Lecture Note: 1-Form via Dot-Product Projection

1. The Curve and Its Tangent Line

Let

$$C = \{(x, y) \in \mathbb{R}^2 : y = f(x)\},$$

and fix a point $p = (a, f(a)) \in C$. The ambient tangent plane is

$$T_p\mathbb{R}^2 = \operatorname{span}\{\, \partial_x|_p,\ \partial_y|_p\} \cong \mathbb{R}^2,$$

and the curve's tangent space at p is the one-dimensional subspace

$$T_pC = \operatorname{span}\{\, \vec{v} = (1, f'(a))^T\}.$$

2. Coordinate Charts

On C. The ambient coordinate projections

$$x, y :\ \mathbb{R}^2 \to \mathbb{R}, \qquad x(x, y) = x,\ y(x, y) = y,$$

restrict to

$$x|_C \colon C \to \mathbb{R}, \quad y|_C \colon C \to \mathbb{R},$$

and assemble into the chart

$$\Phi_C \colon C \to \mathbb{R}^2, \qquad \Phi_C(p) = \big(x|_C(p),\, y|_C(p)\big) = (a, f(a)).$$

On T_pC. The dual projections

$$dx,\, dy :\ T_p\mathbb{R}^2 \to \mathbb{R}, \quad dx(v^1, v^2) = v^1,\ dy(v^1, v^2) = v^2,$$

when restricted to T_pC, yield

$$dx|_{T_pC},\, dy|_{T_pC} :\ T_pC \to \mathbb{R}.$$

Stacking gives the fiber-chart

$$\Phi_{T_pC} \colon T_pC \to \mathbb{R}^2, \qquad \Phi_{T_pC}(v) = \big(dx(v),\, dy(v)\big).$$

In particular, for $v = t\,(1, f'(a))^T$,

$$dx(v) = t, \quad dy(v) = t\, f'(a), \quad \Phi_{T_pC}(v) = \begin{pmatrix} t \\ t\, f'(a) \end{pmatrix}.$$

3. The Projection 1-Form

Fix a unit vector $\mathbf{u} = (\cos\theta, \sin\theta) \in \mathbb{R}^2$. Define the differential 1-form

$$\omega \;\in\; \Omega^1(C)$$

by projecting each tangent vector $v \in T_pC \subset T_p\mathbb{R}^2$ onto $\mathbf{u}$:

$$\forall\, v \in T_pC: \qquad \omega_p(v) = \langle \mathbf{u},\, v\rangle = \cos\theta\; dx(v) + \sin\theta\; dy(v).$$

Linearity in v and smooth dependence on p show that ω is a smooth section of T^*C.

Example. For the canonical tangent $v = (1, f'(a))^T$,

$$\omega_p\big((1, f'(a))^T\big) = \cos\theta \cdot 1 + \sin\theta \cdot f'(a).$$