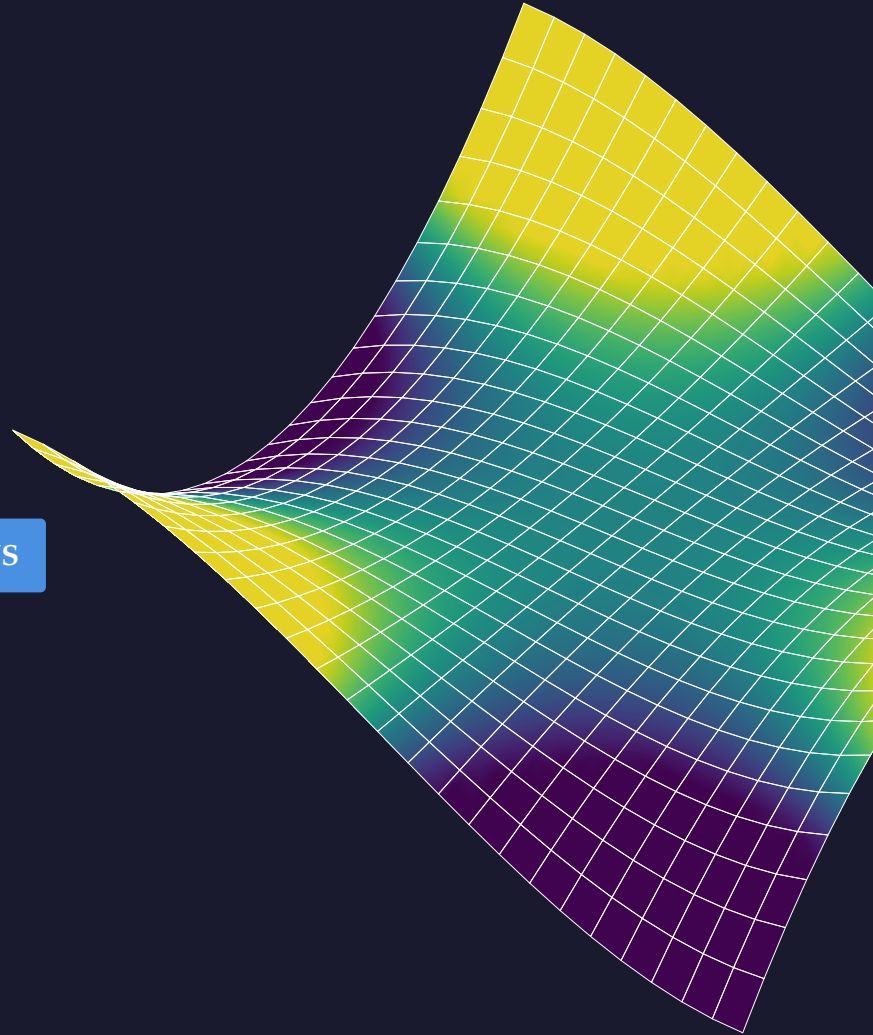


Riemann Surfaces and Algebraic Curves

A framework for understanding Elliptic Curves

Ji, Yonghyeon

PART I — MULTIVARIABLE CALCULUS



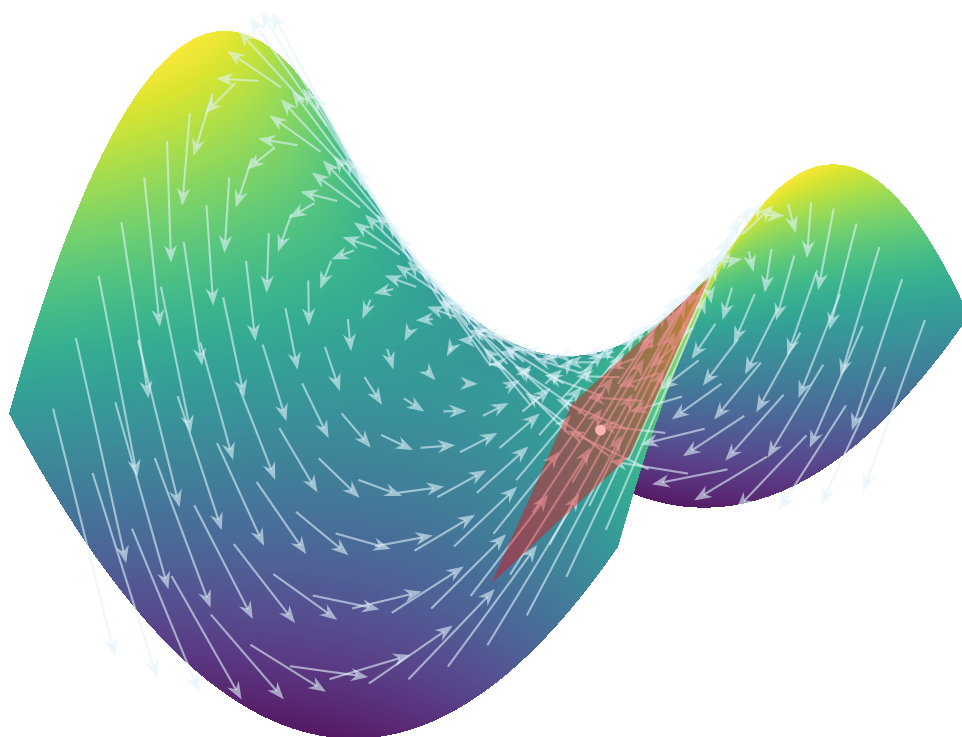
Riemann Surfaces and Algebraic Curves

A Framework for Understanding Elliptic Curves

Part I — Multivariable Calculus

Ji, Yonghyeon

February 11, 2026



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The FTC hierarchy

Name	Formula
FTC I (Accumulation)	$\frac{d}{dx} \left(\int_a^x f(t) dt \right) = f(x).$
FTC II (Evaluation)	$\int_a^b f'(x) dx = f(b) - f(a).$
Fundamental Theorem of Line Integrals	$\int_C \nabla \phi \cdot d\mathbf{r} = \phi(B) - \phi(A).$
Green's Theorem	$\oint_{\partial R} P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$
Stokes' Theorem (3D)	$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS.$
Divergence Theorem	$\iint_{\partial V} \mathbf{F} \cdot \mathbf{n} dS = \iiint_V (\nabla \cdot \mathbf{F}) dV.$
Generalized Stokes	$\int_{\partial \Omega} \omega = \int_{\Omega} d\omega.$

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Changelog

v1.0 2025-12-29 Initial release.

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1 Fundamental Theorem of Calculus

