Complex Analysis

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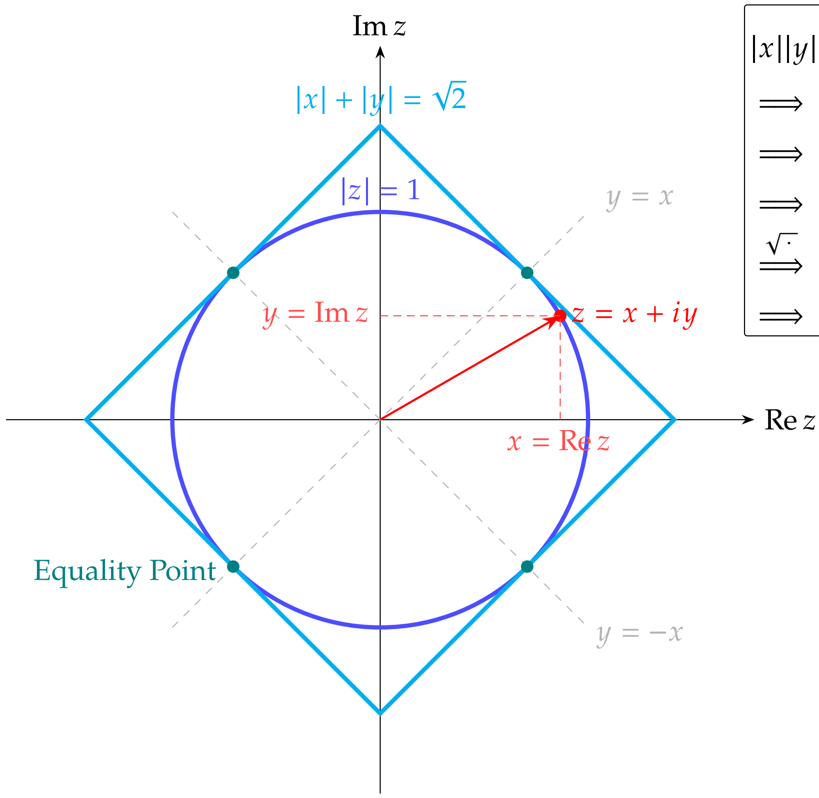
1 Complex numbers

Verify that $\sqrt{2}\,|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$.

Sol. Let $z = x + iy$, so that $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, and $|z| = \sqrt{x^2 + y^2}$. Then

$$
\begin{aligned}
\sqrt{2}\,|z| \;\geq\; |\operatorname{Re} z| + |\operatorname{Im} z| &\iff \sqrt{2}\sqrt{x^2+y^2} \;\geq\; |x| + |y| \\
&\iff 2(x^2+y^2) \;\geq\; (|x|+|y|)^2 \\
&\iff 2(x^2+y^2) \;\geq\; x^2+y^2+2|x||y| \quad (\because\ |x|^2 = x^2,\ |y|^2 = y^2) \\
&\iff x^2+y^2 \;\geq\; 2|x||y| \quad \text{by subtracting } x^2+y^2 \text{ from both sides} \\
&\iff x^2+y^2 \;\geq\; 2\sqrt{x^2y^2} \\
&\iff \frac{x^2+y^2}{2} \;\geq\; \sqrt{x^2y^2} \\
&\iff \frac{a+b}{2} \;\geq\; \sqrt{ab} \quad \text{by setting } a := x^2 \text{ and } b := y^2; \quad \text{(AM-GM inequality)}
\end{aligned}
$$

Hence it holds.



$$
\begin{aligned}
&|x||y| \le \frac{x^2 + y^2}{2} \quad (\text{AM–GM on } x^2, y^2) \\
&\implies (x^2 + y^2) \le (2|x||y|) \\
&\implies 2 \cdot (x^2 + y^2) \le (x^2 + y^2) + (2|x||y|) \\
&\implies (|x| + |y|)^2 \le 2(x^2 + y^2) \\
&\overset{\sqrt{\cdot}}{\implies} |x| + |y| \le \sqrt{2}\,|z| \\
&\implies |\operatorname{Re} z| + |\operatorname{Im} z| \le \sqrt{2}\,|z|.
\end{aligned}
$$

$\square$

By factoring $z^4 - 4z + 3$ into two quadratic factors show that if z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \le \frac{1}{3}.$$

Sol. Since $z^4 - 4z^2 + 3 = (z^2 - 1)(z^2 - 3)$, we have

$$\left|z^4 - 4z^2 + 3\right| = |z^2 - 1|\,|z^2 - 3|.$$

For $|z| = 2$ one has $|z^2| = |z|^2 = 4$. By the triangle inequality,

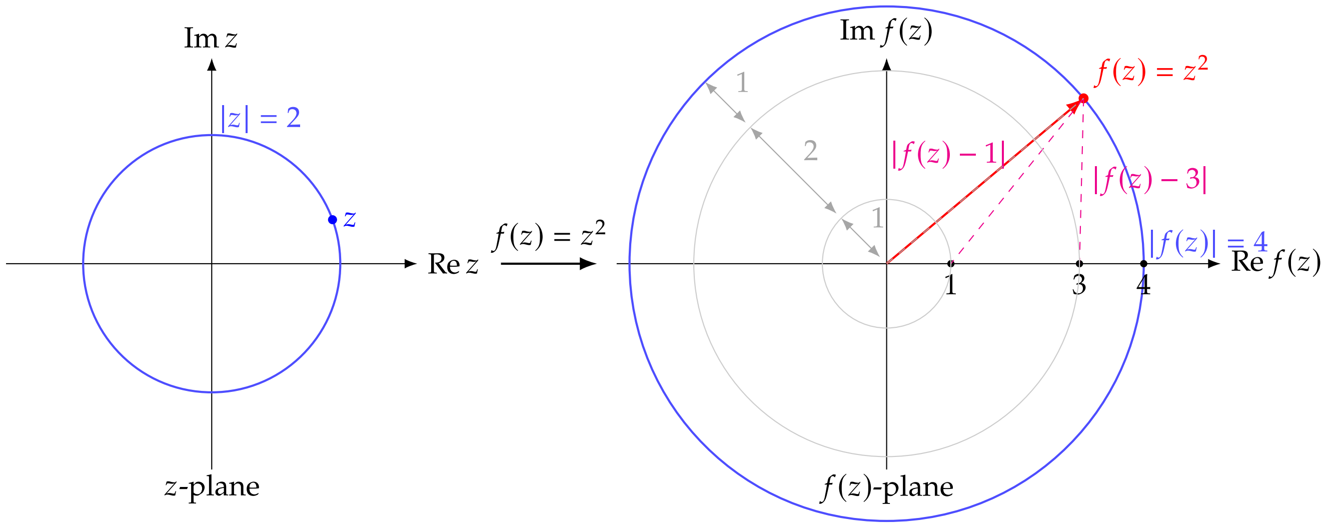
$$|z^2 - 1| \ \ge \ \big||z^2| - |1|\big| = |4 - 1| = 3, \qquad |z^2 - 3| \ \ge \ \big||z^2| - |3|\big| = |4 - 3| = 1.$$

Hence

$$|z^4 - 4z^2 + 3| \ \ge \ 3 \cdot 1 \ = \ 3,$$

and therefore

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \ = \ \frac{1}{|z^4 - 4z^2 + 3|} \ \le \ \frac{1}{3}.$$



□

Prove that the usual formula solves the quadratic equation

$$az^2 + bz + c = 0 \quad (a \neq 0)$$

when the coefficient a,b, and c are complex numbers. Specifically, by completing the square on the left-hand side, derive the **quadratic formula**

$$z = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

where both square roots are to be considered when $b^2 - 4ac \neq 0$. Use this result to find the roots of the equation

$$z^2 + 2z + (1 - i) = 0.$$

Sol. Since

$$az^2 + bz + c = a\left(z^2 + \frac{b}{a}z\right) + c = a\left(z + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c = a\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c,$$

we have

$$a\left(z + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c \quad \Longleftrightarrow \quad \left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking square roots of both sides yields

$$z + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a}, \quad \text{whence} \quad z = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

Consider $z^2 + 2z + (1-i)$ with $a = 1$, $b = 2$, and $c = 1 - i$. The discriminant is

$$\Delta = b^2 - 4ac = 4 - 4(1-i) = 4i.$$

Since

$$\sqrt{i} = \frac{1+i}{\sqrt{2}} \quad \left(\text{indeed, } \left(\frac{1+i}{\sqrt{2}}\right)^2 = \frac{1+2i-1}{2} = i\right),$$

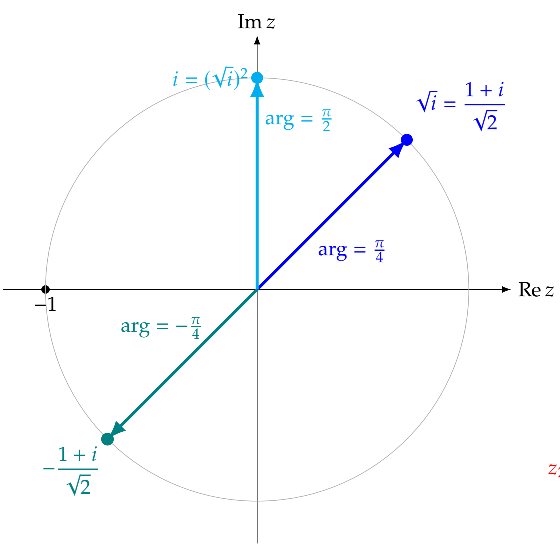
we may take $\sqrt{\Delta} = \sqrt{4i} = 2\sqrt{i} = \sqrt{2}\,(1+i)$. Therefore

