

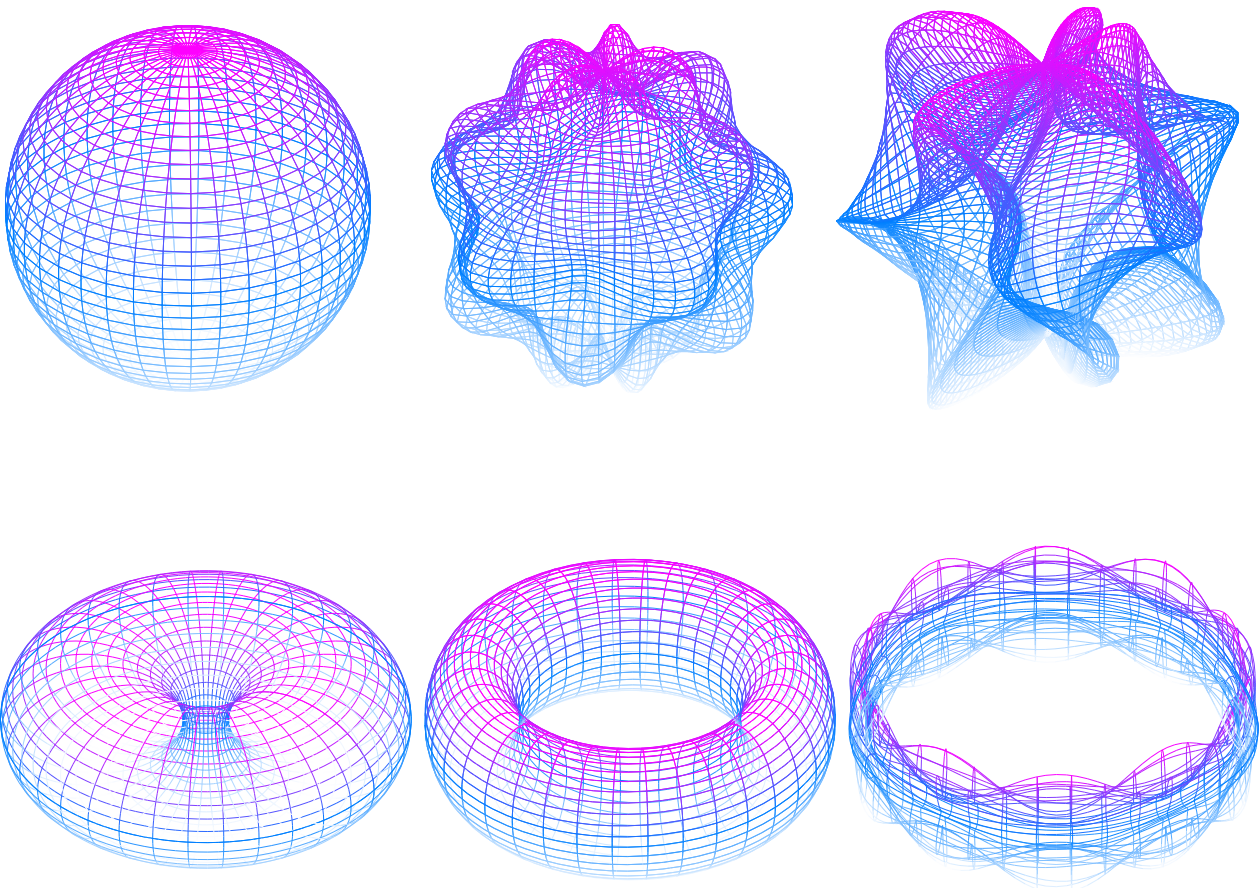
Topology I

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We cover the following topics in this note.

- Topology and Topological Space
 - Open Set
 - Continuous Mapping
 - Distance Function and Metric Space
 - Convergence of Sequences; Continuity of Functions
 - TBA
-



Topology; Topological Space

Definition. Let S be a non-empty set. A **topology**^a on S is a subset $\mathcal{T} \subseteq 2^S$, where 2^S denotes the power set of S , that satisfies the following axioms:

(O1)^b The empty set and the entire set S belong to \mathcal{T} : $S \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.

(O2)^c The union of any collection of elements in \mathcal{T} is also an element of \mathcal{T} :

$$\{U_i\}_{i \in I} \subseteq \mathcal{T} \implies \bigcup_{i \in I} U_i \in \mathcal{T}.$$

(O3)^d The intersection of any finite number of elements in \mathcal{T} is also an element of \mathcal{T} :

$$\{U_i\}_{i=1}^n \subseteq \mathcal{T} \implies \bigcap_{i=1}^n U_i \in \mathcal{T}.$$

The pair (S, \mathcal{T}) is called a **topological space**.

^aThe word “topology” comes from the Greek roots “topos” meaning “place” and “logos” meaning “study”.

^bEmpty set and Whole space

^cClosure under *arbitrary* unions

^dClosure under *finite* intersections

Remark. By mathematical induction, we have

$$O3 \iff [\{U_1, U_2\} \subseteq \mathcal{T} \Rightarrow U_1 \cap U_2 \in \mathcal{T}].$$

Open Set (Topology)

Definition. Let (S, \mathcal{T}) be a topological space. $U \subseteq S$ is an **open set**, or **open** (in S) iff $U \in \mathcal{T}$.

Remark. A subset \mathcal{T} of power set 2^S is a topology on S if and only if

(i) \emptyset and S are open;

(ii) Let $U_1, U_2, \dots \in \mathcal{T}$, i.e., $\{U_i\}_{i \in I} \subseteq \mathcal{T}$. Then $\bigcup_{i \in I} U_i$ is open.

(iii) Let $U_1, U_2, \dots, U_n \in \mathcal{T}$, i.e., $\{U_i\}_{i=1}^n \subseteq \mathcal{T}$. Then $\bigcap_{i=1}^n U_i$ is open.

