Torus and Algebra

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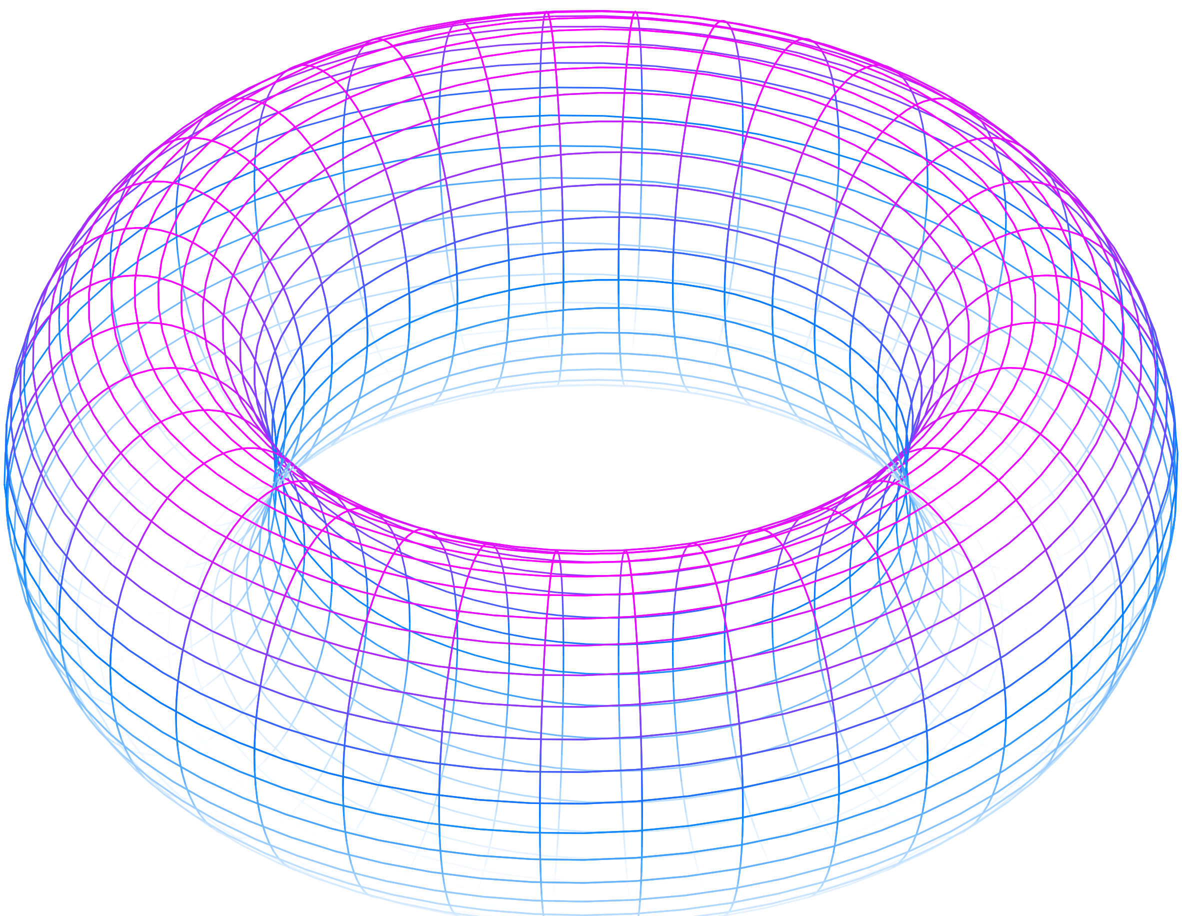
We cover the following topics in this note.

