Fundamental Theorem of Calculus and First Isomorphism Theorem

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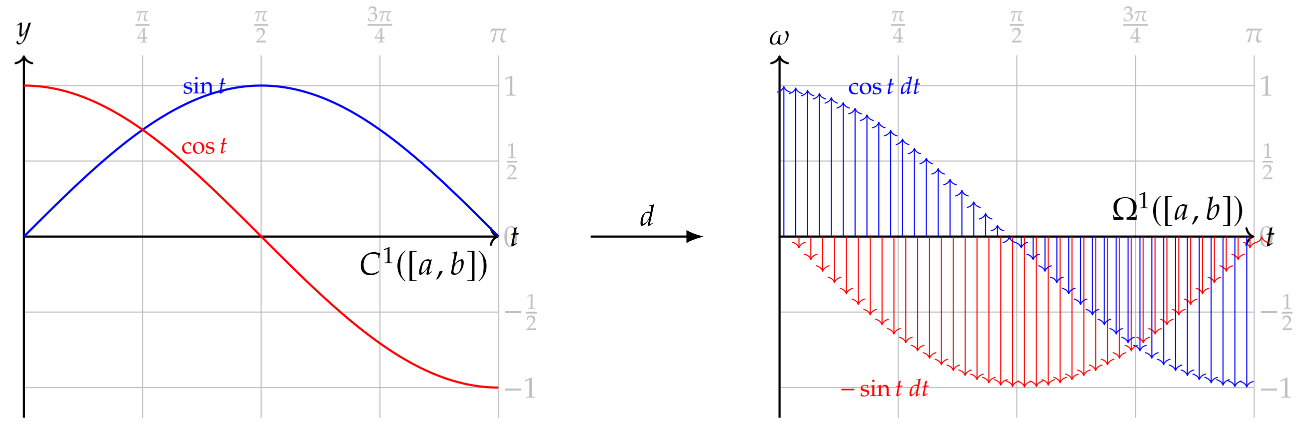
We cover the following topics in this note.

- Fundamental Theorem of Calculus
- First Isomorphism Theorem
- Differential Form



The map

$$\begin{array}{rcccc} d & : & C^1([a,b]) & \longrightarrow & \Omega^1([a,b]) \\ & & f(t) & \longmapsto & d(f(t)) = df \end{array}$$

is defined by $df = f'(t)dt$, where f' is the derivative of f.

Definition 1 (Space of Smooth Functions $C^\infty(\mathbb{R}^n)$)**.** We write $C^\infty(\mathbb{R}^n)$ for the set of all functions

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

that have continuous partial derivatives of every order. Equivalently,

$$C^\infty(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R} \,\middle|\, \text{for all multi-indices } \alpha,\ \partial^\alpha f \text{ exists and is continuous} \right\},$$

where $\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$.

Definition 2 (Space of Smooth 1–Forms on $\mathbb{R}^n$)**.** We denote by $C^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable real-valued functions on $\mathbb{R}^n$. A *smooth* 1*–form* is an expression of the form

$$\omega = f_1(x)\,dx^1 \; + \; f_2(x)\,dx^2 \; + \; \cdots + \; f_n(x)\,dx^n,$$

where each $f_i \in C^\infty(\mathbb{R}^n)$ and dx^i are formal symbols. The collection of all such forms is

$$\Omega^1(\mathbb{R}^n) \; = \; \left\{ \sum_{i=1}^n f_i(x)\,dx^i \;\middle|\; f_i(x) \in C^\infty(\mathbb{R}^n) \right\}.$$

Remark. - Each dx^i is thought of as "dual" to the partial derivative $\partial/\partial x^i$. - Smooth 1–forms can be integrated along curves (line integrals), since dx^i picks out the ith component of a tangent vector. - No notions of "open set" or "manifold" are needed at this level: we work directly in $\mathbb{R}^n$.

Remark 1.

- 0–Forms are just smooth functions: $\Omega^0(U) = C^\infty(U)$.
- A 1–form $\omega = f_1\,dx^1 + \cdots + f_n\,dx^n$ can be naturally integrated along curves, yielding line integrals.
- A 2–form in $\mathbb{R}^3$, $\alpha = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$, encodes a flux density for surface integrals.

