A bibliography on semiseparable matrices

Raf Vandebril
Marc Van Barel
Gene Golub
Nicola Mastronardi

Report TW 412, Dec 2004

Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

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Abstract

Currently there is a growing interest in semiseparable matrices and generalized semiseparable matrices. As it is interesting to know the historical evolution of this concept, we present in this paper an extensive list of publications related to the field of semiseparable matrices. It is interesting to see that semiseparable matrices were investigated in different fields, e.g. integral equations, statistics, vibrational analysis, independently of each other. Also interesting to know is that the leading statisticians at that time used semiseparable matrices, without knowing there inverses to be tridiagonal. During this historical evolution the definition of semiseparable matrices has always been a difficult point leading to misunderstandings, as they were sometimes defined as the inverses of irreducible tridiagonal matrices leading to generator representable matrices, while in other cases they were defined as matrices having low rank blocks below the diagonal.

In this overview we present a list of interesting results which contributed to the evolution of the concept of semiseparable matrices as we know them nowadays.

Keywords : semiseparable matrices, tridiagonal matrices, historical overview
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A bibliography on semiseparable matrices *

Raf Vandebril[†], Marc Van Barel[‡], Gene Golub[‡] and Nicola Mastronardi[§]

June 2004

Abstract

Currently there is a growing interest in semiseparable matrices and generalized semiseparable matrices. As it is interesting to know the historical evolution of this concept, we present in this paper an extensive list of publications related to the field of semiseparable matrices. It is interesting to see that semiseparable matrices were investigated in different fields, e.g. integral equations, statistics, vibrational analysis, independently of each other. Also interesting to know is that the leading statisticians at that time used semiseparable matrices, without knowing there inverses to be tridiagonal. During this historical evolution the definition of semiseparable matrices has always been a difficult point leading to misunderstandings, as they were sometimes defined as the inverses of irreducible tridiagonal matrices leading to generator representable matrices, while in other cases they were defined as matrices having low rank blocks below the diagonal.

In this overview we present a list of interesting results which contributed to the evolution of the concept of semiseparable matrices as we know them nowadays.

Keywords: semiseparable matrices, tridiagonal matrices, historical overview.

1 Semiseparable matrices

As semiseparable matrices were defined in different ways, we will clearly distinguish the different definitions.

Definition 1. *S is called a semiseparable matrix of semiseparability rank r if there exist two matrices R_1 and R_2, both of rank r, such that*

$$S = triu(R_1) + tril(R_2);$$

$triu(R_1)$ and $tril(R_2)$ denote respectively the upper triangular part of the matrix R_1 and the strictly lower triangular part of the matrix R_2. Suppose the semiseparability rank to be equal to 1. This means that R_1 and R_2 are two rank one matrices. Therefore they can both be written as the outer product of two vectors, respectively u and v for R_1 and s and t for R_2. These vectors are also called the generators of the semiseparable matrix S.

In the remainder of the paper, these semiseparable matrices will be addressed as generator representable semiseparable matrices. Generator representable semiseparable matrices can be seen as the inverses of irreducible tridiagonal matrices.

The following definition is a little more general than the previous one in the sense that all symmetric semiseparable matrices satisfying the following definition are block diagonal matrices for which the blocks are generator representable symmetric semiseparable matrices.

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[†]K.U.Leuven, Dept. Computer Science, Email: {raf.vandebril,marc.vanbarel}@cs.kuleuven.ac.be

[‡]SCCM Program, Gates 2B, Stanford University, CA, USA, Email: golub@stanford.edu

[§]Istituto per le Applicazioni del Calcolo M. Picone, sez. Bari, Email:irmanm21@area.ba.cnr.it

Definition 2. *A matrix S is called a lower- (upper-)semiseparable matrix of semiseparability rank r if all submatrices which can be taken out of the lower (upper) triangular part of the matrix S have rank $\leq r$ and there exists at least one submatrix having exact rank r.*

Matrices satisfying this definition will properly be adressed as semiseparable matrices. They can be seen as the inverses of tridiagonal matrices, without any constraints. For example one can easily verify that every diagonal matrix is a semiseparable matrix, but not a generator representable semiseparable matrix (except in special cases).

