

# Set Theory I

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## Terminology.

- Set; Collection; Family.
- Tabular (or Roster) Form

$$A = \{0, 2, 4, 8\}.$$

- Set-builder Form

$$A = \{x : x \text{ is even and } x < 10\}.$$

## Example.

- $\mathbb{N} = \{1, 2, \dots\}$
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- $\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\}, \gcd(p, q) = 1 \right\}$
- $\mathbb{R} = \{x : x \text{ is a real number}\}$
- $\mathbb{C} = \{z : z \text{ is a complex number}\}$

**Exercise.** Show that  $\sqrt{2}$  is irrational.

**Sol.** Assume  $\sqrt{2} \in \mathbb{Q}$ , i.e.,  $\exists p, q \in \mathbb{Z}$  such that  $\sqrt{2}q = p$ ,  $q \neq 0$  and  $\gcd(p, q) = 1$ . Then  $2q^2 = p^2$ .  
Since  $p^2$  is even  $\Rightarrow p$  is even,

$$p = 2k \quad \text{for some } k \in \mathbb{Z}$$

By substituting  $p = 2k$  into  $2q^2 = p^2$ , we have

$$2q^2 = (2k)^2 \implies 2q^2 = 4k^2 \implies q^2 = 2k^2$$

Since  $q^2$  is even  $\Rightarrow q$  is even,

$$q = 2m \quad \text{for some } m \in \mathbb{Z}$$

Thus,  $p$  and  $q$  are both even  $\implies \gcd(p, q) \geq 2$ , which contradicts the assumption  $\gcd(p, q) = 1$ .  $\square$

### Subset and Set Equality

**Definition.** Let  $A$  and  $B$  are sets.

- Subset:  $B \subseteq A \stackrel{\text{def}}{\iff} (x \in B \Rightarrow x \in A)$ .
- Set Equality:

$$\begin{aligned} A = B &\stackrel{\text{def}}{\iff} A \subseteq B \wedge B \subseteq A \\ &\iff (x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A). \end{aligned}$$

### Power Set

**Definition.** The **power set** of a set  $X$  is the set of all subsets of  $X$ .

$$\mathcal{P}(X) = 2^X := \{S : S \subseteq X\}.$$

### Cartesian Product

**Definition.** Let  $A$  and  $B$  are sets. The **cartesian product** of  $A$  and  $B$  is the set

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}.$$

### Union, Intersection and Complement

**Definition.** Let  $U$  is a universal set, and let  $A, B \subseteq U$ .

- The **union** of  $A$  and  $B$  is the set

$$A \cup B := \{x \in U : x \in A \vee x \in B\}.$$

Note that  $x \in A \cup B \iff x \in A \vee x \in B$ .

- The **intersection** of  $A$  and  $B$  is the set

$$A \cap B := \{x \in U : x \in A \wedge x \in B\}.$$

Note that  $x \in A \cap B \iff x \in A \wedge x \in B$ .

- The **complement** of  $A$  is the set

$$A^C := \{x \in U : \neg(x \in A)\} = \{x : x \notin A\}.$$

Note that  $x \in A^C \iff x \notin A$ .

**Proposition 1** Let  $A, B, C \subseteq U$ .

$$(1) A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

$$(2) A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

$$(3) (A \cup B)^C = A^C \cap B^C.$$

$$(4) (A \cap B)^C = A^C \cup B^C.$$

*Proof.* (1) Refer to the Video[1].

(2) Refer to the Video[1].

$$(3) (A \cup B)^C = \{x : \neg [x \in A \vee x \in B]\} = \{x : x \notin A \wedge x \notin B\} = A^C \cap B^C.$$

$$(4) (A \cap B)^C = \{x : \neg [x \in A \wedge x \in B]\} = \{x : x \notin A \vee x \notin B\} = A^C \cup B^C.$$

□

**Exercise.** Let  $A$  has  $n$  elements. Show that  $\mathcal{P}(A)$  has  $2^n$  elements.

**Sol.**

(pf 1) For each element of  $A$ , there are two choices:

1. Include the element in the subset.
2. Exclude the element from the subset.

Since we have two independent choices (include or exclude), the total number of subsets is:

$$\underbrace{2 \times 2 \times \cdots 2}_{n \text{ times}} = 2^n.$$

(pf 2) We use mathematical induction.

