Riemann; Complex Analysis

- HW1 -

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We cover the following topics in this note.

- Vector Fields
- Line Integrals for Vector Fields
- Surface Integrals for Vector Fields
- TBA

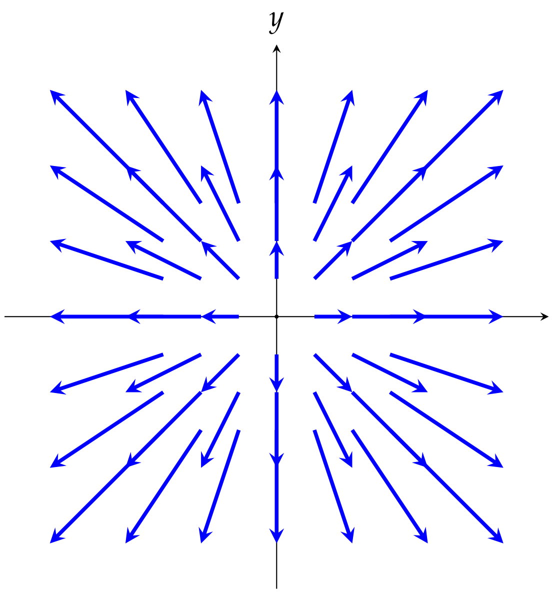
Contents

Scalar Function and Vector Fields

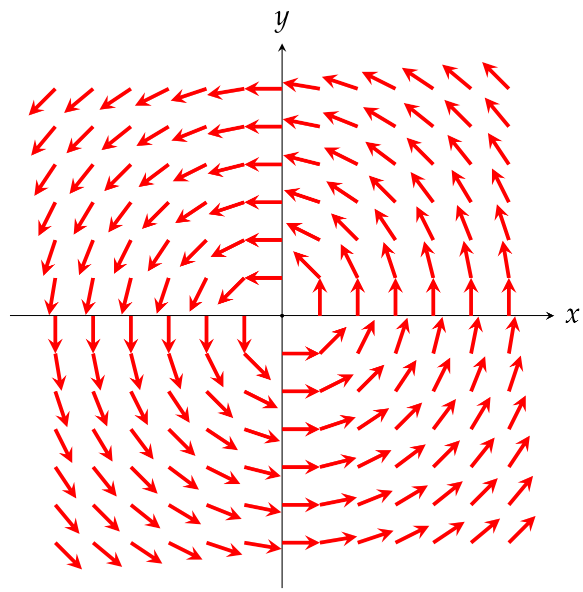
A **scalar function** on $\mathbb{R}^n$ is a real-valued function of an n-tuple; that is,

$$f : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n).$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $f(\mathbf{x}) \in \mathbb{R}$.



The radial field $\mathbf{F} = \langle x, y \rangle = x\mathbf{i} + y\mathbf{j}$



The spin field $\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)^{1/2}$

A **vector field** on $\mathbb{R}^n$ is a function

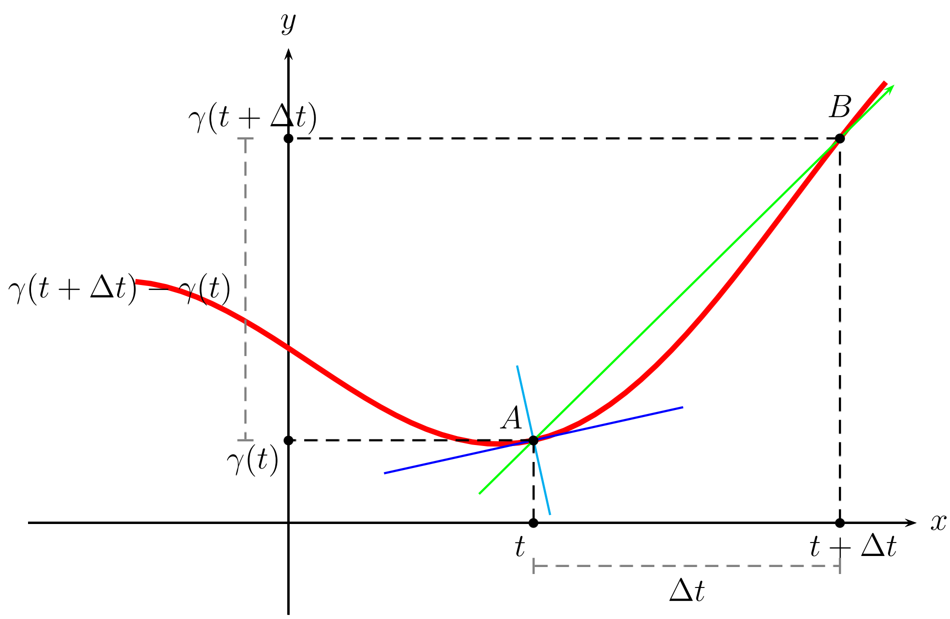
$$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n, \quad \mathbf{x} \mapsto \mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}), \ldots, F_n(\mathbf{x})),$$

