J. Sertić
D. Kozak
R. Scitovski

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LU-DECOMPOSITION FOR SOLVING SPARSE BAND MATRIX SYSTEMS AND ITS APPLICATION IN THIN PLATE BENDING

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Summary

In this paper algorithm for solving sparse band system matrices is proposed. Algorithm is based on LU-decomposition, therefore has good numerical properties. Proposed algorithm is applied within the finite difference method (FDM) on solving thin plates bend problem. In order to compare the solution accuracy obtained by proposed algorithm, the same example has been solved by finite element method.

Key words: *LU-decomposition, sparse band matrix, thin plate bending, finite difference method*

LU-DEKOMPOZICIJA ZA RJEŠAVANJE VRPČASTIH RIJETKO POPUNJENIH MATRIČNIH SUSTAVA I PRIMJENA NA SAVIJANJE TANKIH PLOČA

Sažetak

U ovom se radu predlaže algoritam za rješavanje rijetko popunjenih vrpčastih matričnih sustava. Algoritam se temelji na LU-dekompoziciji, te stoga ima dobra numerička svojstva. Predloženi algoritam primijenjen je uz metodu konačnih diferencija (FDM) na rješavanje problema čvrstoće tankih ploča. Radi usporedbe točnosti rješenja dobivenih pomoću predloženog algoritma, isti primjer riješen je i uz pomoć metode konačnih elemenata.

Ključne riječi: *LU-dekompozicija, rijetko popunjene matrice, savijanje tankih ploča, metoda konačnih diferencija*

1. Introduction

Application of numerical methods in solving practical physical boundary problems includes solving a huge system of linear equations. The best known method for solving boundary problems is Finite Difference Method (FDM) [1]. By using this method, the derivatives of function of one or more variables can be approximated by divided differences. In this way a difference equations system is acquired and it must be solved by a numerical method [1].

One of the very used methods which have good numerical properties is the LU decomposition [1], [2], [3]. This paper deals with LU-decomposition for band matrices and its use in FDM. The application of the specified method is presented on a simple example of solving a biharmonic differential equation of the thin quadratic plate bending with constant thickness and uniform load with constant pressure [6].

FDM is often used in researches related to plate theory. In the paper [4] it is possible to see FDM application in calculation of rectangular plates with non-uniform wall thickness under the influence of arbitrary load. The application of the specified method can be also seen in the paper [5], which gives a contribution in the research of orthotropic plate deformation.

In solving such problems, great and rarely filled system matrices appear. In order to save time and memory of a computer that works with that kind of systems, a special algorithm for LU-decomposition that makes calculations only with elements within the band matrix system was proposed. That algorithm, as well as Crout's [3], gives the matrix decomposition to upper and lower triangular matrix which enable a simple solution.

2. Solving a Biharmonic Differential Equation of Thin Rectangular Plate Bending by Finite Difference Method

2.1. Differential Equation of Thin Rectangular Plate Bending

Differential equation of thin rectangular plate bending is obtained by using equilibrium conditions. Equilibrium conditions for a differential element are obtained by inner forces components: *bending moment* M_x and M_y, *torsional moment* $M_{xy}=M_{yx}$ and *shear forces* Q_{xz} and Q_{yz} [7].

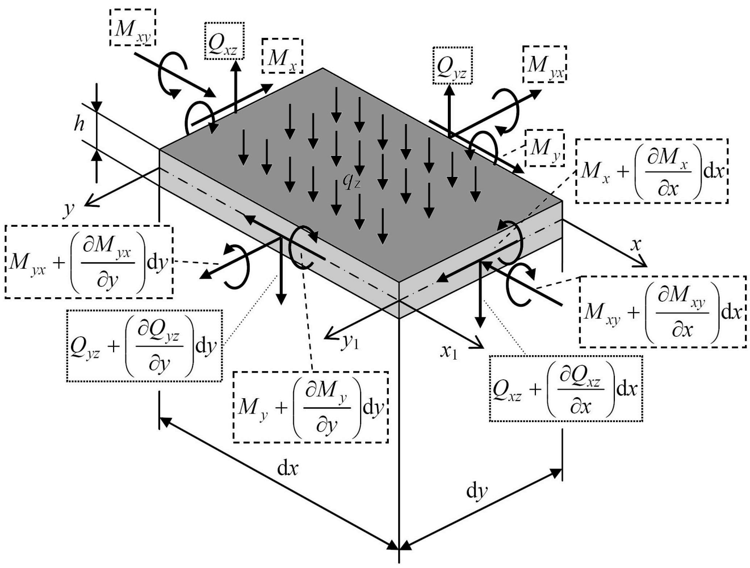

Fig. 1 Load and components of inner forces on plate element [7]

