Riemann; Complex Analysis

- HW1 -

Ji, Yong-hyeon

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We cover the following topics in this note.

- Vector Fields
- Line Integrals for Vector Fields
- Surface Integrals for Vector Fields
- TBA

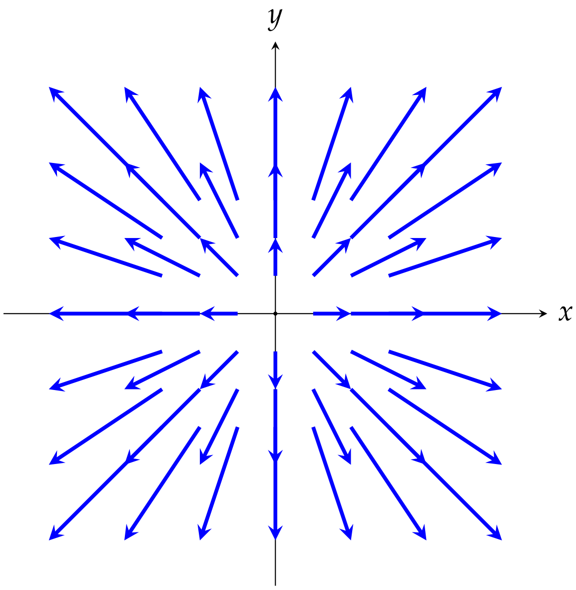
Contents

Scalar Function and Vector Fields

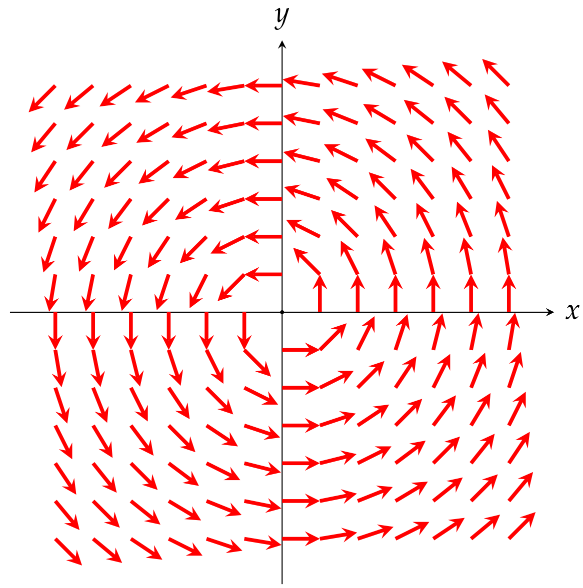
A **scalar function** on $\mathbb{R}^n$ is a real-valued function of an n-tuple; that is,

$$f : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n).$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $f(\mathbf{x}) \in \mathbb{R}$.



The radial field $\mathbf{F} = \langle x, y \rangle = x\mathbf{i} + y\mathbf{j}$



The spin field $\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)^{1/2}$

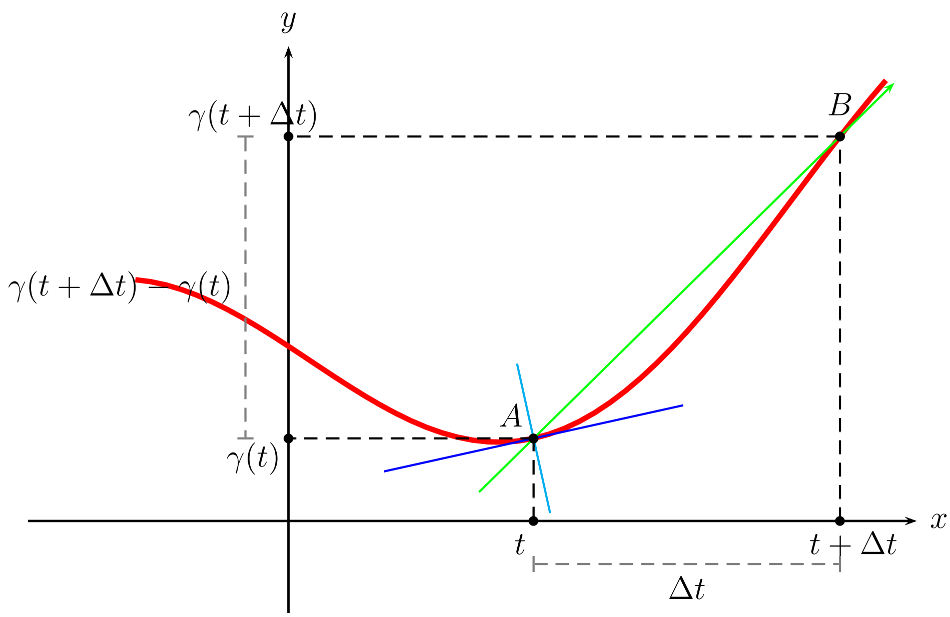
A **vector field** on $\mathbb{R}^n$ is a function

