The Five Miracles of Mirror Descent

Hadar Tal
hadar.tal@mail.huji.ac.il

This paper is a summary of the educational materials and lectures from Professor Sebastian Bubeck, enhanced by Claire Boyer's comprehensive notes, and structured according to Tomer Koren's course on Optimization for Computer Science.

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Contents

Chapter 1

Mathematical Background

1.1 Multivariable Calculus

Definition 1.1.1. *Diffrentiability, single variable*
Let $f : (a, b) \to \mathbb{R}$ be a function. We say that f is differentiable at $x_0 \in (a, b)$ if

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{1.1}$$

exists. If f is differentiable at x_0, then $f'(x_0)$ is the derivative of f at x_0.

Definition 1.1.2. *Diffrentiability, single variable (alternative)*
Let $f : (a, b) \to \mathbb{R}$ be a function. We say that f is differentiable at $x_0 \in (a, b)$ if there exists a number m such that:

$$f(x_0 + h) = f(x_0) + m \cdot h + E(h) \text{ where } \lim_{h \to 0} \frac{E(h)}{h} = 0 \tag{1.2}$$

If f is differentiable at x_0, then $f'(x_0) = m$ is the derivative of f at x_0.

Definition 1.1.3. *Diffrentiability, multivariable*
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. We say that f is differentiable at x_0 if there exists a vector m $\in \mathbb{R}^n$ such that:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - m \cdot h}{||h||} = 0 \tag{1.3}$$

If f is differentiable at x_0, then m is the gradient of f at x_0, denoted $\nabla f(x_0)$.

Suppose the $S \subseteq \mathbb{R}^n$ and $f : S \to \mathbb{R}$ is a function.

Definition 1.1.4. *Limit, multivariate function*
We say that the limit of f at x_0 is L if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all x such that $||x - x_0|| < \delta$, we have $|f(x) - L| < \epsilon$.

Definition 1.1.5. *Diffrentiability, multivariable (alternative)*
We say that f is differentiable at x_0 if there exists a vector $m \in \mathbb{R}^n$ such that:

$$f(x_0 + h) = f(x_0) + m^T \cdot h + E(h) \text{ where } \lim_{h \to 0} \frac{E(h)}{||h||} = 0 \tag{1.4}$$

If f is differentiable at x_0, then m is the gradient of f at x_0, denoted $\nabla f(x_0)$.

Definition 1.1.6. *Partial Derivative*
The partial derivative of f with respect to the i-th variable at x is:

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h \to 0} \frac{f(x + h \cdot e_i) - f(x)}{h} \tag{1.5}$$

where e_i is the i-th standard basis vector.

Theorem 1.1.1. *(Diffrentiability vs. Partial Derivatives)*
If f is differentiable at x, then all partial derivatives of f exist at x and:

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right) \tag{1.6}$$

- If any partial derivative of f does not exist at x, then f is not differentiable at x.
- If all partial derivatives of f exist at x, then f may still not be differentiable at x and the vector $m = \nabla f(x)$ is the only possible vector that satisfies the definition of differentiability.

Definition 1.1.7. *Continuously Differentiable*
We say that f is continuously differentiable or of class C^1 if all partial derivatives of f exist and are continuous at every point in S.

Theorem 1.1.2. *If f is continuously differentiable, then f is differentiable.*

Definition 1.1.8. *The directional derivative*
For a given $x \in S$ and a unit vector $u \in \mathbb{R}^n$, the directional derivative of f at x in the direction of u is:

$$\partial_u f(x) = \lim_{h \to 0} \frac{f(x + h \cdot u) - f(x)}{h} \tag{1.7}$$

Equivalently, $\partial_u f(x) = g'(0)$ where $g(h) = f(x + h \cdot u)$.

Theorem 1.1.3. *If f is differentiable at x, then for all $u \in \mathbb{R}^n$, the directional derivative of f at x in the direction of u exists and is given by:*

$$\partial_u f(x) = \nabla f(x) \cdot u \tag{1.8}$$

Theorem 1.1.4. *Fermat's Theorem*
If f is differentiable at x and x is a local minimum of f, then $\nabla f(x) = 0$.

Theorem 1.1.5. *Suppose that $f : S \to \mathbb{R}$ is differentiable at x. Then $\nabla f(x)$ is orthogonal to the level set of f that passes through x.*

Theorem 1.1.6. *The mean value theorem*
If $f : S \to \mathbb{R}$ is differentiable on the open interval between a and b, then there exists $c \in [a, b]$ such that:

$$f(b) - f(a) = \nabla f(c) \cdot (b - a) \tag{1.9}$$

where $[a, b] = a + t(b - a) | t \in [0, 1]$.