Slika 1. Opterećenje i komponente unutarnjih sila na elementu ploče [7]

From equilibrium conditions of differential element (Figure 1), and by using Hooke's law of plane stress state, a differential equation of thin rectangular plate bending is obtained [7]. That equation has the following form

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+\frac{\partial}{\partial y^2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\frac{q_z}{D}, \tag{1}$$

where:

D – flexural rigidity, [N·mm]

w – plate deflection in z-axis direction, [mm]

q_z – uniform plate loading, [N/mm^2]

x,y – rectangular coordinate system coordinates, [mm].

The equation [1] represents a biharmonic differential equation which can be written shorter as:

$$\nabla^4 w=\frac{q_z}{D}. \tag{2}$$

Flexural rigidity D represents a value which together with the defined material and plate thickness represents a constant. It is calculated by using the following expression

$$D=\frac{E\cdot h^3}{12\cdot\left(1-\nu^2\right)}, \tag{3}$$

where:

E - elasticity modulus, [N/mm^2]

h - plate thickness, [mm]

ν - Poisson's ratio.

2.2. Application of Finite Difference Method

Poisson's differential equation of mathematical physics has the following form

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=f(x,y). \tag{4}$$

If the expression (4) is inserted in (1) that equation takes a new form and it also represents Poisson's differential equation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{q_z}{D}. \tag{5}$$

Solving of differential equation (1) comes to double solving of Poisson's differential equation with appropriate boundary conditions. The equation (5) must be solved first.

The area of function definition f (x,y) is the rectangular plate of width d. The area of function definition f (x,y) is covered by two families of directions [3] which are parallel to coordinate axes x,y

$$x_{\mathrm{i}}=i\cdot t, \qquad y_{\mathrm{j}}=j\cdot t, \qquad i,j\in\mathrm{Z},$$

where t is the given step of the mesh. If the nodes of the mesh are divided to boundary and inner ones, then the number of inner nodes is n^2.

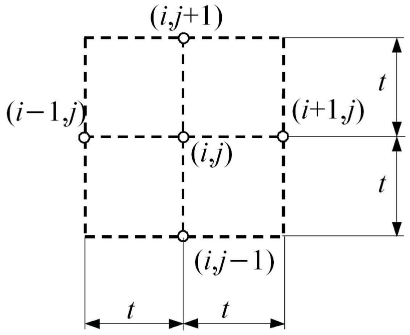

Fig. 2 Pentdotted star in point (i,j)

Slika 2. Peterotočkasta zvijezda u točki (i,j)

According to Taylor's formula, and for the scheme in Figure 2, approximations of other partial derivations of function f are obtained, so that

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{i,j} \approx \frac{f_{i+1,j} - 2\cdot f_{i,j} + f_{i-1,j}}{t^2}, \tag{6}$$

and

$$\left(\frac{\partial^2 f}{\partial y^2}\right)_{i,j} \approx \frac{f_{i,j+1} - 2\cdot f_{i,j} + f_{i,j-1}}{t^2}. \tag{7}$$

If the expressions for partial derivations (6) and (7) are inserted in equation (5) the mesh equation for node (i,j) can be calculated

$$f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4\cdot f_{i,j} = t^2 \cdot \frac{q_{\mathrm{z}}}{D}. \tag{8}$$

The number of linear mesh equations is equal to the number of inner mesh nodes (n^2) which is set on the plate. The set equations for every node can be written in the matrix form

$$\mathbf{H{\cdot}f=g} \tag{9}$$

where is

$$\mathbf{H} = \begin{bmatrix} a_1^1 & a_1^2 & \cdots & a_1^{n+1} & 0 & 0 \\ c_1^1 & a_2^1 & a_2^2 & \cdots & \ddots & 0 \\ \vdots & c_2^1 & a_3^1 & a_3^2 & \cdots & a_{n^2-n}^{n+1} \\ c_1^n & \vdots & c_3^1 & a_4^1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & a_{n^2-1}^2 \\ 0 & 0 & c_{n^2-n}^n & \cdots & c_{n^2-1}^1 & a_{n^2}^1 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ \vdots \\ \vdots \\ g_{n^2} \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ \vdots \\ f_{n^2} \end{bmatrix} \tag{10}$$

and

$$n = \frac{d}{t} - 1. \tag{11}$$

The numbers in the exponent denote the number of the diagonal, and the index denotes the ordinal number of the diagonal element so that certain diagonals can be written as vectors during programming.

