A sufficient condition on successful invasion by the predator

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Abstract

In this paper, we provide a sufficient condition on successful invasion by the predator. Specially, we obtain the persistence of traveling wave solutions of predator-prey system, in which the predator can survive without the predation of the prey. This proof heavily depends on comparison principle of scalar monostable equation, the rescaling method and phase-plane analysis.

Keywords: Predator-prey system; Traveling wave solutions; Persistence

Mathematics Subject Classification: 35K57 · 35C07

1. Introduction

In the natural world, we are more concerned with the ideal scenario in which the predator and the prey may cohabit. As a consequence, many researchers investigate the process of population ecology propagation, which may be well described by traveling wave solutions connecting the prey-present state and the coexistence state, also named invasion waves. At present, the method of dealing with the existence of traveling wave solutions, in general, includes the shoot argument [3], Schauder's fixed-point theorem[6] and Conley index [5]. It is generally accepted that the asymptotic behavior of traveling wave solutions at positive infinity often depends on an appropriate and technicality Lyapunov function, nevertheless, construction of the Lyapunov function is not easy, particularly for the non-monotonic functional response. Just as said in [8], it is sufficient to study the persistence of traveling wave solutions if we only want to know whether the invasion is successful and what the invasion speed is.

Therefore, the purpose of this work is to investigate the persistence of traveling wave solutions to general predator-prey system with diffusion

$$\begin{cases}
 u_t = u_{xx} + uF(u, v), \\
 v_t = dv_{xx} + vG(u, v),
\end{cases}$$
(1.1)

where u and v stand for the population densities of the prey and the predator, respectively. d > 0 denotes the ratio of the diffusion of the predator to that of the prey, F(u, v) and G(u, v)

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describe the interaction between the predator and the prey, and the assumptions for these are given below.

Assumption 1.1. F(u,v) and G(u,v) are C^2 functions and satisfy

- (a) There is μ such that $G(0,v)(v-\mu)<0$ for $v\in[0,\mu)\cup(\mu,+\infty)$.
- (b) There is v_0 such that $G(u, v_0) < 0$ for $u \in [0, 1]$, and

$$G(1,0) \ge G(u,v) \ge G(0,v)$$
 for $(u,v) \in [0,1] \times [0,v_0]$.

(c)
$$F(1,0) = 0$$
, $F(u,0) > 0$ for $u \in [0,1)$, and $F(1,v) < 0$ for $v \in (0,v_0]$.

In view of Assumptions 1.1, u-equation and v-equation are reduced to a scalar monostable reaction-diffusion equation in the absence of the predator and the prey, respectively. And as for the corresponding kinetic system, there are always three boundary equilibria (1,0), (0,0) and $(0,\mu)$. Some classical examples satisfying the above assumptions are as follows

(1) Lotka-Volterra predator-prey system: for given a, r, b > 0

$$F(u, v) = 1 - u - av$$
 and $G(u, v) = r(1 - v + bu)$.

(2) Modified Leslie-Gower predator-prey system: for given $a, e_1, r, e_2 > 0$

$$F(u, v) = 1 - u - \frac{av}{u + e_1}$$
 and $G(u, v) = r\left(1 - \frac{v}{u + e_2}\right)$.

A positive solution is called a traveling wave solution if solution of (1.1) has the form

$$(u, v)(x, t) = (\phi, \psi)(z), z = x + ct,$$

where c is the wave speed. Automatically, $(\phi, \psi)(z)$ satisfies

$$\begin{cases} \phi''(z) - c\phi'(z) + \phi(z)F(\phi,\psi)(z) = 0, \\ d\psi''(z) - c\psi'(z) + \psi(z)G(\phi,\psi)(z) = 0. \end{cases}$$
(1.2)

Traveling wave solution is persistent if there exist two positive constants M_1 and M_2 such that

$$\begin{cases} (\phi, \psi)(-\infty) = (1, 0), \\ M_1 < \liminf_{z \to +\infty} \phi(z) \le \limsup_{z \to +\infty} \phi(z) < M_2, \\ M_1 < \liminf_{z \to +\infty} \psi(z) \le \limsup_{z \to +\infty} \psi(z) < M_2. \end{cases}$$

In this paper, we always assume that $c \geq c_* := 2\sqrt{dG(1,0)}$ and define

$$\lambda_1 := \frac{1}{2d} \left(c - \sqrt{c^2 - 4dG(1, 0)} \right) > 0.$$

Now, we state the main result.