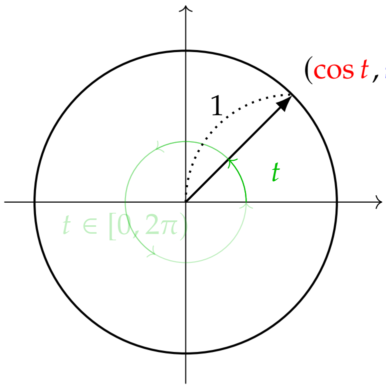
- Unit Circle
- Torus
- Elliptic Curve
- TBA

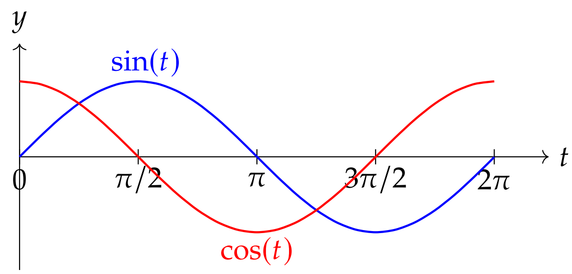


Contents

1 Unit Circle

The set $\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is called the **unit circle**.





The standard parametrization of $\mathbb{S}^1$ is given by

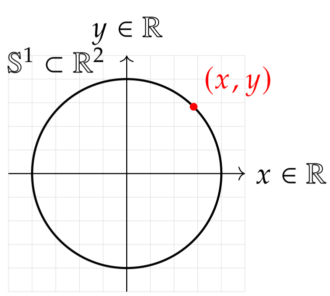
$$t \mapsto (\cos t, \sin t), \quad t \in [0, 2\pi),$$

which implies the *trigonometric identity* $\cos^2 t + \sin^2 t = 1$. The mapping

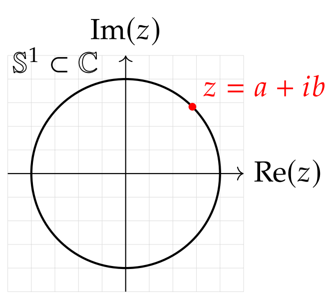
$$\begin{array}{cccc} \varphi & : & [0, 2\pi) & \longrightarrow & \mathbb{S}^1 \\ & & t & \longmapsto & (\cos t, \sin t) \end{array}.$$

provides a bijection between the half-open interval $[0, 2\pi)$ and the unit circle $\mathbb{S}^1$.

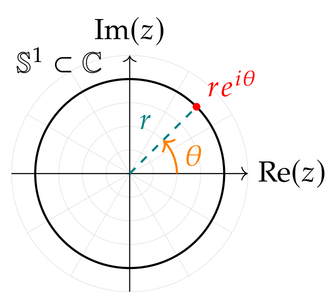
Geometrically, it represents the set of points at a fixed distance 1 from the origin in $\mathbb{R}^2$, while algebraically it can be seen as a group under complex multiplication.



$x^2 + y^2 = 1$



$|z| = 1$



$|r| = 1$ and $\theta \in \mathbb{R}$

The unit circle can be described in several equivalent ways. In $\mathbb{R}^2$, it is given by:

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

In the complex plane, we write:

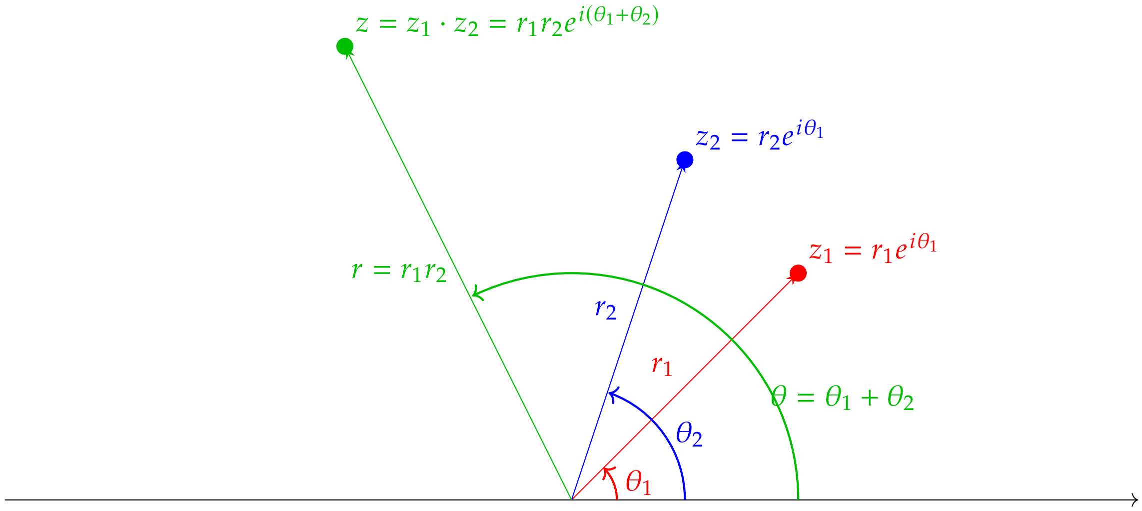
$$\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\} = \{re^{i\theta} : |r| = 1 \text{ and } \theta \in \mathbb{R}\}.$$

