

From Vector Calculus to Differential Forms

Ji, Yong-hyeon

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We cover the following topics in this note.

- Vector calculus (conservative fields, irrotational field)
- Differential forms (exact forms, closed forms)

Vector Calculus (in \mathbb{R}^2 or \mathbb{R}^3)		Differential Forms
Vector Field \mathbf{F}	\iff	1-form ω
Conservative Vector Field ($\mathbf{F} = \nabla f$)	\iff	Exact 1-form ($\omega = df$)
Irrotational Vector Field ($\nabla \times \mathbf{F} = 0$)	\iff	Closed 1-form ($d\omega = 0$)

The Fundamental Implication:

- **Conservative \implies Irrotational, i.e., Exact \implies Closed**
(This is always true.)
- **Irrotational \implies Conservative, i.e., Closed \implies Exact**
(This is only true on “nice” domains, e.g., simply connected ones.)

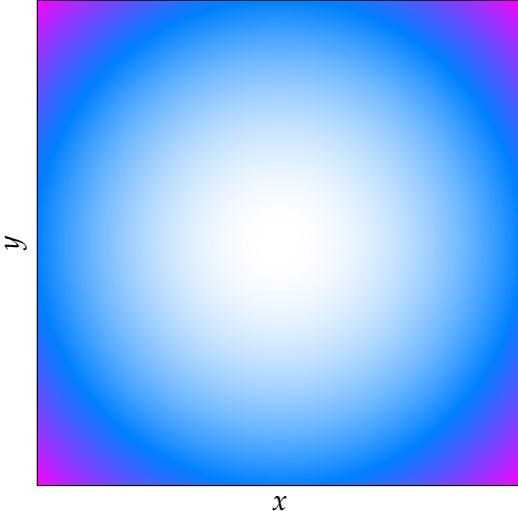
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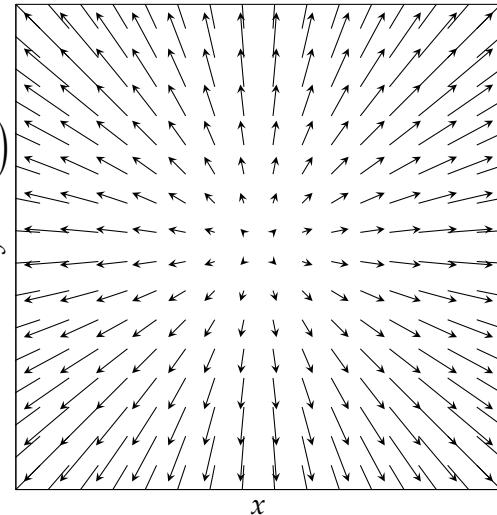
1 Conservative and Exact

Why do we think of the “Conservative Field (or Gradient Field)”?

Scalar field $f(x, y) = x^2 + y^2$



Gradient field $\nabla f = \langle 2x, 2y \rangle$



A vector field \mathbf{F} is **conservative** if it is the **gradient**¹ of some scalar potential function f :

$$\mathbf{F} = \nabla f = \begin{cases} \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle & \text{if } f : U(\subseteq \mathbb{R}^2) \rightarrow \mathbb{R} \\ \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle & \text{if } f : U(\subseteq \mathbb{R}^3) \rightarrow \mathbb{R} . \\ \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle & \text{if } f : U(\subseteq \mathbb{R}^n) \rightarrow \mathbb{R} \end{cases}$$

If \mathbf{F} is conservative, line integrals only depend on the endpoints:

$$\int_{\gamma} \mathbf{F} \cdot d\mathbf{r} = \int_{\gamma} \nabla f \cdot d\mathbf{r} = f(\text{end}) - f(\text{start}),$$

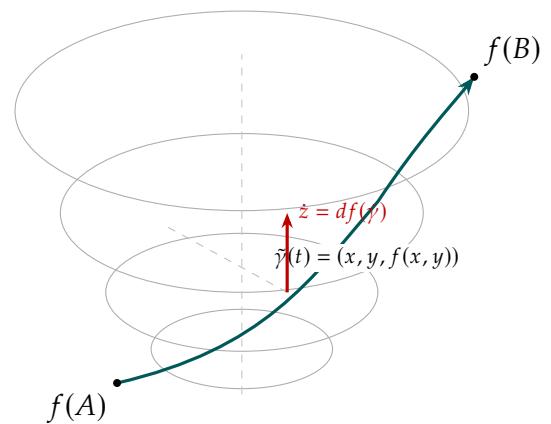
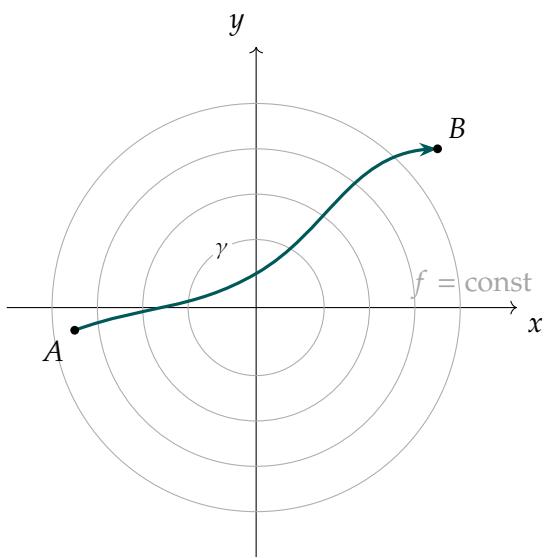
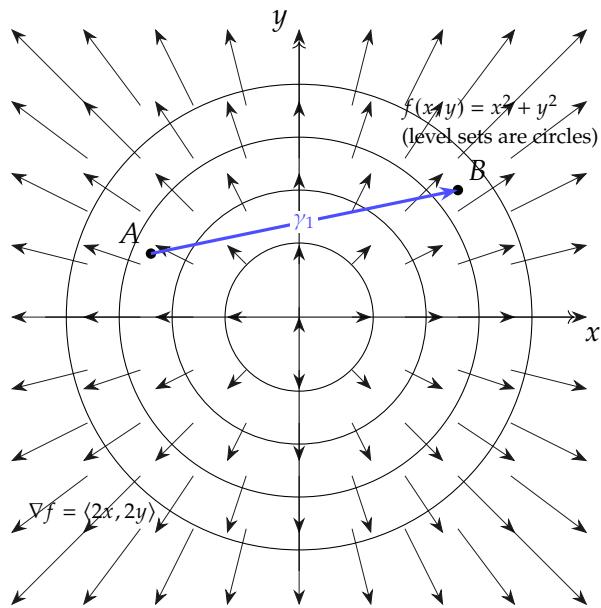
which turns **hard line integrals** into **simple evaluations**. Furthermore every closed loop (start = end) has integral 0.

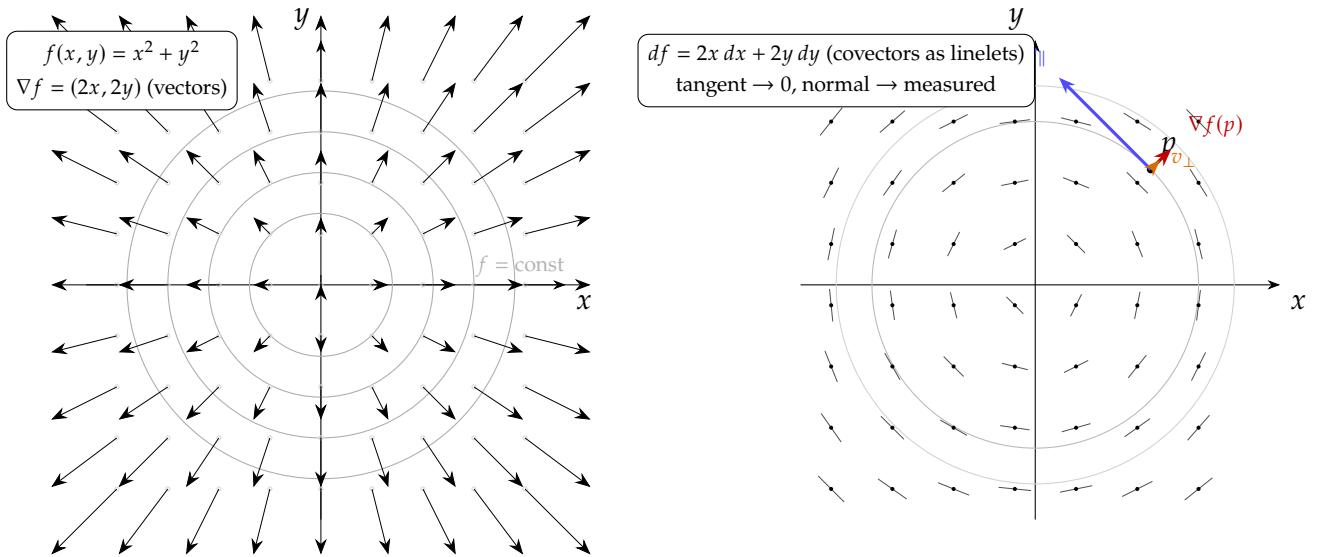
Equivalently (on a nice domain) every line integral of \mathbf{F} is path-independent:

$$\int_{\gamma_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\gamma_2} \mathbf{F} \cdot d\mathbf{r}.$$

whenever γ_1 and γ_2 have the same endpoints.

