compact Riemann surfaces has a holomorphic inverse (since the inverse map is continuous, and by Riemann surface theory it is analytic). So f is a biholomorphism.

Therefore $X \cong \mathbb{CP}^1$ as Riemann surfaces.

$$\boxed{\mathcal{M}(X) \cong \mathbb{C}(z) \implies X \cong \mathbb{CP}^1.}$$

4.4 Conversely, if $X \cong \mathbb{CP}^1$, then $\mathcal{M}(X) \cong \mathbb{C}(z)$

Conversely, if we know $X \cong \mathbb{CP}^1$ (as Riemann surfaces), then by definition there is a biholomorphism $\varphi : X \to \mathbb{CP}^1$. Pullback of meromorphic functions along φ gives a field isomorphism

$$\mathcal{M}(\mathbb{CP}^1) \xrightarrow{\cong} \mathcal{M}(X),$$

and we already know $\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z)$. Hence

$$\mathcal{M}(X) \cong \mathbb{C}(z).$$

4.5 Genus and the "only genus 0 curve" viewpoint

There is a more geometric / topological characterization.

- The *genus* $g(X)$ is the number of "holes" of X as a topological surface: $g(\mathbb{CP}^1) = 0$, $g(\mathbb{C}/\Lambda) = 1$, etc.
- Analytically, $g(X) = \dim_{\mathbb{C}} H^0(X, \Omega^1_X)$, the dimension of the space of holomorphic 1-forms on X.
- One can show: $g(X) = 0$ if and only if $X \cong \mathbb{CP}^1$.
- For such a genus-0 surface X, every meromorphic function behaves like a rational function in some coordinate, so $\mathcal{M}(X) \cong \mathbb{C}(z)$.

Thus we have an equivalence of three properties:

$$X \cong \mathbb{CP}^1 \quad \Longleftrightarrow \quad g(X) = 0 \quad \Longleftrightarrow \quad \mathcal{M}(X) \cong \mathbb{C}(z).$$

5 Final summary

- **Calculus viewpoint:** On the Riemann sphere, any meromorphic function f has finitely many poles. Around each pole we can write a Laurent series, whose principal part coefficients are given by integrals of the 1-form $f(z)\,dz$. Using these principal parts, we build a rational function $R(z)$ with the same local behavior at all poles (finite and infinity). The difference $f - R$ is holomorphic everywhere on the compact sphere, hence constant. Therefore f is rational, and
$$\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z).$$

- **Algebraic / projective viewpoint:** The affine coordinate $z = z_0/z_1$ on $\mathbb{CP}^1$ is a meromorphic function with a simple pole at infinity. Rational functions $R(z) = p(z)/q(z)$ can be expressed via homogeneous polynomials P, Q on $\mathbb{CP}^1$, giving meromorphic maps $F_R : \mathbb{CP}^1 \to \mathbb{CP}^1$. Conversely, any meromorphic map between projective lines arises this way. Thus $\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z)$ as fields.

- **Characterization via function fields:** For any compact Riemann surface X, non-constant meromorphic functions $f : X \to \mathbb{CP}^1$ induce field embeddings $\mathbb{C}(z) \hookrightarrow \mathcal{M}(X)$ and finite extensions of fields. If $\mathcal{M}(X) \cong \mathbb{C}(z)$, the extension has degree 1, so f has degree 1, hence is a biholomorphism $X \cong \mathbb{CP}^1$. Conversely, if $X \cong \mathbb{CP}^1$, then obviously $\mathcal{M}(X) \cong \mathbb{C}(z)$.

Thus both analytically and algebraically, *the Riemann sphere is the unique compact Riemann surface whose function field is* $\mathbb{C}(z)$.

6 Why we use $\mathbb{CP}^1$ (not $\mathbb{C}$) for meromorphic functions

In this section, X denotes a compact Riemann surface.

6.1 Holomorphic maps $X \to \mathbb{C}$ are constant

Let X be a compact Riemann surface and $f : X \to \mathbb{C}$ a holomorphic map. Then f is constant.

Proof. Since f is continuous and X is compact, the image $f(X) \subset \mathbb{C}$ is compact in $\mathbb{C}$. In particular, $f(X)$ is bounded.

But a bounded holomorphic function on a connected open subset of $\mathbb{C}$ must be constant by Liouville's theorem. To connect this with f, proceed as follows.

