GUC
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الجامعة الألمانية بالقاهرة

Numerical Analysis

(ENME 602)

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Lecture 5

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Lecture 5

Interpolation & Polynomial Approximation

5.1 Divided Differences

5.2 Divided Differences Interpolation

5.3 Error of Interpolation using Divided Differences

5.4 Interpolation of Equally-Spaced Points

5.1 Divided Differences

5.1 Divided Differences

Problem:

- We are solving the same problem:
- Given

  $x_0 \ x_1 \qquad\qquad x_n$

  $f_0 \ f_1 \qquad\qquad f_n$

  find a polynomial of degree at most n, $P(x)$, that goes through all the points, that is satisfies:

  $$P(x_k)=f_k$$

- We take a new approach to this problem.

5.1 Divided Differences

- Let $P_n(x)$ be the nth degree interpolating polynomial. We want to rewrite Pn(x) in the form

$$P_n(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+ \\ \ldots.+a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1})$$

  for appropriate constants $a_0,a_1,\ldots,a_n$.

- We want to determine the coefficients $a_0,a_1,\ldots,a_n$.

5.1 Divided Differences

$$P_n(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+ \\ \ldots.+a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1})$$

- Determining a_0 is easy: $a_0=P_n(x_0)=f_0$
- To determine a_1 we compute

$$P_n(x_1)=a_0+a_1(x-x_0)$$

$$f_1=f_0+a_1(x_1-x_0)$$

- Solving for a_1 we have

$$a_1=\frac{f_1-f_0}{x_1-x_0}$$

- This prompts to define the coefficients to be the divided differences.
- The divided differences are defined recursively.

5.1 Divided Differences

- **Definition**: The 0th divided difference of a function f with respect to the point x_i is denoted by $f[x_i]$ and it is defined by

$$f[x_i]=f(x_i)$$

- **Definition:** The first divided difference of f with respect to x_i, x_{i+1} is denoted by $f[x_i,x_{i+1}]$ and it is defined as follows:

$$f[x_i,x_{i+1}]=\frac{f[x_{i+1}]-f[x_i]}{x_{i+1}-x_i}$$

- **Definition**: The second divided difference at the points x_i,x_{i+1},x_{i+2} denoted by $f[x_i,x_{i+1},x_{i+2}]$ is defined as follows:

$$f[x_i,x_{i+1},x_{i+2}]=\frac{f[x_{i+1},x_{i+2}]-f[x_i,x_{i+1}]}{x_{i+2}-x_i}$$

5.1 Divided Differences

- **Definition**: The second divided difference at the points x_i, x_{i+1}, x_{i+2} denoted by $f[x_i, x_{i+1}, x_{i+2}]$ is defined as follows:

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

- **Definition**: If the (k−1)st divided differences

  $f[x_i, \ldots, x_{i+k-1}]$ and $f[x_{i+1}, \ldots, x_{i+k}]$

  are given, the kth divided difference relative to $x_i, \ldots, x_{i+k}$ is given by

$$f[x_i, \ldots, x_{i+k}] = \frac{f[x_{i+1}, \ldots, x_{i+k}] - f[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$$

5.1 Divided Differences

- The divided differences are computed in table:

x	f(x)	Ist DD	IInd DD	IIIrd DD	IVth DD
x_0	f_0				
x_1	f_1	$f[x_0,x_1]$			
x_2	f_2	$f[x_1,x_2]$	$f[x_0,x_1,x_2]$		
x_3	f_3	$f[x_2,x_3]$	$f[x_1,x_2,x_3]$	$f[x_0,x_1,x_2,x_3]$	
x_4	f_4	$f[x_3,x_4]$	$f[x_2,x_3,x_4]$	$f[x_1,x_2,x_3,x_4]$	$f[x_0,x_1,x_2,x_3,x_4]$
…	….	….	…	….	

