Holomorphic 1-Forms, dz, and the Winding Form $\dfrac{dz}{z}$

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Contents

1 What is a 1-Form? (Baby viewpoint)

On an open set of the plane $\mathbb{R}^2$ with coordinates (x, y), a *(real) 1-form* is an expression

$$\alpha = P(x, y)\, dx + Q(x, y)\, dy,$$

which is a rule that eats a tangent vector $v = a\,\partial_x + b\,\partial_y$ and returns a number:

$$\alpha(v) = P(x, y)\, a + Q(x, y)\, b.$$

Here dx, dy are dual to ∂_x, ∂_y:

$$dx(\partial_x) = 1, \quad dx(\partial_y) = 0, \qquad dy(\partial_x) = 0, \quad dy(\partial_y) = 1.$$

In the complex plane $\mathbb{C}$ with coordinate $z = x + iy$, we *complexify* and allow complex coefficients. The most fundamental complex 1-forms are

$$dz := dx + i\, dy, \qquad d\bar{z} := dx - i\, dy.$$

They pair with basis vectors as

$$dz(\partial_x) = 1, \quad dz(\partial_y) = i \qquad \text{and} \qquad d\bar{z}(\partial_x) = 1, \quad d\bar{z}(\partial_y) = -i.$$

2 Holomorphic 1-Forms

Definition 1. *A* holomorphic 1-form *on a domain* $\Omega \subset \mathbb{C}$ *is a 1-form of the shape*

$$\omega = f(z)\,dz,$$

where $f : \Omega \to \mathbb{C}$ *is holomorphic.*

Fact 1 (How to integrate along a curve)**.** *If* $\gamma : [a, b] \to \Omega$ *is a smooth path, then*

$$\int_\gamma \omega = \int_\gamma f(z)\,dz = \int_a^b f\bigl(\gamma(t)\bigr)\,\gamma'(t)\,dt.$$

Examples. $\omega = dz$ (constant coefficient), $\omega = z\,dz$ (simple zero at $z = 0$), $\omega = e^z\,dz$ (no zeros), while $\omega = \overline{z}\,dz$ is *not* holomorphic because $z \mapsto \bar{z}$ is not complex-differentiable.

3 dz in Cartesian and Polar Coordinates

Since $z = x + iy$,

$$dz = dx + i\,dy.$$

In polar coordinates $z = re^{i\theta}$ with $r > 0$, $\theta \in \mathbb{R}$, a standard calculation gives

$$dz = e^{i\theta}\bigl(dr + i\,r\,d\theta\bigr).$$

This splits dz into a *radial* part (dr) and an *angular* part ($r\,d\theta$), rotated by the phase $e^{i\theta}$.

4 The Winding Form $\omega = \dfrac{dz}{z}$

On $\mathbb{C} \setminus \{0\}$ the 1-form

$$\omega := \frac{dz}{z}$$

detects how a path winds about the origin.

4.1 Polar and Cartesian decompositions

In polar coordinates $z = re^{i\theta}$:

$$\boxed{\frac{dz}{z} = \frac{dr}{r} + i\,d\theta}$$

Thus the *real part* measures radial scaling ($d\log r$), and the *imaginary part* measures angular turning ($d\theta$).

In Cartesian coordinates $z = x + iy$,

$$\frac{dz}{z} = \frac{x\,dx + y\,dy}{x^2 + y^2} \;+\; i\,\frac{-y\,dx + x\,dy}{x^2 + y^2}.$$

Here $\operatorname{Re}\frac{dz}{z} = d(\log r)$ annihilates vectors tangent to circles $r = \text{const}$, while $\operatorname{Im}\frac{dz}{z} = d\theta$ annihilates radial vectors.

4.2 Integrating $\frac{dz}{z}$: winding number

Let γ be a smooth closed loop avoiding 0. Then

$$\oint_\gamma \frac{dz}{z} = \oint_\gamma \frac{dr}{r} + i \oint_\gamma d\theta = 0 + i\,(2\pi\,\mathrm{Wind}(\gamma,0)) = 2\pi i\,\mathrm{Wind}(\gamma,0).$$

The term $\int d(\log r)$ vanishes on a closed loop; only the total angle change survives.

Fact 2 (Local primitive vs global obstruction). *Locally on any simply connected region avoiding 0, $\dfrac{dz}{z} = d(\log z)$. Globally, $\log z$ is multi-valued and picks up $2\pi i$ upon circling the origin, hence the nonzero integral around loops.*

5 A Worked Integral and the Winding Number

Let $\gamma(t) = Re^{it}$ for $t \in [0, 2\pi]$ (counterclockwise circle of radius $R > 0$). Then $z = \gamma(t)$, $dz = iRe^{it}dt$, and

$$\oint_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{iRe^{it}}{Re^{it}}\,dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

Reversing orientation gives $-2\pi i$. More generally,

$$\oint_\gamma \frac{dz}{z} = 2\pi i \cdot \mathrm{Wind}(\gamma, 0).$$

6 Summary

- $dz = dx + i\,dy$ complexifies the standard ruler: $dz(\partial_x) = 1$, $dz(\partial_y) = i$.
- Holomorphic 1-forms are $f(z)\,dz$ with f holomorphic; arrows scale by $|f|$ and rotate by $\arg f$.
- $\dfrac{dz}{z} = \dfrac{dr}{r} + i\,d\theta$ splits into radial (scale) and angular (turn) parts.
- On closed loops, only the total turn survives: $\displaystyle\oint \frac{dz}{z} = 2\pi i \cdot \mathrm{Wind}(\gamma, 0)$.

Extra Exercises

Exercise 1. *Show directly from $z = x + iy$ that* $\mathrm{Re}\,\dfrac{dz}{z} = \dfrac{x\,dx + y\,dy}{x^2+y^2} = d(\log r)$ *and* $\mathrm{Im}\,\dfrac{dz}{z} = \dfrac{-y\,dx + x\,dy}{x^2+y^2} = d\theta$.

Exercise 2. *Let f be holomorphic and nonvanishing on a domain. Prove that $d(\log f) = \dfrac{f'(z)}{f(z)}\,dz$ is closed, and integrate it around loops to relate to the argument principle.*