TUTORIAL 3

ROOT FINDING USING NEWTON, SECANT, FALSE POSITION & FIXED POINT METHOD

Numerical Analysis (ENME 602)

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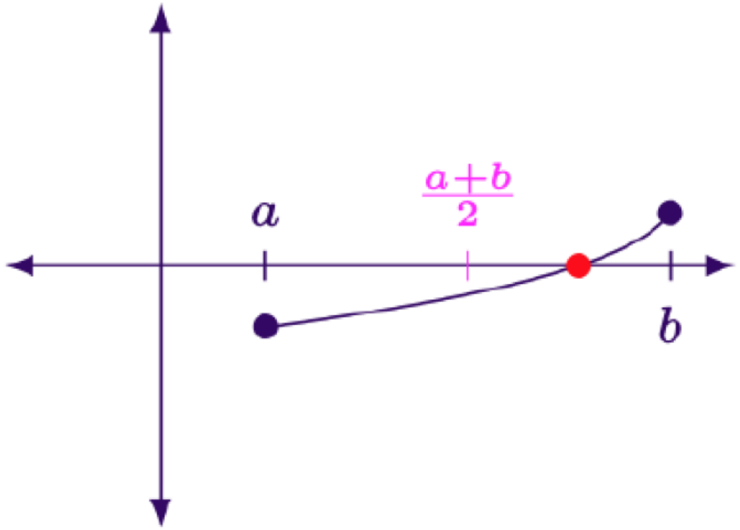
Root Finding Problem Statement

The Root of an Equation

- The value of x that satisfies $f(x) = 0$ is called the root (solution) of the equation $f(x) = 0$ or the zero of the function f.
- Graphically, it is the x-intercept when the graph of f crosses the x-axis.



The Root-finding Problem

is the process of approximating the root of the equation $f(x) = 0$ (to within a given tolerance).

Newton's Method

- Derived from the Taylor series approximation.
- The Newton's method that starts with an initial approximation p_0 and generates a sequence $\{p_n\}_{n=0}^{\infty}$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \qquad \text{for } n \geq 1.$$

NOTE:

- Newton's method cannot be continued if $f'(p_{n-1}) = 0$ for any n.
- This method is most effective when f' is bounded away from zero near p.

Newton's Method: Solution Steps ($f(x) = 0$)

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \leq N_0$ do Steps 3–6.

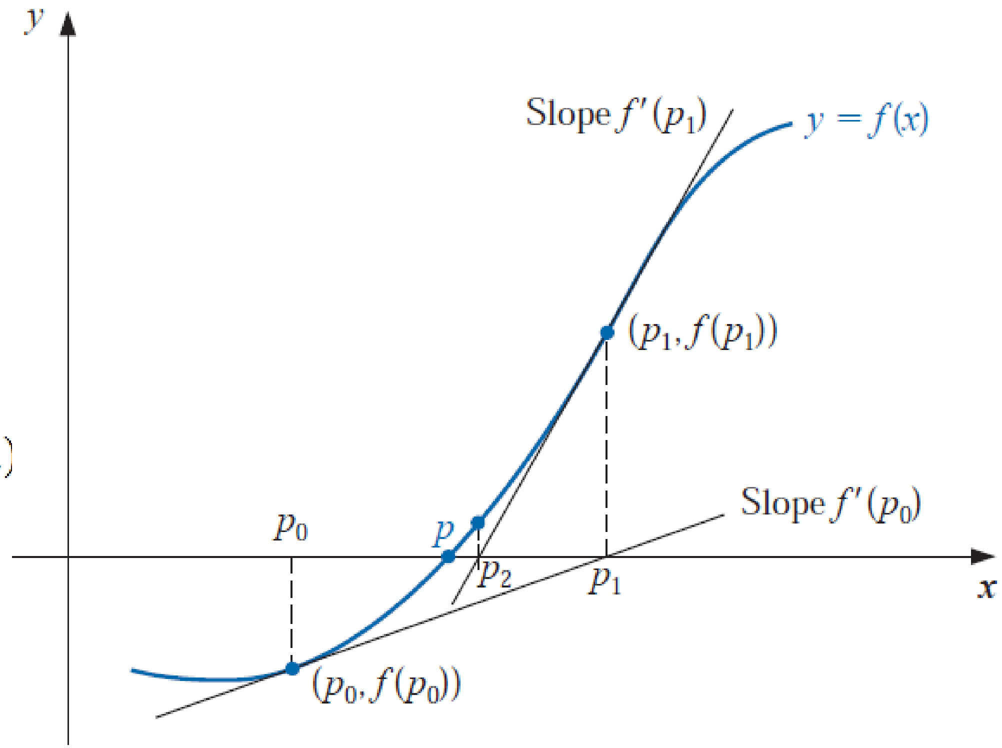
Step 3 Set $p = p_0 - f(p_0)/f'(p_0)$. (*Compute* p_i.)

Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. (*Update* p_0.)

Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP.



Problem 3

4. Use **Newton's** method to find solutions accurate to within 10^{-5} for the following problems:

a) $e^x + 2^{-x} + 2\cos x - 6 = 0, \quad \text{for } 1 \le x \le 2$

Secant Method

- Derived as an approximation of Newton's Method to avoid computing $f'(x)$ since it needs more arithmetic operations and to avoid the problem of dividing by zero.
- The Secant method that starts with initial approximations of p_0 and p_1 and generates a sequence $\{p_n\}_{n=0}^{\infty}$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})} \qquad \text{for } n \geq 2.$$

Secant Method: Solution Steps ($f(x) = 0$)

INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.

Step 2 While $i \le N_0$ do Steps 3–6.

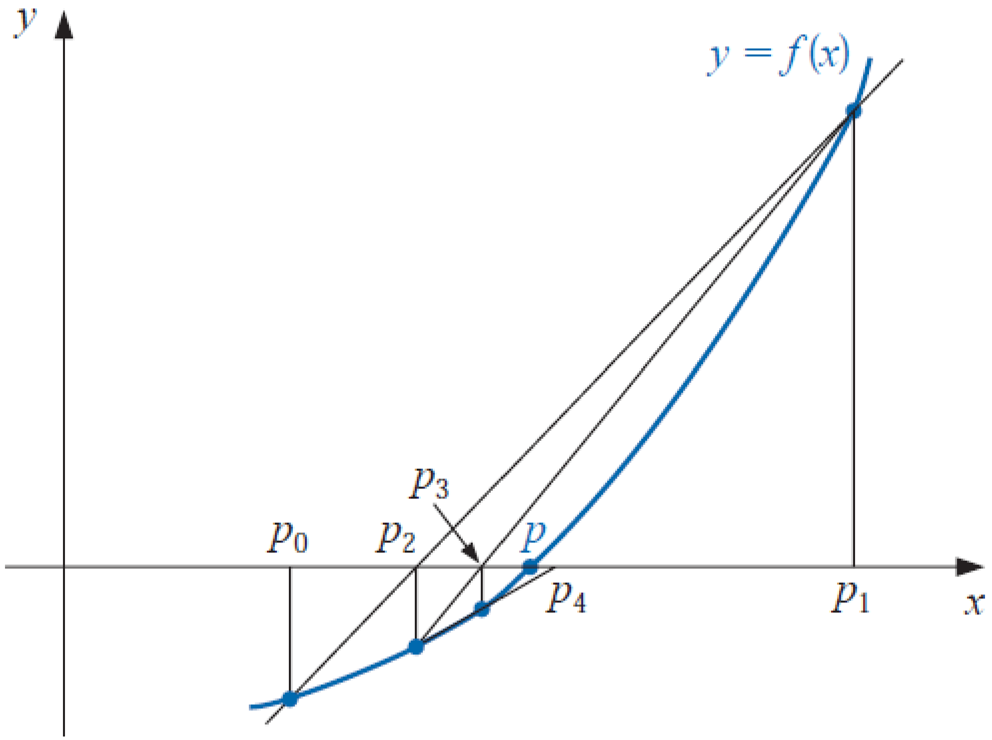
Step 3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute* p_i.)