(Basic Step) Let  $A = \emptyset$  (so  $|A| = 0$ ). Then  $\mathcal{P}(A) = \{\emptyset\}$  and so  $|\mathcal{P}(A)| = |\{\emptyset\}| = 1$ .

(Inductive Step) Assume that  $|\mathcal{P}(A)| = 2^k$  where  $|A| = k$  for some  $k \in \mathbb{Z}_{\geq 0}$ . Let  $A' = A \cup \{x\}$  where  $|A| = k$  and  $x \notin A$ . That is,  $|A'| = k + 1$ . Then

$$\mathcal{P}(A') = \mathcal{P}(A) \cup \{S \cup \{x\} : S \in \mathcal{P}(A)\}.$$

This implies  $|\mathcal{P}(A')| = |\mathcal{P}(A)| + |\mathcal{P}(A)|$ . Therefore, by assumption,  $|\mathcal{P}(A')| = 2^k + 2^k = 2^{k+1}$ .

□

## Function

**Definition.** Let  $A$  and  $B$  are sets. A **function**  $f \subseteq A \times B$  **from**  $A$  **to**  $B$  is a relation on  $A \times B$  satisfying as follows:

- (i) Every element of  $A$  relates to some element of  $B$ .

$$\forall a \in A : \exists b \in B \text{ such that } (a, b) \in f.$$

- (ii) Every element of  $A$  relates to no more than one element of its  $B$ .

$$\forall a \in A : \forall b_1, b_2 \in B : (a, b_1), (a, b_2) \in f \implies b_1 = b_2.$$

**Remark.** A relation  $f \subseteq A \times B$  is a function if  $\forall a \in A : \exists! b \in B : (a, b) \in f$ .

- The **domain** of  $f$  is  $\text{Dom}(f) = A$ .
- The **codomain** of  $f$  is  $\text{Cdm}(f) = B$ .
- The **image** of  $A$  under  $f$  is the set

$$\begin{aligned} \text{Img}(f) &= f[A] := \{b \in B : \exists a \in A \text{ s.t. } (a, b) \in f\} \\ &= \{b \in B : \exists a \in A \text{ s.t. } f(a) = b\} \\ &= \{b \in B : b = f(a) \text{ for at least one } a \in A\}. \end{aligned}$$

Simply we can express it as  $f[A] = \{f(a) \in B : a \in A\}$ .

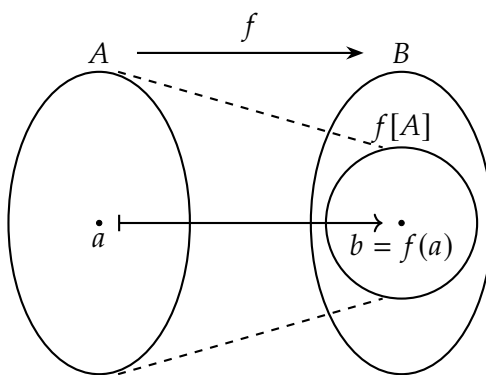


Figure 1: Image of  $A$  under  $f$ .

Note that  $f[A] \subseteq B = \text{Cdm}(f)$  and that

$$b \in f[A] \iff b = f(a) \text{ for some } a \in A.$$

- The **preimage** of  $B_1 \subseteq B$  under  $f$  is the set

$$\begin{aligned} f^{-1}[B_1] &:= \{a \in A : \exists b \in B_1 \text{ s.t. } (a, b) \in f\} \\ &= \{a \in A : \exists! b \in B_1 \text{ s.t. } b = f(a)\} \text{ by def. of a function} \\ &= \{a \in A : f(a) = b \text{ for exactly one } b \in B_1\}. \end{aligned}$$

“Exactly one” ensures a unique assignment for every element of  $A$ , while “at most one” allows no assignment. Simply we can express it as  $f^{-1}[B_1] = \{a \in A : f(a) \in B_1\}$ .

