67678 - Introduction to Control with Learning **Spring 2024**

Linear Dynamical Systems - Summary

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1 Kalman Filter

Definition 1.1 (Kalman Filter State Space)

$$\begin{aligned} x_{i+1} &= Fx_i + Gw_i \\ y_i &= Hx_i + v_i \end{aligned} \tag{1.1}$$

where:

- x_{i+1} *is the state vector at time* $i+1$*,*
- x_i *is the state vector at time* i*,*
- y_i *is the measurement vector at time* i*,*
- w_i *is the process noise (zero mean, uncorrelated),*
- v_i *is the measurement noise (zero mean, uncorrelated).*

Definition 1.2 (Kalman Filter Covariance Matrix)
Formally, the following covariance matrix describes the model:

$$\mathbb{E}\left[\begin{pmatrix} w_i \\ v_i \\ x_0 \end{pmatrix} \begin{pmatrix} w_j^* & v_j^* & x_0^* & 1 \end{pmatrix}\right] = \begin{pmatrix} \begin{pmatrix} Q & S \\ S^* & R \end{pmatrix}\delta_{ij} & 0 & 0 \\ 0 & \Pi_0 & 0 \end{pmatrix}, \tag{1.2}$$

where $\begin{pmatrix} Q & S \\ S^* & R \end{pmatrix}$ *and* Π_0 *are positive semidefinite matrices and* δ_{ij} *equals 1 if* $i = j$ *and is zero otherwise. Note that* w_i *is uncorrelated as a process over time but its coordinates at a fixed time can be correlated via* Q*.*

Markings:

- P_i - The error covariance matrix at time i

$$P_i \triangleq (x_i - \hat{x}_i)(x_i - \hat{x}_i)^T \tag{1.3}$$

- $R_{e,i}$ - The covariance of the innovation (or residual) at time i

$$R_{e,i} \triangleq HP_iH^* + R \tag{1.4}$$

- $K_{p,i}$ - The optimal Kalman gain at time i

$$K_{p,i} \triangleq (FP_iH^* + GS){R_{e,i}}^{-1} \tag{1.5}$$

Kalman Filter Optimality

We suggest the following predictor:

$$\hat{x}_{i+1|i} = F\hat{x}_{i|i-1} + K_{p,i}(y_i - H\hat{x}_{i|i-1}) \tag{1.6}$$

Lemma 1.1

$$\tilde{x}_{i+1} = (F - K_{p,i}H)\tilde{x}_i + (G - K_{p,i})\begin{pmatrix} w_i \\ v_i \end{pmatrix}. \tag{1.7}$$

Proof.

$$\begin{aligned}
\tilde{x}_{i+1} &= x_{i+1} - \hat{x}_{i+1|i} \\
&= (Fx_i + Gw_i) - (F\hat{x}_i + K_{p,i}(y_i - H\hat{x}_i)) \\
&= Fx_i + Gw_i - F\hat{x}_i - K_{p,i}(Hx_i + v_i - H\hat{x}_i) \\
&= Fx_i + Gw_i - F\hat{x}_i - K_{p,i}Hx_i - K_{p,i}v_i + K_{p,i}H\hat{x}_i \\
&= Fx_i - F\hat{x}_i - K_{p,i}Hx_i + K_{p,i}H\hat{x}_i + Gw_i - K_{p_i}v_i \\
&= (F - K_{p,i}H)(x_i - \hat{x}_i) + Gw_i - K_{p,i}v_i \\
&= (F - K_{p,i}H)\tilde{x}_i + Gw_i - K_{p,i}v_i.
\end{aligned}$$

□

Lemma 1.2
For $j < i$, the recursion can be evolved as

$$\begin{aligned}
\tilde{x}_i &= (F - K_{p,i-1}H)\tilde{x}_{i-1} + (G - K_{p,i-1})\begin{pmatrix} w_{i-1} \\ v_{i-1} \end{pmatrix} \\
&= \dots \\
&= \phi_p(i,j)\tilde{x}_j + \xi_i(j),
\end{aligned}$$

where

$$\begin{aligned}
\phi_p(i,j) &= \prod_{k=j}^{i-1}(F - K_{p,k}H), \\
\xi_i(j) &= \sum_{k=j}^{i-1}\phi_p(i,k+1)(Gw_k - K_{p,k}v_k).
\end{aligned}$$