Step 4 If $|p - p_1| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p_1$; (*Update* p_0, q_0, p_1, q_1.)
$q_0 = q_1$;
$p_1 = p$;
$q_1 = f(p)$.

Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP.



Problem 3

5. Use **Secant** method to find solutions accurate to within 10^{-5} for the following problems:

   a) $e^x + 2^{-x} + 2\cos x - 6 = 0, \quad \text{for } 1 \leq x \leq 2$

False Position Method

- Unlike the **Bisection** method, “root bracketing” is not guaranteed for either **Newton’s** and **Secant** Methods.

- The method of **False Position** combines the **secant** method together with the **bisection** bracketing method. The method computes the approximations in the same manner of the secant method that starts with initial approximations of p_0 and p_1 and generates a sequence $\{p_n\}_{n=0}^{\infty}$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})} \qquad p \in [a_n, b_n]$$

- However, the condition of $f(a_n)f(b_n) < 0$ must be satisfied at each iteration.

False Position Method: Solution Steps ($f(x) = 0$)

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INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.

Step 2 While $i \le N_0$ do Steps 3–7.

Step 3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute p_i.*)

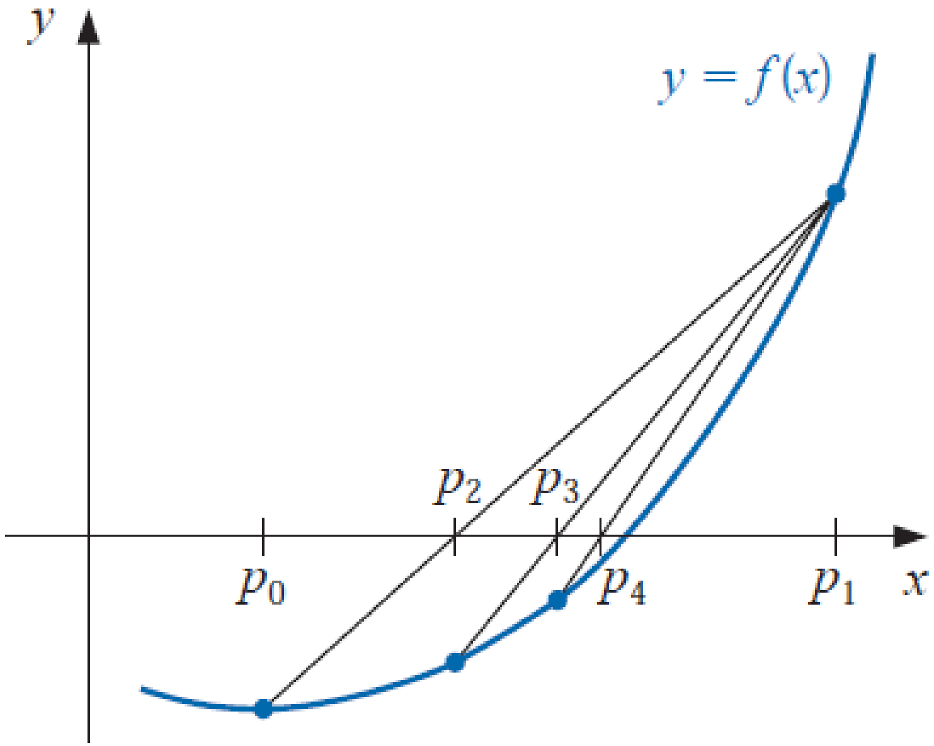
Step 4 If $|p - p_1| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$;
$q = f(p)$.