Definition 1.1.9. *Second-order partial derivatives*
Suppose that f is a C^1 function. If the partial derivatives of f are differentiable, then the second-order partial derivatives of f are:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) \tag{1.10}$$

Equivalently, $\frac{\partial^2 f}{\partial i \partial j} = \partial_j \partial_j f$. If $i = j$ we denote $\frac{\partial^2 f}{\partial x_i^2}$ or $(\partial_i^2 f$

Definition 1.1.10. *The C^2 class*
We say that f is of class C^2 if all second-order partial derivatives of f exist and are continuous.

Theorem 1.1.7. *Clairaut's Theorem*
If f is of class C^2, then $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$.

Definition 1.1.11. *Hessian Matrix*
The Hessian matrix of f at x is the matrix of second-order partial derivatives of f at x:

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} \tag{1.11}$$

Corollary. *The interpretation of the Hessian matrix*
Let $u \in \mathbb{R}^n$ be a unit vector. then

$$\partial_{uu}^2 f(x) = \sum_{i,j=1}^{n} \partial_{ij} f(x) u_i u_j = u^T \nabla^2 f(x) u \tag{1.12}$$

1.2 Taylor series

Definition 1.2.1. *Taylor Series*
Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is k times differentiable at x_0. Then the Taylor series of f at x_0 is given by:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_k(x) \tag{1.13}$$

where $R_k(x) = \frac{f^{(k+1)}(c)}{(k+1)!}(x - x_0)^{k+1}$ for some c between x and x_0.

Definition 1.2.2. *Taylor Series for Multivariable Functions (k=2)*
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function that is C^2 at x_0. Then for any h such that $x_0 + h \in S$, there exists $\theta \in [0, 1]$ such that:

$$f(x_0 + h) = f(x_0) + \nabla f(x_0) \cdot h + \frac{1}{2} h^T \nabla^2 f(x_0 + \theta h) h \tag{1.14}$$

1.3 Important subsets of $\mathbb{R}^n$

Definition 1.3.1. *Open set*
A set $S \subseteq \mathbb{R}^n$ is open if for all $x \in S$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq S$.

Definition 1.3.2. *Closed set*
A set $S \subseteq \mathbb{R}^n$ is closed if its complement is open.

Definition 1.3.3. *Interior point*
A point $x \in S$ is an interior point of S if there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq S$.

Corollary 1.3.1. *Open set characterization*
A set $S \subseteq \mathbb{R}^n$ is open if and only if every point in S is an interior point of S.

Definition 1.3.4. *Boundary point*
A point $x \in S$ is a boundary point of S if for all $\epsilon > 0$, $B(x, \epsilon) \cap S \neq \emptyset$ and $B(x, \epsilon) \cap S^c \neq \emptyset$.

Definition 1.3.5. *Half-space*
A half-space in $\mathbb{R}^n$ is a set of the form $\{x \in \mathbb{R}^n : a^T x \leq b\}$ for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

Definition 1.3.6. *Hyperplane*
A hyperplane in $\mathbb{R}^n$ is a set of the form $\{x \in \mathbb{R}^n : a^T x = b\}$ for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

Definition 1.3.7. *Polyhedron (Polyhedra)*
A polyhedron in $\mathbb{R}^n$ is a set of the form $\{x \in \mathbb{R}^n : Ax \leq b\}$ for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Equivalently, a polyhedron is the intersection of finitely many half-spaces.

Definition 1.3.8. *Polytope*
A polytope in $\mathbb{R}^n$ is a bounded polyhedron - i.e., there exists $r > 0$ such that $\forall x \in \{x \in \mathbb{R}^n : Ax \leq b\} \implies ||x|| \leq r$. Equivalently, a polytope is the convex hull of finitely many points.

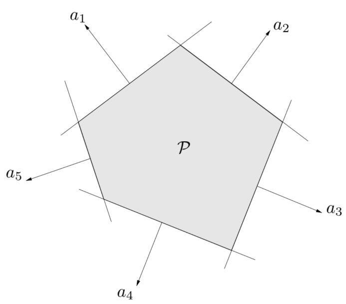


Figure 1.1: Polytope

Definition 1.3.9. *Convex set*
A set $S \subseteq \mathbb{R}^n$ is convex if for all $x, y \in S$ and $\lambda \in [0, 1]$, we have $\lambda t + (1 - \lambda)y \in S$.

