Bound, Equalities and Inequalities

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This paper is a summary of the educational materials and lectures from

- **Wikipedia**
- **3Blue1Brown** YouTube channel

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2.1 Properties of Binomial Coefficients

2.1.1 Symmetry Rule for Binomial Coefficients

Theorem 2.1.1. *For all $n, k \in \mathbb{N}$, the following holds*

$$\binom{n}{k} = \binom{n}{n-k} \tag{2.1}$$

Proof. The proof is by definition of the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{2.2}$$

and the symmetry of the factorial function

$$n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 = 1 \cdot 2 \cdot \ldots \cdot (n-1) \cdot n = n! \tag{2.3}$$

which implies that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k} \tag{2.4}$$

□

Example. *The symmetry rule for binomial coefficients states that the number of ways to choose k elements out of n is the same as the number of ways to choose $n-k$ elements out of n.*

2.1.2 Pascal's Rule for Binomial Coefficients

Theorem 2.1.2. *For all $n, k \in \mathbb{N}$, the following holds*

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{2.5}$$

Proof.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-1-(k-1))!} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{2.6}$$

□

Example. *choosing a team of k players from n candidates: you can either include a specific player in your team and choose the rest k-1 players from the remaining n-1 candidates, or not include that specific player, thus choosing all k players from the remaining n-1 candidates.*

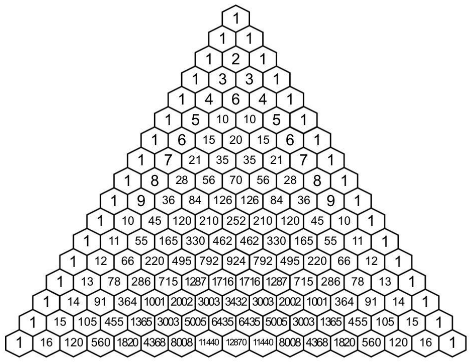

Figure 2.1: Pascal's Triangle

2.1.3 Sum of Binomial Coefficients over Lower Index

Theorem 2.1.3. *For all $n \in \mathbb{N}$, the following holds*

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n \tag{2.7}$$

Proof. The proof is by induction on n. For $n = 0$, the base case is

$$\sum_{i=0}^{0} \binom{0}{i} = \binom{0}{0} = 1 = 2^0 \tag{2.8}$$

For the induction step, assume that the theorem holds for $n = k$. Then

$$\sum_{i=0}^{k+1} \binom{k+1}{i} = \sum_{i=0}^{k+1} \left(\binom{k}{i} + \binom{k}{i-1} \right) = \sum_{i=0}^{k} \binom{k}{i} + \sum_{i=0}^{k} \binom{k}{i-1} = 2^k + 2^k = 2^{k+1} \tag{2.9}$$

□

Example. *The sum of binomial coefficients over the lower index is the same as counting all the subsets of a set of size n which is 2^n.*

2.1.4 Factors of Binomial Coefficient

Theorem 2.1.4. *For all $r \in \mathbb{R}, k \in \mathbb{Z}$, the following holds:*

$$k\binom{r}{k} = r\binom{r-1}{k-1} \tag{2.10}$$

Proof. By definition, the binomial coefficient $\binom{r}{k}$ is given by

$$\binom{r}{k} = \frac{r!}{k!(r-k)!}. \tag{2.11}$$

Multiplying both sides by k, we have

$$k\binom{r}{k} = k\frac{r!}{k!(r-k)!} = \frac{r!}{(k-1)!(r-k)!} = r\frac{(r-1)!}{(k-1)!(r-k)!} = r\binom{r-1}{k-1}. \tag{2.12}$$

□

Example. *Forming a committee of k members from a group of r individuals, with one member to be chosen as chairperson.*

2.1.5 Increasing Sum of Binomial Coefficients

Theorem 2.1.5. *For all $n \in \mathbb{N}$, the following holds*

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1} \tag{2.13}$$

Proof.

$$\sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=1}^{n} n\binom{n-1}{k-1} = n\sum_{k=1}^{n}\binom{n-1}{k-1} = n\sum_{k=0}^{n-1}\binom{n-1}{k} = n \cdot 2^{n-1} \tag{2.14}$$

□

Chapter 3

Inequalities

Theorem 3.0.1. ***Cauchy-Schwarz Inequality***
Let u, v be vectors of an inner product space. Then

$$|\langle u, v\rangle|^2 \leq \|u\| \, \|v\|$$