

Advanced Calculus I

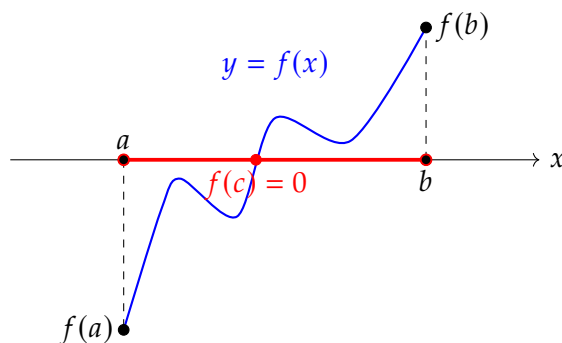
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We cover the following topics in this note.

- Boundedness, Supremum and Infimum
- Least Upper Bound Property (Completeness Axiom)
- Well-Ordering Principle and Mathematical Induction
- Archimedean Property

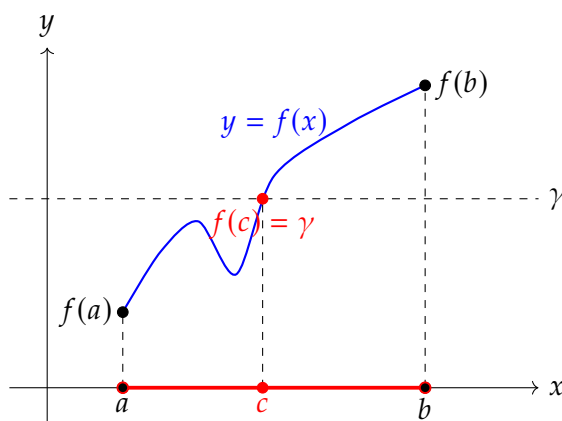
Observation. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Suppose that $f(a)$ and $f(b)$ have opposite signs, i.e., $f(a) \cdot f(b) < 0$. Then, there exists a point $c \in (a, b)$ such that $f(c) = 0$.



Intermediate Value Theorem

Theorem. Let $[a, b] \subseteq \mathbb{R}$ be a real interval, and let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function on $[a, b]$. Let $f(a) < f(b)$. If $\gamma \in \mathbb{R}$ satisfies $f(a) < \gamma < f(b)$, then

$$\exists c \in (a, b) \text{ such that } f(c) = \gamma.$$



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1 Numbers

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$	Natural Numbers
$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \dots\}$	Integers (Zahlen ¹)
$\mathbb{Q} = \left\{ \frac{q}{p} : p, q \in \mathbb{Z}, p \neq 0 \right\}$	Rationals (Quotient ²)
$\mathbb{R} = \{\text{Limit of sequences of rational numbers}\}$	Real Numbers
$\mathbb{C} = \{p + q\sqrt{-1} : p, q \in \mathbb{R}\}$	Complex numbers

Remark. The set $\mathbb{Z}_{\geq 0} := \{0\} \cup \mathbb{N} = \{0, 1, 2, \dots\}$ is called *non-negative integers*.

Remark. Let $n_0 \in \mathbb{Z}$ is given. Then

$$\mathbb{Z}_{\geq n_0} := \{n \in \mathbb{Z} : n \geq n_0\}.$$

¹The integer set is denoted by \mathbb{Z} because it comes from the German word “Zahlen”, meaning “numbers”.

²The rational set is denoted by \mathbb{Q} because it stands for “Quotient”, representing numbers that can be expressed as the quotient of two integers.

2 Least Upper Bound Property of \mathbb{R}

Boundedness

Definition. Let S be a non-empty subset of \mathbb{R} .

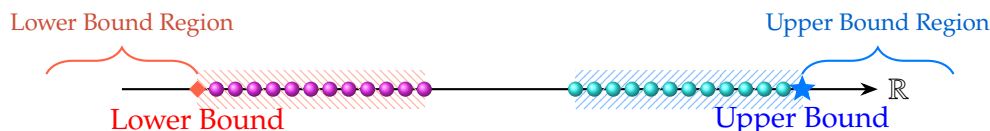
(1) A set S is said to be **bounded above** if $\exists \beta \in \mathbb{R}$ such that for all $x \in S$, $x \leq \beta$.

A real number $\beta \in \mathbb{R}$ is called an **upper bound** of S .

(2) A set S is said to be **bounded below** if $\exists \alpha \in \mathbb{R}$ such that for all $x \in S$, $\alpha \leq x$.

A real number $\alpha \in \mathbb{R}$ is called an **lower bound** of S .

(3) A set S is **bounded** if it is bounded above and below.



Remark (Caution!). It is **not** guaranteed that $\beta \in S$ and $\alpha \in S$.

Remark. Let $\emptyset \neq S \subseteq \mathbb{R}$, and let $\alpha, \beta \in \mathbb{R}$.

S is bounded above (by β) $\iff S$ has an upper bound β

β is an upper bound of S $\iff \forall x \in S, x \leq \beta$

S is bounded below (by α) $\iff S$ has a lower bound α

α is a lower bound of S $\iff \forall x \in S, \alpha \leq x$

Remark.

1. The empty $S = \emptyset$ is bounded.

(i) (\emptyset is bounded above) We need to find a real number $\beta \in \mathbb{R}$ s.t. for all $x \in \emptyset$, $x \leq \beta$. Since " $\forall x \in \emptyset, x \leq \beta$ " is vacuously true, we can choose any real number as β .

(ii) (\emptyset is bounded below) Similarly, we can choose any $\alpha \in \mathbb{R}$ s.t. for all $x \in \emptyset$, $\alpha \leq x$.

2. An upper bound and a lower bound may not be unique. A set $S(\neq \emptyset) \subseteq \mathbb{R}$ may have multiple upper bounds and multiple lower bounds.

Exercise. Show that $A = \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$ has an upper bound and a lower bound.

Sol. The elements of A are: $A = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}$. Let $x \in A$. Then $x = 1 - 1/n$ for some $n \in \mathbb{N}$.

