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Numerical Analysis

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Lecture 6

Dr. Hesham H. Ibrahim
Associate Professor, Mechatronics Department.
hesham.hamed-ahmed@guc.edu.eg
Office C7.04

Lecture 6

Linear Systems: Iterative Methods

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6.1 Introduction

6.1 Introduction

- Earlier in matrix algebra, the linear system $\mathbf{A}\boldsymbol{x} = \mathbf{b}$ is reduced to echelon form using elementary row operations. Solution methods that rely on this strategy (e.g. **LU** factorization) are robust and efficient, and are fundamental tools for solving the systems of linear equations that arise in practice.

- These are known as direct methods, since the solution $\boldsymbol{x}$ is obtained following a single pass through the relevant algorithm.

- In this chapter, we are now going to look at some alternative approaches that fall into the category of iterative methods.

- These techniques can only be applied to square linear systems ($\boldsymbol{n}$ equations in $\boldsymbol{n}$ unknowns), but this is of course a common and important case.

6.1 Introduction

- Iterative methods for $\mathbf{A}\boldsymbol{x} = \mathbf{b}$ begin with an approximation to the solution, $\boldsymbol{x}_0$, then seek to provide a series of improved approximations $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ that converge to the exact solution.

- For engineers, this approach is appealing because it can be stopped as soon as the approximations $\boldsymbol{x}_i$ have converged to an acceptable precision, which might be something as crude as 10^{-3}.

- With a direct method, bailing out early is not an option; the process of elimination and back-substitution has to be carried right through to completion, or else abandoned altogether.

6.1 Introduction

- By far the main attraction of iterative methods, however, is that for certain problems (particularly those where the matrix **A** is large and sparse) they are much faster than direct methods.
- On the other hand, iterative methods can be unreliable; for some problems they may exhibit very slow convergence, or they may not converge at all.
- Those methods required a large number of arithmetic operations, and using finite-digit arithmetic leads only to an approximation to an actual solution of the system.
- To discuss iterative methods for solving linear systems, we first need to determine a way to measure the distance between ***n***-dimensional column vectors.
- This will permit us to determine whether a sequence of vectors converges to a solution of the system.

6.2 Norms of Vectors & Matrices

6.2 Norms of Vectors & Matrices

Vector Norms

- Let $\mathbb{R}^n$ denote the set of all n-dimensional column vectors with real-number components.
- To define a distance in $\mathbb{R}^n$ we use the notion of a norm, which is the generalization of the absolute value on $\mathbb{R}$, the set of real numbers.

Definition: Vector Norm

A vector norm on $\mathbb{R}^n$ is a function, $\|\cdot\|$, from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(i) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$

(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$

(iii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$

(iv) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Vector Norms

Comment

- Vectors in $\mathbb{R}^n$ are column vectors, and it is convenient to use the transpose notation (presented earlier) when a vector is represented in terms of its components.
- For example, the vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

will be written $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$.

6.2 Norms of Vectors & Matrices

Vector Norms

Definition: l_2 and l_∞ Norms

The l_2 and l_∞ norms for the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ are defined by

$$\|\mathbf{x}\|_2 = \left\{ \sum_{i=1}^{n} x_i^2 \right\}^{1/2} \quad \text{and} \quad \|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$$

Note that each of these norms reduces to the absolute value in the case $n = 1$.

Vector Norms

Example

Determine the l_2 norm and the l_∞ norm of the vector $\mathbf{x} = (-1, 1, -2)^t$.

Solution

The vector $\mathbf{x} = (-1, 1, -2)^t$ in $\mathbb{R}^3$ has norms

$$\|\mathbf{x}\|_2 = \sqrt{(-1)^2 + (1)^2 + (-2)^2} = \sqrt{6}$$

and

$$\|\mathbf{x}\|_\infty = \max\{|-1|, |1|, |-2|\} = 2$$

6.2 Norms of Vectors & Matrices

Vector Norms: Distance between Vectors in R^n

- The norm of a vector gives a measure for the distance between an arbitrary vector and the zero vector, just as the absolute value of a real number describes its distance from 0.
- Similarly, the distance between two vectors is defined as the norm of the difference of the vectors just as distance between two real numbers is the absolute value of their difference.