2 The overview

In this section an historical overview of papers and books closely related to the topic of semiseparable matrices is given. For all the references a small summary of the results is included. Also different types of applications and extensions of the class of semiseparable matrices are mentioned. It can be seen, that interesting results are often rediscovered in different fields. Unfortunately we were not able to retrieve the following papers: [133] cited in [66, 89]; the papers cited in [116]; [48]; some citations of [121]; the papers [11, 12].

The name semiseparable matrix finds its origin in the discretization of kernels. A separable kernel $k(x,y)$ means that the kernel is of the following form $k(x,y) = g(x)f(y)$. Associating a matrix with this kernel gives a rank 1 matrix. This matrix can be written as the outer product of two vectors. A semiseparable kernel satisfies the following properties (sometimes also called a Green's kernel [82, p. 110]):

$$k(x,y) = \begin{cases} g(x)f(y) & \text{if} \quad x \leq y \\ g(y)f(x) & \text{if} \quad y \leq x \end{cases}.$$

Associating a matrix with this semiseparable kernel, gives us a generator representable semiseparable matrix.

1937 [59] In this paper of Gantmacher and Krein, they prove that the inverse of a symmetric Jacobi matrix (this corresponds to a symmetric irreducible tridiagonal matrix) is a one-pair matrix (this corresponds to a symmetric generator representable semiseparable matrix) via explicit calculations.

1950 [60] In the book of Gantmacher and Krein, the theory of [59] is included. This book is often referred to as the first book in which the inverse of an irreducible tridiagonal matrix is calculated.

1953 [10] Berger and Saibel provide in this paper an explicit formula for calculating the inverse of a continuant matrix (this is a tridiagonal matrix, not necessarily symmetric), based on the LU-decomposition of the original continuant matrix. Moreover, no conditions of nonzeroness are placed on the elements of the continuant matrix.

1956 [106] In this paper the authors Roy and Sarhan invert very specific matrices arising in statistical applications, e.g., a lower triangular semiseparable matrix, 2 types of specific semiseparable matrices and also a semiseparable plus diagonal matrix are inverted.

1957 [4] In this paper, S. O. Asplund, the father of E. Asplund proves the same as Gantmacher and Krein, by calculating the inverse via techniques for solving finite boundary value problems. A brief remark states that higher order band matrices have as inverses higher order Green's matrices (The Green's matrix can be considered as a generator representable semiseparable matrix).

1957 [3] E. Asplund formulates for the first time a theorem stating that the inverse of an invertible lower $\{p\}$-Hessenberg matrix (with no restrictions on the elements and called a $\{p\}$-band matrix in the paper) is an invertible lower $\{p\}$-Hessenberg-like matrix (with the definition of the ranks of subblocks) and vice versa. A $\{p\}$-Hessenberg-like matrix is referred to as a Green's matrix of order p in the paper. No explicit formulas for the inverse are presented, only theoretical results about the structure. More precisely, he also proves, then if the elements on the pth superdiagonal of the lower $\{p\}$-Hessenberg matrix are different from zero, that the inverse is a generator representable lower $\{p\}$-Hessenberg-like matrix.

1959 [66] In this paper by Greenberg and Sarhan, the papers [133, 106] are generalized and applied to several types of matrices arising in statistical applications. A relation is introduced which needs to be satisfied, such that the inverse of the matrix is a diagonal matrix of type r (these diagonal matrices of type r correspond to band matrices of width r). Different semiseparable matrices are given and the relation is investigated for r equal to $1,2$ and 3, thereby proving that these matrices have a banded inverse.

1960 [105] As the statistical research at that time was interested in fast calculations for so-called patterned matrices, the authors Roy, Greenberg and Sarhan designed an order n algorithm for calculating the determinant. The patterned matrices are semiseparable and semiseparable plus diagonal matrices.

1961 [112] Schechter provides a method for inverting nonsymmetric block tridiagonal matrices for blocks which have the same size and are invertible, based on the LDU-decomposition of these matrices. The applications of this method can be found in the field of partial differential equations.

1964 [117] Ting describes, based on the LU decomposition of a tridiagonal matrix T (not necessarily symmetric) a special decomposition of T^{-1} as a sum of inverses of bidiagonal matrices, multiplied with a diagonal matrix. This can be used for solving various systems of equations with the same tridiagonal matrix T.

