What is a 1-form ?

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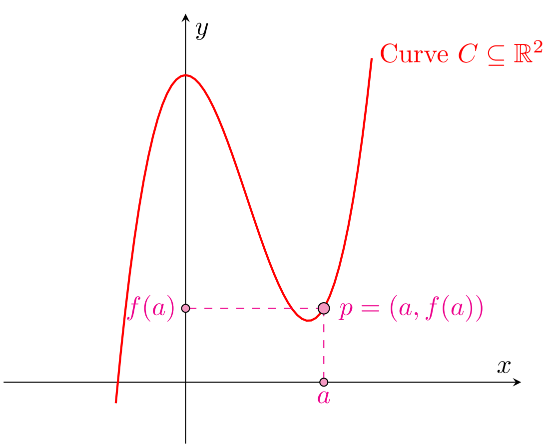
We cover the following topics in this note.

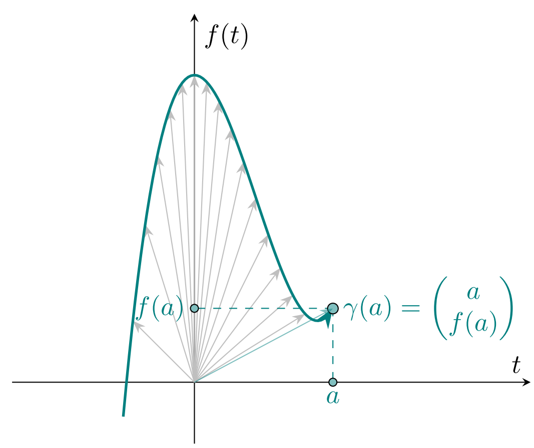
- Point and Tangent Vector
- Tangent Space T_pC
- Coordinates and Differentials
- Differential 1-form

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then its graph

$$C := \{(x, y) \in \mathbb{R}^2 : y = f(x)\}$$

is a smooth curve in the Euclidean plane $\mathbb{R}^2$.

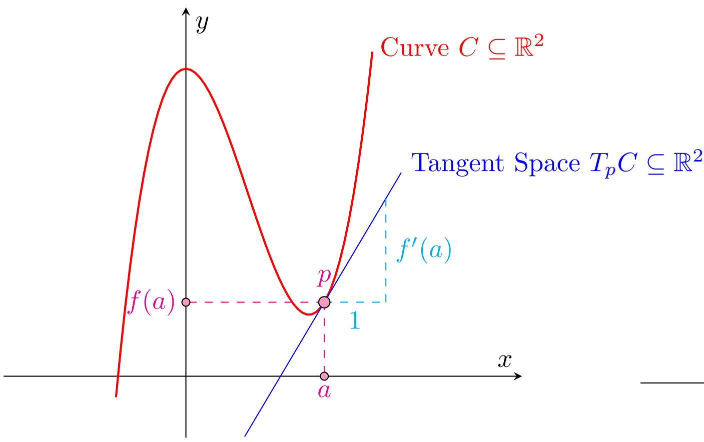


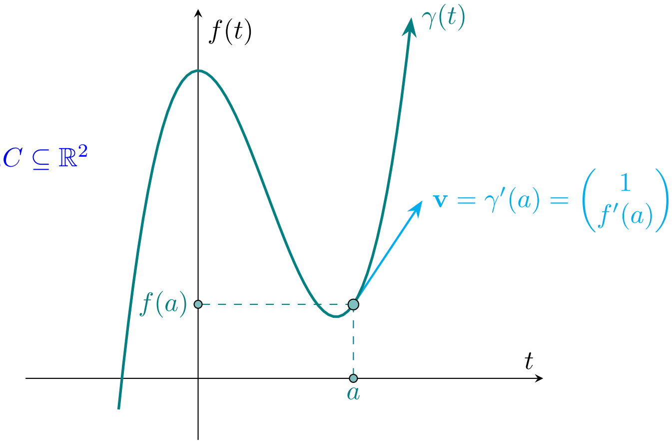


Fix a point $p = (a, f(a)) \in C$, and consider the parametrization

$$\begin{array}{cccc} \gamma & : & \mathbb{R} & \longrightarrow & C\ (\subseteq \mathbb{R}^2) \\ & & t & \longmapsto & \begin{pmatrix} t \\ f(t) \end{pmatrix} \end{array}.$$

The derivative $f'(a)$ is the slope of the tangent to the curve C at $p = (a, f(a))$.



