$O(n)$ Algorithms for Banded Plus Semiseparable Matrices

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Abstract. We present a new representation for the inverse of a matrix that is a sum of a banded matrix and a semiseparable matrix. In particular, we show that under certain conditions, the inverse of a banded plus semiseparable matrix can also be expressed as a banded plus semiseparable matrix. Using this result, we devise a fast algorithm for the solution of linear systems of equations involving such matrices. Numerical results show that the new algorithm competes favorably with existing techniques in terms of computational time.

Keywords. Banded matrix, semiseparable matrix, fast linear algorithms.

1. Introduction

Understanding of structured matrices and computation with them have long been problems of theoretical and practical interest [1]. Recently a class of matrices called semiseparable matrices has received considerable attention [2, 3, 4, 5, 6, 7, 8, 9]. Perhaps the simplest example of a semiseparable matrix is given by the inverse of a symmetric tridiagonal matrix: If $A = A^T$ is irreducible tridiagonal and nonsingular, then it is well-known that A^{-1} can be written as:

$$A^{-1}_{ij} = \begin{cases} u_i v_j & \text{if } i \leq j, \\ u_j v_i & \text{if } i > j. \end{cases} \tag{1}$$

Matrices such as the one in (1) and its generalizations arise in a number of practical applications like integral equations [10, 11], statistics [12], and vibrational analysis [13]. Modeling with a semiseparable matrix evidently offers the potential of reducing the number of parameters describing a matrix by up to an order of magnitude (from $O(n^2)$ to $O(n)$ in the example in (1)). Moreover, it is known that computation with semiseparable matrices requires significantly reduced effort; for example, for a semiseparable matrix of the form in (1), matrix-vector multiplies can be performed in $O(n)$ (as compared to $O(n^2)$ in general).

This work was done when the authors were at Purdue University.

In several practical situations, semiseparable matrices do not arise alone; instead, matrices that are encountered, are a sum of diagonal and a semiseparable matrix or a banded and a semiseparable matrix. Examples where such matrices arise, are in boundary value problems [14, 15], and integral equations [16]. The computation with such matrices has been a subject of considerable interest. Several algorithms have been developed to deal with matrix inversion and linear equation solution with such matrices. Formulae for inversion of diagonal plus semiseparable matrices were first developed in [6]. However, these formulae were valid under the assumption that the matrix is strongly regular, i.e., it has non-vanishing principal minors. These restrictions were later removed in [4]. Recently several algorithms have been developed for solving linear systems of equation

$$Ax = c, \tag{2}$$

where the coefficient matrix A is a sum of diagonal and semiseparable matrix [3, 4, 5, 7]. Fast and numerically stable algorithms for banded plus semiseparable linear system of equations were proposed in [2].

We present two main results in this paper. First, we provide an explicit representation for inverses of banded plus semiseparable matrices. In particular, we show that under certain conditions, the inverse of a banded plus semiseparable matrix is again a banded plus semiseparable matrix. Our second contribution is to provide fast solutions of linear systems of equations with these matrices. A comparison with the state of the art shows that our method is about two times faster than existing solutions of linear system with diagonal plus semiseparable matrices. When banded plus semiseparable matrices are considered, our method is up to twenty times faster than existing solutions.

The remainder of the paper is organized as follows. In §2, we establish mathematical preliminaries and the notation used in the paper, as well as a brief review of the state of the art. In §3 we present formulae for the inverse of banded plus semiseparable matrices. We exploit this result in §4 to provide a fast algorithm for solving linear systems of equations. In §5, we establish the effectiveness of the new algorithm via numerical results. The extension of this algorithm to handle some special cases is presented in an appendix.

