Numerical Analysis

(ENME 602)

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Lecture 3

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Lecture 3

Roots of Non-linear Equations

3.1 Newton's Method and its Extensions

- Newton's Method
- Secant Method
- Method of False Position

3.2 Fixed-Point Method

3.1 Newton's Method and its Extensions

3.1 Newton's Method and its Extensions

"Newton's Method"

3.1 Newton's Method and its Extensions

Taylor's Theorem

Suppose $f \in C^n[a,b]$, that $f^{(n+1)}$ exists on $[a,b]$, and $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $\xi(x)$ between x_0 and x with

$$f(x) = P_n(x) + R_n(x),$$

where

$$\begin{aligned} P_n(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \end{aligned}$$

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}$$

Here $P_n(x)$ is called the n^{th} **Taylor polynomial** for f about x_0, and $R_n(x)$ is called the **remainder term** (or **truncation error**) associated with $P_n(x)$.

Newton's Method

Derivation

- Suppose that $f \in C^2[a, b]$. Let $p_0 \in [a, b]$ be an approximation to p such that $f'(p_0) \neq 0$ and $|p - p_0|$ is "small."
- Consider the first Taylor polynomial for $f(x)$ expanded about p_0 and evaluated at $x = p$.

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)),$$

  where $\xi(p)$ lies between p and p_0.
- Since $f(p) = 0$, this equation gives

$$0 = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)).$$

Newton's Method

$$0 = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)).$$

Derivation (Cont'd)

- Newton's method is derived by assuming that since $|p - p_0|$ is small, the term involving $(p - p_0)^2$ is much smaller, so

$$0 \approx f(p_0) + (p - p_0)f'(p_0).$$

- Solving for p gives

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \equiv p_1.$$

3.1 Newton's Method and its Extensions

Newton's Method

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \equiv p_1.$$

Newton's Method

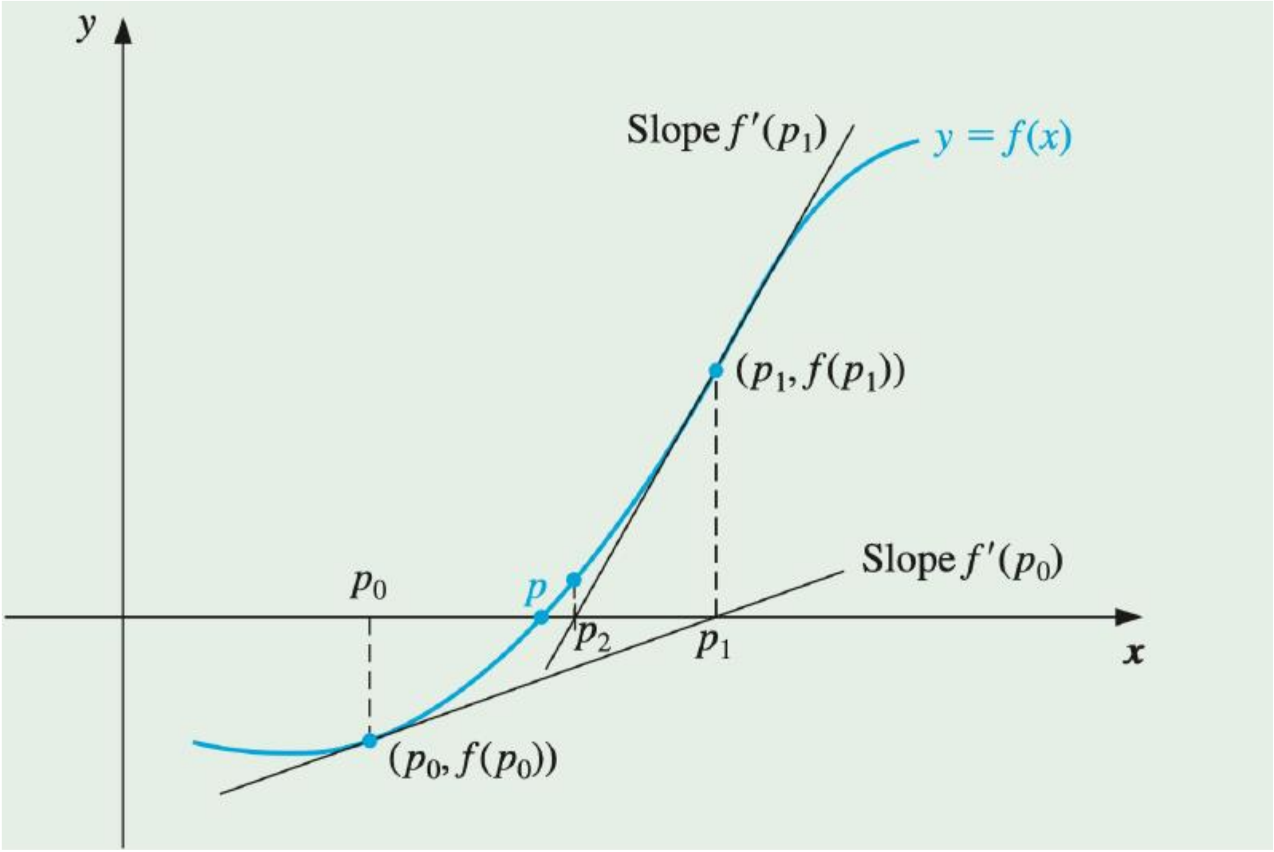
This sets the stage for Newton's method, which starts with an initial approximation p_0 and generates the sequence $\{p_n\}_{n=0}^{\infty}$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \qquad \text{for } n \geq 1$$

It is clear from the equation above that Newton's method cannot be continued if $f'(p_{n-1}) = 0$ for some n. In fact, we will see that the method is most effective when f' is bounded away from zero near p.

3.1 Newton's Method and its Extensions

Newton's Method



Starting with the initial approximation p_0, the approximation p_1 is the x-intercept of the tangent line to the graph of f at $(p_0, f(p_0))$. The approximation p_2 is the x-intercept of the tangent line to the graph of f at $(p_1, f(p_1))$and so on.

3.1 Newton's Method and its Extensions

Newton's Method

Algorithm

To find a solution to $f(x) = 0$ given an initial approximation p_0:

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = p_0 - f(p_0)/f'(p_0)$. (*Compute* p_i.)

Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. (*Update* p_0.)

Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP.

Newton's Method

Stopping Criteria

Various stopping procedures can be applied in Step 3. 3

We can select a tolerance $\epsilon > 0$ and generate $p_1, p_2, \ldots, p_N$, until one of these conditions is met.

$$|p_N - p_{N-1}| < \varepsilon,$$

$$\frac{|p_N - p_{N-1}|}{|p_N|} < \varepsilon, \quad p_N \neq 0,$$

$$|f(p_N)| < \varepsilon.$$

Note that none of these inequalities give precise information about the actual error when the exact solution is used $|p - p_N|$.

