Part II — Building the Weierstrass $\wp$-Function from the Holomorphic Form $dt = \omega$

Roadmap

Starting with the flat holomorphic 1-form dz on $\mathbb{C}$, we pass to the complex torus $X = \mathbb{C}/\Lambda$ and its nowhere-vanishing 1-form ω induced by dz. Fix cycles a, b so that $\int_a \omega = 1$, $\int_b \omega = \tau$ with $\mathrm{Im}\, \tau > 0$. Using the Abel coordinate t defined by $dt = \omega$, we *define* the Weierstrass function by a lattice sum in t, derive its differential equation via Eisenstein series, and identify the elliptic curve $y^2 = 4x^3 - g_2 x - g_3$. TikZ figures illustrate periods, poles, and the cubic.

1 From the torus and its 1-form to the Abel coordinate

Let $\Lambda = \mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$ with $\mathrm{Im}\, \tau > 0$. The quotient $X = \mathbb{C}/\Lambda$ is a complex torus. The standard 1-form dz is Λ-invariant, hence descends to a holomorphic 1-form ω on X that has *no zeros*. Choose homology generators a, b represented by the edges of a fundamental parallelogram. Normalize so that

$$\int_a \omega = 1, \qquad \int_b \omega = \tau.$$

Definition 1 (Abel coordinate)**.** *On the universal cover $\mathbb{C} \to X$, choose a lift near 0 and define t by $dt = \omega$. Thus on the cover one may take $t = z + \mathrm{const}$, so that a- and b-periods are $1, \tau$.*