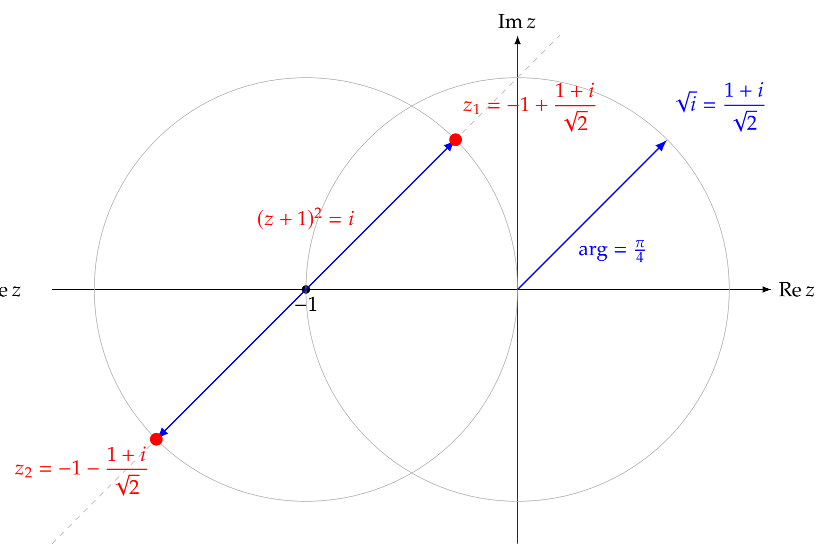
$$z = \frac{-2 \pm \sqrt{4i}}{2} = -1 \pm \sqrt{i} = -1 \pm \frac{1+i}{\sqrt{2}}.$$

Thus the roots are

$$z_1 = -1 + \frac{1+i}{\sqrt{2}}, \qquad z_2 = -1 - \frac{1+i}{\sqrt{2}}.$$

Note that z_1, z_2 are roots of $(z+1)^2 = i$.





$\square$

2 Analytic functions

Show that the following limit does not exitst

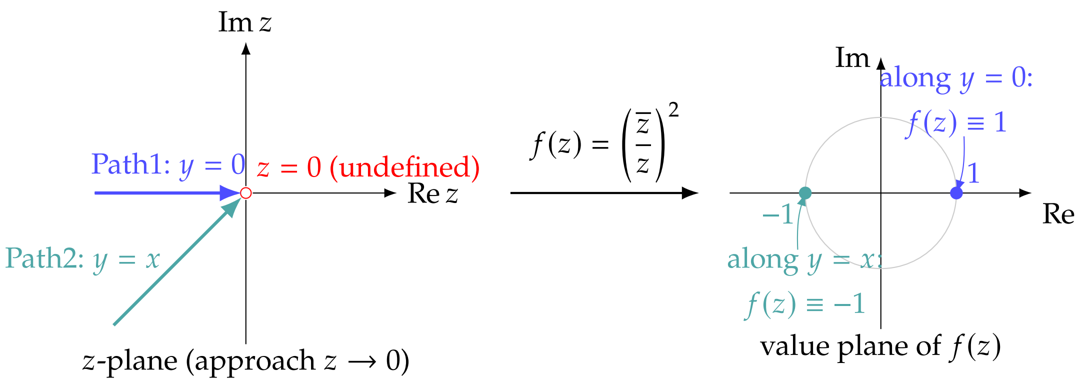
$$\lim_{z\to 0}\left(\frac{\bar z}{z}\right)^2$$

Do this by letting nonzero points $z = (x, 0)$ and $z = (x, x)$ approach the origin. (Note that it is not sufficient to simply consider points $z = (x, 0)$ and $z = (0, y)$.)

Sol. Let $z = x + iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$. Then

$$\left(\frac{\bar z}{z}\right)^2 = \left(\frac{x - iy}{x + iy}\right)^2.$$

If $z = re^{i\theta}$ with $r > 0$, then $\bar z/z = e^{-2i\theta}$, so $|(\bar z/z)^2| = |e^{-4i\theta}| = 1$.



(1) Path 1: approach along the real axis $y = 0$

Let $z = x + 0i = x$ with $x \in \mathbb{R} \setminus \{0\}$ and $x \to 0$. Then $\left(\frac{\bar z}{z}\right)^2 = \left(\frac{x}{x}\right)^2 = 1$.

(2) Path 2: approach along the diagonal $y = x$

Let $z = x + ix = (1 + i)x$ with $x \in \mathbb{R} \setminus \{0\}$ and $x \to 0$. Then

$$\frac{\bar z}{z} = \frac{\overline{(1+i)x}}{(1+i)x} = \frac{(1-i)\,x}{(1+i)\,x} = \frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{1-2i+i^2}{1-i^2} = \frac{1-2i-1}{2} = \frac{-2i}{2} = -i.$$

Hence

$$\left(\frac{\bar z}{z}\right)^2 = (-i)^2 = -1.$$

(3) Conclusion

Since the limits along these two paths are different (namely 1 and −1), the limit cannot exist. □

Let

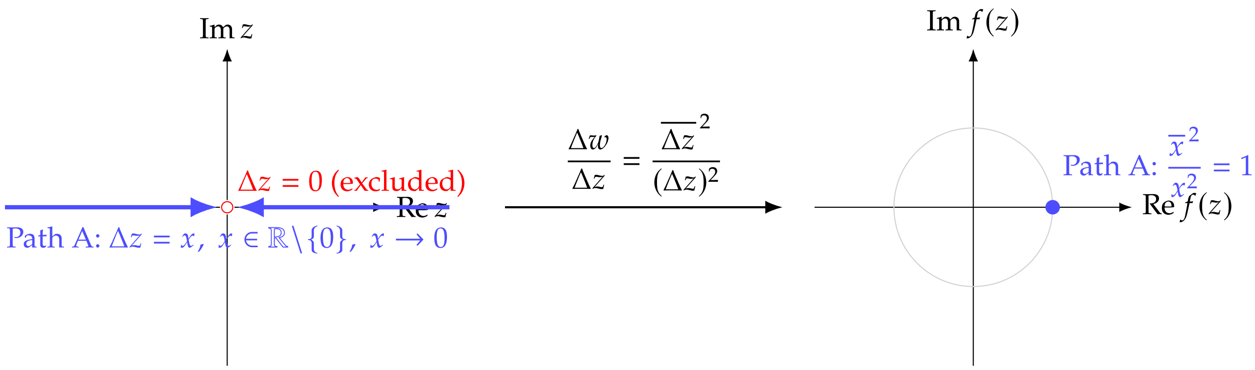
$$f(z) = \begin{cases} \bar{z}^2/z, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Show that if $z = 0$, then $\Delta w/\Delta z = 1$ at eatch nonzero point on the real and imaginary axes in the Δz, or $\Delta x \Delta y$, plane. Then show that $\Delta w/\Delta z = -1$ at each nonzero point $(\Delta x, \Delta y)$ on the line $\Delta y = \Delta x$ in that plane. Conclude from these observations that $f'(0)$ does not exist. Note that to obtain this result, it is not sufficient to consider only horizontal and vertical approaches to the origin in the Δz plane.

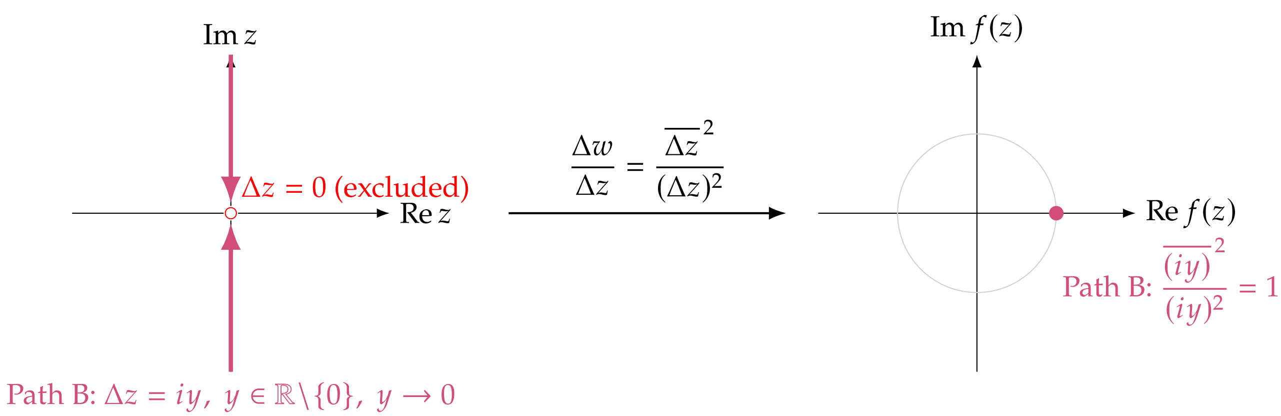
Proof. Let $\frac{\Delta w}{\Delta z} = \frac{f(\Delta z) - f(0)}{\Delta z}$ $(\Delta z \neq 0)$. Since $f(0) = 0$, for $\Delta z \neq 0$, $\dfrac{\Delta w}{\Delta z} = \dfrac{f(\Delta z)}{\Delta z} = \dfrac{\overline{\Delta z}^2}{(\Delta z)^2}$.

