# Linear Algebra II

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We cover the following topics in this note.

- Unique representation as a finite linear combination of the elements of Basis.

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**Proposition.** *Let $V$ be a vector space over a field $F$, and let $\dim V < \infty$, say, $\dim V = n$. Fix a basis $\mathcal{B}$. Then every vector $\mathbf{v} \in V$ has a unique expression of linear combination by $\mathcal{B}$.*

*Proof.* Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis of $V$. Take any $\mathbf{v} \in V (= \operatorname{span} \mathcal{B})$. Then

$$\exists a_1, a_2, \ldots, a_n \in F \quad \text{such that} \quad a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{v}.$$

Suppose that $\exists b_1, b_2, \ldots, b_n \in F$ such that $\sum_{i=1}^{n} b_i\mathbf{v}_i = \mathbf{v}$. Then

$$\begin{aligned} a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots a_n\mathbf{v}_n &= b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n, \\ (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \cdots (a_n - b_n)\mathbf{v}_n &= \mathbf{0}, \end{aligned}$$

and so $a_i = b_i$ for all $i = 1, 2, \ldots, n$ since a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is linearly independent. □

**Coordinate**

**Definition.** Write

$$[\mathbf{v}]_{\mathcal{B}} = (a_1, a_2, \ldots, a_n).$$

is called the coordinate of $\mathbf{v}$ with respect to $\mathcal{B}$.

**Linear Transformation**

**Definition.** We say $\Phi : V \longrightarrow W$ is a linear transformation if $\Phi$ preserves a linearity, i.e.,

$$\Phi(a \cdot \mathbf{v} + b \cdot \mathbf{w}) = a \cdot \Phi(\mathbf{v}) + b \cdot \Phi(\mathbf{w})$$

for any $a, b \in F$ and $\mathbf{v}, \mathbf{w} \in V$. Here, if $\Phi : W \longrightarrow V$ is also a linear transformation the we say $\Phi$ is the vector-space isomorphism.

**Definition.** finite-dimensional vector space $V, W/F$. Then

$$\dim V = \dim W \iff \exists \text{a vector-space isomorphism } \Phi : V \to W, \text{i.e.}, V \simeq W.$$

*Proof.* ($\Rightarrow$) Suppose that $\dim V = \dim W = n \in \mathbb{N}$. Take basis $\mathcal{B}_V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\mathcal{B}_W = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ of $W$. Define

$$\begin{array}{rcccl} \Phi & : & V & \longrightarrow & W \\ & & \mathbf{v} & \longmapsto & a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n \end{array}.$$

We claim that $\Phi$ is one-to-one and onto linear transformation.

($\Leftarrow$) Suppose there exists $\Phi : V \to W$. Take any basis $\mathcal{B}_V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of $V$. Define

$$\mathcal{B}_W := \left\{\Phi(\mathbf{v}_1), \Phi(\mathbf{v}_2), \ldots, \Phi(\mathbf{v}_n)\right\}.$$

We claim that $\mathcal{B}_W$ be a basis of $W$:

(Linearly Independent) Suppose that $a_1\Phi(\mathbf{v}_1) + \cdots a_n\Phi(\mathbf{v}_n) = 0$. Since $\Phi$ is a linear transformation, we have

$$\Phi(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = 0.$$

Since $\Phi$ is one-to-one, and $0 = \Phi(0)$, $a_1\mathbf{v}_1 + a_n\mathbf{v}_n = 0$, $a_1 = a_n = 0$ since $\mathcal{B}_V$ is a basis.

(Spanning Property) Take $\mathbf{w} \in W$. Since $\Phi$ is onto, $\exists\mathbf{v} \in V$ s.t. $\Phi(\mathbf{v}) = \mathbf{w}$. Since $\mathcal{B}_V$ is a basis, $\exists! a_1, a_2, \ldots, a_n$ s.t. $a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v_n} = \mathbf{v}1$, and so

$$\mathbf{w} = \Phi(a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = a_1\Phi(\mathbf{v}_1) + \cdots + a_n\Phi(\mathbf{v}_n)$$

□

## References

[1] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 16. 선형대수학 (c) 차원과 벡터공간의 분류" YouTube Video, 29:08. Published October 11, 2019. URL: `https://www.youtube.com/watch?v=rOKN645fRPs&t=399s`.

[2] 수학의 즐거움, Enjoying Math. "수학 공부, 기초부터 대학원 수학까지, 17. 선형대수학 (d) 선형함수의 행렬 표현" YouTube Video, 29:14. Published October 12, 2019. URL: `https://www.youtube.com/watch?v=Fsy-9KW9-PA`.