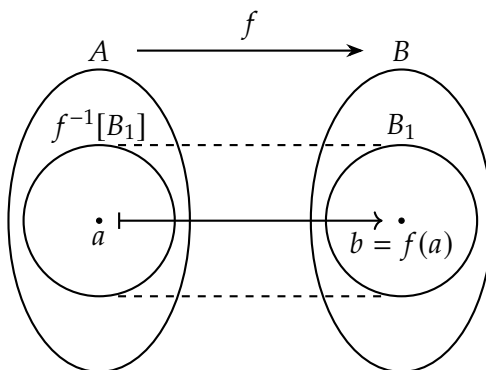


Figure 2: Preimage of  $B_1 \subseteq B$  under  $f$ .

Note that  $f^{-1}[B_1] \subseteq A = \text{Dom}(f)$  and that  $a \in f^{-1}[B_1] \iff f(a) \in B_1$ .

**Proposition 2** Let  $f : A \rightarrow B$  be a function from  $A$  to  $B$ , and let  $A_1, A_2 \subseteq A$ .

(1)  $f[A_1 \cup A_2] = f[A_1] \cup f[A_2]$ .

(2)  $f[A_1 \cap A_2] \subseteq f[A_1] \cap f[A_2]$ .

*Proof.* Recall that

$$b \in f[A] \iff b = f(a) \text{ for some } a \in A.$$

- (1) ( $\subseteq$ ) Let  $b \in f[A_1 \cup A_2]$ . By the definition of the image,  $b = f(a)$  for some  $a \in A_1 \cup A_2$ . Then, either  $a \in A_1$  or  $a \in A_2$ .

(Case 1)  $a \in A_1 \Rightarrow f(a) \in f[A_1]$ .

(Case 2)  $a \in A_2 \Rightarrow f(a) \in f[A_2]$ .

Thus,  $b = f(a) \in f[A_1] \cup f[A_2]$ , and so  $f[A_1 \cup A_2] \subseteq f[A_1] \cup f[A_2]$ .

( $\supseteq$ ) Let  $b \in f[A_1] \cup f[A_2]$ . Then either  $b \in f[A_1]$  or  $b \in f[A_2]$ .

(Case 1)  $b \in f[A_1] \Rightarrow b = f(a_1)$  for some  $a_1 \in A_1$ .

(Case 2)  $b \in f[A_1] \Rightarrow b = f(a_2)$  for some  $a_2 \in A_2$ .

That is,  $\exists a \in A_1 \cup A_2$  such that  $f(a) = b$  and  $a \in \{a_1, a_2\}$ . Thus,  $b \in f[A_1 \cup A_2]$ .

(2) Let  $b \in f[A_1 \cap A_2]$ . By the definition of the image,  $b = f(a)$  for some  $a \in A_1 \cap A_2$ . Since  $a \in A_1 \cap A_2$ , we have  $a \in A_1$  and  $a \in A_2$ . Then *both* of the following hold:

(i)  $a \in A_1 \Rightarrow f(a) \in f[A_1]$

(ii)  $a \in A_2 \Rightarrow f(a) \in f[A_2]$

Therefore,  $b = f(a) \in f[A_1] \cap f[A_2]$ .

□

**Proposition 3** Let  $f : A \rightarrow B$  be a function from  $A$  to  $B$ , and let  $B_1, B_2 \subseteq B$ .

(1)  $f^{-1}[B_1 \cup B_2] = f^{-1}[B_1] \cup f^{-1}[B_2]$ .

(2)  $f^{-1}[B_1 \cap B_2] = f^{-1}[B_1] \cap f^{-1}[B_2]$ .

(3)  $f^{-1}[B_1^C] = (f^{-1}[B_1])^C$ .

*Proof.* Recall that

$$a \in f^{-1}[B] \iff f(a) \in B.$$

(1) ( $\subseteq$ ) Let  $a \in f^{-1}[B_1 \cup B_2]$ . By the definition of the preimage, we have  $f(a) \in B_1 \cup B_2$ . That is, either  $f(a) \in B_1$  or  $f(a) \in B_2$ .

(Case 1)  $f(a) \in B_1 \Rightarrow a \in f^{-1}[B_1]$ .