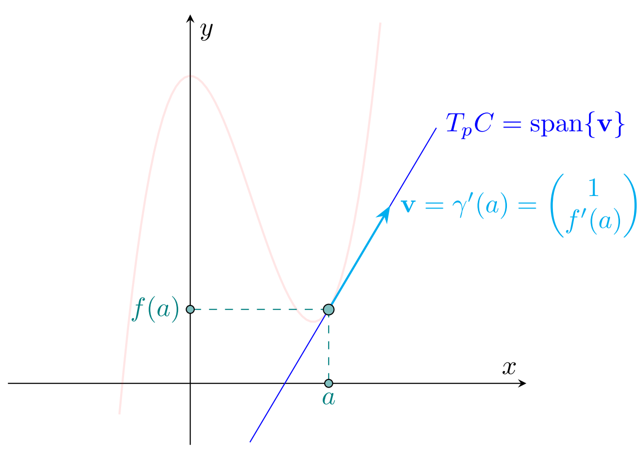


And the velocity of γ is

$$\gamma'(t) = \frac{\mathrm{d}\gamma}{\mathrm{d}t} = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}t \\ \frac{\mathrm{d}}{\mathrm{d}t}f(t) \end{pmatrix} = \begin{pmatrix} 1 \\ f'(t) \end{pmatrix}, \quad \text{and so} \quad \gamma'(a) = \begin{pmatrix} 1 \\ f'(a) \end{pmatrix} = \mathbf{v} \ \in \ T_pC.$$

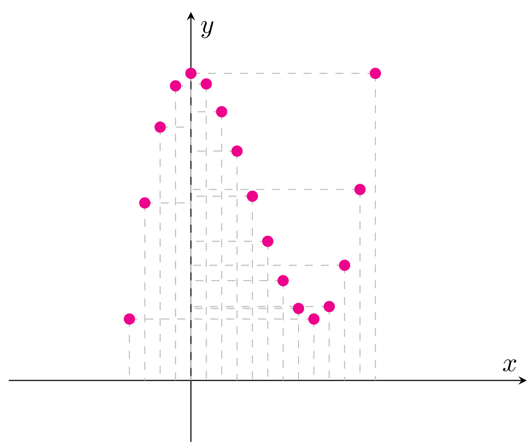
Here, the tangent space T_pC to the curve C at p is the set of all scalar multiples of $\mathbf{v}$:

$$T_pC := \operatorname{span}\{\mathbf{v}\} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ f'(a) \end{pmatrix}\right\} = \left\{t\ \begin{pmatrix} 1 \\ f'(a) \end{pmatrix} : t \in \mathbb{R}\right\} \subseteq \mathbb{R}^2.$$

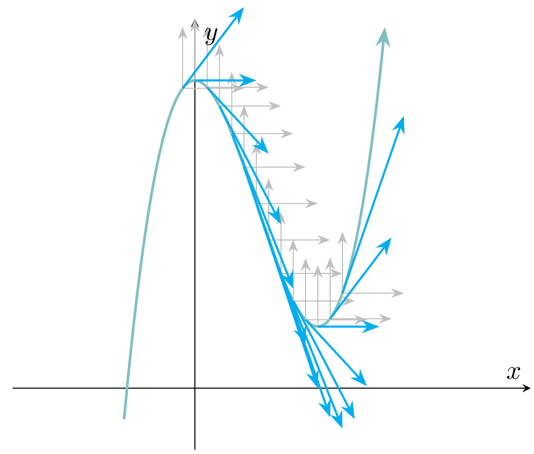


Note that

- A *point* $p = (a, f(a)) \in C$ is a **global location** on the embedded curve $C \subseteq \mathbb{R}^2$; it specifies a particular position in the ambient space.
- A *tangent vector* $\mathbf{v} = \begin{pmatrix} 1 \\ f'(a) \end{pmatrix} \in T_pC \subseteq \mathbb{R}^2$ encodes the curve's **local direction and speed** at p.



