Lecture Notes: Differential 1-Forms and Scalar Projections on Curves

Mathematical Structures of Differential Forms

1. A 1-Form on $\mathbb{R}^2$ as a Scalar Projection

Let $C \subseteq \mathbb{R}^2$ be a smooth curve defined locally as the graph of a function $f : \mathbb{R} \to \mathbb{R}$, i.e.,

$$C = \{(x, f(x)) \mid x \in \mathbb{R}\}.$$

Let $p = (a, f(a)) \in C$ be a fixed point on the curve. The derivative of f at a, denoted $f'(a)$, determines the tangent vector

$$\vec{v} = \langle 1, f'(a) \rangle \in T_pC.$$

Thus, the tangent space to the curve at point p, denoted T_pC, is

$$T_pC = \operatorname{span}\{\langle 1, f'(a) \rangle\}.$$

We aim to describe a differential 1-form on $\mathbb{R}^2$ that projects tangent vectors at p onto the line in direction $\vec{v}$, i.e., the scalar projection.

2. Geometric Identification

Point vs. Tangent Vector Distinction:

- A point $p \in C \subseteq \mathbb{R}^2$ is a geometric location in space. - A vector $\vec{w} \in T_pC \subseteq \mathbb{R}^2$ is an element of the tangent space at p; it encodes a direction and magnitude but is anchored at the point p.

3. Coordinate Systems

Define a coordinate system on C locally near p via the chart:

$$(x, y) : C \to \mathbb{R}^2, \quad q \mapsto (x(q), y(q)) = (x, f(x)).$$

On the tangent space T_pC, the natural dual basis of the cotangent space is given by the differentials:

$$\langle dx, dy \rangle : T_pC \to \mathbb{R}, \quad \vec{w} \mapsto (dx(\vec{w}), dy(\vec{w})).$$

More abstractly:

$$x, y : C \to \mathbb{R}, \quad dx, dy : T_pC \to \mathbb{R}.$$

4. Scalar Projection as a 1-Form

Define a unit vector in the direction of $\vec{v}$:

$$\hat{v} = \frac{1}{\sqrt{1 + (f'(a))^2}} \langle 1, f'(a) \rangle .$$

The differential 1-form $\omega \in \Omega^1(\mathbb{R}^2)$ which projects vectors onto $\vec{v}$ is defined by:

$$\omega = \frac{1}{\sqrt{1 + (f'(a))^2}} \left(dx + f'(a)\, dy \right) .$$

Then for any vector $\vec{w} \in T_pC$, the evaluation $\omega(\vec{w})$ gives the scalar projection of $\vec{w}$ onto the direction of $\vec{v}$.

5. Summary

- The 1-form ω captures the infinitesimal scalar projection onto the tangent line of the curve. - The construction uses the differential of coordinate functions and normalization by the Euclidean norm. - This provides a clear geometric interpretation of a 1-form as a linear functional evaluating directional components of tangent vectors.

Geometric Setting and 1-Form Construction

Let $C \subseteq \mathbb{R}^2$ be a curve given locally by the graph of a smooth function $f(x)$, with

$$C = \{(x, f(x)) \mid x \in \mathbb{R}\}.$$

Let $p = (a, f(a)) \in C$, and let $f'(a)$ be the derivative at $x = a$. Then the tangent vector at p is

$$\vec{v} = \langle 1, f'(a) \rangle .$$

The tangent space is

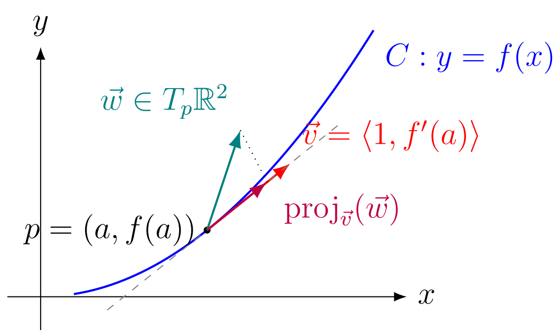
$$T_p C = \operatorname{span}\{\langle 1, f'(a) \rangle\}.$$

We define a 1-form $\omega \in \Omega^1(\mathbb{R}^2)$ by

$$\omega = \frac{1}{\sqrt{1 + (f'(a))^2}} \left(dx + f'(a)\, dy \right),$$

so that for any $\vec{w} \in T_p\mathbb{R}^2$, $\omega(\vec{w})$ gives the scalar projection onto the line in direction $\vec{v}$.

TikZ Visualization



$$\omega = \tfrac{1}{\sqrt{1+f'(a)^2}} (dx + f'(a)\, dy)$$

Interpretation

- The red vector $\vec{v}$ is the tangent to the curve C at point p.
- The vector $\vec{w} \in T_p\mathbb{R}^2$ is arbitrary.
- The 1-form ω computes the scalar projection of $\vec{w}$ onto the direction of $\vec{v}$.
- The 1-form has constant coefficients along the direction of $\vec{v}$, normalized to unit length.