2.3. LU-Decomposition for Band Matrices

The matrix **H** is sparse matrix. If the matrix **H**$\in R^{n,m}$ is the regular square matrix [1], whose main minors are different from zero, a division (**H=LU**) can be made in a simple way. **L** denotes the lower triangular matrix, with number one on the main diagonal, and **U** is the upper triangular matrix, with non-zero diagonal elements.

According to the expected form, **L** and **U** matrices are as follows:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ l_1^2 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & l_2^2 & 1 & \ddots & \ddots & \vdots \\ l_1^{n+1} & \vdots & \ddots & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & l_{n^2-n}^{n+1} & \cdots & l_{n^2-1}^2 & 1 \end{bmatrix}, \tag{12}$$

$$\mathbf{U} = \begin{bmatrix} u_1^1 & u_1^2 & \cdots & u_1^{n+1} & 0 & 0 \\ 0 & u_2^1 & u_2^2 & \cdots & \ddots & 0 \\ 0 & 0 & u_3^1 & u_3^2 & \cdots & u_{n^2-n}^{n+1} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & \ddots & u_{n^2-1}^1 & u_{n^2-1}^2 \\ 0 & 0 & \cdots & \cdots & 0 & u_{n^2}^1 \end{bmatrix}. \tag{13}$$

The members of **L** and **U** matrices can be calculated by using the following algorithm which can be written only in three parts due to the discontinuity in the intervals of the sum:

The first part of the algorithm: for each m=1, ..., n^2-n the following can be applied

$$u_m^p = a_m^p - \sum_{i=1}^{m-1} u_{m-i}^{i+p} \cdot l_{m-i}^{i+1}, \qquad p = 1,\dots,n+1$$

$$l_m^p = \frac{1}{u_m^1}\left(c_m^{p-1} - \sum_{i=1}^{m-1} u_{m-1}^{i+1} \cdot l_{m-i}^{i+p}\right) \qquad p = 2,\dots,n+1. \tag{14}$$

The second part of the algorithm: for each q=1, ..., $n-1$ and for each $m=n^2-n+1$, ..., n^2-1 the following can be applied

$$u_m^p = a_m^p - \sum_{i=1}^{m-1} u_{m-i}^{i+p} \cdot l_{m-i}^{i+1}, \qquad p = 1,\dots,n-q+1$$

$$l_m^p = \frac{1}{u_m^1}\left(c_m^{p-1} - \sum_{i=1}^{m-1} u_{m-i}^{i+1} \cdot l_{m-i}^{i+p}\right). \qquad p = 2,\dots,n-q+1 \tag{15}$$

The third part of the algorithm: the last member of the main diagonal of **U** matrix is determined

$$u_{n^2}^1 = a_{n^2}^1 - \sum_{i=1}^{n^2-1} u_{n^2-i}^{i+1} \cdot l_{n^2-i}^{i+1}. \tag{16}$$

This algorithm can be used to determine LU-decomposition of the band matrix with any number of subordinate diagonals. The advantage of this algorithm compared to *Crout's* [1] algorithm is that during programming only the elements that are next to the main diagonal have to be entered instead of all elements of the system matrix.

Instead of the system **Hf=g**, the system **LUf=g** is observed. That system is solved successively, i.e. the lower triangular system **Lz*=g** is solved first, and then the upper triangular system **Uf=z***. If the vector of unknowns **z*** has the following form

$$\mathbf{z}^{*\mathrm{T}} = \left[z_1^* \; z_2^* \; z_3^* \ldots z_{n^2-1}^* \; z_{n^2}^* \right]. \tag{17}$$

The algorithm for the system Lz*=g:

The first part of the algorithm: for each $m=1, ..., n+1$ the following can be applied

$$z_m^* = g_m - \sum_{i=1}^{m-1} l_{m-i}^{i+1} \cdot z_{m-i}^*, \tag{18}$$

The second part of the algorithm: for each $m=n+2, ..., n^2$ the following can be applied

$$z_m^* = g_m - \sum_{i=1}^{n} l_{m-i}^{i+1} \cdot z_{m-i}^*. \tag{19}$$

The algorithm for the system Uf=z*:

The first part of the algorithm: for each $m=0, ..., n$ the following can be applied

$$f_{n^2-m} = \frac{1}{u_{n^2-m}^1} \left(z_{n^2-m}^* - \sum_{i=1}^{m} u_{n^2-m}^{i+1} \cdot f_{n^2-m+i} \right), \tag{20}$$

The second part of the algorithm: for each $m=n+1, ..., n^2-1$ the following can be applied

$$f_{n^2-m} = \frac{1}{u_{n^2-m}^1} \left(z_{n^2-m}^* - \sum_{i=1}^{n} u_{n^2-m}^{i+1} \cdot f_{n^2-m+i} \right). \tag{21}$$

In this way we can solve the equation (5), i.e. the values of the function f in each node of previously defined mesh with the adequate boundary conditions.