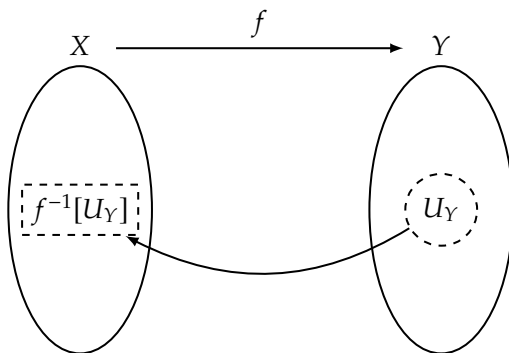
Continuous Mapping by Open Sets

Definition. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are topological spaces. Let $f : X \rightarrow Y$ be a mapping from X to Y .

(1) (Continuous Everywhere) The mapping f is **continuous on X** if and only if

$$U_Y \in \mathcal{T}_Y \implies f^{-1}[U_Y] \in \mathcal{T}_X,$$

where $f^{-1}[U_Y] = \{x \in X : f(x) \in U_Y\}$ is the preimage of U_Y under f .



Note (Preparation for **Example 1**). Let $S \neq \emptyset$ be a set, and let $\{A_\alpha\}_{\alpha \in \Lambda} \subseteq S$. Then

$$\begin{aligned} S \setminus \bigcup_{\alpha \in \Lambda} A_\alpha &= S \setminus \{x \in S : \exists \alpha \in \Lambda \text{ s.t. } x \in A_\alpha\} = \{x \in S : \neg[\exists \alpha \in \Lambda \text{ s.t. } x \in A_\alpha]\} \\ &= \{x \in S : \forall \alpha \in \Lambda, x \notin A_\alpha\} \\ &= \{x \in S : \forall \alpha \in \Lambda, x \in S \setminus A_\alpha\} \\ &= \bigcap_{\alpha \in \Lambda} (S \setminus A_\alpha). \end{aligned}$$

$$\begin{aligned} S \setminus \bigcap_{\alpha \in \Lambda} A_\alpha &= S \setminus \{x \in S : \forall \alpha \in \Lambda, x \in A_\alpha\} = \{x \in S : \neg[\forall \alpha \in \Lambda, x \in A_\alpha]\} \\ &= \{x \in S : \exists \alpha \in \Lambda \text{ s.t. } x \notin A_\alpha\} \\ &= \{x \in S : \exists \alpha \in \Lambda \text{ s.t. } x \in S \setminus A_\alpha\} \\ &= \bigcup_{\alpha \in \Lambda} (S \setminus A_\alpha). \end{aligned}$$

Note (Preparation for **Example 1**).

- (1) A Subset of a Finite Set is Finite.
- (2) The Intersection of Finite Sets is Finite.

Example 1 (Cofinite Topology). Let $S \neq \emptyset$ be a set. Define the cofinite topology $\mathcal{T}_C \subseteq 2^S$ by

$$\begin{aligned}\mathcal{T}_C &:= \{U \subseteq S : S \setminus U \text{ is finite}\} \cup \{\emptyset\} \\ &= \{U \subseteq S : U = \emptyset \text{ or } S \setminus U \text{ is finite}\}.\end{aligned}$$

In other words, U is open in the cofinite topology if U is the empty, or if the complement $S \setminus U$ is a finite set. We claim that \mathcal{T}_C be a topology on S :

(O1) By definition, $\emptyset \in \mathcal{T}_C$. For $U = S$, the complement $S \setminus S = \emptyset$, which is finite, so $S \in \mathcal{T}_C$. Hence, both \emptyset and S are elements of \mathcal{T}_C .

(O2) Let $\{U_i\}_{i \in I} \subseteq \mathcal{T}_C$.

(Case 1) If $U_i = \emptyset$ for all $i \in I$, then $\bigcup_{i \in I} U_i = \emptyset \in \mathcal{T}_C$.

(Case 2) Suppose that there exists $i_0 \in I$ such that $U_{i_0} \neq \emptyset$. Then

$$S \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (S \setminus U_i) \subseteq (S \setminus U_{i_0}).$$

Since $S \setminus U_{i_0}$ is finite, $S \setminus \bigcup_{i \in I} U_i$ is finite, so $\bigcup_{i \in I} U_i \in \mathcal{T}_C$.

(O3) Let $U_1 \in \mathcal{T}_C$ and $U_2 \in \mathcal{T}_C$.

(Case 1) If $U_1 = \emptyset$ or $U_2 = \emptyset$, then $U_1 \cap U_2 = \emptyset \in \mathcal{T}_C$.

(Case 2) Suppose that $U_1 \neq \emptyset$ and $U_2 \neq \emptyset$. Then $S \setminus U_1$ and $S \setminus U_2$ are finite. By the De Morgan law, we have

$$S \setminus (U_1 \cap U_2) = (S \setminus U_1) \cup (S \setminus U_2),$$

which is a finite set. Thus, $U_1 \cap U_2 \in \mathcal{T}_C$.

Example 2 (Discrete Topology). Let $S \neq \emptyset$ be a set, and let $\mathcal{T} = 2^S$ be the power set of S . Then \mathcal{T} is called the **discrete topology** on S and $(S, \mathcal{T}) = (S, 2^S)$ the **discrete (topological) space** on S .

Example 3 (Indiscrete Topology). Let $S \neq \emptyset$ be a set, and let $\mathcal{T} = \{S, \emptyset\}$. Then \mathcal{T} is called the **indiscrete topology** on S and $(S, \mathcal{T}) = (S, \{S, \emptyset\})$ the **indiscrete (topological) space** on S .