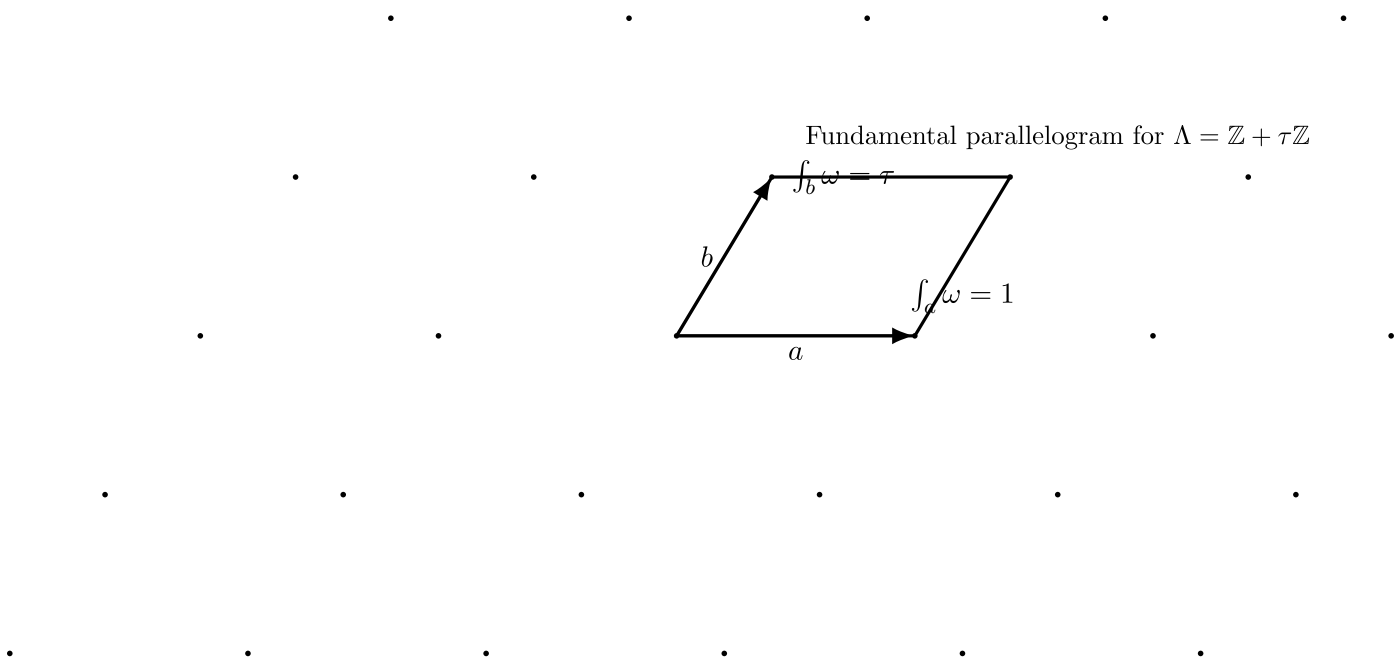

Fig. 1. Periods of ω along a, b in the Abel coordinate t.

2 Defining $\wp$ from $dt = \omega$

Definition 2 (Weierstrass function and its derivative). *With t as above and Λ fixed, define*

$$\boxed{\wp(t;\Lambda) = \frac{1}{t^2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{1}{(t-\lambda)^2} - \frac{1}{\lambda^2}\right), \qquad \wp'(t) = -\frac{2}{t^3} - 2\sum_{\lambda\neq 0} \frac{1}{(t-\lambda)^3}.}$$

Proposition 1 (Basic properties). *$\wp$ is doubly periodic with respect to Λ, even, and has a double pole at each $\lambda \in \Lambda$ with no residues. Its derivative $\wp'$ is odd and vanishes exactly at the three (nonzero) half-periods modulo sign.*

Sketch. Absolute convergence of the modified sum follows by pairing λ and $-\lambda$. Periodicity follows from invariance under $t \mapsto t + \lambda$. Even/odd are immediate from the definitions. Residues vanish by symmetry, hence $\wp$ is elliptic (doubly periodic meromorphic). □

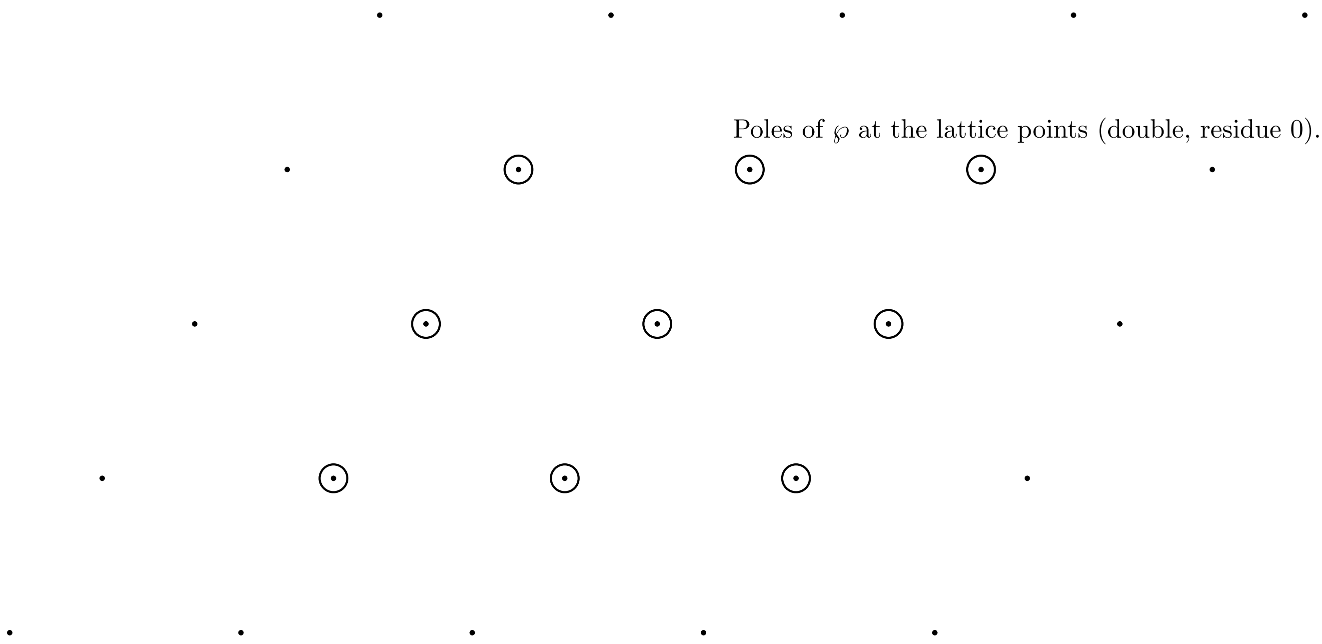

Fig. 2. The pole lattice of $\wp$.