where each component $F_i : \mathbb{R}^n \to \mathbb{R}$ is itself a scalar function.

Line Integrals

Line Integral of Scalar Function over Arc Length

Secant Lines & Tangent as a Limit



For a curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2\colon t \mapsto \big(x(t), y(t)\big)$, the **secant vector** over $[t, t+\Delta t]$ is

$$\frac{\gamma(t+\Delta t)-\gamma(t)}{\Delta t} = \left(\frac{x(t+\Delta t)-x(t)}{\Delta t}, \frac{y(t+\Delta t)-y(t)}{\Delta t}\right).$$

As $\Delta t \to 0$, these secants converge (if γ is smooth) to

$$\begin{aligned}\gamma'(t) = \frac{d}{dt}\gamma(t) = \lim_{\Delta t\to 0}\frac{\gamma(t+\Delta t)-\gamma(t)}{\Delta t} &= \left(\lim_{\Delta t\to 0}\frac{x(t+\Delta t)-x(t)}{\Delta t}, \lim_{\Delta t\to 0}\frac{y(t+\Delta t)-y(t)}{\Delta t}\right)\\ &= \left(\frac{d}{dt}x(t), \frac{d}{dt}y(t)\right)\\ &= (x'(t), y'(t)),\end{aligned}$$

which gives the **tangent vector** at $\gamma(t)$. The tangent vector captures how the curve is moving instantaneously at time t.

By Pythagoras' theorem, the **length moved per unit time** is $\|\gamma'(t)\| = \sqrt{(x'(t))^2+(y'(t))^2}$, and the small arc length traveled between t and $t+\Delta t$ is approximately:

$$\|\gamma'(t)\|\Delta t = \sqrt{\left(\frac{d}{dt}x(t)\right)^2+\left(\frac{d}{dt}y(t)\right)^2}\cdot\Delta t.$$

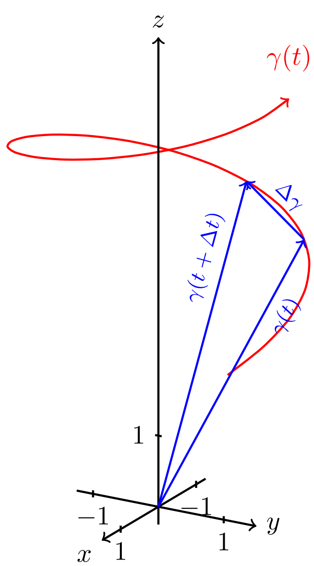
Arc Length of a Parametrized Curve

Definition. Let $C \subset \mathbb{R}^n$ be a piecewise smooth curve, given by a smooth parameterization:

$$\gamma : [a, b] \to \mathbb{R}^n, \quad t \mapsto \gamma(t) = \big(x_1(t), x_2(t), \ldots, x_n(t)\big).$$