(1) Real and imaginary axes.

- Real axis: $\Delta z = x$ with $x \in \mathbb{R} \setminus \{0\}$, $\dfrac{\Delta w}{\Delta z} = \dfrac{\overline{x}^2}{x^2} = \dfrac{x^2}{x^2} = 1$.

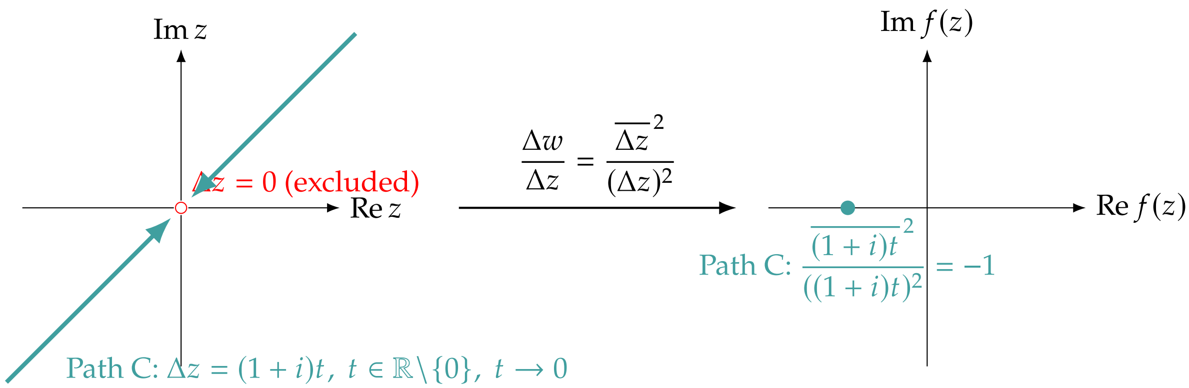


- Imaginary axis: $\Delta z = iy$ with $y \in \mathbb{R} \setminus \{0\}$, $\dfrac{\Delta w}{\Delta z} = \dfrac{\overline{iy}^2}{(iy)^2} = \dfrac{(-iy)^2}{(iy)^2} = \dfrac{-y^2}{-y^2} = 1$.



(2) Line $\Delta y = \Delta x$. Let $\Delta z = (1+i)x$ with $x \in \mathbb{R} \setminus \{0\}$. Then

$$\frac{\Delta w}{\Delta z} = \frac{\overline{(1+i)x}^2}{((1+i)x)^2} = \frac{((1-i)x)^2}{((1+i)x)^2} = \frac{(1-i)^2}{(1+i)^2} = \frac{-2i}{2i} = -1.$$



(3) Conclusion. Since the difference quotient equals 1 along the axes but -1 along the line $\Delta y = \Delta x$, the limit

$$\lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z}$$

depends on the path and therefore does not exist. Consequently, $f'(0)$ does not exist. □

Let

$$f(z) = \bar{z}, \quad f(z) = 2x + ixy^2, \quad f(z) = e^{\bar{z}}$$

Then show that $f'(z)$ does not exists at any point.

Sol. Let $z = x + iy$ and $f(z) = u + iv$ with $x, y, u, v \in \mathbb{R}$. The Cauchy–Riemann equations

$$u_x = v_y, \qquad u_y = -v_x$$

are necessary and sufficient for complex differentiability.

(1) $f_1(z) = \overline{z} = \overline{x+iy} = x - iy$.

Here $u(x,y) = x,\ v(x,y) = -y$. Thus

$$u_x = 1, \quad u_y = 0, \quad v_x = 0, \quad v_y = -1.$$

The CR require $u_x = v_y$, i.e. $1 = -1$, which is impossible. Hence f_1' does not exist anywhere.

(2) $f_2(z) = 2x + i\,xy^2$**.**

Here $u(x,y) = 2x,\ v(x,y) = xy^2$. Thus

$$u_x = 2, \quad u_y = 0, \quad v_x = y^2, \quad v_y = 2xy.$$

The CR demand

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \implies \begin{cases} 2 = 2xy \\ 0 = y^2 \end{cases} \implies \begin{cases} xy = 1 \\ y = 0 \end{cases}.$$

These cannot hold simultaneously for any x. Hence CR fail at every point, so f_2' exists nowhere.

(3) $f_3(z) = e^{\overline{z}}$**.**

Let $\overline{z} = x - iy$. Then $f_3(x,y) = e^{x-iy} = e^x(\cos y - i\sin y)$, so

$$u(x,y) = e^x \cos y, \qquad v(x,y) = -\,e^x \sin y.$$

Compute

$$u_x = e^x \cos y, \quad u_y = -e^x \sin y, \quad v_x = -e^x \sin y, \quad v_y = -e^x \cos y.$$

The CR give

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \implies \begin{cases} e^x \cos y = -e^x \cos y \\ -e^x \sin y = +e^x \sin y \end{cases} \implies \begin{cases} \cos y = 0 \\ \sin y = 0 \end{cases}.$$

These cannot hold simultaneously for any y. Hence CR fail everywhere and f_3' exists nowhere. □

Show that the function

$$f(z) = \ln r + i\theta \quad (r > 0,\ 0 < \theta < 2\pi)$$

is analytic in the indicated domain of definition, with derivative $f'(z) = 1/z$. Then show that th e composite function $g(z) = f(z^2+1)$ is analytic in the quadratic $x > 0, y > 0$ with derivative

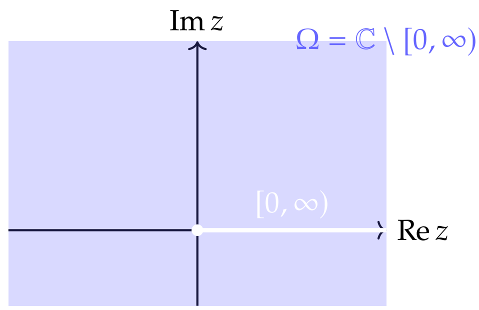
$$g'(z) = \frac{2z}{z^2+1}.$$

(Suggestion: Observe that $\mathrm{Im}(z^2+1) > 0$ when $x > 0, y > 0$)

Sol. Let $z = x + iy = re^{i\theta}$ with $r = \sqrt{x^2+y^2} > 0$ and $0 < \theta < 2\pi$. Define

$$f(z) = \ln r + i\theta.$$

Then f is analytic on the slit plane



$\Omega := \{ z \in \mathbb{C} : r > 0,\ 0 < \theta < 2\pi \} = \mathbb{C} \setminus [0, \infty)$,

and $f'(z) = \frac{1}{z}$ $(z \in \Omega)$.

Write $f = u + iv$ with

$$u(x, y) = \ln r = \frac{1}{2} \ln(x^2 + y^2), \qquad v(x, y) = \theta = \operatorname{Arg}(z) \in (0, 2\pi).$$

On Ω the functions u, v are C^1 and their partials are:

$$u_x = \frac{x}{x^2 + y^2}, \quad u_y = \frac{y}{x^2 + y^2}, \qquad v_x = -\frac{y}{x^2 + y^2}, \quad v_y = \frac{x}{x^2 + y^2}.$$

Hence the Cauchy–Riemann equations hold on Ω:

$$u_x = v_y = \frac{x}{x^2 + y^2}, \qquad u_y = -v_x = \frac{y}{x^2 + y^2}.$$

Since these partials are continuous on Ω, f is analytic there. Its complex derivative is

$$f'(z) = u_x + iv_x = \frac{x}{x^2 + y^2} + i\left(-\frac{y}{x^2 + y^2}\right) = \frac{x - iy}{x^2 + y^2} = \frac{1}{x + iy} = \frac{1}{z}.$$

For $g(z) = f(z^2 + 1)$, compute

$$\begin{aligned} z^2 + 1 &= (x + iy)^2 + 1 \\ &= (x^2 - y^2 + 2ixy) + 1 \\ &= (x^2 - y^2 + 1) + i(2xy). \end{aligned}$$

If $x > 0$ and $y > 0$, then $\operatorname{Im}(z^2 + 1) = 2xy > 0$, so $z^2 + 1$ lies in the open upper half-plane $\mathbb{H}$, in particular in Ω (its argument lies in $(0, \pi) \subset (0, 2\pi)$). Thus g is the composition of analytic functions on the first quadrant Q_1, hence analytic on Q_1. By the chain rule,

$$g'(z) = f'(z^2 + 1) \cdot (2z) = \frac{2z}{z^2 + 1} \qquad (z \in Q_1).$$

□

3 Elementary functions

Show that $f(z) = \exp(\overline{z})$ is not analytic anywhere. *(Hint: use the Cauchy–Riemann equations.)*

Sol. (**Proof via Cauchy–Riemann equations**) Write $z = x + iy$. Then

$$f(z) = e^{\overline{z}} = e^{x-iy} = e^x\big(\cos y - i \sin y\big),$$

so

$$u(x,y) = e^x \cos y, \qquad v(x,y) = -e^x \sin y.$$

Then

$$u_x = e^x \cos y, \quad u_y = -e^x \sin y, \qquad v_x = -e^x \sin y, \quad v_y = -e^x \cos y.$$

If f is complex differentiable at (x, y), the Cauchy–Riemann equations would hold:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

That is,

$$\begin{aligned} u_x = v_y &\implies e^x \cos y = -e^x \cos y && \implies \cos y = 0, \\ u_y = -v_x &\implies -e^x \sin y = e^x \sin y && \implies \sin y = 0. \end{aligned}$$

There is no $y \in \mathbb{R}$ with $\cos y = 0$ and $\sin y = 0$ simultaneously. Hence the Cauchy–Riemann equations fail at every point, so f is nowhere analytic.

