Set Theory II

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We cover the following topics in this note.

- Relations
- Equivalence Relations
- Equivalence Classes
- Partitions

Relation

Definition. Let $A \times B$ be the cartesian product of two sets A and B. A **(binary) relation** on $A \times B$ is a subset $\mathcal{R}$ of $A \times B$. That is,

$$\mathcal{R} \text{ is a relation on } A \times B \iff \mathcal{R} \subseteq A \times B.$$

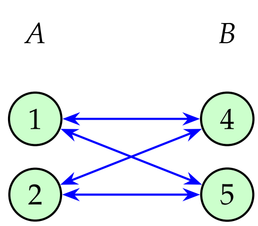
Remark. $\mathcal{R}$ is a relation on $A \iff \mathcal{R} \subseteq A \times A$.

Note (Notation). Let $(s,t) \in \mathcal{R}$. We use the notation $s \ \mathcal{R} \ t$ and we can say "s **is related to** t **by** R". If $(s,t) \notin \mathcal{R}$, we denote as: $s \not\mathcal{R} \ t$.

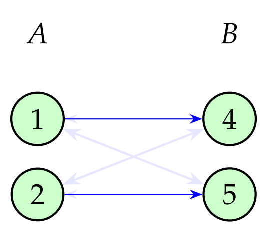
Example. Let $A = \{1,2\}$ and $B = \{4,5\}$. Then

$$A \times B = \left\{(1,4),(1,5),(2,4),(2,5)\right\}.$$

Here, $\mathcal{R} = \left\{(1,4),(2,5)\right\} \subseteq A \times B$ be a relation.



$A \times B$



$\mathcal{R} \subseteq A \times B$

Example. Let A and B are sets, and let $f : A \to B$ be a function from A to B. Then

$$(a,b) \in f \iff a \ f \ b \iff b = f(a).$$

⋆ Equivalence Relation ⋆

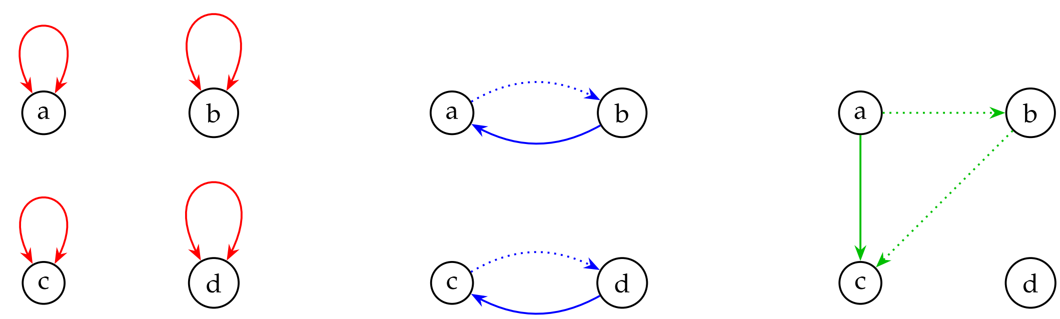
Definition. A binary relation $\mathcal{R}$ on a set S is called an **equivalence relation** if it satisfies the following three properties: for all $a, b, c \in S$,

(i) (Reflexivity) $(a, a) \in \mathcal{R}$;

(ii) (Symmetry) $(a, b) \in \mathcal{R} \implies (b, a) \in \mathcal{R}$;

(iii) (Transitivity) $(a, b) \in \mathcal{R} \wedge (b, c) \in \mathcal{R} \implies (a, c) \in \mathcal{R}$.

Remark.



Reflexivity
(each element is related to itself)

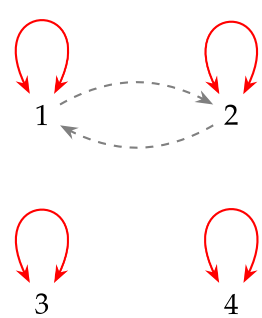
Symmetry
(if a is related to b, then b is related to a)

Transitivity
(if a is related to b and b is related to c, then a is related to c)

Example. Let $A = \{1, 2, 3, 4\}$. Then

$$\mathcal{R} = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1)\}$$

is an equivalence relation on A.



Note. Let A, B, C are sets, and let $f : A \to B$ and $g : B \to C$ are functions.