Then the **arc length** s of the curve C from $t = a$ to $t = b$ is defined by

$$\boxed{s := \int_a^b \|\gamma'(t)\|\, dt}, \quad \text{where } \|\gamma'(t)\| = \sqrt{\left(\frac{d}{dt}x_1(t)\right)^2 + \left(\frac{d}{dt}x_2(t)\right)^2 + \cdots + \left(\frac{d}{dt}x_n(t)\right)^2}.$$



$$\begin{array}{cccl} \gamma & : & \mathbb{R} & \longrightarrow & \mathbb{R}^3 \\ & & t & \longmapsto & \gamma(t) = (x(t), y(t), z(t)) \end{array}$$

$$\gamma'(t) = \frac{d}{dt}\gamma(t) = \lim_{\Delta t \to 0} \frac{\gamma(t + \Delta t) - \gamma(t)}{\Delta t} = (x'(t), y'(t), z'(t))$$

$$s = \int_a^b \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2 + \big(z'(t)\big)^2}\, dt$$

Definition (Line Integral of Scalar Function over Arc Length)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar function, and let C be a piecewise smooth curve in $\mathbb{R}^n$ given by a smooth parameterization:

$$\gamma : [a, b] \to \mathbb{R}^n, \quad t \mapsto \gamma(t) = (x_1(t), x_2(t), \ldots, x_n(t)).$$

The **line integral of the scalar function** f along the curve C with respect to arc length is defined by

$$\int_C f\, ds := \int_a^b f(\gamma(t))\, \|\gamma'(t)\|\, dt.$$

Here, $ds = \|\gamma'(t)\|\, dt$ is the **infinitesimal arc length**.

Line Integral of Vector Fields

Fundamental Theorem of Calculus for Line Integrals

Theorem. *Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ be a C^1–function. Consider its gradient field by*

$$\nabla f = \langle \partial_{x_1} f, \ldots, \partial_{x_n} f \rangle.$$

Let $\gamma : [a, b] \to U$ be any piecewise-C^1 curve with endpoints $\gamma(a) = P$ and $\gamma(b) = Q$. Then

$$\int_\gamma \nabla f \cdot d\mathbf{r} \;=\; \int_a^b \nabla f\big(\gamma(t)\big) \cdot \gamma'(t)\, dt \;=\; f\big(\gamma(b)\big) \;-\; f\big(\gamma(a)\big).$$

Excellent — this is a key question for understanding how **vector calculus** connects with **differential forms** and notation. Let's walk through how to **rigorously deduce**:

$$\int_C F_1\, dx + F_2\, dy \quad \text{from} \quad \int_C \mathbf{F} \cdot d\mathbf{r}$$

for a vector field $\mathbf{F}(x, y) = (F_1(x, y), F_2(x, y))$.

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Step 1: Parametrize the Curve C

Let $\gamma : [a, b] \to \mathbb{R}^2$ be a smooth parameterization of the curve C, given by:

$$\gamma(t) = (x(t), y(t)), \quad t \in [a, b].$$

Then the **derivative of the curve** is:

$$\gamma'(t) = (x'(t), y'(t)),$$

and the **differential displacement** is:

$$d\mathbf{r} = \gamma'(t)\, dt = (dx, dy) = (x'(t)\, dt, y'(t)\, dt).$$

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Step 2: Write the Line Integral of $\mathbf{F}$

The **vector line integral** is defined as:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b \big[F_1(x(t), y(t))\, x'(t) + F_2(x(t), y(t))\, y'(t)\big]\, dt.$$

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Step 3: Recognize the Total Differentials

From calculus, we know:

$$dx = x'(t)\,dt, \quad dy = y'(t)\,dt.$$

Substitute into the integral:

$$\int_a^b F_1(x(t), y(t))\,dx + F_2(x(t), y(t))\,dy.$$

So we have:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b F_1(x(t), y(t))\,dx + F_2(x(t), y(t))\,dy = \int_C F_1\,dx + F_2\,dy.$$

—

Final Deduction

Thus:

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C F_1\,dx + F_2\,dy}$$

where: - $\mathbf{F} = (F_1, F_2)$, - $d\mathbf{r} = (dx, dy)$, - The dot product $\mathbf{F} \cdot d\mathbf{r} = F_1\,dx + F_2\,dy$.