(Case 2)  $f(a) \in B_2 \Rightarrow a \in f^{-1}[B_2]$ .

Thus,  $a \in f^{-1}[B_1] \cup f^{-1}[B_2]$ .

( $\supseteq$ ) Let  $a \in f^{-1}[B_1] \cup f^{-1}[B_2]$ . Then either  $a \in f^{-1}[B_1]$  or  $a \in f^{-1}[B_2]$ .

(Case 1)  $a \in f^{-1}[B_1] \implies f(a) \in B_1$ .

(Case 2)  $a \in f^{-1}[B_2] \implies f(a) \in B_2$ .

That is,  $f(a) \in B_1 \cup B_2$ . Thus,  $a \in f^{-1}[B_1 \cup B_2]$ .

(2) ( $\subseteq$ ) Let  $a \in f^{-1}[B_1 \cap B_2]$ . By the definition of the preimage,  $f(a) \in B_1 \cap B_2$  and so  $f(a) \in B_1$  and  $f(a) \in B_2$ . Then *both* of the following hold:

(i)  $f(a) \in B_1 \implies a \in f^{-1}[B_1]$ .

(ii)  $f(a) \in B_2 \implies a \in f^{-1}[B_2]$ .

Thus,  $a \in f^{-1}[B_1] \cap f^{-1}[B_2]$ .

( $\supseteq$ ) Let  $a \in f^{-1}[B_1] \cap f^{-1}[B_2]$ . Then  $a \in f^{-1}[B_1]$  and  $a \in f^{-1}[B_2]$ . Then *both* of the following hold:

(i)  $a \in f^{-1}[B_1] \implies f(a) \in B_1$ .

(ii)  $a \in f^{-1}[B_2] \implies f(a) \in B_2$ .

That is,  $f(a) \in B_1 \cap B_2$ . Thus,  $a \in f^{-1}[B_1 \cap B_2]$ .

(3) ( $\subseteq$ ) Let  $a \in f^{-1}[B_1^C]$ . By the definition of the preimage,

$$f(a) \in B_1^C \implies f(a) \notin B_1 \implies a \notin f^{-1}[B_1] \implies a \in (f^{-1}[B_1])^C.$$

( $\supseteq$ ) Let  $a \in (f^{-1}[B_1])^C$ . By the definition of the preimage,

$$a \notin f^{-1}[B_1] \implies f(a) \notin B_1 \implies f(a) \in B_1^C \implies a \in f^{-1}[B_1^C].$$

□

**Proposition 4** Let  $f : A \rightarrow B$  be a function from  $A$  to  $B$ . Let  $A_1 \subseteq A$  and  $B_1 \subseteq B$ .

(1)  $f[f^{-1}[B_1]] \subseteq B_1$ .

(2)  $A_1 \subseteq f^{-1}[f[A_1]]$ .

*Proof.* Recall that

$$\begin{aligned} f^{-1}[B_1] &:= \{a \in A : f(a) \in B_1\}, & f[A_1] &:= \{f(a) \in B : a \in A_1\}, \\ f[f^{-1}[B_1]] &:= \{f(a) \in B : a \in f^{-1}[B_1]\}, & f^{-1}[f[A_1]] &:= \{a \in A : f(a) \in f[A_1]\}. \end{aligned}$$

(1) Let  $b \in f[f^{-1}[B_1]]$ . By the definition of the image,

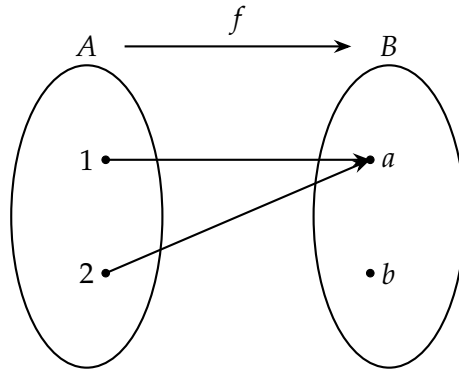
$$\exists a \in f^{-1}[B_1] \text{ such that } b = f(a).$$

From the definition of the preimage,  $a \in f^{-1}[B_1] \Rightarrow f(a) \in B_1$ . Thus  $b = f(a) \in B_1$ .