1965 [118] Published in a Japanese Journal, and translated afterwards to [119].

1966 [119] Torii, provides explicit formulas for calculating the inverse of a nonsymmetric tridiagonal matrix. The stability of solving a tridiagonal system of equations via inversion is investigated.

1967 [25] The author Chandan inverts a symmetric semiseparable plus identity matrix, and concludes that it is again a symmetric semiseparable matrix plus the identity. This matrix results from a statistical problem.

1968 [87] The author Kounias provides explicit formulas for inverting some patterned matrices. For example, a tridiagonal Toeplitz matrix and a matrix completely filled with one element plus a diagonal are inverted.

1969 [122] Uppuluri and Carpenter present an exact formula to compute the inverse of a specific covariance matrix, which is a symmetric tridiagonal Toeplitz matrix.

1970 [2] Allgower provides a method for calculating the inverse of banded Toeplitz matrices.

1970 [84] Kershaw provides bounds between which the elements of the inverse of a tridiagonal matrix with positive off-diagonal elements will lie. The elements of the super and the subdiagonal of the tridiagonal are related in the following way: denote with T the tridiagonal matrix, then we have: $T_{i,i+1} = 1 - T_{i-1,i}$ for $1 < i < n$.

1970 [123] Uppuluri and Carpenter calculate the inverse of a tridiagonal Toeplitz matrix (not necessarily symmetric) via the associated difference equation. The results are based on [87].

1970 [23] Explicit formulas for the inverse of a nonsymmetric tridiagonal matrix are given by Capovani (no demands are placed on the elements of the tridiagonal). The formulas are adapted for the tridiagonal Toeplitz case and the formulas are applicable to block tridiagonal matrices (for which all the blocks have the same size) as well.

1971 [24] Capovani proves that the inverse of a nonsymmetric tridiagonal matrix with the super and subdiagonal elements different from zero, also has as inverse a generator representable matrix, where the upper and lower triangular part have different generators. Using these results the author proves that a pentadiagonal matrix can be written as the product of two tridiagonal matrices, for which one of them is symmetric.

1971 [5] Baranger and Duc-Jacquet prove that the inverse of a generator representable semiseparable matrix (called "une matrice factorisable" in the paper) is a tridiagonal matrix. They explicitely calculate the inverse of a generator representable semiseparable matrix.

1972 [79] Hoskins and Ponzo provide formulas for calculating the inverse of a specific band matrix of dimension $2r+1$, with the binomial coefficients in the expansion of $(x-1)^{2r}$ in each row and column. Such a matrix arises for example in the solution of partial difference equations.

1972 [102] Rehnqvist presents here a method for inverting a very specific Toeplitz band matrix. Via multiplication with another invertible Toeplitz matrix and reordering of the elements one can calculate the inverse of this matrix.

1972 [116] D. Szynal and J. Szynal present two theorems for the existence of the inverse of a Jacobi matrix and two methods for calculating the inverse. The results can be rewritten in block form such that they can be applied to specific band matrices. No specific demands of symmetry or nonzeroness of the elements have to be fulfilled.

1973 [78] Hoskins and Thurgur invert a specific type of band matrix (see [79]). The inversion formulas are obtained by calculating the LU/decomposition of the band matrix and formulas are given to compute the determinant and the infinity norm of this inverse.

1973 [86] Kounadis deduces in this paper a recursive formula for inverting symmetric block tridiagonal matrices, for which all the blocks are of the same dimension. This technique is also applied to the class of tridiagonal matrices.

1973 [21] Bukhberger and Emel'yanenko present in this paper a computational method for inverting symmetric tridiagonal matrices, for which all the elements are different from zero. The theoretical results are applicable in a physical application which studies the motion of charged particles in a particular environment. The results are identical to the ones in the book [60].

1974 [120] Trench presents a method for inverting $\{p,q\}$-banded Toeplitz matrices by exploiting the banded structure.

1976 [13] Bevilacqua and Capovani extend the results of the papers [66] and [1, 2] to band matrices and to block band matrices (not necessarily symmetric). Formulas are presented for inverting band matrices whose elements on the extreme diagonals are different from zero. The results are extended to block band matrices.