Newton's Method

Example 1

(b) Newton's Method for $f(x) = \cos x - x$

- To apply Newton's method to this problem we need

$$f'(x) = -\sin x - 1$$

- Starting again with $p_0 = \pi/4$, we generate the sequence defined, for $n \geq 1$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f(p'_{n-1})} = p_{n-1} - \frac{\cos p_{n-1} - p_{n-1}}{-\sin p_{n-1} - 1}.$$

- This gives the approximations shown in the following table.

Newton's Method

Example

Newton's Method for $f(x) = \cos(x) - x$, $x_0 = \frac{\pi}{4}$

n	p_{n-1}	$f(p_{n-1})$	$f'(p_{n-1})$	p_n	$\lvert p_n - p_{n-1}\rvert$
1	0.78539816	-0.078291	-1.707107	0.73953613	0.04586203
2	0.73953613	-0.000755	-1.673945	0.73908518	0.00045096
3	0.73908518	-0.000000	-1.673612	0.73908513	0.00000004
4	0.73908513	-0.000000	-1.673612	0.73908513	0.00000000

- An excellent approximation is obtained with $n = 3$.
- Because of the agreement of p_3 and p_4 we could reasonably expect this result to be accurate to the places listed.

3.1 Newton's Method and its Extensions

Newton's Method

Convergence Theorem for Newton's Method

- Let $f \in C^2[a, b]$. If $p \in (a, b)$ is such that $f(p) = 0$ and $f'(p) \neq 0$.
- Then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$, defined by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f(p'_{n-1})}$$

converging to p for any initial approximation

$$p_0 \in [p - \delta, p + \delta]$$

- This theorem states that, under reasonable assumptions, Newton's method converges provided a sufficiently accurate initial approximation is chosen. This result is important for the theory of Newton's method, but it is seldom applied in practice because it does not tell us how to determine δ.
- In a practical application, an initial approximation is selected and successive approximations are generated by Newton's method. These will generally either converge quickly to the root, or it will be clear that convergence is unlikely.

3.1 Newton's Method and its Extensions

"Secant Method"

3.1 Newton's Method and its Extensions

Secant Method

- Newton's method is an extremely powerful technique, but it has a major weakness: the need to know the value of the derivative of f at each approximation.

- Frequently, $f'(x)$ is far more difficult and needs more arithmetic operations to calculate than $f(x)$.

- To avoid the problem of the derivative evaluation in Newton's method, the *Secant Method* introduce a slight variation, by replacing the derivative by its quotient approximation.

3.1 Newton's Method and its Extensions
Secant Method

Q1: What happens if the derivative becomes zero in any given iteration of Newton's method?
Secant method is the answer to Newton's method drawback (dividing by zero).
The derivative is replaced by its quotient approximation:

$$f'(p_{n-1}) = \lim_{x \to p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}.$$

Letting $x = p_{n-2}$, we have

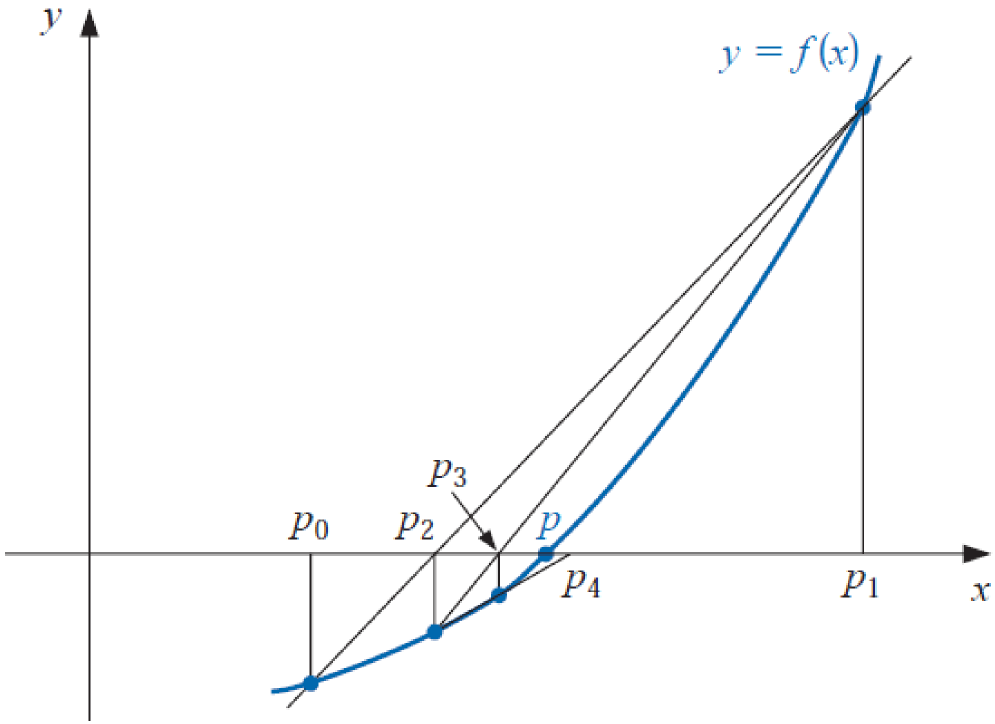
$$f'(p_{n-1}) \approx \frac{f(p_{n-2}) - f(p_{n-1})}{p_{n-2} - p_{n-1}} = \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}.$$

Using this approximation for $f'(p_{n-1})$ in Newton's formula gives

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}.$$

3.1 Newton's Method and its Extensions

Secant Method



- Starting with the two initial approximations p_0 and p_1, the approximation p_2 is the x-intercept of the line joining $(p_0, f(p_0))$and $(p_1, f(p_1))$. The approximation p_3 is the x-intercept of the line joining $(p_1, f(p_1))$and $(p_2, f(p_2))$, and so on. Note that only one function evaluation is needed per step for the Secant method after p_2 has been determined.
- In contrast, each step of Newton's method requires an evaluation of both the function and its derivative.

3.1 Newton's Method and its Extensions

Secant Method

Algorithm

INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute* p_i.)

Step 4 If $|p - p_1| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p_1$; (*Update* p_0, q_0, p_1, q_1.)
$q_0 = q_1$;
$p_1 = p$;
$q_1 = f(p)$.

Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP.

Secant Method

Example 2

Use the Secant method to find a solution to $x = \cos x$, and compare the approximations with those given in Example 1 which applied Newton's method.

Solution

In Example 1 we implemented Newton's method starting with the initial approximation $p_0 = \pi/4$. For the Secant method we need two initial approximations. Suppose we use $p_0 = 0.5$ and $p_1 = \pi/4$. Succeeding approximations are generated by the formula:

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}.$$

$$\downarrow$$

$$p_n = p_{n-1} - \frac{(p_{n-1} - p_{n-2})(\cos p_{n-1} - p_{n-1})}{(\cos p_{n-1} - p_{n-1}) - (\cos p_{n-2} - p_{n-2})}, \quad \text{for } n \geq 2.$$

3.1 Newton's Method and its Extensions

Secant Method

Example 2

Secant	
n	p_n
0	0.5
1	0.7853981635
2	0.7363841388
3	0.7390581392
4	0.7390851493
5	0.7390851332

Newton	
n	p_n
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

- Comparing the results of the Secant method and Newton's method, we see that the Secant method approximation p_5 is accurate to the tenth decimal place, whereas Newton's method obtained this accuracy by p_3.
- For this example, the convergence of the Secant method is much faster than functional iteration but slightly slower than Newton's method. This is generally the case. Newton's method or the Secant method is often used to refine an answer obtained by another technique, such as the Bisection method, since these methods require good first approximations but generally give rapid convergence.