Fundamental Theorem for Gradient Fields

If $\mathbf{F} = \nabla f$ is a conservative vector field and C is a smooth curve from A to B , then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Green's Theorem

For a positively oriented, simple closed curve C bounding a region R in the plane,

$$\oint_C P \, dx + Q \, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Divergence Theorem

Let \mathbf{F} be a vector field defined on a region E with closed boundary surface S (outward-oriented). Then

$$\iiint_E \nabla \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$

Stokes' Theorem

Let S be an oriented surface with boundary curve C , and let \mathbf{F} be a vector field. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS.$$

Triple Integral

To integrate a scalar function $f(x, y, z)$ over a region E in \mathbb{R}^3 ,

$$\iiint_E f(x, y, z) \, dV.$$

1.1 Gradient Vector Fields

Scalar field

Definition 1.1. Let $U \subseteq \mathbb{R}^n$ be an open set. A scalar field on U is a function

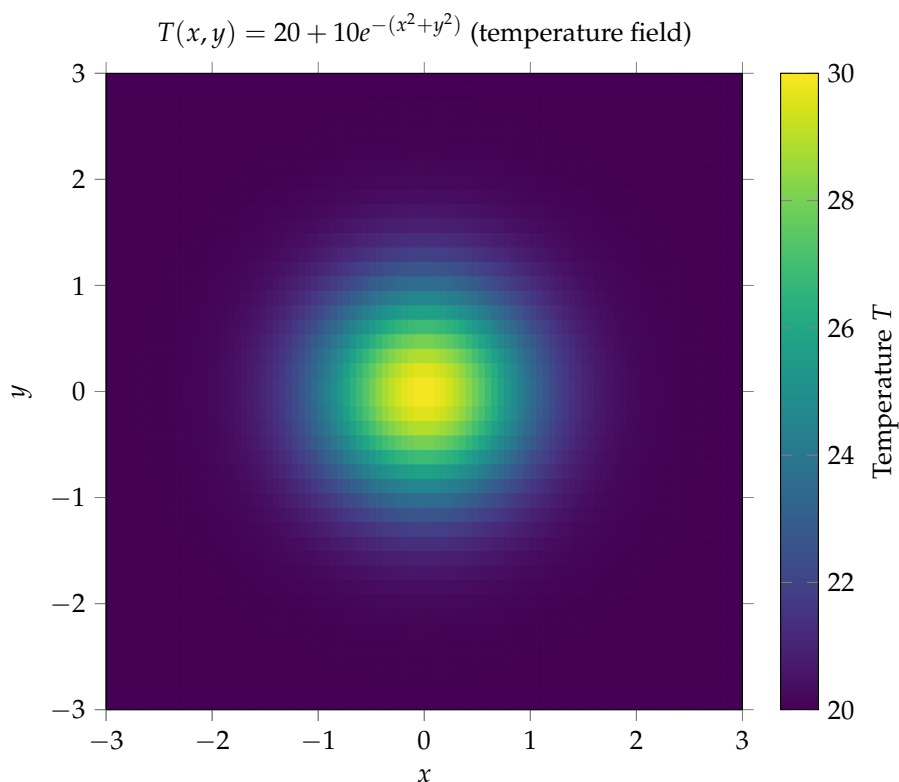
$$\begin{aligned} f &: U \longrightarrow \mathbb{R} \\ \mathbf{x} &\longmapsto f(\mathbf{x}) . \end{aligned}$$

Equivalently, to each point $\mathbf{x} \in U$ the scalar field assigns a real number $f(\mathbf{x})$.

