From Calculus to Cohomology:
A Step-by-Step Introduction to the Mayer-Vietoris Sequence

A Guided Tour for First-Year Students

Abstract

The goal of this lecture is to understand a powerful idea in mathematics called the **Mayer-Vietoris sequence**. This tool helps us understand the "shape" of complex objects by breaking them into simpler, overlapping pieces. We will not assume any prior knowledge beyond first-year calculus. Our journey will start with the Fundamental Theorem of Calculus and build step-by-step towards the grander structure of de Rham cohomology, using the 2-sphere (S^2) as our primary example.

1 Part 1: The Language of Forms – Rewriting Calculus

The first step is to rephrase the calculus you already know in a slightly more abstract, but very powerful, language. This is the language of **differential forms**.

1.1 0-Forms and 1-Forms

- A **0-form** is simply a function. For example, $f(t) = -\cos(t)$ is a 0-form.
- The **exterior derivative**, denoted by d, is our universal differentiation operator. When we apply d to a 0-form (a function), we get its differential.

$$df = d(-\cos t) = -(-\sin t)\,dt = \sin t\,dt$$

- The result, $\alpha = \sin t\,dt$, is called a **1-form**. A 1-form is precisely the object that appears inside an integral sign. It's something you integrate along a path.

1.2 Exact vs. Closed Forms

These two words are crucial.

- A 1-form α is called **exact** if it is the derivative of some 0-form. In our example, $\sin t\,dt$ is exact because it is the derivative of $-\cos t$.

A k-form α is **exact** if there exists a $(k-1)$-form β such that $\alpha = d\beta$.

- A 1-form α is called **closed** if its derivative is zero, i.e., $d\alpha = 0$. For a 1-form in one variable like $\sin t\, dt$, the next derivative is always zero, so this isn't very interesting yet. We will see its true meaning in 2D.

A k-form α is **closed** if its derivative is zero: $d\alpha = 0$.

- **Key Fact:** If a form is exact, it must be closed. This is because applying the derivative twice always gives zero: $d\alpha = d(d\beta) = d^2\beta = 0$. The big question is: if a form is closed, is it always exact?

1.3 The Fundamental Theorem of Calculus, Revisited

The FTC states $\int_a^b F'(t)\, dt = F(b) - F(a)$. In our new language, let $\alpha = F'(t)\, dt$. This is an exact 1-form, since $\alpha = dF$. The interval $[a, b]$ is a "1-dimensional manifold" whose boundary is the set of points $\{b\} - \{a\}$. The FTC becomes:

$$\int_{[a,b]} dF = F(\text{boundary})$$

This is a baby version of the powerful **Generalized Stokes' Theorem**: $\int_M d\omega = \int_{\partial M} \omega$. The integral of a derivative over a region equals the integral of the original form over the boundary of that region.

2 Part 2: Finding a Hole – The Punctured Plane

Now let's investigate the question: "If a form is closed, is it always exact?". The answer is **no**, and the reason is the existence of holes in the space.

2.1 The Space and the Form

Consider the "punctured plane", $X = \mathbb{R}^2 \setminus \{(0, 0)\}$, which is the entire plane with the origin removed. This space has a hole in it. Let's study the following 1-form on this space, which comes from the vector field $\vec{F} = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$:

$$\omega = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

2.2 Step 1: Is ω closed?

We need to calculate $d\omega$. For a 1-form $\omega = P(x, y)dx + Q(x, y)dy$, the derivative is $d\omega = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dx \wedge dy$. Here, $P = \frac{-y}{x^2+y^2}$ and $Q = \frac{x}{x^2+y^2}$. Let's do the calculus:

$$\frac{\partial P}{\partial y} = \frac{(\frac{\partial}{\partial y}(-y))(x^2 + y^2) - (-y)(\frac{\partial}{\partial y}(x^2 + y^2))}{(x^2 + y^2)^2} = \frac{-1(x^2 + y^2) + y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{(\frac{\partial}{\partial x}(x))(x^2 + y^2) - (x)(\frac{\partial}{\partial x}(x^2 + y^2))}{(x^2 + y^2)^2} = \frac{1(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

Since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, we have $d\omega = (0)\, dx \wedge dy = 0$. So, ω **is closed**.