(i) Since $n \in \mathbb{N}$, we have $1/n > 0$. Therefore

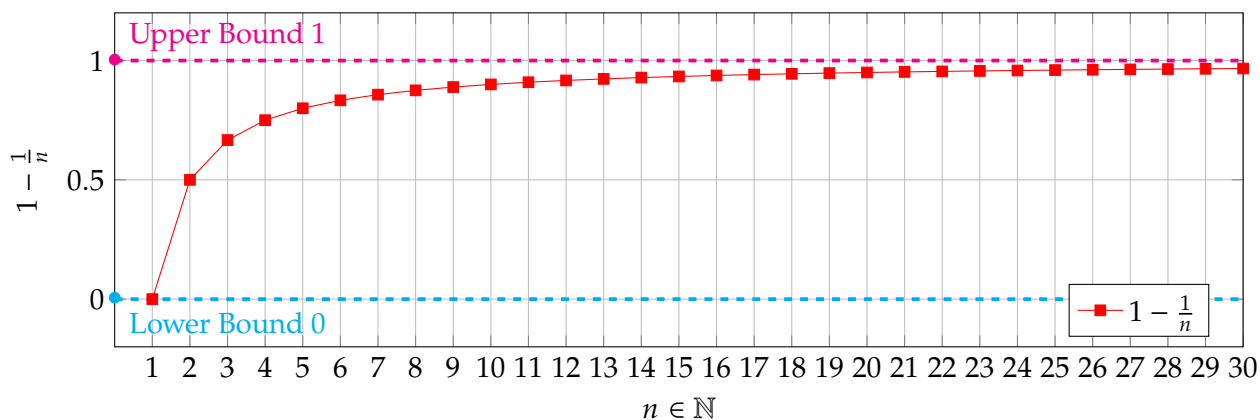
$$1 - \frac{1}{n} < 1.$$

Thus, for all $x = 1 - 1/n \in A$, we have $x \leq 1$. Hence 1 be an upper bound of A .

(ii) Since $n \in \mathbb{N}$, we have $n \geq 1$, so $1/n \leq 1$. Therefore,

$$1 - \frac{1}{n} \geq 1 - 1 = 0.$$

Thus, for all $x = 1 - 1/n \in A$, we have $x \geq 0$. Hence 0 is a lower bound of A .



□

Exercise. Show that \mathbb{N} has a lower bound but does not have an upper bound.

Sol. Let \mathbb{N} is the set of natural numbers.

(i) For each $n \in \mathbb{N}$, we have $n \geq 1$. Therefore, 1 is a lower bound of \mathbb{N} .

(ii) Assume that \mathbb{N} is bounded above by $\beta \in \mathbb{R}$, i.e.,

$$\exists \beta \in \mathbb{R} \text{ such that } \forall n \in \mathbb{N}, n \leq \beta.$$

However, for any $\beta \in \mathbb{R}$, we can always find $n \in \mathbb{N}$ s.t. $\beta < n$, by choosing $n = \lfloor \beta \rfloor + 1 > \beta$.

It contradicts to the our assumption. Thus \mathbb{N} does not have an upper bound.

□

Exercise (★). Consider a set

$$A := \{r \in \mathbb{Q} : r > 0, r^2 < 2\}$$

of positive rational numbers whose squares are less than 2. Then A has a lower bound 0. Prove that A does not have the maximum element.

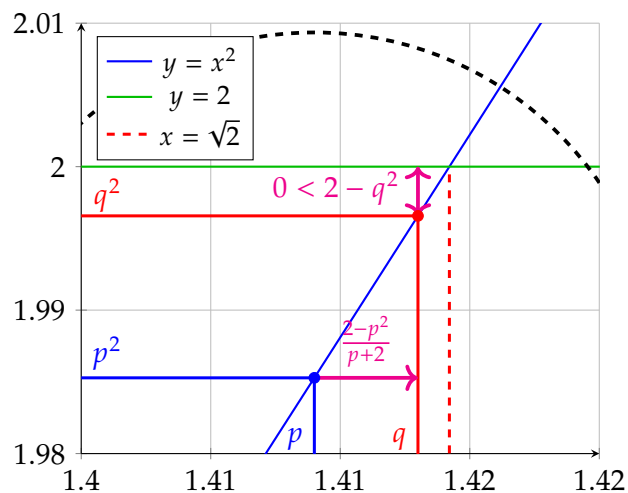
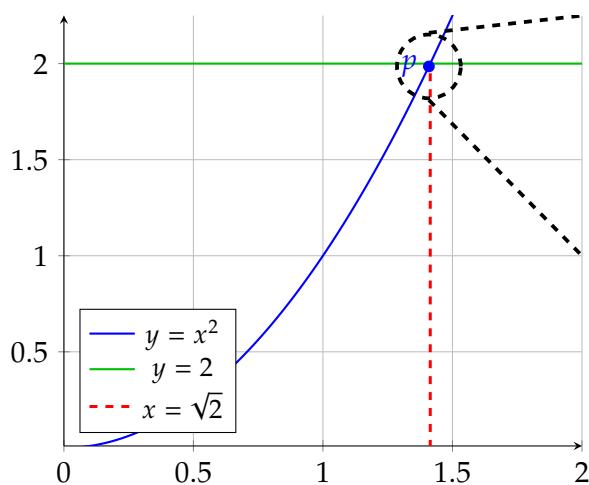
Sol. Note that

$$A := \{r \in \mathbb{Q} : r > 0, r^2 < 2\} = \{\dots, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, 1.4142135, \dots\}.$$

Let $a \in A$. Then

$$a^2 < 2 < 4 \implies a^2 < 4 \implies |a| < 2 \implies a < 2.$$

That is, A is bounded above by 2. Suppose that p is a maximum of A . Since $p \in A$, $p > 0$ and $p^2 < 2$.



Define a rational number $q \in \mathbb{Q}$, for contradiction, by

$$q := p + \frac{2 - p^2}{p + 2} = \frac{2p + 2}{p + 2},$$

where $p < q$. We must show that $q \in A$.

