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On Matrix Trace Inequalities and Related Topics for Products of Hermitian Matrices

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Alternative proofs of some simple matrix trace inequalities of Bellman [*in* "General Inequalities 2, Proceedings, 2nd International Conference on General Inequalities" (E. F. Beckenbach, Ed.), pp. 89–90, Birkhäuser, Basel, 1980], Neudecker [*J. Math. Anal. Appl.* **166** (1992), 302–303], and Yang [*J. Math. Anal. Appl.* **133** (1988), 573–574] are considered and further properties of products of Hermitian and positive (semi)definite matrices are investigated.

MATRIX TRACE INEQUALITIES

THEOREM 1. *For positive semidefinite matrices, A, B, of the same order*

$$0 \le \operatorname{tr}(AB) \le \operatorname{tr}(A)\operatorname{tr}(B). \tag{1}$$

Proof. Since A is positive semidefinite it has a positive semidefinite square root matrix, $A^{1/2}$ with $\operatorname{tr}(AB) = \operatorname{tr}(A^{1/2}BA^{1/2})$ and

$$\begin{aligned} 0 \le \operatorname{tr}(AB) = \operatorname{tr}(A^{1/2}B^{1/2}B^{1/2}A^{1/2}) &= \|A^{1/2}B^{1/2}\|_F^2 \\ &\le \|A^{1/2}\|_F^2\|B^{1/2}\|_F^2 \\ &= \operatorname{tr}(A)\operatorname{tr}(B). \quad \blacksquare \end{aligned}$$

In the above $\|\cdot\|_F$ denotes the Frobenius matrix norm so that $\|C\|_F^2 = \operatorname{tr}(C^HC)$ for any matrix C. If A and B are both positive definite then inequality (1) may be strengthened to $0 < \operatorname{tr} AB \le \operatorname{tr} A \operatorname{tr} B$ because

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$A^{1/2}B^{1/2}$ has full rank so $\|A^{1/2}B^{1/2}\|_F > 0$. Theorem 1 was also established by Neudecker [2] using the "vec" operator, properties of Kronecker products, and "two well-known inequalities" but the above proof seems much clearer.

Both Neudecker [2] and Yang [3] show further than

$$(\operatorname{tr} AB)^{1/2} \le \frac{1}{2}(\operatorname{tr} A + \operatorname{tr} B) \tag{2}$$

when A, B are positive semidefinite, a result conjectured by Bellman [1]. Surprisingly, Bellman [1] had already proved (very neatly) the stronger results that for positive (semi)definite matrices A, B

$$2 \operatorname{tr}(AB) \le \operatorname{tr}(A^2) + \operatorname{tr}(B^2), \tag{3}$$

and

$$\operatorname{tr}(AB) \le (\operatorname{tr} A^2)^{1/2}(\operatorname{tr} B^2)^{1/2}. \tag{4}$$

These last two inequalities clearly imply (1) and (2) since $\operatorname{tr}(A^2) \le (\operatorname{tr} A)^2$ for positive semidefinite A. In fact the property that the trace of a product of two positive definite matrices is positive is not surprising because the much stronger result holds that the eigenvalues of this product matrix are positive.

THEOREM 2. *If A, B are positive definite then AB has positive eigenvalues.*

Proof. AB is similar to the positive definite matrix $A^{1/2}BA^{1/2}$ (or $B^{1/2}AB^{1/2}$). ∎

The eigenvalues of a product of two Hermitian (indefinite) matrices will not usually be real and the diagonal elements will in general be complex. Nevertheless, the following result is true.

THEOREM 3. *The trace of a product of two Hermitian matrices of the same order is real.*

Proof. Let A, B be Hermitian matrices of the same order. Then there exists a real number α such that $C = A + \alpha I$ is positive definite. Thus C has a positive square root matrix $C^{1/2}$ and

$$\operatorname{tr}(AB) = \operatorname{tr}(CB) - \alpha \operatorname{tr}(B) = \operatorname{tr}(C^{1/2}BC^{1/2}) - \alpha \operatorname{tr}(B). \tag{5}$$

The result follows because each of the matrices $C^{1/2}BC^{1/2}$ and B is Hermitian. ∎

In general, none of the Theorems 1, 2, 3 can be extended to products of more than two matrices as the following example shows,

$$A = \begin{bmatrix} 1 & -2i \\ 2i & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}.$$

With A, B, C defined as the positive semidefinite matrices above, each of the products AB, BC, CA have non-negative traces in accord with Theorem 1,

$$AB = \begin{bmatrix} 1 - 2i & 1 - 2i \\ 4 + 2i & 4 + 2i \end{bmatrix}, \quad BC = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix},$$

$$CA = \begin{bmatrix} 4 - 4i & -8 - 8i \\ -2 + 2i & 4 + 4i \end{bmatrix}.$$

However, the matrix ABC does not have a real trace:

$$ABC = \begin{bmatrix} 2 - 4i & -1 + 2i \\ 8 + 4i & -4 - 2i \end{bmatrix}.$$

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