3.1 Newton's Method and its Extensions

"False Position Method"

3.1 Newton's Method and its Extensions

False Position Method

- Each successive pair of approximations in the Bisection method brackets a root p of the equation; that is, for each positive integer n, a root lies between a_n and b_n. This implies that, for each n, the Bisection method iterations satisfy

$$|p_n - p| < \frac{1}{2}|a_n - b_n|,$$

  which provides an easily calculated error bound for the approximations.

- Root bracketing is not guaranteed for either Newton's method or the Secant method.

- In Example 1, Newton's method was applied to $f(x) = \cos x - x$, and an approximate root was found to be 0.7390851332. The iteration table showed that this root was not bracketed by either p_0 and p_1 or p_1 and p_2.

- For the Secant method approximations of the same problem, the initial approximations p_0 and p_1 bracket the root, but the pair of approximations p_3 and p_4 fail to do so.

3.1 Newton's Method and its Extensions

False Position Method

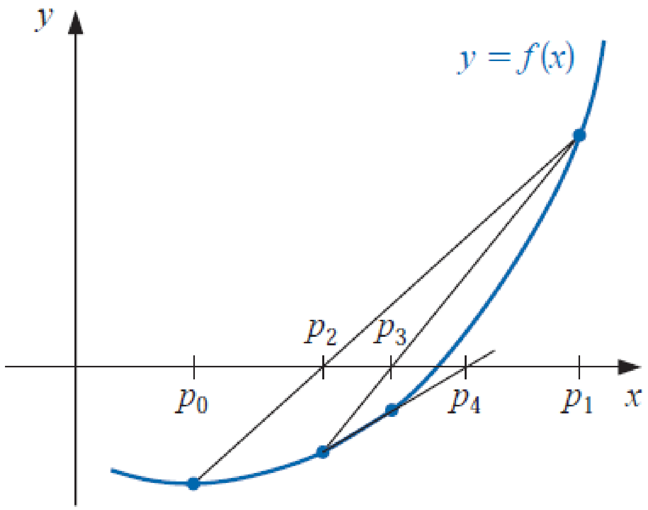
- The method of **False Position** (also called *Regula Falsi*) generates approximations in the same manner as the Secant method, but it includes a test to ensure that the root is always bracketed between successive iterations. Although it is not a method we generally recommend, it illustrates how bracketing can be incorporated.

- First choose initial approximations p_0 and p_1 with $f(p_0) \cdot f(p_1) < 0$. The approximation p_2 is chosen in the same manner as in the Secant method, as the x-intercept of the line joining $(p_0, f(p_0))$and $(p_1, f(p_1))$. To decide which secant line to use to compute p3, consider $f(p2) \cdot f(p_1)$, or more correctly sgn $f(p_2)$ · sgn $f(p_1)$
  - If sgn $f(p_2)$ · sgn $f(p_1) < 0$, then p_1 and p_2 bracket a root. Choose p_3 as the x-intercept of the line joining $(p_1, f(p_1))$ and $(p_2, f(p_2))$.
  - If not, choose p_3 as the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_2, f(p_2))$, and then interchange the indices on p_0 and p_1.

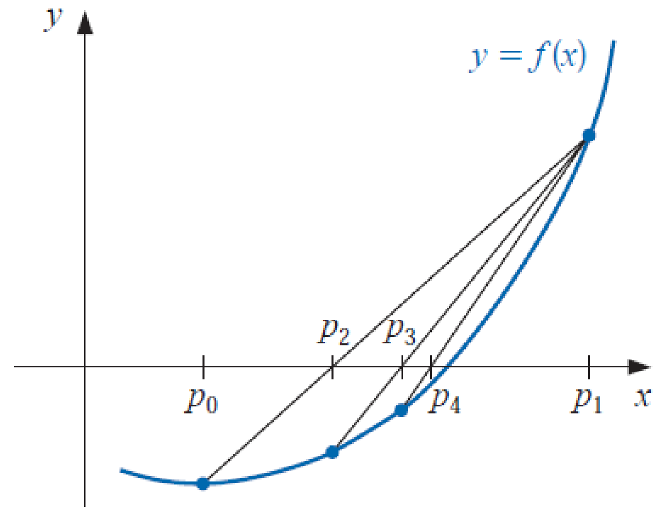
3.1 Newton's Method and its Extensions

False Position Method

Secant Method



Method of False Position



First choose initial approximations p_0 and p_1 with $f\,(p_0)\cdot f\,(p_1) < 0$. The approximation p_2 is chosen in the same manner as in the Secant method, as the x-intercept of the line joining $(p_0, f\,(p_0))$and $(p_1, f\,(p_1))$. To decide which secant line to use to compute p3, consider $f\,(p2)\cdot f\,(p_1)$, or more correctly sgn $f\,(p_2)\cdot$ sgn $f\,(p_1)$

- If sgn $f\,(p_2)\cdot$ sgn $f\,(p_1) < 0$, then p_1 and p_2 bracket a root. Choose p_3 as the x-intercept of the line joining $(p_1, f\,(p_1))$ and $(p_2, f\,(p_2))$.
- If not, choose p_3 as the x-intercept of the line joining $(p_0, f\,(p_0))$ and $(p_2, f\,(p_2))$, and then interchange the indices on p_0 and p_1.

3.1 Newton's Method and its Extensions

False Position Method

To find a solution to $f(x) = 0$ given the continuous function f on the interval $[p_0, p_1]$ where $f(p_0)$ and $f(p_1)$ have opposite signs:

INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.

Step 2 While $i \leq N_0$ do Steps 3–7.

Step 3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute p_i.*)

Step 4 If $|p - p_1| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$;
$q = f(p)$.

Step 6 If $q \cdot q_1 < 0$ then set $p_0 = p_1$;
$q_0 = q_1$.

Step 7 Set $p_1 = p$;
$q_1 = q$.

Step 8 OUTPUT ('Method failed after N_0 iterations, N_0 =', N_0);
(*The procedure unsuccessful.*)
STOP.

3.1 Newton's Method and its Extensions

False Position Method

Example 3

Use the method of False Position to find a solution to $x = \cos x$, and compare the approximations with those given in *Example* 1 which applied Newton's method, and to those found in *Example 2* which applied the Secant method.

	False Position	Secant	Newton
n	p_n	p_n	p_n
0	0.5	0.5	0.7853981635
1	0.7853981635	0.7853981635	0.7395361337
2	0.7363841388	0.7363841388	0.7390851781
3	0.7390581392	0.7390581392	0.7390851332
4	0.7390848638	0.7390851493	0.7390851332
5	0.7390851305	0.7390851332	
6	0.7390851332		

Notice that the False Position and Secant approximations agree through p_3 and that the method of False Position requires an additional iteration to obtain the same accuracy as the Secant method.