- We claim that $\boxed{(g \circ f)[A] = g[f[A]]}$:

$$(g \circ f)[A] = \left\{(g \circ f)(a) : a \in A\right\} = \left\{g(f(a)) : a \in A\right\} = \left\{g(b) : b = f(a) \in f[A]\right\} = g[f[A]].$$

- We claim that $\boxed{f \text{ is surjective} \iff \mathrm{Img}(f) = f[A] = B}$:

$$f : A \twoheadrightarrow B \iff \forall b \in B,\ \exists a \in A \text{ s.t. } f(a) = b \iff f[A] = \left\{f(a) \in B : a \in A\right\} = B.$$

Lemma 1. *Let A, B and C are sets, and let $f : A \to B$ and $g : B \to C$ are functions.*

(1) If f and g are both one-to-one, then $(g \circ f) : A \to C$ is one-to-one.

(2) If f and g are both onto, then $(g \circ f) : A \to C$ is onto.

Proof. (1) Let f and g are both one-to-one. We must show that $(g \circ f) : A \to C$ is one-to-one. Suppose that $(g \circ f)(a) = (g \circ f)(a')$. Then

$$\begin{aligned}
(g \circ f)(a) = (g \circ f)(a') &\implies g(f(a)) = g(f(a')) && \text{by def. of composition}\\
&\implies f(a) = f(a') && \because g \text{ is injective}\\
&\implies a = a'. && \because f \text{ is injective}
\end{aligned}$$

(2) Let f and g are both onto. We must show that $(g \circ f) : A \to C$ is onto, i.e., $(g \circ f)[A] = C$.

$$\begin{aligned}
(g \circ f)[A] &= g[f[A]]\\
&= g[B] && \because f : A \to B \text{ is surjective, i.e., } f[A] = B\\
&= C. && \because g : B \to C \text{ is surjective, i.e., } g[B] = C
\end{aligned}$$

□

Lemma 2. *Let A, B and C are sets, and let $f : A \to B$ and $g : B \to C$ are functions.*

(1) If $(g \circ f) : A \to C$ is one-to-one, then f is one-to-one.

(2) If $(g \circ f) : A \to C$ is onto, then g is onto.

Proof. (1) Let $g \circ f$ is one-to-one. We must show that f is one-to-one. Suppose that $f(a) = f(a')$. Then

$$\begin{aligned} f(a) = f(a') &\implies g(f(a)) = g(f(a')) && \because g \text{ is a function} \\ &\implies (g \circ f)(a) = (g \circ f)(a') && \text{by the def. of composition} \\ &\implies a = a'. && \because g \circ f \text{ is injective} \end{aligned}$$

(2) Let $g \circ f$ is onto, i.e., $(g \circ f)[A] = C$. We must show that g is onto, i.e., $g[B] = C$:

($\subseteq$) $g[B] = \left\{g(b) \in C : b \in B\right\} \subseteq C$;

($\supseteq$) $C = (g \circ f)[A] = g[f[A]] = \left\{g(b) \in C : b \in f[A]\right\} \subseteq g[B]$.

□

Equivalence Relation on 2^A Based on Bijection

Proposition 3. *Let A be a set, and 2^A be its power set. Define a relation $\mathcal{R}$ on 2^A as follows:*

$$X \sim_{\mathcal{R}} Y \iff \exists f \in Y^X \text{ such that } f \text{ is bijective},$$

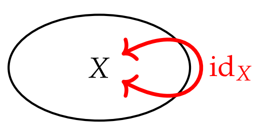
for $X, Y \in 2^A$. In other words,

$$\mathcal{R} := \left\{(X, Y) \in 2^A \times 2^A : \exists \text{ a bijection } f \in Y^X\right\}.$$

Then $\mathcal{R}$ is an equivalence relation on 2^A.

Proof. Let $X, Y, Z \in 2^A$. We must show that $\mathcal{R}$ is reflexive, symmetric and transitive:

(i) (Reflexivity) We NTS[1] that $X \sim_{\mathcal{R}} X$. In other words, we need to find a bijection from X it self.



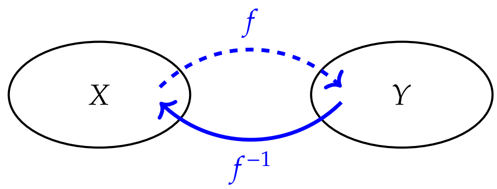
Consider the identity function

$$\begin{aligned} \mathrm{id}_X \quad : \quad X &\longrightarrow X \\ x &\longmapsto x = \mathrm{id}_X(x) \end{aligned}$$

for all $x \in X$. Clearly, id_X is a bijection. Thus, $X \sim_{\mathcal{R}} X$.

