Advanced Calculus III

Ji, Yong-hyeon

January 3, 2025

We cover the following topics in this note.

- Limit of a Function
- TBA

Limit Point (Metric Space)

Definition. Let (X, d) be a metric space. Let $S \subseteq X$ and $\alpha \in X$. A point $p \in X$ is a **limit point** of S if and only if

$$\forall \varepsilon > 0,\ B_{\varepsilon}(\alpha) \cap (S \setminus \{p\}) \neq \varnothing.$$

That is,

$$\forall \varepsilon > 0,\ \{x \in S : 0 < d(x, p) < \varepsilon\} \neq \varnothing.$$

Remark 1. Note that α does not have to be an element of A to be a limit point.

Note. Let (X, τ) be a topological space. For a subset $S \subseteq X$. A point $p \in X$ is a limit point of S if and only if

$$\forall U \in \tau \text{ with } p \in U,\ U \cap (S \setminus \{p\}) \neq \varnothing.$$

⋆ Limit of a Function ($\epsilon - \delta$) ⋆

Definition. Let $f : X \to \mathbb{R}$ be a function defined on a subset X of a topological space, and let $p \in X$ be a limit point of X. We say that $L \in \mathbb{R}$ is the **limit of the function** f **as** x **approaches** p if

$$\boxed{\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x \in X,\ 0 < |x - p| < \delta \implies |f(x) - L| < \varepsilon}.$$

We write

$$\boxed{\lim_{x \to p} f(x) = L}.$$

Remark 2.

$$\lim_{x \to p} f(x) \neq L \iff \exists \varepsilon > 0 : [\forall \delta > 0 : \exists x \in X : 0 < |x - p| < \delta \text{ but } |f(x) - L| > 0].$$

Continuity of a Function

Definition. Let $f : X \to \mathbb{R}$ be a function defined on a subset X of a topological space, and let $p \in X$. The function f is said to be f is **continuous at** p if and only if

$$\lim_{x \to p} f(x) = f(p).$$

That is,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < \left|x - p\right| < \delta \implies |f(x) - f(p)| < \varepsilon.$$

Remark 3 (Continuity of a Set)**.** The function f is continuous on subset $S \subseteq X$ if it it continuous at every point $p \in S$.

Remark 4 (Continuity in a Topological Space)**.** Let (X, τ_X) and (Y, τ_Y) are topological spaces. $f : X \to Y$ is **continuous** if and only if

$$U_Y \in \tau_Y \implies f^{-1}[U_Y] \in \tau_X,$$

where $f^{-1}[U_Y] = \{x \in X : f(x) \in U_Y\}$ is the preimage of U_Y under f.

Note (Subsequence)**.** Let $\{a_n\}$ be a sequence of real numbers, and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a strictly increasing of natural numbers. Then $\{a_{n_k}\}$ is called **subsequence** of $\{a_n\}$.

Proposition. *A sequence a_n of real numbers converges to $L \in \mathbb{R}$ if and only if any subsequence $\{a_{n_k}\}$ of $\{a_n\}$ converges to $L \in \mathbb{R}$. Formally,*

$$\lim_{n\to\infty} a_n = L \iff \lim_{k\to\infty} a_{n_k} = L.$$

Sandwitch Theorem; Squeeze Theorem

Theorem.

Note. TBA

References

[1] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 10. 해석학 개론 (e) 엡실론-델타와 수열의 수렴성” YouTube Video, 25:57. Published September 29, 2019. URL: `https://www.youtube.com/watch?v=2Ml3G_Duffk&t=899s`.