Example 1 (Temperature distribution). Let $U = \mathbb{R}^2$. Define

$$T(x, y) = 20 + 10e^{-(x^2+y^2)}.$$

Then $T : U \rightarrow \mathbb{R}$ is a scalar field. One may interpret $T(x, y)$ as the temperature (in degrees) at the point (x, y) . Notice that $T(0, 0) = 30$ and $T(x, y) \rightarrow 20$ as $x^2 + y^2 \rightarrow \infty$, so the temperature is highest at the origin and decays outward.



Vector field

Definition 1.2. Let $U \subseteq \mathbb{R}^n$ be an open set. A vector field on U is a function

$$\begin{aligned} \mathbf{F} &: U \longrightarrow \mathbb{R}^n \\ \mathbf{x} &\longmapsto \mathbf{F}(\mathbf{x}) \end{aligned}$$

Equivalently, to each point $\mathbf{x} \in U$ the vector field assigns a vector $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^n$. In coordinates, one often writes

$$\mathbf{F}(\mathbf{x}) = \langle F_1(\mathbf{x}), F_2(\mathbf{x}), \dots, F_n(\mathbf{x}) \rangle,$$

where each component function $F_i : U \rightarrow \mathbb{R}$ is a scalar field on U .

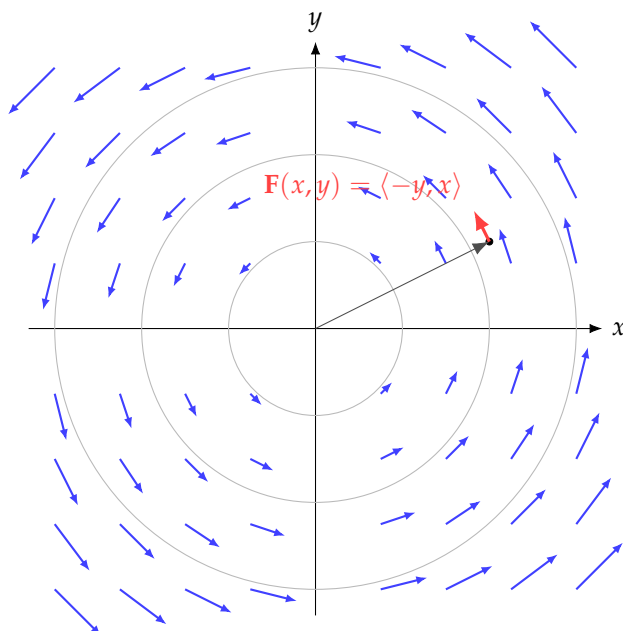
Example 2 (Rotation field). Let $U = \mathbb{R}^2$. Define

$$\mathbf{F}(x, y) = \langle -y, x \rangle.$$

Then $\mathbf{F} : U \rightarrow \mathbb{R}^2$ is a vector field. At each point (x, y) it assigns the vector $(-y, x)$, which is perpendicular to $\langle x, y \rangle$ and tangent to the circle $x^2 + y^2 = \text{constant}$. Moreover,

$$\|\mathbf{F}(x, y)\| = \sqrt{x^2 + y^2},$$

so the magnitude increases linearly with the distance from the origin. This field models a rigid counterclockwise rotational flow about the origin.



Remark.

Definition 1.3 (Conservative vector field). Let $U \subseteq \mathbb{R}^n$ be an open set and let $\mathbf{F} : U \rightarrow \mathbb{R}^n$ be a vector field. We say that \mathbf{F} is conservative on U if there exists a scalar field $f : U \rightarrow \mathbb{R}$ of class C^1 such that

$$\mathbf{F} = \nabla f \quad \text{on } U.$$

In this case, f is called a potential function for \mathbf{F} .

Remark (Equivalent characterization). A vector field \mathbf{F} on U is conservative if and only if for every piecewise C^1 curve C in U with endpoints \mathbf{A}, \mathbf{B} , the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on \mathbf{A} and \mathbf{B} (path independence). Equivalently,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

for every piecewise C^1 closed curve C in U .