We show that **multiplication of complex number is equivalent to addition of angles**: let

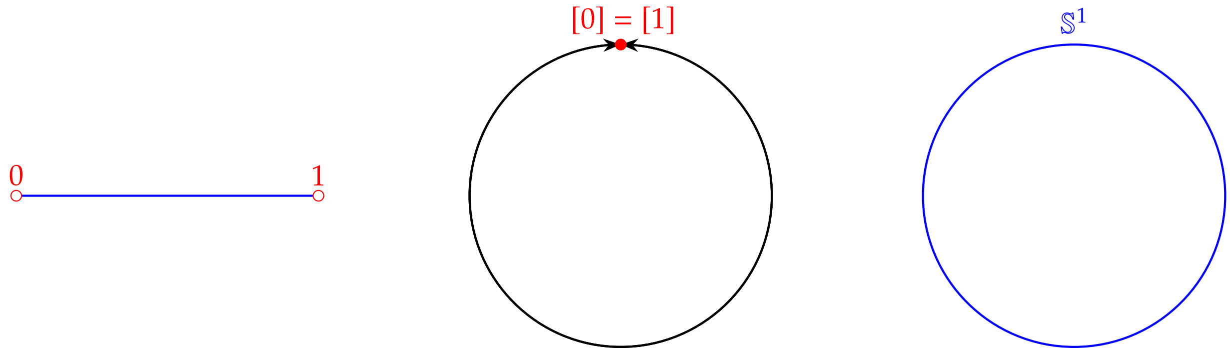
$$z_1 = r_1 e^{i\theta_1} = r_1 (\cos\theta_1 + i\sin\theta_1) \in \mathbb{C} \text{ and}$$
$$z_2 = r_2 e^{i\theta_2} = r_2 (\cos\theta_2 + i\sin\theta_2) \in \mathbb{C}.$$

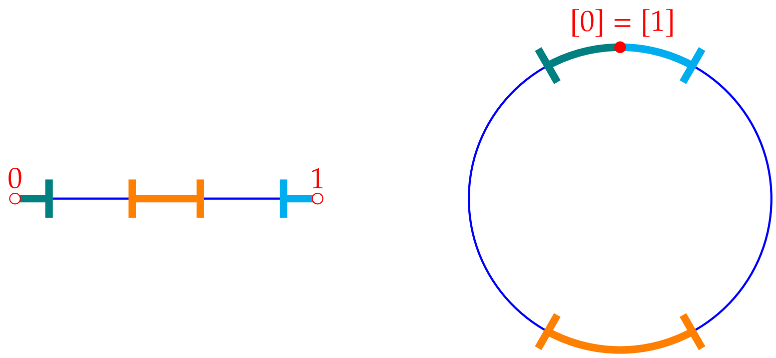
Then

$$\begin{aligned} z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 \left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) \right] \\ &= r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \right] \\ &= r(\cos\theta + \sin\theta) \text{ with } \begin{cases} r = r_1 r_2 \\ \theta = \theta_1 + \theta_2. \end{cases} \end{aligned}$$