(2) Let  $a \in A_1$ . By the definition of the image, we know that  $f(a) \in f[A_1]$ . By the definition of the preimage,  $f(a) \in f[A_1] \Rightarrow \underline{a \in f^{-1}[f[A_1]]}$ .

□

**Example (Counterexample).** Consider a function  $f : A \rightarrow B$ , where  $A = \{1, 2\}$  and  $B = \{a, b\}$ .



(1) Let  $B_1 = \{b\} \subseteq B$ . Then  $f^{-1}[B_1] = \emptyset$  and so

$$f[f^{-1}[B]] = f[\emptyset] = \emptyset \neq \{b\} = B_1.$$

(2) Let  $A_1 = \{1\} \subseteq A$ . Then  $f[A_1] = f[\{1\}] = \{a\}$  and so

$$f^{-1}[f[A_1]] = f^{-1}[\{a\}] = \{1, 2\} \neq \{1\} = A_1.$$



### Injection and Surjection

**Definition.** Let  $f : A \rightarrow B$  is a function from  $A$  to  $B$ .

- A function  $f$  is **an injection** or **injective** (or **one-to-one**) if and only if

$$\forall a_1, a_2 \in A : [f(a_1) = f(a_2) \implies a_1 = a_2].$$

That is, an **injection** is a mapping such that the output uniquely determines its input.

- A function  $f$  is **a surjection** or **surjective** (or **onto**) if and only if

$$\forall b \in B : [\exists a \in A \text{ such that } f(a) = b].$$

That is, a **surjection** is a mapping such that every element of  $B$  is related to by some element of  $A$ .

**Remark.** A function  $f$  is **bijective** if and only if  $f$  is both injective and surjective.

- $f$  is **a bijection** (or **bijective**).
- $f$  is **one-to-one and onto** (or **a one-to-one correspondence**).

### Composition of Functions

**Definition.** Let  $f_1 : A \rightarrow B$  and  $f_2 : B \rightarrow C$  be functions such that  $\text{Cdm}(f_1) = B = \text{Dom}(f_2)$ . The **composition**  $f_2 \circ f_1$  is defined as:

$$(f_2 \circ f_1)(a) := f_2(f_1(a)).$$

for all  $a \in A$ .

**Note** (Diagram).

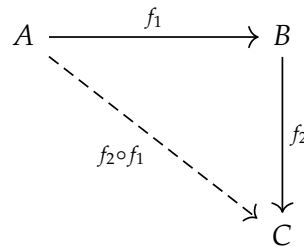


Figure 3: Diagram of  $f_2 \circ f_1$ .