False Position Method

- The added insurance of the method of False Position commonly requires more calculation than the Secant method. Just as the simplification that the Secant method provides over Newton’s method usually comes at the expense of additional iterations.

3.2 Fixed-Point Method

3.2 Fixed-Point Method

Prime Objective

- In what follows, it is important not to lose sight of our prime objective:
- Given a function $f(x)$ where $a \le x \le b$, find values p such that

$$f(p) = 0$$

- Given such a function, $f(x)$, we now construct an auxiliary function $g(x)$ such that

$$p = g(p)$$

whenever $f(p) = 0$ (this construction is not unique).

- The problem of finding p such that $p = g(p)$ is known as the fixed point problem.

Functional (Fixed-Point) Iteration

A Fixed Point

If g is defined on $[a, b]$ and $g(p) = p$ for some $p \in [a, b]$, then the function g is said to have the fixed point p in $[a, b]$.

Note

- The fixed-point problem turns out to be quite simple both theoretically and geometrically.
- The function $g(x)$ will have a fixed point in the interval $[a, b]$ whenever the graph of $g(x)$ intersects the line $y = x$.

3.2 Fixed-Point Method

Functional (Fixed-Point) Iteration

The Equation $f(x) = x - \cos(x) = 0$

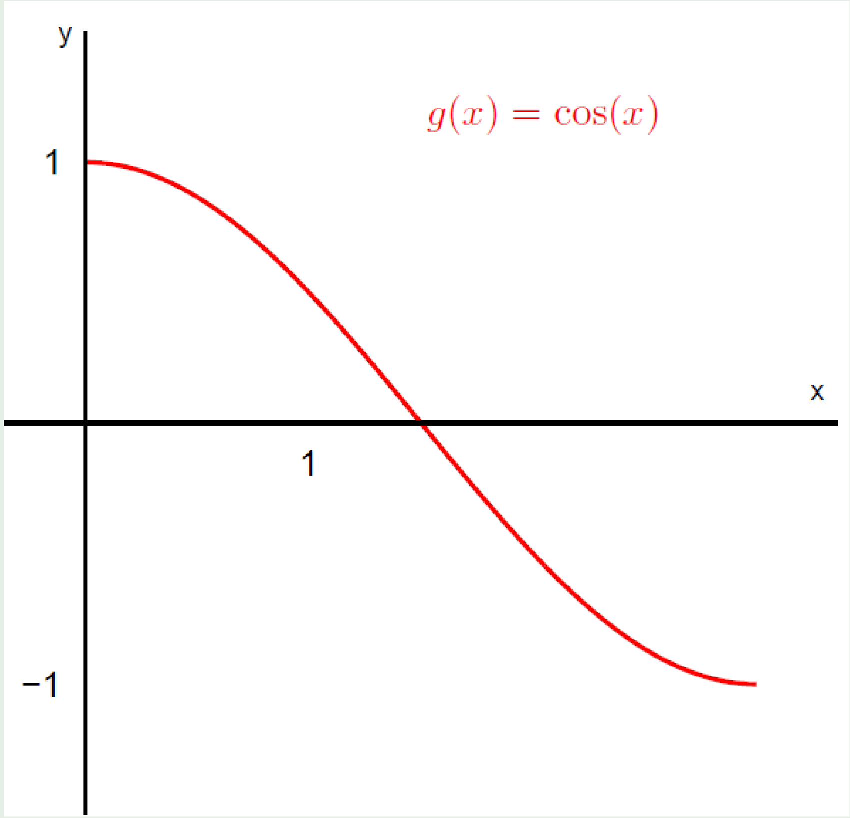
If we write this equation in the form:

$$x = \cos(x)$$

then $g(x) = \cos(x)$.

