Line Integral I

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March 1, 2025

We cover the following topics in this note.

- TBA.

Preliminaries

Let $I = [a, b] \subset \mathbb{R}$ and let

$$\gamma : I \longrightarrow \mathbb{R}^n$$

be a piecewise C^1 mapping whose image $\gamma(I)$ is the curve along which integration is performed. For any partition

$$\mathcal{P} = \{t_0, t_1, \ldots, t_N\} \quad \text{with} \quad a = t_0 < t_1 < \cdots < t_N = b,$$

define the mesh of the partition by

$$\|\mathcal{P}\| := \max_{1 \le i \le N} (t_i - t_{i-1}).$$

For each subinterval $[t_{i-1}, t_i]$, choose a sample point $\xi_i \in [t_{i-1}, t_i]$.

Line Integral of a Scalar Function

Let

$$f : \gamma(I) \longrightarrow \mathbb{R}$$

be a continuous function. Define the Riemann sum by

$$S(\mathcal{P}, \{\xi_i\}) := \sum_{i=1}^{N} f\left(\gamma(\xi_i)\right) \|\gamma(t_i) - \gamma(t_{i-1})\|.$$

Then the *line integral of f along γ with respect to arc length* is defined by

$$\int_\gamma f \, ds := \lim_{\|\mathcal{P}\| \to 0} S(\mathcal{P}, \{\xi_i\}),$$

provided that the limit exists and is independent of the choice of partition $\mathcal{P}$ and sample points $\{\xi_i\}$.

In the case that γ is continuously differentiable, the definition is equivalent to

$$\int_\gamma f\,ds = \int_a^b f\big(\gamma(t)\big)\,\|\gamma'(t)\|\,dt.$$

Line Integral of a Vector Field

Let

$$\mathbf{F} : \gamma(I) \to \mathbb{R}^n$$

be a continuous vector field. Using the same partition $\mathcal{P}$ and sample points $\{\xi_i\}$ as above, define the Riemann sum

$$S(\mathcal{P}, \{\xi_i\}) := \sum_{i=1}^{N} \mathbf{F}\big(\gamma(\xi_i)\big) \cdot \big(\gamma(t_i) - \gamma(t_{i-1})\big),$$

where "$\cdot$" denotes the standard Euclidean dot product. The *line integral of* $\mathbf{F}$ *along* γ is then defined by

$$\int_\gamma \mathbf{F} \cdot d\mathbf{r} := \lim_{\|\mathcal{P}\| \to 0} S(\mathcal{P}, \{\xi_i\}),$$

provided that the limit exists independently of the partition and the choice of sample points.

If γ is continuously differentiable, this definition is equivalent to

$$\int_\gamma \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}\big(\gamma(t)\big) \cdot \gamma'(t)\,dt.$$

Formal Definition (Scalar Line Integral):

Let

$$\gamma : [a,b] \longrightarrow \mathbb{R}^n$$

be a piecewise continuously differentiable (and hence rectifiable) parameterization of a curve $C \subset \mathbb{R}^n$, and let

$$f : C \to \mathbb{R}$$

be a continuous function. Denote by $\|\gamma'(t)\|$ the norm of the derivative of γ at t. Then the *line integral* (or *integral of f with respect to arc length*) is defined by

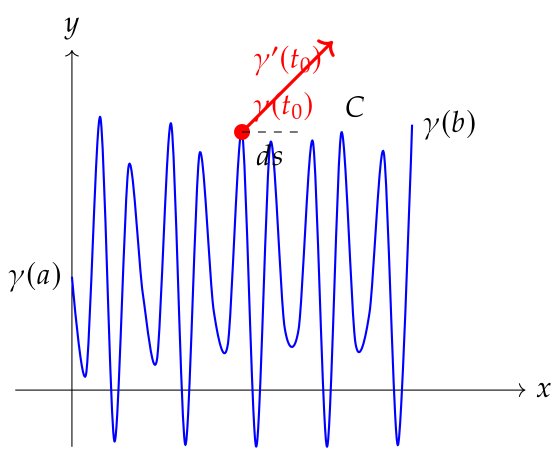
$$\int_C f\,ds \;:=\; \lim_{\|\Delta\|\to 0} \sum_{i=1}^{N} f\Big(\gamma(t_i^*)\Big)\,\Big\|\gamma(t_i) - \gamma(t_{i-1})\Big\|,$$

where $\{t_0, t_1, \ldots, t_N\}$ is an arbitrary partition of $[a,b]$ with mesh $\|\Delta\| = \max_{1\le i\le N}(t_i - t_{i-1})$ and $t_i^* \in [t_{i-1}, t_i]$. Under the hypothesis that γ is continuously differentiable, one may equivalently express the line integral as

$$\int_C f\,ds \;=\; \int_a^b f\big(\gamma(t)\big)\,\|\gamma'(t)\|\,dt.$$

A similar construction yields the definition of the line integral of a vector field $\mathbf{F} : U \subset \mathbb{R}^n \to \mathbb{R}^n$ along C:

$$\int_C \mathbf{F}\cdot d\mathbf{s} \;:=\; \int_a^b \mathbf{F}\big(\gamma(t)\big)\cdot\gamma'(t)\,dt.$$



References

[1] 수학의 즐거움, Enjoying Math. “[리만의 복소해석을 토대로 얻는 내 수학적 시야] 0. 오리엔테이션” YouTube Video, 1:49:27. Published September 4, 2023. URL: https://www.youtube.com/watch?v=EovxcF_DG_k&list=PL4m4z_pFWq2ob-P9m3SQZPyHTaJbbkvdz.

[2] 수학의 즐거움, Enjoying Math. “[리만의 복소해석을 토대로 얻는 내 수학적 시야] 1. 선적분의 정의에 대한 디스커션” YouTube Video, 1:58:19. Published September 11, 2023. URL: https://www.youtube.com/watch?v=zoalSFi1RKo&list=PL4m4z_pFWq2ob-P9m3SQZPyHTaJbbkvdz&index=2.