[1]'NTS' means that "need to show".

(ii) (Symmetry) We NTS that $X \sim_{\mathcal{R}} Y \implies Y \sim_{\mathcal{R}} X$. In other words, if there exists a bijection $f : X \to Y$, then there must exists a bijection $g : Y \to X$.

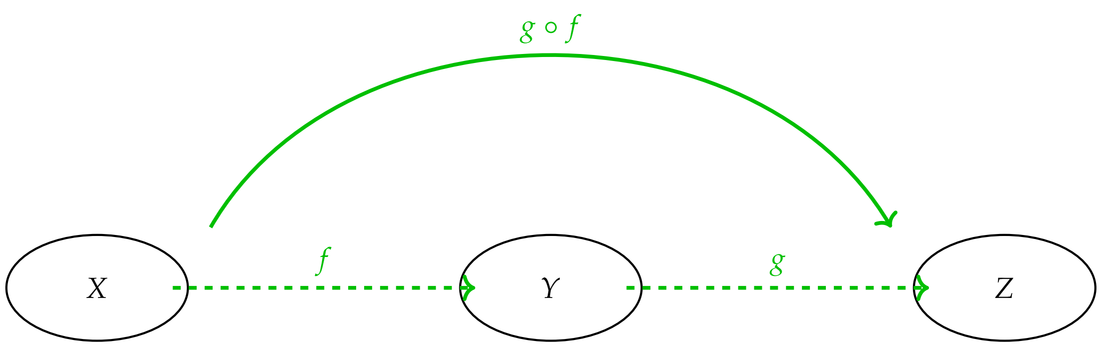


Suppose that $f : X \to Y$ is a bijection. Then it has an inverse function $f^{-1} : Y \to X$, which satisfies:

$$\forall x \in X : f^{-1}(f(x)) = x = \mathrm{id}_X(x) \quad \text{and} \quad \forall y \in X : f(f^{-1}(y)) = y = \mathrm{id}_Y(y).$$

That is, $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$. By **Lemma 2**, f^{-1} must be a bijection since f, id_X and id_Y are bijections. Thus, there is a bijection $g = f^{-1}$.

(iii) We NTS that $X \sim_{\mathcal{R}} Y \wedge Y \sim_{\mathcal{R}} Z \implies X \sim_{\mathcal{R}} Z$. In other words, if there exists two bijection $f : X \to Y$ and $g : Y \to Z$, then there must exists a bijection $h : X \to Z$.



Suppose that $f : X \to Y$ and $g : Y \to Z$ both are bijective. Define the function

$$\begin{array}{cccc} h & : & X & \longrightarrow & Z \\ & & x & \longmapsto & (g \circ f)(x) = h(x) \end{array}$$

for all $x \in X$. By **Lemma 1**, $h = g \circ f$ must be a bijection since f and g are both bijective.

Hence it is proved. □

Indexed Family

Definition. Let I and S are sets. Consider a function $A : I \to S$ defined by $i \mapsto A(i) =: A_i$. The image $\mathrm{Img}(A)$ is called an **indexed family of elements in** S **indexed by** I. We write this indexed family as: $\langle A_i \rangle_{i \in I}$. Note that

$$\mathrm{Img}(A) = \big\{A(i) : i \in I\big\} = \{A_i : i \in I\} = \langle A_i \rangle_{i \in I}\,.$$

Example (Sequence). Let $I = \mathbb{N}$ be an indexing set. Then

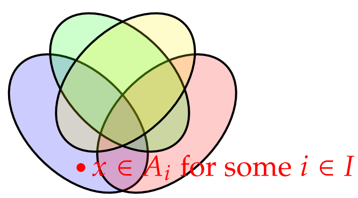
$$S := \{A_1, A_2, A_3, A_4, \cdots\} = \{A_i : i \in \mathbb{N}\} = \langle A_i \rangle_{i \in \mathbb{N}}$$

is an indexed family of elements in S indexed by $\mathbb{N}$.

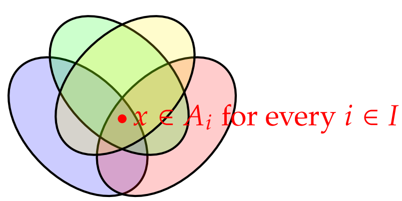
Union and Intersection of an Indexed Family

Definition. Let I and S are sets, and let $\langle A_i \rangle_{i \in I}$ be an indexed family in S.

