From $\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z)$ to $\mathcal{M}(\mathbb{C}/\Lambda) \cong \mathbb{C}(\wp, \wp')$ with Detailed Computations

Contents

1 Reminder: $\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z)$ as the genus 0 prototype

We briefly recall the key fact:

Theorem 1. *Let $\mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}$ be the Riemann sphere with affine coordinate z on $\mathbb{C}$. Then*

$$\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z),$$

i.e. every meromorphic function on $\mathbb{CP}^1$ is a rational function in z.

Sketch (analytic). Let f be meromorphic on $\mathbb{CP}^1$. Then f has finitely many poles $a_1, \ldots, a_N \in \mathbb{C} \cup \{\infty\}$. Around each finite pole $a_j \in \mathbb{C}$, we have a Laurent expansion

$$f(z) = \sum_{n=-m_j}^{\infty} c_{j,n}(z - a_j)^n,$$

and similarly at ∞ in the coordinate $w = 1/z$:

$$f\left(\frac{1}{w}\right) = \sum_{n=-M}^{\infty} b_n w^n.$$

The *principal part* at each pole is a finite sum of negative powers. Using these principal parts, we build a rational function

$$R(z) = P(z) + \sum_{j=1}^{N} \sum_{k=1}^{m_j} \frac{c_{j,-k}}{(z - a_j)^k},$$

where $P(z)$ is the polynomial corresponding to the principal part at infinity. By construction, R has exactly the same principal parts as f at all poles (finite and at infinity). Hence $g := f - R$ has no pole anywhere on $\mathbb{CP}^1$, i.e. g is holomorphic on the compact surface $\mathbb{CP}^1$. Therefore g is constant, so $f = R+$ (constant) is rational in z. □

Thus the sphere is characterized by having a *single global coordinate* z whose field of meromorphic functions is exactly $\mathbb{C}(z)$.

2 General theorem: $\mathcal{M}(X)$ is finite over $\mathbb{C}(f)$

Let X be a compact Riemann surface and $f \in \mathcal{M}(X)$ a nonconstant meromorphic function. Then f defines a holomorphic map

$$f : X \to \mathbb{CP}^1,$$

by

$$f(p) = \begin{cases} [f(p) : 1], & p \text{ not a pole of } f, \\ [1 : 0], & p \text{ a pole of } f. \end{cases}$$

2.1 Degree of f and the field extension

The map $f : X \to \mathbb{CP}^1$ is *finite* and has a well-defined degree $\deg(f) \in \mathbb{N}$: for a generic $w \in \mathbb{CP}^1$, the fiber $f^{-1}(w)$ consists of $\deg(f)$ points counted with multiplicity.

On the level of function fields, f induces an embedding

$$f^* : \mathcal{M}(\mathbb{CP}^1) \hookrightarrow \mathcal{M}(X), \qquad \phi(z) \mapsto \phi\big(f\big).$$

Identifying $\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z)$ (Theorem 1), we get an inclusion

$$\mathbb{C}(z) \hookrightarrow \mathcal{M}(X).$$

Theorem 2 (Standard fact)**.** *Let X be a compact Riemann surface and $f : X \to \mathbb{CP}^1$ a nonconstant meromorphic function. Then:*

1. *$\mathcal{M}(X)$ is a finite extension of $\mathbb{C}(z)$, where z is the affine coordinate on $\mathbb{CP}^1$ (pulled back by f).*

2. *The degree of this field extension equals the degree of the map:*

$$[\mathcal{M}(X) : \mathbb{C}(z)] = \deg(f).$$

Very sketchy idea. One shows that the transcendence degree of $\mathcal{M}(X)$ over $\mathbb{C}$ is 1 (since X is a one-dimensional complex manifold). The map f gives a nonconstant element of $\mathcal{M}(X)$, so $\mathbb{C}(f)$ is isomorphic to $\mathbb{C}(z)$. Then every other meromorphic function $g \in \mathcal{M}(X)$ satisfies a polynomial relation

$$P(f, g) = 0$$

with $P \in \mathbb{C}[T, U]$ not the zero polynomial (this uses compactness and boundedness arguments or more algebraic geometry tools). This shows g is algebraic over $\mathbb{C}(f)$, hence $\mathcal{M}(X)$ is algebraic over $\mathbb{C}(f)$.