(**Proof via Wirtinger derivatives**) Using $\partial/\partial z = \frac{1}{2}(\partial_x - i\,\partial_y)$ and $\partial/\partial\overline{z} = \frac{1}{2}(\partial_x + i\,\partial_y)$, one checks directly that

$$\frac{\partial f}{\partial z} = 0, \qquad \frac{\partial f}{\partial \overline{z}} = e^{\overline{z}} \neq 0 \ \text{ for all } z.$$

A function is holomorphic iff $\partial f/\partial\overline{z} \equiv 0$ on its domain. Since this is not the case, f is nowhere holomorphic. □

Show that $f(z) = \operatorname{Log}(z - i)$ is analytic except on portion $x \le 0$ of the line $y = 1$ and that the function

$$f(z) = \frac{\operatorname{Log}(z + 4)}{z^2 + i}$$

is analytic everywhere except at the points $\pm(1 - i)/\sqrt{2}$ and on the portion $x \le -4$ of the real axis.

Sol. Consider $\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z$, the principal branch of the complex logarithm, with $\operatorname{Arg} z \in (-\pi, \pi)$, so that Log is analytic on

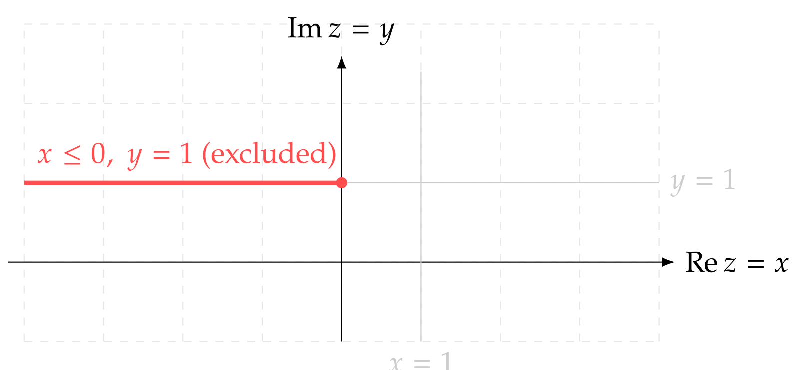
$$\begin{aligned}\mathbb{C}\setminus(-\infty,0] &= \mathbb{C}\setminus\{z\in\mathbb{C} : \operatorname{Re} z \le 0 \wedge \operatorname{Im} z = 0\}\\ &= \{\, z\in\mathbb{C} : \operatorname{Re} z > 0 \vee \operatorname{Im} z \ne 0\,\}.\end{aligned}$$

Then

(1) Since Log is analytic on $\mathbb{C}\setminus(-\infty,0]$ and the map $z\mapsto z-i$ is entire, the composition $z\mapsto \operatorname{Log}(z-i)$ is analytic precisely where $z-i\notin(-\infty,0]$. Equivalently,

$$\begin{aligned} z-i\in(-\infty,0] &\iff \operatorname{Re}(z-i)\le 0 \text{ and } \operatorname{Im}(z-i)=0\\ &\iff \operatorname{Re}(x+i(y-1)) = x\le 0 \text{ and } \operatorname{Im}(x+i(y-1)) = y-1=0.\end{aligned}$$

That is, $f(z) := \operatorname{Log}(z-i)$ is analytic on $\mathbb{C}\setminus\{\, x+iy : \ x\le 0, \ y=1\}$.



$f(z) = \operatorname{Log}(z-i)$

Analytic domain: $\mathbb{C}\setminus\{(x,y)\mid y=1, \ x\le 0\}$.

(2) The numerator $z\mapsto\operatorname{Log}(z+4)$ is analytic wherever $z+4\notin(-\infty,0]$, i.e., $z\notin(-\infty,-4]$. In other words, $\operatorname{Log}(z+4)$ is analytic on

$$\mathbb{C}\setminus(-\infty,-4] = \mathbb{C}\setminus\{z\in\mathbb{C} : \operatorname{Re} z\le -4 \wedge \operatorname{Im} z = 0\} = \{z\in\mathbb{C} : \operatorname{Re} z > -4 \vee \operatorname{Im} z\ne 0\}$$

$\mathbb{C}\setminus(-\infty]$ for $z\notin(-\infty,-4]$, which is the portion $x\le -4$ of the real axis. The denominator z^2+i vanishes exactly at the zeros of $z^2=-i$, namely

$$z = \pm(-i)^{1/2} = \pm e^{-i\pi/4} = \pm\frac{1-i}{\sqrt{2}}.$$

Therefore g is analytic on the domain where the numerator is analytic and the denominator is nonzero, i.e.

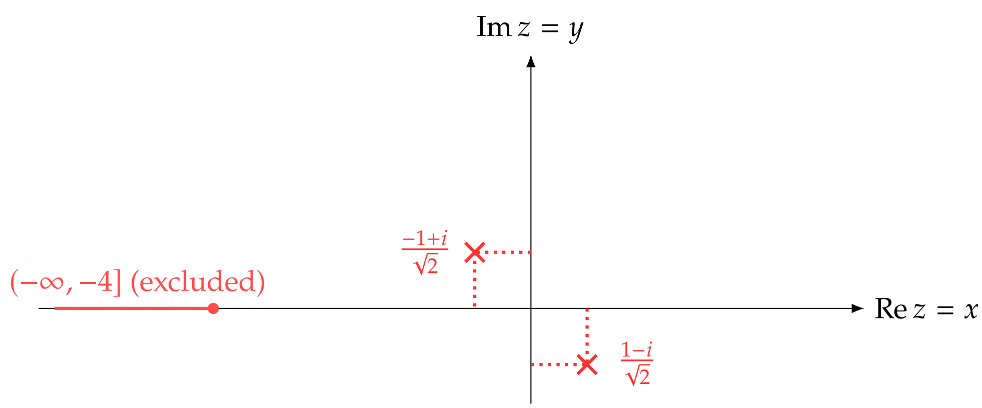
$$\mathbb{C} \setminus \Big((-\infty, -4] \cup \{\pm \tfrac{1-i}{\sqrt{2}}\} \Big),$$

which is exactly the stated set.

$g(z) := \dfrac{\mathrm{Log}(z+4)}{z^2+i}$ is analytic on

$$\mathbb{C} \setminus \Big(\{ x+iy : \ y = 0, \ x \le -4 \} \cup \{\pm \tfrac{1-i}{\sqrt{2}}\} \Big),$$

i.e. everywhere except at the branch cut $x \le -4$ on the real axis and at the two points $\pm(1-i)/\sqrt{2}$.



□

> Show that the function $\ln(x^2+y^2)$ is harmonic in every domain that does not contain the origin.

Sol. For $(x, y) \neq (0, 0)$, we can differentiate:

$$u_x = \frac{\partial}{\partial x} \ln(x^2+y^2) = \frac{2x}{x^2+y^2},$$
$$u_y = \frac{\partial}{\partial y} \ln(x^2+y^2) = \frac{2y}{x^2+y^2}.$$

And then

$$u_{xx} = \frac{2(x^2+y^2) - 2x \cdot 2x}{(x^2+y^2)^2} = \frac{2(x^2+y^2) - 4x^2}{(x^2+y^2)^2} = \frac{-2x^2+2y^2}{(x^2+y^2)^2}$$
$$u_{yy} = \frac{2(x^2+y^2) - 2y \cdot 2y}{(x^2+y^2)^2} = \frac{2(x^2+y^2) - 4y^2}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}.$$

Now compute the Laplacian:

$$u_{xx} + u_{yy} = \frac{-2x^2 + 2y^2}{(x^2 + y^2)^2} + \frac{2x^2 - 2y^2}{(x^2 + y^2)^2} = \frac{(-2x^2 + 2y^2) + (2x^2 - 2y^2)}{(x^2 + y^2)^2} = \frac{0}{(x^2 + y^2)^2} = 0$$

for all $(x, y) \neq (0, 0)$.

(**Proof via Wirtinger-operator**) Let $z = x + iy$ and

$$u(x, y) = \ln(x^2 + y^2) = \ln(|z|^2) = \ln(z\bar{z}).$$

Recall the Wirtinger operators $\partial := \frac{1}{2}(\partial_x - i\,\partial_y)$ and $\bar{\partial} := \frac{1}{2}(\partial_x + i\,\partial_y)$, so that the Laplacian satisfies

$$\Delta = \partial_{xx} + \partial_{yy} = 4\,\partial\bar{\partial} = 4\,\bar{\partial}\partial.$$

On $\mathbb{C} \setminus \{0\}$ the chain rule gives

$$\begin{aligned} \partial u &= \partial\big(\ln(z\bar{z})\big) = \frac{1}{z\bar{z}}\,\partial(z\bar{z}) = \frac{1}{z\bar{z}}\,\bar{z} = \frac{1}{z}, \\ \bar{\partial} u &= \bar{\partial}\big(\ln(z\bar{z})\big) = \frac{1}{z\bar{z}}\,\bar{\partial}(z\bar{z}) = \frac{1}{z\bar{z}}\,z = \frac{1}{\bar{z}}. \end{aligned}$$

Therefore, $\Delta u = 4\,\partial\bar{\partial} u = 4\,\partial\left(\frac{1}{\bar{z}}\right) = 0$ on $\mathbb{C} \setminus \{0\}$. □

Show that $\cosh^2 z - \sinh^2 z = 1$ and $\sinh z + \cosh z = e^z$.