$$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n, \quad \mathbf{x} \mapsto \mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}), \ldots, F_n(\mathbf{x})),$$

where each component $F_i : \mathbb{R}^n \to \mathbb{R}$ is itself a scalar function.

Line Integrals

Line Integral of Scalar Function over Arc Length



For a curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2\colon t \mapsto \langle x(t), y(t)\rangle$, the **secant vector** over $[t, t+\Delta t]$ is

$$\frac{\Delta\gamma}{\Delta t} = \frac{\gamma(t+\Delta t)-\gamma(t)}{\Delta t} = \left\langle \frac{x(t+\Delta t)-x(t)}{\Delta t}, \frac{y(t+\Delta t)-y(t)}{\Delta t} \right\rangle.$$

As $\Delta t \to 0$, these secants converge (if γ is smooth) to

$$\begin{aligned}
\gamma'(t) = \frac{d}{dt}\gamma(t) = \lim_{\Delta t\to 0}\frac{\Delta\gamma}{\Delta t} &= \left\langle \lim_{\Delta t\to 0}\frac{x(t+\Delta t)-x(t)}{\Delta t}, \lim_{\Delta t\to 0}\frac{y(t+\Delta t)-y(t)}{\Delta t} \right\rangle \\
&= \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle \\
&= \langle x'(t), y'(t)\rangle,
\end{aligned}$$

which gives the **tangent vector** at $\gamma(t)$. The tangent vector captures how the curve is moving instantaneously at time t.

By Pythagoras' theorem, the **length moved per unit time** is $\|\gamma'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$, and so the small arc length traveled between t and $t+\Delta t$ is approximately:

$$\|\gamma'(t)\|\Delta t = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\cdot\Delta t.$$

Arc Length of a Parametrized Curve

Definition. Let $C \subset \mathbb{R}^n$ be a piecewise smooth curve, given by a smooth parameterization:

$$\gamma : [a,b] \to \mathbb{R}^n, \quad t \mapsto \gamma(t) = \langle x_1(t), x_2(t), \ldots, x_n(t) \rangle.$$

Then the **arc length** s of the curve C from $t = a$ to $t = b$ is defined by

$$\boxed{s := \int_a^b \|\gamma'(t)\|\ dt}, \quad \text{where } \|\gamma'(t)\| = \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \cdots + \left(\frac{dx_n}{dt}\right)^2}.$$

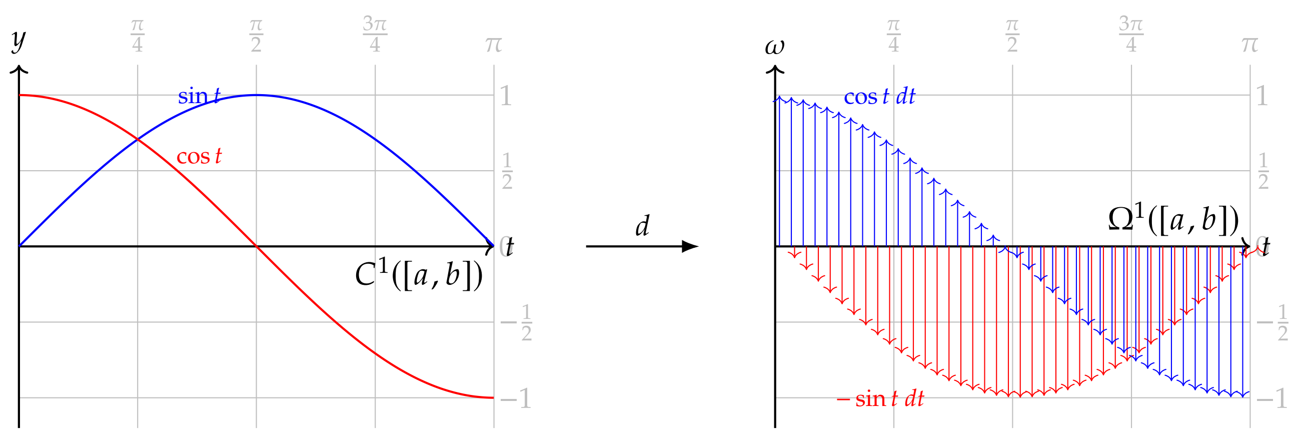
Remark. Let $\gamma : [a,b] \to \mathbb{R}^n$ be a piecewise-C^1 curve, $\gamma(t) = (x_1(t), \ldots x_n(t))$. A arc length function is defined by