Globally, points encode *where* we are



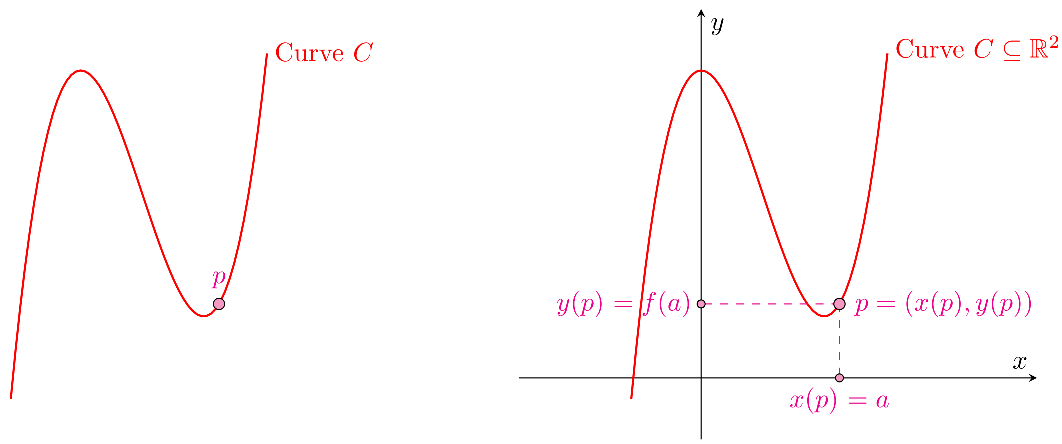
Locally, tangent vectors encode *how* we move through that point

We consider the coordinate projections

$$x, y\colon \mathbb{R}^2 \to \mathbb{R}, \qquad x(a,b) = a, \quad y(a,b) = b,$$

and restrict them to $C \subseteq \mathbb{R}^2$. Thus, we obtain two functions:

$$x : C \to \mathbb{R} \quad \text{and} \quad y : C \to \mathbb{R}.$$



Define a inclusion map

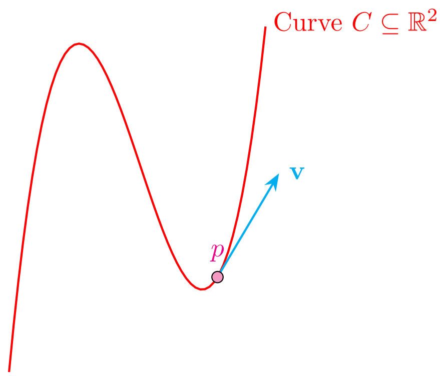
$$\begin{array}{cccc} \Phi_C & : & C & \longrightarrow & \mathbb{R}^2 \\ & & p & \longmapsto & (x(p), y(p)) \end{array}.$$

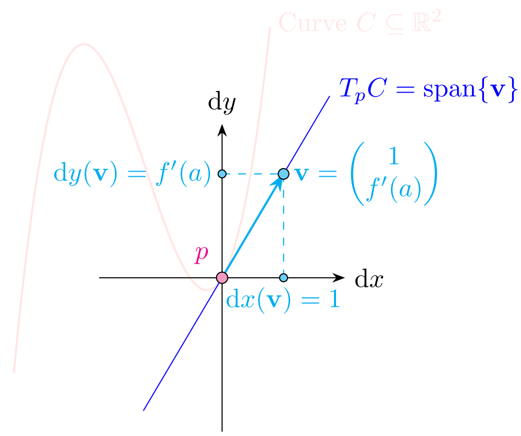
This map Φ_C[1] records the two ambient coordinates of each point in $C \subseteq \mathbb{R}^2$.

[1]Note that $\Phi_C \in (\mathbb{R}^2)^C$, $x|_C \in \mathbb{R}^C$ and $y|_C \in \mathbb{R}^C$. Since $\mathbb{R}^2 \simeq \mathbb{R} \times \mathbb{R}$ we have $(\mathbb{R}^2)^C \simeq (\mathbb{R} \times \mathbb{R})^C \simeq \mathbb{R}^C \times \mathbb{R}^C$

