Proposition 1. *Let X be a Riemann surface and f a holomorphic (resp. meromorphic) function on X. Then $u := \log|f|$ is harmonic on $X \setminus (Z(f) \cup P(f))$. If f is holomorphic and nowhere zero on all of X, then u is harmonic on X; on compact, connected X this forces f (hence u) to be constant.*

Proof. Fix a point $p \in X$ with $f(p) \neq 0$ (and p not a pole if f is meromorphic). Take a small simply connected neighborhood U of p on which f has no zeros or poles. On U the holomorphic 1-form $\frac{f'}{f}\,dz$ is exact, so there exists a holomorphic function g on U with $g' = \frac{f'}{f}$ and $e^g = f$. Write $g = \varphi + i\psi$ with real-valued φ, ψ. Then on U,

$$|f| = |e^g| = e^{\Re g} = e^{\varphi} \qquad \Longrightarrow \qquad u = \log|f| = \varphi.$$

But the real part of a holomorphic function is harmonic (equivalently, $\varphi_{xx} + \varphi_{yy} = 0$ in any local coordinate), hence u is harmonic on U. Since p was arbitrary in $X \setminus (Z(f) \cup P(f))$, u is harmonic there.

If f is holomorphic and nowhere zero on all of X, then the above holds on all of X. If, in addition, X is compact and connected, any harmonic function is constant (e.g. by the maximum principle or by integrating $|\nabla u|^2$ and using Stokes), so u is constant and therefore f has constant modulus. A holomorphic map with constant modulus is constant (by the open mapping theorem), so f is constant. □

Remark (meromorphic case, distributional Laplacian). If f is meromorphic, near a zero or pole at p of order $m \in \mathbb{Z}$ one can write $f(z) = z^m g(z)$ with g holomorphic and $g(0) \neq 0$. Then $\log|f| = m \log|z| + \log|g|$, where $\log|g|$ is harmonic and $\Delta \log|z| = 2\pi\,\delta_0$ (distributionally). Hence

$$\Delta \log|f| \;=\; 2\pi \sum_{p \in X} \mathrm{ord}_p(f)\,\delta_p$$

as distributions. This shows $\log|f|$ is subharmonic globally and harmonic away from zeros and poles.