Theorem 1.1. For $c \ge c_*$, if $F(0, \mu) > 0$, then traveling wave solutions of system (1.1) are persistent.

The proof of Theorem 1.1 is given in Section 2. Firstly, we obtain the existence of a positive solution of system (1.2) from Theorem 2.1 in [1]. Then based on the spreading theory on the scalar monostable equation and comparison principle, we obtain that component $\psi(z)$ of traveling wave solutions is persistent. Meanwhile, inspired by [7], we show that component $\phi(z)$ of traveling wave solutions is persistent by using the rescaling method and phase-plane analysis.

2. The proof of Theorem 1.1

Let us begin this section with the existence result, which may be deduced immediately from Theorem 2.1 in [1].

Lemma 2.1. For $c \geq c_*$, system (1.2) admits a positive solution $(\phi, \psi)(z)$ satisfying

$$(\phi, \phi', \psi, \psi')(-\infty) = (1, 0, 0, 0)$$
 and $0 < \phi(z) < 1, 0 < \psi(z) < v_0$ over \mathbb{R} .

Moreover, $\phi'(z)$ and $\psi'(z)$ are bounded over \mathbb{R} .

To proceed, we require a priori estimate on the inhomogeneous linear equation

$$w''(z) + \alpha w'(z) + f(z)w(z) = h(z), \tag{2.1}$$

where α is a positive constant and f, h are continuous functions on [a, b].

Proposition 2.2. (Lemma 3.3 of [4]) Assume that $w(z) \in C([a,b]) \cap C^2((a,b))$ satisfies (2.1) in (a,b) and

$$-\gamma_1 \le f \le 0$$
 and $|h| \le \gamma_2$ on $[a, b]$,

for some positive constants γ_i , i = 1, 2. If $||w||_{C([a,b])} \leq \gamma_3$ for positive constant γ_3 , then there exists a positive constant $\gamma_4(\alpha, \gamma_1, \gamma_2, \gamma_3, a, b)$ such that

$$||w'||_{C([a,b])} \le \gamma_4(\alpha, \gamma_1, \gamma_2, \gamma_3, a, b).$$

The following two lemmas describe the persistence of travelling wave solutions.

Lemma 2.3. $\liminf_{z\to+\infty}\psi(z)\geq\mu$.

Proof. From Lemma 2.1, we have $\psi(z) > 0$ over \mathbb{R} and $\psi(-\infty) = 0$ for $c \geq c_*$, then there is a constant $\zeta \in (0, \mu)$ such that $\psi(z_1) = \zeta$ for $z_1 \in \mathbb{R}$. Since $(\phi, \psi)(z + a)$ is also traveling wave solution for any $a \in \mathbb{R}$, we consider $z_1 = 0$ without losing generality. Note that $\psi(z)$ is uniformly continuous over \mathbb{R} owing to the boundedness of $\psi'(z)$, there is a $\epsilon > 0$ such that $\psi(z) > \zeta/2$ for $z \in (-\epsilon, \epsilon)$. Now, we consider initial value problem

$$\begin{cases}
 w_t = d\Delta w + wg(w), \\
 w(0, x) = \varphi(x),
\end{cases}$$
(2.2)

where g(w) := G(0, w) and uniformly continuous function $\varphi(x)$ satisfies the conditions:

- (1) $\varphi(x) = \zeta/2$ for $x \in [-\epsilon/2, \epsilon/2]$ and $\varphi(x) = 0$ for $\mathbb{R} \setminus (-\epsilon, \epsilon)$,
- (2) $\varphi(x)$ is increasing for $x \in [-\epsilon, -\epsilon/2]$ and decreasing for $x \in [\epsilon/2, \epsilon]$.

From Assumption 1.1-(a), we know that

$$g(\mu) = 0$$
 and $g(v) > 0$ for $v \in [0, \mu)$.