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Interpretation

- The vector line integral becomes an integral of a **1-form**:

$$\mathbf{F} \cdot d\mathbf{r} \quad \leftrightarrow \quad F_1\,dx + F_2\,dy.$$

- The dot product turns into a sum of **components times differentials**.

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Would you like to go one step further and express this in terms of pullbacks or show how it generalizes to $\mathbb{R}^3$?

Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{F} : U \to \mathbb{R}^n$ a continuous vector field. Suppose $C \subset U$ is a smooth curve parametrized by

$$\mathbf{r}\colon [a,b] \longrightarrow \mathbb{R}^n, \quad t \mapsto \mathbf{r}(t),$$

with nonzero velocity $\mathbf{r}'(t)$. Then the **line integral** of $\mathbf{F}$ along C is defined by

$$\boxed{\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t)\, dt = \int_a^b \sum_{i=1}^n F_i\big(\mathbf{r}(t)\big)\, x_i'(t)\, dt,}$$

where $\mathbf{r}(t) = (x_1(t), \ldots, x_n(t))$ and $\mathbf{F} = (F_1, \ldots, F_n)$.

This integral "accumulates" at each infinitesimal step dt the projection of $\mathbf{F}$ onto the tangent vector $\mathbf{r}'(t)$, yielding a single real number that captures the **circulation** or **work** of $\mathbf{F}$ along C.

Example. Take $n = 2$ and $\mathbf{F}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ on $U = \mathbb{R}^2 \setminus \{(0,0)\}$. Let C be the unit circle $x^2 + y^2 = 1$, counterclockwise. Parametrize $\mathbf{r}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Then

$$\mathbf{r}'(t) = (-\sin t, \cos t), \qquad \mathbf{F}\big(\mathbf{r}(t)\big) = \big(-\sin t, \cos t\big),$$

so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \big(-\sin t, \cos t\big)\cdot(-\sin t, \cos t)\, dt = \int_0^{2\pi} \big(\sin^2 t + \cos^2 t\big)\, dt = 2\pi.$$

Thus the total circulation (or "work") of $\mathbf{F}$ around the unit circle is 2π.

Problem #1 (Line Integral around Unit Circle). Let $C \subset \mathbb{R}^2$ be the unit circle defined by

$$C : x^2 + y^2 = 1,$$

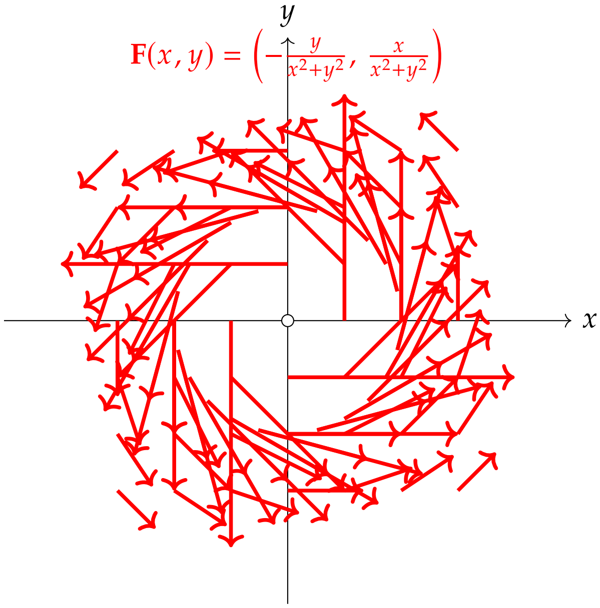
traversed in the **counterclockwise direction**. Let the vector field $\mathbf{F} : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ be defined by

