Lecture Notes: Coordinates and Differentials on a Plane Curve

1 Parametrized Curve and Coordinate Chart on C

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a C^1–function. Define the plane curve

$$C \;=\; \bigl\{(x,y) \in \mathbb{R}^2 \bigm| y = f(x)\bigr\}.$$

We introduce the standard coordinate functions

$$x, y\colon \mathbb{R}^2 \to \mathbb{R}, \qquad x(x,y) = x, \quad y(x,y) = y,$$

and restrict them to C. The resulting chart on C is

$$\Phi_C\colon C \;\longrightarrow\; \mathbb{R}^2, \qquad \Phi_C\bigl(p\bigr) = \bigl(x(p),\, y(p)\bigr).$$

In particular, for

$$p = (a, f(a)) \in C,$$

we have

$$\Phi_C(p) = (a, f(a)).$$

2 Tangent Space and Fiber–Coordinates on T_pC

Parametrize C by

$$\gamma(t) = \bigl(t, f(t)\bigr).$$

At $t = a$ the velocity vector is

$$\gamma'(a) = \begin{pmatrix} 1 \\ f'(a) \end{pmatrix} \;=\; \vec{v} \;\in\; T_pC.$$

Thus the tangent space is

$$T_pC = \operatorname{span}\{\vec{v}\} = \operatorname{span}\Bigl\{\;(1, f'(a))\Bigr\} \subset T_p\mathbb{R}^2 \cong \mathbb{R}^2.$$

On $T_p\mathbb{R}^2$ we employ the dual basis $\{dx, dy\}$ defined by

$$dx(e_1) = 1,\; dx(e_2) = 0, \quad dy(e_1) = 0,\; dy(e_2) = 1,$$

where $e_1 = (1,0)$, $e_2 = (0,1)$. Each tangent vector $v = (v^1, v^2)$ then satisfies

$$dx(v) = v^1, \quad dy(v) = v^2.$$

Restricted to T_pC, we obtain the fiber–chart

$$\Phi_{T_pC} \colon T_pC \longrightarrow \mathbb{R}^2, \qquad \Phi_{T_pC}(\vec{v}) = \begin{pmatrix} dx(\vec{v}) \\ dy(\vec{v}) \end{pmatrix} = \begin{pmatrix} 1 \\ f'(a) \end{pmatrix}.$$

3 Fixed Line and Unit Direction

Choose a nonzero vector

$$w = (w_1, w_2) \in \mathbb{R}^2, \qquad \|w\| \neq 0,$$

and consider the line $L = \operatorname{span}\{w\} \subset \mathbb{R}^2$. Define the unit–direction

$$\hat{w} = \frac{w}{\|w\|} = \left(\tfrac{w_1}{\sqrt{w_1^2 + w_2^2}}, \tfrac{w_2}{\sqrt{w_1^2 + w_2^2}} \right),$$

so that $\|\hat{w}\| = 1$.

4 Definition of the Scalar–Projection 1–Form

Definition 1. *The* scalar–projection 1–form *onto the line L is the mapping*

$$\alpha : T\mathbb{R}^2 \longrightarrow \mathbb{R}, \qquad \alpha_p(v) = \langle \hat{w}, v \rangle,$$

for each $p \in \mathbb{R}^2$ and $v \in T_p\mathbb{R}^2$.

Writing $v = (v^1, v^2)$ in the dual coordinates, one obtains

$$\alpha_p(v) = \hat{w}_1 \, v^1 + \hat{w}_2 \, v^2 = \hat{w}_1 \, dx(v) + \hat{w}_2 \, dy(v).$$

Hence the global 1–form is

$$\boxed{\alpha = \hat{w}_1 \, dx \; + \; \hat{w}_2 \, dy \; = \; \frac{w_1}{\sqrt{w_1^2 + w_2^2}} \, dx \; + \; \frac{w_2}{\sqrt{w_1^2 + w_2^2}} \, dy.}$$

5 Restriction to the Curve C

Pulling back α along the inclusion $i \colon C \hookrightarrow \mathbb{R}^2$ yields

$$i^*\alpha = \hat{w}_1 \, dx\big|_C + \hat{w}_2 \, dy\big|_C,$$

which on T_pC evaluates by

$$\big(i^*\alpha\big)_p(\tau\,(1, f'(a))) = \tau \left(\hat{w}_1 + \hat{w}_2 \, f'(a)\right), \quad \tau \in \mathbb{R}.$$

Summary of Charts and Forms

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\begin{aligned}
&C \subseteq \mathbb{R}^2 \xrightarrow{\Phi_C} \mathbb{R}^2, && p \mapsto (x(p), y(p)) = (a, f(a)),\\
&T_pC \xrightarrow{\Phi_{T_pC}} \mathbb{R}^2, && \vec{v} \mapsto (dx(\vec{v}), dy(\vec{v})) = (1, f'(a)),\\
&\alpha = \hat{w}_1\, dx + \hat{w}_2\, dy, \quad i^*\alpha \in \Omega^1(C).
\end{aligned}
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