Definition 1.4 (Gradient operator). Let $U \subseteq \mathbb{R}^n$ be open. The gradient operator (or nabla operator) is the map

$$\nabla : C^1(U) \longrightarrow C^0(U, \mathbb{R}^n)$$

defined by

$$(\nabla f)(\mathbf{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix}, \quad \mathbf{x} = (x_1, \dots, x_n) \in U.$$

Remark (Directional derivative characterization). For $f \in C^1(U)$ and $\mathbf{x} \in U$, the vector $\nabla f(\mathbf{x}) \in \mathbb{R}^n$ is uniquely characterized by the property that

$$D_{\mathbf{v}}f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{v} \quad \text{for all } \mathbf{v} \in \mathbb{R}^n,$$

where $D_{\mathbf{v}}f(\mathbf{x})$ denotes the directional derivative of f at \mathbf{x} in the direction \mathbf{v} .

[Linearity of ∇] Let $f, g \in C^1(U)$ and $a, b \in \mathbb{R}$. Then

$$\nabla(af + bg) = a \nabla f + b \nabla g.$$

Remark (Jacobian/transpose viewpoint). For $f \in C^1(U)$, the total derivative at $\mathbf{x} \in U$ is the linear map

$$Df(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}, \quad Df(\mathbf{x}) \mathbf{v} = D_{\mathbf{v}}f(\mathbf{x}).$$

In coordinates, $Df(\mathbf{x})$ is represented by the $1 \times n$ Jacobian row matrix

$$Df(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}) \quad \cdots \quad \frac{\partial f}{\partial x_n}(\mathbf{x}) \right),$$

and the gradient is its transpose:

$$\nabla f(\mathbf{x}) = \left(Df(\mathbf{x}) \right)^{\top}.$$

Consequently, for any $\mathbf{v} \in \mathbb{R}^n$,

$$D_{\mathbf{v}}f(\mathbf{x}) = Df(\mathbf{x}) \mathbf{v} = \left(\nabla f(\mathbf{x}) \right)^{\top} \mathbf{v} = \nabla f(\mathbf{x}) \cdot \mathbf{v}.$$

Fundamental Theorem for Gradient Fields

Theorem 1.5. Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \rightarrow \mathbb{R}$ be continuously differentiable ($f \in C^1(U)$). Let C be a piecewise C^1 curve in U with a piecewise C^1 parametrization

$$\mathbf{r} : [a, b] \rightarrow U.$$

If $\mathbf{r}(a) = \mathbf{A}$ and $\mathbf{r}(b) = \mathbf{B}$, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{B}) - f(\mathbf{A}).$$

In particular, the line integral of a gradient field depends only on the endpoints of the curve.

Proof. By definition of the line integral of a vector field along a parametrized curve,

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

Define a scalar-valued function $g : [a, b] \rightarrow \mathbb{R}$ by $g(t) = f(\mathbf{r}(t))$. Since $f \in C^1(U)$ and \mathbf{r} is piecewise C^1 , the composition g is piecewise C^1 . On any subinterval where \mathbf{r} is C^1 , the multivariable chain rule gives

$$g'(t) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

Hence on each such subinterval we have $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = g'(t)$, and summing over the finitely many smooth pieces yields

$$\int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_a^b g'(t) dt.$$

By the (single-variable) Fundamental Theorem of Calculus,

$$\int_a^b g'(t) dt = g(b) - g(a).$$

Substituting back $g(t) = f(\mathbf{r}(t))$ gives

$$g(b) - g(a) = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(\mathbf{B}) - f(\mathbf{A}).$$

Therefore,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{B}) - f(\mathbf{A}),$$

as claimed. □

Consider a small rectangle centered at (x_0, y_0) with side lengths $\Delta x, \Delta y$.

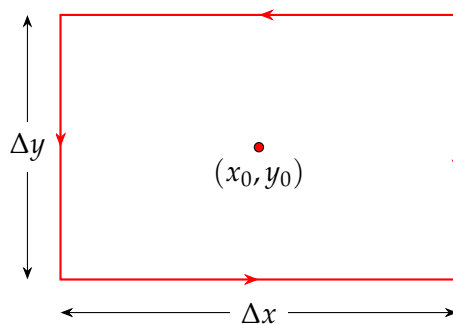


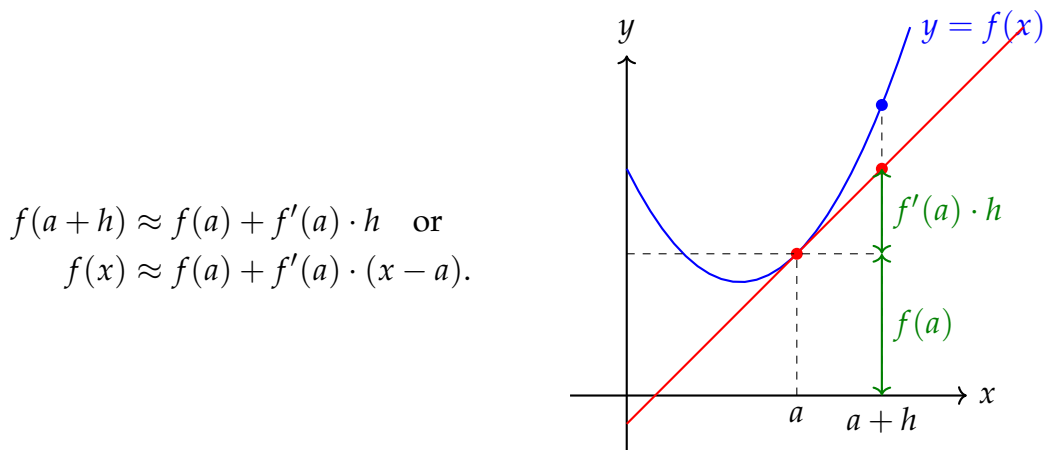
Figure 1: Circulation around an infinitesimal rectangle.