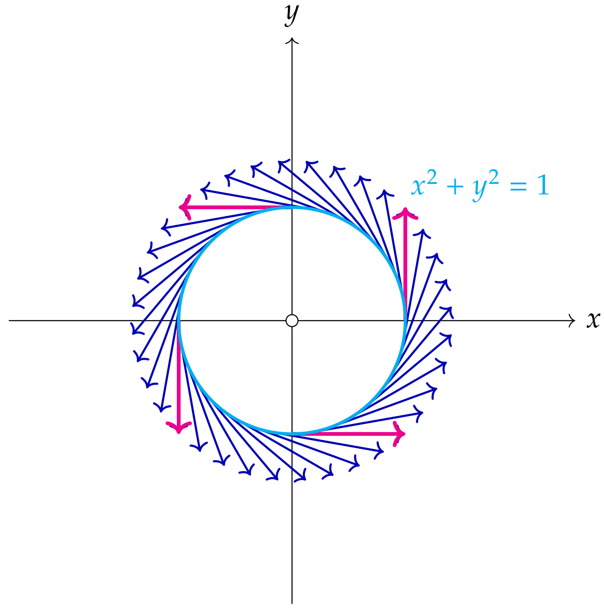
$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

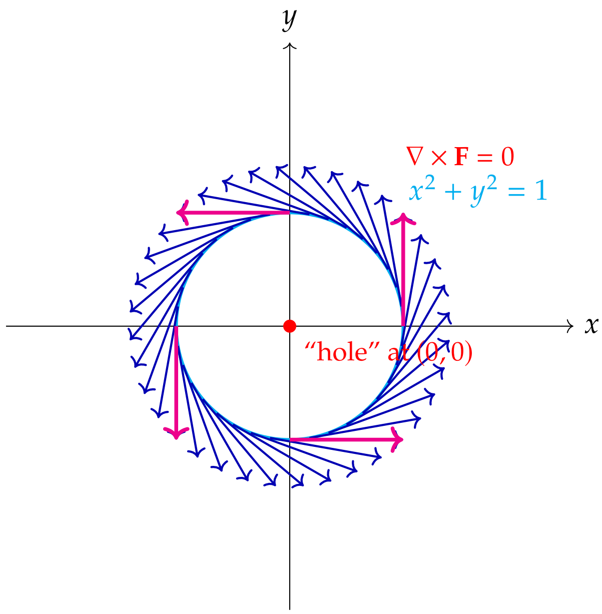
Evaluate the **line integral** of $\mathbf{F}$ along C:

$$\oint_C \mathbf{F}\cdot d\mathbf{r}.$$

Sol.







Consider the vector field:

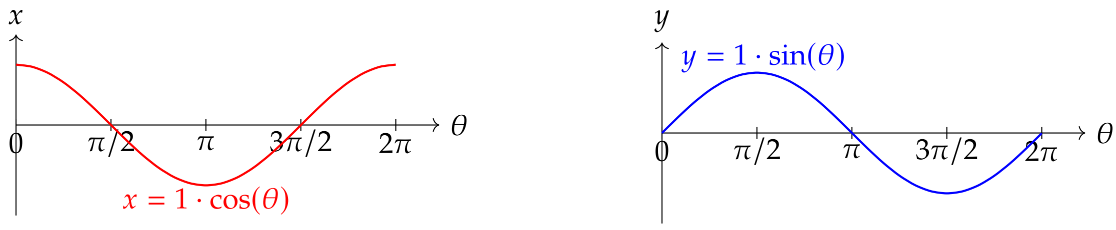
$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right),$$

and the curve C is the unit circle $x^2 + y^2 = 1$, traversed counterclockwise.

Step 1. (Parametrization) Define a function

$$\begin{array}{rccl} \gamma & : & [0, 2\pi] & \longrightarrow & \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\} \\ & & \theta & \longmapsto & \gamma(\theta) = (\cos\theta, \sin\theta) \end{array}.$$

Here, $\frac{d\gamma}{d\theta} = (-\sin\theta, \cos\theta)$.