**Note** (Illustration).

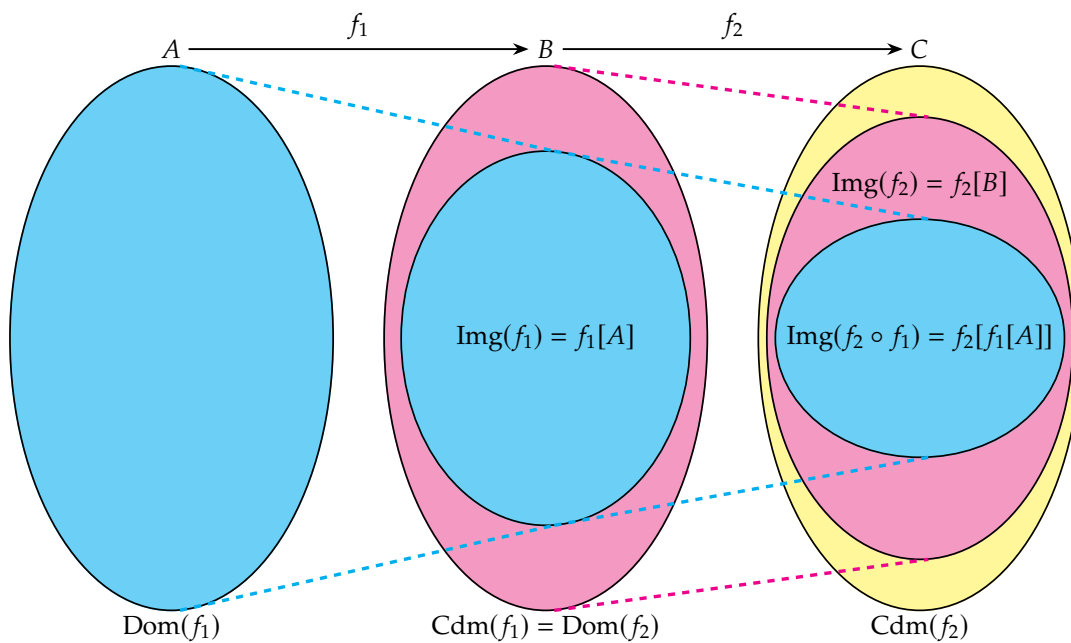


Figure 4: Illustration of  $f_2 \circ f_1$

**Remark.** The composition is associative. For any  $f, g, h \in G$ ,  $(f \circ g) \circ h = f \circ (g \circ h)$ .

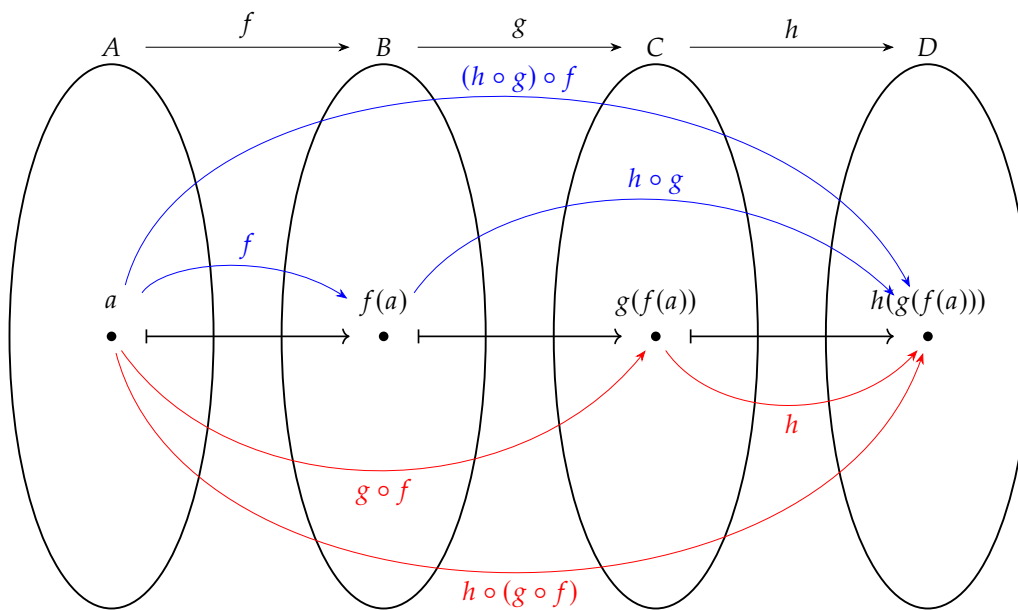


Figure 5: Associativity of Composition.

**Theorem 5** Let  $A$  and  $B$  be sets. Let  $f : A \rightarrow B$  be a function.

(1)  $f$  is one-to-one if and only if there exists the function  $g : B \rightarrow A$  such that  $g \circ f = \text{id}_A$ .

(2)  $f$  is onto if and only if there exists the function  $g : B \rightarrow A$  such that  $f \circ g = \text{id}_B$ .