1.1 Quotient Space





Let

$$\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}, \quad x \mapsto x + \mathbb{Z},$$

be the canonical projection onto the quotient group, where the equivalence relation is given by

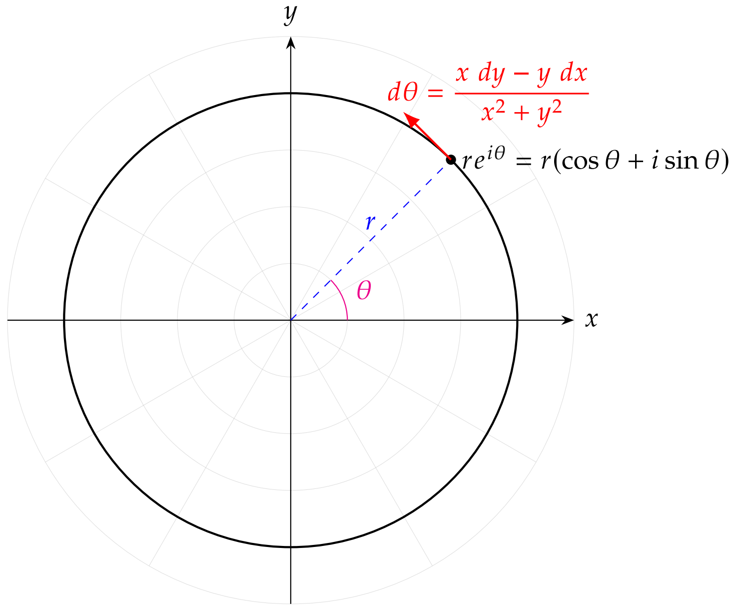
$$x \sim y \quad \Longleftrightarrow \quad x - y \in \mathbb{Z}.$$

Denote by

$$[x] = \{\, y \in \mathbb{R} \mid y \sim x \,\} = x + \mathbb{Z}$$

the equivalence class of x. Then $\mathbb{R}/\mathbb{Z} = \{[x] : x \in \mathbb{R}\}$.

1.2 Total differential of $d\theta$



$d\theta$ is globally defined, whereas θ is local (mod 2π).

Recall that if we express a point in the plane in polar coordinates, then

$$x = r\cos\theta, \quad y = r\sin\theta.$$

One observe that the polar angle θ may be expressed as

$$\tan\theta = \frac{y}{x} \implies \theta = \arctan\left(\frac{y}{x}\right), \quad x \neq 0.$$

Let $\theta(x,y) = \arctan\left(\dfrac{y}{x}\right)$ with $x \neq 0$. We compute the total differential:

$$d\theta = \frac{\partial\theta}{\partial x}\,dx + \frac{\partial\theta}{\partial y}\,dy.$$

Since

$$\frac{\partial\theta}{\partial x} = \frac{\partial}{d\partial}\left[\arctan\left(\frac{y}{x}\right)\right] = \frac{1}{1+(y/x)^2}\cdot\frac{\partial}{\partial x}\left[\frac{y}{x}\right] = \frac{1}{(x^2+y^2)/x^2}\cdot\left(-\frac{y}{x^2}\right) = \frac{x^2}{x^2+y^2}\cdot\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

and

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\frac{\partial}{\partial y}\left[\frac{y}{x}\right] = \frac{x^2}{x^2+y^2}\cdot\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2},$$

we have the total differential of $\theta(x, y)$ is

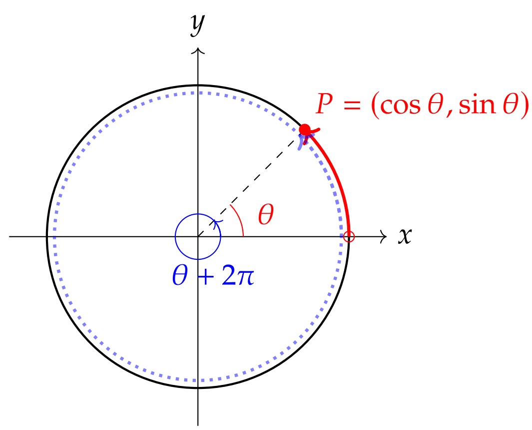
$$d\theta = \frac{\partial\theta}{\partial x}\,dx + \frac{\partial\theta}{\partial y}\,dy = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

This can be neatly written as

$$\boxed{d\theta = \frac{x\,dy - y\,dx}{x^2+y^2}.}$$

1.3 Local Coordinate Function $\theta : U \to \mathbb{R}$

Consider the unit circle defined by $\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.



