Visualizing and Formalizing ∂_z, dz, and Examples of Holomorphic vs. Non-Holomorphic Behavior

Aim

We give a formal derivation of the complex differential operators

$$\partial_z := \tfrac{1}{2}(\partial_x - \mathrm{i}\,\partial_y), \qquad \partial_{\bar z} := \tfrac{1}{2}(\partial_x + \mathrm{i}\,\partial_y),$$

explain their dual pairing with the complex 1-forms

$$dz = dx + \mathrm{i}\,dy, \qquad d\bar z = dx - \mathrm{i}\,dy,$$

and provide *concrete* examples (with computations) that sharply distinguish holomorphic from non-holomorphic functions. TikZ figures illustrate the geometry.

1 Coordinate change and the origin of the factor $\frac{1}{2}$

Set

$$z = x + \mathrm{i}y, \qquad \bar z = x - \mathrm{i}y.$$

Then

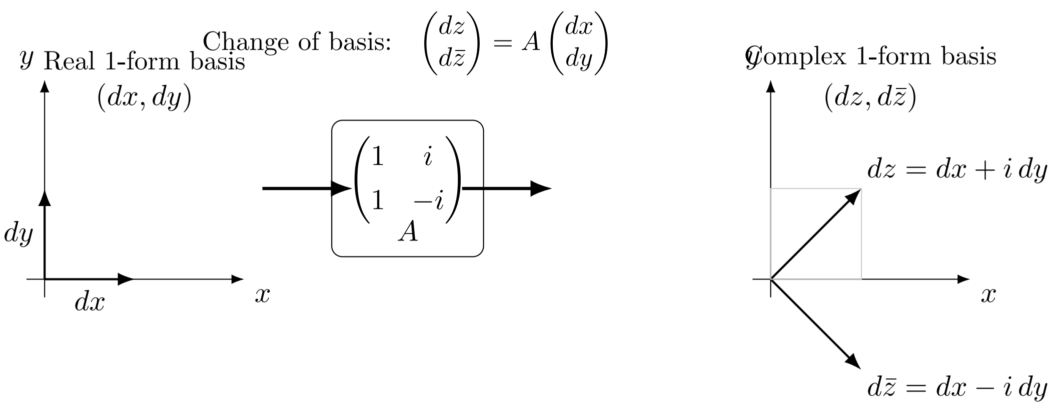
$$\begin{pmatrix} dz \\ d\bar z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}}_{A} \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

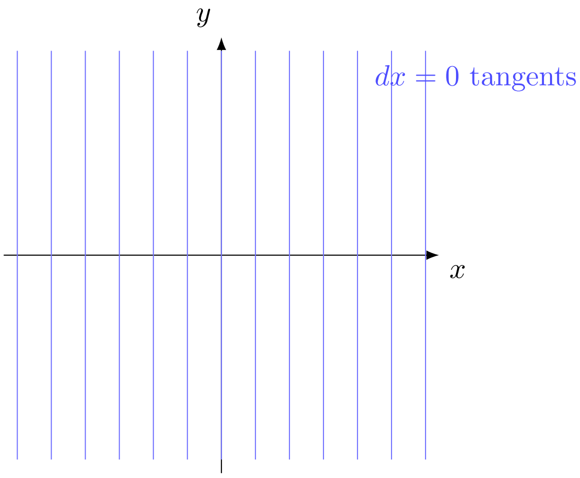
Vector fields transform by the inverse transpose:

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = A^{-T} \begin{pmatrix} \partial_z \\ \partial_{\bar z} \end{pmatrix}.$$

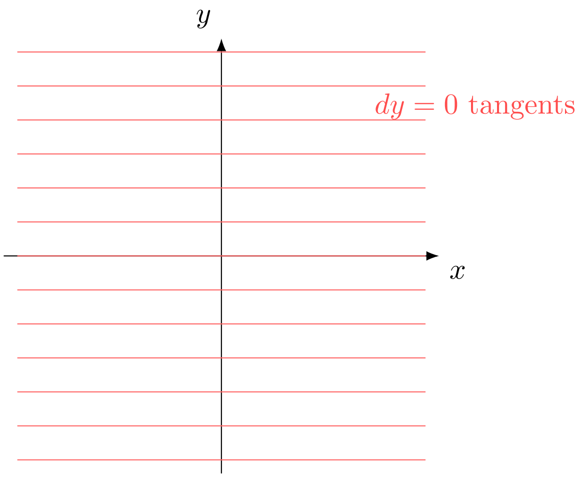
Since $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}$, we have