Step 6 If $q \cdot q_1 < 0$ then set $p_0 = p_1$;
$q_0 = q_1$.

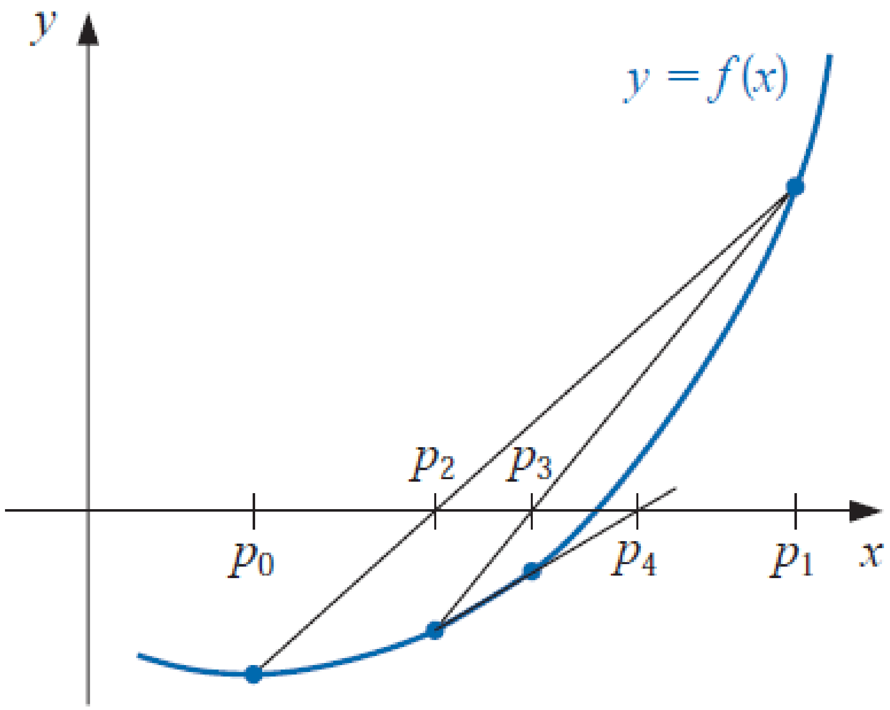
Step 7 Set $p_1 = p$;
$q_1 = q$.

Step 8 OUTPUT ('Method failed after N_0 iterations, N_0 =', N_0);
(*The procedure unsuccessful.*)
STOP.