We consider the coordinate projections

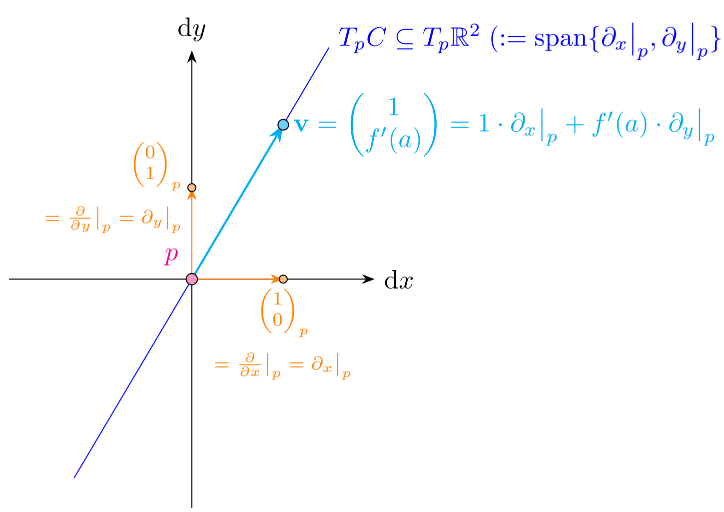
$$\mathrm{d}x, \mathrm{d}y : T_pC \to \mathbb{R}, \quad \mathrm{d}x(\mathbf{v}) = \mathrm{d}x\left(\begin{pmatrix}1 & f'(a)\end{pmatrix}^T\right) = 1, \; \mathrm{d}y(\mathbf{v}) = \mathrm{d}y\left(\begin{pmatrix}1 & f'(a)\end{pmatrix}^T\right) = f'(a).$$





At each $p = (a, f(a)) \in C$, the ambient tangent plane is

$$T_p\mathbb{R}^2 = \mathrm{span}\{\partial_x\big|_p, \partial_y\big|_p\} = \left\{\alpha\partial_x\big|_p + \beta\partial_y\big|_p : \alpha, \beta \in \mathbb{R}\right\} \;\simeq\; \mathbb{R}^2.$$



Define a differential of the inclusion

$$\begin{array}{rccc} \Phi_{T_pC} & : & T_pC & \longrightarrow & \mathbb{R}^2 \\ & & \mathbf{v} & \longmapsto & \begin{pmatrix}\mathrm{d}x(\mathbf{v}) & \mathrm{d}y(\mathbf{v})\end{pmatrix}^T \end{array}$$

This map Φ_{T_pC} records the two components of any tangent vector $\mathbf{v} \in T_p\mathbb{R}^2$.

Let

$$\boxed{\omega = dx + f'(a)\,dy \in T_p^*\mathbb{R}^2}$$

be a **1-form** defined at point $p = (a, f(a)) \in \mathbb{R}^2$, with the direction $\mathbf{v}_p = \begin{pmatrix} 1 \\ f'(a) \end{pmatrix}$. In other words,

$$\begin{array}{rccl} \omega \;:\; & T_p\mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} & \longmapsto & \omega(\mathbf{u}) = u_1 + f'(a)u_2 \end{array}$$

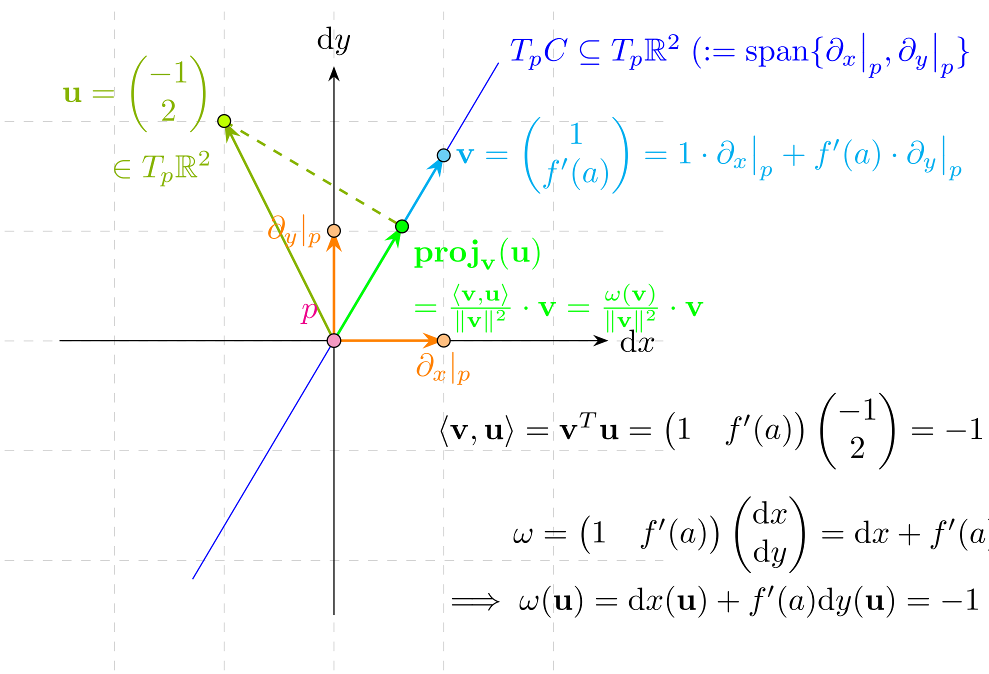
Since $u_1 + f'(a)u_2 = \begin{pmatrix} u_1 & u_2 \end{pmatrix}\begin{pmatrix} 1 \\ f'(a) \end{pmatrix} = \mathbf{u}^T\mathbf{v}_p = \langle \mathbf{u}, \mathbf{v}_p \rangle$,