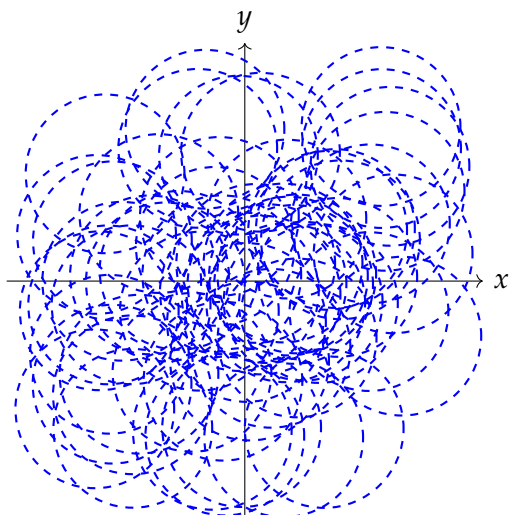
Note.

- (1) Discrete Topology is Finest Topology.
- (2) Indiscrete Topology is Coarsest Topology.

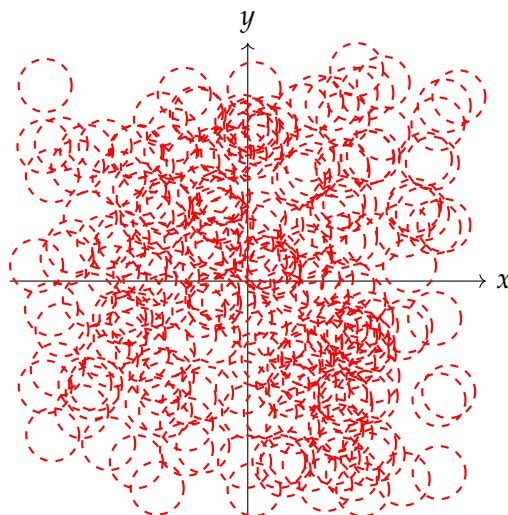
Coarser Topology and Finer Topology

Definition. Let $S \neq \emptyset$ be a set. Let \mathcal{T}_1 and \mathcal{T}_2 be topologies on S .

- (1) \mathcal{T}_1 is said to be **coarser** than \mathcal{T}_2 if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.
- (2) \mathcal{T}_1 is said to be **finer** than \mathcal{T}_2 if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.



Coarser Topology



Finer Topology

Distance Function

Definition. Let S be a set. The real-valued function of two variable

$$d : S \times S \rightarrow \mathbb{R}$$

is called a **distance function** (or **metric**) if it satisfies the following properties:

- (i)^a $\forall x, y \in S, d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$.
- (ii)^b $\forall x, y \in S, d(x, y) = d(y, x)$.
- (iii)^c $\forall x, y, z \in S, d(x, z) \leq d(x, y) + d(y, z)$.

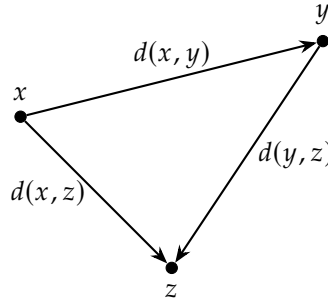
The pair (S, d) is called a **metric space**.

^aNon-negativity and Zero only for identical points

^bSymmetry

^cTriangle inequality

Remark.



Example 4.

- Let $S = \mathbb{R}$, the set of real numbers. Define the function $d : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$d(x, y) = |x - y|$$

for $x, y \in \mathbb{R}$.

- Let $S = \mathbb{R}^n$, the n -dimensional Euclidean space. Define the function $d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=0}^{n-1} |x_i - y_i|^2},$$

where $\mathbf{x} = (x_0, x_1, \dots, x_{n-1})$ and $\mathbf{y} = (y_0, \dots, y_{n-1})$ are vectors in \mathbb{R}^n .

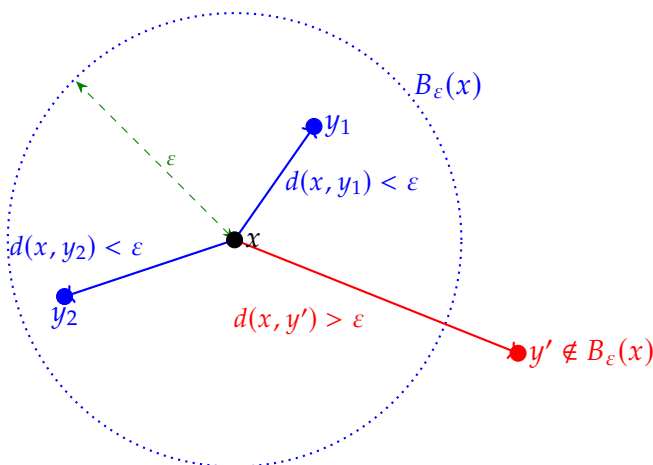
Open Epsilon-Ball

Definition. Let (S, d) be a metric space, where S is a set and $d : S \times S \rightarrow \mathbb{R}$ is a metric. For $x \in S$ and $\varepsilon \in \mathbb{R}_{>0}$, the **open ε -ball^a of x in S** , denoted by $B_\varepsilon(x)$, is defined as

$$B_\varepsilon(x) := \{y \in S : d(x, y) < \varepsilon\}.$$

^aOpen ball with center x and radius ε

Remark.



Epsilon-Neighborhood (Real Analysis)

Definition. Consider the Euclidean space (\mathbb{R}^1, d) . The **ε -neighborhood of $\alpha \in \mathbb{R}$** is defined as the open interval:

$$\mathcal{N}_\varepsilon(\alpha) := \{x \in \mathbb{R} : |x - \alpha| < \varepsilon\} = (\alpha - \varepsilon, \alpha + \varepsilon)$$

where $\varepsilon \in \mathbb{R}_{>0}$.

Neighborhood (Topology)

Definition. Let (S, τ) be a topological space.