According to Proposition 3.1 in [2], there exists a $c^*(d, g(v)) \ge 2\sqrt{dg(0)}$ such that

$$\lim_{t \to +\infty} \inf_{|x| < c^*(d, g(v))t} w(x, t) = \mu.$$

Thanks to $\phi(z) > 0$ and Assumption 1.1-(b), then $v(x,t) := \psi(z)$ satisfies

$$\begin{cases} v_t \ge d\Delta v + vg(v), \\ v(x,0) = \psi(x). \end{cases}$$

Obviously, $\psi(x) \geq \varphi(x)$ over \mathbb{R} . Comparison principle suggests that for $c \geq c_*$

$$\liminf_{z\to +\infty} \psi(z) = \liminf_{z\to +\infty} v(0,z/c) \geq \liminf_{z\to +\infty} w(0,z/c) \geq \lim_{t\to +\infty} \inf_{|x|< c^*(d,g(v))t} w(x,t) = \mu.$$

Hence, we complete the proof.

Lemma 2.4. $\liminf_{z\to +\infty} \phi(z) > 0 \text{ if } F(0,\mu) > 0.$

Proof. For contradiction, we assume that there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ satisfying $z_n \to +\infty$ as $n \to +\infty$ such that $\phi(z_n) \to 0$ as $n \to +\infty$. Let $(\phi_n, \psi_n)(z) := (\phi, \psi)(z_n + z)$, then $\phi''_n(z)$ and $\psi''_n(z)$ are bounded over \mathbb{R} from Lemma 2.1. Differentiating system (1.2), we can express $\phi'''_n(z)$ and $\psi'''_n(z)$ in terms of $\phi_n(z), \phi'_n(z), \phi''_n(z), \psi_n(z), \psi'_n(z)$ and $\psi''_n(z)$, hence $\phi'''_n(z)$ and $\psi'''_n(z)$ are bounded over \mathbb{R} . Therefore, by Arzelà-Ascoli theorem, up to extracting a subsequence, there are some functions $\phi_\infty(z)$ and $\psi_\infty(z)$ such that $\phi_n(z) \to \phi_\infty(z)$ and $\psi_n(z) \to \psi_\infty(z)$ in $C^2_{loc}(\mathbb{R})$ as $n \to +\infty$. Moreover, $\phi_\infty(z)$ satisfies $0 \le \phi_\infty(z) \le 1$ over \mathbb{R} and

$$\phi_{\infty}''(z) - c\phi_{\infty}'(z) + \phi_{\infty}(z)F(\phi_{\infty}, \psi_{\infty})(z) = 0.$$

Note that $\phi_{\infty}(0) = 0$ due to $\phi(z_n) \to 0$ as $n \to +\infty$, we have $\phi_{\infty}(z) \equiv 0$ over \mathbb{R} . Hence, $\psi_{\infty}(z)$ should satisfy $0 \le \psi_{\infty}(z) \le v_0$ over \mathbb{R} and

$$d\psi_{\infty}''(z) - c\psi_{\infty}'(z) + \psi_{\infty}(z)G(0, \psi_{\infty}(z)) = 0.$$

It follows from Lemma 2.3 that $\psi_{\infty}(z) \ge \mu$ for $z \ge 0$, one can readily check that $0 < \psi_{\infty}(z) \le v_0$ over \mathbb{R} . We will show that $(\psi_{\infty}, \psi'_{\infty})(0) = (\mu, 0)$ by phase-plane analysis. In order to do it, we set $\chi_{\infty}(z) := \psi'_{\infty}(z)$ and rewrite ψ_{∞} -equation as a system of first order ODEs in \mathbb{R}^2 ,

$$\begin{cases} \psi_{\infty}'(z) = \chi_{\infty}(z), \\ d\chi_{\infty}'(z) = c\chi_{\infty}(z) - \psi_{\infty}(z)G(0, \psi_{\infty}(z)). \end{cases}$$
 (2.3)