A natural idea is to assign to every point $P = (x, y)$ its angle θ so that

$$P = (x, y) = (\cos\theta, \sin\theta)$$

Both θ and $\theta + 2\pi$ give the same point on the circle, because

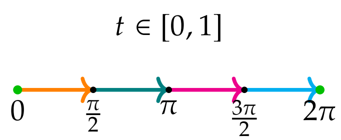
$$(\cos(\theta + 2\pi), \sin(\theta + 2\pi)) = (\cos\theta, \sin\theta)$$

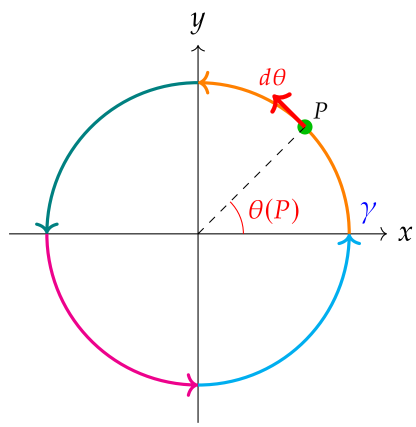
In $U := \mathbb{S}^1 \setminus \big\{(1, 0)\big\}$, we can define an angular coordinate function

$$\begin{array}{cccccc} \theta & : & U(\subsetneq \mathbb{S}^1) & \longrightarrow & \mathbb{R} \\ & & P & \longmapsto & \theta(P) \end{array}.$$

Here, θ is only locally well defined.

1.4 Line Integral $\oint_\gamma d\theta$





Parameterize the Unit Circle We represent the unit circle as

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

A standard parameterization is given by the function

$$\gamma : [0, 2\pi] \to \mathbb{S}^1, \qquad t \mapsto \gamma(t) = (\cos t, \sin t).$$

Differential Form $d\theta$ Since every point on the circle satisfies $x^2 + y^2 = 1$, we have

$$d\theta = \frac{-y\,dx + x\,dy}{x^2 + y^2} = -y\,dx + x\,dy.$$

Since $\gamma(t) = (\cos t, \sin t)$,

(i) The x-coordinate is $x(t) = \cos t$. Then $dx = \dfrac{dx}{dt}\,dt = -\sin t\,dt$.

(ii) The y-coordinate is $y(t) = \sin t$. Then $dy = \dfrac{dy}{dt}\,dt = \cos t\,dt$.

By (i) and (ii), we obtain

$$\begin{aligned} d\theta &= -y\,dx + x\,dy \\ &= -\sin t\,(dx) + \cos t\,(dy) \\ &= -\sin t\,\big(-\sin t\,dt\big) + \cos t\,\big(\cos t\,dt\big) \\ &= (\sin^2 t + \cos^2 t)\,dt \\ &= dt. \end{aligned}$$

Perform the Line Integral

$$\oint_\gamma d\theta = \oint_0^{2\pi} dt = \oint_0^{2\pi} 1\, dt = \Big[t\Big]_{t=0}^{t=2\pi} = 2\pi - 0 = 2\pi.$$

Interpretation The value 2π represents the total angular change as one goes around the circle *once*. In a general situation, if a closed curve γ on $\mathbb{S}^1$ winds around the circle k times, the integral would yield

$$\oint_\gamma d\theta = 2\pi k, \quad \text{where } k \in \mathbb{Z}.$$

Here, $k \in \mathbb{Z}$ is called the **winding number**.