$$\boxed{\omega(\mathbf{u}) = \text{“projection of } \mathbf{u} \text{ onto } \mathbf{v}_p = \begin{pmatrix} 1 \\ f'(a) \end{pmatrix} \text{ direction”.}}$$

Let $\mathbf{proj}_{\mathbf{v}_p}(\mathbf{u}) = \lambda \mathbf{v_p}$. Then

$$(\mathbf{u} - \lambda\mathbf{v_p}) \perp \mathbf{v_p} \implies \langle \mathbf{u} - \lambda\mathbf{v_p}, \mathbf{v}_p \rangle = 0 \implies \langle \mathbf{u}, \mathbf{v_p} \rangle - \lambda\langle \mathbf{v}_p, \mathbf{v}_p \rangle = 0 \implies \lambda = \frac{\langle \mathbf{u}, \mathbf{v}_p \rangle}{\langle \mathbf{v}_p, \mathbf{v}_p \rangle}$$

Thus, $\mathbf{proj}_{\mathbf{v}_p}(\mathbf{u}) = \dfrac{\langle \mathbf{u}, \mathbf{v}_p \rangle}{\langle \mathbf{v}_p, \mathbf{v}_p \rangle}\mathbf{v}_p$:



$$\langle \mathbf{v}, \mathbf{u} \rangle = \mathbf{v}^T\mathbf{u} = \begin{pmatrix} 1 & f'(a) \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = -1 + 2f'(a)$$

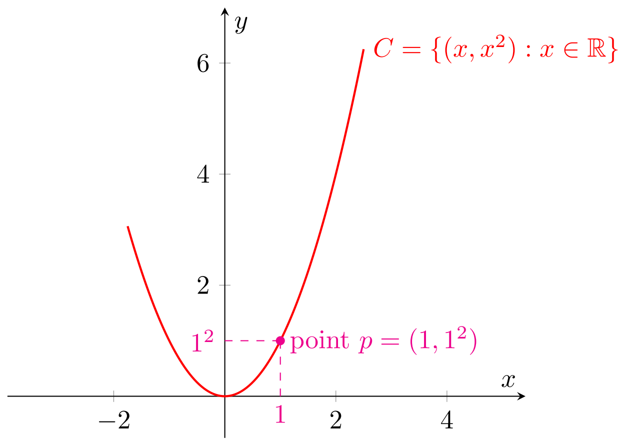
$$\omega = \begin{pmatrix} 1 & f'(a) \end{pmatrix}\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \mathrm{d}x + f'(a)\mathrm{d}y$$

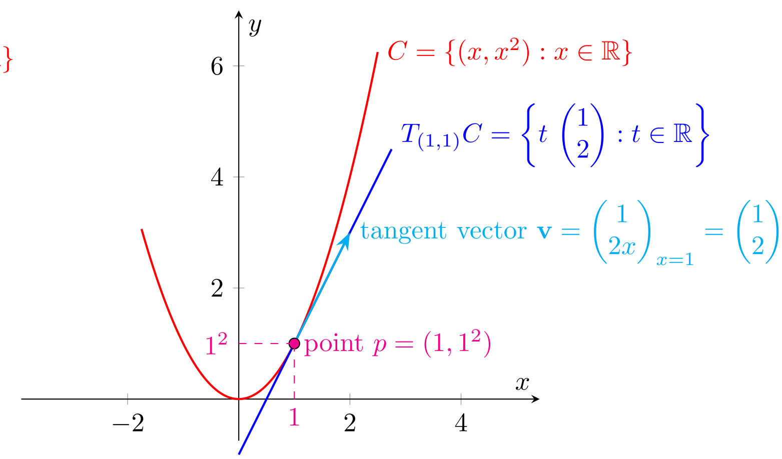
$$\implies \omega(\mathbf{u}) = \mathrm{d}x(\mathbf{u}) + f'(a)\mathrm{d}y(\mathbf{u}) = -1 + 2f'(a)$$