(1) (Neighborhood of a Set) Let $A \subseteq S$. \mathcal{N}_A is a **neighborhood of A** if

$$\exists U \in \tau \text{ such that } A \subseteq U \subseteq \mathcal{N}_A \subseteq S.$$

(2) (Neighborhood of a Point) Consider a singleton $\{a\} = A \subseteq S$, that is, $a \in S$ be a point in S . Then \mathcal{N}_a is a **neighborhood of $a \in S$** if

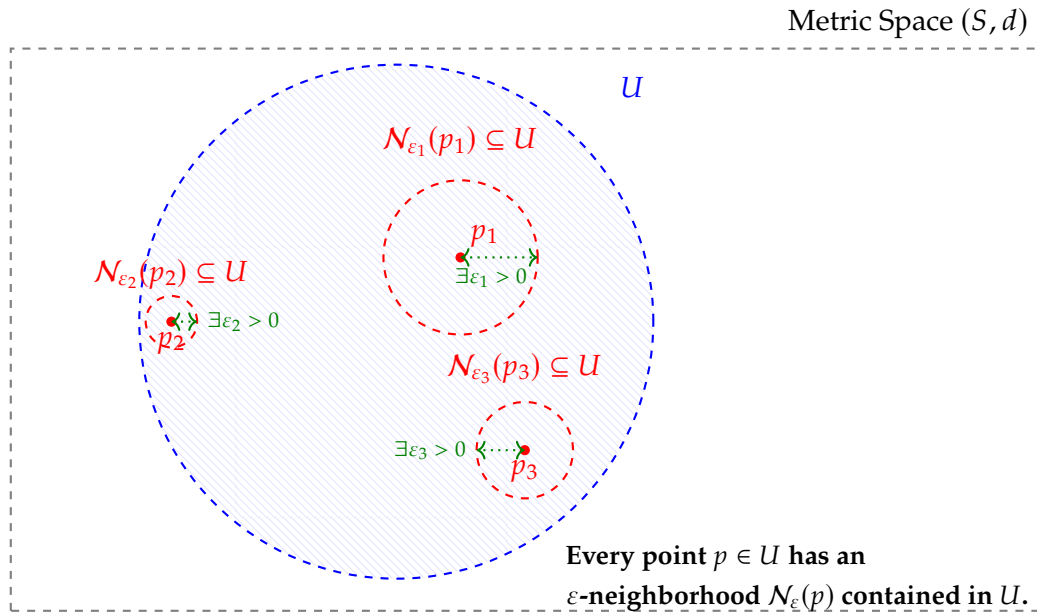
$$\exists U \in \tau \text{ such that } a \in U \subseteq \mathcal{N}_a \subseteq S.$$

Open Set (Metric Space)

Definition. Let (S, d) be a metric space, where S is a set and $d : S \times S \rightarrow \mathbb{R}$ is a metric. Then

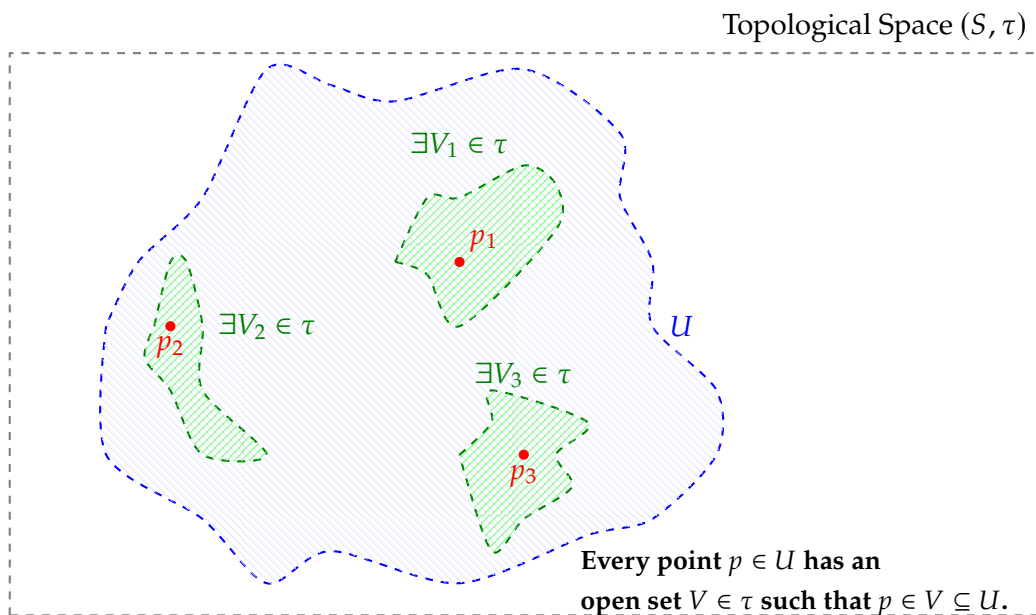
$$U \subseteq S \text{ is } \mathbf{open} \text{ in } S \stackrel{\text{def}}{\iff} \forall p \in U, \exists \varepsilon > 0 \text{ such that } N_\varepsilon(p) \subseteq U.$$

Remark.



Remark. Let (S, τ) be a topological space. Then

$$U \subseteq S \text{ is } \mathbf{open} \text{ in } S \stackrel{\text{def}}{\iff} \forall p \in U, \exists V \in \tau \text{ such that } p \in V \subseteq U.$$



Exercise (Metric Topology). Let (S, d) be a metric space, where S is a set and $d : S \times S \rightarrow \mathbb{R}$ is a metric. Consider the set τ of all open sets of S :

$$\begin{aligned}\tau &:= \{U \subseteq S : U \text{ is open in } S\} \\ &= \{U \subseteq S : \forall p \in U, \exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(p) \subseteq U\}.\end{aligned}$$

We claim that τ is the topology induced by the metric d on the space S :

(O1) $S \in \tau$ and $\emptyset \in \tau$:

($\emptyset \in \tau$) The condition

$$“\forall p \in U, \exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(p) \subseteq U”$$

is vacuously true for $U = \emptyset$. Therefore $\emptyset \in \tau$.

