From Vector Calculus to Differential Forms

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We cover the following topics in this note.

- Vector calculus (conservative fields, irrotational field)
- Differential forms (exact forms, closed forms)

Vector Calculus (in $\mathbb{R}^2$ or $\mathbb{R}^3$)		**Differential Forms**
Vector Field $\mathbf{F}$	$\Longleftrightarrow$	1-form ω
Conservative Vector Field ($\mathbf{F} = \nabla f$)	$\Longleftrightarrow$	Exact 1-form ($\omega = \mathrm{d}f$)
Irrotational Vector Field ($\nabla \times \mathbf{F} = \mathbf{0}$)	$\Longleftrightarrow$	Closed 1-form ($\mathrm{d}\omega = 0$)

The Fundamental Implication:

- **Conservative** $\implies$ **Irrotational**, i.e., **Exact** $\implies$ **Closed**
  (This is always true.)
- **Irrotational** $\implies$ **Conservative**, i.e, **Closed** $\implies$ **Exact**
  (This is only true on “nice” domains, e.g., simply connected ones.)

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1 Conservative and Exact

Why do we think of the "Conservative Field (or Gradient Field)"?

A vector field **F** is **conservative** if it is the **gradient**[1] of some scalar potential function f:

$$\mathbf{F} = \nabla f = \begin{cases} \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle & \text{if } f : U(\subseteq \mathbb{R}^2) \to \mathbb{R} \\ \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle & \text{if } f : U(\subseteq \mathbb{R}^3) \to \mathbb{R} \\ \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle & \text{if } f : U(\subseteq \mathbb{R}^n) \to \mathbb{R} \end{cases}.$$

If **F** is conservative, line integrals only depend on the endpoints:

$$\int_\gamma \mathbf{F} \cdot d\mathbf{r} = \int_\gamma \nabla f \cdot d\mathbf{r} = f(\text{end}) - f(\text{start}),$$

which turns hard line integrals into simple evaluations. Furthermore every closed loop (start = end) has integral 0.

Equivalently (on a nice domain) every line integral of **F** is path-independent:

$$\int_{\gamma_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\gamma_2} \mathbf{F} \cdot d\mathbf{r}.$$

whenever γ_1 and γ_2 have the same endpoints.

From Gradients to Curl

Given a vector field **F**, we wish to determine whether **F** is conservative (i.e., $\mathbf{F} = \nabla f$ for some scalar field f). Trying to guess the potential function f is hard.

We already know that if a field **F** is conservative, it must be the gradient of some potential function f:

$$\mathbf{F} = \langle P(x,y), Q(x,y) \rangle = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \qquad \text{(in 2D)}$$

What happens if we differentiate $P(x,y) = \frac{\partial f}{\partial x}$ with respect to y and $Q(x,y) = \frac{\partial f}{\partial y}$ with respect to x?

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}, \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}.$$

Theorem 1 (Equality of mixed partials; Clairaut's Theorem). *If the partial derivatives $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ exist and are continuous at a point (a,b), then $\frac{\partial^2 f}{\partial y \partial x}(a,b) = \frac{\partial^2 f}{\partial x \partial y}(a,b)$, i.e., second order partial derivatives commute if f is C^2.*

[1]Gradient is a measure of change in a scalar field

If a vector field $\mathbf{F} = \langle P(x,y), Q(x,y) \rangle$ is a gradient, it **must** satisfy the condition

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

The quantity $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ represents the curl of $\mathbf{F} = \langle P, Q \rangle$ and encodes its local rotational behavior. Hence the condition $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ (i.e., $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$) means that the field is **irrotational** (has **zero curl**).

Remark 1. Consider a small rectangle centered at (x_0, y_0) with side lengths $\Delta x, \Delta y$.

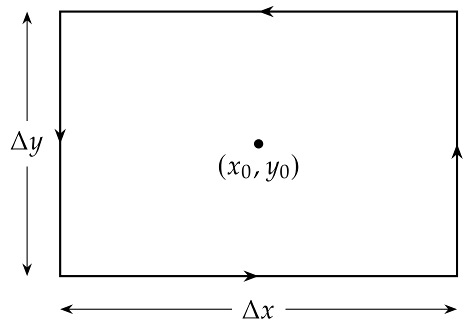


Figure 1: Circulation around an infinitesimal rectangle.