1976 [75] The summary of the paper, as provided by the authors Hoskins and McMaster : For a symmetric positive definite Toeplitz matrix of band width five and order n, those regions where the elements of the inverse alternate in sign are determined.

1977 [76] Hoskins and McMaster investigate properties of the inverse of banded Toeplitz matrices as in [75]. More precisely, they investigate the properties of a band matrix of width 4 and a band matrix of width 5 coming from a boundary value problem.

1977 [9] Berg provides in this paper for the first time an explicit technique for inverting a band matrix that is not symmetric, and has for the upper and lower part different bandwidths. We could already assume these results based on the paper by Asplund, but here explicit formulas are given. A disadvantage is again the strong assumption that the elements on the extreme super and subdiagonals have to be different from zero.

1977 [33] Demko proves theorems bounding the size of the elements of the inverse of band matrices in terms of the norm of the original matrix, the distance towards the diagonal and the bandwidth. In particular it is shown that the size of the elements decays exponentially to zero if one goes further and further away from the diagonal.

1977 [100] Neuman provides algorithms for inverting tri and pentadiagonal matrices (not necessarily symmetric) via the UL decomposition. Under some additional conditions the numerical stability of the algorithm is proved.

1977 [124] Valvi determines several explicit formulas for inverting specific patterned matrices arising in statistical applications. These matrices are specific types of semiseparable and semiseparable plus diagonal matrices.

1978 [77] Bounds on the infinity norm of the inverse of Toeplitz band matrices with band width 5 are derived by Hoskins and McMaster.

1978 [68] Greville provides conditions on band matrices such that the inverse of the band matrix becomes a Toeplitz matrix. This results in Toeplitz matrices which are also semiseparable.

1978 [101] Oohashi proves that the elements of the inverse of a band matrix can be expressed in terms of the solution of a homogeneous difference equation, related to the original band matrix. In this way explicit formulas for calculating the inverse are obtained. The band matrix does not need to have the same lower and upper band size, but the elements on the extreme diagonals need to be different from zero. The results are an extension of the results proved in [119] for tridiagonal matrices.

1978 [7] Barrett and Feinsilver provide a probabilistic proof, stating that the inverse of a symmetric generator representable semiseparable matrix, is a symmetric tridiagonal matrix. Note that the generator representable matrices are not characterized by two generators, but via the rank 1 assumptions in the lower and upper triangular part. The results are restricted to positive definite and symmetric matrices.

1979 [80] Ikebe provides in this paper an algorithm for inverting an upper Hessenberg matrix under the assumption that the subdiagonal elements are different from zero. It is proved that the inverse matrix has the lower triangular part representable with two generators. An extension towards block Hessenberg matrices is included.

1979 [132] Yamammoto and Ikebe propose formulas for inverting band matrices with different bandwidths under the assumption that the elements on the extreme diagonals are different from zero.

1979 [6] Barrett formulates another type of theorem connected to the inverses of tridiagonal matrices. In most of the preceding papers one assumed the sub and superdiagonal elements of the corresponding tridiagonal matrix to be different from zero. In this paper only one condition is left, it assumes that the diagonal elements of the symmetric semiseparable matrix are different from zero. Moreover, the proof is also suitable for nonsymmetric matrices. The theorems presented in this paper are very close to the final version result, stating that the inverse of a tridiagonal matrix is a semiseparable matrix, satisfying the rank definition.

1979 [113] Singh gives explicit formulas for inverting a lower block bidiagonal matrix. The blocks on the diagonal have to be invertible.

1980 [74] The author Haley splits the band system $Bx = u$ into two blocks, which can via recurrence divided once more. This leads to a method for calculating explicit inverses of banded matrices. As example tridiagonal and Toeplitz tridiagonal matrices are considered.

1981 [8] This paper should be considered as one of the most important papers concerning the inverses of band matrices. Barrett and Feinsilver provide a general framework as presented in the first chapter of this thesis. General theorems and proofs considering the vanishing of minors when looking at the matrices and their inverses are given, thereby characterizing the complete class of band and semiseparable matrices, without excluding cases in which there appear zeros. The results are a straightforward consequence of paper [6].