- The **union of** $\langle A_i \rangle_{i \in I}$ is defined by $\boxed{\bigcup_{i \in I} A_i := \{x \in S : \exists i \in I \text{ such that } x \in A_i\}}$.



- The **intersection of** $\langle A_i \rangle_{i \in I}$ is defined by $\boxed{\bigcap_{i \in I} A_i := \{x \in S : \forall i \in I,\ x \in A_i\}}$.



Remark. Let $I = \{1, \dots, n\}$. Then

- $\displaystyle\bigcup_{i \in I} S_i = \bigcup_{i=1}^{n} S_i = S_1 \cup S_2 \cup \cdots \cup S_n$.
- $\displaystyle\bigcap_{i \in I} S_i = \bigcap_{i=1}^{n} S_i = S_1 \cap S_2 \cap \cdots \cap S_n$.

⋆ Partitions ⋆

Definition. Let S be a set, and let the function $A : I \longrightarrow 2^S$ as $i \mapsto A_i := A(i) \subseteq S$, for all $i \in I$. Consider a family of subsets $\langle A_i \rangle_{i\in I}$, where $A_i \subseteq S$ for every index $i \in I$. The family $\langle A_i \rangle_{i\in I}$ is called a **partition of** S if the following conditions are satisfied:

(i) (**Non-empty Subsets**) Each subset A_i is non-empty, i.e., $\boxed{\forall i \in I,\ A_i \neq \varnothing}$.

(ii) (**Pairwise disjoint**) For all distinct $i, j \in I$, the subsets A_i and A_j are disjoint, i.e.,

$$\boxed{\forall i, j \in I,\ i \neq j \implies A_i \cap A_j = \varnothing}.$$

(iii) (**Union covers the entire set**) The union of all subsets A_i covers the whole set S, i.e.,

$$\boxed{\bigcup_{i\in I} A_i = S}.$$

Example. Let $\mathbb{Z}$ be a set of integers. We define an indexed family $\langle A_i \rangle_{i\in\{0,1,2\}}$ of subsets of $\mathbb{Z}$ as follows:

$$\begin{aligned}
A_0 &= \left\{n \in \mathbb{Z} : n \equiv 0 \ (\mathrm{mod} 3)\right\} = \{n \in \mathbb{Z} : n = 3k + 0 \text{ for some } k \in \mathbb{Z}\} =: [0],\\
A_1 &= \left\{n \in \mathbb{Z} : n \equiv 1 \ (\mathrm{mod} 3)\right\} = \{n \in \mathbb{Z} : n = 3k + 1 \text{ for some } k \in \mathbb{Z}\} =: [1],\\
A_2 &= \left\{n \in \mathbb{Z} : n \equiv 2 \ (\mathrm{mod} 3)\right\} = \{n \in \mathbb{Z} : n = 3k + 2 \text{ for some } k \in \mathbb{Z}\} =: [2].
\end{aligned}$$

Then

(i) $[0] \neq \varnothing$, $[1] \neq \varnothing$ and $[2] \neq \varnothing$.

(ii)

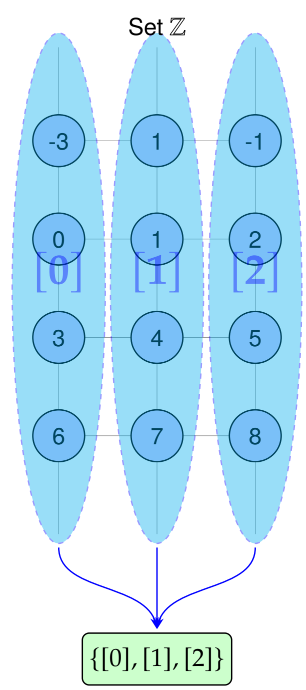
$$\begin{aligned}
[0] \cap [1] &= \varnothing,\\
[1] \cap [2] &= \varnothing,\\
[2] \cap [0] &= \varnothing.
\end{aligned}$$

(iii) $[0] \cup [1] \cup [2] = \mathbb{Z}$.

Thus,

$$\{A_1, A_2, A_3\} = \left\{[0], [1], [2]\right\}$$

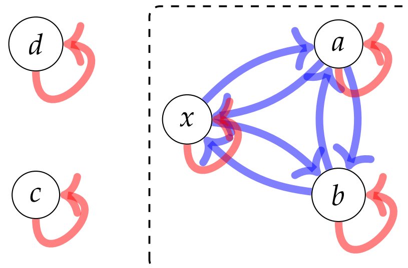
is a partition of $\mathbb{Z}$.