Let $p \in X$. Choose a local coordinate chart

$$\varphi : U \subset X \longrightarrow V \subset \mathbb{C},$$

with $p \in U$ and $\varphi(p) = 0$. In this chart, the restriction of f to U looks like a holomorphic function

$$f \circ \varphi^{-1} : V \to \mathbb{C}$$

which is bounded by the global boundedness of f. By Liouville's theorem on V (considering extensions to entire functions or using the maximum modulus principle locally), this local holomorphic function must be constant on V.

Since X is connected and covered by such coordinate charts, we conclude f is constant on X. $\square$

There is no nonconstant holomorphic map $f : \mathbb{CP}^1 \to \mathbb{C}$.

Proof. The Riemann sphere $\mathbb{CP}^1$ is compact, and the previous lemma applies. $\square$

6.2 Meromorphic functions and maps to $\mathbb{CP}^1$

Recall that a point of $\mathbb{CP}^1$ can be written as $[z_0 : z_1]$. We have the standard affine chart

$$i : \mathbb{C} \hookrightarrow \mathbb{CP}^1, \qquad i(z) = [z : 1],$$

and the point at infinity

$$\infty := [1 : 0] \in \mathbb{CP}^1.$$

A *meromorphic function* on X is a holomorphic map

$$F : X \to \mathbb{CP}^1.$$

Points $p \in X$ with $F(p) \neq \infty$ are called *finite values*, and points p with $F(p) = \infty$ are called *poles* of F.

On the open set $U := F^{-1}(\mathbb{CP}^1 \setminus \{\infty\})$, we can compose with the inverse of i to obtain an honest holomorphic function

$$f := i^{-1} \circ F : U \to \mathbb{C},$$

which is the usual local expression of a meromorphic function.

6.3 Why F usually cannot factor via $i : \mathbb{C} \to \mathbb{CP}^1$

You might try to define meromorphic functions as compositions

$$F = i \circ f, \qquad f : X \to \mathbb{C},$$

with f holomorphic. We now show why this only produces *constant* functions when X is compact.

Let X be a compact Riemann surface and let

$$F : X \to \mathbb{CP}^1$$

be a nonconstant holomorphic map (so F is a nonconstant meromorphic function on X). Then F *does not* factor through the inclusion $i : \mathbb{C} \hookrightarrow \mathbb{CP}^1$; i.e. there is no holomorphic $f : X \to \mathbb{C}$ such that

$$F = i \circ f.$$

Proof. Suppose, for contradiction, that such an f exists:

$$F = i \circ f, \quad f : X \to \mathbb{C} \text{ holomorphic.}$$

Since X is compact, Lemma **??** implies that f is constant, say $f \equiv c \in \mathbb{C}$.

Then

$$F(p) = i(f(p)) = i(c) = [c : 1]$$

for all $p \in X$, so F is constant as a map $X \to \mathbb{CP}^1$. This contradicts the assumption that F is nonconstant.

Hence no nonconstant holomorphic map $F : X \to \mathbb{CP}^1$ can factor through $i : \mathbb{C} \hookrightarrow \mathbb{CP}^1$. □

In other words, if we tried to define

$$\mathcal{M}(X) \stackrel{?}{=} \{\, i \circ f \mid f : X \to \mathbb{C} \text{ holomorphic} \,\}$$

for a compact Riemann surface X, then by the Lemma and Proposition above the right-hand side would consist *only of constants*. This would lose all interesting meromorphic functions.

This is why the correct, global definition of meromorphic function on a compact Riemann surface X necessarily uses $\mathbb{CP}^1$ as the target.

6.4 Correct equivalence: meromorphic functions = maps to $\mathbb{CP}^1$

We now record the standard equivalence in precise form.

Let X be a Riemann surface. Then:

1. If $F : X \to \mathbb{CP}^1$ is holomorphic, then F is a meromorphic function on X in the usual sense (holomorphic except at isolated poles).

2. Conversely, if f is meromorphic on X in the usual sense, there exists a unique holomorphic map $F : X \to \mathbb{CP}^1$ such that

$$F(p) = \begin{cases} [f(p) : 1], & p \text{ not a pole of } f, \\ [1 : 0], & p \text{ a pole of } f. \end{cases}$$

Thus there is a natural one-to-one correspondence:

$$\mathcal{M}(X) \;\cong\; \{\, F : X \to \mathbb{CP}^1 \mid F \text{ holomorphic} \,\}.$$

Proof. (Sketch of (1)) Let $F : X \to \mathbb{CP}^1$ be holomorphic. For a point $p \in X$ with $F(p) \neq [1:0]$ (finite value), choose the affine chart

$$\phi_1 : \mathbb{CP}^1 \setminus \{[1:0]\} \to \mathbb{C}, \quad [z_0 : z_1] \mapsto z_0/z_1,$$

and set $f = \phi_1 \circ F$ on a neighborhood of p. Then f is holomorphic there. If $F(p) = [1:0]$ (infinite value), use the other chart

$$\phi_0 : \mathbb{CP}^1 \setminus \{[0:0:1]\} \to \mathbb{C}, \quad [z_0 : z_1] \mapsto z_1/z_0,$$

and check that in this chart F has a pole. Hence F is meromorphic in the usual sense.