1 Line and Surface Integrals as First Isomorphism Theorems

We work on a smooth domain $U \subset \mathbb{R}^2$ (for line integrals) or $U \subset \mathbb{R}^3$ (for surface integrals), and use the exterior derivative

$$d : \Omega^k(U) \longrightarrow \Omega^{k+1}(U).$$

Proposition 1 (Line Integrals and Exact 1-Forms)**.** *Let $C \subset U$ be a piecewise C^1 closed curve, and define the line-integral functional*

$$I_C : \Omega^1(U) \longrightarrow \mathbb{R}, \qquad I_C(\alpha) = \oint_C \alpha.$$

Then:

1. $\ker I_C = \{\alpha \in \Omega^1(U) : I_C(\alpha) = 0\}$ *coincides with* $\mathrm{im}(d)$*, the space of exact 1-forms, since by the Fundamental Theorem of Line Integrals $I_C(df) = 0$ for every $f \in C^\infty(U)$.*

2. *By the First Isomorphism Theorem for vector spaces,*

$$\Omega^1(U)/\mathrm{im}(d) \;\cong\; I_C\big(\Omega^1(U)\big) \;\subset\; \mathbb{R}.$$

In particular, if C represents a generator of $H_1(U;\mathbb{Z}) \cong \mathbb{Z}$, then $I_C\big(\Omega^1(U)\big) = \mathbb{R}$ and $\Omega^1(U)/\mathrm{im}(d) \cong \mathbb{R}$.

Example. *On $U = \mathbb{R}^2 \setminus \{0\}$, let*

$$\alpha \;=\; -\frac{y}{x^2+y^2}\,dx \;+\; \frac{x}{x^2+y^2}\,dy.$$

One checks $d\alpha = 0$ on U, but $\alpha \notin \mathrm{im}(d)$. The unit circle C generates $H_1(U)$, and

$$I_C(\alpha) = \oint_C \alpha = 2\pi.$$

Thus α represents a nonzero class in $\Omega^1(U)/\mathrm{im}(d) \cong \mathbb{R}$.

Proposition 2 (Surface Integrals and Exact 2-Forms)**.** *Let $D \subset U \subset \mathbb{R}^3$ be a compact oriented surface with boundary ∂D, and define the surface-integral functional*

$$J_D : \Omega^2(U) \longrightarrow \mathbb{R}, \qquad J_D(\beta) = \iint_D \beta.$$

Then:

1. $\ker J_D = \mathrm{im}(d) \subset \Omega^2(U)$*, since Stokes' Theorem gives $J_D(d\gamma) = \iint_D d\gamma = \oint_{\partial D} \gamma$, which vanishes whenever γ has compact support or ∂D is empty.*

2. *By the First Isomorphism Theorem,*

$$\Omega^2(U)/\mathrm{im}(d) \;\cong\; J_D\big(\Omega^2(U)\big) \;\subset\; \mathbb{R}.$$

Example (Divergence Theorem). *In $\mathbb{R}^3$, let $\mathbf{F} = (P, Q, R)$ and identify the 2-form $\beta = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$. Then*

$$d\beta = \big(\partial_x P + \partial_y Q + \partial_z R\big)\,dx \wedge dy \wedge dz$$

is the divergence form. For a compact region D with boundary $S = \partial D$,

$$J_D(d\beta) = \iiint_D \operatorname{div}\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot d\mathbf{S},$$

demonstrating that exactness of forms corresponds exactly to vanishing of flux by Gauss's theorem.

Level 1: The Fundamental Theorem of Calculus

The equation

$$\int_a^b \cos t\,\mathrm{d}t = \sin b - \sin a$$

is a special case of the **Fundamental Theorem of Calculus**, which links differentiation and integration.

- $\displaystyle\int_a^b$: the *definite integral*, representing the signed area under the integrand from $t = a$ to $t = b$.
- $\cos t$: the *integrand*, here the cosine function.
- t: the *variable of integration*.
- $\mathrm{d}t$: the *differential*, indicating integration with respect to t.
- $\sin b - \sin a$: the evaluation of the *antiderivative* $\sin t$ at the endpoints b and a.

Level 2: Line Integral of a Vector Field

The line integral

$$\int_C \left\langle -\frac{y}{r^2},\ \frac{x}{r^2} \right\rangle \cdot \mathrm{d}\mathbf{r}$$

computes the *circulation* of the vector field $\mathbf{F}(x, y) = \langle -y/r^2,\ x/r^2 \rangle$ around the curve C, taken here to be the unit circle $x^2 + y^2 = 1$.

- $\displaystyle\int_C$: integral *along* a curve C.
- $C : x^2 + y^2 = 1$: the *unit circle*.
- $\mathbf{F}(x, y) = \left\langle -y/r^2,\ x/r^2 \right\rangle,\ r^2 = x^2 + y^2$.
- $\mathrm{d}\mathbf{r} = \langle \mathrm{d}x, \mathrm{d}y \rangle$: the infinitesimal tangent vector.
- "$\cdot$": the *dot product*, measuring alignment of $\mathbf{F}$ with $\mathrm{d}\mathbf{r}$.