$$A^{-T} = \frac{1}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}, \quad \Rightarrow \quad \boxed{\partial_z = \tfrac{1}{2}(\partial_x - \mathrm{i}\partial_y), \quad \partial_{\bar z} = \tfrac{1}{2}(\partial_x + \mathrm{i}\partial_y)}.$$

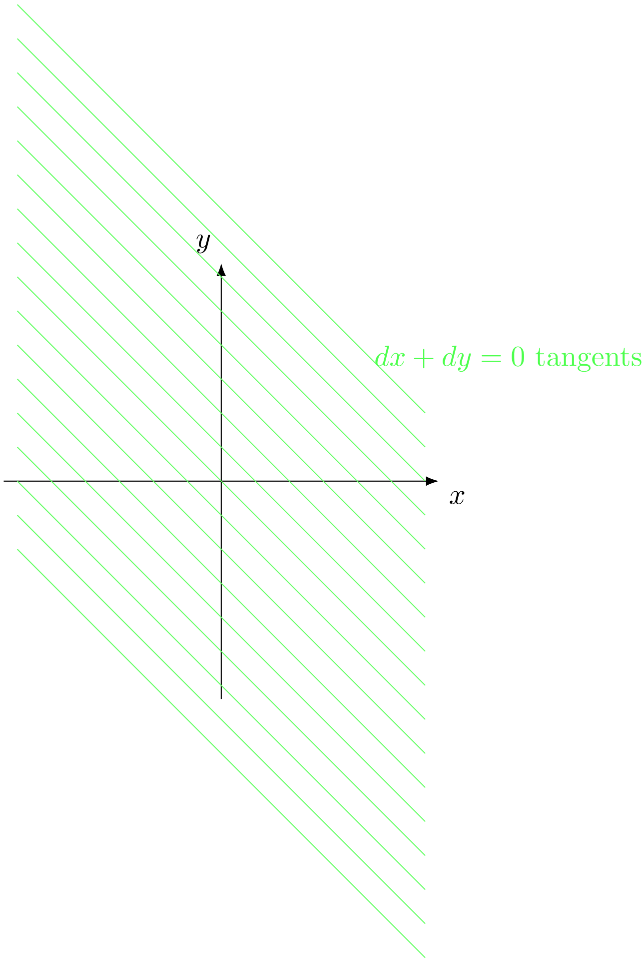


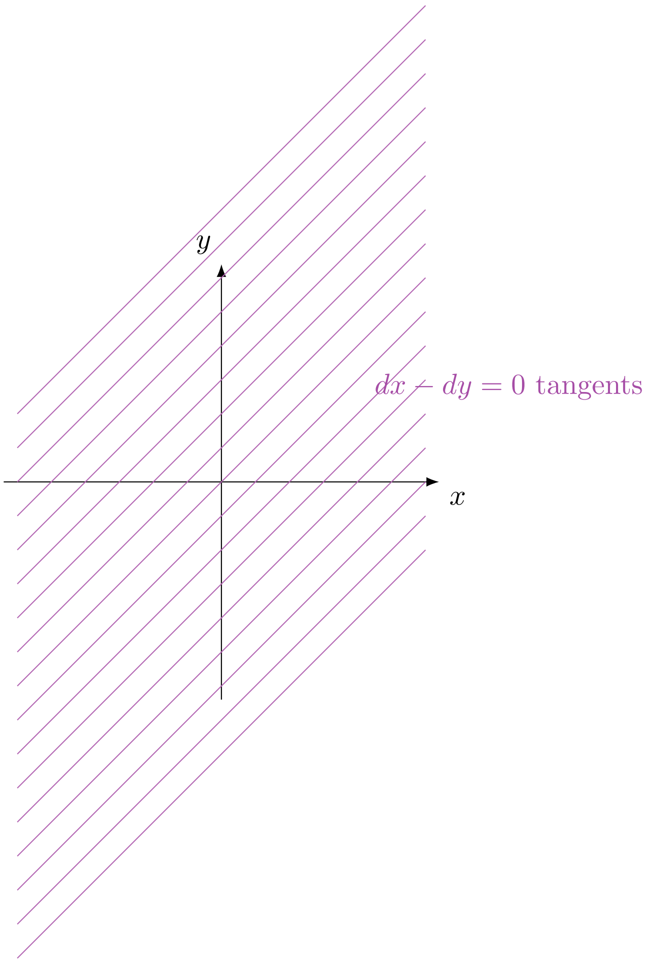


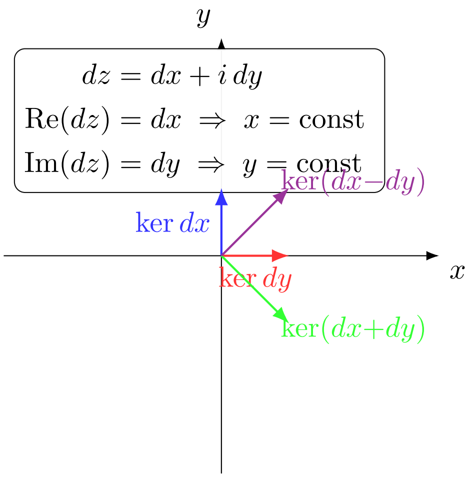
y
$dx = 0$ tangents
x



y
$dy = 0$ tangents
x


y
$dx + dy = 0$ tangents
x


y
$dx - dy = 0$ tangents
x


y
$dz = dx + i\,dy$
$\mathrm{Re}(dz) = dx \;\Rightarrow\; x = \mathrm{const}$
$\mathrm{Im}(dz) = dy \;\Rightarrow\; y = \mathrm{const}$
$\ker(dx - dy)$
$\ker dx$
x
$\ker dy$
$\ker(dx + dy)$

2 Dual pairing and holomorphicity

By construction,

$$dz(\partial_z) = 1, \quad dz(\partial_{\bar z}) = 0, \qquad d\bar z(\partial_{\bar z}) = 1, \quad d\bar z(\partial_z) = 0.$$

For a C^1 function $f\colon \mathbb{C} \to \mathbb{C}$,

$$df = f_x\,dx + f_y\,dy = f_z\,dz + f_{\bar z}\,d\bar z, \quad \text{where} \quad f_z = \tfrac{1}{2}(f_x - \mathrm{i}f_y), \quad f_{\bar z} = \tfrac{1}{2}(f_x + \mathrm{i}f_y).$$

Holomorphic means $f_{\bar z} = 0$, equivalently $df = f_z\,dz$ (no $d\bar z$-part).