Key Point Even though the local function θ is defined only up to an additive constant of 2π, the line integral of its differential $d\theta$ gives a well-defined, global number measuring the rotation.

1.5 Winding Number

Winding Number via the Angular 1-form

Definition. Let

$$\gamma : [0,1] \longrightarrow \mathbb{R}^2 \setminus \{0\}$$

be a piecewise C^1 map with $\gamma(0) = \gamma(1)$; that is, γ is a closed, piecewise smooth curve in $\mathbb{R}^2 \setminus \{0\}$. Define the angular 1-form ω by

$$\omega := \frac{-y\, dx + x\, dy}{x^2 + y^2}.$$

Then the **winding number of γ about the origin** is defined by

$$\mathrm{wind}(\gamma, 0) := \frac{1}{2\pi} \oint_\gamma d\omega.$$

It is a standard result that $\oint_\gamma \omega \in 2\pi\mathbb{Z}$, so that $\mathrm{wind}(\gamma, 0) \in \mathbb{Z}$.

We can indeed define the winding number not just about the origin 0 but relative to any point $p \in \mathbb{R}^2$ (provided that p is not on the image of the curve). In such a case, one writes the winding number as $\mathrm{wind}(\gamma, p)$ rather than $\mathrm{wind}(\gamma, 0)$. The construction is analogous; one "centers" the angular coordinate at the point p instead of at 0.

Let $p \in \mathbb{R}^2$ be a fixed point and let

$$\gamma : [0,1] \to \mathbb{R}^2 \setminus \{p\}$$

be a piecewise C^1 closed curve, that is, $\gamma(0) = \gamma(1)$ and $\gamma(t) \neq p$ for all $t \in [0,1]$. Define the map

$$\widetilde{\gamma}(t) = \frac{\gamma(t) - p}{\|\gamma(t) - p\|},$$

which is a well-defined map from $[0,1]$ to the unit circle

$$S^1 = \{z \in \mathbb{R}^2 : \|z\| = 1\}.$$

Since $\widetilde{\gamma}(0) = \widetilde{\gamma}(1)$, the map $\widetilde{\gamma}$ is a loop in $\mathbb{S}^1$. The *winding number* of γ about p is defined as the degree of $\widetilde{\gamma}$:

$$\mathrm{wind}(\gamma, p) := \deg(\widetilde{\gamma}) \in \mathbb{Z}.$$

Equivalently, if we let ω denote the standard angular 1-form on $\mathbb{R}^2 \setminus \{0\}$,

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2},$$

then by substituting $x' = x - p_1$ and $y' = y - p_2$ (where $p = (p_1, p_2)$), one can define an angular coordinate about p and obtain

$$\mathrm{wind}(\gamma, p) = \frac{1}{2\pi} \int_\gamma \omega_p,$$

where

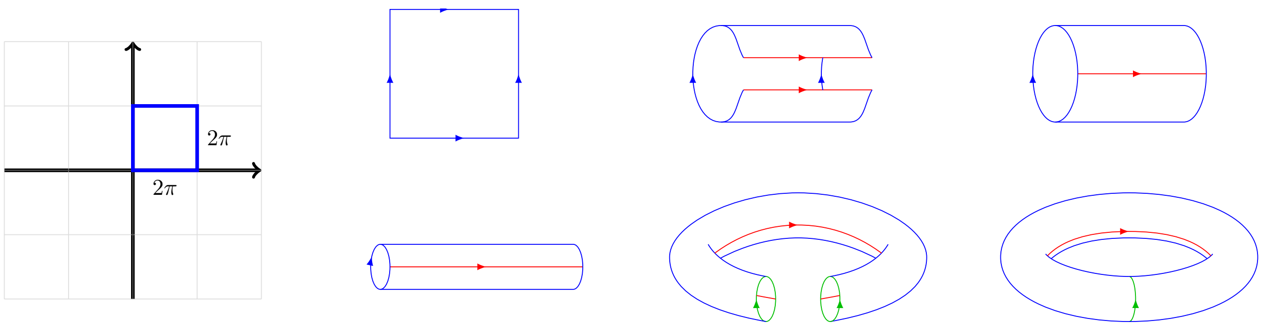
$$\omega_p = \frac{-(y - p_2)\,dx + (x - p_1)\,dy}{(x - p_1)^2 + (y - p_2)^2}.$$

It is a standard result that

$$\int_\gamma \omega_p \in 2\pi\mathbb{Z},$$

so that $\mathrm{wind}(\gamma, p) \in \mathbb{Z}$.