¹Gradient is a measure of change in a scalar field





From Gradients to Curl

Given a vector field \mathbf{F} , we wish to determine whether \mathbf{F} is conservative (i.e., $\mathbf{F} = \nabla f$ for some scalar field f). Trying to guess the potential function f is hard.

We already know that if a field \mathbf{F} is conservative, it must be the gradient of some potential function f :

$$\mathbf{F} = \langle P(x, y), Q(x, y) \rangle = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \quad (\text{in 2D})$$

What happens if we differentiate $P(x, y) = \frac{\partial f}{\partial x}$ with respect to y and $Q(x, y) = \frac{\partial f}{\partial y}$ with respect to x ?

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}.$$

Theorem 1 (Equality of mixed partials; Clairaut's Theorem). *If the partial derivatives $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ exist and are continuous at a point (a, b) , then $\frac{\partial^2 f}{\partial y \partial x}(a, b) = \frac{\partial^2 f}{\partial x \partial y}(a, b)$, i.e., second order partial derivatives commute if f is C^2 .*

If a vector field $\mathbf{F} = \langle P(x, y), Q(x, y) \rangle$ is a gradient, it **must** satisfy the condition

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

The quantity $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ represents the curl of $\mathbf{F} = \langle P, Q \rangle$ and encodes its local rotational behavior. Hence the condition $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ (i.e., $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$) means that the field is **irrotational** (has **zero curl**).

Remark 1. Consider a small rectangle centered at (x_0, y_0) with side lengths $\Delta x, \Delta y$.

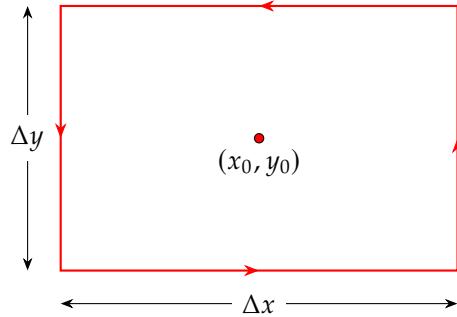


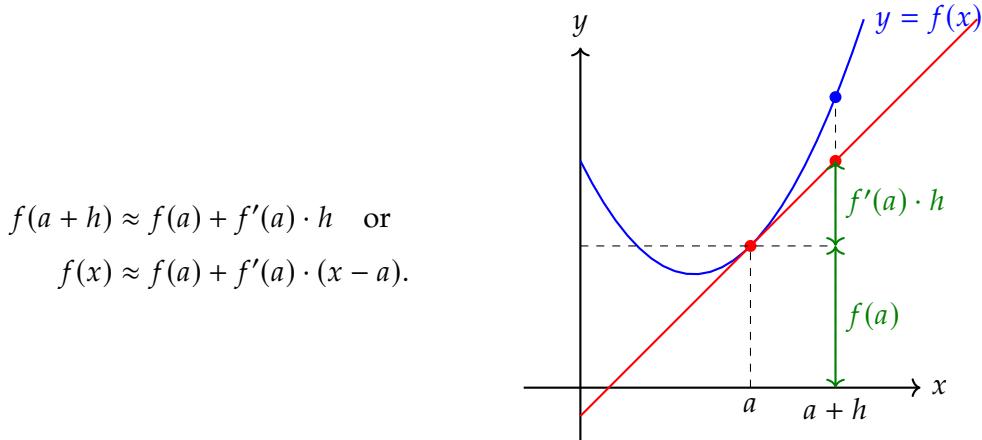
Figure 1: Circulation around an infinitesimal rectangle.

The total counterclockwise circulation is the sum of the line integrals along the four edges:

$$\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \int_{\text{bottom}} P dx + \int_{\text{right}} Q dy + \int_{\text{top}} P dx + \int_{\text{left}} Q dy.$$

We will approximate the value of P or Q along each edge as being constant, equal to its value at the midpoint of that edge. We find this value using a first-order Taylor expansion from the center point (x_0, y_0) .

For a simple function of one variable, $f(x)$, if we know its value at a point a , then we can estimate its value at a nearby point $a + h$ using the tangent line at a :



In words, “New Value \approx Old Value + (Rate of Change) \times (Small Step)”.

For a function of two variables like $P(x, y)$, the idea is identical, but the “rate of change” now has two components (one for each direction), and the “tangent line” becomes a “tangent plane”. The general first-order Taylor expansion for $P(x, y)$ around a center point (x_0, y_0) is

$$P(x_0 + a, y_0 + b) \approx P(x_0, y_0) + \frac{\partial P}{\partial x}(x_0, y_0) \cdot a + \frac{\partial P}{\partial y}(x_0, y_0) \cdot b$$

Here, a is the small step in the x -direction, and b is the small step in the y -direction.

1. The Horizontal Paths These integrals involve the horizontal component of $P(x, y)$.

- **Bottom Path (\rightarrow):**

$$P\left(x, y_0 - \frac{\Delta y}{2}\right) \approx P(x_0, y_0) - \frac{\partial P}{\partial y} \frac{\Delta y}{2} \implies \int_{\text{bottom}} P \, dx \approx \left(P(x_0, y_0) - \frac{\partial P}{\partial y} \frac{\Delta y}{2}\right) (\Delta x)$$

- **Top Path (\leftarrow):**

$$P\left(x, y_0 + \frac{\Delta y}{2}\right) \approx P(x_0, y_0) + \frac{\partial P}{\partial y} \frac{\Delta y}{2} \implies \int_{\text{top}} P \, dx \approx -\left(P(x_0, y_0) + \frac{\partial P}{\partial y} \frac{\Delta y}{2}\right) (\Delta x)$$

Here, we are left with only the parts that describe the *change* in P with respect to y .

$$\int_{\text{bottom}} P \, dx + \int_{\text{top}} P \, dx \approx \left(-\frac{\partial P}{\partial y} \frac{\Delta y}{2}\right) \Delta x - \left(\frac{\partial P}{\partial y} \frac{\Delta y}{2}\right) \Delta x = -\frac{\partial P}{\partial y} \Delta x \Delta y$$

2. The Vertical Paths These integrals involve the vertical component of $Q(x, y)$.

- **Right Path (\uparrow):**

$$Q\left(x_0 + \frac{\Delta x}{2}, y\right) \approx Q(x_0, y_0) + \frac{\partial Q}{\partial x} \frac{\Delta x}{2} \implies \int_{\text{right}} Q \, dy \approx \left(Q(x_0, y_0) + \frac{\partial Q}{\partial x} \frac{\Delta x}{2}\right) (\Delta y)$$

- **Left Path (\downarrow):**

$$Q\left(x_0 - \frac{\Delta x}{2}, y\right) \approx Q(x_0, y_0) - \frac{\partial Q}{\partial x} \frac{\Delta x}{2} \implies \int_{\text{left}} Q \, dy \approx -\left(Q(x_0, y_0) - \frac{\partial Q}{\partial x} \frac{\Delta x}{2}\right) (\Delta y)$$

Here, we are left with only the parts that describe the *change* in Q with respect to x .

$$\int_{\text{right}} Q \, dy + \int_{\text{left}} Q \, dy \approx \left(\frac{\partial Q}{\partial x} \frac{\Delta x}{2}\right) \Delta y + \left(\frac{\partial Q}{\partial x} \frac{\Delta x}{2}\right) \Delta y = \frac{\partial Q}{\partial x} \Delta x \Delta y$$

Now we sum the results from the horizontal and vertical pairs:

$$\begin{aligned}\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} &\approx \left(-\frac{\partial P}{\partial y} \Delta x \Delta y \right) + \left(\frac{\partial Q}{\partial x} \Delta x \Delta y \right) \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \Delta x \Delta y\end{aligned}$$

This shows that the total circulation around the tiny loop is approximately the quantity $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$ multiplied by the area of the loop ($\Delta A = \Delta x \Delta y$).

