Reading Grothendieck's "Riemann–Roch" Doodle (Algebraic Geometry Notes)

Figure 1: Grothendieck's doodle (scan/photo).

Contents

1 What the doodle is about

Grothendieck is gesturing at the *Grothendieck–Riemann–Roch theorem (GRR)*: a compatibility between

- pushforward in *K-theory of coherent sheaves*, and

- pushforward in *intersection theory* (Chow groups / cycles),

after applying a universal "bridge" built from the *Chern character* and the *Todd class.* The joke "the latest craze: the diagram" is that the correct statement is best expressed as a *commuting square*, and setting up the frameworks in full generality is genuinely lengthy.

2 Geometric setting and hypotheses

2.1 The morphism

Let $f\colon X \to Y$ be a morphism of schemes (or varieties) of finite type over a field.

2.2 Properness

Assume f is *proper.* This is the condition ensuring that pushforward of coherent sheaves preserves coherence and that Euler characteristics behave well. Properness is also the hypothesis under which Chow groups admit a natural pushforward.

2.3 Smoothness (recommended for a first pass)

For the cleanest first statement, assume X and Y are *smooth* quasi-projective varieties (or smooth schemes of finite type). Smoothness ensures:

- the tangent bundles T_X and T_Y exist as vector bundles,
- Chern classes and Todd classes are straightforward to define, and
- K-theory of vector bundles and K-theory of coherent sheaves agree (see below).

3 The two worlds GRR connects

3.1 K-theory of coherent sheaves: G_0 and K_0

Two Grothendieck groups. There are two closely related Grothendieck groups:

- $K_0(X)$: the Grothendieck group of *vector bundles* (locally free sheaves) on X.
- $G_0(X)$: the Grothendieck group of *coherent sheaves* on X.

Smooth case identification. If X is smooth (regular), every coherent sheaf admits a finite locally free resolution, and one has a canonical isomorphism $K_0(X) \cong G_0(X)$.

Proper pushforward on G_0. If $f\colon X \to Y$ is proper, there is a natural pushforward

$$f_*\colon G_0(X) \to G_0(Y), \qquad [\mathcal{F}] \longmapsto \sum_{i\geq 0}(-1)^i\,[R^i f_*\mathcal{F}].$$

This is the algebraic-geometric meaning of the doodle's upper horizontal arrow (often written f_*, and sometimes $f_!$ in "wrong-way" notation).

3.2 Chow groups and pushforward

Chow groups. Let $A_k(X)$ denote the Chow group of k-dimensional cycles modulo rational equivalence, and set

$$A_*(X) := \bigoplus_k A_k(X).$$

Because GRR involves denominators (from Chern character and Todd class), we typically work with

$$A_*(X)_{\mathbb{Q}} := A_*(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Proper pushforward on Chow groups. If f is proper, there is a pushforward

$$f_* \colon A_*(X) \to A_*(Y),$$

defined on integral cycles by mapping a subvariety $V \subset X$ to $\deg(V/V') \cdot [V']$ if $f(V) = V' \subset Y$ and $\dim V' = \dim V$, and to 0 otherwise. This is the lower horizontal arrow in the modern diagrammatic statement of GRR.

4 The bridge: Chern character and Todd class

4.1 Chern character

When X is smooth, the Chern character is a ring homomorphism

$$\mathrm{ch} \colon K_0(X) \to A^*(X)_{\mathbb{Q}},$$

where $A^*(X)$ is the Chow ring (codimension grading). It is additive on short exact sequences and multiplicative on tensor products. Concretely, if E has Chern roots x_i, then

$$\mathrm{ch}(E) = \sum_i e^{x_i}.$$

4.2 Todd class

For a vector bundle E on X, the Todd class $\mathrm{Td}(E) \in A^*(X)_{\mathbb{Q}}$ is the multiplicative characteristic class defined (in terms of Chern roots x_i) by

$$\mathrm{Td}(E) = \prod_i \frac{x_i}{1 - e^{-x_i}}.$$

For a smooth X, we write $\mathrm{Td}(T_X)$ for the Todd class of the tangent bundle.

5 Grothendieck–Riemann–Roch (GRR)

5.1 The commuting-square formulation (smooth case)

Let $f \colon X \to Y$ be proper and assume X and Y are smooth. Define the "Riemann–Roch transformation"

$$\tau_X \colon K_0(X) \to A_*(X)_{\mathbb{Q}}, \qquad \tau_X(\alpha) := \mathrm{ch}(\alpha) \cdot \mathrm{Td}(T_X) \cap [X],$$

and similarly τ_Y.