3 Eisenstein series and the cubic differential equation

Define Eisenstein series (absolutely convergent for $k \geq 2$):

$$G_{2k}(\Lambda) = \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{\lambda^{2k}}, \qquad g_2 = 60\, G_4, \quad g_3 = 140\, G_6.$$

Then the Laurent expansion around $t = 0$ is

$$\wp(t) = \frac{1}{t^2} + 3G_4\, t^2 + 5G_6\, t^4 + 7G_8\, t^6 + \cdots.$$

Proposition 2 (Weierstrass equation).

$$\boxed{(\wp'(t))^2 \;=\; 4\,\wp(t)^3 - g_2\,\wp(t) - g_3.}$$

Sketch. The function $F(t) = (\wp')^2 - 4\wp^3 + g_2\wp + g_3$ is elliptic with no poles by checking its Laurent series at $t = 0$. By Liouville for elliptic functions, F is constant, and matching the constant term in the expansion forces the constant to be 0. □

4 Uniformization and the elliptic curve

Let $x = \wp(t)$, $y = \wp'(t)$. Then $(x(t), y(t))$ lies on the cubic

$$E: \quad y^2 = 4x^3 - g_2 x - g_3.$$

Proposition 3 (Differentials match)**.** *Under $t \mapsto (x(t), y(t))$, the holomorphic differential $\omega = dt$ corresponds to*

$$dt = \frac{dx}{y} \quad \text{on } E.$$

Proof. Differentiate $x = \wp(t)$ to get $dx = \wp'(t)\,dt = y\,dt$. □

Remark 1 (Inverse via an elliptic integral)**.** *Conversely,*

$$t = \int^x \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}},$$

recovering the Abel coordinate from the cubic.

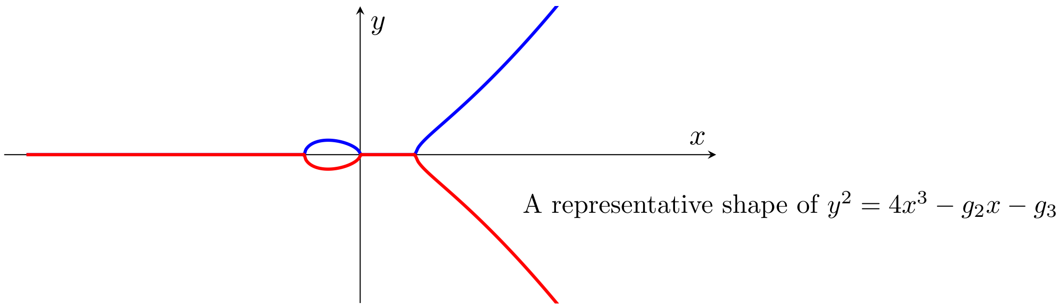

Fig. 3. The elliptic curve; the map $t \mapsto (\wp(t), \wp'(t))$ wraps the torus over it.

5 Half-period values e_1, e_2, e_3

Let e_1, e_2, e_3 be the distinct zeros of $4x^3 - g_2 x - g_3$ (so $e_1 + e_2 + e_3 = 0$). They are exactly the values of $\wp$ at the three nontrivial half-periods $t = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ (order varies with basis). Equivalently,

$$\wp'(t) = 0 \iff t \equiv \tfrac{1}{2}, \tfrac{\tau}{2}, \tfrac{1+\tau}{2} \pmod{\Lambda}, \qquad \wp\big(\tfrac{*}{2}\big) \in \{e_1, e_2, e_3\}.$$

Zeros of $\wp'$ at half-periods; values are e_i

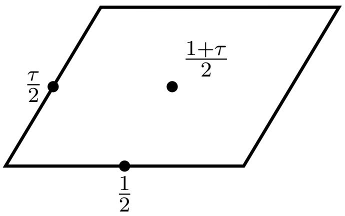

Fig. 4. Half-periods in the fundamental domain; $\wp'$ vanishes there.