Finiteness and the equality of degrees $[\mathcal{M}(X) : \mathbb{C}(f)] = \deg(f)$ can be shown by comparing zeros and poles of pullbacks of rational functions, or using Riemann–Hurwitz. For the purposes of this note, we accept this as a standard result from the theory of compact Riemann surfaces. □

Remark 1. *For $X = \mathbb{CP}^1$ and $f = z$, Theorem 2 tells us $[\mathcal{M}(\mathbb{CP}^1) : \mathbb{C}(z)] = 1$, which matches $\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z)$.*

3 Torus case: $X = \mathbb{C}/\Lambda$ and the Weierstrass $\wp$

Now let $\Lambda \subset \mathbb{C}$ be a lattice generated by ω_1, ω_2 with $\Im(\omega_2/\omega_1) > 0$. The complex torus is

$$X = \mathbb{C}/\Lambda.$$

We denote by $[z]$ the class of $z \in \mathbb{C}$ modulo Λ.

3.1 Definition and basic properties of $\wp$

The Weierstrass $\wp$-function associated with Λ is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

Basic facts (all standard; we list them for context):

- The series converges normally on compact subsets of $\mathbb{C} \setminus \Lambda$, so $\wp$ is meromorphic on $\mathbb{C}$.
- $\wp$ is Λ-periodic:
$$\wp(z + \lambda) = \wp(z), \quad \forall \lambda \in \Lambda.$$
- $\wp$ is even:
$$\wp(-z) = \wp(z).$$
- The only poles of $\wp$ (modulo Λ) are at the lattice points, with a double pole at each $\lambda \in \Lambda$.

Because of periodicity, $\wp$ descends to a meromorphic function on the torus:

$$\wp_X : X \to \mathbb{CP}^1, \quad \wp_X([z]) = \wp(z).$$

We will usually just write $\wp$ for the descended function as well.

3.2 Local expansion of $\wp$ and $\wp'$ near 0

By symmetry and periodicity, it suffices to study $\wp$ near $z = 0$.

One can show (expanding the series or using evenness and the type of pole) that near $z = 0$,

$$\wp(z) = \frac{1}{z^2} + c_2 z^2 + c_4 z^4 + c_6 z^6 + \cdots , \tag{1}$$

for some constants $c_{2k} \in \mathbb{C}$ depending on the lattice.

Differentiating termwise,

$$\wp'(z) = -\frac{2}{z^3} + 2c_2 z + 4c_4 z^3 + 6c_6 z^5 + \cdots .$$

Thus:

- $\wp$ has a double pole at $z = 0$:
$$\mathrm{ord}_0(\wp) = -2.$$
- $\wp'$ has a triple pole at $z = 0$:
$$\mathrm{ord}_0(\wp') = -3.$$
- $\wp$ is even, $\wp'$ is odd: $\wp(-z) = \wp(z)$, $\wp'(-z) = -\wp'(z)$.

Because $\wp$ and $\wp'$ are Λ-periodic, in the quotient $X = \mathbb{C}/\Lambda$ the function $\wp$ has a single double pole at $[0]$, and $\wp'$ has a single triple pole at $[0]$.

3.3 Degree of $\wp : X \to \mathbb{CP}^1$

We now compute the degree of the meromorphic map

$$\wp : X \to \mathbb{CP}^1, \quad [z] \mapsto \wp(z).$$

Fix a generic value $w \in \mathbb{C}$. The equation

$$\wp(z) = w$$

has solutions in pairs $\{z, -z\}$ because $\wp$ is even. For a generic value w (i.e. provided w is not a critical value), we have exactly two solutions modulo Λ.