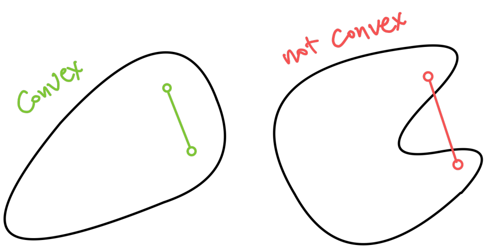


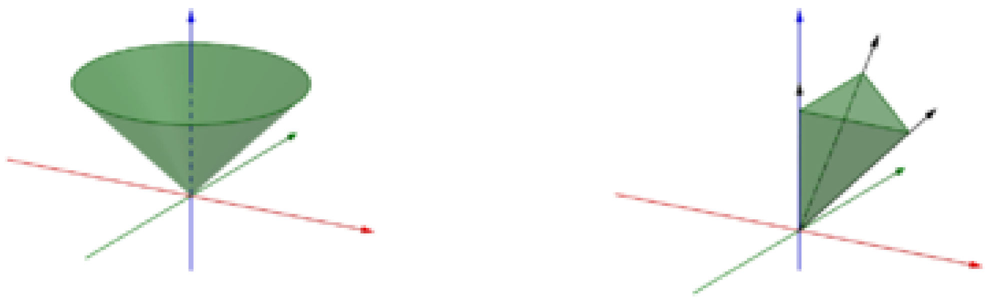
Figure 1.2: Convex set

Definition 1.3.10. *Convex hull*
The convex hull of a set $S \subseteq \mathbb{R}^n$ is the smallest convex set that contains S.

Definition 1.3.11. *Conic combination*
A point $x \in \mathbb{R}^n$ is a conic combination of $y_1, \dots, y_k \in \mathbb{R}^n$ if there exist $\lambda_1, \dots, \lambda_k \geq 0$ such that $x = \sum_{i=1}^k \lambda_i y_i$.

Definition 1.3.12. *Conic hull*
The conic hull of a finite set $S \subseteq \mathbb{R}^n$ is the set of all conic combinations of points in S.

Definition 1.3.13. *Convex cone*
A set $S \subseteq \mathbb{R}^n$ is a convex cone if for all $x \in S$ and $\lambda \geq 0$, we have $\lambda x \in S$.



(a) Convex cone that is not a conic hull of finitely many generators.

(b) Convex cone genrated by the conic combination of three black vectors (conic hull).

Definition 1.3.14. *Normal cone*
The normal cone to a set S at a point x is defined as

$$N_S(x) = \{v \in \mathbb{R}^n : \langle v, y - x \rangle \leq 0 \text{ for all } y \in S\} \tag{1.15}$$

Definition 1.3.15. *Tangent cone*
The tangent cone to a set S at a point x is defined as

$$T_S(x) = \{v \in \mathbb{R}^n : \lim_{t \to 0^+} \frac{x + tv - x}{t} \in S\} \tag{1.16}$$

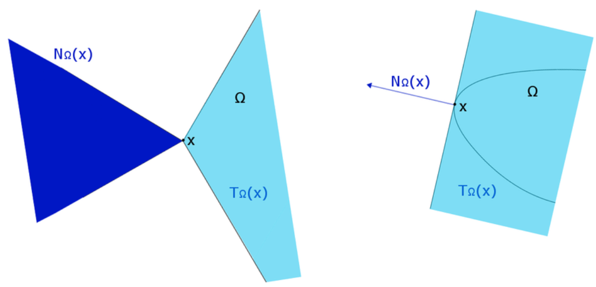


Figure 1.4: Normal and tangent cones

Theorem 1.3.1. *Normal cone of polyhedron*
The normal cone to a polyhedron $S = \{x \in \mathbb{R}^n : \forall j \in [m] \quad a_j \cdot x \leq b_j\}$ at a point x is given by

$$N_S(x) = \{\sum_j \lambda_j a_j : \lambda_j \geq 0 \text{ and } a_j \cdot x = b_j\} \tag{1.17}$$

1.4 Convexity

1.4.1 Definitions and Fundamental Theorems

Definition 1.4.1. *(Convex function): A function $f : S \to \mathbb{R}$ defined on a convex set S is convex if, for all $x, y \in S$ and $\lambda \in [0, 1]$,*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

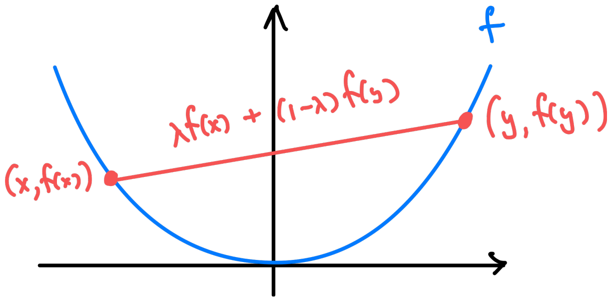


Figure 1.5: Convex function

Theorem 1.4.1. *(Characterization via epigraph): A function $f : S \to \mathbb{R}$ is convex if and only if its epigraph $\{(x, t) \in S \times \mathbb{R} : f(x) \leq t\}$ is a convex set.*