Sol. Recall the exponential definitions (valid for all $z \in \mathbb{C}$):

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}.$$

(1) $\cosh^2 z - \sinh^2 z = 1$**.**

$$\cosh^2 z - \sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2 = \frac{(e^z + e^{-z})^2 - (e^z - e^{-z})^2}{4}.$$

Expanding,

$$(e^z + e^{-z})^2 - (e^z - e^{-z})^2 = e^{2z} + 2 + e^{-2z} - (e^{2z} - 2 + e^{-2z}) = 4,$$

so $\cosh^2 z - \sinh^2 z = \frac{4}{4} = 1$.

(2) $\sinh z + \cosh z = e^z$**.**

$$\sinh z + \cosh z = \frac{e^z - e^{-z}}{2} + \frac{e^z + e^{-z}}{2} = e^z.$$

□

4 Integrals

Let C_0 be the positively oriented circle $|z - z_0| = R$. Show that

$$\int_{C_0} (z - z_0)^{n-1}\,\mathrm{d}z = \begin{cases} 0, & n = \pm 1, \pm 2, \dots \\ 2\pi i, & n = 0. \end{cases}$$

Sol. Parametrize C_0 by $z(t) = z_0 + Re^{it}$ ($t \in [0, 2\pi]$) then $dz = iRe^{it}\,\mathrm{d}t$ and

$$\int_{C_0} (z - z_0)^{n-1}\,\mathrm{d}z = \int_0^{2\pi} (Re^{it})^{n-1}\, iRe^{it}\,\mathrm{d}t = iR^n \int_0^{2\pi} e^{int}\,\mathrm{d}t.$$

(1) If $n \neq 0$, then

$$\int_0^{2\pi} e^{int}\,\mathrm{d}t = \left[\frac{1}{in} e^{int}\right]_0^{2\pi} = \frac{e^{in2\pi} - 1}{in} = \frac{1 - 1}{in} = 0, \qquad \text{so the integral is } 0.$$

(2) If $n = 0$, then $e^{int} \equiv 1$ and the integral equals $iR^0 \int_0^{2\pi} 1\,\mathrm{d}t = 2\pi i$.

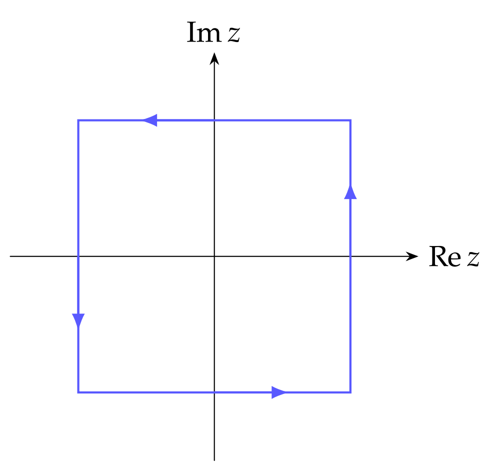
□

Let C be the boundary of the square with sides $x = \pm 2$, $y = \pm 2$, oriented positively. Show that

$$\int_C \frac{\cos z}{z(z^2 + 8)}\,dz = \frac{i\pi}{4}, \qquad \int_C \frac{\cosh z}{z^4}\,dz = 0, \qquad \int_C \frac{\tan(z/2)}{(z - x_0)^2}\,dz = i\pi \sec^2\!\left(\frac{x_0}{2}\right),$$

where $-2 < x_0 < 2$.

Sol. Let C be the positively oriented boundary of the square $\{x + iy : |x| \le 2, |y| \le 2\}$.



(1) $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}\,dz$. The integrand is meromorphic with simple poles at $z = 0$ and $z = \pm 2\sqrt{2}\,i$. Only $z = 0$ is inside C. Around $z = 0$,

$$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \cdots, \qquad \frac{1}{z(z^2+8)} = \frac{1}{z^3+8z} = \frac{1}{8z}\,\frac{1}{1+z^2/8} = \frac{1}{8z}\left(1 - \frac{z^2}{8} + \frac{z^4}{8^2} - \cdots\right).$$

Thus,

$$\operatorname*{Res}_{z=0} \frac{\cos z}{z(z^2+8)} = \operatorname*{Res}_{z=0} \left(\frac{1}{8z}\left(1 - \frac{z^2}{8} + \frac{z^4}{8^2} - \cdots\right)\left(1 - \frac{z^2}{2} + \frac{z^4}{4!} - \cdots\right)\right) = \frac{1}{8}.$$

By the residue theorem,

$$\int_C \frac{\cos z}{z(z^2+8)}\,dz = 2\pi i \cdot \frac{1}{8} = \frac{i\pi}{4}.$$

(2) $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz$. Here the only singularity is at $z = 0$ (order 4). Using

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots, \qquad \frac{\cosh z}{z^4} = \frac{1}{z^4} + \frac{1}{2}\frac{1}{z^2} + \frac{1}{4!} + \cdots,$$

there is no $1/z$ term; hence $\operatorname{Res}_{z=0}(\cosh z/z^4) = 0$, and therefore

$$\int_C \frac{\cosh z}{z^4}\,dz = 0.$$

(3) $\displaystyle\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz$ with $-2 < x_0 < 2$. We know that

$$\tan w = \frac{\sin w}{\cos w},$$

so the poles of $\tan w$ occur exactly where $\cos w = 0$ and $\sin w \neq 0$. The zeros of $\cos w$ are

$$w = \frac{\pi}{2} + k\pi = \frac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z}.$$

Now consider $\tan\left(\frac{z}{2}\right)$. Let $w = \frac{z}{2}$. The poles of $\tan(z/2)$ occur where w is a pole of $\tan w$, i.e. where

$$\frac{z}{2} = \frac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z}.$$

So the poles of $\tan(z/2)$ are precisely at $z = (2k+1)\pi$ with $k \in \mathbb{Z}$. Since

$$|(2k+1)\pi| \geq \pi > 2,$$

none of these poles lie inside or on C. Hence $\tan(z/2)$ is analytic on and inside C. The only singularity of the integrand $\tan(z/2)/(z-x_0)^2$ inside C is at $z = x_0$. Since $\tan(z/2)$ is analytic at $z = x_0$, we may expand it in a Taylor series about x_0:

$$\begin{aligned}\tan\left(\frac{z}{2}\right) &= \tan\left(\frac{x_0}{2}\right) + \frac{\mathrm{d}}{\mathrm{d}z}\tan\left(\frac{z}{2}\right)\bigg|_{z=x_0}(z-x_0)+\cdots \\ &= \tan\left(\frac{x_0}{2}\right) + \frac{1}{2}\sec^2\left(\frac{x_0}{2}\right)(z-x_0)+\cdots.\end{aligned}$$

Dividing by $(z-x_0)^2$ gives the Laurent series

$$\frac{\tan(z/2)}{(z-x_0)^2} = \frac{\tan(x_0/2)}{(z-x_0)^2} + \frac{1}{2}\sec^2\left(\frac{x_0}{2}\right)\frac{1}{z-x_0}+\cdots.$$

Thus, we obtain

$$\mathrm{Res}_{z=x_0}\left(\frac{\tan(z/2)}{(z-x_0)^2}\right) = \frac{1}{2}\sec^2\left(\frac{x_0}{2}\right).$$

By the residue theorem,

$$\int_C \frac{\tan(z/2)}{(z-x_0)^2}\,dz = 2\pi i\cdot\frac{1}{2}\sec^2\left(\frac{x_0}{2}\right) = i\pi\sec^2\left(\frac{x_0}{2}\right).$$

□

5 Series

Show that the limit of a convergent complex sequence is unique by appealing to the corresponding result for a sequence of real numbers.

Sol. We want to show that

"If a complex sequence $\{z_n\}$ converges to both L and M in $\mathbb{C}$, then $L = M$."

Write $z_n = x_n + iy_n$, $L = a + ib$, $M = c + id$ with $x_n, y_n, a, b, c, d \in \mathbb{R}$. Assume that

$$z_n \longrightarrow L \quad \text{and} \quad z_n \longrightarrow M$$

as $n \longrightarrow \infty$. Taking real and imaginary parts,

$$x_n = \operatorname{Re} z_n \longrightarrow \operatorname{Re} L = a \quad \text{and} \quad x_n = \operatorname{Re} z_n \longrightarrow \operatorname{Re} M = c,$$

$$y_n = \operatorname{Im} z_n \longrightarrow \operatorname{Im} L = b \quad \text{and} \quad y_n = \operatorname{Im} z_n \longrightarrow \operatorname{Im} M = d.$$

By the *uniqueness of limits for real sequences*, these imply $a = c$ and $b = d$. Hence

$$L = a + ib = c + id = M.$$

□

Show that

$$\sum_{n=1}^{\infty} z_n = S \implies \sum_{n=1}^{\infty} \overline{z_n} = \overline{S}.$$