(Sketch of (2)) Conversely, if f is meromorphic on X, then on the open set where f is finite, define

$$F(p) = [f(p) : 1],$$

and at poles set $F(p) = [1:0]$. Using local coordinates near a pole, one checks that this F is holomorphic in a neighborhood of each point of X. This gives a holomorphic map $F : X \to \mathbb{CP}^1$. Uniqueness is clear from the defining formula. □

So the right way to think is:

A meromorphic function on X is a holomorphic map $X \to \mathbb{CP}^1$, not a map $X \to \mathbb{C}$ composed with the inclusion $\mathbb{C} \hookrightarrow \mathbb{CP}^1$.

The Riemann sphere as a conic in $\mathbb{CP}^2$

Consider $\mathbb{CP}^1$ as the Riemann sphere with affine coordinate z on the chart $[z:1] \in \mathbb{CP}^1 \setminus \{\infty\}$. On this affine chart we have holomorphic functions

$$f(z) = z, \qquad g(z) = z^2,$$

which satisfy the polynomial relation

$$P(x,y) := y - x^2, \qquad P(f(z), g(z)) = g(z) - f(z)^2 = 0.$$

Thus the map

$$\phi : \mathbb{C} \longrightarrow \mathbb{C}^2, \qquad z \longmapsto (x,y) = (f(z), g(z)) = (z, z^2)$$

has image contained in the affine algebraic curve

$$C_{\text{aff}} := \{(x,y) \in \mathbb{C}^2 \mid y - x^2 = 0\},$$

the parabola $y = x^2$. Conversely, every point on this parabola is of the form $(x,y) = (t,t^2)$, so ϕ is a bijection $\mathbb{C} \simeq C_{\text{aff}}$ of complex manifolds.

Homogenization and the projective conic

Let $[X:Y:Z]$ be homogeneous coordinates on $\mathbb{CP}^2$. On the affine chart $Z \neq 0$ we set

$$x = \frac{X}{Z}, \qquad y = \frac{Y}{Z}.$$

The affine equation $y - x^2 = 0$ becomes

$$\frac{Y}{Z} - \left(\frac{X}{Z}\right)^2 = 0.$$

Multiplying by Z^2 gives the homogeneous equation

$$YZ - X^2 = 0.$$

Thus the projective closure of the parabola is the conic

$$C := \{[X:Y:Z] \in \mathbb{CP}^2 \mid YZ = X^2\}.$$

(Equivalently, one often uses the isomorphic conic $XZ = Y^2$, obtained by renaming coordinates; this is the standard form for the image of the Veronese embedding below.)

The Veronese embedding

We now consider $\mathbb{CP}^1$ with homogeneous coordinates $[u:v]$ and the degree–2 Veronese embedding

$$\nu_2 : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^2, \qquad [u:v] \longmapsto [X:Y:Z] = [u^2 : uv : v^2].$$

A direct computation shows that the image of ν_2 lies on the conic

$$XZ = Y^2.$$

Indeed, for $[X : Y : Z] = [u^2 : uv : v^2]$ we have

$$XZ = (u^2)(v^2) = u^2v^2, \qquad Y^2 = (uv)^2 = u^2v^2,$$

so $XZ - Y^2 = 0$ is identically satisfied.

On the affine chart $Z \neq 0$ (set $Z = 1$), the equation $XZ = Y^2$ becomes $x = y^2$ with $x = X/Z$, $y = Y/Z$. This is (up to swapping x and y) the same parabola $y = x^2$ considered above.

One checks that ν_2 is injective, and as every point of the conic $XZ = Y^2$ has the form $[u^2 : uv : v^2]$ for some $[u : v] \in \mathbb{CP}^1$, the map ν_2 induces a biholomorphism

$$\mathbb{CP}^1 \xrightarrow{\sim} C.$$

In particular, the Riemann sphere $S^2 \simeq \mathbb{CP}^1$ can be viewed equivalently as the smooth projective conic $C \subset \mathbb{CP}^2$ given by $XZ = Y^2$, and the coordinate z on $\mathbb{CP}^1$ corresponds to the parameter that maps z to the point $(x, y) = (z^2, z)$ on the parabola in the affine chart.