$$s : [a,b] \to \mathbb{R}, \quad t \mapsto s(t) = \int_a^t \|\gamma'(u)\|\ du,$$

where $\|\gamma'(u)\| = \sqrt{\sum_{i=1}^n (x_i'(u))^2}$. Define two sets:

$$C^1([a,b]) = \left\{ f \in \mathbb{R}^{[a,b]} : f \text{ is continuously differentiable on } [a,b] \right\}$$
$$\Omega^1([a,b]) = \left\{ \delta(t)\ dt : \delta \in \mathbb{R}^{[a,b]} \text{ is continuous and } t \in [a,b] \right\} = \left\{ \delta(t)\ dt : \delta \in C^0([a,b]) \right\}.$$

Here $s \in C^1([a,b])$ with $s'(t) = \dfrac{d}{dt}\left(\displaystyle\int_a^t \|\gamma'(u)\|\ du \right) \overset{\text{FTC}}{=} \|\gamma'(t)\|$.

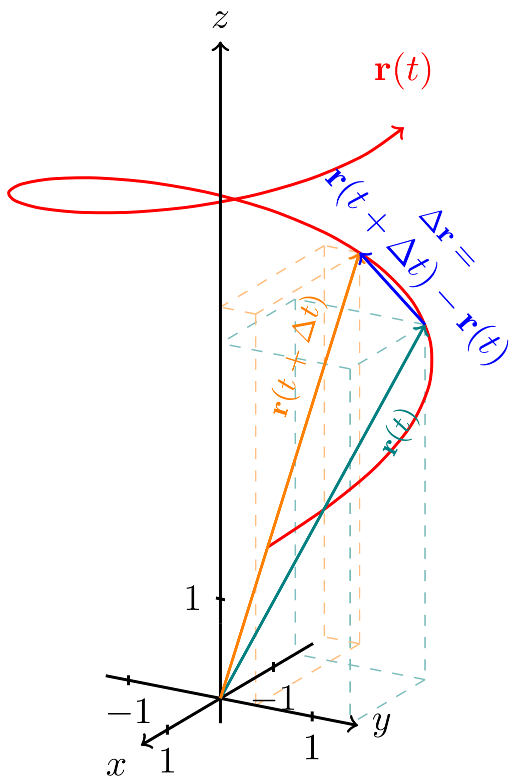


The map

$$\begin{array}{rcccl} d & : & C^1([a,b]) & \longrightarrow & \Omega^1([a,b]) \\ & & f(t) & \longmapsto & d(f(t)) = df \end{array}$$

is defined by $df = f'(t)dt$, where f' is the derivative of f. Thus

$$ds := d(s(t)) = s'(t)\ dt = \|\gamma'(t)\|\ dt.$$



$$\begin{array}{rcccl} \mathbf{r} & : & \mathbb{R} & \longrightarrow & \mathbb{R}^3 \\ & & t & \longmapsto & \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \end{array}$$

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \langle x'(t), y'(t), z'(t) \rangle$$

$$s(t) = \int_a^t \|\mathbf{r}'(t)\| \ dt$$

$$s'(t) = \|\mathbf{r}'(t)\| = \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2 + \big(z'(t)\big)^2}$$

$$ds = d(s(t)) = s'(t)\ dt = \|\mathbf{r}'(t)\| \ dt$$

Line Integral of Scalar Function over Arc Length

Definition. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar function, and let C be a piecewise smooth curve in $\mathbb{R}^n$ given by a smooth parameterization:

$$\gamma : [a, b] \to \mathbb{R}^n, \quad t \mapsto \gamma(t) = \langle x_1(t), x_2(t), \ldots, x_n(t) \rangle \in \mathbb{R}^n = \mathrm{dom}(f).$$