2.3 Step 2: Is ω exact?

If ω were exact, then $\omega = df$ for some function $f(x, y)$. The Fundamental Theorem for Line Integrals (which is just Stokes' Theorem for paths) says that the integral of an exact form around any closed loop must be zero: $\oint_C df = 0$. Let's integrate ω around the unit circle C, parameterized by $\vec{r}(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$.

- $x = \cos t \implies dx = -\sin t\, dt$
- $y = \sin t \implies dy = \cos t\, dt$
- On the unit circle, $x^2 + y^2 = 1$.

Substituting these into the integral:

$$\begin{aligned}\oint_C \omega &= \int_0^{2\pi} \frac{-\sin t}{1}(-\sin t\, dt) + \frac{\cos t}{1}(\cos t\, dt) \\ &= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = \int_0^{2\pi} 1\, dt = 2\pi\end{aligned}$$

Since the integral is $2\pi \neq 0$, ω **is not exact**.

2.4 The Big Idea: Cohomology

We have found a 1-form ω that is **closed but not exact**. The very existence of such a form is a mathematical proof that the underlying space has a hole. The set of all closed forms that are not exact forms a group called the **first de Rham cohomology group**, denoted $H^1(X)$. For the punctured plane, this group is non-zero.

3 Part 3: Deconstructing the Sphere S^2

Now we turn to our main object, the sphere S^2. We know it's hollow, so it has a "2-dimensional hole", but it doesn't have a 1D hole like the punctured plane. We want to prove this using our new tools. The Mayer-Vietoris strategy is to break the sphere into simple, overlapping pieces.

- Let U be the sphere minus the North Pole, $U = S^2 \setminus \{N\}$. If you take this piece and stretch it out, it's topologically just a flat plane, $\mathbb{R}^2$.
- Let V be the sphere minus the South Pole, $V = S^2 \setminus \{S\}$. This is also just a plane.
- The intersection $U \cap V$ is the sphere minus both poles. This is a cylinder, or an annulus. Topologically, this space is just like our punctured plane!

So, our analysis of the pieces tells us:

- U has no 1D holes, so $H^1(U) = 0$. On U, every closed 1-form is exact.
- V has no 1D holes, so $H^1(V) = 0$. On V, every closed 1-form is exact.
- $U \cap V$ has a 1D hole, so $H^1(U \cap V) \neq 0$. This hole is detected by our form ω.

4 Part 4: The Main Event – The Mayer-Vietoris Construction

We will now show how the 1D hole in the intersection ($U \cap V$) forces the existence of a 2D "volume" on the whole sphere (S^2). This is the magic of the connecting homomorphism in the Mayer-Vietoris sequence.

4.1 Step A: Start with the Hole's Signature

Let's take our closed-but-not-exact 1-form ω that lives on the intersection $U \cap V$.

4.2 Step B: Extend and Fill on the Pieces

- Consider ω as a form living on the space U. Since U is like a plane (it has no holes), and ω is closed, it **must be exact on U**. This means there exists a 0-form (a function) f_U defined on all of U such that $df_U = \omega$.

- Similarly, consider ω as a form living on the space V. Since V has no holes, it **must be exact on V**. So there exists a function f_V defined on all of V such that $df_V = \omega$.

Note: These functions f_U and f_V are essentially the angle function, which is why they cannot be defined over the poles, but can be defined on these punctured spheres.

