TUTORIAL 6

LINEAR SYSTEMS: ITERATIVE METHODS

Numerical Analysis (ENME 602)
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Norms of Vectors

- The norm of a vector is a measurement of distance; where it represents the distance between a vector and the zero vector.

- General Form of a P-Norm:

$$||V||_p = (|v_1|^p + \cdots + |v_n|^p)^{1/p}$$

- Any vector norm has some properties. [Lecture 6: Slide 8]

Norms of Vectors

- There are three vector norms that are mostly used for engineering purposes:

1. 1-Norm (l_1): Represents the Manhattan distance.

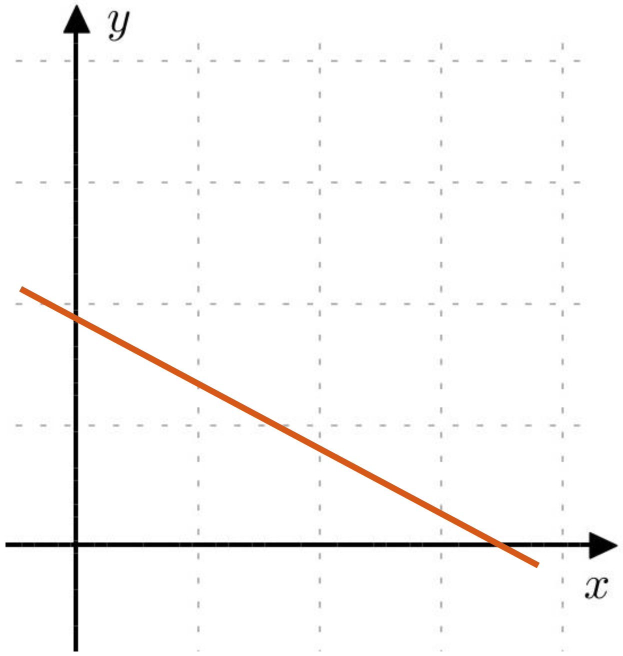
$$||V||_1 = |v_1| + \cdots + |v_n|$$

2. 2-Norm (l_2): Represents the Euclidean (minimum) distance.

$$||V_2|| = \sqrt{|v_1|^2 + \cdots + |v_n|^2}$$

3. ∞-Norm (l_∞): Represents the maximum distance.

$$||V||_\infty = \max |v_i| \qquad 1 \le i \le VectorSize$$



Norms of Vectors

- To find the distance between 2 vectors (V_1 & V_2), vector norms can be used through finding the vector norm of the difference between these 2 vectors.

$$||V||_p = ||V_2 - V_1||_p$$

Solving Linear Systems

- In order to solve the following linear system ($AX = B$):

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1, \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_2, \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= c_3.
\end{aligned}
$$

- Use exact solution: Row-echelon Form.

- Use numerical solutions: shall be discussed in this and the next tutorials.
  - Easier to solve for large problems.
  - Computationally easier to implement.
  - No exact solution in some cases (to be discussed later)

Jacobian Iterative Method

- An iterative method which starting with an initial guess ($X^{(0)}$), the method iterates till it reaches the approximated vector $X^{(n)}$.

- **Solving Steps:**
  1. Make diagonal elements subject from each equation.
  2. Iterate with the values as shown:

$$x_1^{n+1} = \frac{1}{a_{11}}(c_1 - a_{12}x_2^n - a_{13}x_3^n)$$
$$x_2^{n+1} = \frac{1}{a_{22}}(c_2 - a_{21}x_1^n - a_{23}x_3^n)$$
$$x_3^{n+1} = \frac{1}{a_{33}}(c_3 - a_{31}x_1^n - a_{32}x_2^n)$$

  3. Validate that the approximations fit the original set of equations.

NOTES:

- Re-arrange equations to make sure all diagonals have non-zero coefficients.
- To track the accuracy; l_∞ is usually used to find the maximum error between X^n and X^{n-1}.

Gauss-Seidel Iterative Method

- An iterative method which starting with an initial guess $(X^{(0)})$, the method iterates till it reaches the approximated vector $X^{(n)}$.

- **Solving Steps:**
  1. Make diagonal elements subject from each equation.
  2. Iterate with the values as shown:

$$x_1^{n+1} = \frac{1}{a_{11}}(c_1 - a_{12}x_2^n - a_{13}x_3^n)$$
$$x_2^{n+1} = \frac{1}{a_{22}}(c_2 - a_{21}x_1^{n+1} - a_{23}x_3^n)$$
$$x_3^{n+1} = \frac{1}{a_{33}}(c_3 - a_{31}x_1^{n+1} - a_{32}x_2^{n+1})$$

  3. Validate that the approximations fit the original set of equations.

NOTES:

- Re-arrange equations to make sure all diagonals have non-zero coefficients.
- To track the accuracy; l_∞ is usually used to find the maximum error between X^n and X^{n-1}.

Problem 1

2. Use the **Jacobi method** to solve the following linear system, with $TOL = 10^{-3}$ in the l_∞ norm.

$$
\begin{array}{rrrrrrrl}
 & 3x_1 & - & x_2 & + & x_3 & = 1, \\
a. & 3x_1 & + & 6x_2 & + & 2x_3 & = 0, \\
 & 3x_1 & + & 3x_2 & + & 7x_3 & = 4.
\end{array}
$$

Problem 1

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3. Use the **Gauss-Seidel method** to solve the following linear system, with $TOL = 10^{-3}$ in the l_∞ norm.

$$
\text{a.}\quad
\begin{aligned}
3x_1 - x_2 + x_3 &= 1,\\
3x_1 + 6x_2 + 2x_3 &= 0,\\
3x_1 + 3x_2 + 7x_3 &= 4.
\end{aligned}
$$

Thank you!

Appendix

1. **1-Norm: Represents the Manhattan distance.**

$$||V||_1 = |v_1| + \cdots + |v_n|$$

2. **2-Norm: Represents the Euclidean distance.**

$$||V_2|| = \sqrt{|v_1|^2 + \cdots + |v_n|^2}$$

3. **∞-Norm: Represents the maximum distance.**

$$||V||_\infty = \max |v_i| \qquad 1 \le i \le VectorSize$$