Moreover, some sets are given by

$$A_{1} := \{ (\psi_{\infty}, \chi_{\infty})(z) : \psi_{\infty}(z) > \mu, \chi_{\infty}(z) > 0 \},$$

$$A_{2} := \{ (\psi_{\infty}, \chi_{\infty})(z) : \psi_{\infty}(z) = \mu, \chi_{\infty}(z) > 0 \},$$

$$A_{3} := \{ (\psi_{\infty}, \chi_{\infty})(z) : \psi_{\infty}(z) > \mu, \chi_{\infty}(z) = 0 \},$$

$$A_{4} := \{ (\psi_{\infty}, \chi_{\infty})(z) : \psi_{\infty}(z) = \mu, \chi_{\infty}(z) < 0 \},$$

$$A_{5} := \{ (\psi_{\infty}, \chi_{\infty})(z) : \psi_{\infty}(z) > \mu, \chi_{\infty}(z) < 0 \},$$

$$A_{6} := \{ (\psi_{\infty}, \chi_{\infty})(z) : \psi_{\infty}(z) = \mu, \chi_{\infty}(z) = 0 \}.$$

If there exists a $z_0 \in \mathbb{R}$ such that $(\psi_\infty, \chi_\infty)(z_0) \in A_1$, one can see that $\psi_\infty(+\infty) = +\infty$, which contradicts to the boundedness of $\psi_\infty(z)$. Hence we have $(\psi_\infty, \chi_\infty)(z) \notin A_1$ over \mathbb{R} . Meanwhile, if there exists a $z_0 \in \mathbb{R}$ such that $(\psi_\infty, \chi_\infty)(z_0) \in A_2 \cup A_3$, obviously, we have $(\psi_\infty, \chi_\infty)(z_0^+) \in A_1$. Thus, $(\psi_\infty, \chi_\infty)(z) \notin A_1 \cup A_2 \cup A_3$ over \mathbb{R} . Undoubtedly, $(\psi_\infty, \chi_\infty)(0) \notin A_1 \cup A_2 \cup A_3$. Let us continuous to show that $(\psi_\infty, \chi_\infty)(0) \notin A_4 \cup A_5$. If $(\psi_\infty, \chi_\infty)(0) \in A_4$, clearly, $\psi_\infty(0^+) < \mu$, which contradicts to $\psi_\infty(z) \ge \mu$ for $z \ge 0$. On the other hand, if $(\psi_\infty, \chi_\infty)(0) \in A_5$, then $\chi_\infty(z) < 0$ for $z \le 0$ owing to $(\psi_\infty, \chi_\infty)(z) \notin A_3$ over \mathbb{R} , which suggests that for $z \le 0$

$$(\psi_{\infty}, \chi_{\infty})(z) \in \{(\psi_{\infty}, \chi_{\infty})(z) : v_0 \ge \psi_{\infty}(z) > \mu, \chi_{\infty}(z) < 0\}.$$

Thus, monotone bounded principle explains that $\psi_{\infty}(-\infty)$ exists and $\psi_{\infty}(-\infty) > \psi_{\infty}(0) \ge \mu$. In the meantime, we claim that $\vartheta_{-} \le \chi_{\infty}(z) \le \vartheta_{+}$ over \mathbb{R} , where

$$\vartheta_{-} := \min_{\eta \in [0,v_0]} \left\{ \eta G(0,\eta)/c \right\} < 0 < \vartheta_{+} := \max_{\eta \in [0,v_0]} \left\{ \eta G(0,\eta)/c \right\}$$

from Assumptions 1.1-(a). We only show the inequality $\chi_{\infty}(z) \leq \vartheta_{+}$ here since the inequality $\chi_{\infty}(z) \geq \vartheta_{-}$ then can be obtain in same way. For contradiction, we assume that there exists a z_0 such that $\chi_{\infty}(z_0) > \vartheta_{+}$, then one can easily see that $\chi_{\infty}(z) > \vartheta_{+}$ for $z > z_0$ from (2.3), which contradicts to the boundedness of $\psi_{\infty}(z)$. This claim gives the boundedness of $\chi'_{\infty}(z)$, and differentiating χ_{∞} -equation further yields the boundedness of $\chi''_{\infty}(z)$, hence $(\chi_{\infty}, \chi'_{\infty})(-\infty) = 0$ by Barbalat Lemma, which suggests that $G(0, \psi_{\infty}(-\infty)) = 0$. However, this is impossible since $G(0, v)(v - \mu) < 0$ for $v > \mu$ in Assumption 1.1-(a). Therefore, there must be $(\psi_{\infty}, \chi_{\infty})(0) \in A_6$. Note that A_6 is an invariant set of system (2.3), then $\psi_{\infty}(z) \equiv \mu$ over \mathbb{R} . Now, we define