The theorem (diagram form). Then the following square commutes:

$$\begin{tikzcd} K_0(X) \arrow[r, "f_*"] \arrow[d, "\tau_X"'] & K_0(Y) \arrow[d, "\tau_Y"] \\ A_*(X)_{\mathbb{Q}} \arrow[r, "f_*"'] & A_*(Y)_{\mathbb{Q}} \end{tikzcd}$$

In words: translating a K-class to a cycle class via τ and then pushing forward equals pushing forward in K-theory first and translating afterwards.

5.2 Expanded identity

Equivalently, for every $\alpha \in K_0(X)$,

$$\mathrm{ch}(f_*\alpha)\,\mathrm{Td}(T_Y) \cap [Y] \;=\; f_*(\mathrm{ch}(\alpha)\,\mathrm{Td}(T_X) \cap [X])\,.$$

5.3 A relative variant (optional)

One often rewrites GRR using the *relative Todd class*

$$\mathrm{Td}(T_f) := \frac{\mathrm{Td}(T_X)}{f^*\mathrm{Td}(T_Y)} \in A^*(X)_{\mathbb{Q}},$$

so that

$$\mathrm{ch}(f_*\alpha) = f_*(\mathrm{ch}(\alpha)\,\mathrm{Td}(T_f) \cap [X]) \qquad \text{(after identifying targets appropriately).}$$

This is a convenient form when comparing to classical statements.

6 How GRR recovers classical Riemann–Roch

6.1 Curves

Let X be a smooth projective curve and let $f\colon X \to \mathrm{Spec}(k)$ be the structure morphism. For a line bundle $\mathcal{L}$, the pushforward $f_*[\mathcal{L}]$ in $K_0(k) \cong \mathbb{Z}$ computes the Euler characteristic:

$$f_*[\mathcal{L}] = \chi(X, \mathcal{L}) := \sum_i (-1)^i \dim_k H^i(X, \mathcal{L}).$$

GRR expresses $\chi(X, \mathcal{L})$ as an intersection-theoretic number built from $\mathrm{ch}(\mathcal{L})$ and $\mathrm{Td}(T_X)$; evaluating gives

$$\chi(X, \mathcal{L}) = \deg(\mathcal{L}) + 1 - g,$$

the classical Riemann–Roch theorem for curves.

7 Mapping the doodle's symbols to modern notation

Grothendieck's doodle uses historically flavored notation (and some intentionally informal shorthand). A modern algebraic-geometry decoding is:

- $K'(X)$: read as $G_0(X)$ (coherent K-theory), or as $K_0(X)$ when X is smooth.
- "Gr": a hint that K-theory has natural filtrations (e.g. by codimension) whose associated graded relates to cycle groups; morally: "linearize to Chow."

- τ: the Riemann–Roch natural transformation $\alpha \mapsto \mathrm{ch}(\alpha)\mathrm{Td}(T_X) \cap [X]$.
- ch: the Chern character, explicitly present in the doodle.
- Tensoring with $\mathbb{Q}$: needed because ch and Td introduce denominators.

Thus, the punchline "i.e. commutative!" is precisely the GRR commuting square.

8 A minimal prerequisite roadmap

If you want to understand GRR efficiently, a good order is:

1. Coherent sheaves and derived pushforward Rf_*.
2. Grothendieck groups $G_0(X)$, $K_0(X)$; exact sequence relations.
3. Chow groups $A_*(X)$ and proper pushforward of cycles.
4. Chern classes; then Chern character ch.
5. Todd class Td and why rational coefficients appear.
6. Statement and meaning of GRR; then compute special cases (curves, surfaces).

9 Mental model: why the theorem *must* be a diagram

K-theory is the natural home for exact sequences and derived functors (like Rf_*). Chow groups are the natural home for intersection products and characteristic classes. GRR says that, after translating K-classes to cycle classes via ch and Td, pushforward becomes compatible:

$$\tau_Y \circ f_* \;=\; f_* \circ \tau_X.$$

In other words: the correct formulation is, inevitably, a commuting diagram.

Optional appendix: a clean "modern diagram" block (copy-paste)