2. Preliminaries

For $k = 1,\ldots,a$, and $r = 1,\ldots,b$, let $u_k = \{u_k(i)\}_{i=1}^n$, $v_k = \{v_k(i)\}_{i=1}^n$, $p_r = \{p_r(i)\}_{i=1}^n$, and $q_r = \{q_r(i)\}_{i=1}^n$, be specified vectors. Then S_a^b, an $n \times n$ semiseparable matrix of order (a,b), is characterized as follows:

$$S_{ij} = \begin{cases} \sum_{k=1}^{a} u_k(i)v_k(j) & \text{if } i \le j, \\ \sum_{r=1}^{b} p_r(j)q_r(i) & \text{if } i > j. \end{cases} \tag{3}$$

We use $\mathbb{S}_n$ as the generic notation for the class of semiseparable matrices of size n.

We use $\mathbb{B}_n$ to denote the class of banded matrices of size n. $B_l^m = \left\{B_{ij}\right\}_{i,j=1}^n$ is used to denote a banded matrix with l non-zero diagonals strictly above the main diagonal and m non-zero diagonals strictly below the main diagonal, i.e., if $B_l^m \in \mathbb{B}_n$ then $B_{ij} = 0$ if $i-j > m$ or $j-i > l$. The numbers l and m are called respectively the upper and lower bandwidths of a banded matrix B_l^m. $\mathbb{D}_n$ is used to denote the class of diagonal matrices.

$D = \mathrm{diag}(d)$ is used to denote a diagonal matrix with d as the main diagonal. We now define a proper banded matrix.

Definition 2.1. A nonsingular banded matrix B_l^m is said to be proper if any sub-matrix obtained by deleting $r(= \max(l,m))$ consecutive rows and r consecutive columns is nonsingular.

It is well-known that the inverses of banded matrices are semiseparable matrices.

Theorem 2.2 ([17]**).** *Let B_l^m be a $n \times n$ proper banded matrix. Then its inverse can be written as*

$$(B_l^m)^{-1} = S_l^m, S_l^m \in \mathbb{S}_n.$$

Remark 2.3. The above semiseparable representation, though elegant and compact, suffers from numerical instabilities [18], making it of limited practical use. Hence, the above representation will be only used as a theoretical tool, and not in any numerical implementations.

We next present a brief review of the state of the art for solving linear system of equation in (2), where A is a sum of banded and semiseparable matrices. We first consider the case when A is a sum of a diagonal and a semiseparable matrix. Two algorithms for solving such systems were developed in [3]. The first step is the same with both algorithms, where $A \in n \times n$ is reduced to a upper Hessenberg matrix H via $n-1$ Givens rotations:

$$A = \underbrace{G_2^T G_3^T \cdots G_n^T}_{G_{2,\ldots,n}^T} H.$$

It was shown that $G_{2,\ldots,n}^T$ is a lower Hessenberg matrix, whose upper triangular part is the upper triangular part of a unit-rank matrix. The upper triangular part of H was shown to be the upper triangular part of a matrix of rank two. The second step of both algorithms is to reduce the upper Hessenberg matrix H into an upper triangular matrix via $n-1$ Givens rotations. Two different algorithms were obtained by either applying the Givens rotations on the left of H, leading to QR algorithm, or applying Givens rotations to the right of H, obtaining the URV algorithm. Exploiting the low rank structure of $G_{2,\ldots,n}^T$ and H, both algorithms were shown to require $54n-44$ flops as compared to $58n$ flops for the algorithm in [5] and $59n$ flops for the one in [2].

The authors in [2] proposed a fast and numerical stable algorithm for the more general case of the solution of (2) when A is a sum of banded and semiseparable matrix. The basic idea is to compute a two-sided decomposition of the matrix A such that $A = WLH$. Here L is a lower triangular matrix, and both W and H can be written as a product of elementary matrices. An efficient algorithm for solving (2) is obtained by inverting W, L, and H on the fly. The matrices W and H can be either obtained via standard Gaussian elimination, or by using Givens rotations and Householder reflection matrices. The algorithm based on Gaussian elimination was shown to be marginally better in computational time than the one based on Givens rotations and Householder reflections, with the latter algorithm performing better with respect to accuracy. For a banded matrix with upper and lower bandwidth l and m and a semiseparable matrix of size n and order (a,b), the algorithm based on Givens rotation and Householder reflection was shown to have an operation count of $\left(11a^2+2\left(2l+2m+3b+5a\right)\left(l+a\right)\right)n$ flops, while the algorithm based on Gaussian elimination requires $\left(9a^2+2\left(l+2m+2b+2a\right)\left(l+a\right)\right)n$ flops.