6 Two concrete symmetry cases

(A) Lemniscatic (square) lattice $\tau = \mathrm{i}$

By quarter-turn symmetry $z \mapsto \mathrm{i}z$, one shows $G_6 = 0$, hence $g_3 = 0$. The cubic is

$$y^2 = 4x^3 - g_2 x, \qquad \text{roots } \{e_1, 0, -e_1\}.$$

One of the half-period values is 0 (indeed $\wp(\frac{1}{2} + \frac{\mathrm{i}}{2}) = 0$ up to basis choice). The curve has symmetric "lemniscatic" shape.

(B) Equianharmonic (hexagonal) lattice $\tau = \mathrm{e}^{\mathrm{i}\pi/3}$

By 60° rotational symmetry, $G_4 = 0$, hence $g_2 = 0$. The cubic is

$$y^2 = 4x^3 - g_3, \qquad \text{roots } \omega\sqrt[3]{\tfrac{g_3}{4}},\ \omega^2\sqrt[3]{\tfrac{g_3}{4}},\ \sqrt[3]{\tfrac{g_3}{4}}, \quad (\omega^3 = 1,\ \omega \neq 1),$$

so the three e_i are the three vertices of a rotated equilateral triple on the x-line (in the appropriate real model).

7 Local expansions and quick computations

From the Laurent series,

$$\wp(t) = \frac{1}{t^2} + 3G_4t^2 + 5G_6t^4 + \cdots, \qquad \wp'(t) = -\frac{2}{t^3} - 6G_4t - 20G_6t^3 - \cdots.$$

Example 1 (Sanity check near $t = 0$)**.** *Take a small t and keep the first two nontrivial terms: $\wp(t) \approx t^{-2} + 3G_4t^2$, $\wp'(t) \approx -2t^{-3} - 6G_4t$. Plug into $y^2 = 4x^3 - g_2x - g_3$; using $g_2 = 60G_4$, $g_3 = 140G_6$, the first non-canceling terms match identically.*

8 Why ω determines everything (and vice versa)

- Input: (X, ω) with $\int_a \omega = 1$, $\int_b \omega = \tau$. - Output: $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, Abel coordinate t with $dt = \omega$, then $\wp, \wp'$, then g_2, g_3, then the cubic E. - Conversely, given $E : y^2 = 4x^3 - g_2x - g_3$, the invariant differential $\dfrac{dx}{y}$ has periods $1, \tau$, recovering ω and Λ (up to scaling).

Worked "recipe"

1. Pick τ with $\operatorname{Im} \tau > 0$, set $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$.
2. Define t by $dt = \omega$ and periods $1, \tau$.
3. Build $\wp, \wp'$ by lattice sums; compute G_4, G_6, g_2, g_3.
4. Verify $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ and locate half-period zeros of $\wp'$.
5. Map $t \mapsto (\wp(t), \wp'(t))$ onto $E :\ y^2 = 4x^3 - g_2x - g_3$; inverse is $t = \displaystyle\int \frac{dx}{y}$.

Exercises (with short nudges)

1) Prove absolute convergence of the modified series for $\wp$ by pairing λ and $-\lambda$.
2) Show that if f is entire and Λ-periodic then f is constant (Liouville on a fundamental domain).
3) For $\tau = \mathrm{i}$, prove $G_6 = 0$ by symmetry: the six directions of shortest nonzero lattice vectors cancel in the 6-th power sum.
4) For $\tau = \mathrm{e}^{\mathrm{i}\pi/3}$, prove $G_4 = 0$ similarly.
5) Check that dt equals $\dfrac{dx}{y}$ by differentiating $x = \wp(t)$.