1982 [89] Lewis provides an explicit formula which can be used to compute the inverse of tridiagonal matrices. The matrix does not necessarily need to be symmetric, nor all elements have to be different from zero. Interesting is also a new kind of representation for nonsymmetric semiseparable matrices, which does not use 4 but three vectors x,y and z, where the elements of S are of the following form:

$$s_{ij} = \begin{cases} x_i y_j z_j, & i \geq j, \\ y_i x_j z_j, & i \leq j. \end{cases}$$

The explicit formula is a generalization of the theorem for symmetric matrices by Yoshimasa [133].

1984 [34] Decay rates for the inverse of band matrices are obtained by Demko, Moss and Smith. It is shown that the decay rate depends on the spectrum of the matrix AA^T, for a band matrix A.

1984 [103] Rizvi derives, based on the formulas proposed by Barrett and Feinsilver and on the LU-decomposition, formulas for inverting a quasi-tridiagonal matrix, which is in fact a block tridiagonal matrix (for which all the blocks have the same size). Necessary and sufficient conditions, for which a special block matrix has as inverse a quasi-tridiagonal matrix are derived.

1985 [63] Gohberg, Kailath and Koltracht provide a method for solving higher order semiseparable plus diagonal systems of equations. The class of matrices needs to be the strongly regular.

1986 [93] Mattheij and Smooke provide in this paper a method for deriving the explicit inverse of tridiagonal matrices, as coming from nonlinear boundary value problems. The formulas are also suitable for the block tridiagonal case.

1986 [104] Romani investigates the demands on a symmetric band matrix, such that its inverse can be written as the sum of inverses of irreducible tridiagonal matrices, because this is not true in general.

1986 [107] Rózsa investigates sufficient conditions to be placed on a semiseparable matrix, in order to have a nonsingular $\{p,q\}$-semiseparable matrix. This paper is to our knowledge the first paper in which there appeared the now well-known name of semiseparable matrices. A theorem is included stating that the inverse of a strict $\{p,q\}$-band matrix is a generator representable $\{p,q\}$-semiseparable matrix.

1986 [56] In this paper, Fiedler translates the abstract formulation in [71] towards the matrix case. The resulting theorem is called the nullity theorem in this thesis. This theorem is very useful in the field of semiseparable and tridiagonal matrices as all possible relations connected to the inverse fit in this nullity theorem.

1986 [22] A generalization of the papers [80, 132] is presented by Cao and Stewart, for Hessenberg matrices with a larger bandwidth and block Hessenberg matrices, for which the blocks do not necessary need to have the same dimension. The implementation described is suitable for parallel computations.

1987 [108] Rózsa proves, based on linear difference equations, the same results as in [107]. A small section is dedictated to band Toeplitz matrices.

1987 [57] In this paper Fiedler proves that the off-diagonal rank of a matrix is maintained under inversion.

1988 [14] Bevilacqua, Codenotti and Romani present a method to solve a block tridiagonal system in a parallel way, by exploiting the structure of the inverse of the block tridiagonal matrix.

1988 [45] Eijkhout and Polman provide decay rates for the inverse of band matrices which are close to Toeplitz matrices.

1989 [109] The authors Rózsa, Bevilacqua, Favati and Romani, present generalizations of their previous papers towards methods for computing the inverse of block tridiagonal and block band matrices. The results are very general and the blocks do not all need to be square or of the same dimension. The paper contains a lot of interesting references connected to the theory of semiseparable and tridiagonal matrices.

1990 [16] The authors Bevilacqua, Lotti and Romani, present two types of algorithms for reducing the total amount of storage locations needed by the inverse of a band matrix.

1990 [17] The authors Bevilacqua, Romani and Lotti provide a method for parallel inversion of band matrices. No other assumption is required than the nonsingularity of some of the principal submatrices.

1991 [53] Favati, Lotti, Romani and Rózsa provide theorems and formulas for the inverse of generalized block band matrices. The paper covers many different cases in a very general framework.

1991 [110] A new proof is included by Rózsa, Bevilacqua, Romani and Favati stating that the inverse of a $\{p,q\}$-semiseparable matrix is a strict $\{p,q\}$-band matrix. This new proof leads to a recursive scheme for calculating the inverse.