The **line integral of the scalar function** f along the curve C with respect to arc length is defined by

$$\int_C f \ ds := \int_a^b f(\gamma(t)) \, \|\gamma'(t)\| \ dt.$$

Line Integral of Vector Fields

Line Integral of a Vector Field in $\mathbb{R}^2$

Definition. Let C be a smooth curve parametrized by

$$\gamma : [a,b] \to \mathbb{R}^2, \quad t \mapsto \gamma(t) = \langle x(t), y(t) \rangle.$$

Let $\mathbf{F} = \langle F_1, F_2 \rangle$ be a smooth vector field on $\mathbb{R}^2$. The **line integral of the vector field** $\mathbf{F} = (F_1, F_2)$ along the curve γ is defined by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\, dt.$$

Alternatively,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (F_1, F_2) \cdot (dx, dy) = \int_C F_1 dx + F_2 dy.$$

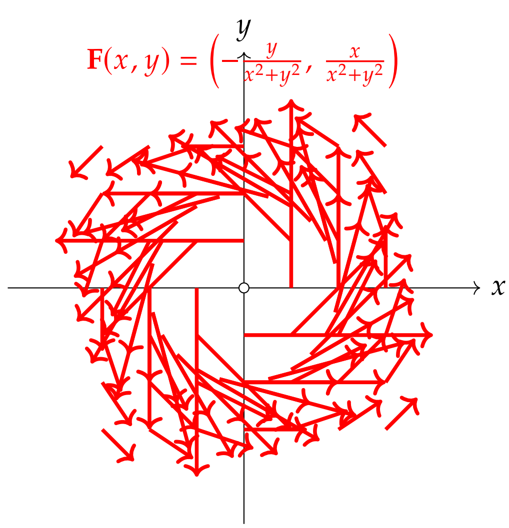
Problem #1 **(Line Integral around Unit Circle).** Let $C \subset \mathbb{R}^2$ be the unit circle defined by $C : x^2 + y^2 = 1$, traversed in the **counterclockwise direction**. Let the vector field $\mathbf{F} : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ be defined by

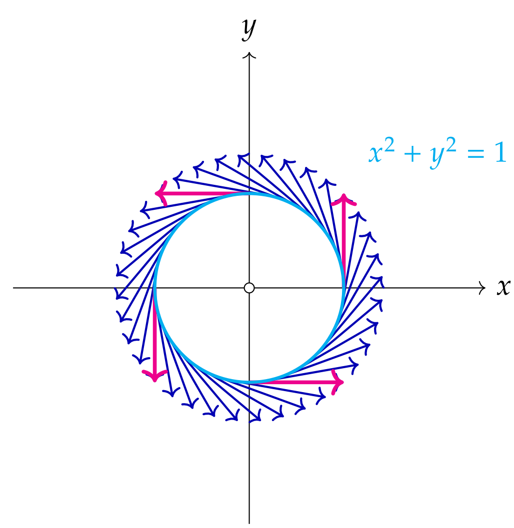
$$\mathbf{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$

Evaluate the **line integral** of $\mathbf{F}$ along C:

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

Sol.



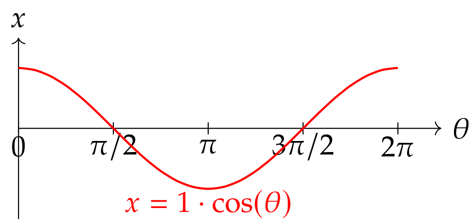


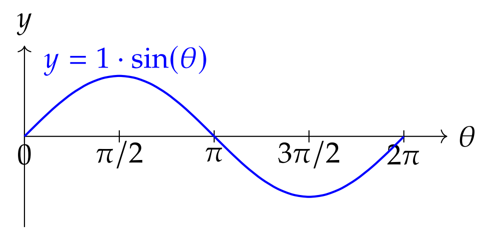
Consider the vector field $\mathbf{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$, and the curve C is the unit circle $x^2+y^2=1$, traversed counterclockwise.

(Parametrization) Define a function

$$\begin{array}{rcccl} \gamma & : & [0,2\pi] & \longrightarrow & \left\{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\right\} \\ & & \theta & \longmapsto & \gamma(\theta) = (\cos\theta, \sin\theta) \end{array}.$$

Here, $\frac{d\gamma}{d\theta} = (-\sin\theta, \cos\theta)$.