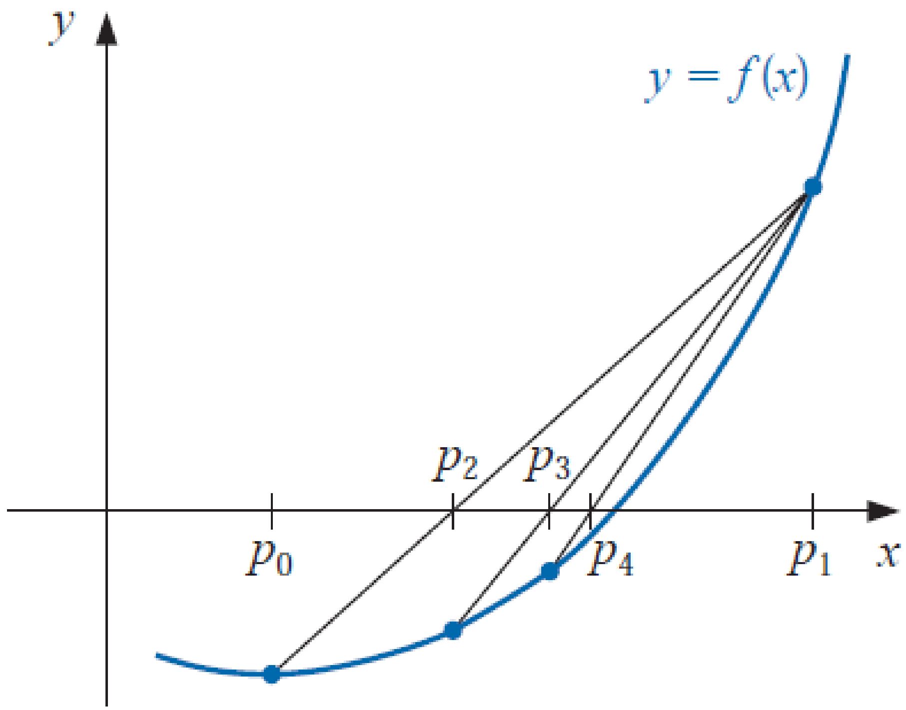


Secant vs. False Position Methods:

Secant Method



Method of False Position



Problem 3

6. Use **False Position** method to find solutions accurate to within 10^{-5} for the following problems:

a) $e^x + 2^{-x} + 2\cos x - 6 = 0, \quad \text{for } 1 \leq x \leq 2$

Fixed Point Method

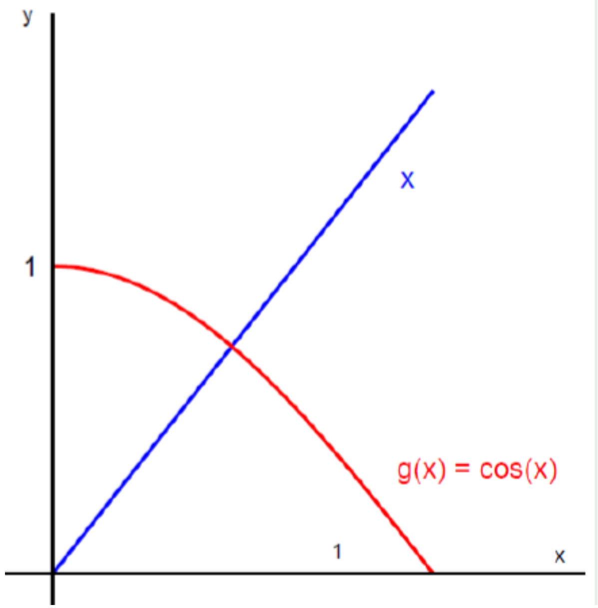
Given a root-finding problem $f(x) = 0$; where a $\leq x \leq b$. we can define function g with a fixed point at p as $g(p) = p$.

Example:

Find the root of the equation $f(x) = x - \cos(x)$ for $x\epsilon[0,1]$

$$\mathrm{g}(x) = x = \cos(x)$$

- The root of $f(x) = x - \cos(x)$ is at the intersection of $y = x$ and $y = \cos(x)$.



Fixed Point Method

- If $x \in [a, b]$, $g(x)$ has at least one fixed point in the interval $[a, b]$; if

$$g(a) \in [a, b] \text{ and } g(b) \in [a, b]$$

- In order to make sure that only one fixed point exists within the given range,

$$|g'(a)| \leq k < 1 \text{ and } |g'(b)| \leq k < 1$$

Fixed Point Method: Solution Steps ($f(x) = 0$)