1992 [95] In this paper by Meurant many references connected to the inverse of semiseparable and band matrices are included. Some results concerning the inverse of symmetric tridiagonal and block tridiagonal matrices are reviewed, based on the Cholesky decomposition. Also results concerning the decay of the elements of the inverse are obtained.

1993 [54] Fiedler formulates general theorems connected to the structured ranks of matrices and their inverses.

1993 [20] The authors Brualdi and Massey generalize some of the results of [54], for structures in which the diagonal is not included.

1994 [111] Rózsa, Romani and Bevilacqua provide results similar to the ones in [53]. But the results are now proved via the nullity theorem, which is also proved in this paper.

1996 [38] Eidelman and Gohberg present an order $O(n)$ algorithm for calculating the inverse of a generator representable plus diagonal semiseparable matrix.

1997 [40] Eidelman and Gohberg present a look ahead recursive algorithm to compute the triangular factorization of generator representable semiseparable matrices plus a diagonal.

1997 [39] Eidelman and Gohberg present a fast and numerically stable algorithm for inverting semiseparable plus diagonal matrices. No conditions, except nonsingularity, are placed on the matrices.

1998 [94] McDonald, Nabben, Neumann, Schneider and Tsatsomeros pose properties on the class of generator semiseparable matrices such that their tridiagonal inverses belong to the class of Z-matrices.

1999 [46] Elsner investigates in more detail the inverses of band and Hessenberg matrices. The different cases, concerning generator representable semiseparable matrices and semiseparable matrices for which the special subblocks have low rank are included.

1997 [85] Koltracht provides in this paper a new method for solving higher order semiseparable systems of equations. The semiseparable matrix is thereby transformed into a narrow band matrix.

1999 [99] Nabben provides upper and lower bounds for the entries of the inverse of diagonally dominant tridiagonal matrices.

1999 [98] Decay rates for the inverse of special tridiagonal and band matrices are given by Nabben.

2000 [28] Chandrasekaran and Gu present an algorithm to transform a generator representable semiseparable matrix plus band matrix into a similar tridiagonal matrix.

2000 [121] Tyrtyshnikov expands the class of generator representable matrices towards a class called weakly semiseparable matrices. He proves that the inverse of a (p,q)-band matrix is a (p,q)-weakly semiseparable matrix, but the converse does not necessarily holds. This means that the class of (p,q)-weakly semiseparable matrices is a little more general than our class of (p,q)-semiseparable matrices.

2001 [50] This paper by Fasino and Gemignani provides for semiseparable matrices, also singular ones, a sparse structured representation. More precisely, it is shown that a semiseparable matrix A, always can be written as the inverse of a block tridiagonal matrix plus a sparse, low rank matrix Z.

2001 [51] Fasino and Gemignani, study in this paper the direct and the inverse eigenvalue problem of diagonal plus semiseparable matrices.

2001 [90] Mastronardi, Chandrasekaran and Van Huffel present an order $O(n)$ algorithm to solve a system of equations where the coefficient matrix is a generator representable semiseparable plus diagonal matrix. The algorithm is suitable for an implementation on two processors.

2002 [128] Van Camp, Van Barel and Mastronardi provide two fast algorithms for solving diagonal plus generator representable semiseparable systems of equations. The solution method consists of an effective calculation of the QR-factorization of this type of matrices.

2002 [125] In this paper, Van Barel, Fasino, Gemignani and Mastronardi investigate the relation between orthogonal rational functions and generator representable semiseparable plus diagonal matrices.

2002 [97] Mullhaupt and Riedel derive properties for matrices having weakly lower triangular rank equal to d. Decompositions of these matrices as a product of a unitary matrix and an upper $\{p\}$-Hessenberg matrix are provided.

2002 [61] Fasino and Gemignani describe an order $O(n)$ solver for banded plus semiseparable systems of equations. The algorithm exploits the structure of the inverse of the semiseparable matrix.

2003 [30] Chandrasekaran and Gu present a method for solving systems whose coefficient matrix is a semiseparable plus band matrix.