Definition: Distance between Vectors

If $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^t$ are vectors in $\mathbb{R}^n$, the l_2 and l_∞ distances between $\mathbf{x}$ and $\mathbf{y}$ are defined by

$$\|\mathbf{x} - \mathbf{y}\|_2 = \left\{ \sum_{i=1}^{n} (x_i - y_i)^2 \right\}^{1/2} \quad \text{and} \quad \|\mathbf{x} - \mathbf{y}\|_\infty = \max_{1 \le i \le n} |x_i - y_i|$$

6.2 Norms of Vectors & Matrices

Vector Norms: Distance between Vectors in R^n

Example

The linear system

$$\begin{aligned} 3.3330x_1 + 15920x_2 - 10.333x_3 &= 15913 \\ 2.2220x_1 + 16.710x_2 + 9.6120x_3 &= 28.544 \\ 1.5611x_1 + 5.1791x_2 + 1.6852x_3 &= 8.4254 \end{aligned}$$

has the exact solution $\mathbf{x} = (x_1, x_2, x_3)^t = (1, 1, 1)^t$, and Gaussian elimination, performed using five-digit rounding arithmetic and partial pivoting, produces the approximate solution

$$\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)^t = (1.2001, 0.99991, 0.92538)^t$$

Determine the l_2 and l_∞ distances between the exact and approximate solutions.

Vector Norms: Distance between Vectors in R^n

Solution

Measurements of $\mathbf{x} - \tilde{\mathbf{x}}$ are given by

$$\begin{aligned} \|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty &= \max\{|1 - 1.2001|, |1 - 0.99991|, |1 - 0.92538|\} \\ &= \max\{0.2001, 0.00009, 0.07462\} = 0.2001 \end{aligned}$$

and

$$\begin{aligned} \|\mathbf{x} - \tilde{\mathbf{x}}\|_2 &= \left[(1 - 1.2001)^2 + (1 - 0.99991)^2 + (1 - 0.92538)^2\right]^{1/2} \\ &= [(0.2001)^2 + (0.00009)^2 + (0.07462)^2]^{1/2} = 0.21356 \end{aligned}$$

Although the components $\tilde{x}_2$ and $\tilde{x}_3$ are good approximations to x_2 and x_3, the component $\tilde{x}_1$ is a poor approximation to x_1, and $|x_1 - \tilde{x}_1|$ dominates both norms.

6.2 Norms of Vectors & Matrices

Matrix Norms & Distances

Definition: Matrix Norm

A matrix norm on the set of all $n \times n$ matrices is a real-valued function, $\|\cdot\|$, defined on this set, satisfying for all $n \times n$ matrices A and B and all real numbers α:

(i) $\|A\| \geq 0$

(ii) $\|A\| = 0$, if and only if A is O, the matrix with all 0 entries

(iii) $\|\alpha A\| = |\alpha|\|A\|$

(iv) $\|A + B\| \leq \|A\| + \|B\|$

(v) $\|AB\| \leq \|A\|\|B\|$

The distance between $\boldsymbol{n \times n}$ matrices A and B with respect to this matrix norm is $\|A - B\|$.

Matrix Norms & Distances

The l_∞ norm of a matrix can be easily computed from the entries of the matrix.

Theorem

If $A = (a_{ij})$ is an $n \times n$ matrix, then

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

Matrix Norms & Distances

Example

Determine $\|A\|_\infty$ for the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 5 & -1 & 1 \end{bmatrix}$$

Solution

We have

$$\sum_{j=1}^{3} |a_{1j}| = |1| + |2| + |-1| = 4, \quad \sum_{j=1}^{3} |a_{2j}| = |0| + |3| + |-1| = 4$$

$$\sum_{j=1}^{3} |a_{3j}| = |5| + |-1| + |1| = 7$$

So the previous theorem implies that $\|A\|_\infty = \max\{4, 4, 7\} = 7$.

6.3 The Spectral Radius of a Matrix

6.3 The Spectral Radius of a Matrix

The Characteristic Polynomial of a Matrix

Definition: Characteristic Polynomial

If A is a square matrix, the characteristic polynomial of A is defined by

$$p(\lambda) = \det(A - \lambda I)$$

Definition: Eigenvalues & Eigenvectors

- If p is the characteristic polynomial of the matrix A, the zeros of p are eigenvalues, or characteristic values, of the matrix A.
- If λ is an eigenvalue of A and $\mathbf{x} \neq \mathbf{0}$ satisfies

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

then $\mathbf{x}$ is an eigenvector, or characteristic vector, of A corresponding to the eigenvalue λ.