3 Geometric action of dz

For a real tangent vector $v = a\,\partial_x + b\,\partial_y$,

$$dz(v) = a + \mathrm{i}b.$$

Thus dz converts a real direction into a complex number whose modulus is speed and whose argument is direction.

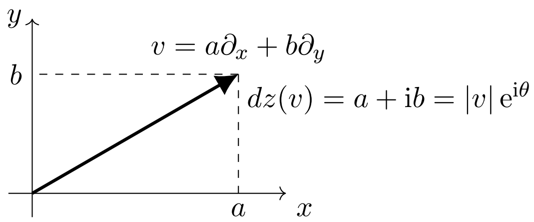


Fig. 1. The 1-form dz encodes magnitude/direction as a complex number.

4 Penetrating examples

We now compute $f_z, f_{\bar z}$ and df explicitly and interpret the results geometrically.

Example A (holomorphic): $f(z) = z^2$

Write $f(x, y) = (x + \mathrm{i}y)^2 = x^2 - y^2 + 2\mathrm{i}xy$. Then

$$f_x = 2x + 2\mathrm{i}y, \qquad f_y = -2y + 2\mathrm{i}x.$$

Hence

$$f_z = \tfrac{1}{2}(f_x - \mathrm{i}f_y) = \tfrac{1}{2}\big[(2x + 2\mathrm{i}y) - \mathrm{i}(-2y + 2\mathrm{i}x)\big] = 2(x + \mathrm{i}y) = 2z,$$
$$f_{\bar z} = \tfrac{1}{2}(f_x + \mathrm{i}f_y) = \tfrac{1}{2}\big[(2x + 2\mathrm{i}y) + \mathrm{i}(-2y + 2\mathrm{i}x)\big] = 0.$$

Thus $df = 2z\,dz$ and f is holomorphic. Along any vector v,

$$df(v) = 2z \cdot dz(v) = 2z\,(a + \mathrm{i}b).$$

Geometric meaning: the complex directional derivative is the complex number $2z$ times the complex encoding of v.

