From Vector Calculus to Differential Forms: A Dictionary for Conservative and Curl-Free Fields

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Abstract

This lecture note establishes a rigorous correspondence between fundamental concepts in vector calculus on $\mathbb{R}^2$ and $\mathbb{R}^3$ and the language of differential forms. We explore the relationship between conservative vector fields and exact 1-forms, and between curl-free vector fields and closed 1-forms. The critical role of the domain's topology, particularly simple connectedness, is highlighted through the Poincaré Lemma.

1 The Core Dictionary

The translation between vector calculus and differential forms provides both computational power and deeper theoretical insight. The primary correspondences are as follows:

Vector Calculus	$\Longleftrightarrow$	Differential Forms
Vector field $\mathbf{F}$ on $\mathbb{R}^n$ $(n=2,3)$	$\Longleftrightarrow$	1-form ω
Conservative field: $\mathbf{F} = \nabla f$	$\Longleftrightarrow$	Exact 1-form: $\omega = \mathrm{d}f$
Curl-free field: $\nabla \times \mathbf{F} = \mathbf{0}$	$\Longleftrightarrow$	Closed 1-form: $\mathrm{d}\omega = 0$

This dictionary is governed by a fundamental logical implication structure.

Proposition 1.1. *For any smooth 1-form ω on an open domain $U \subseteq \mathbb{R}^n$:*

$$\omega \text{ is exact} \implies \omega \text{ is closed.}$$

The converse holds if the domain U is simply connected (e.g., star-shaped, convex, or all of $\mathbb{R}^n$).

2 Definitions and Interpretations

2.1 Conservative Fields and Exact Forms

Definition 2.1. A vector field $\mathbf{F} : U \to \mathbb{R}^n$ on an open set $U \subseteq \mathbb{R}^n$ is **conservative** if there exists a scalar function $f : U \to \mathbb{R}$, called a scalar potential, such that $\mathbf{F} = \nabla f$.

Definition 2.2. A 1-form ω on an open set $U \subseteq \mathbb{R}^n$ is **exact** if there exists a smooth function (a 0-form) $f : U \to \mathbb{R}$ such that $\omega = \mathrm{d}f$.

In coordinates, if $\mathbf{F} = \langle F_1, \ldots, F_n \rangle$, its corresponding 1-form is $\omega = \sum_{i=1}^{n} F_i \, \mathrm{d}x_i$. The condition $\mathbf{F} = \nabla f = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle$ is identical to $\omega = \mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i$.

Remark 2.1 (Physical Significance). The primary utility of a conservative field is the path-independence of its line integral, a consequence of the Fundamental Theorem for Line Integrals:

$$\int_{\gamma} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\gamma} \nabla f \cdot \mathrm{d}\mathbf{r} = f(\gamma(b)) - f(\gamma(a)).$$

This implies that for any closed loop γ, $\oint_{\gamma} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$.

2.2 Curl-Free Fields and Closed Forms

Definition 2.3. A vector field $\mathbf{F} = \langle P, Q, R \rangle$ on $U \subseteq \mathbb{R}^3$ is **curl-free** (or irrotational) if its curl is the zero vector: $\nabla \times \mathbf{F} = \mathbf{0}$.

Definition 2.4. A differential form ω is **closed** if its exterior derivative is zero: $\mathrm{d}\omega = 0$.

For a 1-form $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$ in $\mathbb{R}^3$, its exterior derivative is the 2-form:

$$\mathrm{d}\omega = (\partial_y R - \partial_z Q)\,\mathrm{d}y \wedge \mathrm{d}z + (\partial_z P - \partial_x R)\,\mathrm{d}z \wedge \mathrm{d}x + (\partial_x Q - \partial_y P)\,\mathrm{d}x \wedge \mathrm{d}y.$$

The coefficients of the basis 2-forms are precisely the components of $\nabla \times \mathbf{F}$ where $\mathbf{F} = \langle P, Q, R \rangle$. Thus, $\mathrm{d}\omega = 0$ is equivalent to $\nabla \times \mathbf{F} = \mathbf{0}$.