($S \in \tau$) For $p \in S$, the ε -neighborhood of p is defined as

$$\mathcal{N}_\varepsilon(p) = \{q \in S : d(p, q) < \varepsilon\} \subseteq S.$$

Since S is the entire space, $\mathcal{N}_\varepsilon(p) \subseteq S$ for any $\varepsilon > 0$.

(O2) τ is closed under arbitrary unions:

Let $\{U_i\}_{i \in I}$ be an arbitrary collection of sets in τ . Let $p \in \bigcup_{i \in I} U_i$. Then

$$\exists i_0 \in I \text{ such that } p \in U_{i_0}.$$

Since $U_{i_0} \in \tau$, there exists $\varepsilon > 0$ such that $\mathcal{N}_\varepsilon(p) \subseteq U_{i_0}$. Then

$$\mathcal{N}_\varepsilon(p) \subseteq U_{i_0} \subseteq \bigcup_{i \in I} U_i.$$

Thus, $\bigcup_{i \in I} U_i \in \tau$.

(O3) τ is closed under finite intersections:

Let $U_1, U_2 \in \tau$, and let $p \in (U_1 \cap U_2)$. Then

$$\exists \varepsilon_1 > 0 \text{ such that } \mathcal{N}_{\varepsilon_1}(p) \subseteq U_1,$$

$$\exists \varepsilon_2 > 0 \text{ such that } \mathcal{N}_{\varepsilon_2}(p) \subseteq U_2.$$

Define $\varepsilon := \min(\varepsilon_1, \varepsilon_2)$. Then

$$\mathcal{N}_\varepsilon(p) \subseteq \mathcal{N}_{\varepsilon_i}(p) \subseteq U_i \text{ for } i = 1, 2.$$

Thus $\mathcal{N}_\varepsilon(p) \subseteq U_1 \cap U_2$, and so $U_1 \cap U_2 \in \tau$.

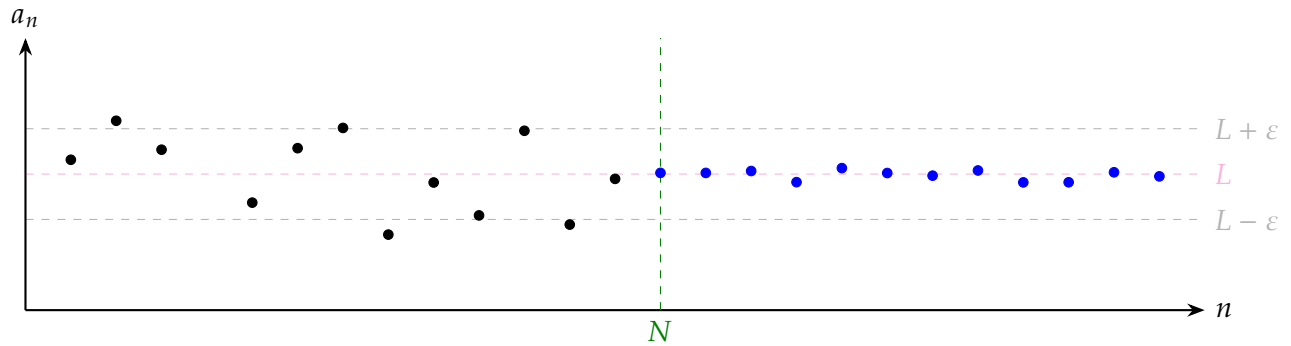
Note (Convergence of Sequences). We consider the topological space (\mathbb{R}, τ) where

$$\tau = \left\{ U \subseteq \mathbb{R} : U = \bigcup_{i \in I} (a_i, b_i) \right\}$$

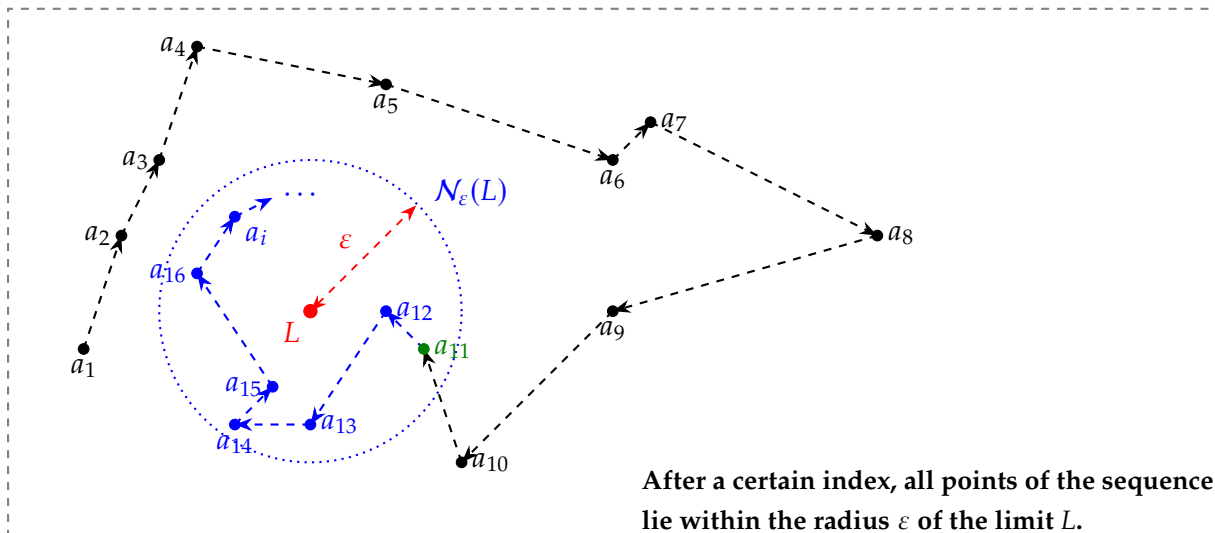
where each (a_i, b_i) is an open interval with $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$, that is, τ consists of all open intervals (and unions of such intervals).