Example B (non-holomorphic): $f(z) = |z|^2 = z\bar{z} = x^2 + y^2$

Here

$$f_x = 2x, \qquad f_y = 2y, \quad f_z = \tfrac{1}{2}(2x - \mathrm{i}\cdot 2y) = x - \mathrm{i}y = \bar{z}, \quad f_{\bar{z}} = \tfrac{1}{2}(2x + \mathrm{i}\cdot 2y) = x + \mathrm{i}y = z.$$

So

$$df = \bar{z}\,dz + z\,d\bar{z},$$

and $f_{\bar{z}} = z \neq 0$ unless $z = 0$: *not holomorphic.* Note that the gradient in real terms points radially; the presence of a $d\bar{z}$-piece records the anti-holomorphic contribution.

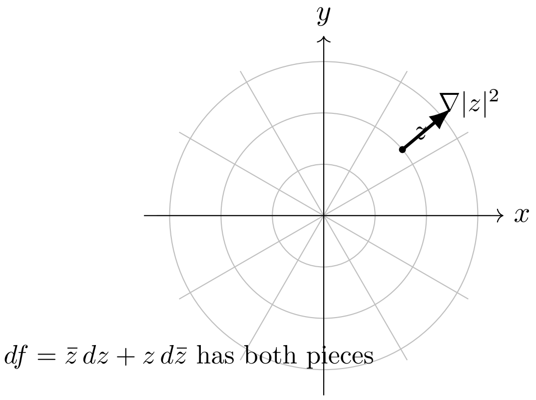


Fig. 2. For $f = |z|^2$, level sets are circles; df has a $d\bar{z}$-part, so f is not holomorphic.

Example C (anti-holomorphic): $f(z) = \bar{z}$

We have $f_x = 1$, $f_y = -\mathrm{i}$, so

$$f_z = \tfrac{1}{2}(1 - \mathrm{i}\cdot(-\mathrm{i})) = 0, \qquad f_{\bar{z}} = \tfrac{1}{2}(1 + \mathrm{i}\cdot(-\mathrm{i})) = 1,$$

hence $df = d\bar{z}$. This is *purely* anti-holomorphic: dz is annihilated and only $d\bar{z}$ survives.

Example D (logarithmic/winding): $\omega = \dfrac{dz}{z}$ on $\mathbb{C}^\times$

Write $z = re^{\mathrm{i}\theta}$ with $r > 0$. Then

$$\frac{dz}{z} = \frac{d(re^{\mathrm{i}\theta})}{re^{\mathrm{i}\theta}} = \frac{dr}{r} + \mathrm{i}\,d\theta = d(\log r) + \mathrm{i}\,d\arg z.$$

$\mathrm{Re}(\omega)$ measures radial change, $\mathrm{Im}(\omega)$ measures angular change. For the circle $\gamma(t) = R\,e^{\mathrm{i}t}$, $t \in [0, 2\pi]$,

$$\oint_\gamma \frac{dz}{z} = \mathrm{i}\int_0^{2\pi} dt = 2\pi\mathrm{i}.$$

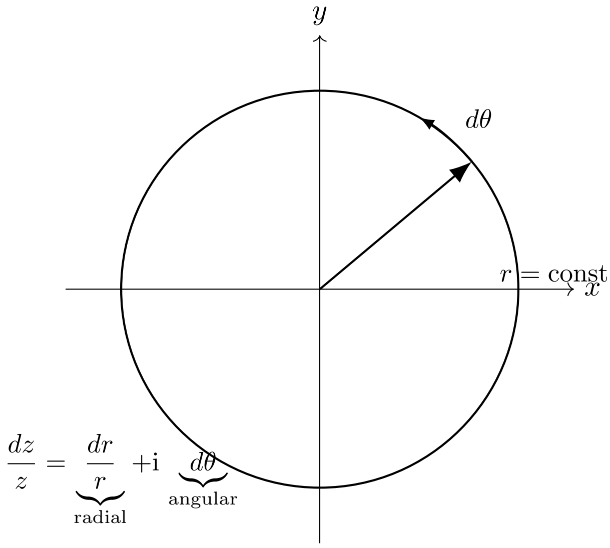


Fig. 3. The decomposition of $\frac{dz}{z}$ as radial + angular.