2003 [31] Chandrasekaran and Gu present a divide and conquer method to calculate the eigendecomposition of a symmetric generator representable semiseparable plus a block diagonal matrix.

2003 [127] The authors, Van Barel, Vandebril and Mastronardi construct in this paper an orthogonal similarity transformation to transform symmetric matrices in semiseparable form.

2003 [129] The same authors as above construct an implicit QR-algorithm for semiseparable matrices, in order to compute their eigendecomposition.

2003 [91] Mastronardi, Chandrasekaran and Van Huffel, provide an algorithm to transform a symmetric generator representable semiseparable plus diagonal matrix into a similar tridiagonal one. Also a second algorithm to reduce an unsymmetric generator representable semiseparable plus diagonal matrix to a bidiagonal one by means of orthogonal transformations is included.

2003 [92] Mastronardi, Van Camp and Van Barel present a divide and conquer algorithm to compute the eigendecomposition of diagonal plus generator representable semiseparable matrices.

2003 [52] Fasino, Mastronardi and Van Barel propose two new algorithms for transforming diagonal plus generator representable semiseparable matrices to tridiagonal or bidiagonal form. See also [91].

2003 [55] Fiedler investigates the properties of so-called basic matrices, these matrices have weakly lower and weakly upper triangular rank 1, in fact this class is closely related to the matrices.

2003 [130] Vandebril, Van Barel and Mastronardi translate in this paper the traditional algorithm for computing the singular values via bidiagonal matrices towards their analogue in the semiseparable case.

2003 [18] Bini, Gemignani and Pan derive an algorithm for performing a step of QR on a generalized semiseparable matrix. The lower part of this generalized matrix is in fact lower triangular semiseparable. The authors present an alternative representation, consisting of 3 vectors, to represent this part.

2003 [15] Bevilacqua and Del Corso investigate in this report the existence and the uniqueness of a unitary similarity transformation of a symmetric matrix into semiseparable form. Also the implementation of a *QR*-step on a semiseparable matrix without shift is investigated.

2004 [115] Strang and Nguyen investigate in this paper the nullity theorem as provided by Fiedler and Markham, and also the theoretical results from Barrett and Feinsilver.

2004 [126] Van Barel, Van Camp and Mastronardi provide an orthogonal similarity transformation to reduce symmetric matrices into a similar semiseparable one of rank k.

2004 [131] Vandebril, Van Barel and Mastronardi give a clear overview of the differences between generator representable semiseparable matrices and semiseparable matrices. Also a new type of representation for semiseparable matrices is presented.

2004 [58] Fiedler investigates the properties of a class called Generalized Hessenberg matrices, these matrices have weakly lower triangular rank 1.

2004 [49] Fasino proves that any Hermitian matrix, with pairwise distinct eigenvalues can be transformed into a similar diagonal plus semiseparable matrix, with a prescribed diagonal. Moreover, it is proved that the unitary transformation in this similarity reduction is the unitary matrix in the *QR*-factorization of a suitably defined Krylov subspace.

In the remaining part of this section we will briefly mention some new classes of matrices closely related to semiseparable matrices. Currently the class of sequentially semiseparable matrices is investigated thoroughly. (This class is sometimes also called quasiseparable, or recursively semiseparable.) Interesting algorithms and the definition of this class of matrices can be found in [26, 27, 29, 35, 36, 37, 41, 43]. Recently a new class of matrices, $\mathcal{H}$-matrices was introduced. This class is also closely related to the class of semiseparable and sequentially semiseparable matrices (see [19, 72, 73] and the references therein).

Semiseparable matrices also appear in various types of applications, e.g., the field of integral equations [64, 69, 81], operator theory [42, 44, 70], boundary value problems [64, 67, 88, 114], in the theory of Gauss-Markov processes [83], time varying linear systems [35, 62], in statistics [65], acoustic and electromagnetic scattering theory [32], signal processing [96, 97], numerical integration and differentiation [114] and rational interpolation [125]. Also in the biological field applications exist resulting directly in semiseparable matrices [47].

3 Conclusions

We hope the list we presented here gives a clear overview of publications related to semiseparable matrices. We are well aware that it is not possible to trace all related publications. Therefore we appreciate missing references and feedback. In this way we can maintain and update our reference list at our website.

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