Homework2: One-to-one Correspondence between Linear Transformations and Matrices

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Let U, V, W are vector spaces with basis

$$\mathcal{B}_U = \{u_1, \dots, u_\ell\},\ \mathcal{B}_V = \{v_1, \dots, v_n\},\ \mathcal{B}_W = \{w_1, \dots, w_m\},$$

respectively. Consider linear transformations $T_1 : U \to V$, $T_2 : V \to W$, and $T_2 \circ T_1 : U \to W$. Let $A = [T_1]_{\mathcal{B}_U}^{\mathcal{B}_V} \in M_{n\times\ell}$ and $B = [T_2]_{\mathcal{B}_V}^{\mathcal{B}_W} \in M_{m\times n}$. Prove that

$$[T_2 \circ T_1]_{\mathcal{B}_U}^{\mathcal{B}_W} = [T_2]_{\mathcal{B}_V}^{\mathcal{B}_W}\,[T_1]_{\mathcal{B}_U}^{\mathcal{B}_V} = BA \in M_{m\times\ell}.$$

Proof.

Column-wise computation. Define

$$A := [T_1]_{\mathcal{B}_U}^{\mathcal{B}_V} = \begin{bmatrix} | & & | \\ [T_1(u_1)]_{\mathcal{B}_V} & \cdots & [T_1(u_\ell)]_{\mathcal{B}_V} \\ | & & | \end{bmatrix} \in \mathbb{R}^{n\times\ell}, \quad B := [T_2]_{\mathcal{B}_V}^{\mathcal{B}_W} = \begin{bmatrix} | & & | \\ [T_2(v_1)]_{\mathcal{B}_W} & \cdots & [T_2(v_n)]_{\mathcal{B}_W} \\ | & & | \end{bmatrix} \in \mathbb{R}^{m\times n}.$$

The matrix of the composition will be

$$C := [T_2 \circ T_1]_{\mathcal{B}_U}^{\mathcal{B}_W} = \begin{bmatrix} | & & | \\ [(T_2 \circ T_1)(u_1)]_{\mathcal{B}_W} & \cdots & [(T_2 \circ T_1)(u_\ell)]_{\mathcal{B}_W} \\ | & & | \end{bmatrix} \in \mathbb{R}^{m\times\ell}.$$

For each $j \in \{1, \dots, \ell\}$, by linearity and the definition of A and B,

$$[(T_2 \circ T_1)(u_j)]_{\mathcal{B}_W} = [T_2\big(T_1(u_j)\big)]_{\mathcal{B}_W} = [T_2]_{\mathcal{B}_V}^{\mathcal{B}_W}\,[T_1(u_j)]_{\mathcal{B}_V} = B \cdot (\text{the } j\text{-th column of } A).$$

Hence the j-th column of C equals B times the j-th column of A, i.e.,

$$C = BA.$$

Entry-wise (index) verification. Write

$$A = \left[a_{ij}\right]_{1\le i\le n,\,1\le j\le \ell}, \qquad B = \left[b_{ki}\right]_{1\le k\le m,\,1\le i\le n}.$$

Then for each j,

$$T_1(u_j) = \sum_{i=1}^{n} a_{ij} v_i, \quad T_2(v_i) = \sum_{k=1}^{m} b_{ki} w_k,$$

so

$$(T_2 \circ T_1)(u_j) = T_2\left(\sum_{i=1}^{n} a_{ij} v_i\right) = \sum_{i=1}^{n} a_{ij} T_2(v_i) = \sum_{i=1}^{n} a_{ij} \sum_{k=1}^{m} b_{ki} w_k = \sum_{k=1}^{m}\left(\sum_{i=1}^{n} b_{ki} a_{ij}\right) w_k.$$

Thus the k-th coordinate (in $\mathcal{B}_W$) of $(T_2 \circ T_1)(u_j)$ is

$$c_{kj} := \sum_{i=1}^{n} b_{ki} a_{ij}.$$

Therefore $C = [c_{kj}]$ with

$$C = BA = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1\ell} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n\ell} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} b_{1i} a_{i1} & \cdots & \sum_{i=1}^{n} b_{1i} a_{i\ell} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} b_{mi} a_{i1} & \cdots & \sum_{i=1}^{n} b_{mi} a_{i\ell} \end{bmatrix}.$$

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