Clearly $q > 0$. We claim that $q^2 < 2$:

$$\begin{aligned}
 2 - q^2 &= 2 - \left(p + \frac{2 - p^2}{p + 2} \right)^2 = 2 - \left(p^2 + \frac{2p(2 - p^2)}{p + 2} + \frac{(2 - p^2)^2}{(p + 2)^2} \right) \\
 &= 2 - p^2 - \frac{2p(2 - p^2)}{p + 2} - \frac{(2 - p^2)^2}{(p + 2)^2} \\
 &= \frac{(2 - p^2)(p + 2)^2 - 2p(2 - p^2)(p + 2) - (2 - p^2)^2}{(p + 2)^2} \\
 &= \frac{1}{(p + 2)^2} \left[(2 - p^2)(p^2 + 4p + 4) + (2p^2 + 4p)(p^2 - 2) + (-p^4 + 4p^2 - 4) \right] \\
 &= \frac{1}{(p + 2)^2} \left[\cancel{(-1 + 2 - 1)}p^4 + \cancel{(-4 + 4)}p^3 + (2 - 4 - \cancel{4} + \cancel{4})p^2 + \cancel{(8 - 8)}p + (8 - 4) \right] \\
 &= \frac{1}{(p + 2)^2} \left[-2p^2 + 4 \right] = \frac{2(2 - p^2)}{(p + 2)^2} > 0.
 \end{aligned}$$

Thus, $q \in A$ with $p < q$. It contradicts to the assumption that $p = \max A$. Hence it is proved. \square

Note (Existence of $\sqrt{2}$). There exists $x \in \mathbb{R}$ such that $x^2 = 2$. We write $x = \sqrt{2} > 0$.

Proof. Consider a set

$$A = \left\{ x \in \mathbb{Q} : x > 0, x^2 < 2 \right\} \subseteq \mathbb{R}.$$

Since $1 \in A$, we know $A \neq \emptyset$. Clearly, A is bounded above (by 2). By the completeness axiom, A has the supremum $\lambda = \sup A \in \mathbb{R}$. Define a number $\mu \in \mathbb{R}$ by

$$\mu := \lambda + \frac{2 - \lambda^2}{\lambda + 2}, \text{ and then } \mu^2 - 2 = \frac{2(\lambda^2 - 2)}{(\lambda + 2)^2}.$$

(Case I) Let $\lambda^2 < 2$. Then $\lambda < \mu$ and $\mu^2 < 2$. That is, $\mu \in A$ with $\lambda < \mu$. It contradicts to the fact that λ is an upper bound of A .

(Case II) Let $\lambda^2 > 2$. Then

$$\begin{aligned}
 \mu < \lambda \text{ and } 2 < \mu^2 &\implies \forall x \in A, x^2 < 2 < \mu^2 \implies \forall x \in A, x < \mu \\
 &\implies \mu \text{ is an upper bound of } A \text{ with } \mu < \lambda.
 \end{aligned}$$

It contradicts to the fact that λ is an upper bound of A .

By trichotomy, we obtain $\lambda^2 = 2$ with $\lambda \in \mathbb{R}$. \square

Supremum and Infimum

Definition. Let $\emptyset \neq S \subseteq \mathbb{R}$.

(1) Let S is bounded above. The number $\beta \in \mathbb{R}$ is the **supremum** (or the **least upper bound**) of S if and only if

(i) β is an upper bound of S , i.e., $\forall x \in S, x \leq \beta$;

(ii) u is any upper bound of $S \implies \beta \leq u$.

We write $\beta = \sup S \in \mathbb{R}$.

(2) Let S is bounded below. The number $\alpha \in \mathbb{R}$ is the **infimum** (or the **greatest lower bound**) of S if and only if

(i) α is a lower bound of S , i.e., $\forall x \in S, \alpha \leq x$;

(ii) if ℓ is any lower bound of S then $\ell \leq \alpha$.

We write $\alpha = \inf S \in \mathbb{R}$.

Remark (Caution!). It is **not** guaranteed that $\sup S \in S$ and that $\inf S \in S$.

Remark. Let $\emptyset \neq S \subseteq \mathbb{R}$.

(1) Suppose S is bounded above. Then

$$\begin{aligned} \beta = \sup S &\iff (i) \forall x \in S, x \leq \beta; \\ &\quad (ii) \forall \text{upper bound } u \in S, \beta \leq u. \end{aligned}$$

(2) Suppose S is bounded below. Then

$$\begin{aligned} \alpha = \inf S &\iff (i) \forall x \in S, \alpha \leq x; \\ &\quad (ii) \forall \text{lower bound } \ell \in S, \ell \leq \alpha. \end{aligned}$$

Remark.

$$\begin{aligned} [u \text{ is any upper bound of } S \implies \beta \leq u] &\iff [u < \beta \implies u \text{ is NOT an upper bound of } S] \\ &\iff \beta \leq u \text{ for all upper bound } u \text{ of } S. \end{aligned}$$

$$\begin{aligned} [\ell \text{ is any lower bound of } S \implies \ell \leq \alpha] &\iff [\alpha < \ell \implies \ell \text{ is NOT a lower bound of } S] \\ &\iff \ell \leq \alpha \text{ for all lower bound } \ell \text{ of } S. \end{aligned}$$

Remark (Uniqueness of Supremum and Infimum).

(Proof by Trichotomy) Let $\emptyset \neq S \subseteq \mathbb{R}$ and S is bounded above. Suppose that $\sup S = a$ and that $\sup S = b$ also. By trichotomy, exactly one of the following holds:

$$a = b, \quad a < b, \quad \text{or} \quad b < a.$$

However, $a < b$ and $b < a$ are impossible, as a and b are upper bounds, respectively. Hence $a = b$. Similarly, infimum also is unique. \square

(Proof by Anti-symmetry³ of \leq) Let $\emptyset \neq T \subseteq \mathbb{R}$ and T is bounded above. Suppose that $\sup T = a$ and that $\sup T = b$ also. Then

(a) a is an upper bound of T in \mathbb{R} ; (b) a is a supremum of T in \mathbb{R} ;

(c) b is an upper bound of T in \mathbb{R} ; (d) b is a supremum of T in \mathbb{R} .

Then

$$(a) \text{ and } (d) \implies b \leq a, \quad (b) \text{ and } (c) \implies a \leq b.$$

By the anti-symmetry of \leq , we obtain $a = b$. Similarly, infimum also is unique. \square

Unbounded Sets

Definition. Let $\emptyset \neq S \subseteq \mathbb{R}$.

(1) If S is unbounded above, then we write $\sup S = \infty$.

(2) If S is unbounded below, then we write $\inf S = -\infty$.

(3) $\sup \emptyset := -\infty$ and $\inf \emptyset := \infty$.