Remark 2.2. In 2D, for $\mathbf{F} = \langle P, Q \rangle$ and $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$, the condition $\mathrm{d}\omega = 0$ simplifies to $(\partial_x Q - \partial_y P)\,\mathrm{d}x \wedge \mathrm{d}y = 0$, which is the familiar scalar curl condition $\partial_x Q = \partial_y P$.

3 Practical Tests for Exactness

Given a 1-form $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$ on a domain $U \subseteq \mathbb{R}^2$.

1. **Local Test (Mixed Partials):** Check if ω is closed. This involves testing the equality of mixed partial derivatives:

$$\mathrm{d}\omega = 0 \iff \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

   If the domain U is simply connected, the Poincaré Lemma guarantees that closedness implies exactness.

2. **Global Test (Path Independence):** The form ω is exact if and only if its integral over any closed loop $\gamma \subset U$ is zero: $\oint_\gamma \omega = 0$. This is equivalent to stating that the integral $\int_\gamma \omega$ depends only on the start and end points of the path γ.

3. **Constructive Test (Potential Recovery):** To find a potential function $f(x, y)$ such that $\mathrm{d}f = \omega$, one can proceed by integration.

   (a) Integrate $P = \frac{\partial f}{\partial x}$ with respect to x:

   $$f(x, y) = \int P(x, y)\,\mathrm{d}x + h(y),$$

   where $h(y)$ is an unknown function of y.

   (b) Differentiate this expression for f with respect to y and set it equal to Q:

   $$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int P(x, y)\,\mathrm{d}x\right) + h'(y) = Q(x, y).$$

   (c) Solve for $h'(y)$ and integrate to find $h(y)$. The potential f is unique up to an additive constant.

4 The Role of Domain Topology

The distinction between closed and exact forms is entirely topological.

Definition 4.1. A path-connected set $U \subseteq \mathbb{R}^n$ is **simply connected** if every simple closed curve in U can be continuously shrunk to a point within U. Intuitively, U has no "holes."

- **Simply Connected Domains:** $\mathbb{R}^n$, convex sets, star-shaped domains.

- **Non-Simply Connected Domains:** The punctured plane $\mathbb{R}^2 \setminus \{0\}$, an annulus, $\mathbb{R}^3$ minus a line (e.g., the z-axis).

Lemma 4.1 (Poincaré Lemma)**.** *On a simply connected domain $U \subseteq \mathbb{R}^n$, every closed form is exact.*

5 Illustrative Examples

Example 5.1 (Exact on $\mathbb{R}^2$)**.** Consider the 1-form $\omega = (2xy + ye^{xy}) \, \mathrm{d}x + (x^2 + xe^{xy}) \, \mathrm{d}y$ on $\mathbb{R}^2$. We identify $P = 2xy + ye^{xy}$ and $Q = x^2 + xe^{xy}$.

- **Closedness:** We check the mixed partials:

$$\frac{\partial Q}{\partial x} = 2x + (e^{xy} + xye^{xy}) = 2x + e^{xy}(1 + xy)$$
$$\frac{\partial P}{\partial y} = 2x + (e^{xy} + yxe^{xy}) = 2x + e^{xy}(1 + xy)$$

Since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, the form is closed.

- **Exactness:** Since the domain $\mathbb{R}^2$ is simply connected, ω must be exact. We can construct the potential $f(x, y) = x^2 y + e^{xy}$, for which $\mathrm{d}f = \omega$.

Example 5.2 (Closed but Not Exact on $\mathbb{R}^2 \setminus \{0\}$)**.** Consider the 1-form $\omega = \frac{-y}{x^2 + y^2} \, \mathrm{d}x + \frac{x}{x^2 + y^2} \, \mathrm{d}y$ on the punctured plane $U = \mathbb{R}^2 \setminus \{(0, 0)\}$.

- **Closedness:** A direct calculation shows $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$, so ω is closed.