2 Torus



Note. The torus rotation parameter is set to θ_1 and θ_2, each θ expresses

$$d\theta = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

globally (where θ itself is only a local parameter), and θ is not a gradient vector field; hence, the integral value over a circle is expressed by a nonzero integer multiple (of 2π).

These two (normalized) 1-forms constitute a basis of $\mathbb{R}^2$ when one identifies the collection of closed differential forms modulo exact forms.

2.1 Quotient and Its Lattice Structure

Consider

$$\mathbb{R}^2 = \big\{(x, y) : x, y \in \mathbb{R}\big\} \quad \text{and} \quad \mathbb{Z}^2 = \{(m, n) : m, n \in \mathbb{Z}\}.$$

The quotient space $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ is defined by the equivalence relation on $\mathbb{R}^2$

$$(x, y) \sim (x', y') \quad \Longleftrightarrow \quad (x - x', y - y') \in \mathbb{Z}^2.$$

Consider

$$\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2,$$

where the subgroup $\mathbb{Z}^2$ is regarded as a free (abelian) $\mathbb{Z}$–module of rank 2. That is, there exist vectors

$$v_1, v_2 \in \mathbb{R}^2$$

such that every element of $\mathbb{Z}^2$ is uniquely expressible as

$$m\,v_1 + n\,v_2, \quad m, n \in \mathbb{Z}.$$

Consequently, the torus possesses two independent cycles which, topologically, are isomorphic to $\mathbb{S}^1$. In particular, one may write

$$\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1.$$

[$\mathbb{Z}^2$ as Free $\mathbb{Z}$-Module of Rank 2] Consider the abelian group

$$\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z} := \{(a, b) : a, b \in \mathbb{Z}\}$$

with addition defined coordinate-wise: $(a, b) + (c, d) = (a + c, b + d)$. Since every element of $\mathbb{Z}^2$ can be uniquely expressed in the form

$$(a, b) = a \cdot (1, 0) + b \cdot (0, 1),$$

$\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$ is a free $\mathbb{Z}$-module with basis $\{(1, 0), (0, 1)\}$ (rank 2).

2.2 Local Angular Coordinates and Their Differential Forms

On each S^1 factor of T^2 one can introduce a local angular coordinate. More precisely, let

$$\theta_j : U_j \to \mathbb{R}, \quad j = 1, 2,$$

be smooth functions defined on open subsets $U_j \subset S^1$ such that for a point P in the corresponding open set we have the (local) identification

$$P = (\cos\theta_j(P), \sin\theta_j(P)).$$

Due to the periodicity of the trigonometric functions,

$$e^{i\theta} = e^{i(\theta+2\pi k)} \quad \text{for all } k \in \mathbb{Z},$$

the function θ_j is defined only locally; its values are determined modulo 2π. Thus, while θ_j serves as a local coordinate (or parameter) on S^1, it cannot be defined globally as a real-valued function on S^1.

However, the exterior derivative of θ_j,

$$d\theta_j,$$

is independent of the additive ambiguity. To be explicit, if $\widetilde{\theta}_j$ is any other local angular coordinate with

$$\widetilde{\theta}_j = \theta_j + 2\pi k, \quad k \in \mathbb{Z},$$

then

$$d\widetilde{\theta}_j = d(\theta_j + 2\pi k) = d\theta_j,$$

since the exterior derivative of a constant is zero. This shows that the 1-form $d\theta_j$ is uniquely defined on each S^1 factor, and thereby on the torus T^2.