**Remark.**

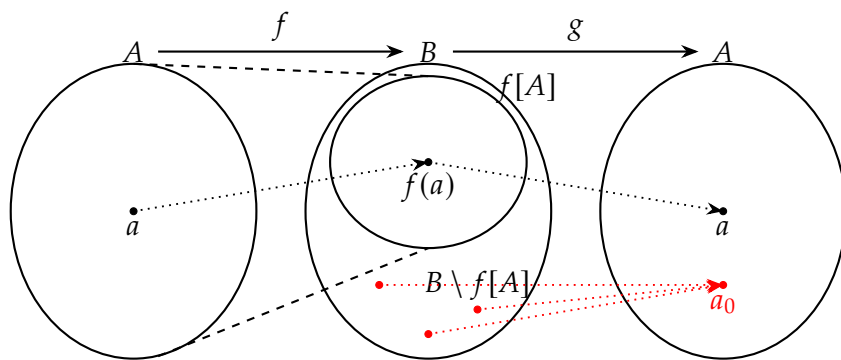
$$(1) \quad A \xrightarrow{f} B \xrightarrow{\exists g} A$$

$$g \circ f = \text{id}_A$$

$$(2) \quad B \xrightarrow{\exists g} A \xrightarrow{f} B$$

$$f \circ g = \text{id}_B$$

*Proof.* (1)  $(\Rightarrow)$  Assume that  $f : A \rightarrow B$  is injective. We need to construct a function  $g : B \rightarrow A$  such that  $g \circ f = \text{id}_A$ .



We define a function  $g : B \rightarrow A$  given by

$$g(b) = \begin{cases} a & \text{if } \exists! a \in A \text{ such that } f(a) = b \\ a_0 & \text{if } b \notin f[A] \end{cases}$$

for all  $b \in B$ , where  $a_0 \in A$  is an arbitrary element of  $A$ . Since  $f$  is one-to-one,  $g$  is well-defined. For any  $a \in A$ , we have  $f(a) \in B$ . By the definition of  $g$ , we obtain  $g(f(a)) = a$ . Thus,

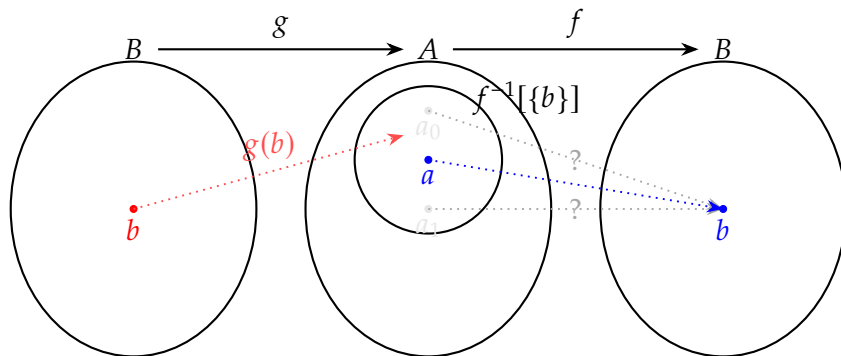
$$(g \circ f)(a) = g(f(a)) = a = \text{id}_A(a)$$

for all  $a \in A$ .

$(\Leftarrow)$  Assume that there exists  $g : B \rightarrow A$  such that  $g \circ f = \text{id}_A$ . Suppose that  $f(a_1) = f(a_2)$  for any  $a_1, a_2 \in A$ . Then

$$\begin{aligned} f(a_1) = f(a_2) &\implies g(f(a_1)) = g(f(a_2)) \quad \text{by def. of a function} \\ &\implies a_1 = a_2. \quad \because g \circ f = \text{id}_A \end{aligned}$$

- (2) ( $\Rightarrow$ ) Assume  $f : A \rightarrow B$  is surjective. Then, for every  $b \in B$ , there exists at least one  $a \in A$  such that  $f(a) = b$ . We need to construct a function  $g : B \rightarrow A$  such that  $f \circ g = \text{id}_B$ , i.e.,  $f(g(b)) = b$  for every  $b \in B$ .



The **Axiom of Choice**<sup>1</sup> allows us to define  $g : B \rightarrow A$  given by

$$g(b) = a \in f^{-1}[\{b\}]$$

for each  $b \in B$ . Thus,

$$\begin{aligned} (f \circ g)(b) &= f(g(b)) && \text{by def. of composition} \\ &= f(a) && \text{by def. of } g \\ &= b && \text{by assumption} \\ &= \text{id}_B(b) \end{aligned}$$

for all  $b \in B$ . That is,  $f \circ g = \text{id}_B$ . Without the Axiom of Choice, we cannot always guarantee the existence of such a selection function, especially when the sets  $f^{-1}[\{b\}]$  are uncountable.