The approach we take in this paper is to first provide an explicit representation for inverses of banded plus semiseparable matrices. We then exploit these results to come up with a fast algorithm for solving linear systems of equations.

3. Structure for inverses of banded plus semiseparable matrices

We begin this section by considering semiseparable matrices of order (1,1). We first present a theorem on multiplicative structure of inverses of banded plus semiseparable matrices of order $(1,1)$.

Theorem 3.1. *Let B_l^m be a $n \times n$ banded matrix and S_1^1 be a $n \times n$ semiseparable matrix of order (1,1). Then the inverse of their sum has the following multiplicative structure:*

$$\left(B_l^m+S_1^1\right)^{-1}=DL^T\left(B_{l+1}^{m+1}\right)^{-1}L\tilde{D},$$

where D, $\tilde{D}\in\mathbb{D}_n$, $B_{l+1}^{m+1}\in\mathbb{B}_n$, and L is a lower bidiagonal matrix.

Proof: Let

$$\left(S_1^1\right)_{ij}=\begin{cases} u_i v_j & \text{if } i\le j, \\ p_i q_j & \text{if } i>j. \end{cases}$$

Assume $v_1(i)\neq 0$, $q_1(i)\neq 0$ for this and the next section. The results have been extended for the more general case in the appendix. Let $D_v=\mathrm{diag}(v)$ and $D_q=\mathrm{diag}(q)$ be diagonal matrices. Let L be a lower bidiagonal matrix, which is defined as follows:

$$L=\begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix}. \tag{4}$$

It can be easily verified that

$$LD_qS_1^1D_vL^T=\hat{L}, \tag{5}$$

$$D_vL^T\hat{S}_1^1LD_q=\tilde{S}_1^1+D, \tag{6}$$

$$LD_q\hat{S}_1^1D_vL^T=\tilde{S}_1^1+\tilde{L}, \tag{7}$$

where $\hat{L}$, $\tilde{L}$ are lower bidiagonal matrices, $D\in\mathbb{D}_n$, and $\hat{S}_1^1$, $\tilde{S}_1^1\in\mathbb{S}_n$. Now, from (5)

$$LD_q\left(B_l^m+S_1^1\right)D_vL^T=LD_qB_l^mD_vL^T+\hat{L}.$$

It can be readily verified that matrix $LD_qB_l^mD_vL^T$ is banded with lower and upper bandwidth $l+1$ and $m+1$ respectively, i.e., $LD_vB_l^mD_q^TL^T = \tilde{B}_{l+1}^{m+1} \in \mathbb{B}_n$. Moreover, as $\hat{L}$ is lower bidiagonal, we have

$$LD_q\left(B_l^m + S_1^1\right)D_vL^T = B_{l+1}^{m+1}$$

Thus

$$\left(B_l^m + S_1^1\right)^{-1} = D_vL^T\left(B_{l+1}^{m+1}\right)^{-1}LD_q.$$

□

Remark 3.2. Suppose the banded matrix B_{l+1}^{m+1} in Theorem 3.1 is proper. Then using Theorem 2.2 and (6), we have the following elegant additive structure for the inverse of a banded and semiseparable matrix of order $(1,1)$:

$$\begin{aligned}\left(B_l^m + S_1^1\right)^{-1} &= D_vL^T\tilde{S}_{l+1}^{m+1}LD_q\\ &= S_{l+1}^{m+1} + D,\end{aligned}$$

where $S_{l+1}^{m+1} \in \mathbb{S}_n$, and $D \in \mathbb{D}_n$.

More generally, the following theorem characterizes the inverses of general banded plus semiseparable matrices.