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

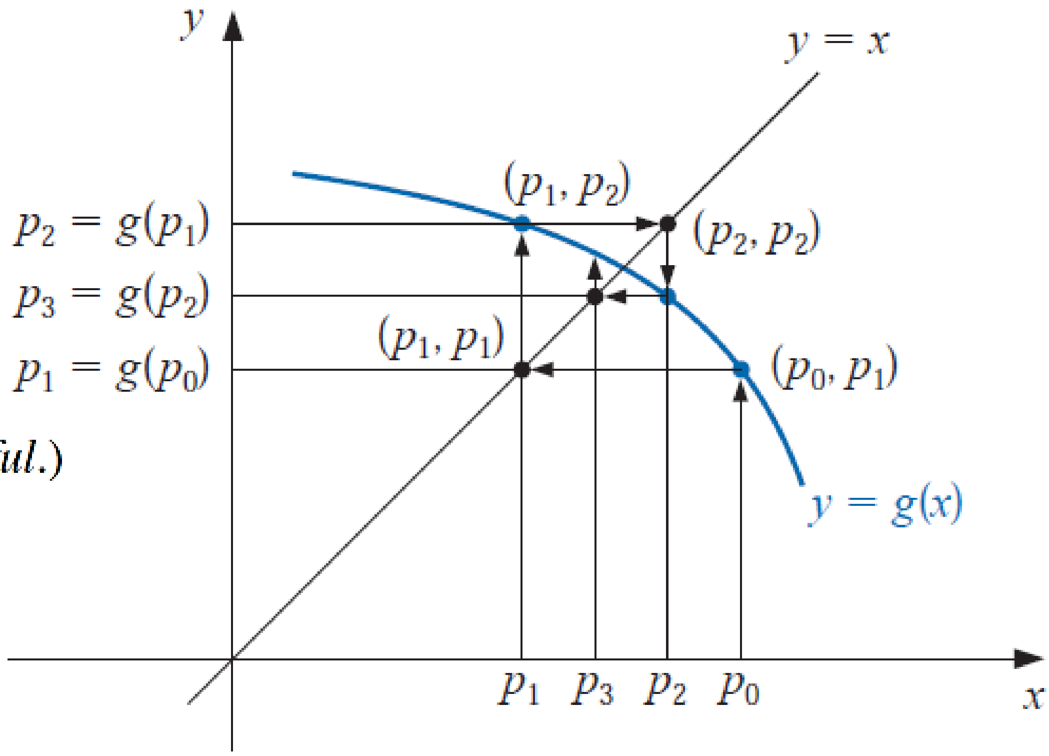
Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = g(p_0)$. *(Compute p_i.)*

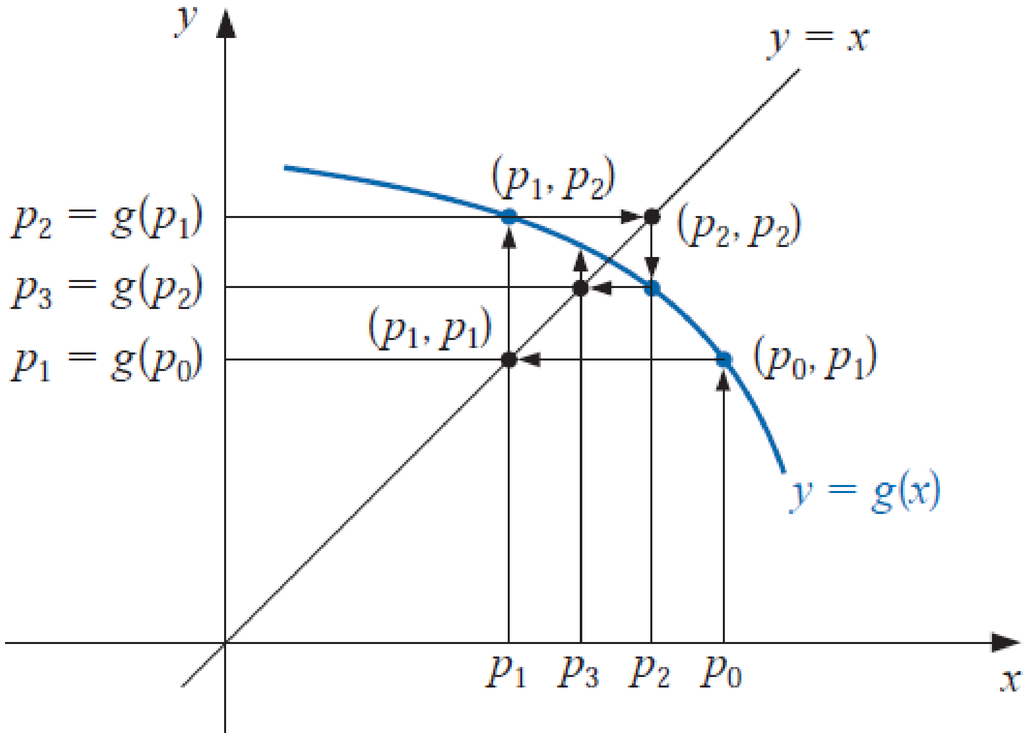
Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); *(The procedure was successful.)*
STOP.