$$\omega_n(z) = \frac{\phi_n(z)}{\phi(z_n)} = \frac{\phi(z+z_n)}{\phi(z_n)} = \exp\left\{\int_{z_n}^{z_n+z} \frac{\phi'(z)}{\phi(z)} dz\right\}. \tag{2.4}$$

Clearly, $\omega_n(z) > 0$ over \mathbb{R} . It follows from ϕ_n -equation that

$$\omega_n''(z) - c\omega_n'(z) + \omega_n(z)F(\phi_n, \psi_n)(z) = 0.$$
(2.5)

We claim that function $\varpi(z) = \phi'(z)/\phi(z)$ is bounded over \mathbb{R} . In fact, since $\phi(z)$ and $\psi(z)$ are bounded, there exists a m > 0 such that $F(\phi, \psi)(z) \ge -m$ over \mathbb{R} . Automatically,

$$\varpi'(z) = c\varpi(z) - \varpi^2(z) - F(\phi, \psi)(z)$$

$$\leq c\varpi(z) - \varpi^2(z) + m = -(\varpi(z) - \pi_+)(\varpi(z) - \pi_-),$$

where

$$\pi_{+} = \frac{1}{2} \left(c + \sqrt{c^2 + 4m} \right) \text{ and } \pi_{-} = \frac{1}{2} \left(c - \sqrt{c^2 + 4m} \right).$$

Owing to $\varpi(-\infty) = 0$, there exists a $z_0 \ll -1$ such that $\varpi(z) < \pi_+$ for $z \in (-\infty, z_0]$. Since function $\varpi_+(z) := \pi_+$ satisfies $\varpi'_+(z) = -(\varpi_+(z) - \pi_+)(\varpi(z)_+ - \pi_-)$, we have $\varpi(z) \leq \varpi_+(z)$ for $z \in (z_0, +\infty]$ by comparison principle. Hence $\varpi(z) \leq \pi_+$ over \mathbb{R} . Let us continuous to show $\varpi(z) > -\pi_+$ over \mathbb{R} . For contradiction, we assume that $\varpi(z_1) \leq -\pi_+$ for $z_1 \in \mathbb{R}$. Let $\varpi_-(z)$ be solution of the following equation

$$\begin{cases}
\varpi'_{-}(z) = c\varpi_{-}(z) - \varpi_{-}^{2}(z) + m, \\
\varpi_{-}(z_{1}) = \varpi(z_{1}).
\end{cases}$$
(2.6)

By solving (2.6), we get

$$\varpi_{-}(z) = \frac{\pi_{+} - \pi_{-}e^{-\sqrt{c^{2}+4m}(z-z_{2})}}{1 - e^{-\sqrt{c^{2}+4m}(z-z_{2})}},$$

where

$$z_2 = z_1 + \frac{1}{\sqrt{c^2 + 4m}} \ln \left(\frac{\pi_+ - \varpi_-(z_1)}{\pi_- - \varpi_-(z_1)} \right) > z_1.$$

Hence $\varpi_{-}(z) \to -\infty$ as $z \to z_{2}^{-}$. On the other hand, comparison principle yields that $\varpi_{-}(z) \geq \varpi(z)$ for $z > z_{1}$, which implies that $\varpi(z) \to -\infty$ as $z \to z_{3}^{-}$ with $z_{3} \in (z_{1}, z_{2}]$. However, $\varpi(z)$ is defined in all $z \in \mathbb{R}$, hence the claim is valid. According to the above claim, $\omega_{n}(z)$ is locally uniformly bounded. Let $\upsilon_{n}(z) := \omega_{n}(-z)$, then for any positive constant β

$$\upsilon_n''(z) + c\upsilon_n'(z) - \beta\upsilon_n(z) = -\upsilon_n(z) \left(F(\phi_n, \psi_n)(-z) + \beta \right).$$