6.3 The Spectral Radius of a Matrix

The Characteristic Polynomial of a Matrix

6.3 The Spectral Radius of a Matrix

eigenvalues and eigenvectors

Example

Determine the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix}$$

Solution (1/4)

The characteristic polynomial of A is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ 1 & -1 & 4-\lambda \end{bmatrix} \\ &= -(\lambda^3 - 7\lambda^2 + 16\lambda - 12) \\ &= -(\lambda - 3)(\lambda - 2)^2 \end{aligned}$$

so there are two eigenvalues of A: $\lambda_1 = 3$ and $\lambda_2 = 2$.

eigenvalues and eigenvectors

Solution (2/4)

An eigenvector $\mathbf{x}_1$ corresponding to the eigenvalue $\lambda_1 = 3$ is a solution to the vector-matrix equation $(A - 3 \cdot I)\mathbf{x}_1 = \mathbf{0}$, so

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

which implies that $x_1 = 0$ and $x_2 = x_3$.

Any nonzero value of x_3 produces an eigenvector for the eigenvalue $\lambda_1 = 3$. For example, when $x_3 = 1$ we have the eigenvector $\mathbf{x}_1 = (0, 1, 1)^t$, and any eigenvector of A corresponding to $\lambda = 3$ is a nonzero multiple of $\mathbf{x}_1$.

eigenvalues and eigenvectors

Solution (3/4)

An eigenvector $\mathbf{x} \neq \mathbf{0}$ of A associated with $\lambda_2 = 2$ is a solution of the system $(A - 2 \cdot I)\mathbf{x} = \mathbf{0}$, so

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In this case the eigenvector has only to satisfy the equation

$$x_1 - x_2 + 2x_3 = 0$$

which can be done in various ways.

eigenvalues and eigenvectors

Solution (4/4)

- For example, when $x_1 = 0$ we have $x_2 = 2x_3$, so one choice would be $\mathbf{x}_2 = (0, 2, 1)^t$.
- We could also choose $x_2 = 0$, which requires that $x_1 = -2x_3$.
- Hence $\mathbf{x}_3 = (-2, 0, 1)^t$ gives a second eigenvector for the eigenvalue $\lambda_2 = 2$ that is not a multiple of $\mathbf{x}_2$.

6.3 The Spectral Radius of a Matrix

Definition: Spectral Radius

The spectral radius $\rho(A)$ of a matrix A is defined by

$$\rho(A) = \max |\lambda|, \quad \text{where } \lambda \text{ is an eigenvalue of } A$$

(For complex $\lambda = \alpha + \beta i$, we define $|\lambda| = (\alpha^2 + \beta^2)^{1/2}$.)

For the matrix in the previous example, namely

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix}$$

note that

$$\rho(A) = \max\{2, 3\} = 3$$

6.3 The Spectral Radius of a Matrix

The spectral radius is closely related to the norm of a matrix, as shown in the following theorem.

Theorem

If A is an $n \times n$ matrix, then

(i) $\|A\|_2 = [\rho(A^t A)]^{1/2}$

(ii) $\rho(A) \leq \|A\|$

for any natural norm $\|\cdot\|$

6.3 The Spectral Radius of a Matrix

Example

Determine the l_2 norm of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Note: We will apply part (i) of the theorem, namely that

$$\|A\|_2 = [\rho(A^t A)]^{1/2}$$

Solution (1/3)

We first need the eigenvalues of $A^t A$, where

$$A^t A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & 5 \end{bmatrix}$$

$$\|A\|_2 = [\rho(A^t A)]^{1/2}$$

Solution (2/3)

If

$$\begin{aligned} 0 = \det(A^t A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{bmatrix} \\ &= -\lambda^3 + 14\lambda^2 - 42\lambda \\ &= -\lambda(\lambda^2 - 14\lambda + 42) \end{aligned}$$

then $\lambda = 0$ or $\lambda = 7 \pm \sqrt{7}$.

$$||A||_2 = [\rho(A^t A)]^{1/2}$$

Solution (3/3)

By part (i) of the theorem, we have

$$\begin{aligned} ||A||_2 &= \sqrt{\rho(A^t A)} \\ &= \sqrt{\max\{0,\ 7-\sqrt{7},\ 7+\sqrt{7}\}} \\ &= \sqrt{7+\sqrt{7}} \\ &\approx 3.106 \end{aligned}$$

6.4 The Jacobi Iterative Method

6.4 The Jacobi Iterative Method

- The Jacobi method is the simplest iterative method for solving a (square) linear system $\mathbf{Ax} = \mathbf{b}$. Before developing a general formulation of the algorithm, it is instructive to explain the basic workings of the method with reference to a small example such as

$$\begin{bmatrix} 4 & 2 & 3 \\ 3 & -5 & 2 \\ -2 & 3 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -14 \\ 27 \end{bmatrix}$$