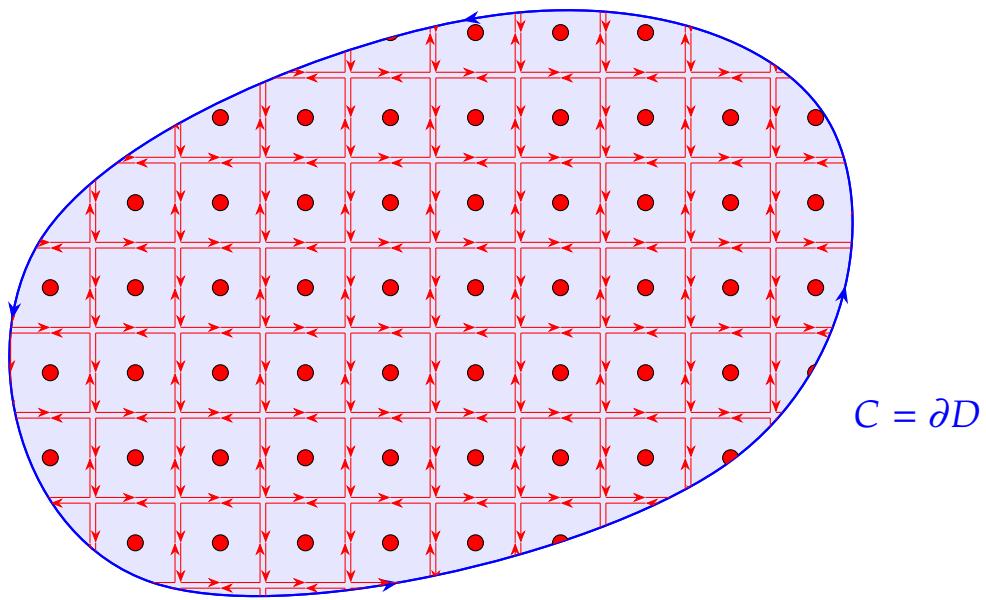
To get the property *at the point* (x_0, y_0) , we find the circulation **density**. We divide by the area and take the limit as the rectangle shrinks to zero.

$$\lim_{\Delta A \rightarrow 0} \frac{1}{\Delta A} \oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \frac{\partial Q}{\partial x}(x_0, y_0) - \frac{\partial P}{\partial y}(x_0, y_0)$$

This is why we call the scalar quantity $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ the **curl**: it is the circulation per unit area at a point, which measures the local rotational tendency of the field.

Remark 2. If $C = \partial D$ is a positively oriented simple closed curve enclosing a region D , Green's theorem states

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{Line Integral (Total Circulation)}} = \underbrace{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA}_{\text{Double Integral (Sum of Local Curls)}}$$



Example 1: Rigid rotation and angular velocity

Consider the rigid rotation field with angular speed ω :

$$\mathbf{F}(x, y) = \langle -\omega y, \omega x \rangle.$$

Then

$$\frac{\partial Q}{\partial x} = \omega, \quad \frac{\partial P}{\partial y} = -\omega \quad \Rightarrow \quad \operatorname{curl} \mathbf{F} = Q_x - P_y = 2\omega.$$

This shows curl equals twice the angular velocity. For a circle of radius R , parametrize $r(t) = (R \cos t, R \sin t)$, $dr = (-R \sin t, R \cos t) dt$. Then

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \omega R^2 dt = 2\pi\omega R^2.$$

Meanwhile, $\iint_D (2\omega) dA = 2\omega \cdot \pi R^2 = 2\pi\omega R^2$, agreeing with Green's theorem.

Example 2: Curl-free but not conservative (topology matters)

On $\mathbb{R}^2 \setminus \{(0, 0)\}$, define

$$\mathbf{F}(x, y) = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

A direct calculation shows $Q_x - P_y = 0$ wherever defined (curl-free). However, the circulation around the unit circle is

$$\oint \mathbf{F} \cdot d\mathbf{r} = 2\pi \neq 0.$$

Hence there is no global potential function; the puncture creates a topological obstruction. This illustrates that $\operatorname{curl} \mathbf{F} = 0$ captures **local** rotation, while global circulation can persist in domains with holes.

Summary checklist

- $Q_x - P_y$ is the infinitesimal (per-area) circulation density.
- Green's theorem sums local curl to give total circulation.
- Rigid rotation: $\operatorname{curl} = 2\omega$ (twice angular velocity).
- $\operatorname{curl} = 0$ can still have nonzero loop integrals if the domain has holes.

Test 1: Equality of Mixed Partial

Test 2: Path Independence

Test 3: Potential Recovery

2 Del to Differential ($\nabla \rightarrow d$)

2.1 What is a covector?

Observation. For $\alpha = \begin{bmatrix} 2 & 1 \end{bmatrix} \in (\mathbb{R}^2)^*$ and any $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$,

$$\alpha(\mathbf{v}) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (2)(x) + (1)(y) = 2x + y.$$

Definition 1. Let $V = \mathbb{R}^n$. A **covector** (or **linear functional**) is a linear map

$$\alpha : V \rightarrow \mathbb{R}, \quad \mathbf{v} \mapsto \alpha(\mathbf{v}) = \begin{bmatrix} \alpha_1 & \cdots & \alpha_n \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \sum_{i=0}^n \alpha_i v^i.$$

The set of all covectors is the dual space $V^* = \text{Hom}(V, \mathbb{R})$.

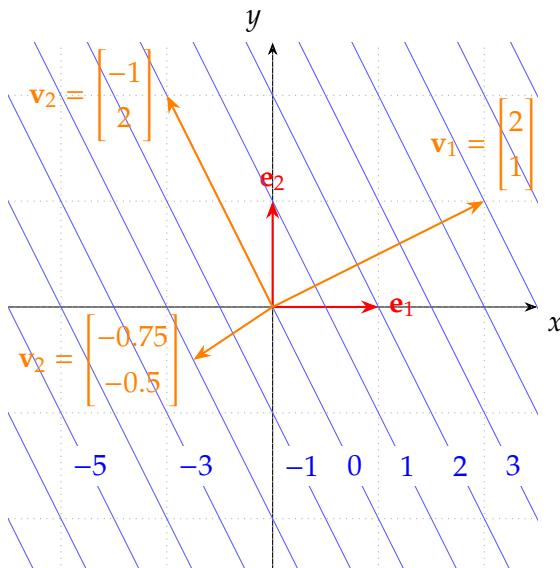
Remark 3. For covectors $\alpha, \beta \in V^*$ and $c \in \mathbb{R}$,

$$(\alpha + \beta)(\mathbf{v}) = \alpha(\mathbf{v}) + \beta(\mathbf{v}), \quad (c\alpha)(\mathbf{v}) = c\alpha(\mathbf{v}) \quad \text{for all } \mathbf{v} \in V.$$

So V^* is a vector space with these operations.