- ( $\Leftarrow$ ) Assume that there exists  $g : B \rightarrow A$  such that  $f \circ g = \text{id}_B$ . Let  $b \in B$ . Since  $f \circ g = \text{id}_B$ , we have  $f(g(b)) = \text{id}_B(b) = b$ . Thus, for every  $b \in B$ ,

$$\exists a = g(b) \in A \quad \text{such that} \quad f(a) = f(g(b)) = b.$$

□

<sup>1</sup>Here,  $\mathbb{S} = \{f^{-1}[\{b\}] \subseteq A : b \in B\}$  and  $\bigcup \mathbb{S} = \bigcup_{b \in B} f^{-1}[\{b\}] = A$ . That is, there is a choice function  $F : \mathcal{P}(A) \setminus \{\emptyset\} \rightarrow A$ .

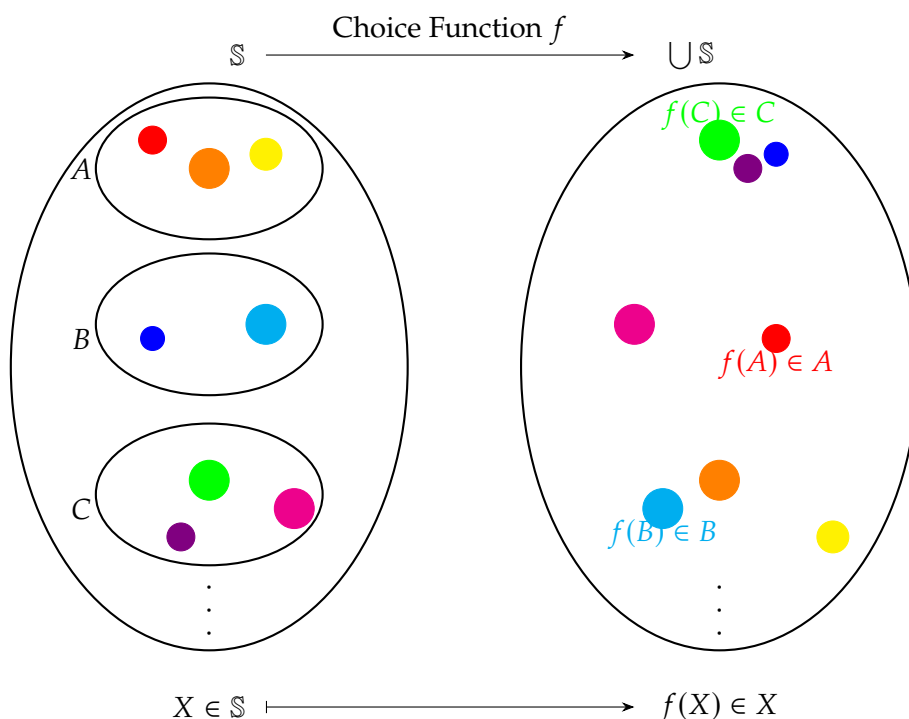
**Note** (Axiom of Choice). Let  $\mathbb{S}$  be a set of non-empty sets.

“It is always possible to assert the existence of a choice function that selects one element from each member of the set.”

Formally,

$$\forall \mathbb{S} : \left[ \emptyset \notin \mathbb{S} \implies \exists \left( f : \mathbb{S} \rightarrow \bigcup \mathbb{S} \right) \text{ s.t. } \forall X \in \mathbb{S} : [f(X) \in X] \right].$$

For example, let  $\mathbb{S} = \{A, B, C, \dots\}$  and  $\bigcup \mathbb{S} = A \cup B \cup C \cup \dots$ .



## References

- [1] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 1. 집합론 기초 (a).” YouTube Video, 30:55. Published September 05, 2019. URL: [https://www.youtube.com/watch?v=9HUK8zays2E&list=PL4m4z\\_pFWq2pLwFsWf0KJX\\_uMNo-jktN5&index=132](https://www.youtube.com/watch?v=9HUK8zays2E&list=PL4m4z_pFWq2pLwFsWf0KJX_uMNo-jktN5&index=132).
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