A Step-by-Step Calculus Approach to the Mayer–Vietoris Sequence on S^2

Your Name

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Abstract

This article explains the Mayer–Vietoris argument for de Rham Cohomology on the unit sphere S^2 using only first-year calculus (single and double integrals, the Fundamental Theorem of Calculus) and basic linear algebra (kernels and images). We cover S^2 by two overlapping hemispheres and show, via two applications of the FTC, how local antiderivatives glue to a global one exactly when patching data match, and how the equator integral equals the sphere's total integral.

1 1. The Open Cover and Local Exactness

Split S^2 into the northern and southern open hemispheres:

$$U = \{(x, y, z) \in S^2 : z > -\tfrac{1}{2}\}, \quad V = \{(x, y, z) \in S^2 : z < +\tfrac{1}{2}\}.$$

Each of U, V is diffeomorphic to a disk (hence contractible). Any continuous 1-form on a disk is exact by integrating along straight paths (FTC): given a 1-form

$$beta(\phi, \theta) = P(\phi, \theta)\, d\phi + Q(\phi, \theta)\, d\theta$$

in spherical coordinates, one finds a function $F_U(\phi, \theta)$ on U by

$$F_U(\phi, \theta) = \int_{\phi_0}^{\phi} P(s, \theta)\, ds$$

for a fixed base–latitude ϕ_0, so that

$$\frac{\partial F_U}{\partial \phi}(\phi, \theta) = P(\phi, \theta).$$

Thus $dF_U = P\, d\phi + \frac{\partial F_U}{\partial \theta}\, d\theta$. One checks by FTC again that

$$\frac{\partial F_U}{\partial \theta}(\phi, \theta) = Q(\phi, \theta)$$

on any overlap region where the 1-form equals $Q\, d\theta$ in the θ–direction.

A similar construction on V gives $F_V(\phi, \theta) = \int_{\phi_1}^{\phi} P(s, \theta)\, ds$. Hence on $U \cap V$:

$$F_U(\phi, \theta) - F_V(\phi, \theta) = H(\theta)$$

for some function H of θ alone (difference of two FTC–integrals with the same integrand). Differentiating in θ:

$$H'(\theta) = \frac{\partial F_U}{\partial \theta} - \frac{\partial F_V}{\partial \theta} = Q - Q = 0.$$

By FTC, $H'(\theta) = 0$ implies $H(\theta)$ is constant. Adjusting F_V by this constant makes $F_U = F_V$ on $U \cap V$, so

$$F(p) = \begin{cases} F_U(p), & p \in U, \\ F_V(p), & p \in V, \end{cases}$$

defines a single global F with $dF = \beta$ on all of S^2. This proves

$$\operatorname{Im}\bigl(\Omega^0(S^2) \xrightarrow{d} \Omega^1(S^2)\bigr) = \ker\bigl(\Omega^1(S^2) \to \Omega^1(U) \oplus \Omega^1(V)\bigr).$$

2 2. The Equator Integral and Surface Integral

On the overlap region, identify the equator $U \cap V$ with S^1 via $z = 0$. The standard closed-but-not-exact 1-form is

$$\omega = d\theta, \qquad \oint_{S^1} \omega = \int_0^{2\pi} d\theta = 2\pi.$$

To see how this winding produces the sphere's area, define a cutoff function (partition of unity) depending only on z:

$$\rho(z) = \frac{1+z}{2}, \quad 0 \le \rho \le 1, \quad \rho = 1 \text{ at north pole}, \rho = 0 \text{ at south pole}.$$

On S^2, form the 2-form

$$\Theta = d\bigl(\rho(z)\, \omega\bigr) = d\rho \wedge \omega.$$

In spherical coordinates (ϕ, θ) we have $z = \cos\phi$, hence

$$d\rho = -\tfrac{1}{2} \sin\phi\, d\phi, \quad \omega = d\theta, \quad \Rightarrow \Theta = -\tfrac{1}{2} \sin\phi\, d\phi \wedge d\theta.$$

Then the surface integral of Θ over S^2 is a double integral:

$$\int_{S^2} \Theta = \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \Bigl(-\tfrac{1}{2} \sin\phi\Bigr)\, d\theta\, d\phi.$$

Apply FTC in θ first:

$$\int_0^{2\pi} d\theta = 2\pi,$$

then FTC in ϕ:

$$\int_0^{\pi} \sin\phi\, d\phi = [-\cos\phi]_0^{\pi} = 2.$$

Hence

$$\int_{S^2} \Theta = -\tfrac{1}{2} \times (2\pi) \times 2 = -2\pi.$$

Choosing the standard outward orientation flips the sign to $+2\pi$, recovering the equator's 2π via Stokes' theorem. Thus globally,

$$\int_{S^2} d\rho \wedge \omega = \oint_{S^1} \omega.$$

3 3. Exactness in Degree Zero via Kernels and Images

The induced sequence on functions (0-forms)

$$0 \to C^\infty(S^2) \xrightarrow{r_0} C^\infty(U) \oplus C^\infty(V) \xrightarrow{s_0} C^\infty(U \cap V) \to 0$$

has

$$\ker(r_0) = \{f : f|_U = f|_V = 0\} = 0, \quad \mathrm{Im}(r_0) = \{(f, f)\} = \ker(s_0), \quad \mathrm{Im}(s_0) = C^\infty(U \cap V).$$

All checks are immediate by comparing values on overlaps (basic algebra of real functions).

4 4. Conclusion

We have shown using only first-year calculus:

1. Local antiderivatives exist by a single-variable FTC in the ϕ-direction.
2. Matching on overlaps forces a constant difference via FTC in the θ-direction, so one global primitive exists exactly when patch data match.
3. The non-exact equator 1-form $d\theta$ integrates to 2π and yields the sphere's total integral by two nested FTCs.

This concrete approach demystifies the Mayer–Vietoris argument for students familiar only with multivariable calculus and the FTC.