Observation (Level sets). The value of α is constant on the **level sets**

$$2x + y = c, \quad c \in \mathbb{R}.$$

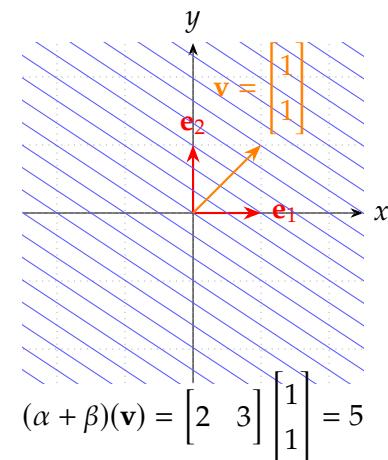
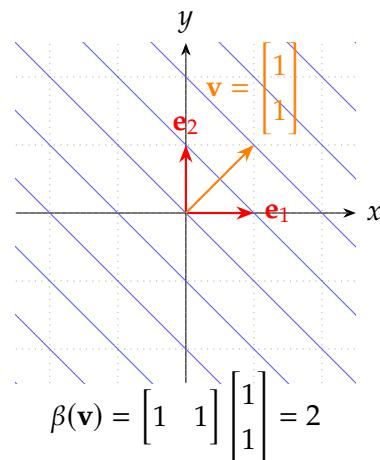
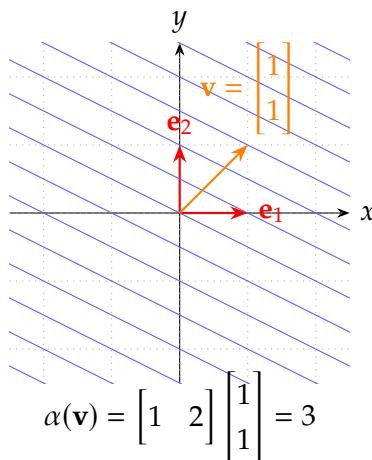
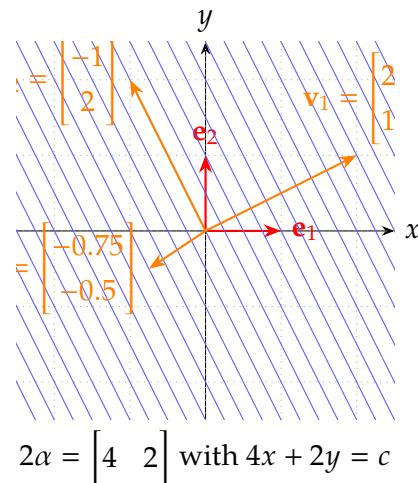
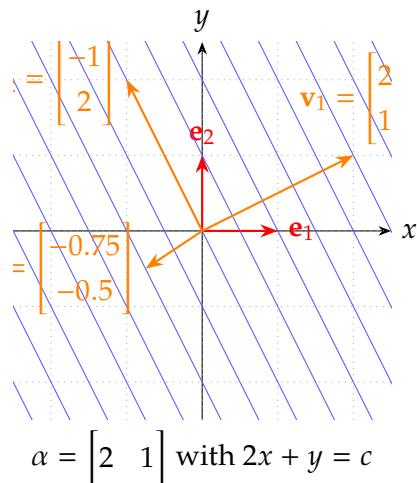
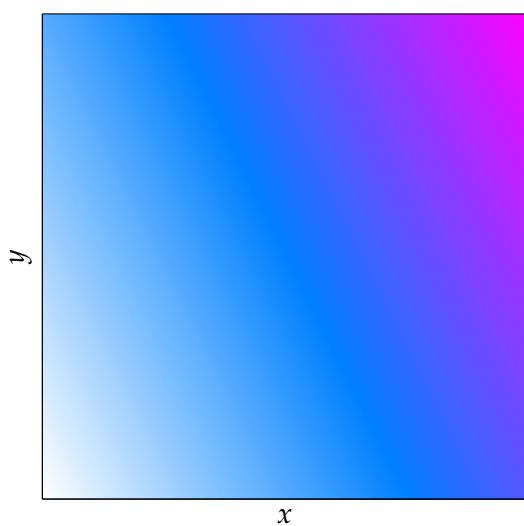
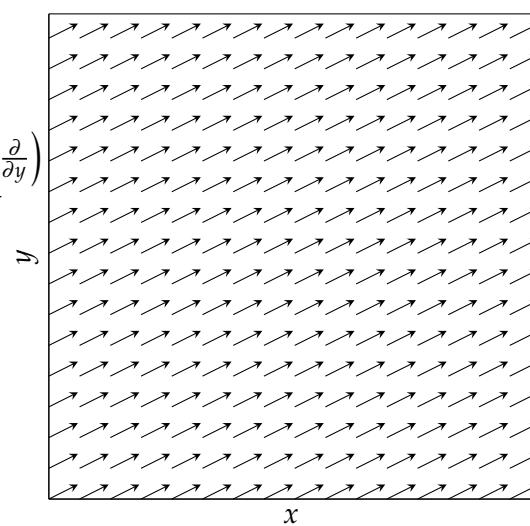


$$2x + y = c \Leftrightarrow y = -2x + c \text{ (level sets of } \alpha\text{)}$$

$$\alpha(\mathbf{v}_1) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5$$

$$\alpha(\mathbf{v}_2) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0$$

$$\alpha(\mathbf{v}_3) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} -0.75 \\ 0.5 \end{bmatrix} = -2$$

Scalar field $f(x, y) = 2x + y$ Gradient field $\nabla f = \langle 2, 1 \rangle$ 

2.2 Dual basis

Let V be a \mathbb{R} -vector space with basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. A **covector** is a linear functional

$$\alpha : V \rightarrow \mathbb{R}, \quad \alpha(a\mathbf{v}_1 + b\mathbf{v}_2) = a\alpha(\mathbf{v}_1) + b\alpha(\mathbf{v}_2).$$

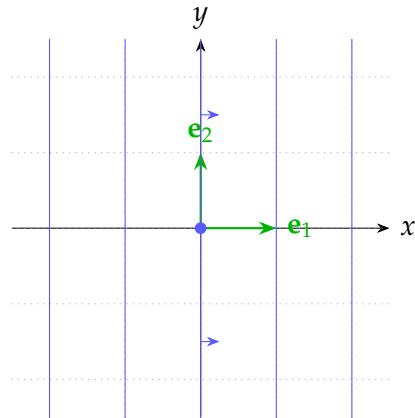
The collection of all covectors is the **dual space** $V^* = \text{Hom}(V, \mathbb{R})$.

We cannot use a basis of V to measure covectors directly. To measure covectors, we use the dual basis of V^* . Given a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of V , the **dual basis** $\{\varepsilon^1, \varepsilon^2\} \subset V^*$ is defined by

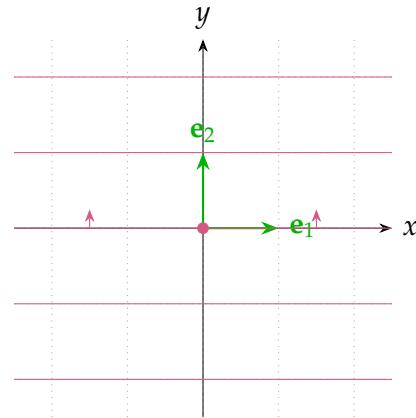
$$\varepsilon^i(\mathbf{e}_j) = \delta_{ij} = \begin{cases} 1 & : i = j \\ 0 & : i \neq j \end{cases} \quad (i, j \in \{1, 2\}),$$

where δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$ if $i = j$, and 0 otherwise). Concretely,

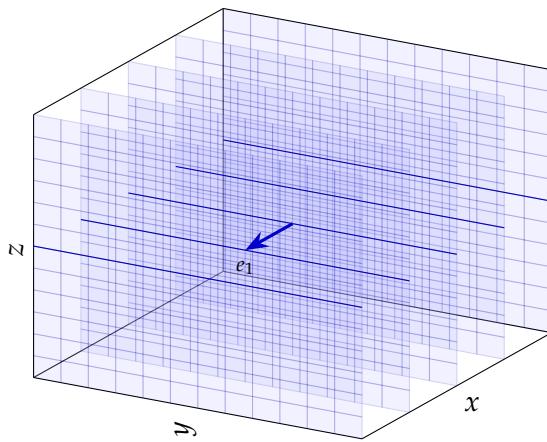
$$\varepsilon^1(\mathbf{e}_1) = 1, \quad \varepsilon^1(\mathbf{e}_2) = 0, \quad \varepsilon^2(\mathbf{e}_1) = 0, \quad \varepsilon^2(\mathbf{e}_2) = 1.$$