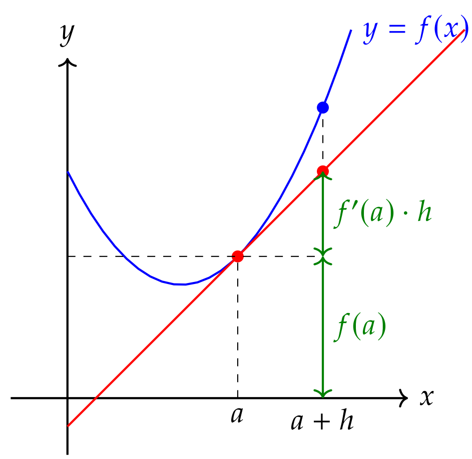
The total counterclockwise circulation is the sum of the line integrals along the four edges:

$$\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \int_{\text{bottom}} P\,dx + \int_{\text{right}} Q\,dy + \int_{\text{top}} P\,dx + \int_{\text{left}} Q\,dy.$$

We will approximate the value of P or Q along each edge as being constant, equal to its value at the midpoint of that edge. We find this value using a first-order Taylor expansion from the center point (x_0, y_0).

For a simple function of one variable, $f(x)$, if we know its value at a point a, then we can estimate its value at a nearly point $a + h$ using the tangent line at a:

$$f(a+h) \approx f(a) + f'(a) \cdot h \quad \text{or}$$
$$f(x) \approx f(a) + f'(a) \cdot (x - a).$$



In words, "New Value ≈ Old Value + (Rate of Change) × (Small Step)".

For a function of two variables like $P(x, y)$, the idea is identical, but the "rate of change" now has two components (one for each direction), and the "tangent line" becomes a "tangent plane". The general first-order Taylor expansion for $P(x, y)$ around a center point (x_0, y_0) is

$$P(x_0 + a, y_0 + b) \approx P(x_0, y_0) + \frac{\partial P}{\partial x}(x_0, y_0) \cdot a + \frac{\partial P}{\partial y}(x_0, y_0) \cdot b$$

Here, a is the small step in the x-direction, and b is the small step in the y-direction.

1. The Horizontal Paths These integrals involve the horizontal component of $P(x, y)$.

- **Bottom Path (⟶):**

$$P\left(x, y_0 - \frac{\Delta y}{2}\right) \approx P(x_0, y_0) - \frac{\partial P}{\partial y}\frac{\Delta y}{2} \implies \int_{\text{bottom}} P \,\mathrm{d}x \approx \left(P(x_0, y_0) - \frac{\partial P}{\partial y}\frac{\Delta y}{2}\right)(\Delta x)$$

- **Top Path (⟵):**

$$P\left(x, y_0 + \frac{\Delta y}{2}\right) \approx P(x_0, y_0) + \frac{\partial P}{\partial y}\frac{\Delta y}{2} \implies \int_{\text{top}} P \,\mathrm{d}x \approx -\left(P(x_0, y_0) + \frac{\partial P}{\partial y}\frac{\Delta y}{2}\right)(\Delta x)$$

Here, we are left with only the parts that describe the *change* in P with respect to y.

$$\int_{\text{bottom}} P \,\mathrm{d}x + \int_{\text{top}} P \,\mathrm{d}x \approx \left(-\frac{\partial P}{\partial y}\frac{\Delta y}{2}\right)\Delta x - \left(\frac{\partial P}{\partial y}\frac{\Delta y}{2}\right)\Delta x = -\frac{\partial P}{\partial y}\Delta x \Delta y$$

2. The Vertical Paths These integrals involve the vertical component of $Q(x, y)$.

- **Right Path (↑):**

$$Q\left(x_0 + \frac{\Delta x}{2}, y\right) \approx Q(x_0, y_0) + \frac{\partial Q}{\partial x}\frac{\Delta x}{2} \implies \int_{\text{right}} Q \,\mathrm{d}y \approx \left(Q(x_0, y_0) + \frac{\partial Q}{\partial x}\frac{\Delta x}{2}\right)(\Delta y)$$

- **Left Path (↓):**

$$Q\left(x_0 - \frac{\Delta x}{2}, y\right) \approx Q(x_0, y_0) - \frac{\partial Q}{\partial x}\frac{\Delta x}{2} \implies \int_{\text{left}} Q \,\mathrm{d}y \approx -\left(Q(x_0, y_0) - \frac{\partial Q}{\partial x}\frac{\Delta x}{2}\right)(\Delta y)$$

Here, we are left with only the parts that describe the *change* in Q with respect to x.

$$\int_{\text{right}} Q \,\mathrm{d}y + \int_{\text{left}} Q \,\mathrm{d}y \approx \left(\frac{\partial Q}{\partial x}\frac{\Delta x}{2}\right)\Delta y + \left(\frac{\partial Q}{\partial x}\frac{\Delta x}{2}\right)\Delta y = \frac{\partial Q}{\partial x}\Delta x \Delta y$$

Now we sum the results from the horizontal and vertical pairs:

$$\begin{aligned}\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} &\approx \left(-\frac{\partial P}{\partial y}\Delta x \Delta y\right) + \left(\frac{\partial Q}{\partial x}\Delta x \Delta y\right) \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\Delta x \Delta y\end{aligned}$$

This shows that the total circulation around the tiny loop is approximately the quantity $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$ multiplied by the area of the loop ($\Delta A = \Delta x \Delta y$).