- **Non-Exactness:** We test the integral around a closed loop that encloses the origin. Let γ be the unit circle, parameterized by $\mathbf{r}(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$.

$$\oint_\gamma \omega = \int_0^{2\pi} \left(\frac{-\sin t}{\cos^2 t + \sin^2 t} (-\sin t) + \frac{\cos t}{\cos^2 t + \sin^2 t} (\cos t) \right) \mathrm{d}t$$
$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t) \, \mathrm{d}t = \int_0^{2\pi} 1 \, \mathrm{d}t = 2\pi.$$

Since the integral over a closed loop is non-zero, ω cannot be exact on U. The "hole" at the origin is a topological obstruction.

Remark 5.1. In polar coordinates (r, θ), this form is simply $\omega = \mathrm{d}\theta$. This makes it clear that it is locally exact, but the potential function θ is not globally single-valued on $\mathbb{R}^2 \setminus \{0\}$, preventing global exactness.

6 The Fundamental Identity: $d^2 = 0$

The fact that exact forms are always closed is a direct consequence of a fundamental property of the exterior derivative.

Theorem 6.1. *For any smooth k-form α, the exterior derivative of its exterior derivative is zero: $\mathrm{d}(\mathrm{d}\alpha) = 0$. This is often written as $\mathrm{d}^2 = 0$.*

Proof for 0-forms. If ω is an exact 1-form, then $\omega = \mathrm{d}f$ for some 0-form (function) f. Applying the exterior derivative again yields:
$$\mathrm{d}\omega = \mathrm{d}(\mathrm{d}f).$$
In coordinates on $\mathbb{R}^n$, $\mathrm{d}f = \sum_i \frac{\partial f}{\partial x_i} \mathrm{d}x_i$. Then

$$\mathrm{d}(\mathrm{d}f) = \sum_{j,i} \frac{\partial^2 f}{\partial x_j \partial x_i} \mathrm{d}x_j \wedge \mathrm{d}x_i.$$

By the symmetry of mixed partial derivatives ($\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$) and the anti-symmetry of the wedge product ($\mathrm{d}x_j \wedge \mathrm{d}x_i = -\mathrm{d}x_i \wedge \mathrm{d}x_j$), the terms in the sum cancel pairwise. For example, the term involving $\mathrm{d}x_1 \wedge \mathrm{d}x_2$ is

$$\left(\frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial^2 f}{\partial x_2 \partial x_1} \right) \mathrm{d}x_1 \wedge \mathrm{d}x_2 = 0.$$

Hence, $\mathrm{d}\omega = 0$. In vector calculus terms, this identity is equivalent to $\nabla \times (\nabla f) = \mathbf{0}$ (the curl of a gradient is always zero). $\qquad\square$

7 Exercises

1. Let $\omega = (3x^2 y + ye^{xy})\,\mathrm{d}x + (x^3 + xe^{xy})\,\mathrm{d}y$ on $\mathbb{R}^2$. Show that ω is closed and find a scalar potential f such that $\omega = \mathrm{d}f$.
   Answer sketch: Check $\partial_x Q = \partial_y P = 3x^2 + e^{xy}(1 + xy)$. The potential is $f(x, y) = x^3 y + e^{xy} + C$.

2. Let $\mathbf{F}(x, y) = \langle -y, x \rangle$ on $\mathbb{R}^2$. Determine if this field is conservative.
   Answer sketch: The corresponding 1-form is $\omega = -y\,\mathrm{d}x + x\,\mathrm{d}y$. Then $\mathrm{d}\omega = (\partial_x(x) - \partial_y(-y))\,\mathrm{d}x \wedge \mathrm{d}y = 2\,\mathrm{d}x \wedge \mathrm{d}y \neq 0$. The form is not closed, and therefore not exact (conservative).

3. Consider the vector field $\mathbf{F}(x, y, z) = \langle yz, xz, xy \rangle$ on $\mathbb{R}^3$.

   (a) Show that $\mathbf{F}$ is curl-free.
   (b) Since the domain $\mathbb{R}^3$ is simply connected, find a scalar potential f such that $\mathbf{F} = \nabla f$.

   Answer sketch: (a) $\nabla \times \mathbf{F} = \mathbf{0}$. (b) The potential is $f(x, y, z) = xyz + C$.