Example. $\sup \mathbb{N} = \infty$ and $\inf \mathbb{Z} = -\infty$.

Remark. Suppose that $\emptyset \neq S \subseteq \mathbb{R}$ is unbounded above. Then

$$\neg[\exists \beta \in \mathbb{R} \text{ s.t. } \forall x \in S, x \leq \beta], \quad \text{i.e.,} \quad [\forall \beta \in \mathbb{R}, \exists x \in S \text{ s.t. } \beta < x].$$

Suppose that $\emptyset \neq T \subseteq \mathbb{R}$ is unbounded below. Then

$$\neg[\exists \alpha \in \mathbb{R} \text{ s.t. } \forall x \in T, \alpha \leq x], \quad \text{i.e.,} \quad [\forall \alpha \in \mathbb{R}, \exists x \in T \text{ s.t. } x < \alpha].$$

³A relation \mathcal{R} on a set S is anti-symmetric if, for $a, b \in S$, $a \mathcal{R} b \wedge b \mathcal{R} a \implies a = b$.

Approximation Property for Supremum and Infimum I

Proposition 1.

(1) Let $\emptyset \neq S \subseteq \mathbb{R}$ which is bounded above, and let λ be an upper bound of S in \mathbb{R} .

$$\lambda = \sup S \iff \forall \varepsilon > 0, \exists x_\varepsilon \in S \text{ s.t. } \lambda - \varepsilon < x_\varepsilon \leq \lambda.$$

(2) Let $\emptyset \neq T \subseteq \mathbb{R}$ which is bounded below, and let γ be a lower bound of T in \mathbb{R} .

$$\gamma = \inf T \iff \forall \varepsilon > 0, \exists x_\varepsilon \in T \text{ s.t. } \gamma \leq x_\varepsilon < \gamma + \varepsilon.$$

Proof. (1) (\Rightarrow) Let $\lambda = \sup S$. Let $\varepsilon > 0$. Suppose that

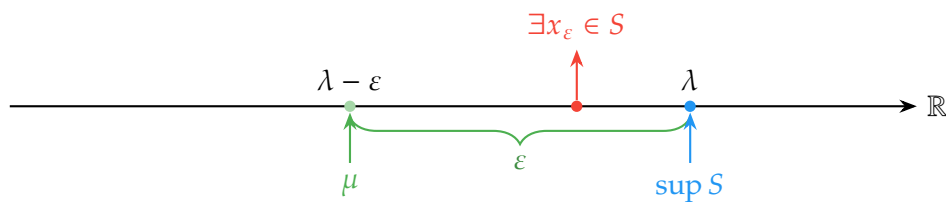
$$\neg[\exists x_\varepsilon \in S \text{ s.t. } \lambda - \varepsilon < x_\varepsilon], \quad \text{i.e.,} \quad [\forall x_\varepsilon \in S, x_\varepsilon \leq \lambda - \varepsilon].$$

Then $\lambda - \varepsilon$ be an upper bound of S . Since $\varepsilon > 0$, we have $\lambda - \varepsilon < \lambda$. It contradicts to the assumption that λ is the *least* upper bound of S . Thus

$$\exists x_\varepsilon \in S \text{ s.t. } \lambda - \varepsilon < x_\varepsilon.$$

Since $x_\varepsilon \in S$ and λ is an upper bound of S , $x_\varepsilon \leq \lambda$ holds.

(\Leftarrow) Let the RHS holds. We claim that λ is the *least* upper bound of S :



Assume, for contradiction, that there exists a smaller upper bound $\mu \in \mathbb{R}$ s.t. $\mu < \lambda$.

Let $\varepsilon = \lambda - \mu > 0$. Then

$$\exists x_\varepsilon \in S \text{ such that } \lambda - \varepsilon < x_\varepsilon \leq \lambda,$$

and so

$$\mu = \lambda - \varepsilon < x_\varepsilon \leq \lambda \implies \mu < x_\varepsilon \leq \lambda.$$

It contradicts to the assumption that μ is an upper bound of S . Thus, λ be the least upper bound of S .

(2) (\Rightarrow) Let $\gamma = \inf T$. Let $\varepsilon > 0$. Suppose that

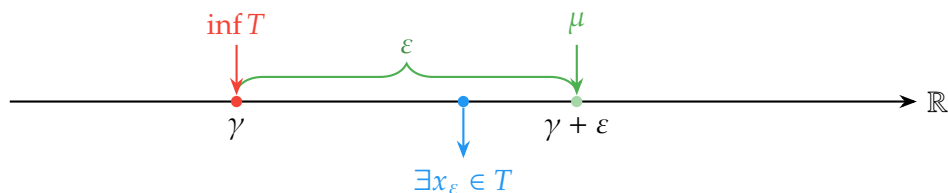
$$\neg[\exists x \in T \text{ s.t. } x < \gamma + \varepsilon], \quad \text{i.e.,} \quad [\forall x \in T, \gamma + \varepsilon \leq x].$$

Then $\gamma + \varepsilon$ be a lower bound of T . Since $\varepsilon > 0$, we have $\gamma < \gamma + \varepsilon$. It contradicts to the assumption that γ is the *greatest* lower bound of T . Thus,

$$\exists x_\varepsilon \in T \text{ s.t. } x_\varepsilon < \gamma + \varepsilon.$$

Since $x_\varepsilon \in T$ and γ is a lower bound of T , $\gamma \leq x_\varepsilon$ holds.

(\Leftarrow) Let the RHS holds. We claim that γ is the *greatest* lower bound of T :



Assume, for contradiction, that there exists a greater lower bound $\gamma < \mu$.

Let $\varepsilon = \mu - \gamma > 0$. Then

$$\exists x_\varepsilon \in T \text{ such that } \gamma \leq x_\varepsilon < \gamma + \varepsilon,$$

and so $\gamma \leq x_\varepsilon < \mu$. It contradicts to the assumption that γ is an lower bound of T . Thus, γ be the greatest lower bound of T .

□

Remark. See **Approximation Property for Supremum and Infimum II**.

Least Upper Bound Property (Completeness Axiom) of Real Number

Axiom. Every non-empty subset of \mathbb{R} that is bounded above has the supremum in \mathbb{R} .

