Theorem 1 (Harmonic $\Rightarrow$ constant on a compact Riemann surface, elementary proof)**.** *Let X be a compact, connected Riemann surface. If $u \in C^\infty(X, \mathbb{R})$ is harmonic, then u is constant.*

Set-up (what "harmonic" means here). A Riemann surface admits *conformal* local coordinates (x, y) in which the Laplacian is the *flat* one up to a positive factor. Thus u is harmonic iff in any such chart

$$u_{xx} + u_{yy} = 0.$$

(All we need below is harmonicity in each chart.)

Key local identity (just product rule). Fix one conformal chart (x, y) on an open set $U \subset X$. Define

$$P := -u\, u_y, \qquad Q := u\, u_x.$$

Then a direct computation gives

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = u_x^2 + u_y^2 + u\,(u_{xx} + u_{yy}) = |\nabla u|^2 + u\,\Delta u. \tag{$*$}$$

(Here $|\nabla u|^2 := u_x^2 + u_y^2$ and $\Delta u := u_{xx} + u_{yy}$ in this chart.)

Green's theorem in the chart. For a *compactly supported* smooth function ρ in U, apply Green's theorem to the vector field

$$(P_\rho, Q_\rho) := \big(-\rho\, u\, u_y,\ \rho\, u\, u_x\big).$$

On U we have, by the product rule and $(*)$,

$$\frac{\partial Q_\rho}{\partial x} - \frac{\partial P_\rho}{\partial y} = \rho\big(|\nabla u|^2 + u\,\Delta u\big) \;+\; u\, u_x\, \rho_x + u\, u_y\, \rho_y. \tag{$**$}$$

If U' is a slightly larger coordinate patch containing $\operatorname{supp}\rho$ and with smooth boundary, then Green's theorem gives

$$\int_{U'} \Big(\frac{\partial Q_\rho}{\partial x} - \frac{\partial P_\rho}{\partial y}\Big)\, dx\, dy = \int_{\partial U'} P_\rho\, dx + Q_\rho\, dy = 0,$$

because $\rho \equiv 0$ on $\partial U'$ (so $P_\rho = Q_\rho = 0$ there).

Make it global with a partition of unity. Cover X by finitely many conformal coordinate discs $U_1, \ldots, U_N$. Choose a smooth partition of unity $\{\rho_j\}_{j=1}^N$ subordinate to $\{U_j\}$; that is, each $\rho_j \geq 0$ has compact support in U_j, $\sum_j \rho_j \equiv 1$, and $\sum_j \rho_{j,x} \equiv 0 \equiv \sum_j \rho_{j,y}$ in overlapping coordinates. Apply the previous step in each U_j and sum:

$$0 = \sum_{j=1}^N \int_{U_j'} \Big(\frac{\partial Q_{\rho_j}}{\partial x} - \frac{\partial P_{\rho_j}}{\partial y}\Big)\, dx\, dy = \int_X \sum_{j=1}^N \Big[\rho_j\big(|\nabla u|^2 + u\,\Delta u\big) + u\, u_x\, \rho_{j,x} + u\, u_y\, \rho_{j,y}\Big]\, dx\, dy.$$

Using $\sum_j \rho_j \equiv 1$ and $\sum_j \rho_{j,x} = \sum_j \rho_{j,y} \equiv 0$, we simplify to

$$0 \;=\; \int_X \big(|\nabla u|^2 + u\,\Delta u\big)\, dx\, dy.$$

Finish. Since u is harmonic, $\Delta u = 0$ in every chart, hence

$$0 = \int_X |\nabla u|^2 \, dx \, dy.$$

The integrand is pointwise nonnegative, so $|\nabla u| \equiv 0$ on X. Therefore u is locally constant, and by connectedness of X, u is constant.

$\square$

Notes.

- This proof only used the product rule, Green's theorem (no wedge products), and a partition of unity.
- The same argument works for any compact oriented surface endowed with a conformal atlas.