5 Why ∂_z and $\partial_{\bar{z}}$ matter

Given a C^1 function f,

$$\partial_z f = f_z, \qquad \partial_{\bar{z}} f = f_{\bar{z}}.$$

Holomorphicity is the single linear condition $\partial_{\bar{z}} f = 0$, equivalently, df has *no* $d\bar{z}$ component. This compresses the Cauchy–Riemann equations into a basis statement: "df is complex-linear."

6 Operational "recipe" (ready to use)

Let $f(x, y)$ be given.

$$\boxed{\begin{aligned} &f_z = \tfrac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right), \qquad f_{\bar{z}} = \tfrac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right), \\ &df = f_z\, dz + f_{\bar{z}}\, d\bar{z}, \quad f \text{ holomorphic} \iff f_{\bar{z}} \equiv 0. \end{aligned}}$$

7 A visual of the complex coframe

Although $\partial_z, \partial_{\bar{z}}$ are complex combinations of real directions, the *pairing* with $dz, d\bar{z}$ is exact: one kills the other.

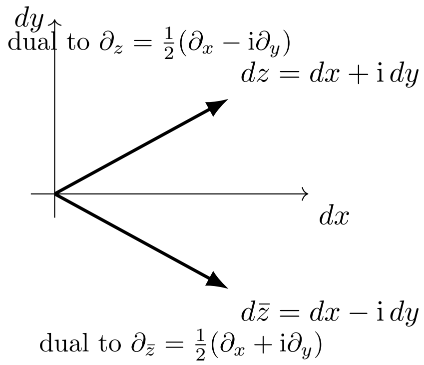


Fig. 4. Heuristic picture: dz detects ∂_z and kills $\partial_{\bar{z}}$; $d\bar{z}$ does the opposite.

Appendix: fully worked micro-examples

Example 1 (Directional values of dz) *Along ∂_x, $dz(\partial_x) = 1$; along ∂_y, $dz(\partial_y) = \mathrm{i}$; along $\partial_x + \partial_y$, $dz = 1 + \mathrm{i}$.*

Example 2 ($f(z) = \mathrm{e}^z$ is holomorphic) *$f_x = \mathrm{e}^x \cos y + \mathrm{i}\,\mathrm{e}^x \sin y,\ f_y = -\mathrm{e}^x \sin y + \mathrm{i}\,\mathrm{e}^x \cos y$ so $f_z = \frac{1}{2}(f_x - \mathrm{i} f_y) = \mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^z,\ f_{\bar{z}} = 0,$ and $df = \mathrm{e}^z\,dz$.*

Example 3 ($f(z) = \overline{z}^2$ is anti-holomorphic) *$f_x = 2x,\ f_y = -2\mathrm{i}y$ so $f_z = 0,\ f_{\bar{z}} = 2\bar{z}$, hence $df = 2\bar{z}\,d\bar{z}$.*

Exercises (with short answers)

1) Show directly from definitions that $dz(\partial_{\bar{z}}) = 0$ and $d\bar{z}(\partial_z) = 0$.
2) For $f(z) = x^3 - 3xy^2 + \mathrm{i}(3x^2y - y^3)$, compute $f_z, f_{\bar{z}}$ and determine holomorphicity. (Ans: $f_{\bar{z}} = 0$: this is the cubic z^3.)
3) Let $\gamma(t) = R\mathrm{e}^{\mathrm{i}t}$, $t \in [0, 2\pi]$. Compute $\oint_\gamma dz$ and $\oint_\gamma dz/z$. (Ans: 0 and $2\pi\mathrm{i}$.)