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} \qquad f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

Example 1

- Compute the divided differences with following data:

x	f(x)			
0	3			
1	4			
2	7			
4	19			

5.1 Divided Differences

Example 1

- Completing the table:

x	f(x)	Ist DD	IInd DD	IIIrd DD
0	3			
1	4	1		
2	7	3	1	
4	19	6	1	0

5.2 Divided Differences Interpolation

5.2 Divided Differences Interpolation

- So the nth interpolating polynomial becomes:
  $P_n(x)=f[x_0]+f[x_1,x_1](x-x_0)+f[x_0,x_1,x_2](x-x_0)(x-x_1)+\ldots+f[x_0,\ldots,x_n](x-x_0)\ldots(x-x_{n-1})$
- **Definition**: This formula is called Newton's interpolatory forward divided difference formula.

5.2 Divided Differences Interpolation

Example 2

Construct the interpolating polynomial of degree 4 for the points:

x	0.0	0.1	0.3	0.6	1.0
f(x)	-6.0000	-5.89483	-5.65014	-5.17788	-4.28172

$P_n(x)=f[x_0]+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2](x-x_0)(x-x_1)+\ldots+f[x_0,\ldots,x_n](x-x_0)\ldots(x-x_{n-1})$

- We construct the divided difference table

x	f(x)	Ist DD	IInd DD	IIIrd DD	IVth DD
0.0	-6.00000				
0.1	-5.89483	1.0517			
0.3	-5.65014	1.22345	0.5725		
0.6	-5.17788	1.5742	0.7015	0.215	
1.0	-4.28172	2.2404	0.9517	0.278	0.063

- Then, Newton's forward polynomial is:

$$P_4(x)=-6+1.0517x+0.5725x(x-0.1)+$$
$$+0.215x(x-0.1)(x-0.3)+$$
$$+0.063x(x-0.1)(x-0.3)(x-0.6)$$

5.2 Divided Differences Interpolation

Example 2

- (B) Add the point $f(1.1)=-3.99583$ to the table, and construct the polynomial of degree five.

x	f(x)	Ist DD	IInd DD	IIIrd DD	IVth DD	Vth DD
0.0	-6.00000					
0.1	-5.89483	1.0517				
0.3	-5.65014	1.22345	0.5725			
0.6	-5.17788	1.5742	0.7015	0.215		
1.0	-4.28172	2.2404	0.9517	0.278	0.063	
1.1	-3.99583	2.8589	1.237	0.356625	0.078625	0.0142

- Newton's polynomial: $P_5(x)=P_4(x)+$

$$+0.0142x(x-0.1)(x-0.3)(x-0.6)(x-1)$$

5.2 Divided Differences Interpolation

Newton's Backward Formula

- If the interpolating nodes are reordered as

  $x_n, x_{n-1}, \ldots x_1, x_0$

  a formula similar to the Newton's forward divided difference formula can be established.

- $P_n(x) = f[x_n] + f[x_n, x_{n-1}](x - x_n) + \ldots$
  $+ f[x_n, \ldots, x_0](x - x_n)\ldots(x - x_1)$

- **Definition**: This formula is called Newton's backward divided difference formula.

5.2 Divided Differences Interpolation

Newton's Backward Formula

- If the interpolating nodes are reordered as

  $x_n, x_{n-1}, \ldots x_1, x_0$

  a formula similar to the Newton's forward divided difference formula can be established.
- $P_n(x) = f[x_n] + f[x_n, x_{n-1}](x - x_n) + \ldots$

  $+ f[x_n, \ldots, x_0](x - x_n) \ldots (x - x_1)$
- **Definition**: This formula is called Newton's backward divided difference formula.

5.2 Divided Differences Interpolation

Example 3

- Construct the interpolating polynomial of degree four using Newton's backward divided difference formula using the data:

0.0	-6.00000				
0.1	-5.89483	1.0517			
0.3	-5.65014	1.22345	0.5725		
0.6	-5.17788	1.5742	0.7015	0.215	
1.0	-4.28172	2.2404	0.9517	0.278	0.063

$$
\begin{aligned}
P_4(x)=-4.28172+2.2404(x-1)+ \\
+0.9517(x-1)(x-0.6)+ \\
+0.278(x-1)(x-0.6)(x-0.3) \\
+0.063(x-1)(x-0.6)(x-0.3)(x-0.1)
\end{aligned}
$$

$$
\begin{aligned}
P_n(x)=f[x_n]+f[x_n,x_{n-1}](x-x_n)+\ldots \\
+f[x_n,\ldots,x_0](x-x_n)\ldots(x-x_1)
\end{aligned}
$$