Sol. Let $s_N := \sum_{n=1}^{N} z_n$ be the partial sums. By hypothesis $s_N \longrightarrow S$ as $N \longrightarrow \infty$. Consider the conjugated partial sums

$$\overline{s_N} = \overline{\sum_{n=1}^{N} z_n} = \sum_{n=1}^{N} \overline{z_n},$$

so $\{\overline{s_N}\}$ are the partial sums of $\sum_{n=1}^{\infty} \overline{z_n}$. Since complex conjugation is continuous (indeed, an isometry: $|\overline{w} - \overline{z}| = |w - z|$), we have $\overline{s_N} \longrightarrow \overline{S}$. Therefore the series $\sum_{n=1}^{\infty} \overline{z_n}$ converges and

$$\sum_{n=1}^{\infty} \overline{z_n} = \lim_{N\to\infty} \sum_{n=1}^{N} \overline{z_n} = \lim_{N\to\infty} \overline{s_N} = \overline{S}.$$

□

Derive the Taylor series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}, \qquad |z-i| < \sqrt{2}.$$

Sol. Note that

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1-\left(\frac{z-i}{1-i}\right)}.$$

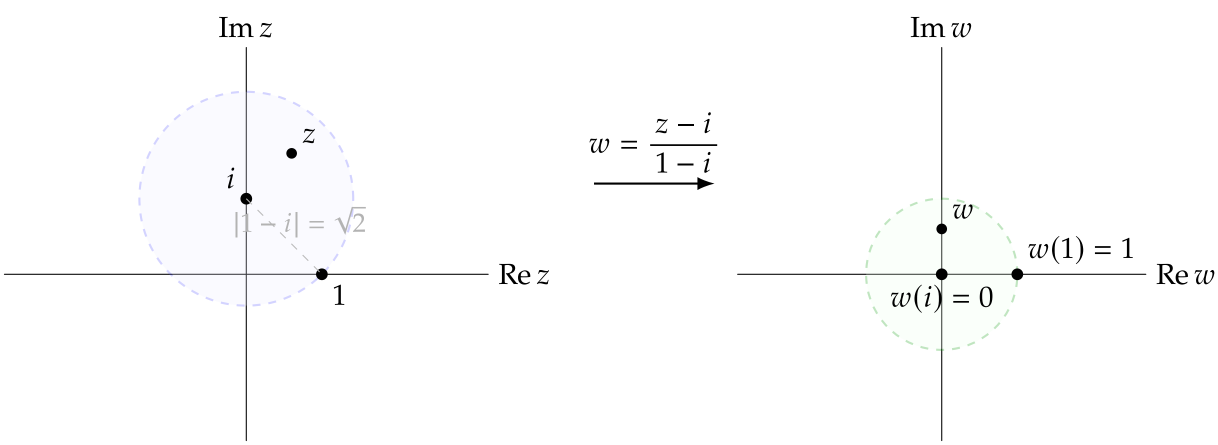
For $\left|\frac{z-i}{1-i}\right| < 1$ (i.e. $|z-i| < |1-i| = \sqrt{2}$), expand the geometric series:

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n \quad (|w| < 1), \qquad w = \frac{z-i}{1-i}.$$

Hence

$$\frac{1}{1-z} = \frac{1}{1-i} \sum_{n=0}^{\infty} \left(\frac{z-i}{1-i}\right)^n = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}},$$

which converges for $|z-i| < \sqrt{2}$.



z-plane: expansion about $z = i$ $\qquad$ w-plane: geometric series region

□

Show that the two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

are

$$\sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1} + \frac{1}{z} \quad (0 < |z| < 1), \qquad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}} \quad (1 < |z| < \infty).$$

Sol. (1) $(0 < |z| < 1)$ Since

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n} \qquad (|z| < 1),$$

we have

$$\begin{aligned}\frac{1}{z(1+z^2)} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n z^{2n} &= \sum_{n=0}^{\infty}(-1)^n z^{2n-1} \\ &= \frac{1}{z} + (-z) + z^3 + (-z^5) + z^7 + \cdots \\ &= \frac{1}{z} + \sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1}.\end{aligned}$$

Therefore the Laurent series on $0 < |z| < 1$ is $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1} + \frac{1}{z}$.

(2) $(1 < |z| < \infty)$ Since

$$\frac{1}{1+z^2} = \frac{1}{z^2}\frac{1}{1+z^{-2}} = \frac{1}{z^2}\frac{1}{1-(-z^{-2})} = \frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n z^{-2n} \qquad (|z| > 1),$$

we obtain

$$\begin{aligned}\frac{1}{z(1+z^2)} = \frac{1}{z}\cdot\frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n z^{-2n} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+3}} \\ &= \frac{1}{z^3} + \frac{-1}{z^5} + \frac{1}{z^7} + \frac{-1}{z^9} + \cdots \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}},\end{aligned}$$

Hence the Laurent series is $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}$ on $1 < |z| < \infty$.

□

Let $a \in \mathbb{R}$, where $-1 < a < 1$. Then the Laurent series representation $a/(z-a)$ is

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n}, \qquad |a| < |z| < \infty.$$

After writing $z = e^{i\theta}$ in the above equation, equate real parts and then imaginary parts on each side of the result to derive the summation formulas:

$$\sum_{n=1}^{\infty} a^n \cos(n\theta) = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin(n\theta) = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2}.$$

Sol. For $|a| < |z|$, we know that

$$\frac{a}{z-a} = \frac{a}{z}\frac{1}{1-a/z} = \frac{a}{z}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n = \sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^{n+1} = \sum_{n=1}^{\infty}\frac{a^n}{z^n}.$$

Set $z = e^{i\theta}$ (so $|a| < |z| = 1$). Then

$$\frac{a}{e^{i\theta}-a} = \sum_{n=1}^{\infty} a^n e^{-in\theta} = \sum_{n=1}^{\infty} a^n \big(\cos(n\theta) - i\sin(n\theta)\big).$$

Note that

$$\begin{aligned}
\frac{a}{e^{i\theta}-a} = \frac{e^{-i\theta}}{e^{-i\theta}}\cdot\frac{a}{e^{i\theta}-a} = \frac{ae^{-i\theta}}{1-ae^{-i\theta}} = \frac{ae^{-i\theta}(1-ae^{i\theta})}{(1-ae^{i\theta})(1-ae^{-i\theta})} &= \frac{ae^{-i\theta}(1-ae^{i\theta})}{1-a(e^{i\theta}+e^{-i\theta})+a^2e^{i\theta-i\theta}} \\
&= \frac{a\,(e^{-i\theta}-a)}{1-2a\cos\theta+a^2} \\
&= \frac{a\,(\cos\theta - i\sin\theta - a)}{1-2a\cos\theta+a^2} \\
&= \frac{a(\cos\theta - a) - i\,a\sin\theta}{1-2a\cos\theta+a^2}.
\end{aligned}$$

Thus, we obtain

$$\sum_{n=1}^{\infty} a^n \big(\cos(n\theta) - i\sin(n\theta)\big) = \frac{a}{e^{i\theta}-a} = \frac{a(\cos\theta - a) - i\,a\sin\theta}{1-2a\cos\theta+a^2}.$$

Therefore

$$\sum_{n=1}^{\infty} a^n \cos(n\theta) = \frac{a\cos\theta - a^2}{1-2a\cos\theta+a^2}, \qquad \sum_{n=1}^{\infty} a^n \sin(n\theta) = \frac{a\sin\theta}{1-2a\cos\theta+a^2},$$

valid for $-1 < a < 1$ (indeed $1-2a\cos\theta+a^2 = (1-ae^{i\theta})\overline{(1-ae^{i\theta})} = |1-ae^{i\theta}|^2 > 0$). □

With the aid of series, show that the function f defined by means of the equations

$$f(z) = \begin{cases} (\sin z)/z & : z \neq 0 \\ 1 & : z = 0 \end{cases}$$

is entire. Use this result to establish the limit

$$\lim_{z \to 0} \frac{\sin z}{z} = 1.$$

Sol. The Maclaurin series of $\sin z$ (entire) is

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

For $z \neq 0$, divide by z:

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots.$$

This is a power series with infinite radius of convergence, hence defines an entire function

$$F(z) := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}.$$

Note that $F(0) = 1$, and for $z \neq 0$ we have $F(z) = \sin z / z$. Therefore $f \equiv F$ on $\mathbb{C}$; in particular, f is entire (the singularity at 0 is removable). By continuity of F at 0,

$$\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} F(z) = F(0) = 1.$$

□

6 Residues and Poles

Use Cauchy's residue theorem to evaluate integral of each these functions around the circle $|z| = 3$ in the positive sense:

$$\frac{e^{-z}}{z^2}, \qquad \frac{e^{-z}}{(z-1)^2}, \qquad z^2 \exp\left(\frac{1}{z}\right), \qquad \frac{z+1}{z^2-2z}.$$

(Answers: $-2\pi i, -2\pi i/e, \pi i/3, 2\pi i$.)

Sol. All integrals are $\displaystyle\int_{|z|=3} (\cdot)\, dz$ with positive orientation.