For this field, the result is 2π, indicating one full rotation around the origin.

Level 3: Surface Integral of a Vector Field

The surface integral

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} \ = \ \iint_D \mathbf{F}\big(T(u, v)\big) \ \cdot \ \left(\frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v} \right) \mathrm{d}u\, \mathrm{d}v$$

measures the *flux* of $\mathbf{F}$ through a parametrized surface $S = T(D)$.

- $\displaystyle\iint_S$: surface integral over $S \subset \mathbb{R}^3$.
- $\mathrm{d}\mathbf{S}$: the vector area element, normal to S.
- $T(u, v)$: a smooth *parametrization* of S by $(u, v) \in D \subset \mathbb{R}^2$.
- $\frac{\partial T}{\partial u}, \frac{\partial T}{\partial v}$: tangent vectors to S.
- "$\times$": cross product, giving the normal vector with magnitude equal to the area element.
- $\mathrm{d}u\, \mathrm{d}v$: area element in parameter domain D.

Level 4: Abstract View with Differential Forms

We summarize classical integrals in the language of differential forms and duality:

Concept	Domain→Target	Role
Differential d	$C^\infty(\mathbb{R}) \to \Omega^1(\mathbb{R})$	$f \mapsto f'(x)\,\mathrm{d}x$
Antiderivative I_{x_0}	$\Omega^1(\mathbb{R}) \to C^\infty(\mathbb{R})$	$\omega = f(x)\,\mathrm{d}x \mapsto \int_{x_0}^x f(t)\,\mathrm{d}t$
Integration $\mathcal{I}_{[a,b]}$	$\Omega^1(\mathbb{R}) \to \mathbb{R}$	$\omega \mapsto \int_a^b \omega$
Dual module	$\Omega^1(\mathbb{R}) \to X(\mathbb{R})$	1-forms $\leftrightarrow$ vector fields

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Integration $\mathcal{I}_{[a,b]}$	$\Omega^1(\mathbb{R}) \to \mathbb{R}$	$\omega \mapsto \displaystyle\int_a^b \omega$
Line integral I_C	$\Omega^1(\mathbb{R}^2) \to \mathbb{R}$	$\alpha = P\,dx + Q\,dy \mapsto \displaystyle\oint_C \alpha = \oint_C P\,dx + Q\,dy$
Surface integral J_S	$\Omega^2(\mathbb{R}^3) \to \mathbb{R}$	$\beta = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy \mapsto \displaystyle\iint_S \beta = \iint_S \mathbf{F} \cdot d\mathbf{S}$
Dual module	$\Omega^1(\mathbb{R}) \to X(\mathbb{R})$	1-forms $\leftrightarrow$ vector fields

$C^\infty(\mathbb{R})$: smooth functions. $\Omega^1(\mathbb{R})$: space of 1-forms. $X(\mathbb{R})$: space of vector fields.

Remark 2 (FTC as a First Isomorphism Theorem for Differential Forms)**.** Consider the exterior derivative

$$d : C^\infty(\mathbb{R}) \longrightarrow \Omega^1(\mathbb{R}),$$

which sends a smooth function f to the 1-form $df = f'(x)\,dx$. Then

$$\ker(d) = \{ f \in C^\infty(\mathbb{R}) : df = 0\} = \{\text{constant functions}\}, \quad \operatorname{im}(d) = \{ \omega \in \Omega^1(\mathbb{R}) : \exists f,\ \omega = df\},$$

the *exact* 1-forms. The *First Isomorphism Theorem* for the ring homomorphism d gives

$$C^\infty(\mathbb{R}) / \ker(d) \ \cong\ \operatorname{im}(d).$$

On the other hand, the *Fundamental Theorem of Calculus* asserts that for any exact form df,

$$\int_a^b df \ =\ f(b) - f(a),$$

and that this assignment depends only on the class of f modulo constants. Thus the map

$$C^\infty(\mathbb{R}) / \ker(d) \ \longrightarrow\ \mathbb{R}, \qquad [f] \ \longmapsto\ f(b) - f(a),$$

is precisely an inverse to the inclusion $\mathrm{im}(d) \hookrightarrow \Omega^1(\mathbb{R})$ composed with integration. Equivalently, the FTC is the realization of the first isomorphism theorem in the category of differential forms.

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