A sequence $\{a_n\}_{n=1}^{\infty} (\subseteq \mathbb{R})$ is **converge** to $L \in \mathbb{R}$ if and only if

$$\begin{aligned} & \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } [n \geq N \implies |a_n - L| < \varepsilon] \\ \iff & \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } [n \geq N \implies d(a_n, L) < \varepsilon] \\ \iff & \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } [n \geq N \implies a_n \in (L - \varepsilon, L + \varepsilon)] \\ \iff & \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } [n \geq N \implies a_n \in \mathcal{N}_{\varepsilon}(L)] \\ \iff & \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } [n \geq N \implies a_n \in U_{\varepsilon}] \\ \iff & \forall U \in \tau \text{ with } L \in U, \exists N \in \mathbb{N} \text{ such that } [n \geq N \implies a_n \in U] \end{aligned}$$



Metric Space



Continuity of Functions

Definition. Let $S \subseteq \mathbb{R}$ be a non-empty subset of \mathbb{R} . Let $f : S \rightarrow \mathbb{R}$ be a real-valued function, and let $a \in S$. We say that f is **continuous at a** if and only if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

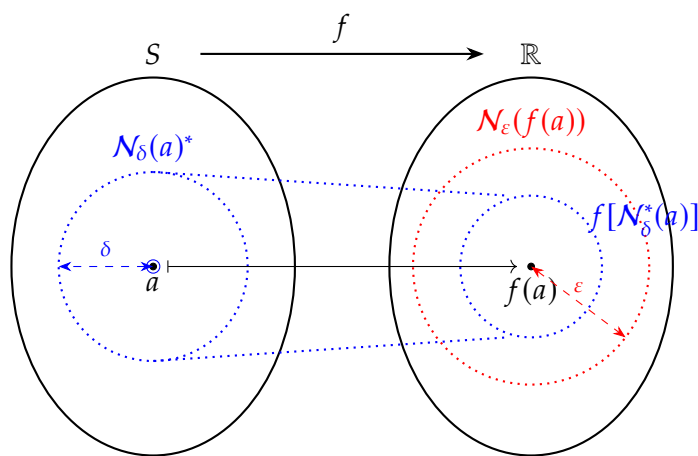
That is,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

If f is continuous on every point of S , then f is called a **continuous function on S** .

Remark.

$$\begin{aligned} & \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon \\ \iff & \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } x \in (a - \delta, a) \cup (a, a + \delta) \implies f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon) \\ \iff & \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } x \in \mathcal{N}_\delta(a) \setminus \{a\} \implies f(x) \in \mathcal{N}_\varepsilon(f(a)) \\ \iff & \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } f(x) \in f[\mathcal{N}_\delta^*(a)] \implies f(x) \in \mathcal{N}_\varepsilon(f(a)) \quad \because f[\mathcal{N}_\delta^*(a)] = \{f(x) : x \in \mathcal{N}_\delta^*(a)\} \\ \iff & \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } f[\mathcal{N}_\delta^*(a)] \subseteq \mathcal{N}_\varepsilon(f(a)). \end{aligned}$$



Remark. f is discontinuous at a if and only if

$$\begin{aligned} & \exists \varepsilon > 0 \text{ such that } \forall \delta > 0, |x - a| < \delta \text{ but } |f(x) - f(a)| \geq \varepsilon \\ \iff & \exists \varepsilon > 0 \text{ such that } \forall \delta > 0, \mathcal{N}_\varepsilon(f(a)) \not\subseteq f[\mathcal{N}_\delta^*(a)]. \end{aligned}$$

Note. Consider a topological space (\mathbb{R}, τ_d) , where

$$\tau_d := \{U \subseteq \mathbb{R} : \forall p \in U, \exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(p) \subseteq U\}.$$

A sequence $\{a_n\}_{n=1}^\infty \subseteq \mathbb{R}$ converges to $L \in \mathbb{R}$ if and only if

$$\begin{aligned} & \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } [n \geq N \implies |a_n - L| < \varepsilon] \\ \iff & \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } [n \geq N \implies a_n \in (L - \varepsilon, L + \varepsilon)] \\ \iff & \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } [n \geq N \implies a_n \in \mathcal{N}_\varepsilon(L)] \\ \iff & \forall U \in \tau_d \text{ with } L \in U, \exists N \in \mathbb{N} \text{ such that } [n \geq N \implies a_n \in U]. \end{aligned}$$

Limit Theorem (Topology)

Theorem. Consider a topological space (\mathbb{R}, τ_d) , where

$$\tau_d := \{U \subseteq \mathbb{R} : \forall p \in U, \exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(p) \subseteq U\}.$$

Let $\{a_n\} \subseteq \mathbb{R}$ and $\{b_n\} \subseteq \mathbb{R}$. Let $\lim_{n \rightarrow \infty} a_n = \alpha \in \mathbb{R}$, $\lim_{n \rightarrow \infty} b_n = \beta \in \mathbb{R}$, and $k \in \mathbb{R}$. Then

- (1) $\lim_{n \rightarrow \infty} ka_n = k\alpha = k \lim_{n \rightarrow \infty} a_n$.
- (2) $\lim_{n \rightarrow \infty} a_n \pm b_n = \alpha \pm \beta = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n$.
- (3) $\lim_{n \rightarrow \infty} a_n b_n = \alpha\beta = \lim_{n \rightarrow \infty} a_n \lim_{n \rightarrow \infty} b_n$.
- (4) $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\alpha}{\beta} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$. (Here, $\beta \neq 0$ and $b_n \neq 0$)

Proof.