- Noting that there are no zeros on the leading diagonal, we can solve the first row for x, the second for y and the third for z:

$$x = (8 - 2y - 3z)/4$$
$$y = (-14 - 3x - 2z)/(-5)$$
$$z = (27 + 2x - 3y)/8$$

6.4 The Jacobi Iterative Method

$$x = (8 - 2y - 3z)/4$$
$$y = (-14 - 3x - 2z)/(-5)$$
$$z = (27 + 2x - 3y)/8$$

- This rearrangement is the key to the method, since it puts the equations in a form that is amenable to iteration. At each stage of the process, new values of x, y and z will be obtained by substituting the old values into the expressions on the right-hand side of the Equation. In other words, we will apply the simple iterative scheme

$$x_{n+1} = (8 - 2y_n - 3z_n)/4$$
$$y_{n+1} = (-14 - 3x_n - 2z_n)/(-5)$$
$$z_{n+1} = (27 + 2x_n - 3y_n)/8$$

6.4 The Jacobi Iterative Method

$$x_{n+1} = (8 - 2y_n - 3z_n)/4$$
$$y_{n+1} = (-14 - 3x_n - 2z_n)/(-5)$$
$$z_{n+1} = (27 + 2x_n - 3y_n)/8$$

- In the absence of better information, we will begin with $x_0 = y_0 = z_0 = 0$ as our initial approximation. This makes the first iteration easy:

$$\begin{aligned} x_1 &= (8 - 2\times 0 - 3\times 0)/4 &&= 2 \\ y_1 &= (-14 - 3\times 0 - 2\times 0)/(-5) &&= 2.8 \\ z_1 &= (27 + 2\times 0 - 3\times 0)/8 &&= 3.375 \end{aligned}$$

6.4 The Jacobi Iterative Method

The second iteration gives

$$\begin{aligned} x_2 &= (8 - 2\times 2.8 - 3\times 3.375)/4 &&= -1.931 \\ y_2 &= (-14 - 3\times 2 - 2\times 3.375)/(-5) &&= 5.350 \\ z_2 &= (27 + 2\times 2 - 3\times 2.8)/8 &&= 2.825 \end{aligned}$$

The third iteration gives

$$\begin{aligned} x_2 &= (8 - 2\times 5.350 - 3\times 2.825)/4 &&= -2.794 \\ y_2 &= (-14 - 3\times(-1.931) - 2\times 2.825)/(-5) &&= 2.771 \\ z_2 &= (27 + 2\times(-1.931) - 3\times 5.350)/8 &&= 0.886 \end{aligned}$$

6.4 The Jacobi Iterative Method

- Although the early iterations don't look promising in terms of convergence, things do eventually settle down:

Iter.	x	y	z
1	2.000	2.800	3.375
2	-1.931	5.350	2.825
3	-2.794	2.771	0.886
4	-0.050	1.478	1.637
5	0.033	3.425	2.808
6	-1.819	3.943	2.099
7	-1.546	2.548	1.442
8	-0.355	2.449	2.033
9	-0.749	3.400	2.368
10	-1.476	3.297	1.913
11	-1.083	2.680	1.770
12	-0.667	2.858	2.099
13	-1.003	3.240	2.137
14	-1.222	3.053	1.909
15	-0.958	2.830	1.925
16	-0.859	2.995	2.074
17	-1.053	3.114	2.037
18	-1.085	2.983	1.944
19	-0.949	2.926	1.985
20	-0.952	3.024	2.040
21	-1.042	3.045	2.003
22	-1.025	2.976	1.973
23	-0.967	2.974	2.003
24	-0.989	3.021	2.018
25	-1.024	3.014	1.995

Iter.	x	y	z
26	-1.003	2.984	1.989
27	-0.984	2.994	2.005
28	-1.001	3.012	2.006
29	-1.011	3.002	1.995
30	-0.997	2.992	1.997
31	-0.993	3.000	2.004
32	-1.003	3.006	2.002
33	-1.004	2.999	1.997
34	-0.997	2.996	1.999
35	-0.998	3.001	2.002
36	-1.002	3.002	2.000
37	-1.001	2.999	1.999
38	-0.998	2.999	2.000
39	-1.000	3.001	2.001
40	-1.001	3.001	2.000
41	-1.000	2.999	1.999
42	-0.999	3.000	2.000
43	-1.000	3.001	2.000
44	-1.001	3.000	2.000
45	-1.000	3.000	2.000
46	-1.000	3.000	2.000
47	-1.000	3.000	2.000
48	-1.000	3.000	2.000
49	-1.000	3.000	2.000
50	-1.000	3.000	2.000

6.5 Gauss-Seidel Method

6.5 Gauss-Seidel Method

- The Gauss-Seidel method is a variant of the Jacobi method that usually improves the rate of convergence. In the previous section, the procedure for conducting a Jacobi iteration was

$$x_{n+1} = (8 - 2y_n - 3z_n)/4$$
$$y_{n+1} = (-14 - 3x_n - 2z_n)/(-5)$$
$$z_{n+1} = (27 + 2x_n - 3y_n)/8$$

- Note that in the second line, when working out y_{n+1}, we used the old value of x (i.e. x_n) even though a new and presumably more accurate value (i.e. x_{n+1}) had just been worked out in the first line.