⋆ Equivalence Class ⋆

Definition. Let $\mathcal{R} \subseteq S \times S$ be an equivalence relation on S. The **equivalence class of** $x \in S$ **under** $\mathcal{R}$ is the set
$$[x]_{\mathcal{R}} = \{y \in S : x \,\mathcal{R}\, y\}.$$

Note. Note that $\alpha \,\mathcal{R}\, x \iff \alpha \in [x]_{\mathcal{R}} \iff x \,\mathcal{R}\, \alpha$.



Equivalence class of x

Lemma 4. *Let $\mathcal{R}$ be an equivalence relation on a set S. For any $x, y \in S$, let $[x]$ and $[y]$ represent the equivalence classes of x and y, respectively, under $\mathcal{R}$.*

(1) $\forall x \in S,\ x \in [x]$.

(2) $x \,\mathcal{R}\, y \iff [x] = [y]$.

(3) $x \,\not\mathcal{R}\, y \iff [x] \cap [y] = \varnothing$.

Proof. (1) Let $x \in S$. Since $\mathcal{R}$ is reflexive, we have $x \,\mathcal{R}\, x$, i.e., $x \in [x]$.

(2) ($\Rightarrow$) Let $x \,\mathcal{R}\, y$. We NTS that $[x] = [y]$:

($\subseteq$) Let $\alpha \in [x]$, i.e., $\alpha \,\mathcal{R}\, x$. Then

$$\begin{aligned} \alpha \,\mathcal{R}\, x &\implies \alpha \,\mathcal{R}\, y && \because x \,\mathcal{R}\, y \text{ and } \mathcal{R} \text{ is transitive} \\ &\implies \alpha \in [y]. \end{aligned}$$

($\supseteq$) Let $\beta \in [y]$, i.e., $y \,\mathcal{R}\, \beta$. Then

$$\begin{aligned} y \,\mathcal{R}\, \beta &\implies x \,\mathcal{R}\, \beta && \because x \,\mathcal{R}\, y \text{ and } \mathcal{R} \text{ is transitive} \\ &\implies \beta \in [x]. \end{aligned}$$

($\Leftarrow$) Let $[x] = [y]$. Then

$$x \in S \overset{\text{by (1)}}{\implies} x \in [x] = [y] \implies x \in [y] \implies x \,\mathcal{R}\, y.$$

(3) ($\Rightarrow$) Let $x \not\mathcal{R}\, y$. Suppose that $[x] \cap [y] \neq \varnothing$ then $\exists \gamma \in S$ such that $\gamma \in [x] \cap [y]$. Then

$$\gamma \in [x] \cap [y] \implies \gamma \in [x] \wedge \gamma \in [y] \implies x \,\mathcal{R}\, \gamma \wedge \gamma \,\mathcal{R}\, y \implies x \,\mathcal{R}\, y \lightning.$$

($\Leftarrow$) Let $[x] \cap [y] = \varnothing$. Suppose that $x \,\mathcal{R}\, y$. By (1) and (2), we have $x \in [x] = [y] \lightning$.

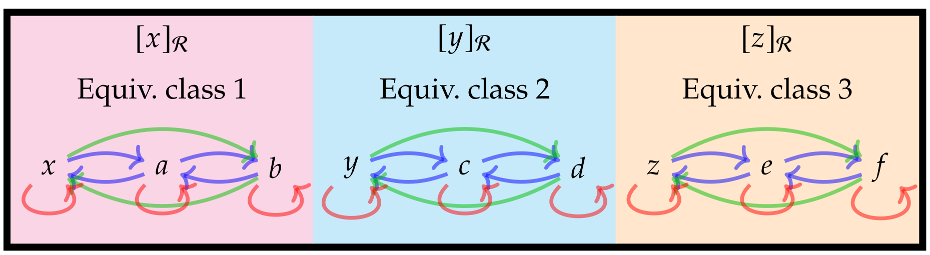
□

★ ★ ★ Fundamental Theorem on Equivalence Relations ★ ★ ★

Theorem 5. *Let S be a set and let $\mathcal{R}$ be an equivalence relation on S. Define the set of equivalence classes*

$$\mathcal{P} := \left\{[x]_{\mathcal{R}} : x \in S\right\}, \text{ where } [x]_{\mathcal{R}} = \left\{y \in S : x \,\mathcal{R}\, y\right\}.$$

Then $\mathcal{P}$ forms a partition of S.