Example $C : y = x^2$

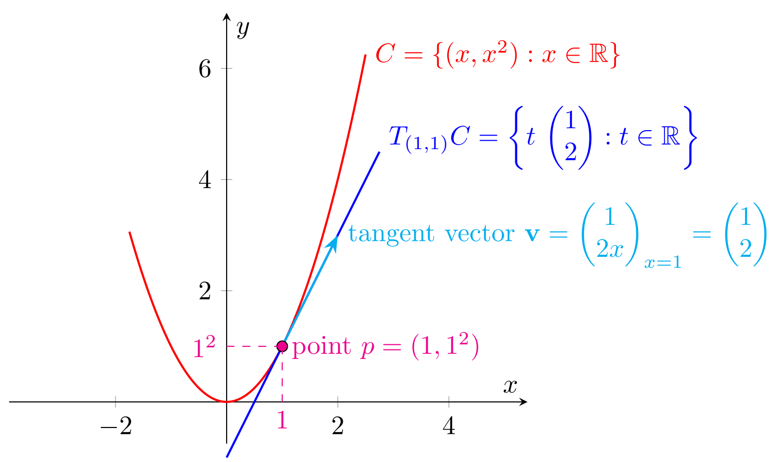
We take

$$C = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}.$$





Set $a = 1$. Then the point $p = (1, 1)$ lies on C.



Since $f'(x) = 2x$, we have

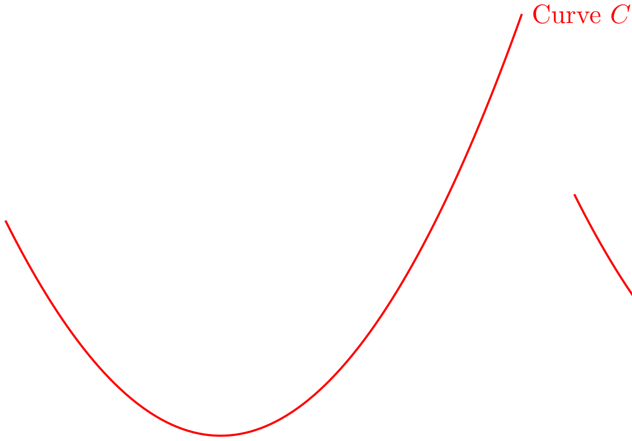
$$\mathbf{v} = \begin{pmatrix}1\\f'(1)\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}.$$

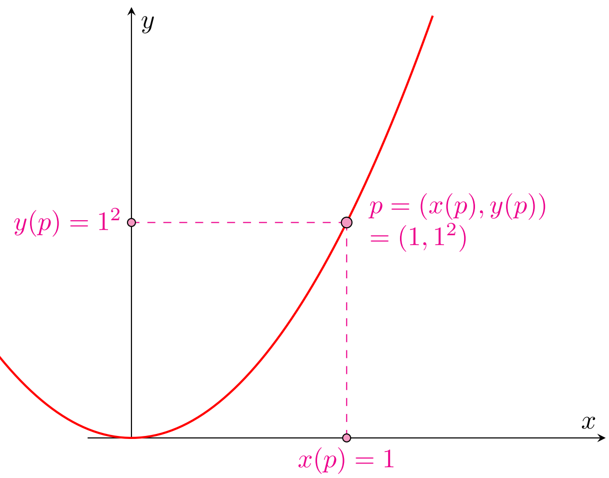
Thus

$$T_pC = \operatorname{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\} = \left\{t\begin{pmatrix}1\\2\end{pmatrix} \ : \ t \in \mathbb{R}\right\} \subseteq \mathbb{R}^2.$$