- Similarly, in the third line, when working out z_{n+1}, we used the old values of x and y (i.e. x_n and y_n) even though updated values (i.e. x_{n+1} and y_{n+1}) had just been worked out in previous lines.

6.5 Gauss-Seidel Method

- The Gauss-Seidel method implements the strategy of always using the latest available value of a particular variable. For our small example, therefore, it gives the iterative scheme

$$x_{n+1} = (8 - 2y_n - 3z_n)/4$$
$$y_{n+1} = (-14 - 3x_{n+1} - 2z_n)/(-5)$$
$$z_{n+1} = (27 + 2x_{n+1} - 3y_{n+1})/8$$

- These expressions should be compared very carefully with their Jacobi counterparts, where only 'old' variable values (subscript = n) appear on the right-hand sides. In the Gauss-Seidel method, we use 'new' variable values (subscript = n + 1) wherever possible.

6.5 Gauss-Seidel Method

- To clarify the operation of the Gauss-Seidel method, we will go through the first few iterations of the example, again starting from $x_0 = y_0 = z_0 = 0$ as the initial approximation. The first iteration gives

$$\begin{aligned} x_1 &= (8 - 2\times 0 - 3\times 0)/4 &&= 2 \\ y_1 &= (-14 - 3\times 2 - 2\times 0)/(-5) &&= 4 \\ z_1 &= (27 + 2\times 2 - 3\times 4)/8 &&= 2.375 \end{aligned}$$

- which is already different from before. The second iteration gives

$$\begin{aligned} x_2 &= (8 - 2\times 4 - 3\times 2.375)/4 &&= -1.781 \\ y_2 &= (-14 - 3\times(-1.781) - 2\times 2.375)/(-5) &&= 2.681 \\ z_2 &= (27 + 2\times(-1.781) - 3\times 2.681)/8 &&= 1.924 \end{aligned}$$

6.5 Gauss-Seidel Method

- The third iteration gives

$$\begin{aligned} x_3 &= (8 - 2\times 2.681 - 3\times 1.924)/4 &&= -0.784 \\ y_3 &= (-14 - 3\times(-0.784) - 2\times 1.924)/(-5) &&= 3.099 \\ z_3 &= (27 + 2\times(-0.784) - 3\times 3.099)/8 &&= 2.017 \end{aligned}$$

- Bearing in mind that the exact solution is $x = -1$, $y = 3$, $z = 2$, things are looking a great deal better than they were at the comparable stage of the Jacobi iterations, where we had $x = -2.794$, $y = 2.771$ and $z = 0.886$.

6.5 Gauss-Seidel Method

- If we carry on with the Gauss-Seidel iterations, we obtain:

Iter.	x	y	z
1	2.000	4.000	2.375
2	-1.781	2.681	1.924
3	-0.784	3.099	2.017
4	-1.062	2.969	1.996
5	-0.982	3.009	2.001
6	-1.006	2.997	2.000
7	-0.998	3.001	2.000
8	-1.001	3.000	2.000
9	-1.000	3.000	2.000
10	-1.000	3.000	2.000

- Convergence to three decimal places in all components is achieved after just 9 iterations, whereas the Jacobi method took 45 iterations to achieve the same level of accuracy. It would be wrong, however, to assume that Gauss-Seidel is always superior to Jacobi; occasionally it is worse!

6.5 Gauss-Seidel Method

- Intuitively, the Gauss-Seidel method seems more natural than the Jacobi method. If the solution is converging and updated information is available for some of the variables, surely it makes sense to use that information!
- From a programming point of view, the Gauss-Seidel method is definitely more convenient, since the old value of a variable can be overwritten as soon as a new value becomes available. With the Jacobi method, the values of all variables from the previous iteration need to be retained throughout the current iteration, which means that twice as much storage is needed.
- On the other hand, the Jacobi method is perfectly suited to parallel computation, whereas the Gauss-Seidel method is not. Because the Jacobi method updates or 'displaces' all of the variables at the same time (at the end of each iteration) it is often called the method of simultaneous displacements. The Gauss-Seidel method updates the variables one by one (during each iteration) so its corresponding name is the method of successive displacements.

Thank You

"Compact Formulation of Linear Systems & Convergence Criteria"