Lecture Notes: Coordinates and Differentials on a Plane Curve

1 Setup: The Plane Curve and Its Tangent

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a C^1–function, and define the plane curve

$$C \;=\; \big\{(x,y) \in \mathbb{R}^2 \mid y = f(x)\big\} \subseteq \mathbb{R}^2.$$

Fix a point

$$p = (a, f(a)) \;\in\; C,$$

and consider the standard parametrization

$$\Phi\colon \mathbb{R} \longrightarrow C, \qquad \Phi(t) = \big(t,\; f(t)\big).$$

At $t = a$, the tangent (velocity) is

$$\Phi'(a) = \begin{pmatrix} 1 \\ f'(a) \end{pmatrix} = \vec{v} \;\in\; T_pC,$$

so the tangent space is the one–dimensional subspace

$$T_pC \;=\; \operatorname{span}\{\vec{v}\} = \operatorname{span}\Big\{\,(1,\; f'(a))\Big\} \;\subset\; T_p\mathbb{R}^2 \cong \mathbb{R}^2.$$

2 Points vs. Tangent Vectors

- A *point* $p \in C$ is an element of the set $C \subseteq \mathbb{R}^2$. We denote it by unadorned parentheses, e.g. $p = (a, f(a))$.
- A *tangent vector* $v \in T_pC$ is an element of the tangent space (a copy of $\mathbb{R}^2$) at the point p. We denote it in bold or arrow notation, e.g. $\vec{v} = \big(1, f'(a)\big)^T$.

3 Coordinates on the Curve C

Define the ambient coordinate functions

$$x, y \,:\; \mathbb{R}^2 \;\longrightarrow\; \mathbb{R}, \qquad x(x,y) = x, \quad y(x,y) = y.$$

Restricted to C, these give a chart

$$(C \subseteq \mathbb{R}^2) \;\longrightarrow\; \mathbb{R}^2, \qquad p \;\mapsto\; \big(x(p),\, y(p)\big) = \big(a,\; f(a)\big).$$

Thus the *point–coordinates* of p are $(x(p), y(p))$.

4 Coordinates on the Tangent Line T_pC

On the ambient tangent plane $T_p\mathbb{R}^2 \cong \mathbb{R}^2$, introduce the dual basis $\{dx, dy\}$ defined by

$$dx\big(e_1\big) = 1,\ dx(e_2) = 0, \quad dy(e_1) = 0,\ dy(e_2) = 1,$$

where $e_1 = (1, 0)$, $e_2 = (0, 1)$. Then for any $v = (v^1, v^2)^T$,

$$dx(v) = v^1, \quad dy(v) = v^2.$$

Restricted to the tangent line $T_pC = \operatorname{span}\{(1, f'(a))\}$, these give the fiber–chart

$$T_pC \longrightarrow \mathbb{R}^2, \qquad \vec{v} \mapsto \begin{pmatrix} dx(\vec{v}) \\ dy(\vec{v}) \end{pmatrix} = \begin{pmatrix} 1 \\ f'(a) \end{pmatrix}.$$

Thus the *vector–coordinates* of $\vec{v}$ in the basis $\{\partial_x, \partial_y\}$ are $\big(dx(\vec{v}),\, dy(\vec{v})\big)$.

5 Summary of Maps

$$\begin{aligned} \underbrace{C \subseteq \mathbb{R}^2}_{\text{curve}} &\longrightarrow \mathbb{R}^2, \qquad p \longmapsto (x(p), y(p)) = (a, f(a)); \\ \underbrace{T_pC}_{\text{tangent line}} &\longrightarrow \mathbb{R}^2, \qquad \vec{v} \longmapsto \big(dx(\vec{v}),\, dy(\vec{v})\big) = (1, f'(a)). \end{aligned}$$