Example. \mathbb{Q} does NOT hold completeness axiom. We already showed that $\{x \in \mathbb{Q} : x > 0, x^2 < 2\}$ has NO supremum in \mathbb{Q} .

Infimum Property

Axiom. Every non-empty subset of \mathbb{R} that is bounded below has the infimum in \mathbb{R} .

3 Well-Ordering Principle and Mathematical Induction

Well-Ordering Principle (Principle of the Least Element)

Axiom. Every non-empty subset S of \mathbb{N} has a least element, i.e.,

$$\emptyset \neq S \subseteq \mathbb{N} \implies \exists n \in S \text{ s.t. } \forall k \in S, n \leq k.$$

In other words, $[\emptyset \neq S \subseteq \mathbb{N} \implies \exists n \in S \text{ s.t. } n = \min(S)]$.

Remark (general version). $\emptyset \neq S \subseteq \mathbb{Z}_{\geq n_0} \implies \exists n \in S \text{ s.t. } n = \min S \geq n_0$.

Principle of Mathematical Induction

Axiom. Suppose that $S \subseteq \mathbb{N}$ satisfies the following two conditions:

1. (Basic Step) $1 \in S$, and
2. (Inductive Step) $n \in S \implies n + 1 \in S$.

Then $S = \mathbb{N}$.

Remark (general version). Let $n_0 \in \mathbb{Z}$ be given, and let $S \subseteq \mathbb{Z}_{\geq n_0}$. Suppose that S satisfies the following two conditions:

1. (Basic Step) $n_0 \in S$, and
2. (Inductive Step) $\forall n \in \mathbb{Z}_{\geq n_0} : [n \in S \implies n + 1 \in S]$.

Then $\forall n \in \mathbb{Z}_{\geq n_0} : n \in S$, i.e., $S = \mathbb{Z}_{\geq n_0}$.

Remark. To show that a mathematical statement $P(n)$ (property for n) holds for $n \in \mathbb{N}$, simply verify that the set

$$S := \{n \in \mathbb{N} : P(n) \text{ holds}\}$$

satisfies the following conditions:

(Step 1) Show that $P(1)$ holds.

(Step 2) Show that $P(n + 1)$ holds with the assumption $P(n)$ holds.

Equivalence of Well-Ordering Principle and Induction

Theorem.

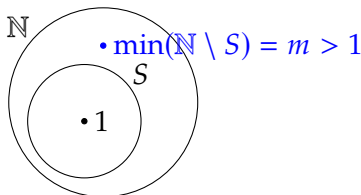
The Well-Ordering Principle and Principle of Mathematical Induction are equivalent.

Proof. (WOP \Rightarrow MI) Let $S \subseteq \mathbb{N}$ satisfy the followings: (i) $1 \in S$ and (ii) $k \in S \Rightarrow k + 1 \in S$. We want to establish that $S = \mathbb{N}$ by the Well-Ordering Principle (WOP).

Assume for contradiction that $S \neq \mathbb{N}$. Then $S \subsetneq \mathbb{N}$, which means $\mathbb{N} \setminus S \neq \emptyset$. By the WOP,

$$\exists m \in \mathbb{N} \setminus S \text{ s.t. } m = \min(\mathbb{N} \setminus S).$$

Since $1 \in S$, we have $1 \notin \mathbb{N} \setminus S$, so $m \neq 1$ and thus $m > 1$ (or $m \geq 2$).



Then

$$m = \min(\mathbb{N} \setminus S) \xRightarrow{\text{by minimality of } m} m - 1 \notin \mathbb{N} \setminus S \implies m - 1 \in S \xRightarrow{\text{by (ii)}} m \in S \text{ } \nabla.$$

Hence $S = \mathbb{N}$.

(MI \Rightarrow WOP) Suppose that $\emptyset \neq S \subseteq \mathbb{N}$ has no least element. Define the set $T \subseteq \mathbb{N}$ by

$$T := \{n \in \mathbb{N} : 1, 2, 3, \dots, n \notin S\}.$$

For example, if $3 \in T$ then $1, 2, 3 \notin S$; conversely, if $1, 2, 3 \notin S$ then $3 \in T$. We claim that T satisfies the condition of MI:

- (i) (Basic Step) Since S has no least element, $1 \notin S$. Therefore, $1 \in T$.
- (ii) (Inductive Step) Suppose that $k \in T$. This means that $1, 2, \dots, k \notin S$. Since S has no least element, $k + 1 \notin S$ (otherwise $k + 1$ would be a least element of S). Therefore

$$1, 2, \dots, k, k + 1 \notin S, \text{ i.e., } k + 1 \in T.$$

By the Principle of Mathematical Induction, we have $T = \mathbb{N}$. It follows that no natural number is in S , which contradicts $S \neq \emptyset$. Hence it is proved. \square

4 Archimedean Principle

Archimedean Property (The Unboundedness of Natural Numbers)

Theorem. Let $x \in \mathbb{R}$. Then

$$\exists n \in \mathbb{N} \text{ such that } x < n.$$



Proof. Assume, for contradiction, that

$$\forall n \in \mathbb{N}, n \leq x.$$

That is $\mathbb{N} \subseteq \mathbb{R}$ is bounded above by $x \in \mathbb{R}$. By the completeness axiom of \mathbb{R} ,

$$\exists \sup \mathbb{N} =: s \in \mathbb{R}.$$

By **Proposition 1**, for $\varepsilon = 1 > 0$, $\exists n \in \mathbb{N}$ s.t. $s - 1 < n$. Then

$$s - 1 < n \implies s < n + 1 \stackrel{n+1 \in \mathbb{N}}{\implies} s < n + 1 \leq \sup \mathbb{N} = s \quad \text{?}$$

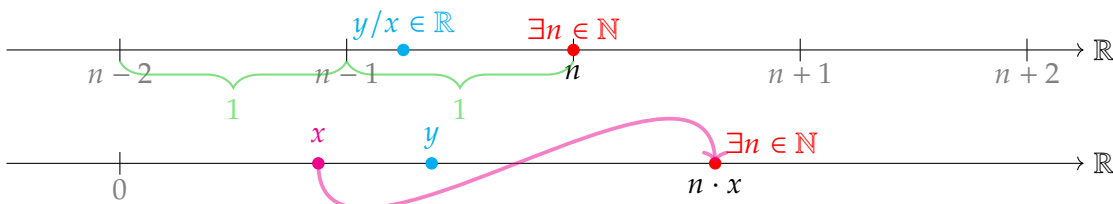
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Corollary. Let $x, y \in \mathbb{R}$ with $x > 0$. Then $\exists n \in \mathbb{N}$ such that $y < n \cdot x$.