So the map $\wp : X \to \mathbb{CP}^1$ has degree

$$\deg(\wp) = 2.$$

By Theorem 2, the induced inclusion

$$\mathbb{C}(x) \hookrightarrow \mathcal{M}(X), \quad x := \wp,$$

makes $\mathcal{M}(X)$ a field extension of degree 2 over $\mathbb{C}(x)$:

$$[\mathcal{M}(X) : \mathbb{C}(\wp)] = 2.$$

Thus $\mathcal{M}(X)$ is a *quadratic extension* of the rational function field $\mathbb{C}(\wp)$, exactly analogous to an algebraic function field $\mathbb{C}(x, y)$ with a relation of the form $y^2 = \cdots$.

4 The algebraic relation $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$

We next derive the key differential equation satisfied by $\wp$ and $\wp'$.

4.1 Constructing an elliptic function with no poles

Consider the function

$$F(z) := \big(\wp'(z)\big)^2 - 4\big(\wp(z)\big)^3 + g_2\,\wp(z) + g_3,$$

where $g_2, g_3 \in \mathbb{C}$ are some constants we will choose.

We want to show that for suitable g_2, g_3, this F is identically zero. The strategy is:

- Note that F is an elliptic (doubly periodic) meromorphic function.
- Show that F has no poles (for appropriate g_2, g_3).
- Conclude that F must be constant.
- Analyze the constant by looking at the expansion near 0; choose g_2, g_3 so that the constant is 0.

4.2 Local expansion of $(\wp')^2$ and $\wp^3$ near $z = 0$

From the expansions in (1), write

$$\wp(z) = z^{-2} + a_2z^2 + a_4z^4 + a_6z^6 + \cdots,$$

$$\wp'(z) = -2z^{-3} + b_1z + b_3z^3 + b_5z^5 + \cdots,$$

where $a_{2k}, b_{2k-1} \in \mathbb{C}$.

Compute $(\wp'(z))^2$ near 0. We have

$$\begin{aligned}(\wp'(z))^2 &= \big(-2z^{-3} + b_1z + b_3z^3 + \cdots\big)^2\\ &= 4z^{-6} - 4b_1z^{-2} + (\text{higher powers in } z).\end{aligned}$$

More precisely, we can expand:

$$(\wp'(z))^2 = 4z^{-6} + c_{-4}z^{-4} + c_{-2}z^{-2} + c_0 + c_2z^2 + \cdots.$$

Compute $4(\wp(z))^3$ near 0. First

$$\wp(z)^3 = (z^{-2} + a_2z^2 + a_4z^4 + \cdots)^3.$$

Expanding:

$$\wp(z)^3 = z^{-6} + 3a_2z^{-2} + (\text{terms with } z^0, z^2, \ldots).$$

Then

$$4\wp(z)^3 = 4z^{-6} + 12a_2z^{-2} + (\text{higher powers}).$$

So near $z = 0$:

$$(\wp'(z))^2 - 4\wp(z)^3 = \left(4z^{-6} + c_{-4}z^{-4} + c_{-2}z^{-2} + c_0 + \cdots\right) - \left(4z^{-6} + 12a_2 z^{-2} + \cdots\right)$$
$$= c_{-4}z^{-4} + (c_{-2} - 12a_2)z^{-2} + c_0 + \cdots .$$

The remarkable fact (which can be checked more systematically from the series definition of $\wp$) is that $c_{-4} = 0$, so the z^{-4} term vanishes.

Even more structure: the combination

$$(\wp'(z))^2 - 4\wp(z)^3$$

has at worst a z^{-2} term in its Laurent series. We then *define*

$$g_2 := -20a_2, \qquad g_3 := \text{(suitable constant to kill the } z^0 \text{ term)},$$

so that

$$(\wp'(z))^2 - 4\wp(z)^3 + g_2\wp(z) + g_3$$

has no $z^{-6}, z^{-4}, z^{-2}, z^0$ terms in its Laurent expansion at $z = 0$, and in fact its Laurent expansion at $z = 0$ starts at some positive power z^2 or higher.

4.3 No poles $\Rightarrow$ constant $\Rightarrow 0$

Because $\wp$ and $\wp'$ are elliptic, the function

$$F(z) := \left(\wp'(z)\right)^2 - 4\left(\wp(z)\right)^3 + g_2\,\wp(z) + g_3$$

is elliptic (meromorphic and Λ-periodic). We have arranged the coefficients g_2, g_3 so that *near* $z = 0$ the Laurent expansion of F has no negative powers. By periodicity, this is also true near any other lattice point (as the expansions are the same up to translation). Hence F has no poles on $\mathbb{C}$.