5.3 Error of Interpolation using Divided Differences

5.3 Error of Interpolation using Divided Differences

- The nth degree polynomial generated by the Newton's divided difference formula is the **exact same polynomial** generated by Lagrange formula. Thus, the error is the same:

$$E_n(x,f)=\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)\ldots(x-x_n)$$

- Recall also that

$$E_n(x,f)=f(x)-P_n(x)$$

Example 4

- For the function

$$f(x) = x^2 e^{\frac{-x}{2}}$$

  - Construct the divided difference table for the points
    $x_0=1.1$ $x_1=2$ $x_2=3.5$ $x_3=5$ $x_4=7.1$
  - Find the Newton's forward divided difference polynomials of degree 1, 2 and 3.
  - Find the errors of the interpolates for f(1.75).
  - Find the error bound for $E_1(x,f)$.

5.3 Error of Interpolation using Divided Differences

Example 4

- The divided difference table is:

1.75

x	f(x)	Ist DD	IInd DD	IIIrd DD	IVth DD
1.1	0.6981				
2	1.4715	0.8593			
3.5	2.1287	0.4381	-0.1755		
5	2.0521	-0.0511	-0.1631	0.0032	
7.1	1.4480	-0.2877	-0.0657	0.0191	0.0027

- $P_1(x) = 0.6981+0.8593(x-1.1)$
- $P_2(x) = P_1(x)-0.1755(x-1.1)(x-2)$
- $P_3(x) = P_2(x)+0.0032(x-1.1)(x-2)(x-3.5)$

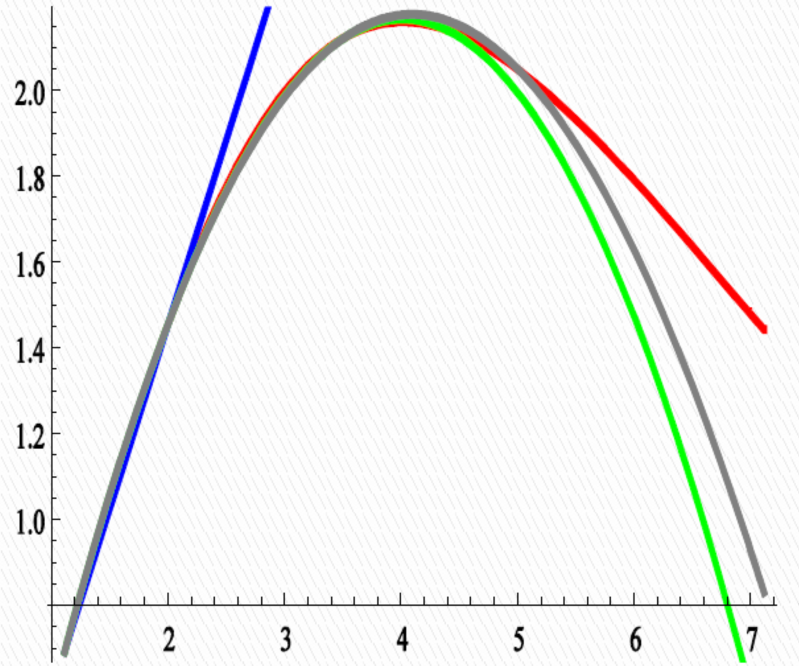
5.3 Error of Interpolation using Divided Differences

Example 4

- $f(1.75)=1.2766$

Degree	$P_n(1.75)$	Actual error
1	1.25665	0.01995
2	1.2852	−0.0086
3	1.2861	−0.0095

- Typically we can expect that a higher degree polynomial will approximate better but here $P_2(x)$ approximates better than $P_3(x)$.
- Difference is small.