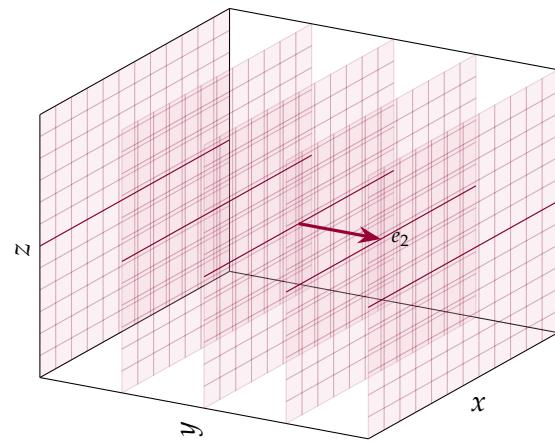
Covector $\varepsilon^1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ with lines $x = c$



Covector $\varepsilon^2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ with lines $y = c$



Covector $\varepsilon^1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ with planes $x = c$



Covector $\varepsilon^2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ with planes $y = c$

Every vector $\mathbf{v} \in V$ has a unique expansion $\mathbf{v} = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2$. The dual basis **reads off** these coordinates:

$$\begin{aligned}\varepsilon^1(\mathbf{v}) &= \varepsilon^1(v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2) = v^1 \varepsilon^1(\mathbf{e}_1) + v^2 \varepsilon^1(\mathbf{e}_2) = v_1, \\ \varepsilon^2(\mathbf{v}) &= \varepsilon^2(v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2) = v^1 \varepsilon^2(\mathbf{e}_1) + v^2 \varepsilon^2(\mathbf{e}_2) = v_2.\end{aligned}$$

In matrix form,

$$\varepsilon^1(\mathbf{v}) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = v^1, \quad \varepsilon^2(\mathbf{v}) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = v^2.$$

For any $\alpha \in V^*$ and $\mathbf{v} = v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2$,

$$\alpha(\mathbf{v}) = \alpha(v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2) = v^1 \alpha(\mathbf{e}_1) + v^2 \alpha(\mathbf{e}_2) = \varepsilon^1(\mathbf{v}) \alpha(\mathbf{e}_1) + \varepsilon^2(\mathbf{v}) \alpha(\mathbf{e}_2).$$

Set

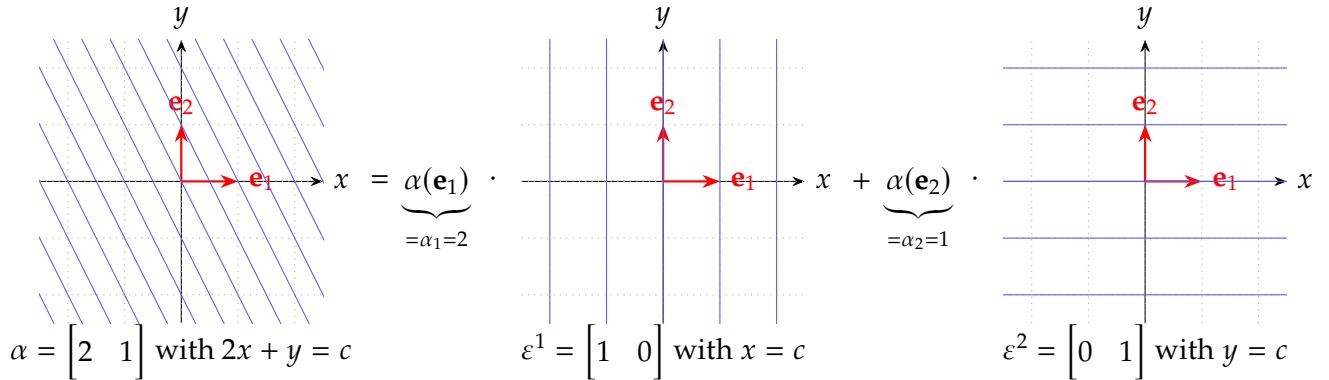
$$\alpha(\mathbf{e}_1) = \alpha_1 \in \mathbb{R}, \quad \alpha(\mathbf{e}_2) = \alpha_2 \in \mathbb{R}.$$

Then for every $\mathbf{v} \in V$,

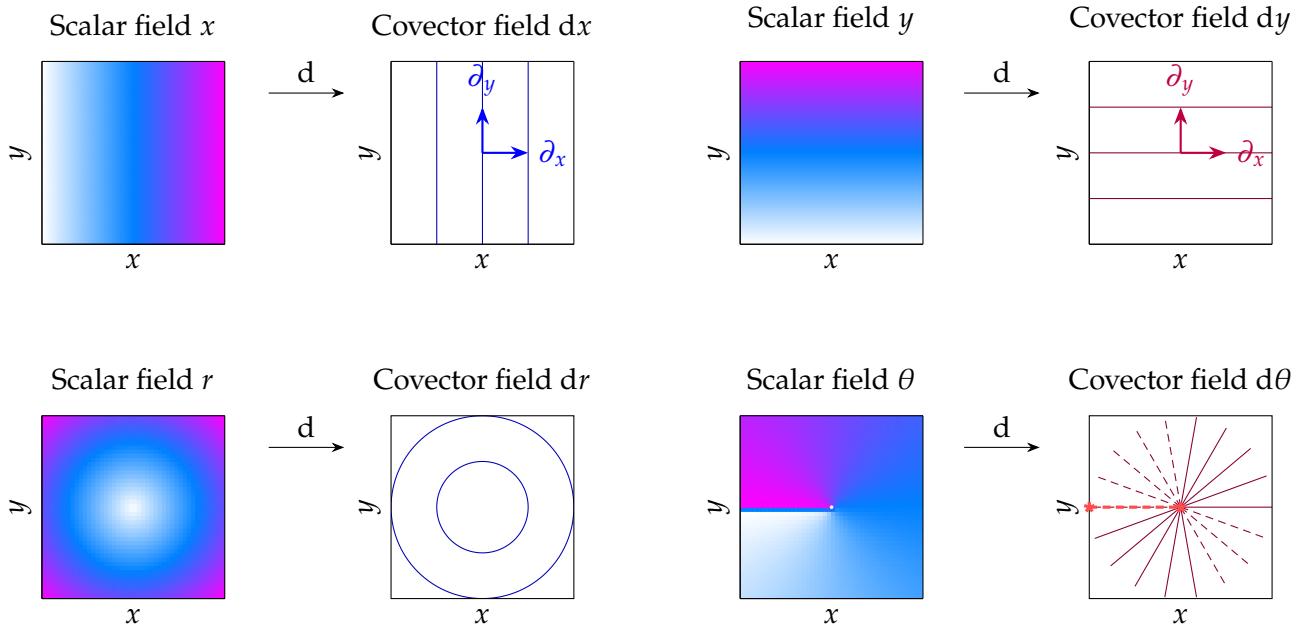
$$\alpha(\mathbf{v}) = \alpha_1 \varepsilon^1(\mathbf{v}) + \alpha_2 \varepsilon^2(\mathbf{v}).$$

Since this holds for all \mathbf{v} , we identify α as the linear combination

$$\boxed{\alpha = \alpha_1 \varepsilon^1 + \alpha_2 \varepsilon^2 \in V^*}$$



2.3 dx, dy and $dr, d\theta$



Remark 4 (Parametrization for scalar field $\theta(x, y)$). Let

$$x = r \cos t, \quad y = r \sin t, \quad r > 0, t \in (-\pi, \pi].$$

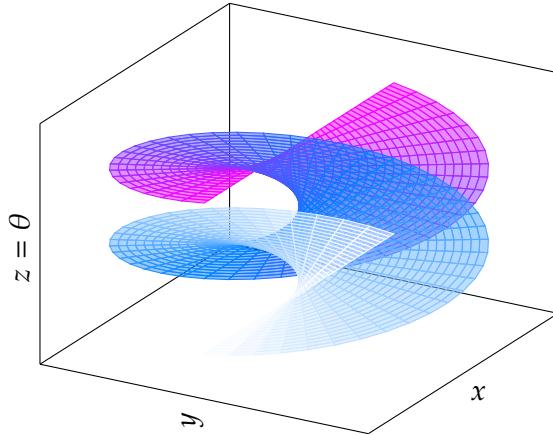
The scalar field $z = \theta(x, y)$ becomes the **helicoid** parametrized by

$$(r, t) \mapsto (x, y, z = \theta(x, y)) = \left(r \cos t, r \sin t, \theta(r \cos t, r \sin t) = \arctan \left(\frac{\sin t}{\cos t} \right) = \arctan(\tan(t)) = t \right).$$