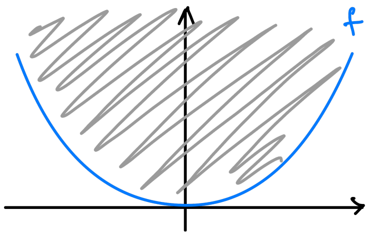


Figure 1.6: Epigraph of a convex function

claim 1.4.1. *(Convexity of sublevel sets): If $f : S \to \mathbb{R}$ is convex, then the sublevel set $S_t = \{x \in S : f(x) \leq t\}$ is convex for any $t \in \mathbb{R}$.*

1.4.2 Inequalities and Characterizations

Theorem 1.4.2. *(Jensen's inequality): If f is a convex function, then for any $x_1, x_2, \ldots, x_n \in S$ and any non-negative weights α_i such that $\sum_{i=1}^n \alpha_i = 1$,*

$$f\left(\sum_{i=1}^n \alpha_i x_i\right) \leq \sum_{i=1}^n \alpha_i f(x_i).$$

Theorem 1.4.3. *(First-order characterization, aka "the gradient inequality"): If f is a differentiable convex function on an open set S, then for all $x, y \in S$,*

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x).$$

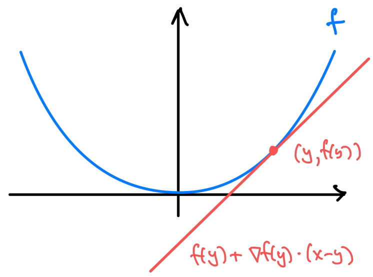


Figure 1.7: First-order characterization of convexity

Theorem 1.4.4. *(Jensen's inequality, generalized for expectation): If f is a convex function and X is a random variable over S, then*

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)].$$

Theorem 1.4.5. *(Second-order characterization of convexity): A twice differentiable function f is convex on an open set S if and only if the Hessian matrix of f is positive semidefinite at every point in S.*

1.4.3 Optimization and Projection

Definition 1.4.2. *(Convex optimization): The problem of minimizing a convex function over a convex set.*

Theorem 1.4.6. *(Optimality conditions, unconstrained): If f is convex and differentiable, x^* is a local minimum of f $\Leftrightarrow$ x^* is a global minimum of f $\Leftrightarrow$ $\nabla f(x^*) = 0$.*

Theorem 1.4.7. *(Optimality conditions, constrained): If f is differentiable and C is a convex set, x^* is a local minimum of f on C if and only if $\langle \nabla f(x^*), x - x^* \rangle \geq 0$ for all $x \in C$.*

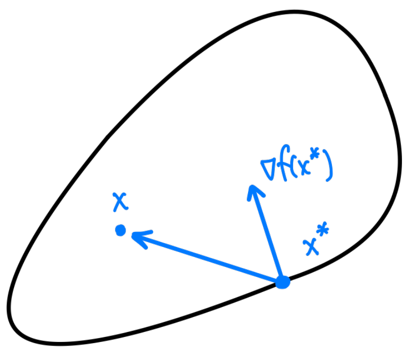


Figure 1.8: Optimality conditions, constrained

Corollary 1.4.1. *Optimality conditions, constrained (alternative)*
If f is differentiable and C is a convex set, then x^* is a local minimum of f on C if and only if $-\nabla f(x^*) \in N_C(x^*)$.

Definition 1.4.3. *(Projection): The projection of a point x onto a convex set S is defined as* $\Pi_S(x) = \arg\min_{y \in S} \|y - x\|$.

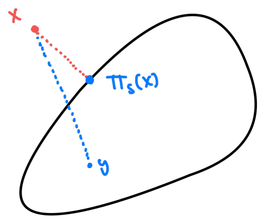


Figure 1.9: Projection

Theorem 1.4.8. *Generalized cosine theorem*
Let $S \subseteq \mathbb{R}^d$ be convex and $x \in \mathbb{R}^d$. Then the projection $\Pi_S[x]$ is unique and satisfies:

$$\|x - \Pi_S[x]\|^2 + \|\Pi_S[x] - y\|^2 \leq \|x - y\|^2, \quad \forall y \in S. \tag{1.18}$$

In particular:

$$\|\Pi_S[x] - y\| \leq \|x - y\|, \quad \forall y \in S. \tag{1.19}$$

1.5 Properties of Convex Functions

1.6 Important Inequalities

1.6.1 $\mathbf{1 + x \leq e^x}$