In standard Cartesian coordinates on $\mathbb{R}^2 \setminus \{0\}$, a direct computation yields the coordinate expression

$$d\theta = \frac{-y\,dx + x\,dy}{x^2 + y^2}.$$

When restricted to the unit circle (or to any circle via rescaling), this expression computes the infinitesimal change in the angle.

2.3 Non-Exactness and the Integral of $d\theta$

If $d\theta$ were exact—that is, if there existed a global smooth function f on S^1 such that $df = d\theta$—then by the exactness property the integral over any closed curve would vanish:

$$\int_\gamma df = f(\gamma(1)) - f(\gamma(0)) = 0,$$

where γ is any closed loop. However, for a loop γ which represents a full rotation around the circle (or a nontrivial element in $H_1(S^1;\mathbb{Z})$), one has

$$\int_\gamma d\theta = 2\pi k,$$

with k being the winding number (typically nonzero). Hence, $d\theta$ is closed (since $d(d\theta) = 0$ identically) but not exact. The non-exactness of $d\theta$ reflects the nontrivial topology of S^1 and, by extension, of T^2.

3 Elliptic Curve

An **elliptic curve** E over a field K (like $\mathbb{C}$) is given in Weierstrass form by a cubic polynomials in two variables:

$$E: \quad y^2 = x^3 + ax + b$$

where $a, b \in K$ satisfy the nonvanishing discriminant condition $\Delta = -16(4a^3 + 27b^2) \neq 0$.

A Calculus

A.1 Differentiation of Arctangent

Compute $\dfrac{d}{dx}\ \tan^{-1} u$:

$$\begin{aligned} y &= \tan^{-1}(u), \\ \tan y &= u\ , \\ \frac{d}{dx}\ \tan y &= \frac{d}{dx}\ u\ , \\ \sec^2 y\ \frac{dy}{dx} &= \frac{du}{dx}, \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y}\frac{du}{dx} \\ &= \frac{1}{1+\tan^2 y}\frac{du}{dx} = \frac{1}{1+u^2}\frac{du}{dx}. \end{aligned}$$

Thus,

$$\frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}, \quad \text{i.e.,} \quad \frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}.$$

A.2 Generalizing Differentials

In single-variable calculus, we learn:

$$y = f(x) \quad \Longrightarrow \quad \frac{dy}{dx} = f'(x) \quad \Longrightarrow \quad dy = f'(x)\,dx.$$

If y depends on x alone, then a small change dx in x produces a corresponding small change dy in y. The amount of change in y is given by $f'(x)\,dx$.

Intuitive Idea. The derivative $f'(x)$ tells us how fast y changes when x changes, and dx tells you how much x changed. We multiply the two to get the approximate change in y.

Extending to Two Variables. Now suppose x depends on *two* quantities, say r and θ. We write:

$$x = x(r, \theta).$$

We want to figure out what happens if *both* r and θ change a little bit. How does x change?

1. (Think of r changing while θ is frozen.) If we imagine θ held fixed, then x is effectively a one-variable function of r. So a small change in r (call it dr) would change x by

$$\big(\text{rate of change w.r.t. } r\big) \times dr \;=\; \frac{\partial x}{\partial r}\, dr.$$

   The partial derivative of x with respect to r, $\frac{\partial x}{\partial r}$, tells us how fast x changes if only r changes and θ stays fixed.

2. (Think of θ changing while r is frozen.) Similarly, if r is held fixed, then x is effectively a one-variable function of θ. So a small change in θ (call it $d\theta$) would change x by

$$\big(\text{rate of change w.r.t. } \theta\big) \times d\theta \;=\; \frac{\partial x}{\partial \theta}\, d\theta.$$

3. (Add the two contributions together.) If *both* r and θ change at the same time, then the *total* change in x, which we call dx, is the sum of the two partial changes:

$$dx \;=\; \frac{\partial x}{\partial r}\, dr \;+\; \frac{\partial x}{\partial \theta}\, d\theta.$$