Hence $v'_n(z)$ is locally uniformly bounded by using Proposition 2.2, in other words, $\omega'_n(z)$ is locally uniformly bounded. ω_n -equation further gives that $\omega''_n(z)$ is locally uniformly bounded, so is its derivative by differentiating (2.5). Therefore, with the aid of Arzelà-Ascoli theorem, up to extracting a subsequence, there is a function $\omega_{\infty}(z)$ such that $\omega_n(z) \to \omega_{\infty}(z)$ in $C^2_{loc}(\mathbb{R})$ as $n \to +\infty$. Hence we obtain a linear equation

$$\omega_{\infty}''(z) - c\omega_{\infty}'(z) + F(0, \mu)\omega_{\infty}(z) = 0$$
(2.7)

due to $(\phi_{\infty}, \psi_{\infty})(z) \equiv (0, \mu)$ over \mathbb{R} . Note that $\omega_n(z) > 0$ over \mathbb{R} , we have $\omega_{\infty}(z) \geq 0$ over \mathbb{R} , meanwhile, $\omega_{\infty}(z) > 0$ over \mathbb{R} thanks to $\omega_{\infty}(0) = 1$. Firstly, let us consider $c_0 := 2\sqrt{F(0, \mu)} > c$. At the moment, we have

$$\omega_{\infty}(z) = e^{z\operatorname{Re}(\eta^{+})} \left[\cos \left(z\operatorname{Im}(\eta^{+}) \right) + \kappa \sin \left(z\operatorname{Im}(\eta^{+}) \right) \right] \text{ with } \kappa \in \mathbb{R},$$

where

$$\eta^{\pm} = \frac{1}{2} \left(c \pm \sqrt{c^2 - 4F(0, \mu)} \right).$$

Hence, $\omega_{\infty}(z_0) < 0$ for some $z_0 \in \mathbb{R}$, and we get a contradiction with the positivity of $\omega_{\infty}(z)$. On the other hand, we consider $c \geq c_0$. At the moment, there are constants κ_1 and κ_2 such that

$$\omega_{\infty}(z) = \begin{cases} \kappa_1 e^{\eta^{-}z} + (1 - \kappa_1)e^{\eta^{+}z}, & \text{if } c > c_0, \\ e^{\eta^{+}z}(1 + \kappa_2 z), & \text{if } c = c_0. \end{cases}$$

Using the positivity of $\omega_{\infty}(z)$ again, we obtain $\kappa_1 \in [0,1]$ and $\kappa_2 = 0$. And

$$\omega_{\infty}'(z) = \begin{cases} \kappa_1 \eta^- e^{\eta^- z} + (1 - \kappa_1) \eta^+ e^{\eta^+ z}, & \text{if } c > c_0, \\ \eta^+ e^{\eta^+ z}, & \text{if } c = c_0. \end{cases}$$

Therefore, we have $\omega'_{\infty}(z) > 0$ over \mathbb{R} , which implies that $\omega'_{n}(z) > 0$ in any bounded closed set for large enough n. However, since $\phi(-\infty) = 1$ and $0 < \phi(z) < 1$ over \mathbb{R} , then there is a $z^* \in \mathbb{R}$ such that $\phi'(z) \leq 0$ for $z \leq z^*$, it follows from (2.4) that $\omega'_{n}(z) \leq 0$ for $z \in [z^* - 1, z^*]$. Thus, we get a contradiction, which finish proof of this lemma.

Remark 2.5. If $F(0,\mu) = 0$, then $\omega_{\infty}(z) = \kappa + (1-\kappa)e^{cz}$ for $\kappa \in [0,1]$ owing $\omega_{\infty}(0) = 1$ and $\omega_{\infty}(z) > 0$ over \mathbb{R} . Clearly, $\omega'_{\infty}(z) > 0$ or $\omega_{\infty}(z) \equiv 1$ over \mathbb{R} . We are unable to judge whether traveling wave solutions are persistent for $c \geq c_*$ since the latter cannot be excluded.

Finally, we finish the proof of Theorem 1.1.

Lemma 2.6. $\limsup_{z \to +\infty} \phi(z) < 1$ and $\limsup_{z \to +\infty} \psi(z) < v_0$.