The total counterclockwise circulation is the sum of the line integrals along the four edges:

$$\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \int_{\text{bottom}} P dx + \int_{\text{right}} Q dy + \int_{\text{top}} P dx + \int_{\text{left}} Q dy.$$

We will approximate the value of P or Q along each edge as being constant, equal to its value at the midpoint of that edge. We find this value using a first-order Taylor expansion from the center point (x_0, y_0) .

For a simple function of one variable, $f(x)$, if we know its value at a point a , then we can estimate its value at a nearby point $a + h$ using the tangent line at a :



In words, “New Value \approx Old Value + (Rate of Change) \times (Small Step)”.

For a function of two variables like $P(x, y)$, the idea is identical, but the “rate of change” now has two components (one for each direction), and the “tangent line” becomes a “tangent plane”. The general first-order Taylor expansion for $P(x, y)$ around a center point (x_0, y_0) is

$$P(x_0 + a, y_0 + b) \approx P(x_0, y_0) + \frac{\partial P}{\partial x}(x_0, y_0) \cdot a + \frac{\partial P}{\partial y}(x_0, y_0) \cdot b$$

Here, a is the small step in the x -direction, and b is the small step in the y -direction.

1. The Horizontal Paths These integrals involve the horizontal component of $P(x, y)$.

• **Bottom Path (\rightarrow):**

$$\begin{aligned} P\left(x, y_0 - \frac{\Delta y}{2}\right) &\approx P(x_0, y_0) - \frac{\partial P}{\partial y}(x_0, y_0) \frac{\Delta y}{2} \\ \Rightarrow \int_{\text{bottom}} P \, dx &\approx \int_{-\Delta x/2}^{\Delta x/2} \left(P_0 + P_x s - P_y \frac{\Delta y}{2}\right) ds \quad (x = x_0 + s, \, dx = ds) \\ \Rightarrow \int_{\text{bottom}} P \, dx &\approx \left(P(x_0, y_0) - \frac{\partial P}{\partial y}(x_0, y_0) \frac{\Delta y}{2}\right) (\Delta x) \end{aligned}$$

Note that

$$\int_{-\Delta x/2}^{\Delta x/2} P_x s \, ds = P_x \left[\frac{s^2}{2} \right]_{-\Delta x/2}^{\Delta x/2} = P_x \left(\frac{(\Delta x/2)^2}{2} - \frac{(\Delta x/2)^2}{2} \right) = 0.$$

• **Top Path (\leftarrow):**

$$P\left(x, y_0 + \frac{\Delta y}{2}\right) \approx P(x_0, y_0) + \frac{\partial P}{\partial y} \frac{\Delta y}{2} \Rightarrow \int_{\text{top}} P \, dx \approx - \left(P(x_0, y_0) + \frac{\partial P}{\partial y} \frac{\Delta y}{2}\right) (\Delta x)$$

Here, we are left with only the parts that describe the *change* in P with respect to y .

$$\int_{\text{bottom}} P \, dx + \int_{\text{top}} P \, dx \approx \left(-\frac{\partial P}{\partial y} \frac{\Delta y}{2}\right) \Delta x - \left(\frac{\partial P}{\partial y} \frac{\Delta y}{2}\right) \Delta x = -\frac{\partial P}{\partial y} \Delta x \Delta y$$

2. The Vertical Paths These integrals involve the vertical component of $Q(x, y)$.

• **Right Path (\uparrow):**

$$Q\left(x_0 + \frac{\Delta x}{2}, y\right) \approx Q(x_0, y_0) + \frac{\partial Q}{\partial x} \frac{\Delta x}{2} \Rightarrow \int_{\text{right}} Q \, dy \approx \left(Q(x_0, y_0) + \frac{\partial Q}{\partial x} \frac{\Delta x}{2}\right) (\Delta y)$$

• **Left Path (\downarrow):**

$$Q\left(x_0 - \frac{\Delta x}{2}, y\right) \approx Q(x_0, y_0) - \frac{\partial Q}{\partial x} \frac{\Delta x}{2} \Rightarrow \int_{\text{left}} Q \, dy \approx - \left(Q(x_0, y_0) - \frac{\partial Q}{\partial x} \frac{\Delta x}{2}\right) (\Delta y)$$