The same procedure can be used to get final solutions for the plate deflection according to the equation (4) that has the following discretization form

$$w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1} - 4 \cdot w_{i,j} = f_{i,j}, \tag{22}$$

The equation (22) has the following matrix form

$$\mathbf{H} \cdot \mathbf{w} = \mathbf{f}. \tag{23}$$

The free coefficients vector is the solution for the equation (5) in this case, i.e. of the function f in some nodes.

2.4. Application of the Method on Bending of Thin Quadratic Plate that has Joint Connection along the Edge (Freely Supported)

Boundary Conditions:

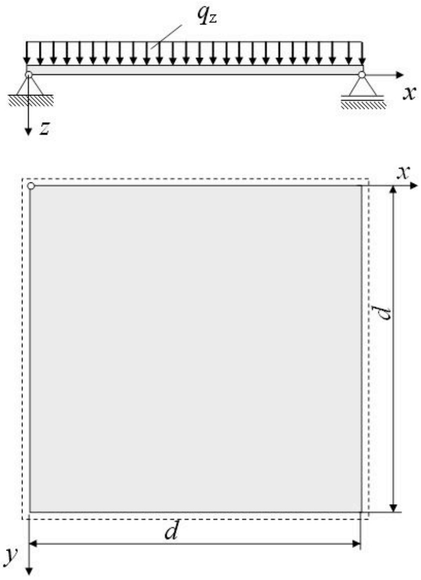

Fig. 3 Quadratic plate with constant load and joint connection along the edges (freely supported)

Slika 3. Pravokutna ploča opterećena jednoliko i zglobno vezana duž rubova (slobodno oslonjena)

The following can be applied for the quadratic plate with joint connection

$w(0,y)=0, \qquad w(x,0)=0,$

$w(d,y)=0, \qquad w(x,d)=0,$

i.e. for:

$$x=0 \text{ and } x=d \Rightarrow M_x=0 \Rightarrow \frac{M_{\mathrm{x}}}{D} = \frac{\partial^2 w}{\partial x^2} + \nu \cdot \frac{\partial^2 w}{\partial y^2} = 0, [6]$$

$$\frac{\partial^2 w}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 w}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \Rightarrow f(0,y)=0 \text{ and } f(d,y)=0,$$

$$y=0 \text{ and } y=d \Rightarrow M_y=0 \Rightarrow \frac{M_{\mathrm{y}}}{D} = \frac{\partial^2 w}{\partial y^2} + \nu \cdot \frac{\partial^2 w}{\partial x^2} = 0, [6]$$

$$\frac{\partial^2 w}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 w}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \Rightarrow f(x,0)=0 \text{ and } f(x,d)=0,$$

where M_x and M_y are the bending moments in the direction of the coordinate axes x and y. By using the finite difference method (as in 2.2) and LU-decomposition for the band matrices (as in 2.3) and by taking into account previously mentioned boundary conditions, we can get the distribution of the thin rectangular plate deflection that is under the influence of constant load.

Example:

If the quadratic plate has the dimensions $d \cdot d$=1000·1000 mm, and if it has constant pressure q_z=0,1 MPa, with plate thickness h=20 mm, then the elastic properties of the plate are ν=0,3 and E=210 GPa.

By using previously described model (FDM-LU) and discretization t=50 mm the deflections on quadratic plate symmetric line have been obtained. The calculation of deflections was done also by using the finite element method in ANSYS 7.0 programme [8] for the same plate geometry and elastic material properties. This calculation was done in order to compare the results that were acquired by the suggested method. For this purpose finite element SHELL63 from ANSYS library was applied [9]. This element has both bending and membrane capabilities and is defined by four nodes with six degrees of freedom at each node: translations in the nodal x, y, and z directions and rotations about the nodal x, y, and z axes. Finite element dimensions are in full concordance with discretization used during FDM. The testing gave almost the same results. Solutions comparison is given on Figure 4. and Table 1.