Step 2. (Evaluate $\mathbf{F}(\gamma(\theta))$ and the dot product) We have

$$\mathbf{F}(\gamma(\theta)) = \mathbf{F}(\cos\theta, \sin\theta) \overset{\sin^2\theta+\cos^2\theta=1}{=} \left(\frac{-\sin\theta}{1}, \frac{\cos\theta}{1}\right) = (-\sin\theta, \cos\theta).$$

and

$$\mathbf{F}(\gamma(\theta)) \cdot \frac{d\gamma}{d\theta} = (-\sin\theta)(-\sin\theta) + (\cos\theta)(\cos\theta) = \sin^2\theta + \cos^2\theta = 1.$$

Step 3. (Integral)

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\gamma(\theta)) \cdot \frac{d\gamma}{d\theta}\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi.$$

□

Surface Integral for Vector Fields

Surface Integral of a Vector Field

Definition. Let $S \subset \mathbb{R}^3$ be a smooth, oriented surface, and let

$$T : D \subseteq \mathbb{R}^2 \longrightarrow S, \qquad (u,v) \longmapsto T(u,v),$$

be a regular C^1 parametrization whose orientation agrees with that of S. Consider the partial-derivative (tangent) vectors

$$T_u(u,v) = \frac{\partial T}{\partial u}(u,v), \quad T_v(u,v) = \frac{\partial T}{\partial v}(u,v),$$

and the induced normal-vector field

$$N(u,v) = T_u(u,v) \times T_v(u,v) \in \mathbb{R}^3,$$

which is everywhere nonzero on D and points according to the chosen orientation. Now let

$$\mathbf{F} : U \ (\supseteq S) \longrightarrow \mathbb{R}^3$$

be a continuous (or C^1) vector field defined on an open neighborhood U of S. Then **the surface integral of F over** S is defined by the formula

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}\big(T(u,v)\big) \cdot N(u,v)\, du\, dv = \iint_D \mathbf{F}\big(T(u,v)\big) \cdot \big(T_u(u,v) \times T_v(u,v)\big)\, du\, dv.}$$

Remark. 1. **Geometric meaning.** At each point $T(u,v) \in S$, the vector $N(u,v)\, du\, dv$ represents the oriented area element $d\mathbf{S}$. Thus $\mathbf{F} \cdot d\mathbf{S}$ measures how much $\mathbf{F}$ "flows through" that little patch of surface.

2. **Independence of parametrization.** If $\widetilde{T} : \widetilde{D} \to S$ is any other orientation-preserving C^1 parametrization, then a change-of-variables argument shows

$$\iint_D \mathbf{F} \cdot \big(T_u \times T_v\big)\, du\, dv = \iint_{\widetilde{D}} \mathbf{F} \cdot \big(\widetilde{T}_u \times \widetilde{T}_v\big)\, d\widetilde{u}\, d\widetilde{v}.$$

3. **Special case (scalar area).** Taking $\mathbf{F} = (0,0,1)$ recovers the usual surface-area integral $\text{Area}(S) = \iint_S dS = \iint_D \|T_u \times T_v\|\, du\, dv$.

This definition is the standard one found in graduate-level treatments of differential geometry and vector calculus.

Problem #2 (Surface-Flux). Let $S \subset \mathbb{R}^3$ be the smooth surface parametrized by

$$\mathbf{r}\colon [0,1]\times[0,1] \longrightarrow \mathbb{R}^3, \qquad \mathbf{r}(u,v) = \big(u+2v,\ 2u+v,\ 3uv\big),$$

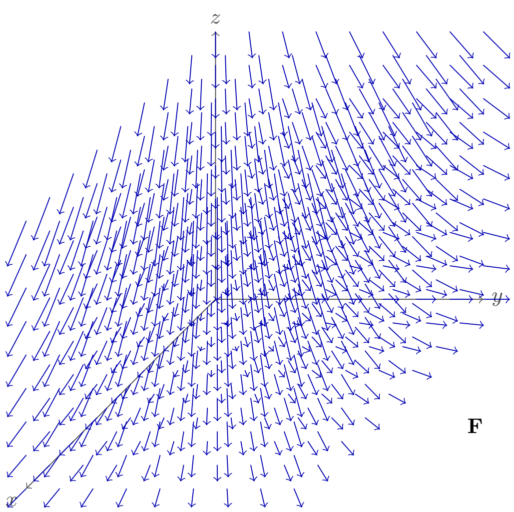
equipped with the orientation induced by the parametrization (so that the unit normal points in the direction of $\mathbf{r}_u \times \mathbf{r}_v$). Let the vector field