Proof. For contradiction, we assume that there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ satisfying $z_n \to +\infty$ as $n \to +\infty$ such that $\phi(z_n) \to 1$ as $n \to +\infty$. Let $(\phi_n, \psi_n)(z) := (\phi, \psi)(z_n + z)$, similar to proof in Lemma 2.4, there are some functions $\phi_{\infty}(z)$ and $\psi_{\infty}(z)$ satisfy

$$\phi_{\infty}''(z) - c\phi_{\infty}'(z) + \phi_{\infty}(z)F(\phi_{\infty}, \psi_{\infty})(z) = 0,$$

$$d\psi_{\infty}''(z) - c\psi_{\infty}'(z) + \psi_{\infty}(z)G(\phi_{\infty}, \psi_{\infty})(z) = 0.$$
(2.8)

One can readily verify that $0 < \phi_{\infty} \le 1$ and $0 < \psi_{\infty} \le v_0$ over \mathbb{R} . Note that $\phi_{\infty}(0) = 1$ due to $\phi(z_n) \to 1$ as $n \to +\infty$, we have $\phi'_{\infty}(0) = 0$ and $\phi''_{\infty}(0) \le 0$. Thus, from Assumption 1.1-(c),

$$0 \ge \phi_{\infty}''(0) = -F(1, \psi_{\infty}(0)) > 0,$$

hence we conclude $\limsup_{z\to+\infty}\phi(z)<1$. A similar discussion yields that

$$0 \ge d\psi_{\infty}''(0) = -v_0 G(\phi_{\infty}(0), v_0).$$

From Assumption 1.1-(b), we have $\phi_{\infty}(0) > 1$, which leads to a contradiction. Hence we also conclude $\limsup_{z\to+\infty} \psi(z) < v_0$. Therefore, we complete the proof.

Remark 2.7. Our result can also be applied to Holling-Tanner system in [9]

$$\begin{cases}
 u_t = u_{xx} + u \left(1 - u - \frac{\alpha u^{m-1} v}{1 + \beta_1 u^m + \beta_2 v^m} \right), \\
 v_t = dv_{xx} + rv \left(1 - \frac{v}{u} \right),
\end{cases}$$
(2.9)

where $\alpha, d, r > 0$, $\beta_1, \beta_2 \geq 0$ and $m \geq 1$. We follow the idea of [1] to consider

$$\begin{cases}
 u_t = u_{xx} + u \left(1 - u - \frac{\alpha u^{m-1} v}{1 + \beta_1 u^m + \beta_2 v^m} \right), \\
 v_t = dv_{xx} + rv \left(1 - \frac{v}{\sigma_{\varepsilon}(u)} \right),
\end{cases}$$
(2.10)

where

$$\sigma_{\varepsilon}(u) = \begin{cases} u, & u \ge \varepsilon, \\ u + \varepsilon e^{\frac{1}{u - \varepsilon}}, & 0 \le u < \varepsilon. \end{cases}$$

Clearly, $\mu = \varepsilon (1/e)^{1/\varepsilon}$ and

$$F(0,\mu) = \begin{cases} 1, & m > 1, \\ 1 - \frac{\alpha \varepsilon (1/e)^{1/\varepsilon}}{1 + \beta_2 \varepsilon (1/e)^{1/\varepsilon}}, & m = 1. \end{cases}$$

Thus, $F(0,\mu) > 0$ as long as ε is small enough. In view of Theorem 1.1, we conclude that traveling wave solution $(\phi_{\varepsilon}, \psi_{\varepsilon})(z)$ of system (2.10) is persistent for $c \geq 2\sqrt{dr}$, then there exists $\delta > 0$ such that $\phi_{\varepsilon}(z) > \delta$ over \mathbb{R} . Consequently, $\sigma_{\varepsilon}(\phi_{\varepsilon}) = \phi_{\varepsilon}$ for small enough ε , and $(\phi, \psi)(z) := (\phi_{\varepsilon}, \psi_{\varepsilon})(z)$ is a traveling wave solution of system (2.9).

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Statements and Declarations

Authors have no conflict of interest to declare.

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