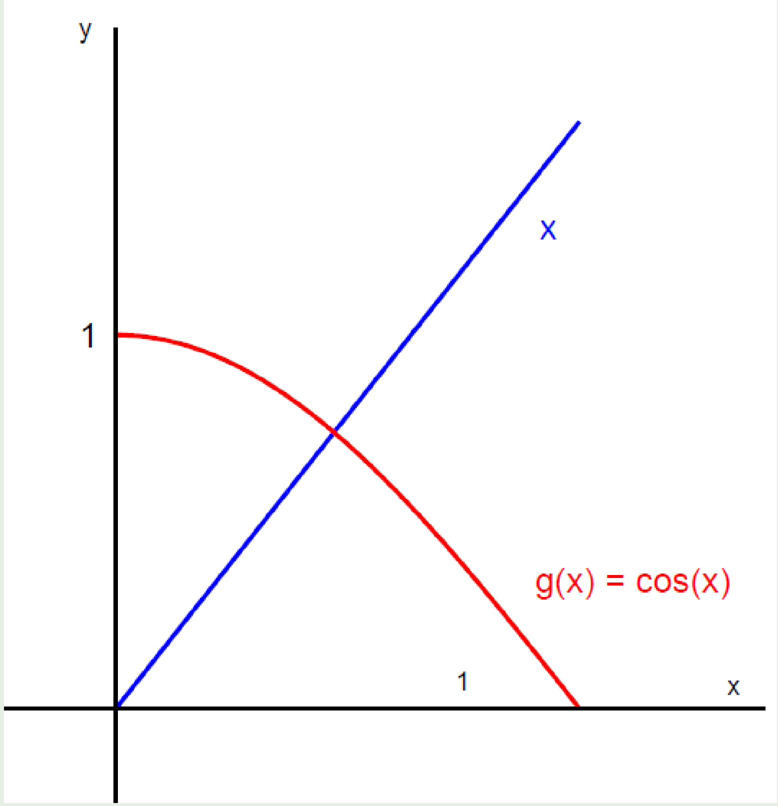
3.2 Fixed-Point Method

Single Nonlinear Equation $f(x) = x - \cos(x) = 0$



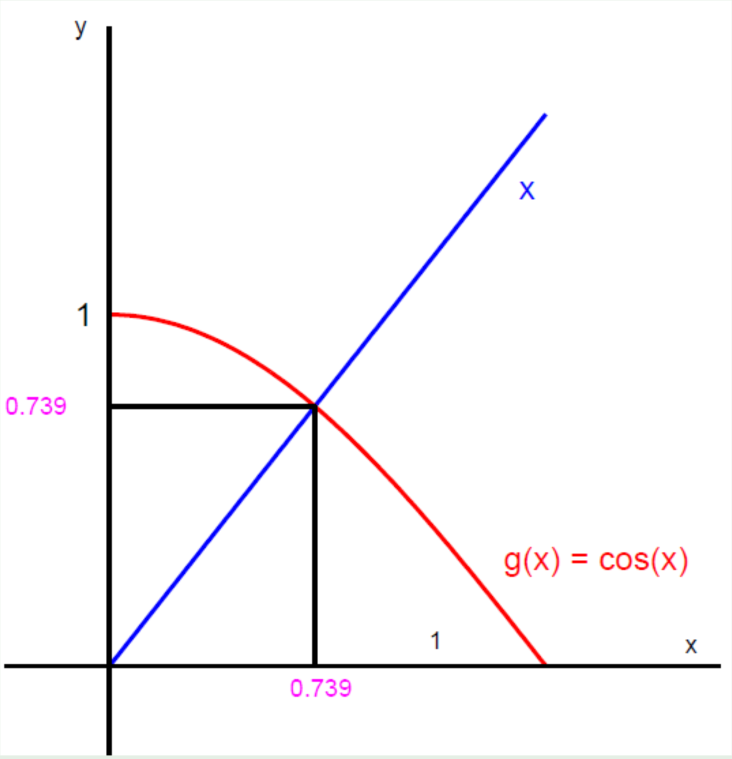
Functional (Fixed-Point) Iteration

$$x = \cos(x)$$



Functional (Fixed-Point) Iteration

$$p = \cos(p) \qquad\qquad p \approx 0.739$$

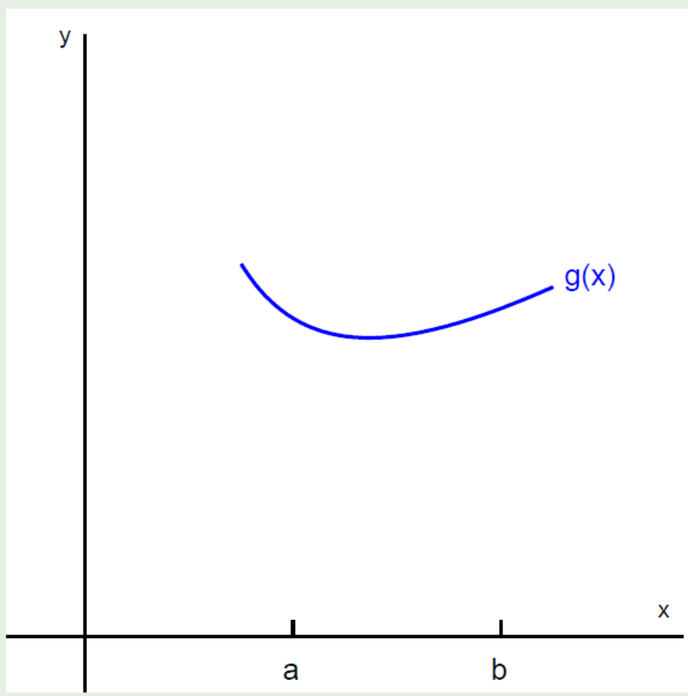


3.2 Fixed-Point Method

Existence of a Fixed Point

If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$ then the function g has a fixed point in $[a, b]$.

$g(x)$ is Defined on $[a, b]$

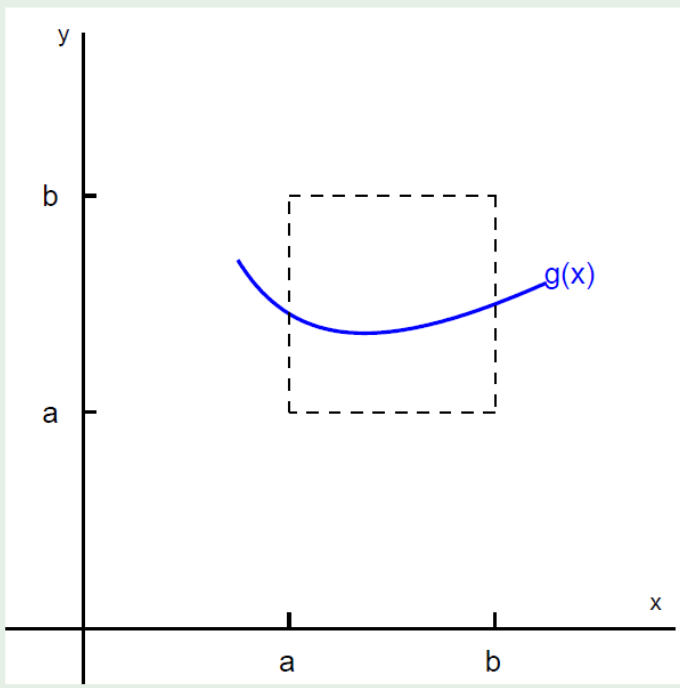


3.2 Fixed-Point Method

Existence of a Fixed Point

If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$ then the function g has a fixed point in $[a, b]$.

$g(x) \in [a, b]$ for all $x \in [a, b]$

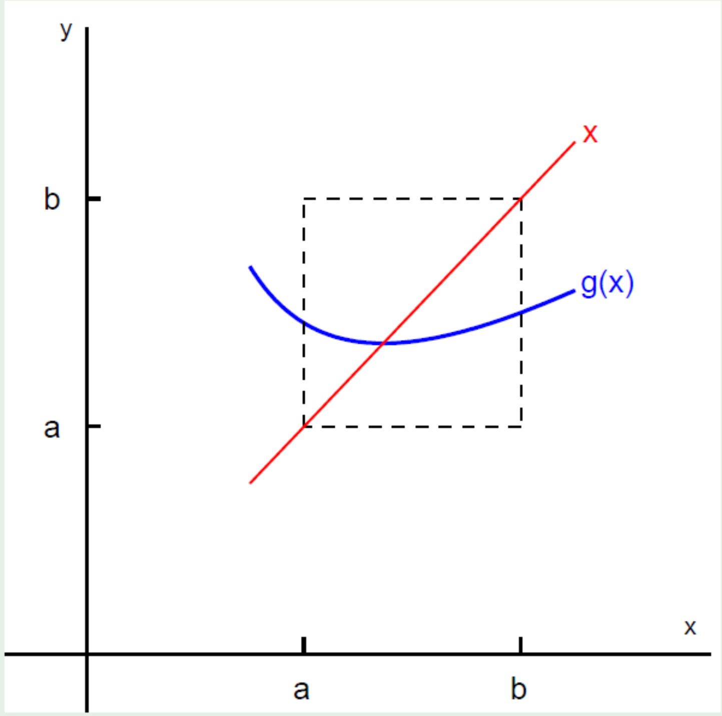


3.2 Fixed-Point Method

Existence of a Fixed Point

If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$ then the function g has a fixed point in $[a, b]$.

