Advanced Calculus II

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We cover the following topics in this note.

- Convergence of Sequences
- Inequality Rule for Reals
- ~~Algebraic Property of Limit of Sequence~~

Sequence

Definition. Let $A \subseteq \mathbb{N}$ and $X \subseteq \mathbb{R}$. A **sequence** is a function

$$a : A \to X,$$

with domain A and range in X.

Remark. A function a is a real sequence if

$$\begin{array}{cccc} a & : & \mathbb{N} & \longrightarrow & \mathbb{R} \\ & & n & \longmapsto & a(n) =: a_n \end{array}$$

for $n = 1, 2, \cdots$. We write

$$\{a_n\}_{n=1}^{\infty}, \quad \{a_n\}_{n\in\mathbb{N}}, \quad (a_n)_{n\in\mathbb{N}}, \quad \text{or} \quad \langle a_n\rangle_{n\in\mathbb{N}}.$$

Convergence of Sequence

Definition. A real sequence $\{a_n\}_{n=1}^{\infty}$ $(\subseteq \mathbb{R})$ is said to **converge** to $L \in \mathbb{R}$ if and only if

$$\forall \varepsilon > 0,\ \exists N_\epsilon \in \mathbb{N} \text{ such that } \left[n \geq N_\epsilon \implies |a_n - L| < \varepsilon\right].$$

Remark. A real number $L \in \mathbb{R}$ is called **the limit**. When a sequence $\{a_n\}_{n=1}^{\infty}$ has the limit L, we will use the notation

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty.$$

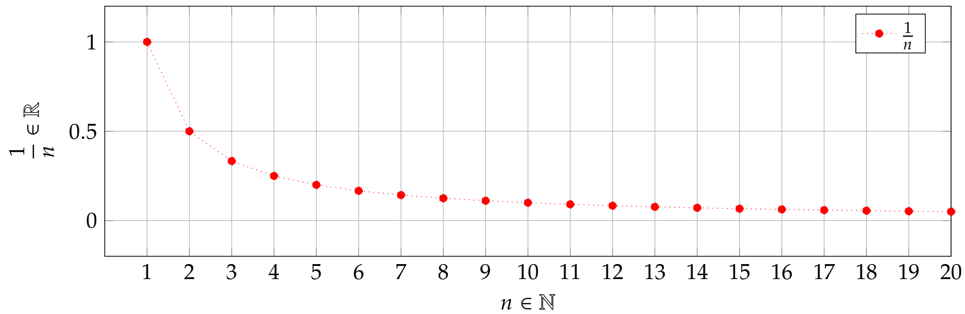
That is,

$$\lim_{n\to\infty} a_n = L \iff \forall \varepsilon > 0 : \exists N \in \mathbb{N} : \left[n \geq N \implies |a_n - L| < \varepsilon\right].$$

Note. If a sequence has a limit, we say that the sequence is **convergent**; if it has no limit, we say that the sequence is **divergent**.

Example. Consider the sequence defined by $a_n = 1/n$ for each $n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$



Proof. Let $\varepsilon > 0$. By the Archimedean property, we obtain

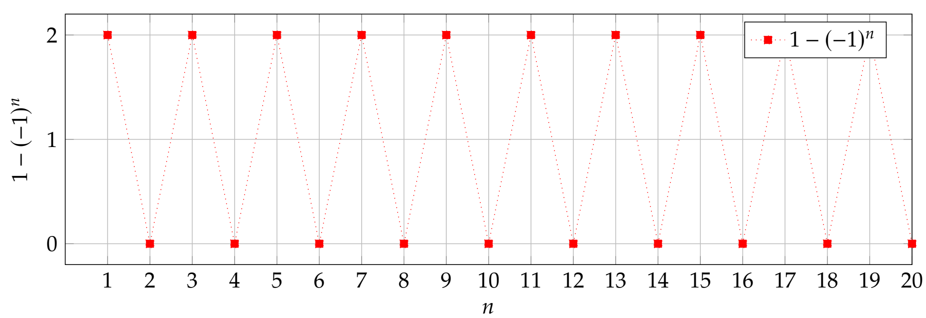
$$\exists N_\epsilon \in \mathbb{N} \quad \text{s.t.} \quad 1 < \varepsilon \cdot N_\epsilon, \text{ i.e.,} \ \frac{1}{N_\epsilon} < \varepsilon.$$

Assume that $n \geq N_\epsilon$ then

$$|a_n - 0| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N_\epsilon} < \varepsilon.$$

Hence $\lim_{n\to\infty} \frac{1}{n} = 0.$ □

Example. Consider the sequence defined by $b_n = 1 - (-1)^n$ for all $n \in \mathbb{N}$. Prove that b_n does not converge.



Proof. Suppose that $\{b_n\}_{n=1}^{\infty}$ converges to $\beta \in \mathbb{R}$. Let $\varepsilon \in (0, 2)$. Then if $n \geq N_\varepsilon$,

$$\begin{aligned} |b_n - \beta| &= |b_n - b_{n+1} + b_{n+1} - \beta| \\ &\leq |b_n - b_{n+1}| + |b_{n+1} - \beta| \\ &= 2 + |b_{n+1} - \beta| \end{aligned}$$

□

Absolute Value in Reals

Definition. Let $x \in \mathbb{R}$. A **absolute value** $|x|$ of x is defined by

$$|x| := \begin{cases} x & : x \geq 0 \\ -x & : x < 0 \end{cases}$$

Proposition. *Let $x, y \in \mathbb{R}$.*

(a) $|x| = |-x| = \sqrt{x^2}$

(b) $|xy| = |x||y|$

(c) *For each $r > 0$,*

$$|x| < r \iff -r < x < r$$

(d)

$$\delta < |x| \iff \delta < x \text{ or } x < -\delta$$

(e) $-|x| \leq x \leq |x|$

(f) *(Triangle Inequality)*

$$|x + y| \leq |x| + |y|$$

Proof. (a)

□

Boundedness of Sequence

Definition. Let $\{a_n\}$ is a real sequence. $\{a_n\}$ is said to be **bounded** when

$$\exists M \in \mathbb{R} \text{ such that } \forall n \in \mathbb{N}, |a_n| \leq M.$$

Proposition. *A convergent sequence is bounded.*

Proof. Let $\lim_{n\to\infty} a_n = L$. For $\varepsilon = 1$, $\exists N \in \mathbb{N}$ such that $n \geq N \implies |a_n - L| < 1$. Then we see that

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| < 1 + |L|.$$

Let $M := \max\left\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |L|\right\}$. Then

$$|a_n| \leq M$$

for all $n \in \mathbb{N}$. That is, $\{a_n\}$ is bounded. $\square$

References

[1] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 6. 해석학 개론 (c) 수열의 수렴성.” YouTube Video, 26:29. Published September 20, 2019. URL: https://www.youtube.com/watch?v=jwLfzJyIxmU.

[2] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 7. 해석학 개론 (d) 극한 정리” YouTube Video, 26:46. Published September 26, 2019. URL: https://www.youtube.com/watch?v=1TRD34QbIaw.