2.4 The surface $z = \theta(x, y)$

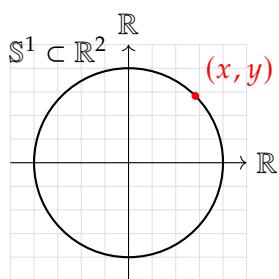
We visualize the scalar field $z = \theta(x, y)$ as a surface:

Surface $z = \theta(x, y)$ via $(r, t) \mapsto (r \cos t, r \sin t, t)$

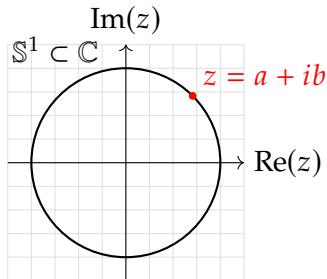


We identify the plane \mathbb{R}^2 with the complex line \mathbb{C} via

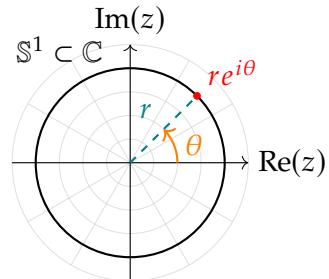
$$(x, y) \longleftrightarrow z := x + iy.$$



$$x^2 + y^2 = 1$$



$$|z| = 1$$



$$|r| = 1 \text{ and } \theta \in \mathbb{R}$$

The one-argument arctangent

$$\arctan : \mathbb{R} \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

is single-valued but **cannot** return angles outside that range. The correct globally defined angle $(\bmod 2\pi)$, with a chosen principal branch, is given by the two-argument arctangent:

$$\theta = \text{atan2}(y, x) \in (-\pi, \pi],$$

which is defined piecewise by

$$\theta = \begin{cases} \arctan\left(\frac{y}{x}\right), & x > 0, \\ \arctan\left(\frac{y}{x}\right) + \pi, & x < 0, y \geq 0, \\ \arctan\left(\frac{y}{x}\right) - \pi, & x < 0, y < 0, \quad (\text{undefined at } x = y = 0). \\ \frac{\pi}{2}, & x = 0, y > 0, \\ -\frac{\pi}{2}, & x = 0, y < 0, \end{cases}$$

Hence:

$\text{Global angle (principal): } \theta(x, y) = \text{atan2}(y, x).$

On the right half-plane $x > 0$ this reduces to the simple formula $\arctan(y/x)$.

3. Complex viewpoint: Arg of a complex number

Write $z = x + iy \in \mathbb{C}^\times$. The (principal) argument $\text{Arg } z \in (-\pi, \pi]$ is defined by the **polar form**

$$z = |z| e^{i \text{Arg } z}.$$

Equivalently,

$$\frac{z}{|z|} = \cos(\text{Arg } z) + i \sin(\text{Arg } z).$$

Thus the geometric angle $\theta(x, y)$ is the principal argument:

$\theta(x, y) = \text{Arg}(x + iy) = \text{atan2}(y, x).$

This is the cleanest global statement; it automatically handles all quadrants and the axes (except the origin).

4. When does $\arctan(y/x)$ equal the true angle?

Let $\theta = \operatorname{Arg}(x + iy) \in (-\pi, \pi]$. Then:

$$\arctan\left(\frac{y}{x}\right) = \theta \iff \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ (i.e. } x > 0).$$

In general,

$$\arctan\left(\frac{y}{x}\right) = \theta - \pi N, \quad N = \left\lfloor \frac{\theta}{\pi} + \frac{1}{2} \right\rfloor \in \{-1, 0, 1\}.$$

The subtracted multiple of π compensates for the quadrant.

5. Analytic formula via the complex logarithm

For a **complex** variable w , the complex arctangent is defined (on a log-branch domain) by

$$\arctan w = \frac{1}{2i} \left(\log(1 + iw) - \log(1 - iw) \right),$$

where \log is a chosen branch of the complex logarithm. This identity is obtained by differentiating both sides and matching values at $w = 0$. Its branch cuts are typically placed so that $1 \pm iw$ avoid the logarithm's cut.

Specialization to $w = \frac{y}{x}$ (real ratio). When $x \neq 0$ and $w = y/x \in \mathbb{R}$, the above produces the real $\arctan(w) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. But to recover the **global** geometric angle you must still correct by $\pm\pi$ according to the sign of x (i.e. use $\operatorname{atan2}$), or, equivalently, use $\operatorname{Arg}(x + iy)$ directly.

6. Branch cuts and continuity of the angle

Any single-valued selection of the angle on $\mathbb{R}^2 \setminus \{0\}$ must jump by 2π somewhere (a **branch cut**). The standard choice (principal branch) places the cut along the negative real axis:

$$\operatorname{Arg} : \mathbb{C} \setminus (-\infty, 0] \longrightarrow (-\pi, \pi].$$

Crossing the cut increases/decreases the angle by 2π . This is why a color plot of $\theta(x, y)$ shows a “seam” along $x < 0, y = 0$.

7. Differential of the angle and the 1-form $d\theta$

From

$$z = |z|e^{i\theta} \implies d(\log z) = \frac{dz}{z} = d(\ln|z|) + i d\theta,$$

and writing $z = x + iy$, $dz = dx + i dy$, we get

$$\frac{dz}{z} = \frac{dx + i dy}{x + iy} = \frac{(x dx + y dy) + i(x dy - y dx)}{x^2 + y^2}.$$

Taking imaginary parts:

$$d\theta = \frac{x dy - y dx}{x^2 + y^2} \quad (x, y) \neq (0, 0).$$

Geometrically:

- $d\theta$ vanishes on **radial** motion (along rays $\theta = \text{const}$).
- $d\theta$ measures **angular** motion (tangent to circles).

For a closed loop γ avoiding 0,

$$\int_{\gamma} d\theta = 2\pi \text{Ind}(\gamma, 0),$$

the net number of turns around the origin (winding number).

8. Examples and edge cases

- $(x, y) = (1, 1)$: $\arctan(y/x) = \arctan(1) = \pi/4$; $\text{Arg}(1+i) = \pi/4$ (agree).
- $(x, y) = (-1, 1)$: $\arctan(y/x) = \arctan(-1) = -\pi/4$ but the true angle is $3\pi/4$. Here $\text{atan2}(1, -1) = 3\pi/4 = \arctan(-1) + \pi$.
- $(x, y) = (0, -2)$: $\arctan(y/x)$ undefined; $\text{atan2}(-2, 0) = -\pi/2$; $\text{Arg}(-2i) = -\pi/2$.
- $(x, y) = (-3, 0)$: slope 0 so $\arctan(0) = 0$ (misleading); $\text{atan2}(0, -3) = \pi$; $\text{Arg}(-3) = \pi$ (principal).

9. A precise dictionary

- Local/right half-plane formula:

$$\theta = \arctan(y/x) \quad (\text{valid for } x > 0).$$

- Global/principal angle:

$$\theta = \text{atan2}(y, x) = \text{Arg}(x + iy) \in (-\pi, \pi].$$

- **Complex-analytic identity:** for $w \in \mathbb{C}$,

$$\arctan w = \frac{1}{2i} \left(\log(1 + iw) - \log(1 - iw) \right)$$

(on a suitable log-branch domain).

- **Differential:**

$$d\theta = \frac{x \, dy - y \, dx}{x^2 + y^2} = \operatorname{Im}\left(\frac{dz}{z}\right).$$

10. Common pitfalls (and remedies)

- Using $\arctan(y/x)$ globally **without** correcting the quadrant. Remedy: use $\operatorname{atan2}(y, x)$ or $\operatorname{Arg}(x + iy)$.
- Forgetting the point $(0, 0)$ is excluded; θ and $d\theta$ are not defined there.
- Expecting a continuous single-valued θ on the punctured plane **without** a branch cut. Any single-valued angle has a jump by 2π along some cut.
- Confusing degrees and radians: in plotting packages $\operatorname{atan2}$ often returns **degrees**; multiply by $\pi/180$ to get radians.

