Penetrating Examples in Differential Forms

Manifolds: S^1, S^2, and T^2

September 22, 2025

These examples get to the very heart of how a space's **topology** (its shape and number of "holes") influences the behavior of calculus on it. The key concept we'll illustrate is the difference between **closed** and **exact** differential forms, which is the sophisticated version of the difference between a **curl-free** and a **conservative** vector field.

A form ω is:

- **Closed** if its derivative is zero ($d\omega = 0$). This is a *local* property, like being curl-free.
- **Exact** if it is the derivative of another form ($\omega = d\alpha$). This is a *global* property, meaning a single potential function α exists across the whole space.

The crucial rule is: **Exact always implies Closed**. The interesting question is when the reverse is true. On spaces with "holes," it often isn't.

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1 The Circle (S^1)

The circle is the simplest space with a hole. This hole prevents a specific, natural 1-form from being exact.

The Topology S^1 has one 1-dimensional hole. You can't shrink a loop drawn on it to a point without leaving the circle. Because of this, we expect to find a 1-form that is closed but not exact.

The Penetrating Example The "angle form," $\omega = d\theta$. If we view S^1 as the unit circle in the plane, this form is written as $\omega = -y\,dx + x\,dy$.

Analysis

1. **Is it closed?** Yes. Trivially, $d\omega = d(d\theta) = 0$. Any form on a 1-dimensional space is automatically closed because its derivative would be a 2-form, which must be zero.
2. **Is it exact?** Let's test it using the fundamental theorem of calculus. If ω were exact, its integral around any closed loop must be zero. Let's integrate it once around the circle. We parameterize the circle with $x = \cos\theta, y = \sin\theta$.

$$\begin{aligned}\oint_{S^1} \omega &= \int_0^{2\pi} (-\sin\theta)(-\sin\theta\,\mathrm{d}\theta) + (\cos\theta)(\cos\theta\,\mathrm{d}\theta) \\ &= \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\,\mathrm{d}\theta = \int_0^{2\pi} 1\,\mathrm{d}\theta = 2\pi\end{aligned}$$

The Insight The integral is $2\pi \neq 0$. Therefore, ω **is not exact**. The angle θ is a perfect potential function *locally*, but it fails globally because it's not single-valued (it jumps from 2π back to 0). This failure is a direct measurement of the hole in the circle.

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2 The Sphere (S^2)

The sphere is the opposite. It has no 1-dimensional holes; any loop you draw on its surface can be shrunk down to a single point.

The Topology S^2 is **simply connected**. It lacks the kind of hole found in the circle. This means its topology guarantees that **every closed 1-form on the sphere is also exact**. There are no interesting 1-form examples here because the phenomenon of "closed but not exact" cannot happen.

The Penetrating Example (for 2-forms) The sphere's "hole" is the 3D volume it encloses. This 2-dimensional hole is detected by a **2-form**: the **area form** σ.

Analysis

1. **Is it closed?** Yes. Any 2-form on a 2-dimensional space is trivially closed because its derivative, a 3-form, must be zero.

2. **Is it exact?** If σ were exact, it would be the derivative of some 1-form, $\sigma = d\omega$. By Stokes' Theorem, the integral of σ over the entire sphere would have to be zero, because the sphere has no boundary ($\partial S^2 = \emptyset$).

$$\int_{S^2} \sigma = \int_{S^2} d\omega = \oint_{\partial S^2} \omega = 0$$

But we know the integral of the area form is simply the surface area of the sphere:

$$\int_{S^2} \sigma = \text{Area}(S^2) = 4\pi r^2$$

The Insight Since the integral is $4\pi r^2 \neq 0$, the area form σ **is not exact**. The non-zero area of the sphere, a closed surface, acts as the obstruction. This proves that the sphere, while having no 1D holes, encloses a 2D hole which is detected by a 2-form.

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3 The Torus (T^2)

The torus is richer than the circle because it has two distinct types of holes.

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Figure 1: The two fundamental loops on a torus, γ_θ and γ_ϕ.

The Topology The torus ($T^2 = S^1 \times S^1$) has two independent 1-dimensional holes: one going around the "long way" (θ) and one going through the "hole of the donut" (ϕ). We therefore expect to find two independent 1-forms that are closed but not exact.

The Penetrating Examples The differentials of the two angle coordinates, $\omega_\theta = d\theta$ and $\omega_\phi = d\phi$.

Analysis

1. **Are they closed?** Yes, for the same reason as on the circle: $d(d\theta) = 0$ and $d(d\phi) = 0$.

2. **Are they exact?** We test them by integrating over the two fundamental loops.

   - Let γ_θ be the "long way" loop where θ goes from 0 to 2π and ϕ is constant.
   - Let γ_ϕ be the "short way" loop where ϕ goes from 0 to 2π and θ is constant.

   Let's integrate ω_θ around both loops:

$$\oint_{\gamma_\theta} \omega_\theta = \int_0^{2\pi} d\theta = 2\pi \neq 0 \implies \omega_\theta \text{ is not exact.}$$

$$\oint_{\gamma_\phi} \omega_\theta = \int_0^{2\pi} 0 = 0$$

   Now let's integrate ω_ϕ around both loops:

$$\oint_{\gamma_\theta} \omega_\phi = \int_0^{2\pi} 0 = 0$$

$$\oint_{\gamma_\phi} \omega_\phi = \int_0^{2\pi} d\phi = 2\pi \neq 0 \implies \omega_\phi \text{ is not exact.}$$

The Insight We have found two distinct closed forms, ω_θ and ω_ϕ. Neither is exact. Each one successfully "detects" one of the torus's holes (by having a non-zero integral around it) while ignoring the other. This shows how different non-exact forms can probe the distinct topological features of a space.