# Measure Theory

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# Chapter 1

# Measure Theory

## 1.1 $\sigma$-algebras and measures

**Definition 1.1.1**
*($\sigma$-algebra)*
*Let $\Omega$ be a set. A collection $\mathcal{F}$ of subsets of $\Omega$ is called a $\sigma$-algebra if:*

1. $\Omega \in \mathcal{F}$.
2. *If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.*
3. *If $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.*

**Definition 1.1.2**
*(Measure)*
*Let $\Omega$ be a set and $\mathcal{F}$ be a $\sigma$-algebra on $\Omega$. A function $\mu : \mathcal{F} \to [0, \infty]$ is called a measure if:*

1. $\mu(\emptyset) = 0$.
2. *If $A_1, A_2, \ldots \in \mathcal{F}$ are pairwise disjoint, then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.*

**Definition 1.1.3**
*(Measure Space)*
*A measure space is a triple $(\Omega, \mathcal{F}, \mu)$, where $\Omega$ is a set, $\mathcal{F}$ is a $\sigma$-algebra on $\Omega$, and $\mu$ is a measure on $\mathcal{F}$.*

## 1.2 Order of spaces

### 1.2.1 1. Topological Space (Most General)

**Axioms:** A topological space is a set $X$ together with a collection $\mathcal{T}$ of subsets of $X$ (called open sets) such that:

1. The empty set and $X$ itself are in $\mathcal{T}$.
2. Any union of sets in $\mathcal{T}$ is also in $\mathcal{T}$.
3. Any finite intersection of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

**Why it's more general:** Topological spaces are the most general because they only require a definition of open sets, without any need for distances or measures. They provide a flexible framework for discussing continuity, compactness, and connectedness.

### 2. Measurable Space

**Axioms:** A measurable space is a set $X$ together with a collection $\mathcal{F}$ of subsets of $X$ (called measurable sets) such that:

1. The empty set and $X$ itself are in $\mathcal{F}$.
2. $\mathcal{F}$ is closed under complements: if $A \in \mathcal{F}$, then $X \setminus A \in \mathcal{F}$.
3. $\mathcal{F}$ is closed under countable unions: if $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

**Why it's less general than a topological space:** Measurable spaces are less general because they require the collection of measurable sets to be closed under complements and countable unions, forming a $\sigma$-algebra. This imposes more structure compared to the open sets in a topological space, which only need to be closed under arbitrary unions and finite intersections.

### 3. Metric Space (Most Specific)

**Axioms:** A metric space is a set $X$ together with a function $d : X \times X \to \mathbb{R}$ (called a metric) such that:

1. $d(x, y) \geq 0$ for all $x, y \in X$ (non-negativity).
2. $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles).
3. $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry).
4. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (triangle inequality).

**Why it's less general than a measurable space:** Metric spaces are less general because they require the existence of a metric that defines distances between points. This metric induces a topology, where open sets are defined in terms of open balls around points. While every metric space is a topological space (with the topology induced by the metric), not every topological space can be given a metric that defines its open sets. Furthermore, metric spaces do not inherently involve measurable sets or $\sigma$-algebras.

## 1.3 Borel $\sigma$-algebra

**Definition 1.3.1**
*(Borel $\sigma$-algebra)*
*Let $X$ be a topological space. The Borel $\sigma$-algebra on $X$, denoted by $\mathcal{B}(X)$, is the smallest $\sigma$-algebra that contains all the open sets of $X$. Formally, $\mathcal{B}(X)$ is the $\sigma$-algebra generated by the collection of open sets $\mathcal{T}$ in $X$:*

$$\mathcal{B}(X) = \sigma(\mathcal{T}),$$

*where $\sigma(\mathcal{T})$ denotes the $\sigma$-algebra generated by $\mathcal{T}$. This means that $\mathcal{B}(X)$ is the smallest collection of subsets of $X$ that contains all open sets and is a $\sigma$-algebra.*

Examples of Borel $\sigma$-algebras include:

- **Borel $\sigma$-algebra on** $\mathbb{R}$: The Borel $\sigma$-algebra on the real line $\mathbb{R}$, denoted by $\mathcal{B}(\mathbb{R})$, is the $\sigma$-algebra generated by all open intervals in $\mathbb{R}$. This includes:
  - All open intervals $(a, b)$ where $a < b$.
  - All closed intervals $[a, b]$ where $a < b$.

- All half-open intervals $(a, b]$ and $[a, b)$ where $a < b$.
- All singletons $\{a\}$ where $a \in \mathbb{R}$.
- All countable unions and intersections of such sets.

- **Borel $\sigma$-algebra on $\mathbb{R}^n$**: The Borel $\sigma$-algebra on $\mathbb{R}^n$, denoted by $\mathcal{B}(\mathbb{R}^n)$, is the $\sigma$-algebra generated by all open sets in the Euclidean space $\mathbb{R}^n$. This includes:
  - All open balls $B(x, r) = \{y \in \mathbb{R}^n \mid \|y - x\| < r\}$ where $x \in \mathbb{R}^n$ and $r > 0$.
  - All open rectangles $(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$ where $a_i < b_i$ for each $i$.
  - All closed sets and other sets formed by countable unions and intersections of open sets in $\mathbb{R}^n$.

## 1.4 Measurable Functions

**Definition 1.4.1**
*(Measurable Function)*
*Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be measurable spaces. A function $f : \Omega_1 \to \Omega_2$ is called measurable (with respect to $\mathcal{A}_1$ and $\mathcal{A}_2$) if:*

$$f^{-1}(A_2) \in \mathcal{A}_1 \quad \textit{for all } A_2 \in \mathcal{A}_2.$$

### 1.4.1 Examples

1. $(\Omega, \mathcal{A})$, $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

   - **Characteristic Function**:
     $$\chi_A : \Omega \to \mathbb{R},$$
     where
     $$\chi_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$$
     For all measurable $A \in \mathcal{A}$, $\chi_A$ is a measurable map.
     - $\chi_A^{-1}(\varnothing) = \varnothing \subset \Omega$
     - $\chi_A^{-1}(\mathbb{R}) = \Omega \in A$
     - $\chi_A^{-1}(1) = A \in \mathcal{A}$
     - $\chi_A^{-1}(0) = A^c \in \mathcal{A}$ by the closure of $\mathcal{A}$ under complements.

2. $(\Omega_1, \mathcal{A}_1)$, $(\Omega_2, \mathcal{A}_2)$, $(\Omega_3, \mathcal{A}_3)$ are measurable spaces

   - If $f : \Omega_1 \to \Omega_2$ is measurable and $g : \Omega_2 \to \Omega_3$ is measurable, then $g \circ f : \Omega_1 \to \Omega_3$ is measurable.
     $$(g \circ f)^{-1}(A_3) = f^{-1}(g^{-1}(A_3)) \in \mathcal{A}_1 \quad \text{for all } A_3 \in \mathcal{A}_3$$

### Important

$(\Omega, \mathcal{A})$, $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

- $f, g$ measurable $\Rightarrow f + g, f \cdot g$ measurable
- $|f|$ measurable