From Calculus to Mayer–Vietoris in de Rham Cohomology of S^2

Lecture Notes

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1 Calculus Preliminaries

1.1 Fundamental Theorem of Calculus

For any $a, b \in \mathbb{R}$,

$$\int_a^b \cos t\, dt = \sin b - \sin a,$$

since $\frac{d}{dt}(\sin t) = \cos t$. Equivalently,

$$d(\sin t) = \cos t\, dt, \quad d(-\cos t) = \sin t\, dt,$$

so

$$\int_a^b \cos t\, dt = \int_a^b d(\sin t) = \sin b - \sin a, \quad \int_a^b \sin t\, dt = -\cos b + \cos a.$$

1.2 Fundamental Theorem for Gradients (Line Integrals)

Let

$$\mathbf{F}(x, y) = \Big(-\frac{y}{x^2+y^2},\ \frac{x}{x^2+y^2}\Big),$$

and let C be the unit circle $x^2 + y^2 = 1$ oriented counterclockwise. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \Big(-\tfrac{y}{r^2}, \tfrac{x}{r^2}\Big) \cdot (dx, dy) = 2\pi.$$

Although $\mathbf{F}$ is not a gradient globally, one computes via parameterization $\mathbf{r}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$:

$$dx = -\sin t\, dt, \quad dy = \cos t\, dt, \quad r^2 = 1,$$

so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \Big(-\sin t, \cos t\Big) \cdot \big(-\sin t, \cos t\big)\, dt = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = 2\pi.$$

1.3 Surface Integrals (Flux) – Stokes' Theorem

Let $D = [0,1]^2$ in the uv-plane, and parameterize a surface S by $\mathbf{T}(u, v)$. Suppose

$$\mathbf{F}(u, v) = (27\, u\, v, 0, 0),$$

so that the 2-form is $\beta = 27\, u\, v\, du \wedge dv$. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D 27\, u\, v\, du\, dv = 27 \int_0^1 \int_0^1 u\, v\, du\, dv = 27 \Big(\tfrac{1}{2}\Big)\Big(\tfrac{1}{2}\Big) = \frac{27}{4}.$$

2 Differential Forms on $\mathbb{R}^n$

Definition. The space of *smooth functions* is

$$C^\infty(\mathbb{R}^n) = \big\{ f : \mathbb{R}^n \to \mathbb{R} \mid f \text{ has continuous derivatives of all orders} \big\}.$$

A *smooth* 1*-form* on $\mathbb{R}^n$ is

$$\omega = f_1(x)\, dx^1 + \cdots + f_n(x)\, dx^n, \quad f_i \in C^\infty(\mathbb{R}^n),$$

and the collection of all of them is denoted $\Omega^1(\mathbb{R}^n)$.

The *exterior derivative* $d : C^\infty(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$ is

$$d\big(f(x)\big) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}\, dx^i.$$

3 de Rham Cohomology of the 2–Sphere S^2

Cover $S^2 \subset \mathbb{R}^3$ by two charts:

$$U = S^2 \setminus \{\text{north pole}\}, \quad V = S^2 \setminus \{\text{south pole}\}.$$

Each of U and V is diffeomorphic to $\mathbb{R}^2$. Their intersection $U \cap V$ is diffeomorphic to $\mathbb{R} \times S^1$.

3.1 Mayer–Vietoris Sequence in de Rham Theory

For the cover $\{U, V\}$ of S^2, the Mayer–Vietoris sequence in de Rham cohomology begins

$$0 \longrightarrow \Omega^0(S^2) \xrightarrow{\phi} \Omega^0(U) \oplus \Omega^0(V) \xrightarrow{\psi} \Omega^0(U \cap V) \xrightarrow{\delta} \Omega^1(S^2) \xrightarrow{\phi'} \Omega^1(U) \oplus \Omega^1(V) \rightarrow \cdots$$

where

$$\phi(f) = (f|_U,\ f|_V), \quad \psi(g_U, g_V) = g_U|_{U\cap V} - g_V|_{U\cap V}.$$

3.2 Computations

Degree 0.

$$\Omega^0(S^2) = C^\infty(S^2), \quad \Omega^0(U) \cong C^\infty(\mathbb{R}^2), \quad \Omega^0(U \cap V) \cong C^\infty(\mathbb{R} \times S^1).$$

Exactness at $\Omega^0(S^2)$ and $\Omega^0(U) \oplus \Omega^0(V)$ shows

$$H^0_{\text{dR}}(S^2) \cong \mathbb{R}$$

(smooth functions constant on connected S^2).

Connecting Map δ. Given $h \in \Omega^0(U \cap V)$, one constructs a 1-form on S^2 whose difference of restrictions equals dh. Explicitly this uses a partition of unity, but on first-year level one may think of $\delta(h)$ as "gluing data" producing a closed 1-form on S^2.

Degree 1. Exactness at $\Omega^1(S^2)$ and $\Omega^1(U) \oplus \Omega^1(V)$ shows

$$H^1_{\text{dR}}(S^2) = 0,$$

since every closed 1-form on S^2 is exact (no "holes" in S^2).

Conclusion

We have built step by step from standard calculus—FTC, line integrals, surface integrals—to the language of differential forms, and used the Mayer–Vietoris sequence to compute

$$H^0_{\text{dR}}(S^2) \cong \mathbb{R}, \quad H^1_{\text{dR}}(S^2) = 0.$$