$g(x)$ has a Fixed Point in $[a, b]$



3.2 Fixed-Point Method

Illustration

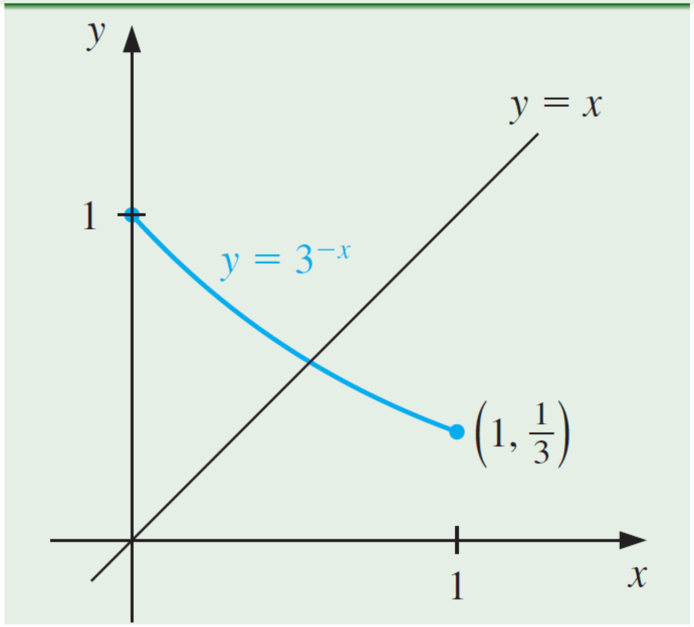
- Consider the function $g(x) = 3^{-x}$ on $0 \leq x \leq 1$. $g(x)$ is continuous and since

$$g'(x) = -3^{-x} \log 3 < 0 \qquad \text{on } [0,1]$$

  $g(x)$ is decreasing on $[0,1]$.
- Hence

$$g(1) = \frac{1}{3} \leq g(x) \leq 1 = g(0)$$

  i.e. $g(x) \in [0,1]$ for all $x \in [0,1]$ and therefore, by the preceding result, $g(x)$ must have a fixed point in $[0,1]$.



3.2 Fixed-Point Method

An Important Observation

- It is fairly obvious that, on any given interval $I = [a, b]$, $g(x)$ may have many fixed points (or none at all).
- In order to ensure that $g(x)$ has a unique fixed point in I, we must make an additional assumption that $g(x)$ does not vary too rapidly.
- Thus we have to establish a uniqueness result.

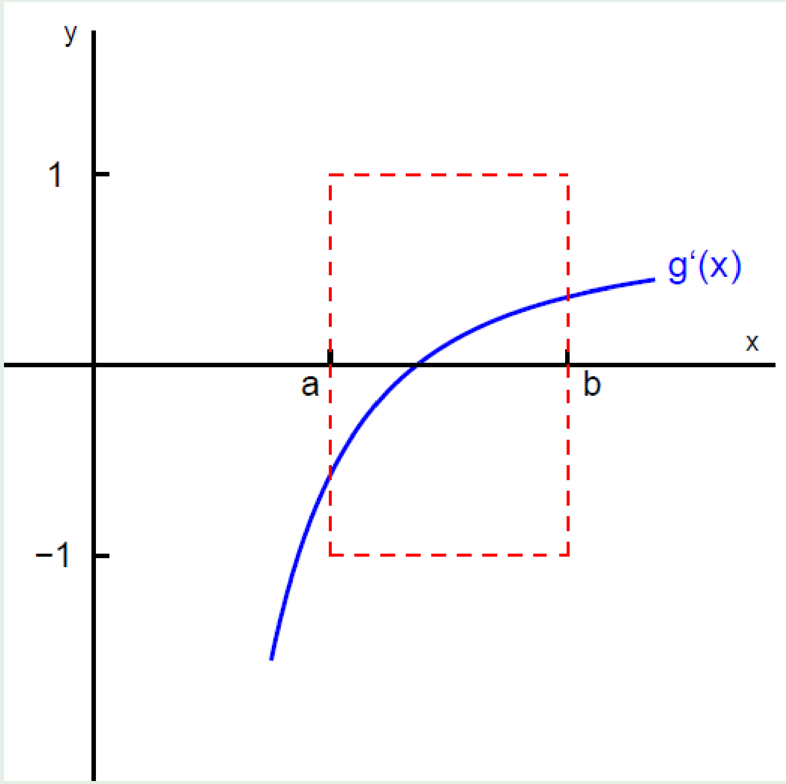
Uniqueness Result

Let $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$. Further if $g'(x)$ exists on (a, b) and

$$|g'(x)| \leq k < 1, \quad \forall\, x \in [a, b],$$

then the function g has a unique fixed point p in $[a, b]$.

Unique Fixed Point: $|g'(x)| \le 1$ for all $x \in [a, b]$



3.2 Fixed-Point Method

Fixed-Point Algorithm

To find the fixed point of g in an interval $[a, b]$, given the equation $x = g(x)$ with an initial guess $p_0 \in [a, b]$:

1. $n = 1$;
2. $p_n = g(p_{n-1})$;
3. If $|p_n - p_{n-1}| < \epsilon$ then 5;
4. $n \rightarrow n + 1$; go to 2.
5. End of Procedure.

3.2 Fixed-Point Method

Model Problem

Consider the quadratic equation:

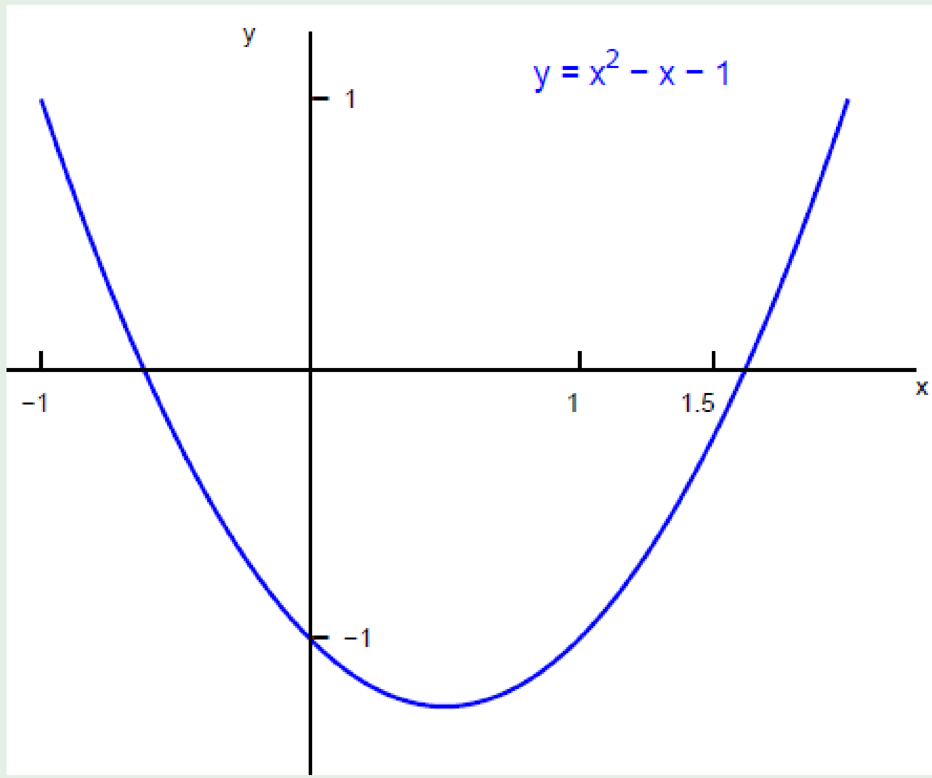
$$x^2 - x - 1 = 0$$

Positive Root

The positive root of this equations is:

$$x = \frac{1 + \sqrt{5}}{2} \approx 1.618034$$

Single Nonlinear Equation $f(x) = x^2 - x - 1 = 0$



We can convert this equation into a fixed-point problem.

