Introduction to Machine Learning (67577)

Exercise II – Mathematical Background

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**Theoretical Questions**

– m rows (samples) & d column (features)

– response vector corresponding to samples in .

Solutions of the Normal Equations

1. Prove that

In the first direction, let , therefore . Multiply both sides with we have which means by definition

In the other direction, let , therefore . Multiply both sides by we have . Since we have

1. Let , prove that .

In the first direction, let . If so, s.t . We wish to show that

Let us consider some arbitrary . From the definition of the kernel, such satisfies . Now we calculate :

From here we conclude that and are orthogonal, therefore

In the second direction, let . We wish to show that . This is the same as showing that . In other words, assuming , we must show that there exists some vector s.t . From here we will conclude that , because for to be in it must satisfy , yet we have found one vector for which the claim will not hold. If so, let us find such :

Assuming , it must have some component. Let be that component, i.e., we can choose (note that for such we have because ). This is orthogonal to any vector in

If so,

For a vector we have that , and since is orthogonal to any vector in , it is orthogonal to , therefore the product is

1. Let where non-invertible. Show that the system has solutions

Now, since is invertible than has either 0 or solutions, yet since there is at least one solution, there are . Such condition can be rewritten as

Which finishes the proof.

1. Prove that the normal linear system can only have unique solution (if is invertible) or infinitely many solutions (otherwise):

For the case where is invertible we can write a unique solution for as followed

For the case where is not invertible

Indeed, for some we have , therefore

Projection Matrices

, = orthogonal basis of .

1. A. P is symmetric (:

where is due to the fact that the transpose of a sum is the sum of transposes

B. the eigenvalues of are and are the eigenvectors corresponding to :

We can use the fact that (proof in 5.D) as followed: the eigenvalues and vectors associated with satisfy the equation

Therefore, we can multiply both sides with and get

Now using

Combining this with the above equation we get

In section 5.C we see that , therefore for the claim also holds. That is, are the eigenvectors corresponding to the eigenvalue

C. :

If then we can write as a linear combination of : for some . Therefore,

D. :

For we can use section C:

Otherwise, we must generalize for all (which also holds for :

Since is an orthonormal basis we get , so we can re-write as

E. :

Least Squares

1. Show that if is invertible, :
2. Show that is invertible iff :

We know from definition that , and in class we saw that . For the set to span it must be of dimension , which in turn means that . From here we can use the fact that has dimension , therefore it is invertible by definition iff

Simply written, we have

1. show that for any other solution , , where (this is the case where is not invertible):

we know that for , and for = , and could by any value . This means that the norm satisfies