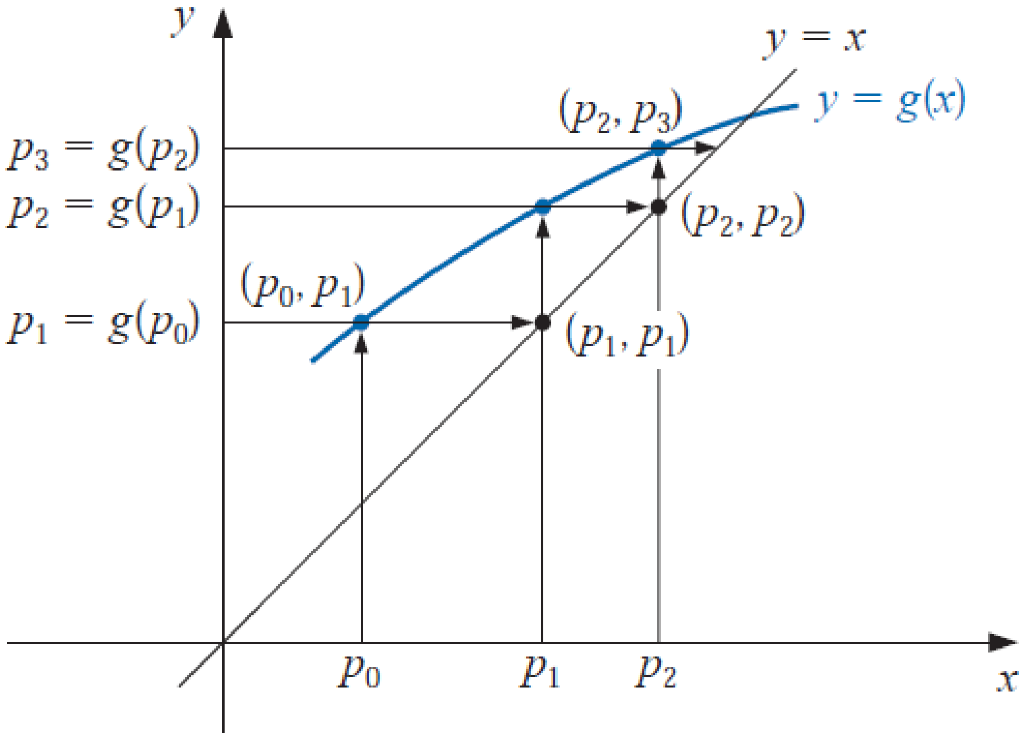
Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. *(Update p_0.)*



Fixed Point Method: **Different initial conditions**





Problem 5

1. Use algebraic manipulation to show that each of the following functions has a fixed point at p precisely when $f(p) = 0$, where$f(x) = x^4 + 2x^2 - x - 3$.

a) $g_1(x) = (3 + x - 2x^2)^{1/4}$

d) $g_4(x) = \dfrac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$

Problem 8

An object falling vertically through the air is subjected to viscous resistance as well as to the force of gravity. Assume that an object with mass m is dropped from a height s_0 and that the height of the object after t seconds is

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2 g}{k^2}(1 - e^{-kt/m}),$$

where $g = 32.17$ ft/s^2 and k represents the coefficient of air resistance in lbs/ft. Suppose $s_0 = 300$ ft, $m = 0.25$ lb and $k = 0.1$ lbs/ft . Find, to within 0.01 s, the time it takes this quarter-pounder to hit the ground.


Thank you!