(1) $\displaystyle\int_{|z|=3} \frac{e^{-z}}{z^2}\, \mathrm{d}z$. Let $f(z) = e^{-z}$. Then f is entire (analytic everywhere), and the only singularity of the integrand inside $|z| = 3$ is a pole of order 2 at $z = 0$. By Cauchy's integral formula for the first derivative,

$$f'(z_0) = \frac{1}{2\pi i} \int_{|z|=3} \frac{f(z)}{(z-z_0)^2}\, \mathrm{d}z. \implies \int_{|z|=3} \frac{f(z)}{(z-z_0)^2}\, \mathrm{d}z = 2\pi i\, f'(z_0).$$

With $z_0 = 0$,

$$\int_{|z|=3} \frac{e^{-z}}{z^2}\, \mathrm{d}z = 2\pi i\, f'(0) = 2\pi i \cdot \frac{\mathrm{d}}{\mathrm{d}z} e^{-z}\Big|_{z=0} = 2\pi i \cdot -e^{-z}\Big|_{z=0} = 2\pi i \cdot (-1) = -2\pi i.$$

(2) $\displaystyle\int \frac{e^{-z}}{(z-1)^2}\, \mathrm{d}z$. Let $f(z) = e^{-z}$, which is entire. The integrand has a pole of order 2 at $z = 1$. Since $|1| < 3$, this singularity lies inside the circle $|z| = 3$, and there are no other singularities inside the contour. By Cauchy's integral formula for the first derivative, with $z_0 = 1$,

$$\int_{|z|=3} \frac{f(z)}{(z-1)^2}\, \mathrm{d}z = 2\pi i \cdot f'(1) = 2\pi i \cdot (-e^{-z})\Big|_{z=1} = 2\pi i \cdot \left(-\frac{1}{e}\right) = \frac{-2\pi i}{e}.$$

(3) $\displaystyle\int_{|z|=3} z^2 \exp\left(\frac{1}{z}\right)\, \mathrm{d}z$. The only singularity of the integrand is at $z = 0$, due to the factor $e^{1/z}$. This is an essential singularity at $z = 0$, which lies inside the contour $|z| = 3$. We use the residue theorem:

$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k).$$

Here there is only one singularity at $z = 0$, so

$$\int_{|z|=3} z^2 \exp\left(\frac{1}{z}\right)\, \mathrm{d}z = 2\pi i\, \mathrm{Res}\left(z^2 e^{1/z}, 0\right).$$

To find the residue, expand $\exp\left(\frac{1}{z}\right)$ in a Laurent series around $z = 0$:

$$\begin{aligned}
\exp\left(\frac{1}{z}\right) &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots, \\
z^2 \exp\left(\frac{1}{z}\right) &= z^2\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right) \\
&= z^2 + z + \frac{1}{2!} + \frac{1}{3!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^2} + \cdots.
\end{aligned}$$

The residue at $z = 0$ is $\operatorname{Res}\left(z^2 e^{1/z}, 0\right) = \frac{1}{3!} = \frac{1}{6}$. Therefore

$$\int_{|z|=3} z^2 \exp\left(\frac{1}{z}\right) \mathrm{d}z = 2\pi i \cdot \frac{1}{6} = \frac{\pi i}{3}.$$

(4) $\displaystyle\int_{|z|=3} \frac{z+1}{z^2-2z}\,\mathrm{d}z = \int \frac{z+1}{z(z-2)}\,\mathrm{d}z.$ The singularities are simple poles at $z = 0$ and $z = 2$. We can either compute residues directly or use partial fraction decomposition:

$$\begin{aligned}
\frac{z+1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2} &\implies z + 1 = A(z-2) + Bz \\
&\implies z + 1 = (A+B)z - 2A \\
&\implies \begin{cases} A + B = 1, \\ -2A = 1. \end{cases} \\
&\implies A = -1/2 \quad \text{and} \quad B = 3/2.
\end{aligned}$$

Thus

$$\frac{z+1}{z^2-2z} = -\frac{1}{2z} + \frac{3}{2(z-2)}.$$

Now the integral becomes

$$\int \frac{z+1}{z(z-2)}\,\mathrm{d}z = \int_{|z|=3}\left(-\frac{1}{2z} + \frac{3}{2(z-2)}\right) dz = -\frac{1}{2}\int_{|z|=3}\frac{1}{z}\,dz + \frac{3}{2}\int_{|z|=3}\frac{1}{z-2}\,dz.$$

Note that

$$\int_{|z|=3}\frac{1}{z}\,dz = 2\pi i \quad \text{and} \quad \int_{|z|=3}\frac{1}{z-2}\,dz = 2\pi i,$$

since both $z = 0$ and $z = 2$ lie inside $|z| = 3$. Therefore

$$\int \frac{z+1}{z(z-2)}\,\mathrm{d}z = -\frac{1}{2}\cdot 2\pi i + \frac{3}{2}\cdot 2\pi i = -\pi i + 3\pi i = 2\pi i.$$

□

Show that the singular point of each of the following functions is a pole.

$$f(z) = \frac{1-\cosh z}{z^3}, \quad g(z) = \frac{1-\exp(2z)}{z^4}, \quad h(z) = \frac{\exp(2z)}{(z-1)^2}.$$

Determine the order m of that pole and the corresponding residue B.
(Answers: $f(z)$: $m = 1$, $B = -1/2$; $g(z)$: $m = 3$, $B = -4/3$; $h(z)$: $m = 2$, $B = 2e^2$.)

Sol.

(1) $f(z) = \dfrac{1-\cosh z}{z^3}$ at $z = 0$. Note that $\cosh z = 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \frac{1}{6!}z^6 + \cdots$, and so

$$\begin{aligned} 1-\cosh z &= -\frac{1}{2}z^2 - \frac{1}{4!}z^4 - \frac{1}{6!}z^6 - \cdots, \\ \frac{1-\cosh z}{z^3} &= -\frac{1}{2}\frac{1}{z} - \frac{1}{4!}z + \frac{1}{6!}z^3 + \cdots. \end{aligned}$$

Hence the singularity is a *simple pole* ($m = 1$) with residue $B = \operatorname{Res}_{z=0} f = -\frac{1}{2}$.

(2) $g(z) = \dfrac{1-\exp(2z)}{z^4}$ at $z = 0$. Note that

$$\begin{aligned} \exp(2z) &= 1 + 2z + \frac{(2z)^2}{2!} + \frac{(2z)^3}{3!} + \frac{(2z)^4}{4!} + \cdots \\ &= 1 + 2z + 2z^2 + \frac{4}{3}z^3 + \frac{2}{3}z^4 + \cdots, \\ 1-\exp(2z) &= -\left(2z + 2z^2 + \frac{4}{3}z^3 + \frac{2}{3}z^4 + \cdots\right), \\ \frac{1-\exp(2z)}{z^4} &= -\frac{2}{z^3} - \frac{2}{z^2} - \frac{4}{3}\frac{1}{z} - \frac{2}{3} + \cdots. \end{aligned}$$

Thus we have a pole of order 3 ($m = 3$) with residue $B = \operatorname{Res}_{z=0} g = -\frac{4}{3}$.

(3) $h(z) = \dfrac{\exp(2)}{(z-1)^2}$ at $z = 1$. Let $w := z - 1$, i.e., $z = 1 + w$. Then

$$\begin{aligned} \exp(2z) &= \exp(2(1+w)) = \exp(2)\exp(2w) \\ &= \exp(2)\left(1 + 2w + \frac{2^2}{2!}w^2 + \frac{2^3}{3!}w^3 + \cdots\right), \\ \frac{\exp(2z)}{(z-1)^2} &= \frac{\exp(2)}{w^2}\left(1 + 2w + \frac{2^2}{2!}w^2 + \cdots\right) = \exp(2)\left(\frac{1}{w^2} + \frac{2}{w} + \frac{2^2}{2!} + \cdots\right). \end{aligned}$$

Therefore the singularity is a pole of order 2 ($m = 2$) with residue $B = \operatorname{Res}_{z=1} h = 2\exp(2)$.

□

Show that

$$\operatorname*{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \frac{1+i}{\sqrt{2}} \qquad (|z|>0,\ 0<\arg z<2\pi),$$
$$\operatorname*{Res}_{z=i} \frac{\operatorname{Log}\ z}{(z^2+1)^2} = \frac{\pi+2i}{8},$$
$$\operatorname*{Res}_{z=i} \frac{z^{1/2}}{(z^2+1)^2} = \frac{1-i}{8\sqrt{2}} \qquad (|z|>0,\ 0<\arg z<2\pi).$$

Sol.

(1) **(Residue of $f_1(z) = \dfrac{z^{1/4}}{z+1}$ at $z=-1$)** We work with the branch

$$|z|>0, \quad 0<\arg z<2\pi,$$

so the branch cut is along the positive real axis, and

$$z^{1/4} = \exp(\frac{1}{4}\operatorname{Log} z), \qquad \operatorname{Log}\ z = \ln|z| + i\arg z, \quad 0<\arg z<2\pi.$$

At $z=-1$ we have $|-1|=1$ and $\arg(-1)=\pi$, hence

$$\operatorname{Log}(-1) = \ln 1 + i\pi = i\pi,$$

and therefore

$$z^{1/4}\big|_{z=-1} = (-1)^{1/4} = \exp\left(i\frac{\pi}{4}\right) = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1+i}{\sqrt{2}}.$$

The integrand $f_1(z) = \dfrac{z^{1/4}}{z+1}$ has a simple pole at $z=-1$, and $z^{1/4}$ is analytic at $z=-1$ on this branch. Consider

$$g(z) := (z+1)\,f_1(z) = (z+1)\,\frac{z^{1/4}}{z+1} = z^{1/4}.$$

Since $z^{1/4}$ is analytic at $z=-1$, the function $(z+1)f_1(z)$ is analytic at $z=-1$. Therefore it has a Taylor expansion around $z=-1$:

$$(z+1)f_1(z) = z^{1/4} = a_0 + a_1(z+1) + a_2(z+1)^2 + \cdots, \quad \text{with} \quad a_n = \frac{g^{(n)}(-1)}{n!}\ (n=0,1,2,\ldots),$$

valid for z near -1. Dividing both sides by $(z+1)$, we get a Laurent expansion for f_1 at $z=-1$:

$$f_1(z) = \frac{a_0}{z+1} + a_1 + a_2(z+1) + \cdots.$$

By definition, the residue of f_1 at $z=-1$ is $\operatorname{Res}_{z=-1} f_1(z) = a_0$. Thus

$$\operatorname*{Res}_{z=-1} \frac{z^{1/4}}{z+1} = a_0 = \frac{g^{(0)}(-1)}{0!} = \lim_{z\to -1}(z+1)\frac{z^{1/4}}{z+1} = (-1)^{1/4} = \frac{1+i}{\sqrt{2}}.$$