Therefore F is an elliptic function with no poles, hence entire and bounded (on the fundamental domain), thus constant. Let C be this constant:

$$F(z) \equiv C.$$

Looking at the Laurent expansion at $z = 0$, one finds that the constant term must be 0 (we chose g_3 exactly to kill the constant term). Thus

$$C = 0,$$

and we have the fundamental relation

$$\boxed{\left(\wp'(z)\right)^2 = 4\left(\wp(z)\right)^3 - g_2\wp(z) - g_3.}$$

Set

$$x := \wp(z), \qquad y := \wp'(z).$$

Then

$$y^2 = 4x^3 - g_2 x - g_3.$$

5 Function field of the torus: $\mathcal{M}(\mathbb{C}/\Lambda) = \mathbb{C}(\wp, \wp')$

We already know $\mathbb{C}(\wp) \hookrightarrow \mathcal{M}(X)$ and that

$$[\mathcal{M}(X) : \mathbb{C}(\wp)] = 2.$$

From the relation

$$y^2 = 4x^3 - g_2 x - g_3,$$

we see that $\wp'$ is *algebraic* of degree 2 over $\mathbb{C}(\wp)$, since it satisfies the quadratic equation

$$Y^2 - (4x^3 - g_2 x - g_3) = 0.$$

Thus

$$\mathbb{C}(\wp, \wp') = \mathbb{C}(\wp)\big(\wp'\big)$$

is a degree ≤ 2 extension of $\mathbb{C}(\wp)$. On the other hand, we already know from the degree of the map $\wp : X \to \mathbb{CP}^1$ that

$$[\mathcal{M}(X) : \mathbb{C}(\wp)] = 2.$$

Hence

$$\mathbb{C}(\wp, \wp') = \mathcal{M}(X).$$

In more algebraic-geometric language, the pair $(x, y) = (\wp, \wp')$ gives an embedding of the torus as the complex curve

$$Y^2 = 4X^3 - g_2 X - g_3$$

in the affine plane, and the function field of that curve is exactly

$$\mathbb{C}(X, Y)/(Y^2 - 4X^3 + g_2 X + g_3) \cong \mathbb{C}(\wp, \wp').$$

Theorem 3. *Let $X = \mathbb{C}/\Lambda$ be a complex torus, and $\wp, \wp'$ the associated Weierstrass functions. Then:*

$$\mathcal{M}(X) = \mathbb{C}(\wp, \wp'),$$

with the single algebraic relation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3.$$

6 Comparing with the sphere case

We can now directly compare:

- **Sphere**: $X = \mathbb{CP}^1$ with coordinate z and function field

  $$\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z).$$

  This is the *rational function field* in one variable.

- **Torus**: $X = \mathbb{C}/\Lambda$ with Weierstrass functions

  $$x := \wp(z), \quad y := \wp'(z),$$

  and function field

  $$\mathcal{M}(\mathbb{C}/\Lambda) = \mathbb{C}(\wp, \wp') = \mathbb{C}(x, y)/(y^2 - 4x^3 + g_2 x + g_3),$$

  i.e. a quadratic extension of the rational function field $\mathbb{C}(x)$.

Thus the Weierstrass function $\wp$ plays the role of a "base coordinate" (like z on $\mathbb{CP}^1$), and the full function field is an *extension* of $\mathbb{C}(\wp)$ obtained by adjoining the algebraic function $\wp'$.

This is exactly the sense in which one can think of the torus case as an *extended* version of the sphere case $\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z)$:

$$\boxed{\begin{array}{ll} \text{Genus } 0: & \mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(z), \\ \text{Genus } 1: & \mathcal{M}(\mathbb{C}/\Lambda) = \mathbb{C}(\wp, \wp'), \quad (\wp')^2 = 4\wp^3 - g_2\wp - g_3. \end{array}}$$