From Gradient Fields to Exact Forms

Notes for a Vector Calculus Student

September 21, 2025

You understand the most important concept in vector calculus: if a vector field $\mathbf{F}$ is the **gradient** of a potential function f (written $\mathbf{F} = \nabla f$), line integrals become simple. The language of **differential forms** offers a new perspective on this idea, helping us understand exactly when a field is a gradient.

1 A New Test from an Old Idea

Let's start with what you know. If a 2D vector field $\mathbf{F} = \langle P, Q \rangle$ is a gradient, then it comes from a potential function $f(x, y)$, and:

- $P = \frac{\partial f}{\partial x}$
- $Q = \frac{\partial f}{\partial y}$

Now, think back to partial derivatives. You learned about the **equality of mixed partials** (Clairaut's Theorem), which states that for a nice function f, the order of differentiation doesn't matter: $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$.

Let's apply this to P and Q:

- Differentiate P with respect to y: $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$
- Differentiate Q with respect to x: $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

Because the right-hand sides are equal, the left-hand sides must be too! This gives us a powerful and necessary condition:

If a vector field $\mathbf{F} = \langle P, Q \rangle$ is a gradient, it ***must*** *satisfy the condition $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.*

This gives us a simple test to check if a field might be conservative.

2 The Dictionary: A Quick Translation

Vector Calculus (Your Current Language)	**Differential Forms (The New Language)**
Vector Field $\mathbf{F} = \langle P, Q \rangle$	**1-Form** $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$
Conservative Field ($\mathbf{F} = \nabla f$)	**Exact** Form ($\omega = \mathrm{d}f$)
Mixed Partials Test ($\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$)	**Closed** Form ($\mathrm{d}\omega = 0$)

A form is called **"closed"** if it passes the mixed partials test. The big question is: if a form is closed, is it always exact?

Answer: Only on domains without "holes" (called **simply connected** domains).

3 The Classic Example: When the Test Isn't Enough

Let's look at a field on the plane with the origin removed, $\mathbb{R}^2 \setminus \{(0,0)\}$. This domain has a hole.

$$\mathbf{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

As a 1-form, this is:

$$\omega = \frac{-y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$$

3.1 Step 1: Does it pass our test?

Let's check the mixed partials. Here, $P = \frac{-y}{x^2+y^2}$ and $Q = \frac{x}{x^2+y^2}$.

$$\frac{\partial P}{\partial y} = \frac{(-1)(x^2+y^2) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$
$$\frac{\partial Q}{\partial x} = \frac{(1)(x^2+y^2) - (x)(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

They are equal! The test passes, so the form is **closed**. This means the field has the *local properties* of a gradient field.

3.2 Step 2: Is it actually a gradient field (conservative)?

You know that if a field is conservative, its integral around **any closed loop must be zero**. Let's test this by integrating around the unit circle, $\gamma(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$.

- $x = \cos t \implies \mathrm{d}x = -\sin t\,\mathrm{d}t$
- $y = \sin t \implies \mathrm{d}y = \cos t\,\mathrm{d}t$
- $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

The line integral is:

$$\oint_\gamma \omega = \int_0^{2\pi} \left(\frac{-\sin t}{1}(-\sin t\,\mathrm{d}t) + \frac{\cos t}{1}(\cos t\,\mathrm{d}t) \right)$$
$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,\mathrm{d}t = \int_0^{2\pi} 1\,\mathrm{d}t = 2\pi$$

3.3 The Punchline

The integral is 2π, which is **not zero**.

Conclusion: Even though the field passed our mixed-partials test (it's **closed**), it fails the path independence test. Therefore, the field is <u>not</u> conservative (it's **not exact**).

The **hole at the origin** is the culprit. The mixed partials test is a local check, and it's blind to the global problem of the hole. The hole allows the field to have a "global circulation" that prevents a single, consistent potential function $f(x, y)$ from existing over the whole domain.