(2) **(Residue of $f_2(z) = \dfrac{\operatorname{Log} z}{(z^2+1)^2}$ at $z=i$)** We factor $z^2+1 = (z-i)(z+i)$, so near $z=i$,

$$(z^2+1)^2 = (z-i)^2(z+i)^2.$$

Hence

$$f_2(z) = \frac{\operatorname{Log} z}{(z-i)^2(z+i)^2} = \frac{g(z)}{(z-i)^2}, \qquad g(z) := \frac{\operatorname{Log} z}{(z+i)^2}.$$

The function g is analytic at $z=i$. Thus $z=i$ is a double pole of f of the form $f_2(z) = g(z)/(z-i)^2$ with g analytic at i. Then

$$\operatorname*{Res}_{z=i} f_2(z) = \operatorname*{Res}_{z=i} \frac{g(z)}{(z-i)^2} = \frac{g^{(1)}(i)}{1!} = g'(i).$$

Compute $g'(i)$:

$$\begin{aligned}
g'(i) = \frac{d}{dz} g(z)\bigg|_{z=i} &= \frac{d}{dz}\left((\operatorname{Log} z)(z+i)^{-2}\right)\bigg|_{z=i} \\
&= \left[\frac{1}{z}(z+i)^{-2} - 2\operatorname{Log} z\,(z+i)^{-3}\right]_{z=i} \\
&= \frac{1}{i}\cdot\frac{1}{-4} - 2\operatorname{Log}(i)\cdot\frac{1}{-8i} \\
&= \frac{-1}{4i} + \frac{1}{4i}\cdot(\ln|i| + i\arg(i)) \\
&= \frac{i}{4} + \frac{-i}{4}\cdot\left(0 + \frac{\pi i}{2}\right) \\
&= \frac{2i}{8} + \frac{\pi}{8} \\
&= \frac{\pi + 2i}{8}.
\end{aligned}$$

(3) **(Residue of $f_3(z) = \dfrac{z^{1/2}}{(z^2+1)^2}$ at $z=i$)** As before, $(z^2+1)^2 = (z-i)^2(z+i)^2$, so

$$f_3(z) = \frac{z^{1/2}}{(z-i)^2(z+i)^2} = \frac{h(z)}{(z-i)^2}, \qquad h(z) := \frac{z^{1/2}}{(z+i)^2},$$

and $h(z)$ is analytic at $z=i$. Thus $z=i$ is again a double pole of f_3, and $\operatorname{Res}_{z=i} f(z) = h'(i)$.

Compute $h'(i)$:

$$\begin{aligned}
h'(i) = \frac{d}{dz}h(z)\Big|_{z=i} &= \frac{d}{dz}\left(z^{1/2}(z+i)^{-2}\right)\Big|_{z=i} \\
&= \left[\frac{1}{2}z^{-1/2}(z+i)^{-2} - 2z^{1/2}(z+i)^{-3}\right]_{z=i} \\
&= \frac{1}{2}\cdot i^{-1/2}\cdot\frac{1}{-4} - 2\cdot i^{1/2}\cdot\frac{1}{-8i}.
\end{aligned}$$

We need the branch values of $z^{1/2}$ and $z^{-1/2}$ at $z = i$ for $0 < \arg z < 2\pi$:

$$i^{1/2} = \exp(\frac{1}{2}\operatorname{Log} i) = \exp(\frac{1}{2}(\ln|i| + i\arg(i))) = \exp(\frac{1}{2}\left(\frac{\pi i}{2}\right)) = \exp(i\pi/4) = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1+i}{\sqrt{2}},$$
$$i^{-1/2} = \frac{1}{i^{1/2}} = \frac{1}{\frac{1+i}{\sqrt{2}}} = \frac{\sqrt{2}}{1+i} = \frac{\sqrt{2}(1-i)}{(1+i)(1-i)} = \frac{\sqrt{2}(1-i)}{2} = \frac{1-i}{\sqrt{2}}.$$

Therefore

$$\begin{aligned}
h'(i) &= \frac{1}{2}\cdot i^{-1/2}\cdot\frac{1}{-4} - 2\cdot i^{1/2}\cdot\frac{1}{-8i} \\
&= \frac{-1}{8}\left(\frac{1-i}{\sqrt{2}}\right) + \frac{-i}{4}\left(\frac{1+i}{\sqrt{2}}\right) \\
&= \frac{-(1-i)+2(1-i)}{8\sqrt{2}} \\
&= \frac{(1-i)}{8\sqrt{2}}
\end{aligned}$$

Thus

$$\operatorname*{Res}_{z=i}\frac{z^{1/2}}{(z^2+1)^2} = h'(i) = \frac{1-i}{8\sqrt{2}}.$$

□

Find the value of the integral

$$\int_{|z|=3} \frac{z^3\, e^{1/z}}{1+z^3}\, dz$$

taken CCW around the circle $|z| = 3$.
(Answer: $2\pi i$.)

Sol. Let

$$f(z) = \frac{z^3 e^{1/z}}{1+z^3}.$$

Set

$$w = \frac{1}{z} \quad\Longrightarrow\quad z = \frac{1}{w}, \quad dz = -\frac{1}{w^2}\, dw.$$

The circle $|z| = 3$ corresponds to $|w| = 1/3$. As z runs CCW, $w = 1/z$ runs clockwise. Therefore,

$$\begin{aligned}
\int_{|z|=3} f(z)\, dz &= \underbrace{\int_{|w|=1/3} f\left(\frac{1}{w}\right)\left(-\frac{1}{w^2}\, dw\right)}_{\text{CW}} \\
&= -\underbrace{\int_{|w|=1/3} f\left(\frac{1}{w}\right)\left(-\frac{1}{w^2}\, dw\right)}_{\text{CCW}} \\
&= \int_{|w|=1/3} \frac{f(1/w)}{w^2}\, dw.
\end{aligned}$$

Since

$$f\left(\frac{1}{w}\right) = \frac{(1/w)^3 e^w}{1+(1/w)^3} = \frac{\frac{e^w}{w^3}}{\frac{w^3+1}{w^3}} = \frac{e^w}{w^3+1},$$

we have

$$\int_{|z|=3} f(z)\, dz = \int_{|w|=1/3} \left(\frac{1}{w^2} \cdot f\left(\frac{1}{w}\right)\right) dw = \int_{|w|=1/3} \frac{e^w}{w^2(w^3+1)}\, dw.$$

The only singularity of

$$h(w) = \frac{e^w}{w^2(w^3+1)}$$

inside the circle $|w| = 1/3$ is at $w = 0$, since the roots of $w^3 + 1 = 0$ are

$$w = -1, \quad w = e^{i\pi/3}, \quad w = e^{-i\pi/3},$$

all of which satisfy $|w| = 1$. Hence, by the residue theorem,

$$\int_{|z|=3} f(z)\, dz = 2\pi i \operatorname*{Res}_{w=0} h(w).$$

We compute the residue via series expansion. Write

$$h(w) = \frac{1}{w^2} \cdot \frac{e^w}{1+w^3}.$$

Since

$$\frac{1}{1+w^3} = 1 - w^3 + w^6 - w^9 + \cdots \quad (|w| < 1) \quad \text{and}$$
$$e^w = 1 + w + \frac{w^2}{2} + \frac{w^3}{6} + \cdots,$$

we have

$$\frac{e^w}{1+w^3} = (1 + w + \frac{w^2}{2} + \frac{w^3}{6} + \cdots)(1 - w^3 + w^6 - \cdots),$$

and so

$$\begin{aligned}
h(w) &= \frac{1}{w^2} \cdot \left(\left(1 + w + \frac{w^2}{2} + \frac{w^3}{6} + \cdots\right)(1 - w^3 + w^6 - \cdots)\right) \\
&= \frac{1}{w^2}\left(\left(1 + w + \frac{w^2}{2} + \frac{w^3}{6} + \cdots\right) - \left(w^3 + w^4 + \frac{w^5}{2} + \frac{w^6}{6} + \cdots\right) + \left(w^6 + w^7 + \frac{w^8}{2} + \frac{w^9}{6} + \cdots\right) - \cdots\right) \\
&= \left(\frac{1}{w^2} + \frac{1}{w} + \frac{1}{2} + \frac{w}{6} + \cdots\right) - \left(w^1 + w^2 + \frac{w^3}{2} + \frac{w^4}{6} + \cdots\right) + \left(w^4 + w^5 + \frac{w^6}{2} + \frac{w^7}{6} + \cdots\right) - \cdots.
\end{aligned}$$

Thus $\operatorname{Res}_{w=0} h(w) = 1$. By the residue theorem,

$$\int_{|z|=3} \frac{z^3 e^{1/z}}{1+z^3}\, dz = 2\pi i \operatorname*{Res}_{w=0} h(w) = 2\pi i \cdot 1 = 2\pi i.$$

□

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