$S = \left\{x, y, z, a, b, c, d, e, f\right\}$

$\mathcal{P} = \{[x]_{\mathcal{R}}, [y]_{\mathcal{R}}, [z]_{\mathcal{R}}\}$

Proof. We must show that the set of equivalence classes $\left\{[x]_{\mathcal{R}} : x \in S\right\}$ satisfies the three conditions of a partition:

(i) (Equivalence Class is not Empty) By (1) of **Lemma 4**, it is proved.

(ii) (Equivalence Classes are Disjoint) By (2) and (3) of **Lemma 4**, it is proved.

(iii) (Union of Equivalence Classes is Whole Set) We NTS that $\bigcup \left\{[x]_{\mathcal{R}} : x \in S\right\} = S$:

($\subseteq$) Since $[x]_{\mathcal{R}} \subseteq S$, we have

$$\bigcup \left\{[x]_{\mathcal{R}} : x \in S\right\} = \bigcup_{x \in S} [x]_{\mathcal{R}} \subseteq S.$$

($\supseteq$) Let $\alpha \in S$. We want to show that $\alpha \in \bigcup_{x \in S} [x]_{\mathcal{R}}$, i.e.,

$$\exists x \in S \text{ such that } \alpha \in [x].$$

By (1) of **Lemma 4**, we obtain $\alpha \in [\alpha]$. Thus, for every $\alpha \in S$, $\alpha \in \bigcup_{x \in S} [x]_{\mathcal{R}}$.

□

⋆ Relation Induced by Partition is Equivalence ⋆

Theorem 6. *Let S be a set and $\mathcal{P} = \langle P_i \rangle_{i \in I}$ be a partition of S. We define a relation $\mathcal{R}$ on S:*

$$x \sim_{\mathcal{R}} y \iff \exists i \in I \text{ such that } x, y \in P_i$$

for all $x, y \in S$. That is, x is related to y under $\mathcal{R}$ if and only if x and y belong to the same subset P_i in the partition. Then $\mathcal{R}$ is the equivalence relation induced by a partition $\mathcal{P}$.

Proof. Let $\langle P_i \rangle_{i \in I}$ be a partition of S. That is,

$$\text{(a) } P_i \neq \varnothing \text{ for all } i \in I; \qquad \text{(b) } P_i \cap P_j = \varnothing \text{ for } i \neq j; \qquad \text{(c) } \bigcup_{i \in I} P_i = S.$$

Let $x, y \in S$. Note that

$$\mathcal{R} := \left\{ (x, y) \in S \times S : \exists i \in I \text{ s.t. } x \in P_i \wedge y \in P_i \right\}.$$

We NTS that $\mathcal{R}$ is reflexive, symmetric and transitive:

(i) (Reflexivity) We NTS that $x \sim_{\mathcal{R}} x$:

$$x \in S \overset{\text{by (c)}}{\Longrightarrow} x \in \bigcup_{i \in I} P_i \implies \exists i \in I \text{ s.t. } x \in P_i \implies \exists i \in I \text{ s.t. } x \in P_i \wedge x \in P_i \implies x \sim_{\mathcal{R}} x.$$

(ii) (Symmetry) We NTS that $x \sim_{\mathcal{R}} y \implies y \sim_{\mathcal{R}} x$:

$$x \sim_{\mathcal{R}} y \implies \exists i \in I \text{ s.t. } x \in P_i \wedge y \in P_i \implies \exists i \in I \text{ s.t. } y \in P_i \wedge x \in P_i \implies y \sim_{\mathcal{R}} x.$$

(iii) (Transitivity) We NTS that $x \sim_{\mathcal{R}} y \wedge y \sim_{\mathcal{R}} z \implies x \sim_{\mathcal{R}} z$:

$$\begin{cases} x \sim_{\mathcal{R}} y \\ y \sim_{\mathcal{R}} z \end{cases} \implies \begin{cases} \exists i \in I \text{ s.t. } x \in P_i \wedge {\color{red} y \in P_i} \\ \exists j \in I \text{ s.t. } {\color{red} y \in P_j} \wedge z \in P_j \end{cases} \overset{\text{by (b), } i = j}{\Longrightarrow} \exists i = j \in I \text{ s.t. } x \in P_i \wedge z \in P_{j=i} \implies x \sim_{\mathcal{R}} z.$$

□

References

[1] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 3. 집합론 기초 (c)." YouTube Video, 35:04. Published September 07, 2019. URL: `https://www.youtube.com/watch?v=2gM-Vh8CY8I&t=1607s`.