Consider a inclusion map

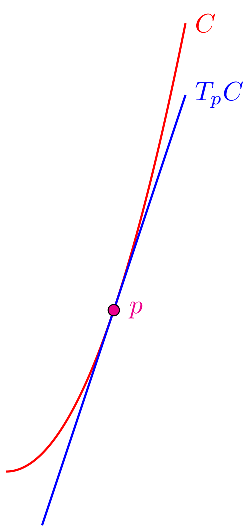
$$\begin{array}{rccc} \Phi_C & : & C & \longrightarrow & \mathbb{R}^2 \\ & & p & \longmapsto & (x(p), y(p)) \end{array}.$$

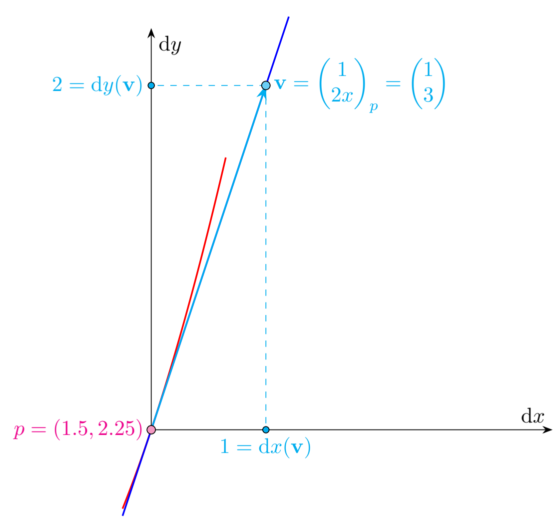




Then a differential of the inclusion map is

$$\begin{array}{rccc} \Phi_{T_pC} & : & T_pC & \longrightarrow & \mathbb{R}^2 \\ & & \mathbf{v} & \longmapsto & \begin{pmatrix} \mathrm{d}x(\mathbf{v}) \\ \mathrm{d}y(\mathbf{v}) \end{pmatrix} \end{array}$$





Consider the differential 1-form on $\mathbb{R}^2 \setminus \{0\}$:

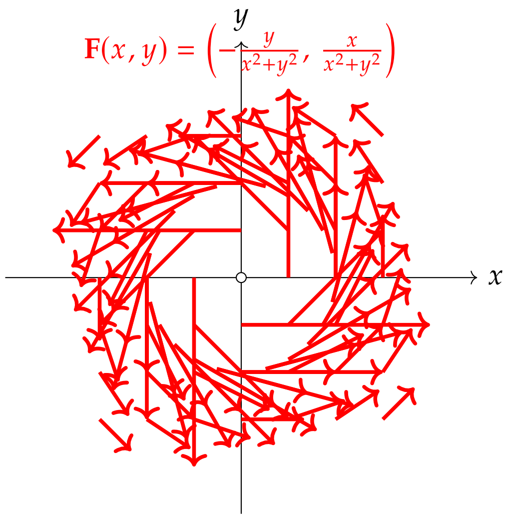
$$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

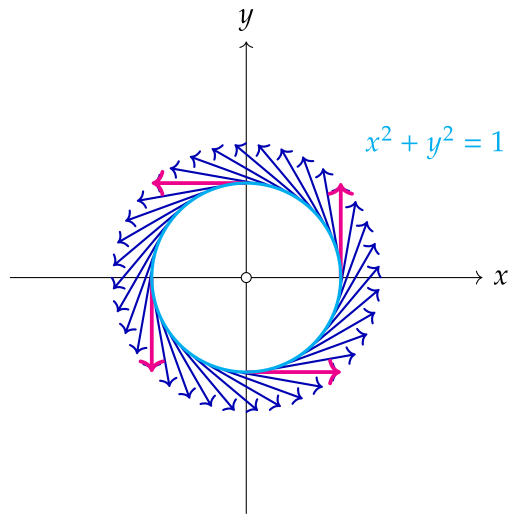
This 1-form corresponds to the angular differential $d\theta$ in polar coordinates.

Let $C \subseteq \mathbb{R}^2$ be the unit circle centered at the origin, parametrized by:

$$\gamma(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \theta \in [0, 2\pi],$$

with counterclockwise orientation.





TBA...

Degree k	**Object**	**Vector Space**
0-form	Scalar function	$f(p) \in \mathbb{R}$
1-form	Linear Functional	$(T_p\mathbb{R}^n \longrightarrow \mathbb{R}) \in T_p^*\mathbb{R}^n$
2-form	TBA	TBA
TBA		