Here, we are left with only the parts that describe the *change* in Q with respect to x .

$$\int_{\text{right}} Q \, dy + \int_{\text{left}} Q \, dy \approx \left(\frac{\partial Q}{\partial x} \frac{\Delta x}{2} \right) \Delta y + \left(\frac{\partial Q}{\partial x} \frac{\Delta x}{2} \right) \Delta y = \frac{\partial Q}{\partial x} \Delta x \Delta y$$

Now we sum the results from the horizontal and vertical pairs:

$$\begin{aligned} \oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} &\approx \left(-\frac{\partial P}{\partial y} \Delta x \Delta y \right) + \left(\frac{\partial Q}{\partial x} \Delta x \Delta y \right) \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \Delta x \Delta y \end{aligned}$$

This shows that the total circulation around the tiny loop is approximately the quantity $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$ multiplied by the area of the loop ($\Delta A = \Delta x \Delta y$).

To get the property *at the point* (x_0, y_0) , we find the circulation **density**. We divide by the area and take the limit as the rectangle shrinks to zero.

$$\lim_{\Delta A \rightarrow 0} \frac{1}{\Delta A} \oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \frac{\partial Q}{\partial x}(x_0, y_0) - \frac{\partial P}{\partial y}(x_0, y_0)$$

This is why we call the scalar quantity $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ the **curl**: it is the circulation per unit area at a point, which measures the local rotational tendency of the field.

If $C = \partial D$ is a positively oriented simple closed curve enclosing a region D , Green's theorem states

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\substack{\text{Line Integral} \\ \text{(Total Circulation)}}} = \underbrace{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA}_{\substack{\text{Double Integral} \\ \text{(Sum of Local Curls)}}$$

1. Let $\mathbf{F} = \langle 2x, 2y \rangle$. Show that \mathbf{F} is conservative and compute

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is any path from $(0,0)$ to $(1,1)$.

Sol. Let $\mathbf{F} = \langle P, Q \rangle = \langle 2x, 2y \rangle$. Since

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2x) = 0 \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(2y) = 0,$$

we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere in \mathbb{R}^2 , a simply connected domain. Therefore \mathbf{F} is conservative.

To find a potential function f with $\nabla f = \mathbf{F}$, we solve

$$f_x = 2x \quad \Rightarrow \quad f(x, y) = \int 2x \, dx = x^2 + g(y),$$

for some function $g(y)$. Then

$$f_y = g'(y) = 2y \quad \Rightarrow \quad g(y) = y^2 + C.$$

Hence a potential function is

$$f(x, y) = x^2 + y^2 \quad (\text{constant irrelevant}).$$

By the Fundamental Theorem for Line Integrals, for any path C from $(0,0)$ to $(1,1)$,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1) - f(0,0) = (1^2 + 1^2) - (0^2 + 0^2) = 2.$$

□

2. Determine whether the vector field $\mathbf{F} = \langle y, x \rangle$ is conservative. If so, find a potential function.

Sol. Let $\mathbf{F} = \langle P, Q \rangle = \langle y, x \rangle$. Compute the mixed partials:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y) = 1, \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x) = 1.$$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere in \mathbb{R}^2 (a simply connected domain), \mathbf{F} is conservative.

To find a potential function f such that $\nabla f = \mathbf{F}$, we solve

$$f_x = P = y.$$

Integrating with respect to x gives

$$f(x, y) = \int y \, dx = xy + g(y),$$

where g is a function of y only. Differentiate with respect to y :

$$f_y = x + g'(y).$$

But f_y must equal $Q = x$, so $g'(y) = 0$, hence $g(y) = C$.

Therefore, a potential function is

$$f(x, y) = xy \quad (\text{up to an additive constant}).$$

Let $\mathbf{F} = \langle P, Q \rangle = \langle y, x \rangle$ on an open set $U \subseteq \mathbb{R}^2$. Since $P, Q \in C^1(U)$, \mathbf{F} is conservative on U if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on U .

Compute:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y) = 1, \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x) = 1.$$

Thus $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere on U , so \mathbf{F} is conservative.

To find a potential function f with $\nabla f = \mathbf{F}$, we require

$$f_x = y, \quad f_y = x.$$

Integrate $f_x = y$ with respect to x :

$$f(x, y) = xy + g(y),$$

for some function g of y alone. Differentiate with respect to y :

$$f_y(x, y) = x + g'(y).$$

Set this equal to x (since $f_y = x$):

$$x + g'(y) = x \Rightarrow g'(y) = 0 \Rightarrow g(y) = C.$$

Therefore a potential function is

$$f(x, y) = xy + C.$$

(Any two potential functions differ by an additive constant.)