Proof. Since \mathbb{R} is a field, we know $\frac{y}{x} \in \mathbb{R}$. By the Archimedean property,

$$\exists n \in \mathbb{N} \text{ such that } \frac{y}{x} < n, \text{ i.e., } y < n \cdot x.$$

□

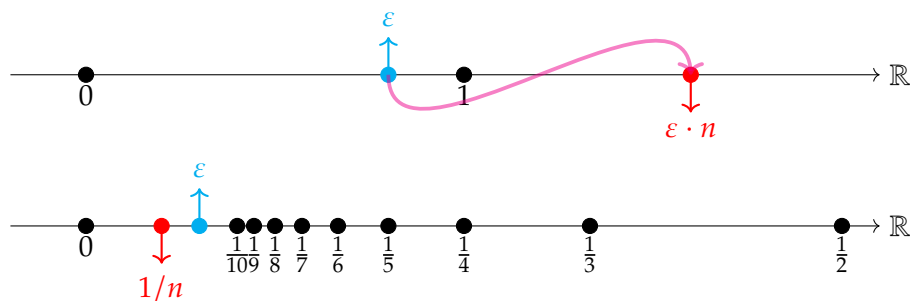


Corollary. $\forall \varepsilon > 0, \exists n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

Proof. For $\varepsilon \in \mathbb{R}^+$ and $1 \in \mathbb{R}$, by Archimedean property, we have

$$\exists n \in \mathbb{N} \text{ such that } 1 < \varepsilon \cdot n, \text{ i.e., } \frac{1}{n} < \varepsilon.$$

□



Note (Archimedean Property in Number Theory). Let $a, b \in \mathbb{N}$. Then $\exists n \in \mathbb{N}$ such that $b < na$.

Proof. Suppose that $\exists a, b \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, na \leq b.$$

Define a set S by

$$S := \{b - na \geq 0 : n \in \mathbb{N}\} \subseteq \mathbb{Z}_{\geq 0}.$$

By the well-ordering principle, $\exists s := \min S$. Since $s = \min S \in S$, we have

$$s = b - ma \text{ for some } m \in \mathbb{N}.$$

Since $m + 1 \in \mathbb{N}$ also, we have $b - (m + 1)a \in S$, and so

$$\begin{aligned} b - (m + 1)a &= b - ma - a \\ &< b - ma \quad \because a \in \mathbb{N}, \text{ i.e., } a > 0 \\ &= \min S \nmid. \end{aligned}$$

Hence it is proved.

□

Approximation Property for Supremum and Infimum II

Proposition 2.

(1) Let $\emptyset \neq S \subseteq \mathbb{R}$ which is bounded above, and let λ be an upper bound of S in \mathbb{R} .

$$\begin{aligned} \lambda = \sup S &\iff \forall \varepsilon > 0, \exists x_\varepsilon \in S \text{ s.t. } \lambda - \varepsilon < x_\varepsilon \leq \lambda \\ &\iff \forall n \in \mathbb{N}, \exists x_n \in S \text{ s.t. } \lambda - \frac{1}{n} < x_n \leq \lambda. \end{aligned}$$

(2) Let $\emptyset \neq T \subseteq \mathbb{R}$ which is bounded below, and let γ be a lower bound of T in \mathbb{R} .

$$\begin{aligned} \gamma = \inf T \in \mathbb{R} &\iff \forall \varepsilon > 0, \exists x_\varepsilon \in T \text{ s.t. } \gamma \leq x_\varepsilon < \gamma + \varepsilon \\ &\iff \forall n \in \mathbb{N}, \exists x_n \in T \text{ s.t. } \gamma \leq x_n < \gamma + \frac{1}{n}. \end{aligned}$$

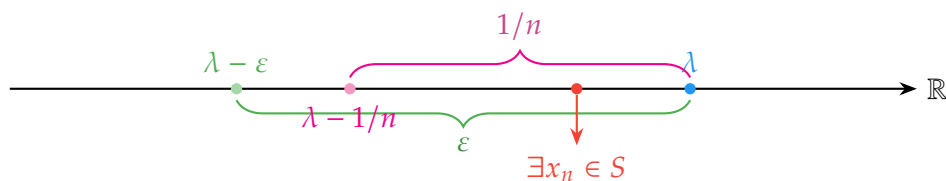
Proof. (1) We NTS that

$$[\forall \varepsilon > 0, \exists x_\varepsilon \in S \text{ s.t. } \lambda - \varepsilon < x_\varepsilon \leq \lambda] \iff [\forall n \in \mathbb{N}, \exists x_n \in S \text{ s.t. } \lambda - \frac{1}{n} < x_n \leq \lambda].$$

(\Rightarrow) Assume that the LHS holds. Let $n \in \mathbb{N}$. Since $1/n > 0$, we can take $\varepsilon = 1/n$ for each $n \in \mathbb{N}$. By assumption, for this choice of ε ,

$$\exists x_n \in S \text{ s.t. } \lambda - \frac{1}{n} < x_n \leq \lambda.$$

(\Leftarrow) Assume that the RHS holds.



Let $\varepsilon > 0$. By Archimedean property, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$. By assumption, for this n ,

$$\lambda - \frac{1}{n} < x_n \leq \lambda.$$

Since $1/n < \varepsilon$, we have $\lambda - \varepsilon < \lambda - \frac{1}{n}$, and so

$$\lambda - \varepsilon < \lambda - \frac{1}{n} < x_n \leq \lambda, \text{ i.e., } \lambda - \varepsilon < x_n \leq \lambda.$$

Letting $x_n = x_\varepsilon$, we see that such an $x_\varepsilon \in S$ exists for any $\varepsilon > 0$.