Theorem 3.3. *Let B_l^m be a $n\times n$ banded matrix and S_a^b be a $n\times n$ semiseparable matrix. Then the inverse of their sum has the following multiplicative structure:*

$$\left(B_l^m + S_a^b\right)^{-1} = D_1'L^T\cdots D_b'L^T\left(B_{l+a}^{m+b}\right)^{-1}LD_a\cdots LD_1,$$

where $D_1, D_1', D_2, D_2', \ldots, D_a, D_a', D_b, D_b' \in \mathbb{D}_n$, $B_{l+a}^{m+b} \in \mathbb{B}_n$, and L is a lower bidiagonal matrix.

Proof: Without loss of generality, assume $b \geq a$. Now, from (5) and (7)

$$\begin{aligned}LD_1\left(B_l^m + S_a^b\right)D_1'L^T &= B_{l+1}^{m+1} + S_{a-1}^{b-1}\\ LD_a\cdots LD_1\left(B_l^m + S_a^b\right)D_1'L^T\cdots D_a'L^T &= B_{l+a}^{m+a} + S_0^{b-a}\\ LD_a\cdots LD_1\left(B_l^m + S_a^b\right)D_1'L^T\cdots D_a'L^T\cdots D_b'L^T &= B_{l+a}^{m+b},\end{aligned}$$

where $D_1, D_1', D_2, D_2', \ldots, D_b, D_b' \in \mathbb{D}_n$. Thus

$$\left(B_l^m + S_a^b\right)^{-1} = D_1'L^T\cdots D_a{}'L^T\cdots D_b'L^T\left(B_{l+a}^{m+b}\right)^{-1}LD_a\cdots LD_1.$$

□

Remark 3.4. Suppose the banded matrix B_{l+a}^{m+b} in Theorem 3.3 is proper. Then using Theorem 2.2 and (6), we have the following elegant additive structure for the inverse of a banded and semiseparable matrix:

$$\begin{aligned}\left(B_l^m + S_a^b\right)^{-1} &= D_1'L^T\cdots D_a'L^T\cdots D_b'L^T\tilde{S}_{l+a}^{m+b}LD_a\cdots LD_1\\ &= B_{a-1}^{b-1} + S_{l+a}^{m+b},\end{aligned}$$

where $B_{a-1}^{b-1} \in \mathbb{B}_n$, and $S_{l+a}^{m+b} \in \mathbb{S}_n$.

Corollary 3.5. *The inverse of a diagonal plus semiseparable matrix is again a diagonal plus semiseparable matrix.*

4. Fast solution of $Ax = c$

In this section, we consider the problem of finding the solution of linear systems of equation, $Ax = c$, where the coefficient matrix, A is a sum of banded and semiseparable matrix, i.e.,

$$A = B_l^m + S_a^b,\ B_l^m \in \mathbb{B}_n,\ S_a^b \in \mathbb{S}_n.$$

Without loss of generality assume $b \geq a$.

$$x = A^{-1}c = \left(B_l^m + S_a^b\right)^{-1} c = D_1^{'} L^T \cdots D_a^{'} L^T \cdots D_b^{'} L^T \left(B_{l+a}^{m+b}\right)^{-1} LD_a \cdots LD_1 c. \tag{8}$$

`function` x = Ainvc(S_a^b, B_l^m, c)
1. Calculate $D_1, D_1^{'}, \ldots, D_a, D_a^{'}, \ldots, D_b, D_b^{'},$ and B_{l+a}^{m+b} as described in proof of Theorem 3.3;
2. $z = LD_a \cdots LD_1 c$;
3. $y = B_{l+a}^{m+b} \setminus z$ (Solve $B_{l+a}^{m+b} y = z$);
4. $x = D_1^{'} L^T \cdots D_a^{'} L^T \cdots D_b^{'} L^T y$;

`return` x;

Note that for solving $Ax = c$, we do not need the condition of the banded matrices being proper.