- (1) Let $\varepsilon > 0$. Since $\lim_{n \rightarrow \infty} a_n = \alpha$, we know

$$\exists N \in \mathbb{N} \text{ s.t. } n \geq N \implies |a_n - \alpha| < \frac{\varepsilon}{|k| + 1}$$

Thus, if $n \geq N$ then

$$\begin{aligned} |ka_n - k\alpha| &= |k(a_n - \alpha)| \\ &= |k||a_n - \alpha| \quad \because |xy| = |x||y| \\ &< |k| \cdot \frac{\varepsilon}{|k| + 1} \\ &< \varepsilon. \end{aligned}$$

- (1) Let $U \in \tau_d$ with $k\alpha \in U$. Since $k\alpha \in U$, by definition of τ_d , we have

$$\exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(k\alpha) \subseteq U.$$

Since $\lim_{n \rightarrow \infty} a_n = \alpha$, we know

$$\exists N \in \mathbb{N} \text{ s.t. } n \geq N \implies a_n \in \left(\alpha - \frac{\varepsilon}{|k| + 1}, \alpha + \frac{\varepsilon}{|k| + 1} \right).$$

Thus, if $n \geq N$ then

$$\begin{aligned} ka_n &\in \left(k\alpha - k \cdot \frac{\varepsilon}{|k| + 1}, k\alpha + k \cdot \frac{\varepsilon}{|k| + 1} \right) \\ &\subseteq (k\alpha - \varepsilon, k\alpha + \varepsilon) \\ &= \mathcal{N}_\varepsilon(k\alpha) \subseteq U. \end{aligned}$$

(2) Let $\varepsilon > 0$. Since $\lim_{n \rightarrow \infty} a_n = \alpha$ and $\lim_{n \rightarrow \infty} b_n = \beta$, we know

$$\exists N_1 \in \mathbb{N} \text{ s.t. } n \geq N_1 \Rightarrow |a_n - \alpha| < \frac{\varepsilon}{2},$$

$$\exists N_2 \in \mathbb{N} \text{ s.t. } n \geq N_2 \Rightarrow |b_n - \beta| < \frac{\varepsilon}{2}.$$

Let $N = \max \{N_1, N_2\}$. If $n \geq N$ then

$$\begin{aligned} |(a_n + b_n) - (\alpha + \beta)| &= |(a_n - \alpha) + (b_n - \beta)| \\ &\leq |a_n - \alpha| + |b_n - \beta| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

and

$$\begin{aligned} |(a_n - b_n) - (\alpha - \beta)| &= |(a_n - \alpha) + (-b_n + \beta)| \\ &\leq |a_n - \alpha| + |b_n - \beta| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

(2) Let $U \in \tau_d$ with $\alpha \pm \beta \in U$. Since $\alpha \pm \beta \in U$, by definition of τ_d , we have

$$\exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(\alpha \pm \beta) \subseteq U.$$

Since $\lim_{n \rightarrow \infty} a_n = \alpha$ and $\lim_{n \rightarrow \infty} b_n = \beta$,

$$\exists N_1 \in \mathbb{N} \text{ s.t. } n \geq N_1 \Rightarrow a_n \in \left(\alpha - \frac{\varepsilon}{2}, \alpha + \frac{\varepsilon}{2}\right),$$

$$\exists N_2 \in \mathbb{N} \text{ s.t. } n \geq N_2 \Rightarrow b_n \in \left(\beta - \frac{\varepsilon}{2}, \beta + \frac{\varepsilon}{2}\right).$$

Let $N = \max \{N_1, N_2\}$. if $n \geq N$ then

$$\begin{aligned} a_n + b_n &\in \left(\alpha - \frac{\varepsilon}{2} + \beta - \frac{\varepsilon}{2}, \alpha + \frac{\varepsilon}{2} + \beta + \frac{\varepsilon}{2}\right) \\ &= (\alpha + \beta - \varepsilon, \alpha + \beta + \varepsilon) \\ &= \mathcal{N}_\varepsilon(\alpha + \beta) \subseteq U \end{aligned}$$

and

$$\begin{aligned} a_n + (-b_n) &\in \left(\alpha - \frac{\varepsilon}{2} - \beta - \frac{\varepsilon}{2}, \alpha + \frac{\varepsilon}{2} - \beta + \frac{\varepsilon}{2}\right) \\ &= (\alpha - \beta - \varepsilon, \alpha - \beta + \varepsilon) \\ &= \mathcal{N}_\varepsilon(\alpha - \beta) \subseteq U. \end{aligned}$$

(3) Let $\varepsilon > 0$. Since $\{a_n\}$ is bounded,

$$\exists M > 0 \text{ such that } \forall n \in \mathbb{N}, |a_n| \leq M.$$

$$\text{Since } \lim_{n \rightarrow \infty} a_n = \alpha \text{ and } \lim_{n \rightarrow \infty} b_n = \beta,$$

$$\exists N_1 \in \mathbb{N} \text{ s.t. } n \geq N_1 \Rightarrow |a_n - \alpha| < \frac{\varepsilon}{2|\beta| + 1},$$

$$\exists N_2 \in \mathbb{N} \text{ s.t. } n \geq N_2 \Rightarrow |b_n - \beta| < \frac{\varepsilon}{2M}.$$

Let $N = \max\{N_1, N_2\}$. If $n \geq N$ then

$$\begin{aligned} |a_n b_n - \alpha \beta| &= |a_n b_n - \alpha \beta + a_n \beta - a_n \beta| \\ &= |a_n(b_n - \beta) + \beta(a_n - \alpha)| \\ &\leq |a_n||b_n - \beta| + |\beta||a_n - \alpha| \\ &< M \cdot \frac{\varepsilon}{2M} + \frac{|\beta| \cdot \varepsilon}{2|\beta| + 1} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

$$\text{Note that } 2|\beta| < 2|\beta| + 1 \Leftrightarrow \frac{|\beta|}{2|\beta| + 1} < \frac{1}{2}.$$