(2) We NTS that

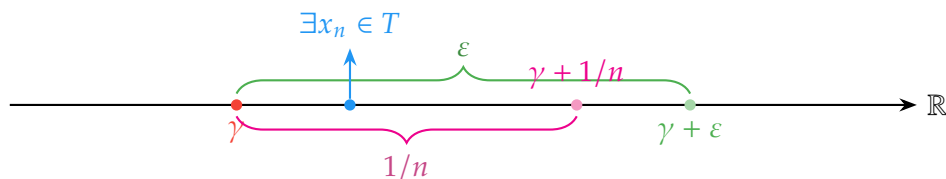
$$[\forall \varepsilon > 0, \exists x_\varepsilon \in T \text{ s.t. } \gamma \leq x_\varepsilon < \gamma + \varepsilon] \iff [\forall n \in \mathbb{N}, \exists x_n \in T \text{ s.t. } \gamma \leq x_n < \gamma + \frac{1}{n}].$$

(\Rightarrow) Assume that the LHS holds. Let $n \in \mathbb{N}$. Since $1/n > 0$, we can take $\varepsilon = 1/n$ for each $n \in \mathbb{N}$.

By assumption, for this choice of ε ,

$$\exists x_n \in T \text{ s.t. } \gamma \leq x_n < \gamma + \varepsilon.$$

(\Leftarrow) Assume that the RHS holds.



Let $\varepsilon > 0$. By Archimedean property, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$. By assumption, for this n ,

$$\gamma \leq x_n < \gamma + \frac{1}{n}.$$

Since $1/n < \varepsilon$, we have $\gamma + \frac{1}{n} < \gamma + \varepsilon$, and so

$$\gamma \leq x_n < \gamma + \frac{1}{n} < \gamma + \varepsilon, \text{ i.e., } \gamma \leq x_n < \gamma + \varepsilon.$$

Letting $x_n = x_\varepsilon$, we see that such an $x_\varepsilon \in T$ exists for any $\varepsilon > 0$.

□

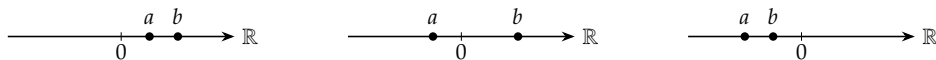
Remark. See **Approximation Property for Supremum and Infimum I**.

Density of the Rationals

Theorem. Let $a, b \in \mathbb{R}$.

$$a < b \implies \exists q \in \mathbb{Q} \text{ such that } a < q < b.$$

Proof. Let $a, b \in \mathbb{R}$. Suppose that $a < b$.



We consider the following two cases:

(Case I) ($0 \leq a$) Since $b - a > 0$, we have $\frac{1}{b-a} \in \mathbb{R}$. By the **Archimedean property**,

$$\exists n \in \mathbb{N} \quad \text{s.t.} \quad \frac{1}{b-a} < n, \text{ i.e., } na + 1 < nb.$$

Clearly $na \in \mathbb{R}$. Define a set A by

$$A := \{k \in \mathbb{N} : na < k\} \subseteq \mathbb{N}.$$

By the Archimedean property, $\exists k \in \mathbb{N}$ such that $na < k$. That is, $A \neq \emptyset$. By the **well-ordering principle**, $\exists m = \min A$. By the minimality of m , we know $m - 1 \notin A$, i.e., $m - 1 \leq na$, and so $m \leq na + 1$. Thus, we obtain

$$na \stackrel{m \in A}{<} m \leq na + 1 \stackrel{\text{by A.P.}}{<} nb.$$

Therefore

$$na < m < ny \implies a < \frac{m}{n} < b.$$

Thus, $q := \frac{m}{n} \in \mathbb{Q}$ satisfies $a < q < b$.

(Case II) ($a < 0$) Note that $-a \in \mathbb{R}^+$. By the Archimedean property,

$$\exists n \in \mathbb{N} : -a < n, \text{ i.e., } 0 < a + n.$$

By (Case I), we have

$$\exists q \in \mathbb{Q} : a + n < q < b + n$$

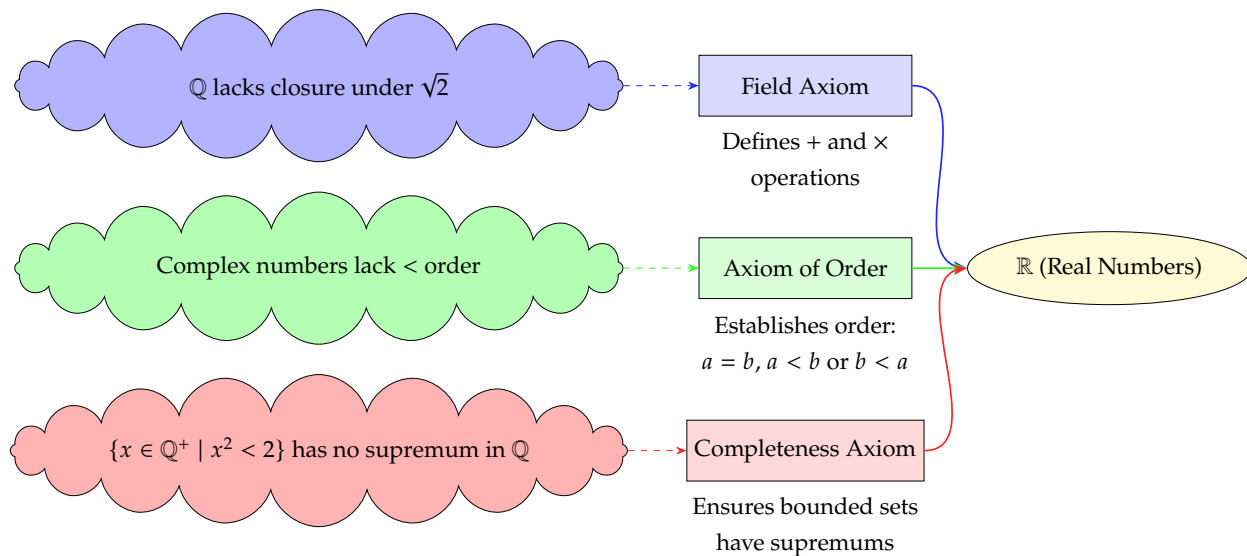
since $0 < a + n < b + n$. Let $q' = q - n \in \mathbb{Q}$. Then $a < q' < b$.