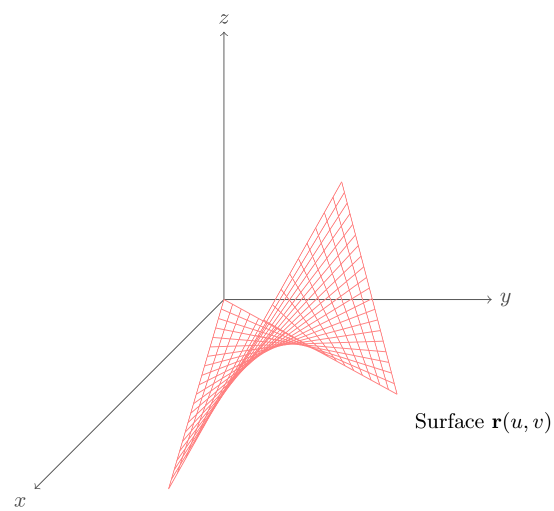
$$\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^3, \qquad \mathbf{F}(x,y,z) = (\,x,\ y,\ -z\,).$$

Compute

$$\iint_S \mathbf{F}\cdot d\mathbf{S}.$$

Sol. We compute in five steps.



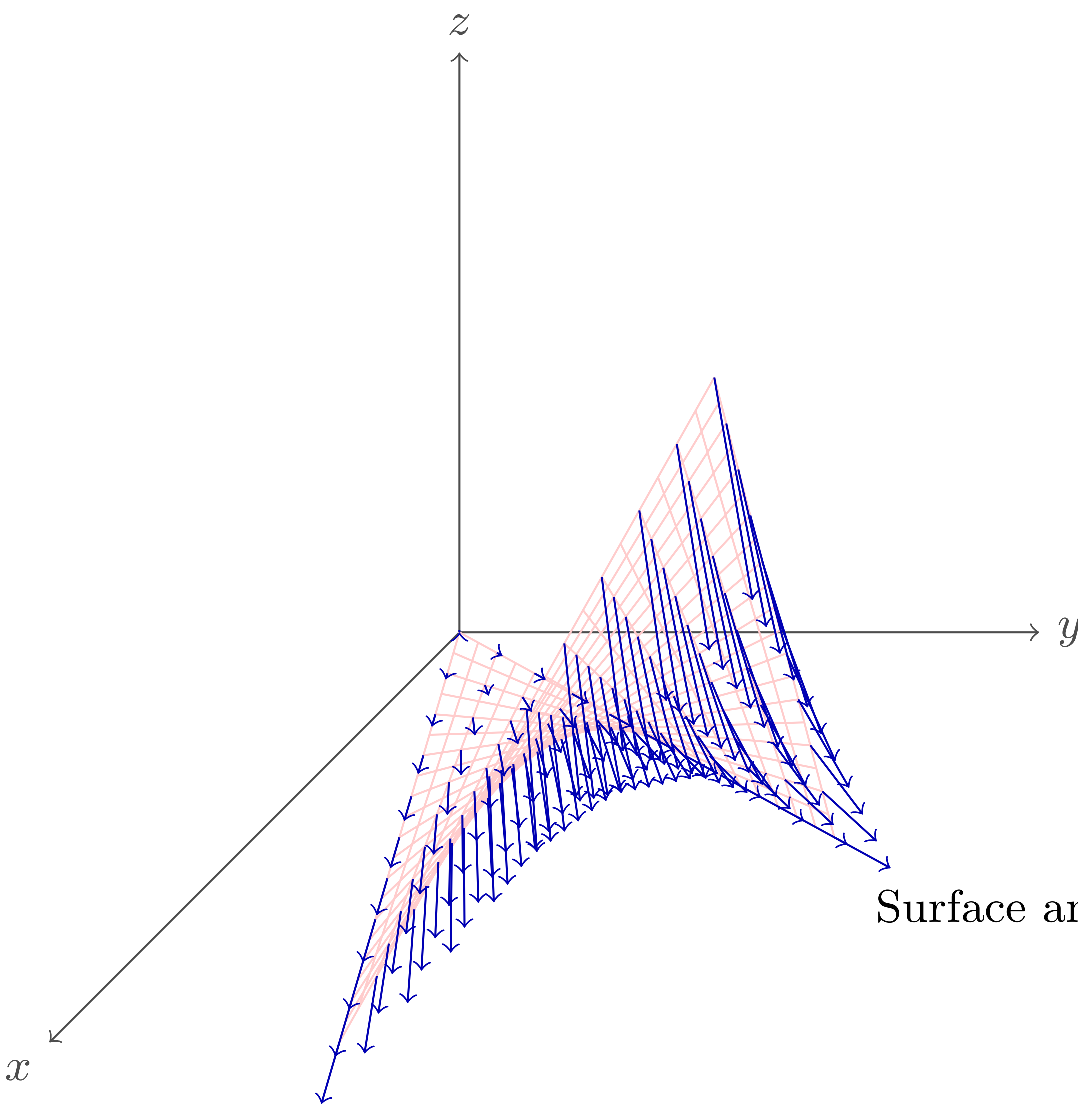


1. **Parametrization and partials.** The surface is

$$S = \mathbf{r}\big([0,1]^2\big), \qquad \mathbf{r}(u,v) = (\,u+2v,\ 2u+v,\ 3uv\,),$$

and hence

$$\mathbf{r}_u = (1,\ 2,\ 3v), \qquad \mathbf{r}_v = (2,\ 1,\ 3u).$$



Surface and $\mathbf{F}$

2. **Oriented normal.** The induced normal vector is the cross-product

$$
\begin{aligned}
\mathbf{r}_u \times \mathbf{r}_v &= \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3v \\ 2 & 1 & 3u \end{vmatrix} \\
&= \det \begin{vmatrix} 2 & 3v \\ 1 & 3u \end{vmatrix} \mathbf{i} - \det \begin{vmatrix} 1 & 3v \\ 2 & 3u \end{vmatrix} \mathbf{j} + \det \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \mathbf{k} \\
