Lecture Notes: Coordinates and Differentials on a Plane Curve

Lecture Note: 1-Form as Scalar Projection onto a Fixed Direction

1. Curve and Its Tangent Line

Let

$$C \;=\; \{(x, y) \in \mathbb{R}^2 : y = f(x)\}$$

be a smooth curve. Fix a point

$$p = (a, f(a)) \in C$$

and write the tangent direction at p as

$$\vec{v} = \begin{pmatrix} 1 \\ f'(a) \end{pmatrix} \in T_p\mathbb{R}^2.$$

Then the one–dimensional tangent space is

$$T_pC \;=\; \operatorname{span}\{\vec{v}\} \;\subset\; T_p\mathbb{R}^2.$$

2. Coordinate System on C

On the ambient plane $\mathbb{R}^2$ we have the standard projections

$$x, y \;:\; \mathbb{R}^2 \longrightarrow \mathbb{R}, \qquad x(x, y) = x, \quad y(x, y) = y.$$

Restricting to C yields two functions

$$x\big|_C \;:\; C \to \mathbb{R}, \quad y\big|_C \;:\; C \to \mathbb{R}.$$

Define the chart

$$\Phi_C \;:\; C \;\longrightarrow\; \mathbb{R}^2, \qquad \Phi_C(p) = \big(x|_C(p),\, y|_C(p)\big) = (a, f(a)).$$

3. Coordinate Projections on T_pC

At each $p \in C$, the ambient tangent plane is

$$T_p\mathbb{R}^2 = \operatorname{span}\bigl\{\partial_x|_p,\ \partial_y|_p\bigr\} \ \cong\ \mathbb{R}^2.$$

Its dual coordinates are

$$dx,\, dy\ :\ T_p\mathbb{R}^2 \to \mathbb{R}, \qquad dx\bigl((v^1, v^2)^T\bigr) = v^1, \quad dy\bigl((v^1, v^2)^T\bigr) = v^2.$$

Restrict these to the line $T_pC = \operatorname{span}\{(1, f'(a))^T\}$ to obtain

$$dx\big|_{T_pC},\ dy\big|_{T_pC}\ :\ T_pC \ \longrightarrow\ \mathbb{R}.$$

Stacking gives the fiber–chart

$$\Phi_{T_pC}\ :\ T_pC \ \longrightarrow\ \mathbb{R}^2, \qquad \Phi_{T_pC}(v) = \begin{pmatrix} dx(v) \\ dy(v) \end{pmatrix}.$$

Concretely, if $v = t\,(1, f'(a))^T \in T_pC$, then

$$dx(v) = t, \quad dy(v) = t\, f'(a), \quad \Phi_{T_pC}(v) = \begin{pmatrix} t \\ t\, f'(a) \end{pmatrix}.$$

4. The 1-Form of Scalar Projection

Fix a unit direction

$$\mathbf{u} = (\cos\theta, \sin\theta)\ \in\ \mathbb{R}^2.$$

Define a differential 1-form

$$\omega\ \in\ \Omega^1(C)$$

by declaring its action on each tangent vector $v \in T_pC \subset T_p\mathbb{R}^2$ to be the scalar projection onto $\mathbf{u}$:

$$\forall\, v \in T_pC: \qquad \omega_p(v) = \langle \mathbf{u}, v\rangle = \cos\theta\ dx(v) + \sin\theta\ dy(v).$$

Linearity in v and smooth dependence on p show that ω is indeed a smooth section of the cotangent bundle T^*C. In particular, for the canonical basis vector $\vec{v} = (1, f'(a))^T$,

$$\omega_p(\vec{v}) = \cos\theta \cdot 1 + \sin\theta \cdot f'(a).$$

Abstract Formulation. A *differential* 1-*form* on C is by definition a section $\omega \in \Gamma(T^*C)$. The above ω arises from the ambient inner–product by pulling back the functional $v \mapsto \langle \mathbf{u}, v\rangle$ along the inclusion $T_pC \hookrightarrow T_p\mathbb{R}^2$, thereby encoding the scalar projection onto the fixed direction $\mathbf{u}$ at each point of the curve.