Why Nonconstant Holomorphic Functions Don't Exist on Compact Riemann Surfaces

A proof via the winding (logarithmic) form

0. Setup and the winding form

Let X be a compact Riemann surface (connected, without boundary). If $f : X \to \mathbb{C}$ is holomorphic and *nonvanishing* on some open set, we may define on that open set the *logarithmic derivative* (winding form)

$$\omega_f \ := \ d\log f \ = \ \frac{f'}{f}\, dz,$$

where z is any local holomorphic coordinate. This is a well-defined meromorphic 1-form on X with the following basic property:

Lemma 1 (Local argument principle)**.** *If $p \in X$ is a zero of f of order $m \geq 1$ (so locally $f(z) = u(z)(z-p)^m$ with $u(p) \neq 0$), then*

$$\mathrm{Res}_p\left(\frac{f'}{f}\, dz\right) = m.$$

If p is a pole of order $n \geq 1$ (so $f(z) = u(z)(z-p)^{-n}$ with $u(p) \neq 0$), then

$$\mathrm{Res}_p\left(\frac{f'}{f}\, dz\right) = -\, n.$$

Proof. Write $f(z) = u(z)(z-p)^m$ with u holomorphic and nowhere zero; then

$$\frac{f'}{f} = \frac{u'}{u} + \frac{m}{z-p},$$

so the residue is m. The pole case is identical with $m = -n$. □

1. Residue theorem on a compact surface

On any compact Riemann surface, the sum of residues of a meromorphic 1-form is zero:

Proposition 1 (Residue theorem, global form)**.** *If η is a meromorphic 1-form on a compact Riemann surface X, then*

$$\sum_{p \in X} \mathrm{Res}_p(\eta) = 0.$$

Idea. Cover X by coordinate discs avoiding poles and small punctured discs around each pole; apply Stokes' theorem on the complement of the punctures and use that small positively oriented circles around the punctures bound with negative orientation; the boundary integrals are $2\pi i$ times the residues and sum to 0. □

2. The winding-form proof of Liouville on compact surfaces

Theorem 1. *If X is compact and $f : X \to \mathbb{C}$ is holomorphic, then f is constant.*

Proof via the winding (logarithmic) form. Consider $\omega_f = \dfrac{f'}{f}\,dz$, which is meromorphic on X. Since f is holomorphic on X, it has *no poles*; thus, by the local argument principle, the only possible residues of ω_f occur at the zeros of f, and each such residue equals the zero's multiplicity (a nonnegative integer).

By the residue theorem (Proposition 1),

$$\sum_{p\in X} \operatorname{Res}_p(\omega_f) = 0.$$

But each term is ≥ 0. Hence every term must be 0, so f has *no zeros*. Therefore ω_f is in fact holomorphic everywhere on X (no poles anywhere).

Now take real parts:

$$d\log|f| \;=\; \Re\!\left(\frac{f'}{f}\,dz\right).$$

Since ω_f is holomorphic, its real part is a *closed* 1-form; locally it is the differential of the harmonic function $\log|f|$. Globally, $\log|f|$ is thus a harmonic function on the *compact* surface X. A harmonic function on a compact manifold attains its max/min and hence is constant. Therefore $|f|$ is constant on X.

Finally, if $|f| \equiv c > 0$ is constant and f is holomorphic, then $f\overline{f} = c^2$. Differentiating in z,

$$0 = \partial_z(f\overline{f}) = f'\,\overline{f},$$

so $f' \equiv 0$ and f is constant. $\square$

Remark (Equivalent phrasing via the argument principle). Equivalently, on a compact surface, the divisor of a meromorphic function has degree 0:

$$\sum_p \operatorname{ord}_p(f) = 0.$$

When f is holomorphic (no poles), this forces $\operatorname{ord}_p(f) = 0$ for all p, i.e. no zeros. Then $\log|f|$ is globally harmonic and must be constant, hence f is constant.

3. Two standard one-line proofs (for comparison)

(A) Maximum modulus principle. A continuous image of a compact space is compact, so $|f|$ attains a maximum on X. A holomorphic function with an interior maximum is constant. Hence f is constant.

(B) Open mapping + compactness. If f is nonconstant holomorphic, it is open; thus $f(X)$ is an open subset of $\mathbb{C}$. But X is compact, so $f(X)$ is compact. The only subset of $\mathbb{C}$ that is both open and compact is empty; contradiction. Hence f is constant.

4. Intuition: winding numbers and $\frac{dz}{z}$

On $\mathbb{C}^\times$, the *winding form* $\dfrac{dz}{z}$ satisfies

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z} \in \mathbb{Z},$$

the winding number of γ about 0. For a holomorphic f, $\dfrac{f'}{f}dz = d\log f$ pulls this counting to the domain:

$$\frac{1}{2\pi i}\int_{\partial U} \frac{f'}{f}dz = \sum_{p\in U} \mathrm{ord}_p(f) - \sum_{p\in U} \mathrm{ord}_p(\text{poles of } f),$$

the *argument principle*. On a compact surface with $U = X$, the boundary term vanishes and the global count must be 0. This forces holomorphic f to have neither zeros nor poles; then $\log|f|$ is harmonic and constant.

5. Tiny exercise set

- Show directly that if f is holomorphic and nowhere zero on a compact Riemann surface, then $\log|f|$ is harmonic and hence constant; conclude f is constant.
- Let ω be a meromorphic 1-form on a compact Riemann surface. Prove $\sum_p \mathrm{Res}_p(\omega) = 0$ using Stokes' theorem.
- (Argument principle) For a domain $U \Subset X$ with smooth boundary avoiding zeros/poles of a meromorphic f, prove

$$\frac{1}{2\pi i}\int_{\partial U} \frac{f'}{f}dz = \sum_{p\in U} \mathrm{ord}_p(f) - \sum_{p\in U} \mathrm{ord}_p(1/f).$$