We now present a complexity analysis of the above mentioned procedure of solving $Ax = c$. We will assume $1 << a, b, l, m << n$ to make the complexity analysis simpler. The flop count of the overall algorithm is dominated by the cost of steps 1 and 3. Total cost of step-1, i.e., calculating $D_1, D_1^{'}, \ldots, D_a, D_a^{'}, \ldots, D_b, D_b^{'},$ and B_{l+a}^{m+b} is

$$\left(2\,(l+m)\max\,(a,b) + 8\left(a^2 + b^2\right)\right) n$$

flops. In step-3 the cost of solving a banded system of equations is $2(l+a)(m+b)n$ flops. Hence the total complexity of the proposed algorithm is

$$\left(2(l+a)(m+b) + 2\,(l+m)\max\,(a,b) + 8\left(a^2 + b^2\right)\right) n$$

operations. For diagonal plus semiseparable systems, the complexity reduces to $24n$ operations.

5. Numerical Results

In this section, we compare the results for computational time and accuracy for solving a linear system of equations of form $Ax = b$, for several algorithms. We compare the results for following four algorithms:

- Algorithm I: *QR* algorithm for solving a diagonal plus semiseparable system, as given in [3].
- Algorithm II: *URV* algorithm for solving a diagonal plus semiseparable system, as given in [3].
- Algorithm III: Chandrasekaran-Gu algorithm for banded plus semiseparable system of equations, as given in [2].
- Algorithm IV: The new algorithm as described in §4.

All numerical experiments were performed in MATLAB running on a 4 CPU 1.5GHz Intel® Pentium® machine. For Algorithm I and II, we used the author's implementation, taken directly from [19]. All the matrix entries are randomly generated, drawn from a Gaussian distribution with zero mean and unit variance.

5.1. *A* is a sum of diagonal and semiseparable matrix

We first present the computational requirements and the accuracy of our approach against Algorithm I and II from [3]. The matrix A comprises of a diagonal plus a semiseparable matrix. Table 1 summarizes the results. Error in solving x in $Ax = b$ is defined as $\frac{\|(Ax-b)\|_\infty}{\|A\|_\infty \|x\|_\infty}$. As expected all three algorithms are linear in computational time. Algorithm IV is faster than and comparable in accuracy to Algorithms I and II. For a system with size 320000, Algorithm IV is 1.9× faster than Algorithm I and 2.4× faster than Algorithm II. This supports the theoretical complexities mentioned previously, where Algorithm IV takes $24n$ operations as compared to $54n-44$ operations taken by Algorithm I and II.

Size	Error = $\frac{\|(Ax-b)\|_\infty}{\|A\|_\infty \|x\|_\infty}$			Time (in sec)		
	Alg I [3]	Alg II [3]	Alg IV	Alg I [3]	Alg II [3]	Alg IV
10000	3.33×10^{-19}	5.32×10^{-19}	3.27×10^{-18}	.34	.43	.14
20000	8.01×10^{-19}	4.66×10^{-19}	1.40×10^{-17}	.68	.85	.27
40000	9.24×10^{-19}	1.80×10^{-18}	5.63×10^{-18}	1.39	1.74	.65
80000	1.47×10^{-18}	8.27×10^{-19}	2.15×10^{-17}	2.77	3.50	1.41
160000	1.29×10^{-19}	5.49×10^{-19}	5.65×10^{-19}	5.59	7.00	3.05
320000	2.34×10^{-19}	5.24×10^{-19}	2.30×10^{-18}	11.06	13.92	5.86
640000	6.53×10^{-20}	1.71×10^{-19}	8.91×10^{-18}	22.02	27.79	11.86
1280000	8.30×10^{-20}	8.40×10^{-19}	2.88×10^{-18}	44.17	55.89	24.39

TABLE 1. Error values and Computational time as compared with Algorithms I and II.