&= (6u - 3v,\ -3u + 6v,\ -3).
\end{aligned}
$$

3. **Pullback of the field.** The given field is $\mathbf{F}(x, y, z) = (x, y, -z)$. Along the patch,

$$\mathbf{F}\big(\mathbf{r}(u, v)\big) = \big(u + 2v,\ 2u + v,\ -3uv\big).$$

4. **Integrand.** Taking the dot-product,

$$\begin{aligned}\mathbf{F}(\mathbf{r}) \cdot (\mathbf{r}_u \times \mathbf{r}_v) &= (u + 2v)(6u - 3v) \ + \ (2u + v)(-3u + 6v) \ + \ (-3uv)(-3) \\ &= 6u^2 - 3uv + 12uv - 6v^2 \ - \ 6u^2 + 12uv - 3uv + 6v^2 \ + \ 9uv \\ &= (-3uv + 12uv + 12uv - 3uv + 9uv) \ = \ 27\, u\, v.\end{aligned}$$

5. **Double integral.** Thus the flux is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} \ = \ \iint_{[0,1]^2} 27\, u\, v\ du\, dv \ = \ 27 \int_0^1 \int_0^1 u\, v\ du\, dv \ = \ 27\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) \ = \ \frac{27}{4}.$$

Hence

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} \ = \ \frac{27}{4}}.$$

□

A Differential Geometry