□

References

- [1] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 4. 해석학 개론 (a) 완비성 공리.” YouTube Video, 32:20. Published September 09, 2019. URL: <https://www.youtube.com/watch?v=pHIImTBdBRs>.
- [2] 수학의 즐거움, Enjoying Math. “수학 공부, 기초부터 대학원 수학까지, 5. 해석학 개론 (b) 유리수의 조밀성과 실수, 자연수 공리” YouTube Video, 30:51. Published September 11, 2019. URL: <https://www.youtube.com/watch?v=RYjhQyXxTpQ&t=2s>.

A Axioms of the Real Numbers



The following axioms define the real numbers \mathbb{R} as a complete ordered field.

A.1 Field Axioms

Addition:

1. **Closure under addition:** $\forall a, b \in \mathbb{R}, a + b \in \mathbb{R}$
2. **Associativity of addition:** $\forall a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$
3. **Commutativity of addition:** $\forall a, b \in \mathbb{R}, a + b = b + a$
4. **Existence of additive identity:** $\exists 0 \in \mathbb{R}$ such that $\forall a \in \mathbb{R}, a + 0 = a$
5. **Existence of additive inverses:** $\forall a \in \mathbb{R}, \exists -a \in \mathbb{R}$ such that $a + (-a) = 0$

Multiplication:

1. **Closure under multiplication:** $\forall a, b \in \mathbb{R}, a \cdot b \in \mathbb{R}$
2. **Associativity of multiplication:** $\forall a, b, c \in \mathbb{R}, (a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. **Commutativity of multiplication:** $\forall a, b \in \mathbb{R}, a \cdot b = b \cdot a$
4. **Existence of multiplicative identity:** $\exists 1 \in \mathbb{R}, 1 \neq 0$, such that $\forall a \in \mathbb{R}, a \cdot 1 = a$
5. **Existence of multiplicative inverses:** $\forall a \in \mathbb{R}, a \neq 0, \exists a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = 1$

Distributive law:

1. **Distributivity of multiplication over addition:** $\forall a, b, c \in \mathbb{R}, a \cdot (b + c) = a \cdot b + a \cdot c$

A.2 Axiom of Order

A relation $<$ defined on \mathbb{R} satisfy the followings:

1. **Trichotomy:** For $a, b \in \mathbb{R}$, exactly one of the following holds:

$$a = b, \quad a < b, \quad \text{or} \quad b < a.$$

2. **Transitivity:** For $a, b, c \in \mathbb{R}$,

$$a < b \text{ and } b < c \implies a < c$$

3. **Additive compatibility:** For $a, b, c \in \mathbb{R}$,

$$a < b \implies a + c < b + c$$

4. **Multiplicative compatibility:** For $a, b \in \mathbb{R}$ and $c \in \mathbb{R}^+$,

$$a < b \implies a \cdot c < b \cdot c$$

A.3 Completeness Axiom

The least upper bound property (or supremum property):

$$\forall S \subseteq \mathbb{R}, S \neq \emptyset, \text{ if } S \text{ is bounded above then } \exists \sup(S) \in \mathbb{R}$$

B Application of Well-Ordering Principle

Theorem. $\sqrt{2}$ is irrational, i.e., $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

Proof. Suppose $\sqrt{2} \in \mathbb{Q}$. That is, $\exists p, q \in \mathbb{N}$ s.t. $p\sqrt{2} = q$. Define a set S by

$$S := \{k\sqrt{2} \in \mathbb{N} : k \in \mathbb{N}\} \subseteq \mathbb{N}.$$

Since $p\sqrt{2} = q \in \mathbb{N}$, we have $S \neq \emptyset$. By the Well-Ordering Principle,

$$\exists s = \min(S) \in S.$$

Then $s = t\sqrt{2}$ for some $t \in \mathbb{N}$. Define a number

$$r := s\sqrt{2} - s.$$

(Claim 1) $r \in S$:

$$\begin{aligned} r &= s\sqrt{2} - s \\ &= s\sqrt{2} - t\sqrt{2} \\ &= (s - t)\sqrt{2} \\ &\in S \end{aligned} \quad \because s = t\sqrt{2} > t \Rightarrow s - t > 0 \Rightarrow s - t \in \mathbb{N}.$$

(Claim 2) $r < s = \min(S)$:

$$\begin{aligned} r &= s\sqrt{2} - s \\ &= s(\sqrt{2} - 1) \\ &< s = \min S \end{aligned} \quad \because s \in \mathbb{N} \text{ and } 1 < \sqrt{2} < 2 \Rightarrow 0 < \sqrt{2} - 1 < 1.$$

It is a contradiction. Hence $\sqrt{2} \notin \mathbb{Q}$. □

C The 2nd Principle of Mathematical Induction

The 2nd Principle of Mathematical Induction

Theorem. Suppose that $T \subseteq \mathbb{N}$ satisfies the following two conditions:

1. (Basic Step) $1 \in T$, and
2. (Inductive Step) $1, 2, \dots, n \in T \implies n + 1 \in T$.

Then $T = \mathbb{N}$.

Proof. We use the first principle of mathematical induction. Define the set T' by

$$T' := \{n \in \mathbb{N} : 1, 2, \dots, n \in T\} \subseteq \mathbb{N}.$$

For example, if $1, 2, 3 \in T$ then $3 \in T'$; conversely, if $3 \in T'$ then $1, 2, 3 \in T$. Since $n \in T' \implies n \in T$, we have $T' \subseteq T \subseteq \mathbb{N}$. We claim that T' satisfies the condition of MI:

- (i) (Basic Step) Clearly $1 \in T'$.
- (ii) (Inductive Step) Suppose that $k \in T'$. This means that $1, 2, \dots, k \in T$. By condition 2,

$$1, 2, \dots, k, k + 1 \in T, \text{ i.e., } k + 1 \in T'.$$

Therefore by the first principle of mathematical induction, $T' = \mathbb{N}$. That is,

$$\mathbb{N} = T' \subseteq T \subseteq \mathbb{N} \implies T = \mathbb{N}.$$

Hence it is proved. □

Remark. To show that a mathematical statement $P(n)$ (property for n) holds for $n \in \mathbb{N}$, verify that the set

$$S := \{n \in \mathbb{N} : P(n) \text{ holds}\}$$

satisfies the following conditions:

(Step 1) Show that $P(1)$ holds.

(Step 2) Show that $P(n + 1)$ holds assuming $P(k)$ holds for all $k \leq n$.