5.2. *A* is a sum of banded and semiseparable matrix

We now present the computational requirements and the accuracy of our approach against Algorithm III from [2]. The matrix A comprises of a banded plus semiseparable matrix. The three variables of interest of the coefficient matrix are the bandwidth of the banded matrix, order of the semiseparable matrix and size of the system. We first give results by varying one of the quantities at a time, keeping other two constant. Table 2 shows the results for increasing sizes of the linear systems. The upper and lower bandwidth of the banded matrix in all cases is 5. The order of the semiseparable matrix is also kept constant at $(5,5)$. Algorithm IV performs favorably in terms of computational time, being $10\times$ faster than Algorithm IV for the size of 8000. Similar results are seen in Table 3 and 4, where we are varying the bandwidth and the order respectively. Algorithm III exhibits better accuracy than Algorithm IV, as it relies heavily on Givens rotations.

We now give results when all three variables are varied at the same time. Table 5 shows the results for increasing sizes of the linear systems. The upper and lower bandwidth of banded matrices as well as the order of semiseparable matrices are varied as $\frac{n}{250}$, where n denotes the size of the system. Algorithm IV is faster, but at the expense of numerical accuracy. The computational times are consistent with the theoretical flop count. For $a = b = l = m = r$, flop count of Algorithm III reduces to $59r^2n$. For the same case, flop count of the proposed Algorithm is $28r^2n$.

Size	Error = $\frac{\\|(Ax-b)\\|_\infty}{\\|A\\|_\infty \\|x\\|_\infty}$		Time (in sec)	
	Alg III [2]	Alg IV	Alg III [2]	Alg IV
1000	8.58×10^{-17}	3.07×10^{-13}	1.33	0.13
2000	6.41×10^{-17}	8.42×10^{-13}	2.65	0.26
3000	1.78×10^{-16}	2.68×10^{-13}	3.96	0.39
4000	7.14×10^{-17}	1.57×10^{-12}	5.29	0.53
5000	1.84×10^{-16}	1.87×10^{-12}	6.57	0.65
6000	1.20×10^{-16}	1.25×10^{-12}	7.98	0.80
7000	9.05×10^{-17}	8.48×10^{-13}	9.27	0.96
8000	3.73×10^{-17}	1.03×10^{-12}	10.62	1.12

TABLE 2. Error values and Computational time as compared with Algorithm III; $l = m = a = b = 5$.

l=m	Error = $\frac{\\|(Ax-b)\\|_\infty}{\\|A\\|_\infty \\|x\\|_\infty}$		Time (in sec)	
	Alg III [2]	Alg IV	Alg III [2]	Alg IV
11	7.51×10^{-17}	3.34×10^{-13}	2.24	0.16
21	2.33×10^{-16}	2.40×10^{-13}	3.57	0.24
31	8.64×10^{-16}	4.82×10^{-13}	4.90	0.35
41	6.88×10^{-16}	1.22×10^{-12}	5.54	.46
51	4.32×10^{-16}	5.61×10^{-13}	5.60	.55
61	1.33×10^{-15}	2.33×10^{-12}	7.24	.67
71	4.95×10^{-16}	1.26×10^{-12}	8.19	.71
81	9.71×10^{-16}	2.87×10^{-12}	9.61	.81

TABLE 3. Error values and Computational time as compared with Algorithm III; $n = 4000$, $a = b = 1$.

a=b	Error = $\frac{\|(Ax-b)\|_\infty}{\|A\|_\infty \|x\|_\infty}$		Time (in sec)	
	Alg III [2]	Alg IV	Alg III [2]	Alg IV
11	2.41×10^{-16}	2.70×10^{-10}	8.96	1.09
21	3.52×10^{-16}	2.88×10^{-10}	15.47	3.20
31	9.79×10^{-16}	1.82×10^{-8}	25.29	6.04
41	6.37×10^{-16}	2.48×10^{-9}	43.19	9.40
51	1.16×10^{-15}	3.75×10^{-9}	61.07	14.83
61	1.01×10^{-15}	6.06×10^{-9}	117.37	21.32
71	1.68×10^{-15}	1.62×10^{-7}	188.40	27.26
81	1.70×10^{-15}	1.03×10^{-7}	288.77	34.94

TABLE 4. Error values and Computational time as compared with Algorithm III; $n = 4000$, $l = m = 1$.