□

Theorem 1.6 (Curl test for conservativeness in \mathbb{R}^2). *Let $U \subseteq \mathbb{R}^2$ be an open and simply connected set, and let*

$$\mathbf{F} = \langle P, Q \rangle : U \rightarrow \mathbb{R}^2$$

be a C^1 vector field. Then the following are equivalent:

- (a) \mathbf{F} is conservative on U , i.e. there exists a C^1 function $f : U \rightarrow \mathbb{R}$ such that $\nabla f = \mathbf{F}$.
- (b) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere on U .

Proof. (1) \Rightarrow (2). Assume \mathbf{F} is conservative, so $\mathbf{F} = \nabla f$ for some $f \in C^1(U)$. In coordinates,

$$P = f_x, \quad Q = f_y.$$

If moreover $f \in C^2(U)$ (which holds, for instance, if $P, Q \in C^1$ and f is constructed as in the converse direction), then

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(f_x) = f_{xy}, \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(f_y) = f_{yx}.$$

By Clairaut's theorem (equality of mixed partials for C^2 functions), $f_{xy} = f_{yx}$, hence

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{on } U.$$

(2) \Rightarrow (1). Assume $P_y = Q_x$ on U . Let C be any positively oriented, piecewise C^1 , simple closed curve in U bounding a region $D \subseteq U$. By Green's Theorem,

$$\oint_C P dx + Q dy = \iint_D (Q_x - P_y) dA.$$

Since $Q_x - P_y = 0$ on U , it follows that

$$\oint_C P dx + Q dy = 0$$

for every such curve C .

Path independence. Fix two piecewise C^1 curves C_1 and C_2 in U with the same endpoints A and B . Consider the closed curve $C = C_1 \cup (-C_2)$ obtained by traversing C_1 from A to B and then C_2 from B back to A . Because U is simply connected, this closed curve can be decomposed into finitely many simple closed curves each bounding a region contained in U . Since the integral around each such simple closed curve is 0, additivity of line integrals yields

$$\oint_C P dx + Q dy = 0.$$

Therefore,

$$\int_{C_1} P dx + Q dy = \int_{C_2} P dx + Q dy,$$

so the line integral depends only on endpoints (path independence).

Construction of a potential. Fix a base point $A_0 \in U$. Define $f : U \rightarrow \mathbb{R}$ by

$$f(x, y) = \int_{C_{A_0 \rightarrow (x, y)}} P dx + Q dy,$$

where $C_{A_0 \rightarrow (x, y)}$ is any piecewise C^1 curve in U from A_0 to (x, y) . By path independence, f is well-defined.

Verification that $\nabla f = \mathbf{F}$. Let $(x, y) \in U$ and choose h small so that the horizontal segment from (x, y) to $(x + h, y)$ lies in U . Using the definition of f and path independence,

$$f(x + h, y) - f(x, y) = \int_x^{x+h} P(s, y) ds.$$

Divide by h and let $h \rightarrow 0$ to obtain

$$f_x(x, y) = P(x, y).$$

Similarly, using a vertical segment,

$$f(x, y + h) - f(x, y) = \int_y^{y+h} Q(x, t) dt,$$

so

$$f_y(x, y) = Q(x, y).$$

Hence $\nabla f = \langle f_x, f_y \rangle = \langle P, Q \rangle = \mathbf{F}$, and \mathbf{F} is conservative on U . □

3. Let $f(x, y, z) = xyz$. Compute ∇f and evaluate the line integral of ∇f over the path from $(1, 0, 0)$ to $(1, 2, 3)$.

Sol. Given $f(x, y, z) = xyz$, its gradient is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle yz, xz, xy \rangle.$$

To evaluate the line integral of ∇f over any path C from $(1, 0, 0)$ to $(1, 2, 3)$, use the Fundamental Theorem for Line Integrals:

$$\int_C \nabla f \cdot d\mathbf{r} = f(1, 2, 3) - f(1, 0, 0).$$

Compute the endpoint values:

$$f(1, 2, 3) = (1)(2)(3) = 6, \quad f(1, 0, 0) = (1)(0)(0) = 0.$$

Hence,

$$\int_C \nabla f \cdot d\mathbf{r} = 6 - 0 = 6.$$

□

4. Let $\mathbf{F} = \nabla f$ for $f(x, y) = x^2 + y^2$. Compute the line integral over a circular path from $(1, 0)$ to $(0, 1)$ and explain the result.