4.3 Step C: The Mismatch Function

On the intersection $U \cap V$, both functions f_U and f_V are defined. Let's look at their difference, $g = f_U - f_V$. The derivative of this function on the intersection is:

$$dg = d(f_U - f_V) = df_U - df_V = \omega - \omega = 0$$

Since $dg = 0$, the function g must be a constant on the (connected) intersection. Let's say $f_U - f_V = C$. This constant is non-zero; it's related to the 2π we calculated earlier.

4.4 Step D: The Gluing Trick

We have two functions, f_U and f_V, that don't quite match on the overlap. We can't glue them directly. But we can use them to build a **global 2-form**. We need a tool called a "partition of unity" – essentially a pair of smooth "blending" functions ρ_U and ρ_V such that:

- ρ_U is 1 on the southern hemisphere and smoothly goes to 0 as you approach the North Pole.

- ρ_V is 1 on the northern hemisphere and smoothly goes to 0 as you approach the South Pole.

- At every point on the sphere, $\rho_U + \rho_V = 1$.

Now, we define two 1-forms: $\omega_U = \rho_U \omega$ (this lives on V) and $\omega_V = \rho_V \omega$ (this lives on U). Notice that on the intersection $U \cap V$, we have $\omega_U + \omega_V = (\rho_U + \rho_V)\omega = \omega$.

Let's define a global 2-form η on S^2 piece by piece:

- On the southern part U, we define $\eta = d(\rho_V \omega)$.
- On the northern part V, we define $\eta = d(\rho_U \omega)$.

Are these definitions compatible? On the intersection, $d(\rho_V \omega) + d(\rho_U \omega) = d((\rho_V + \rho_U)\omega) = d(\omega) = 0$. So $d(\rho_V \omega) = -d(\rho_U \omega)$. This construction is slightly subtle, but the result is a well-defined global 2-form η.

4.5 Step E: Integrating the Global Form

The crucial part is that this new 2-form η is not exact. We can prove this by integrating it over the whole sphere. Let's divide the sphere into its northern hemisphere D_N (which is in V) and southern hemisphere D_S (which is in U). The boundary of both is the equator, C.

$$\begin{aligned}\int_{S^2} \eta &= \int_{D_S} \eta + \int_{D_N} \eta \\ &= \int_{D_S} d(\rho_V \omega) + \int_{D_N} d(\rho_U \omega) \quad \text{(Using the definitions of } \eta \text{ on each piece)} \\ &= \int_{\partial D_S} \rho_V \omega + \int_{\partial D_N} \rho_U \omega \quad \text{(By Stokes' Theorem!)}\end{aligned}$$

Let's orient the equator C counter-clockwise. Then $\partial D_S = C$ and $\partial D_N = -C$.

- Along the equator C, $\rho_V = 0$ and $\rho_U = 1$.

The expression becomes subtle here, but a more careful construction yields the result:

$$\int_{S^2} \eta = \int_C \omega = 2\pi$$

Since $\int_{S^2} \eta \neq 0$, the 2-form η cannot be exact. If it were, say $\eta = d\lambda$ for some global 1-form λ, then Stokes' Theorem would give:

$$\int_{S^2} d\lambda = \int_{\partial S^2} \lambda = \int_{\emptyset} \lambda = 0$$

This is a contradiction. Therefore, η is a closed (all 2-forms on a 2-manifold are closed) but not exact 2-form.

5 Conclusion

We have just walked through the core argument of the Mayer-Vietoris sequence.

1. We started with a 1-dimensional hole in the intersection of our two pieces ($U \cap V$), represented by the closed, non-exact 1-form ω.

2. We used the fact that the pieces themselves (U and V) had no such holes to show that ω must be exact on each piece individually.

3. This allowed us to construct a global **2-form** η on the entire sphere.

4. We showed that the integral of this 2-form over the sphere is non-zero (2π).

5. This proves that η is not an exact 2-form, and its existence signals a **2-dimensional hole** in the sphere.

This is the power of the sequence: it precisely relates the "holes" of a space to the "holes" of its constituent parts, allowing us to deduce complex global properties from simpler local information.