Size	Error = $\frac{\|(Ax-b)\|_\infty}{\|A\|_\infty \|x\|_\infty}$		Time (in sec)	
	Alg III [2]	Alg IV	Alg III [2]	Alg IV
1000	7.00×10^{-17}	1.21×10^{-12}	1.31	0.13
2000	6.68×10^{-17}	7.69×10^{-12}	4.49	0.90
3000	2.69×10^{-16}	1.69×10^{-11}	8.84	2.71
4000	2.51×10^{-16}	1.46×10^{-9}	16.46	6.53
5000	5.14×10^{-16}	6.25×10^{-10}	26.36	11.62
6000	2.32×10^{-16}	1.11×10^{-10}	41.16	19.50
7000	2.55×10^{-16}	2.67×10^{-9}	63.70	30.93
8000	6.37×10^{-16}	3.39×10^{-10}	96.16	45.39

TABLE 5. Error values and Computational time as compared with Algorithm III; $l = m = a = b = \frac{n}{250}$.

6. Conclusions

We have presented a representation for inverse of banded plus semiseparable matrices. We have also presented fast algorithms for solving linear system of equations with these matrices. Numerical results show that the proposed approach competes favorably with the state of the art algorithms in terms of computational efficiency.

Appendix

In the discussion till now, we have assumed we are given a semiseparable matrix S_a^b, as defined in (3), such that for all i,k $v_k(i) \neq 0, q_k(i) \neq 0$. Now, we will give procedure to modify the proposed methods when such assumptions do not hold true. We will only consider the symmetric case, i.e., $p = u,\ q = v$. In addition, we assume that the order of semiseparable matrices in consideration is $(1,1)$, i.e., $l = m = 1$. Hence

$$S_{ij} = \begin{cases} u_i v_j & \text{if } i \le j, \\ u_j v_i & \text{if } i > j. \end{cases}$$

The general case can be handled in a similar fashion.

Let us assume $v_k = 0,$ and $v_i \neq 0$ if $i \neq k$. The technique that we propose next can be easily extended to handle the case when for more than one $i,\ v_i$ is zero. Consider the matrices L, and D_v as defined in §3. In addition, let $L(k,k-1) = 0$, and $D_v(k,k) = 1$. Then it can be easily verified that

$$L D_v S_1^1 D_v L^T = M_{u,v},$$

where M is defined as follows:

$$M_{u,v} = \begin{pmatrix} \alpha_1 & & & & \alpha_1 & & \\ & \alpha_2 & & & \alpha_2 & & \\ & & \ddots & & \vdots & & \\ & & & 0 & \alpha_k & & \\ \alpha_1 & \alpha_2 & \dots & \alpha_k & \alpha_{k+1} & & \\ & & & & & \ddots & \\ & & & & & & \alpha_n \end{pmatrix},$$

where $\alpha_1 = u_1 v_1,\ \alpha_k = u_k,\ \alpha_{k+1} = u_{k+1} - u_k,\ \alpha_i = u_i v_i - u_{i-1} v_{i-1}$ for all $i \notin \{1,k,k+1\}$. We can now do a tridiagonal decomposition of $M_{u,v}$ as

$$PMP^T = T,$$

where T is a tridiagonal matrix, and P is given by

$$P = \begin{pmatrix} 1 & -\frac{\alpha_1}{\alpha_2} & & & \\ & 1 & -\frac{\alpha_2}{\alpha_3} & & \\ & & \ddots & \ddots & \\ & & & 1 & -\frac{\alpha_{n-1}}{\alpha_n} \\ & & & & 1 \end{pmatrix}$$

Proceeding as with the proof of Theorem 1, we can show that

$$\left(B_l^m + S_1^1\right)^{-1} = D L^T \left(B_{l+2}^{m+2}\right)^{-1} L \tilde{D},$$

where $D, \tilde{D} \in \mathbb{D}_n, B_{l+2}^{m+2} \in \mathbb{B}_n$, and L is a lower bidiagonal matrix.

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