1. Argument as a scalar field on \mathbb{C}^\times

For $z = x + iy \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, write $z = re^{i\theta}$ with

$$r = |z| = \sqrt{x^2 + y^2}, \quad \theta = \arg z \in \mathbb{R}/2\pi\mathbb{Z}.$$

Any **single-valued** choice of θ requires a **branch cut**. The **principal argument** is

$$\operatorname{Arg} z \in (-\pi, \pi], \quad \text{cut usually along } (-\infty, 0) \subset \mathbb{R}.$$

In coordinates, $\operatorname{Arg}(x+iy)$ coincides with the quadrant-aware angle $\operatorname{atan2}(y, x)$. (Using $\arctan(y/x)$ without quadrant care is ambiguous on $x < 0$.)

Level sets. For fixed θ_0 , the set $\{z \neq 0 : \arg z = \theta_0\}$ is the **ray** $\{re^{i\theta_0} : r > 0\}$.

2. Relation to the complex logarithm

On a simply connected domain avoiding 0 and the cut, define a holomorphic branch of the log:

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z = u + iv, \quad u = \ln r, v = \operatorname{Arg} z.$$

Then $\theta = \operatorname{Arg} z = \operatorname{Im}(\operatorname{Log} z)$ and

$$d(\operatorname{Log} z) = \frac{dz}{z}, \quad \operatorname{Im}(d(\operatorname{Log} z)) = d\theta.$$

3. Differential and gradient

With $z = x + iy$ and $dz = dx + i dy$,

$$\frac{dz}{z} = \frac{dx + i dy}{x + iy} = \frac{(x dx + y dy) + i(x dy - y dx)}{x^2 + y^2}.$$

Hence

$$d\theta = \operatorname{Im}\left(\frac{dz}{z}\right) = \frac{x dy - y dx}{x^2 + y^2} \quad \text{on } \mathbb{C}^\times.$$

In the Euclidean metric, the gradient of θ is

$$\nabla\theta(x, y) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right), \quad \|\nabla\theta\| = \frac{1}{\sqrt{x^2 + y^2}} = \frac{1}{r}.$$

Thus $\nabla\theta$ is **tangent to circles** and **orthogonal to the radius**.

4. Winding number via a line integral

For a closed C^1 curve γ avoiding 0,

$$\int_{\gamma} d\theta = \operatorname{Im} \int_{\gamma} \frac{dz}{z} = 2\pi \operatorname{Ind}(\gamma, 0),$$

the net angle (in radians) swept by the radius vector as you traverse γ . Equivalently, $\int_{\gamma} \frac{dz}{z} = 2\pi i \operatorname{Ind}(\gamma, 0)$.

Example (unit circle). Let $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. Then $dz/z = i dt$, so

$$\int_{\gamma} d\theta = \operatorname{Im} \int_0^{2\pi} i dt = \operatorname{Im}[2\pi i] = 2\pi,$$

which is the expected one full turn around the origin.

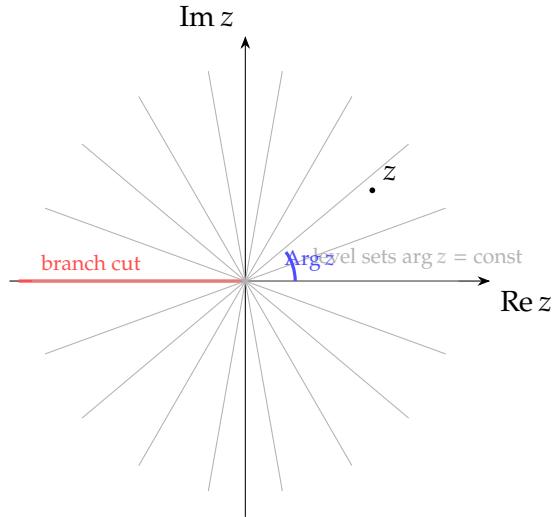
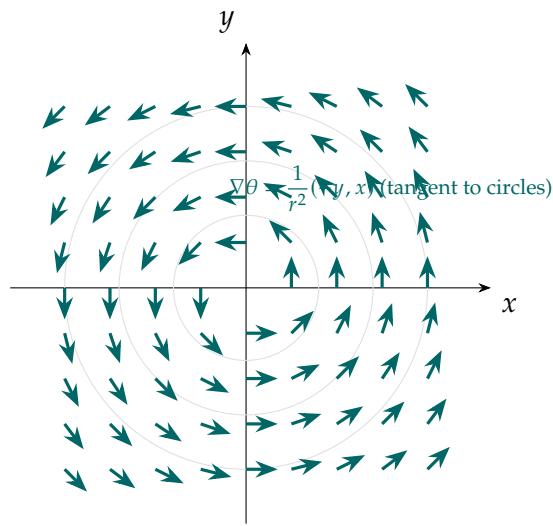
5. Harmonic conjugates and Cauchy–Riemann

On a branch domain, $u = \ln r$ and $v = \theta$ satisfy Cauchy–Riemann and are harmonic:

$$\Delta u = 0, \quad \Delta v = 0 \quad \text{on } \mathbb{C}^{\times}, \quad \nabla v = J \nabla u, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

So $\nabla(\ln r) = \frac{1}{r^2}(x, y)$ is radial, while $\nabla\theta = \frac{1}{r^2}(-y, x)$ is the $+\pi/2$ rotation of it.

6. Two compact TikZ visuals

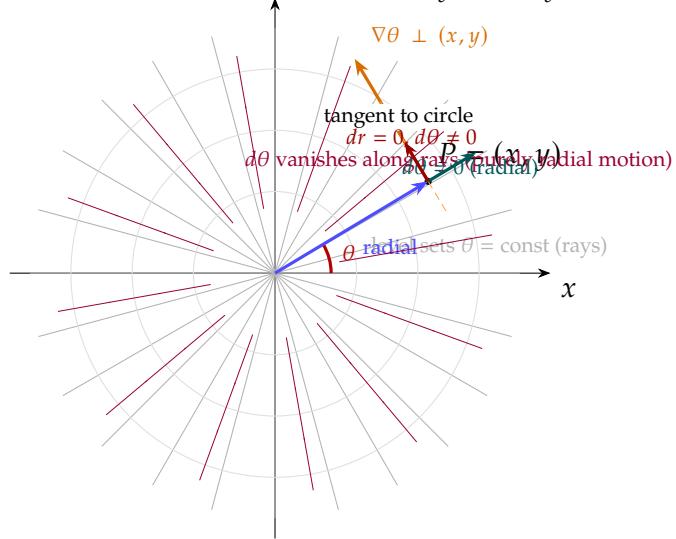
(A) Branch cut and level sets of θ (rays)(B) Gradient field $\nabla\theta$ (tangent to circles)

7. Quick dictionary

- $\arg z$: multi-valued angle; $\text{Arg } z$ is a single-valued branch (needs a cut).
- $\text{Log } z = \ln |z| + i \text{ Arg } z$ (holomorphic on a branch domain), $d \text{ Log } z = \frac{dz}{z}$.
- $d\theta = \text{Im} \left(\frac{dz}{z} \right) = \frac{x dy - y dx}{x^2 + y^2}$.

- $\int_{\gamma} d\theta = 2\pi \text{Ind}(\gamma, 0)$ (net turning / winding number).
- $u = \ln r$ and $v = \theta$ are harmonic conjugates: $u + iv = \text{Log } z$.

$$\theta(x, y) = \arctan\left(\frac{y}{x}\right), \quad \nabla\theta = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right), \quad d\theta = \frac{-y \, dx + x \, dy}{x^2 + y^2}$$



1. 1-forms (covectors) and what they measure

A **1-form** at a point is a linear functional on tangent vectors. In coordinates:

$$dx, dy \in T_p^*\mathbb{R}^2 \quad \text{and} \quad dr, d\theta \in T_p^*(\mathbb{R}^2 \setminus \{0\})$$

act on an infinitesimal displacement v by returning the corresponding coordinate rate: if $v = v^x \partial_x + v^y \partial_y$ then

$$dx(v) = v^x, \quad dy(v) = v^y.$$

In polar coordinates (r, θ) with $v = v^r \partial_r + v^\theta \partial_\theta$,

$$dr(v) = v^r, \quad d\theta(v) = v^\theta.$$

(Be mindful: with the Euclidean metric, $\|\partial_\theta\| = r$, so $d\theta$ measures **angular** rate, not arclength; the arclength 1-form along circles is $r d\theta$.)