Sol. Given $f(x, y) = x^2 + y^2$, we have

$$\mathbf{F} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2x, 2y \rangle.$$

Since \mathbf{F} is a gradient field, it is conservative. Therefore, by the Fundamental Theorem for Line Integrals, for any smooth path C from $(1, 0)$ to $(0, 1)$ (including a circular arc),

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0, 1) - f(1, 0).$$

Compute:

$$f(0, 1) = 0^2 + 1^2 = 1, \quad f(1, 0) = 1^2 + 0^2 = 1.$$

Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 1 - 1 = 0.$$

Explanation. The line integral depends only on the endpoints because $\mathbf{F} = \nabla f$ is conservative. Here both endpoints lie on the same level curve of f (indeed, on the circle $x^2 + y^2 = 1$), so f has the same value at $(1, 0)$ and $(0, 1)$; thus the net change in potential is zero, and the work done by \mathbf{F} along the circular path is 0. \square

1.2 Green's Theorem

1. Use Green's Theorem to evaluate

$$\oint_C x \, dy - y \, dx$$

where C is the unit circle oriented counterclockwise.

Sol. Write the integral in Green's Theorem form:

$$\oint_C P \, dx + Q \, dy,$$

where here $P(x, y) = -y$ and $Q(x, y) = x$, since

$$x \, dy - y \, dx = (-y) \, dx + x \, dy.$$

By Green's Theorem (with C positively oriented),

$$\oint_C P \, dx + Q \, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

where D is the unit disk. Compute:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x) = 1, \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(-y) = -1.$$

Thus

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2.$$

Therefore,

$$\oint_C x \, dy - y \, dx = \iint_D 2 \, dA = 2 \cdot \text{Area}(D) = 2 \cdot \pi(1)^2 = 2\pi.$$

□

2. Let $\mathbf{F} = \langle y^2, 2xy \rangle$. Use Green's Theorem to evaluate the line integral around the boundary of the square $[0, 1] \times [0, 1]$.

Sol. Let $\mathbf{F} = \langle P, Q \rangle = \langle y^2, 2xy \rangle$, and let C be the positively oriented (counterclockwise) boundary of the square

$$D = [0, 1] \times [0, 1].$$

By Green's Theorem,

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y, \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y^2) = 2y.$$

Hence,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y - 2y = 0.$$

Therefore,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy = \iint_D 0 dA = 0.$$

□

3. Evaluate

$$\oint_C (x + y)dx + (x - y)dy$$

where C is the triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ oriented counterclockwise.

Sol. Write the line integral as $\oint_C P dx + Q dy$ with

$$P(x, y) = x + y, \quad Q(x, y) = x - y.$$

Since C is positively oriented (counterclockwise), Green's Theorem gives

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

where D is the triangular region with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$.

Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x - y) = 1, \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x + y) = 1.$$

Thus,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 1 = 0.$$

Therefore,

$$\oint_C (x + y) dx + (x - y) dy = \iint_D 0 dA = 0.$$

□

4. Determine if

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

for $\mathbf{F} = \langle y, -x \rangle$ around a circle of radius r centered at the origin.

Sol. Let $\mathbf{F} = \langle P, Q \rangle = \langle y, -x \rangle$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy = \oint_C y dx - x dy,$$

where C is the circle of radius r centered at the origin, oriented counterclockwise.

Using Green's Theorem,

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

where D is the disk of radius r . Compute

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(-x) = -1, \quad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(y) = 1.$$

Hence

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -1 - 1 = -2.$$

Therefore,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (-2) dA = -2 \text{Area}(D) = -2 \cdot \pi r^2 = -2\pi r^2.$$

So the integral is *not* zero (except in the degenerate case $r = 0$). For clockwise orientation, the value would be $+2\pi r^2$. \square

1.3 Divergence Theorem

1. Let $\mathbf{F} = \langle x, y, z \rangle$. Use the Divergence Theorem to compute the flux across the surface of the unit sphere.

Sol. content...

□

2. Let $\mathbf{F} = \langle x^2, y^2, z^2 \rangle$. Compute both the divergence and the surface integral over the unit cube $[0, 1]^3$.

Sol. content...



3. Use the Divergence Theorem to find the outward flux of $\mathbf{F} = \langle yz, xz, xy \rangle$ through the unit cube.

Sol. content...



4. Let $\mathbf{F} = \langle x, -y, z \rangle$. Verify the Divergence Theorem on the upper hemisphere of radius 1 centered at the origin.

Sol. content...



1.4 Stokes' Theorem

1. Let $\mathbf{F} = \langle -y, x, 0 \rangle$. Use Stokes' Theorem to compute the circulation around the boundary of the disk $x^2 + y^2 \leq 1$ in the xy -plane.

Sol. content...

□

2. Let $\mathbf{F} = \langle z, 0, x \rangle$. Use Stokes' Theorem on the triangular surface with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$.

Sol. content...



3. Compute both sides of Stokes' Theorem for $\mathbf{F} = \langle y, z, x \rangle$ on the surface $z = 0$ bounded by the unit circle.

Sol. content...



4. Use Stokes' Theorem to show that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

if \mathbf{F} is the gradient of some scalar field f .

Sol. content...

□

2 Differential Forms

TBA

3 Winding Numbers and Complexification

TBA