# From Gradients to Curl: A Natural Introduction

Motivating the Three Tests for Conservative Fields

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The core problem is this: given a random vector field $\mathbf{F}$, how can we tell if it's a **conservative** (or gradient) field? Trying to guess the potential function $f$ is hard. We need a more systematic approach, starting only from what we know about gradients.

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## 1 Why You Should Naturally Think About "Curl"

You already know that if a field $\mathbf{F}$ is conservative, it must be the gradient of some potential function $f$. In 2D, this means:

$$\mathbf{F} = \langle P, Q \rangle = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

This gives us two direct relationships: $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$.

Now, let's ask a simple question. What happens if we differentiate $P$ with respect to $y$ and $Q$ with respect to $x$?

- Differentiate $P$ with respect to $y$: $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$
- Differentiate $Q$ with respect to $x$: $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

From calculus, you know that for any well-behaved function $f$, the order of differentiation doesn't matter (Clairaut's Theorem on mixed partials). This means $\frac{\partial^2 f}{\partial y \partial x}$ must equal $\frac{\partial^2 f}{\partial x \partial y}$.

This leads to a profound conclusion: if a field $\mathbf{F} = \langle P, Q \rangle$ is truly a gradient field, it is **necessary** that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

This specific quantity, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, measures the microscopic "rotation" or "swirl" of a vector field at a point. It's so important that it gets its own name: the **curl**. The condition we just derived, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$, is simply the statement that the field must have **zero curl**.

So, you should naturally think about "zero curl" not as a random new concept, but as a **simple, necessary consequence of a field being a gradient**, derived directly from the equality of mixed partials.

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## 2 A Natural Reason for the Three Tests

Based on this, we can approach the problem of identifying conservative fields from three different, very natural angles. Each "test" is really just a different question we're asking.

## 2.1 The Local Test (Equality of Mixed Partials)

- **The Question:** "Is there a fast, upfront check to see if a field is *disqualified* from being conservative?"
- **The Reason:** This is the "zero curl" test we just derived ($\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$). It's called the **Local Test** because it checks the field's rotational property at every single point. If this test fails, the field cannot be a gradient, and we can stop. It's our first, essential checkpoint.

## 2.2 The Global Test (Path Independence)

- **The Question:** "What is the most important *physical consequence* of a field being conservative?"
- **The Reason:** You know that for a conservative field, the line integral only depends on the start and end points. This is the defining feature! This test, called the **Global Test**, checks this very property. It's "global" because it depends on the entire path, not just local behavior.

## 2.3 The Constructive Test (Potential Recovery)

- **The Question:** "If a field passes the local test, how can I *prove* it's conservative and find its potential function $f$?"
- **The Reason:** This is the most direct approach. You try to **build** the potential function $f$ by reversing the process of the gradient—that is, by integrating. If you can successfully construct $f$, you have provided definitive proof. This is the **Constructive Test**.

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# 3 An Example That Permeates All Three Tests

Let's investigate the vector field $\mathbf{F}(x, y) = \langle 2xy, x^2 + 3y^2 \rangle$. Is it conservative? Let's ask our three questions.

## 3.1 Test 1: The Fast Check (Local Test)

Does it have the necessary "zero curl" property? Here, $P = 2xy$ and $Q = x^2 + 3y^2$.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x$$
$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3y^2) = 2x$$

Yes, $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. The field passes our first check. It *could* be conservative.

## 3.2 Test 2: Checking the Physical Consequence (Global Test)

Is the line integral from $A = (0, 0)$ to $B = (1, 2)$ path-independent?

- **Path 1 (Along the axes):** $(0, 0) \to (1, 0) \to (1, 2)$.
  - Segment 1 ($(0, 0) \to (1, 0)$): $y = 0, \mathrm{d}y = 0$. Integral is $\int_0^1 2x(0)\,\mathrm{d}x = 0$.
  - Segment 2 ($(1, 0) \to (1, 2)$): $x = 1, \mathrm{d}x = 0$. Integral is $\int_0^2 (1^2 + 3y^2)\,\mathrm{d}y = [y + y^3]_0^2 = 10$.

  **Total for Path 1 = 10.**

- **Path 2 (Straight line):** Parameterize as $\mathbf{r}(t) = \langle t, 2t\rangle$ for $t \in [0, 1]$. This gives $x = t, y = 2t, \mathrm{d}x = \mathrm{d}t, \mathrm{d}y = 2\,\mathrm{d}t$.

$$\begin{aligned}\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y &= \int_0^1 \big((2(t)(2t))\,\mathrm{d}t + (t^2 + 3(2t)^2)(2\,\mathrm{d}t)\big) \\ &= \int_0^1 (4t^2 + 2(13t^2))\,\mathrm{d}t = \int_0^1 30t^2\,\mathrm{d}t \\ &= [10t^3]_0^1 = 10.\end{aligned}$$

  **Total for Path 2 = 10.**

Both paths give the same answer! This demonstrates the global property of path independence.

## 3.3 Test 3: Building the Proof (Constructive Test)

Let's find the potential function $f(x, y)$ by reversing the gradient.

1. **Integrate $P$ with respect to $x$:** We know $\frac{\partial f}{\partial x} = 2xy$.

$$f(x, y) = \int 2xy\,\mathrm{d}x = x^2y + h(y)$$

   The "constant" of integration is an unknown function of $y$.

2. **Differentiate with respect to $y$ and match to $Q$:** We know $\frac{\partial f}{\partial y} = x^2 + 3y^2$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y + h(y)) = x^2 + h'(y)$$

   Setting our two expressions for $\frac{\partial f}{\partial y}$ equal:

$$x^2 + h'(y) = x^2 + 3y^2$$

3. **Solve for $h(y)$:**

$$h'(y) = 3y^2 \implies h(y) = \int 3y^2\,\mathrm{d}y = y^3$$

We have successfully built the potential function: $f(x, y) = x^2y + y^3$. Since we found a potential, the field is definitively **conservative**.