2. From a scalar field f to its differential df

A smooth scalar field $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ has differential

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = f_x dx + f_y dy.$$

In polar coordinates:

$$df = f_r dr + f_\theta d\theta, \quad f_r = \frac{\partial f}{\partial r}, \quad f_\theta = \frac{\partial f}{\partial \theta}.$$

With the Euclidean metric, $df(\cdot) = \nabla f \cdot (\cdot)$, and

$$\nabla f = f_x \mathbf{e}_x + f_y \mathbf{e}_y = f_r \mathbf{e}_r + \frac{1}{r} f_\theta \mathbf{e}_\theta.$$

3. Level sets and what the basic 1-forms “kill”

A 1-form is naturally visualized by its **level sets** (where the associated coordinate is constant):

- dx : level sets $x = c$ are vertical lines. Motion **along** these lines is tangent (horizontal value 0 for dx); motion across them gives nonzero dx .
- dy : level sets $y = c$ are horizontal lines. dy kills horizontal motion, measures vertical motion.
- dr : level sets $r = c$ are circles. dr kills tangential (angular) motion along circles; it measures radial motion.

- $d\theta$: level sets $\theta = c$ are rays from the origin. $d\theta$ kills radial motion; it measures angular motion (rate). The arclength form along circles is $r d\theta$.

4. Cartesian \leftrightarrow Polar (for $r > 0$)

Let $x = r \cos \theta$, $y = r \sin \theta$. Then

$$\begin{aligned} dx &= \cos \theta dr - r \sin \theta d\theta, \\ dy &= \sin \theta dr + r \cos \theta d\theta, \end{aligned} \quad \begin{aligned} dr &= \cos \theta dx + \sin \theta dy, \\ d\theta &= \frac{-\sin \theta dx + \cos \theta dy}{r}. \end{aligned}$$

Thus $df = f_x dx + f_y dy = f_r dr + f_\theta d\theta$ with $f_r = f_x \cos \theta + f_y \sin \theta$ and $f_\theta = -rf_x \sin \theta + rf_y \cos \theta$.

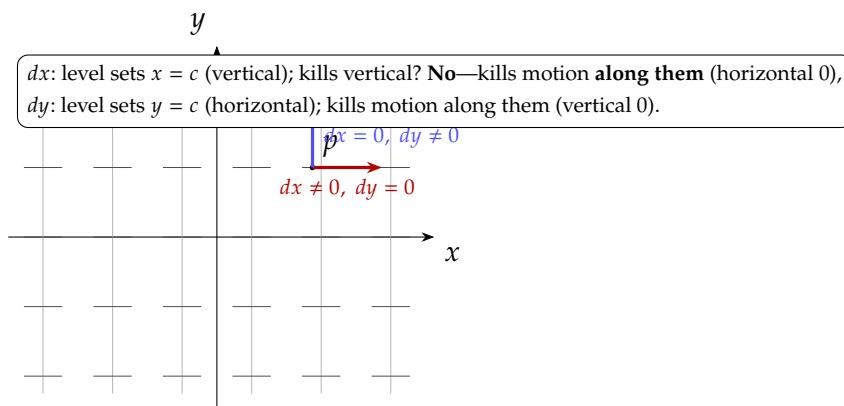
5. Example: $f(x, y) = x^2 + y^2 = r^2$

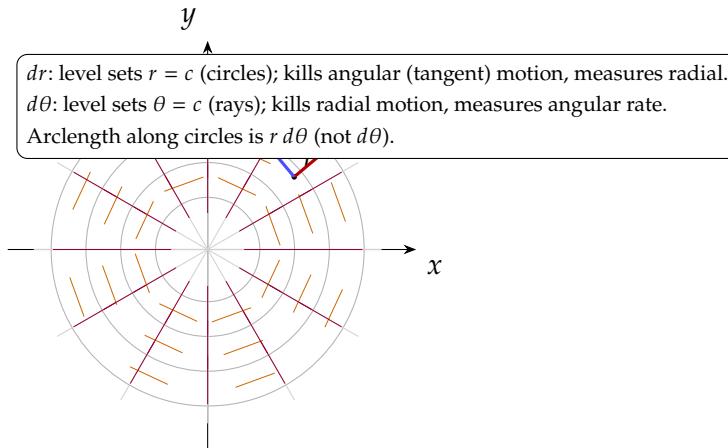
$$df = 2x dx + 2y dy = 2r dr, \quad \text{level sets: } x^2 + y^2 = c \iff r = \sqrt{c}.$$

Here df kills the angular direction (since $df = 2r dr$) and measures only the radial component.

6. Tiny TikZ sketches (1-forms as linelets)

(a) Cartesian: dx and dy



(b) Polar: dr and $d\theta$ 

Big picture mappings

- **Variable** $x \rightarrow dx$ (small change): If x is a coordinate on \mathbb{R} (or part of a coordinate chart), then dx is its **differential**—a linear map that takes a tangent vector (an infinitesimal displacement) and returns the rate of change of x along that displacement. Formally, $dx \in T_p^*M$ is a covector.
- **Scalar field** $f \rightarrow df$ (covector field): A smooth $f : M \rightarrow \mathbb{R}$ determines its **differential** df , a **1-form** (covector field) defined pointwise by

$$df_p(v) = \left. \frac{d}{dt} \right|_{t=0} f(\gamma(t)), \quad \gamma(0) = p, \quad \dot{\gamma}(0) = v.$$

In coordinates x^1, \dots, x^n , $df = \sum_i \frac{\partial f}{\partial x^i} dx^i$.

- **Function** $f \rightarrow \text{level sets}$: For each constant $c \in \mathbb{R}$, the **level set** $f^{-1}(c) = \{p \in M : f(p) = c\}$ collects points where f has the same value. When $\nabla f \neq 0$, level sets are hypersurfaces orthogonal (via a metric) to ∇f .
- **0-form** \rightarrow **1-form**: A smooth function f is a **0-form**. Its exterior derivative d sends 0-forms to 1-forms: $d : \Omega^0(M) \rightarrow \Omega^1(M)$, namely $f \mapsto df$.

Concrete example on \mathbb{R}^2

Let $f(x, y) = x^2 + y^2$.

$$df = 2x \, dx + 2y \, dy, \quad \nabla f = (2x, 2y) \quad (\text{Euclidean metric}).$$

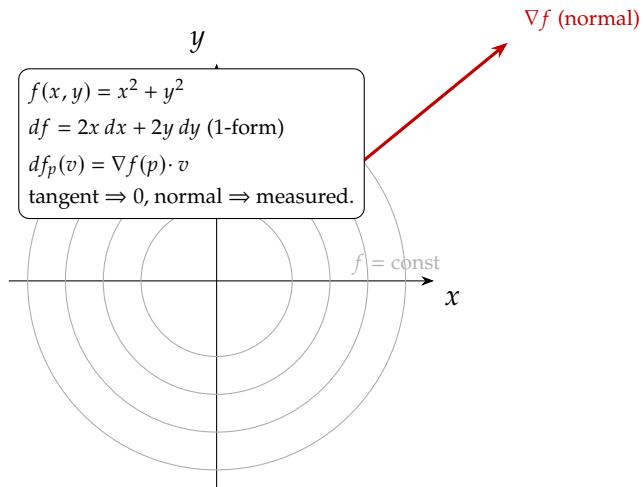
At a point $p = (x, y)$, for a small displacement $v = (v_x, v_y)$,

$$df_p(v) = 2x v_x + 2y v_y = \nabla f(p) \cdot v.$$

Level sets are circles $x^2 + y^2 = c$ (concentric about the origin). Tangent motion along a level set contributes nothing to df ; only the component of motion across level sets (along the gradient) contributes.

One-line summary

$x \mapsto dx$ (basis 1-forms);	$f \mapsto df$ (1-form);	$f \mapsto \{f = c\}$ (level sets);	$0\text{-form} \xrightarrow{d} 1\text{-form}$
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Tiny TikZ sketch (level sets and df)

3 Zero Curl and Closed