(Evaluate $\mathbf{F}(\gamma(\theta))$ and the dot product) We have

$$\mathbf{F}(\gamma(\theta)) = \mathbf{F}(\cos\theta, \sin\theta) \overset{\sin^2\theta+\cos^2\theta=1}{=} \left\langle \frac{-\sin\theta}{1}, \frac{\cos\theta}{1} \right\rangle = (-\sin\theta, \cos\theta).$$

and

$$\mathbf{F}(\gamma(\theta)) \cdot \frac{d\gamma}{d\theta} = (-\sin\theta)(-\sin\theta) + (\cos\theta)(\cos\theta) = \sin^2\theta + \cos^2\theta = 1.$$

(Integral)

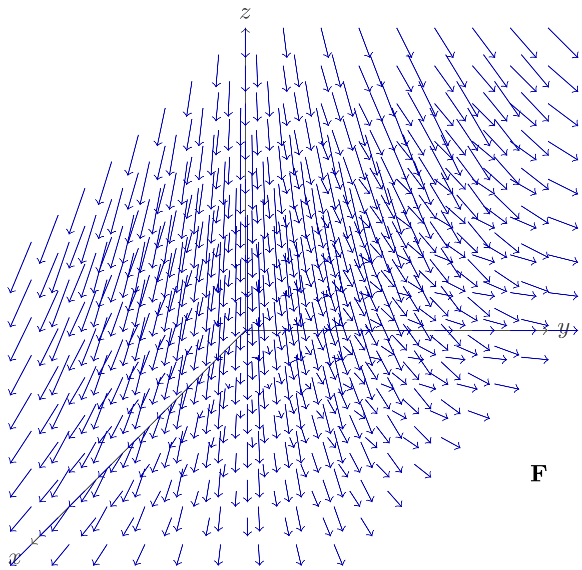
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\gamma(\theta)) \cdot \frac{d\gamma}{d\theta}\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi.$$

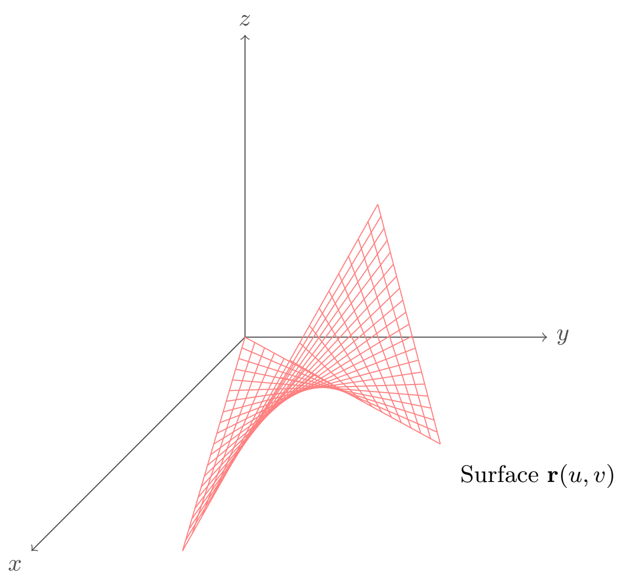
□

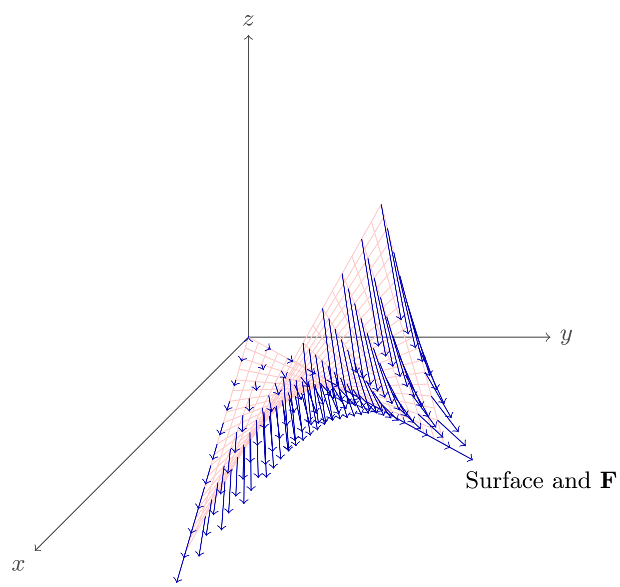
Surface Integral for Vector Fields

Problem #2 (Surface-Flux).

Sol.







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