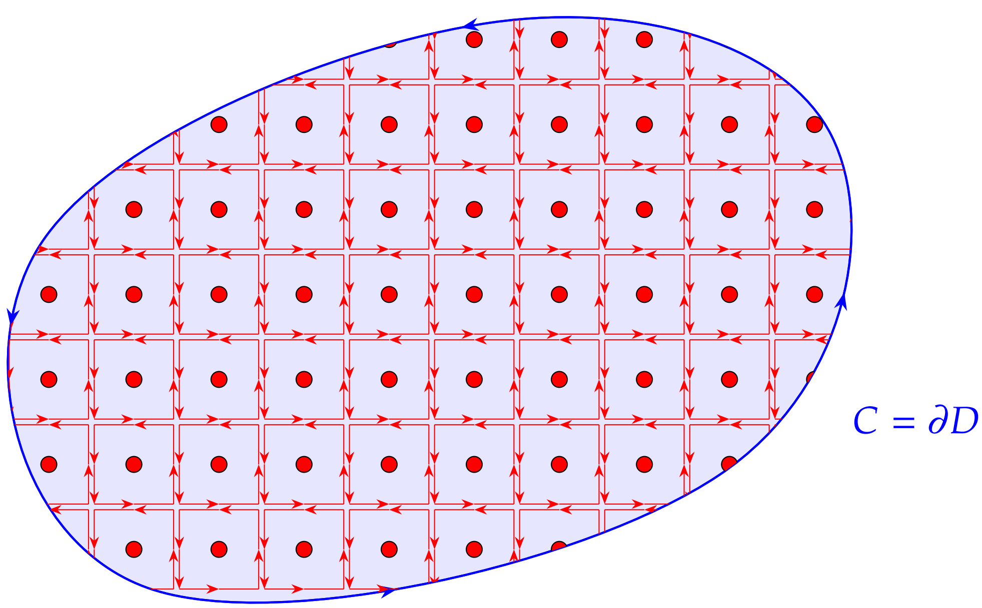
To get the property *at the point* (x_0, y_0), we find the circulation **density**. We divide by the area and take the limit as the rectangle shrinks to zero.

$$\lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint_{\partial R} \mathbf{F} \cdot \mathbf{dr} = \frac{\partial Q}{\partial x}(x_0, y_0) - \frac{\partial P}{\partial y}(x_0, y_0)$$

This is why we call the scalar quantity $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ the **curl**: it is the circulation per unit area at a point, which measures the local rotational tendency of the field.

Remark 2. If $C = \partial D$ is a positively oriented simple closed curve enclosing a region D, Green's theorem states

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{dr}}_{\substack{\textbf{Line Integral} \\ \textbf{(Total Circulation)}}} = \underbrace{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A}_{\substack{\textbf{Double Integral} \\ \textbf{(Sum of Local Curls)}}}$$



Example 1: Rigid rotation and angular velocity

Consider the rigid rotation field with angular speed ω:

$$\mathbf{F}(x, y) = \langle -\omega y,\ \omega x \rangle.$$

Then

$$\frac{\partial Q}{\partial x} = \omega, \qquad \frac{\partial P}{\partial y} = -\omega \quad \Rightarrow \quad \operatorname{curl}\mathbf{F} = Q_x - P_y = 2\omega.$$

This shows curl equals twice the angular velocity. For a circle of radius R, parametrize $r(t) = (R\cos t, R\sin t)$, $dr = (-R\sin t, R\cos t)\,dt$. Then

$$\oint \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \omega R^2\,dt = 2\pi\omega R^2.$$

Meanwhile, $\iint_D (2\omega)\,dA = 2\omega\cdot\pi R^2 = 2\pi\omega R^2$, agreeing with Green's theorem.

Example 2: Curl-free but not conservative (topology matters)

On $\mathbb{R}^2 \setminus \{(0,0)\}$, define

$$\mathbf{F}(x, y) = \left\langle -\frac{y}{x^2+y^2},\ \frac{x}{x^2+y^2} \right\rangle.$$

A direct calculation shows $Q_x - P_y = 0$ wherever defined (curl-free). However, the circulation around the unit circle is

$$\oint \mathbf{F}\cdot d\mathbf{r} = 2\pi \neq 0.$$

Hence there is no global potential function; the puncture creates a topological obstruction. This illustrates that $\operatorname{curl}\mathbf{F} = 0$ captures **local** rotation, while global circulation can persist in domains with holes.

Summary checklist

- $Q_x - P_y$ is the infinitesimal (per-area) circulation density.
- Green's theorem sums local curl to give total circulation.
- Rigid rotation: curl $= 2\omega$ (twice angular velocity).
- Curl $= 0$ can still have nonzero loop integrals if the domain has holes.

Test 1: Equality of Mixed Partials

Test 2: Path Independence

Test 3: Potential Recovery

2 Zero Curl and Closed