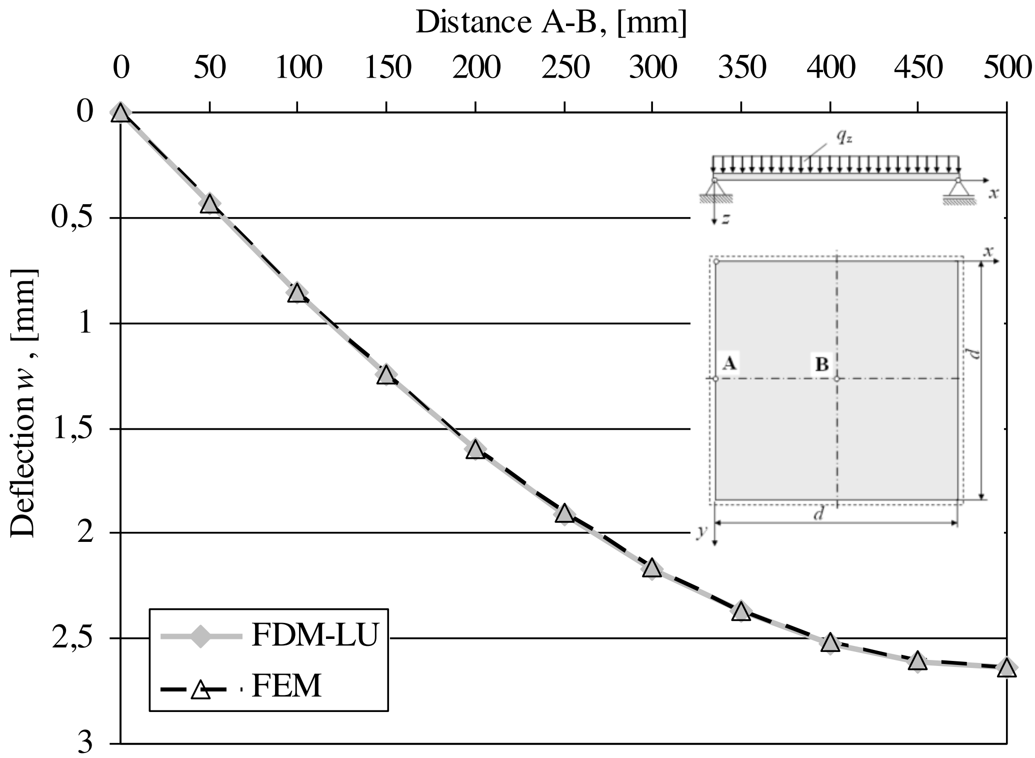

Fig. 4. Deflections of freely supported and constant loaded quadratic plate

Slika 4. Progibi slobodno oslonjene pravokutne ploče opterećene jednoliko

Distance A-B, [mm]	Deflection w, [mm]		$w_{FDM\text{-}LU} - w_{FEM}$, [mm]
	FDM-LU	FEM	
0	0,0000	0,0000	0,00000
50	0,43569	0,43390	0,00179
100	0,85553	0,85247	0,00306
150	1,2471	1,2432	0,00390
200	1,6009	1,5964	0,00450
250	1,9098	1,9049	0,00490
300	2,1684	2,1632	0,00520
350	2,3729	2,3676	0,00530
400	2,5206	2,5153	0,00530
450	2,6099	2,6045	0,00540
500	2,6398	2,6344	0,00540

Table 1. Deflections of freely supported and constant loaded quadratic plate

Tablica 1. Progibi slobodno oslonjene pravokutne ploče opterećene jednoliko

3. Conclusion

Finite difference method is a very practical method for approximate solution of boundary problems. It is based on approximation of derivation by finite differences by evolving into Taylor's order. A number of linear equations is acquired in this way and they have to be solved by some numerical method. The matrix of linear difference equations system has the properties of a rarely filled band matrix. It is a well-known fact that LU-decomposition has better numerical properties than Gauss method which is often applied [3]. Therefore, a special algorithm for solving the system of linear equations by using LU-decomposition is suggested in this paper. Compared to Crout's algorithm, LU-decomposition uses only the elements of the matrix system that are located in the zone next to the main diagonal.

The suggested method was used for solving the boundary problem of thin plate bending. These results were compared with the results that were acquired by finite element method, by using ANSYS 7.0. The results acquired in such a way are almost identical with the results that were acquired by the suggested method.

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Josip Sertić, dipl.ing.stroj.
Prof. dr. sc. Dražan Kozak
Mechanical Engineering Faculty in Slavonski Brod
J. J. Strossmayer University of Osijek

Prof. dr. sc. Rudolf Scitovski
Department of Mathematics
J. J. Strossmayer University of Osijek