Single Nonlinear Equation $f(x) = x^2 - x - 1 = 0$

One Possible Formulation for $g(x)$

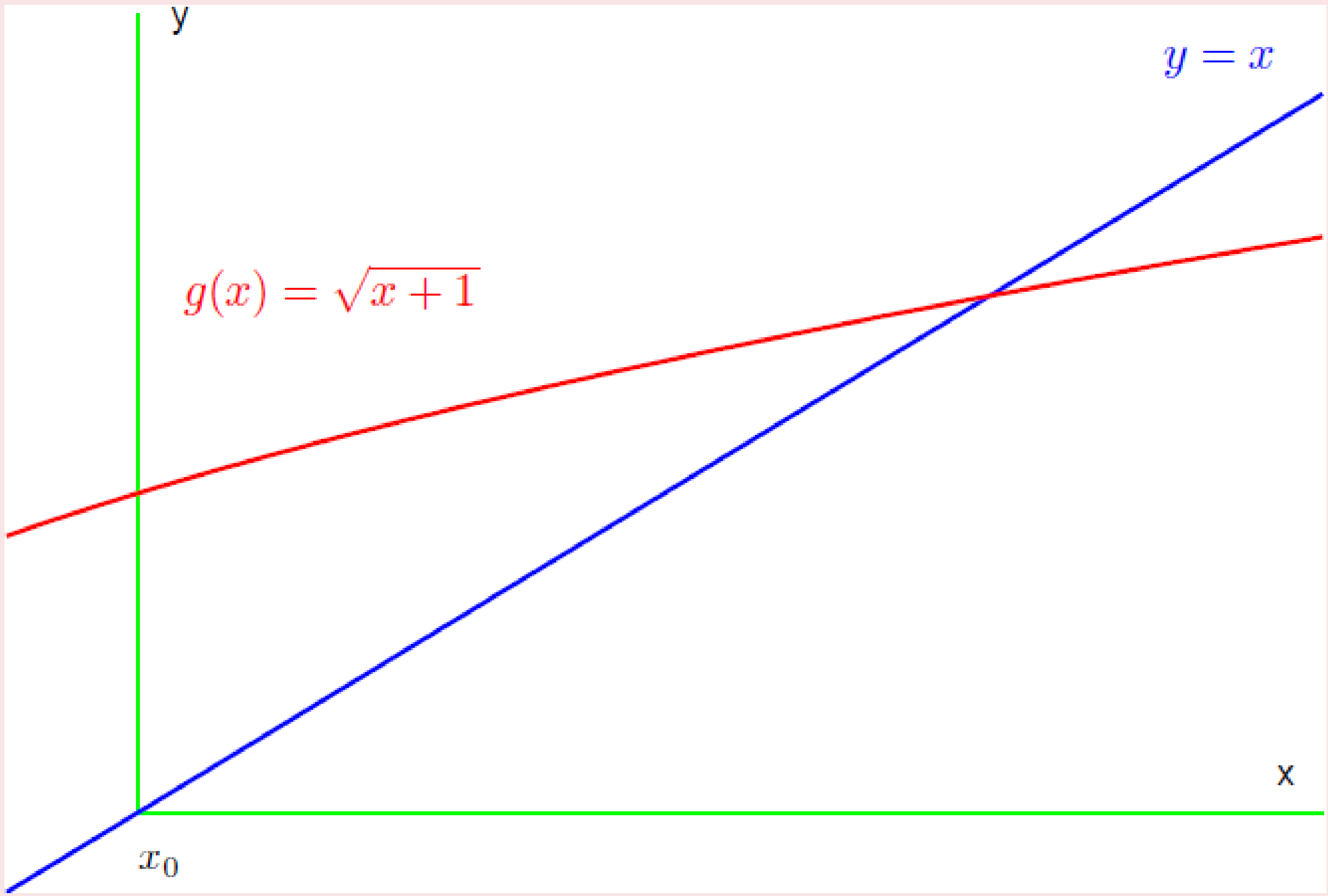
Transpose the equation $f(x) = 0$ for variable x:

$$\begin{aligned} x^2 - x - 1 &= 0 \\ \Rightarrow \qquad x^2 &= x + 1 \\ \Rightarrow \qquad x &= \pm\sqrt{x+1} \end{aligned}$$

$$g(x) = \sqrt{x+1}$$

3.2 Fixed-Point Method

$x_{n+1} = g(x_n) = \sqrt{x_n + 1}$ with $x_0 = 0$



3.2 Fixed-Point Method

Fixed Point: $g(x) = \sqrt{x+1}$ $x_0 = 0$

n	p_n	p_{n+1}	$\lvert p_{n+1} - p_n \rvert$
1	0.000000000	1.000000000	1.000000000
2	1.000000000	1.414213562	0.414213562
3	1.414213562	1.553773974	0.139560412
4	1.553773974	1.598053182	0.044279208
5	1.598053182	1.611847754	0.013794572

Single Nonlinear Equation $f(x) = x^2 - x - 1 = 0$

A Second Formulation for $g(x)$

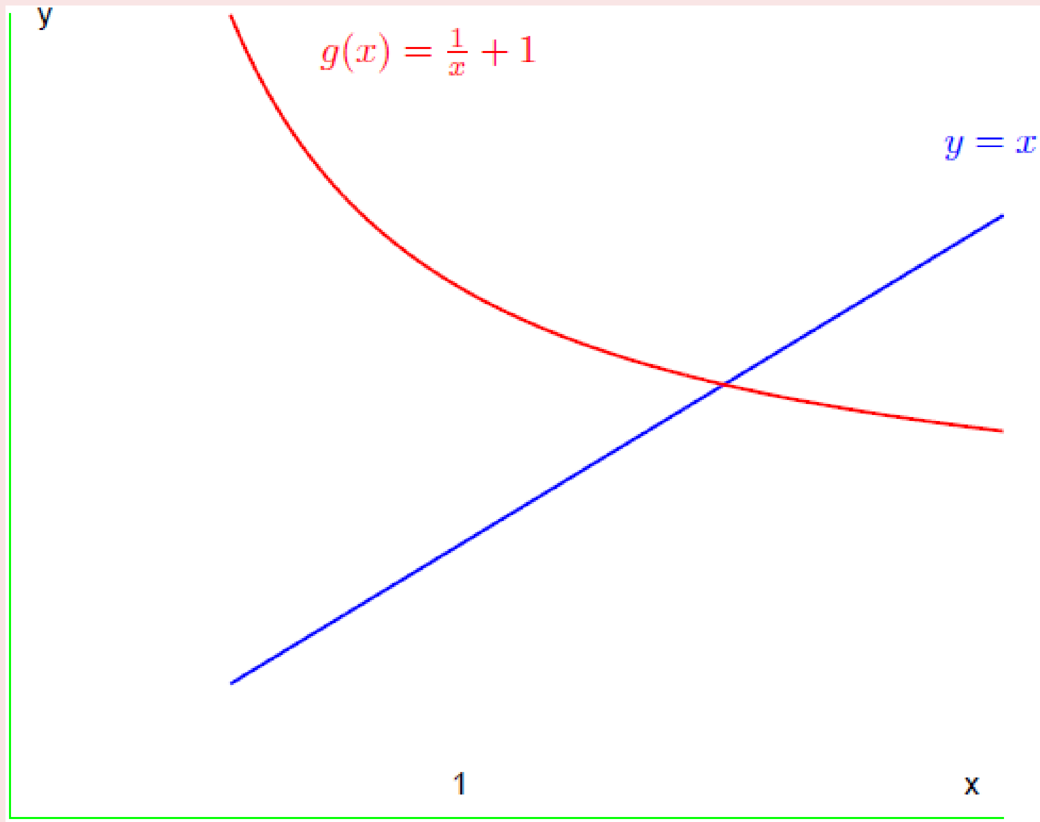
Transpose the equation $f(x) = 0$ for variable x:

$$\begin{aligned} x^2 - x - 1 &= 0 \\ \Rightarrow \qquad x^2 &= x + 1 \\ \Rightarrow \qquad x &= 1 + \frac{1}{x} \end{aligned}$$

$$g(x) = 1 + \frac{1}{x}$$

3.2 Fixed-Point Method

$$x_{n+1} = g(x_n) = \frac{1}{x_n} + 1 \text{ with } x_0 = 1$$



3.2 Fixed-Point Method

Single Nonlinear Equation $f(x) = x - \cos(x) = 0$

n	p_n
0	0.7853981635
1	0.7071067810
2	0.7602445972
3	0.7246674808
4	0.7487198858
5	0.7325608446
6	0.7434642113
7	0.7361282565

Fixed-Point Method

	False Position	Secant	Newton
n	p_n	p_n	p_n
0	0.5	0.5	0.7853981635
1	0.7853981635	0.7853981635	0.7395361337
2	0.7363841388	0.7363841388	0.7390851781
3	0.7390581392	0.7390581392	0.7390851332
4	0.7390848638	0.7390851493	0.7390851332
5	0.7390851305	0.7390851332	
6	0.7390851332		

3.2 Fixed-Point Method

Example 4

The equation

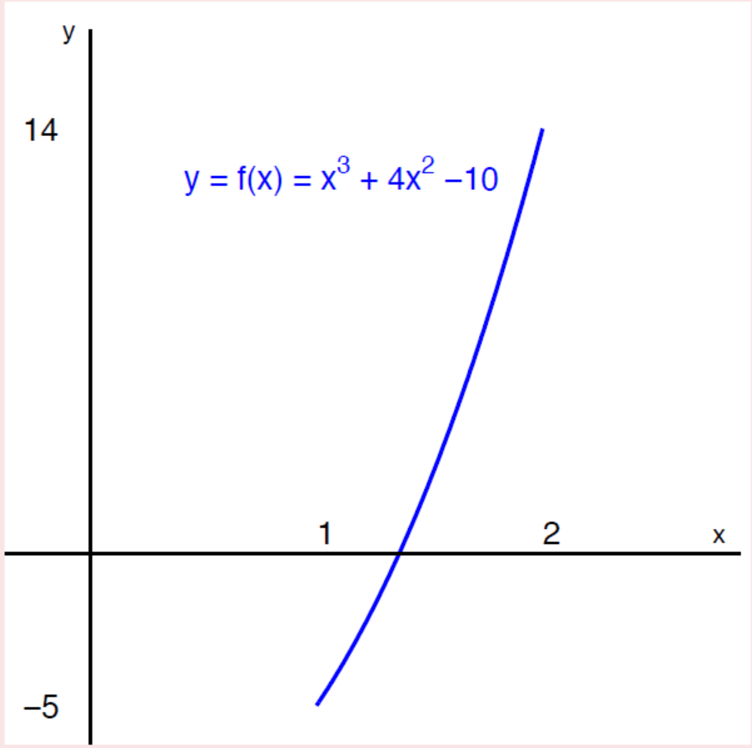
$$x^3 + 4x^2 - 10 = 0$$

has a unique root in $[1, 2]$. Its value is approximately 1.365230013.

3.2 Fixed-Point Method

Example 4

$$f(x) = x^3 + 4x^2 - 10 = 0 \text{ on } [1, 2]$$



3.2 Fixed-Point Method

Example 4

Possible Choices for $g(x)$

- There are many ways to change the equation to the fixed-point form $x = g(x)$ using simple algebraic manipulation.
- For example, to obtain the function g described in part (c), we can manipulate the equation $x^3 + 4x^2 - 10 = 0$ as follows:

$$4x^2 = 10 - x^3, \quad \text{so} \quad x^2 = \frac{1}{4}(10 - x^3), \quad \text{and} \quad x = \pm\frac{1}{2}(10 - x^3)^{1/2}.$$

- We will consider 5 such rearrangements and, later in this section, provide a brief analysis as to why some do and some not converge to $p = 1.365230013$.

Example 4

5 Possible Transpositions to $x = g(x)$

$$x = g_1(x) = x - x^3 - 4x^2 + 10$$

$$x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$$

$$x = g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$$

$$x = g_4(x) = \sqrt{\frac{10}{4 + x}}$$

$$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$

3.2 Fixed-Point Method

Example 4

Numerical Results for $f(x) = x^3 + 4x^2 - 10 = 0$

n	g_1	g_2	g_3	g_4	g_5
0	1.5	1.5	1.5	1.5	1.5
1	-0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	-469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

3.2 Fixed-Point Method

Example 4

$x = g(x)$ with $x_0 = 1.5$

$x = g_1(x) = x - x^3 - 4x^2 + 10$	Does not Converge
$x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$	Does not Converge
$x = g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$	Converges after 31 Iterations
$x = g_4(x) = \sqrt{\frac{10}{4 + x}}$	Converges after 12 Iterations
$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$	Converges after 5 Iterations

3.2 Fixed-Point Method

Example 4

Iteration for $x = g_1(x)$ Does Not Converge

Since

$$g_1'(x) = 1 - 3x^2 - 8x \qquad g_1'(1) = -10 \qquad g_1'(2) = -27$$

there is no interval $[a, b]$ containing p for which $|g_1'(x)| < 1$. Also, note that

$$g_1(1) = 6 \qquad \text{and} \qquad g_2(2) = -12$$

so that $g(x) \notin [1, 2]$ for $x \in [1, 2]$.

Example 4

Iteration for $x = g_1(x)$ Does Not Converge

Since

$$g_1'(x) = 1 - 3x^2 - 8x \qquad g_1'(1) = -10 \qquad g_1'(2) = -27$$

there is no interval $[a, b]$ containing p for which $|g_1'(x)| < 1$. Also, note that

$$g_1(1) = 6 \qquad \text{and} \qquad g_2(2) = -12$$

so that $g(x) \notin [1, 2]$ for $x \in [1, 2]$.

Thank You

"Interpolation and Polynomial Approximation"