(3) Let $U \in \tau_d$ with $\alpha\beta \in U$. Since $\alpha\beta \in U$, by definition of τ_d , we have

$$\exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(\alpha\beta) \subseteq U.$$

Since $\{a_n\}$ is bounded,

$$\exists M > 0 \text{ such that } \forall n \in \mathbb{N}, |a_n| \leq M.$$

$$\text{Since } \lim_{n \rightarrow \infty} a_n = \alpha \text{ and } \lim_{n \rightarrow \infty} b_n = \beta,$$

$$\exists N_1 \in \mathbb{N} \text{ s.t. } n \geq N_1 \Rightarrow a_n \in \left(\alpha - \frac{\varepsilon}{2|\beta| + 1}, \alpha + \frac{\varepsilon}{2|\beta| + 1} \right),$$

$$\exists N_2 \in \mathbb{N} \text{ s.t. } n \geq N_2 \Rightarrow b_n \in \left(\beta - \frac{\varepsilon}{2M}, \beta + \frac{\varepsilon}{2M} \right).$$

Let $N = \max\{N_1, N_2\}$, and let $n \geq N$. Then

$$a_n b_n - \alpha \beta = a_n b_n - a_n \beta + a_n \beta = \underbrace{a_n(b_n - \beta)}_{(i)} + \underbrace{\beta(a_n - \alpha)}_{(ii)}.$$

$$\begin{aligned} \text{(i)} \quad b_n &\in \left(\beta - \frac{\varepsilon}{2M}, \beta + \frac{\varepsilon}{2M} \right) \\ \Rightarrow b_n - \beta &\in \left(-\frac{\varepsilon}{2M}, \frac{\varepsilon}{2M} \right) \\ \Rightarrow a_n(b_n - \beta) &\in \left(-M \cdot \frac{\varepsilon}{2M}, M \cdot \frac{\varepsilon}{2M} \right) = \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2} \right). \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad a_n &\in \left(\alpha - \frac{\varepsilon}{2|\beta| + 1}, \alpha + \frac{\varepsilon}{2|\beta| + 1} \right) \\ \Rightarrow a_n - \alpha &\in \left(-\frac{\varepsilon}{2|\beta| + 1}, \frac{\varepsilon}{2|\beta| + 1} \right) \\ \Rightarrow \beta(a_n - \alpha) &\in \left(-\frac{|\beta| \cdot \varepsilon}{2|\beta| + 1}, \frac{|\beta| \cdot \varepsilon}{2|\beta| + 1} \right) \subseteq \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2} \right). \end{aligned}$$

Thus,

$$\begin{aligned} a_n b_n - \alpha \beta &= a_n(b_n - \beta) + \beta(a_n - \alpha) \in (-\varepsilon, \varepsilon) \\ \Rightarrow a_n b_n &\in (\alpha\beta - \varepsilon, \alpha\beta + \varepsilon) = \mathcal{N}_\varepsilon(\alpha\beta) \subseteq U. \end{aligned}$$

(4) Let $\varepsilon > 0$. Note that

$$|y| - |x| \leq |y - x|$$

for any $x, y \in \mathbb{R}$. Since $\lim_{n \rightarrow \infty} b_n = \beta$, for $\frac{1}{2}|\beta| > 0$, $\exists N_1 \in \mathbb{N}$ such that if $n \geq N_1$

$$|\beta| - |b_n| \leq |\beta - b_n| = |b_n - \beta| < \frac{1}{2}|\beta|.$$

Thus, we obtain that

$$|\beta| - |b_n| < \frac{1}{2}|\beta| \implies \frac{1}{2}|\beta| < |b_n| \implies \frac{1}{b_n} < \frac{2}{|\beta|}$$

And

$$\exists N_2 \in \mathbb{N} : \left[n \geq N_2 \implies |b_n - \beta| < \frac{\beta^2}{2} \varepsilon \right].$$

Let $N = \max \{N_1, N_2\}$. If $n \geq N$ then

$$\begin{aligned} \left| \frac{1}{b_n} - \frac{1}{\beta} \right| &= \left| \frac{\beta - b_n}{\beta b_n} \right| \\ &= \frac{|b_n - \beta|}{|\beta| |b_n|} \\ &< \varepsilon \cdot \frac{\beta^2}{2} \cdot \frac{1}{|\beta|} \cdot \frac{2}{|\beta|} = \varepsilon. \end{aligned}$$

(4) Let $U \in \tau_d$ with $1/\beta \in U$. Since $1/\beta \in U$, by definition of τ_d , we have

$$\exists \varepsilon > 0 \text{ such that } \mathcal{N}_\varepsilon(1/\beta) \subseteq U.$$

Since $\lim_{n \rightarrow \infty} b_n = \beta$, for $|\beta|/2$, we know that

$$\exists N_1 \in \mathbb{N} \text{ s.t. } n \geq N_1 \implies b_n \in \left(\beta - \frac{|\beta|}{2}, \beta + \frac{|\beta|}{2} \right),$$

$$\exists N_2 \in \mathbb{N} \text{ s.t. } n \geq N_2 \implies b_n \in \left(\beta - \frac{\beta^2 \cdot \varepsilon}{2}, \beta + \frac{\beta^2 \cdot \varepsilon}{2} \right).$$

Let $N = \max \{N_1, N_2\}$. If $n \geq N$ then

$$\frac{1}{b_n} - \frac{1}{\beta} = \frac{\beta - b_n}{\beta \cdot b_n}$$

and that Thus, if $n \geq N$ then

$$\begin{aligned} k\alpha_n &\in \left(k\alpha - k \cdot \frac{\varepsilon}{|k|+1}, k\alpha + k \cdot \frac{\varepsilon}{|k|+1} \right) \\ &\subseteq (k\alpha - \varepsilon, k\alpha + \varepsilon) \\ &= \mathcal{N}_\varepsilon(k\alpha) \subseteq U. \end{aligned}$$

□

Note. TBA

References

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