$f(x)$ in red, $P_1(x)$ in blue, $P_2(x)$ in green, $P_3(x)$ in gray

5.3 Error of Interpolation using Divided Differences

Example 4

- The error of $P_1(x)$ is

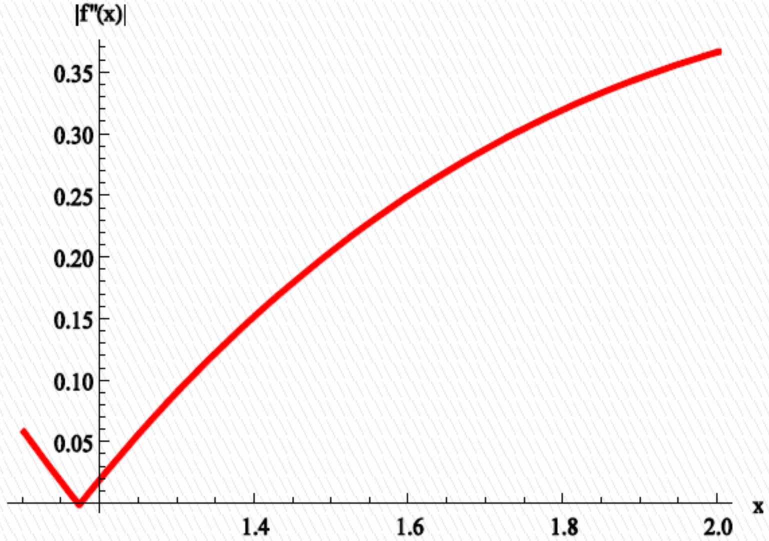
$$E_1(x,f)=\frac{f''(\xi(x))}{2!}(x-1.1)(x-2)$$

- We find the derivatives

$$f(x)=x^2e^{-\frac{x}{2}}$$

$$f'(x)=(2x-\frac{x^2}{2})e^{-\frac{x}{2}}$$

$$f''(x)=(2-2x+\frac{x^2}{4})e^{-\frac{x}{2}}$$



$\max_x |f''(x)| \le |f''(2)| = 0.3679$

Plot of $|f''(x)|$ on [1.1,2]

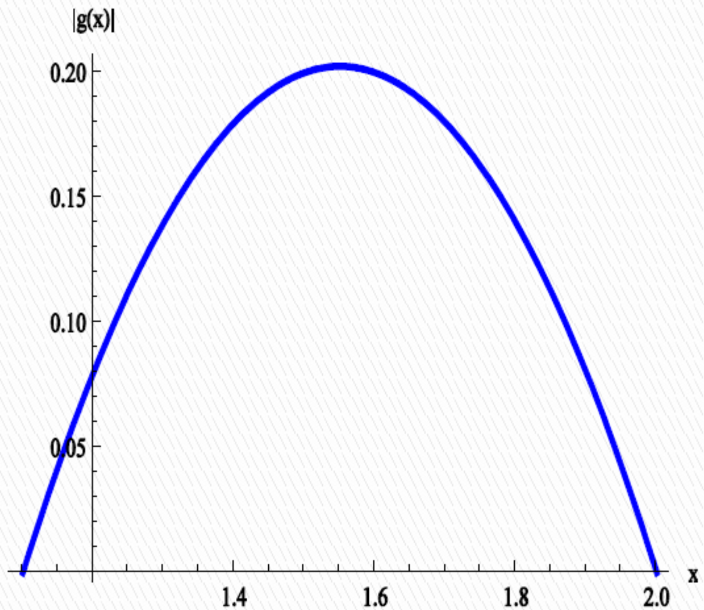
5.3 Error of Interpolation using Divided Differences

Example 4

- $g(x)=(x-1.1)(x-2)$
- The maximum of $|g(x)|$ is attained at the midpoint of the interval $[1.1,2]$:
- $p_m=(1.1+2)/2=1.55$
- $|g(x)|\le|g(1.55)|=0.2025$
- Error bound:

$$|E_1(x,f)|=\frac{|f''(\xi(x))|}{2!}|(x-1.1)(x-2)|$$

$$\le\frac{0.3679}{2}0.2025=0.03725$$



Plot of $|g(x)|$ on $[1.1,2]$.

5.3 Error of Interpolation using Divided Differences

How Does Divided Difference Relate to the Derivative?

- Notice that

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

- The Mean Value Theorem says that if $f'(x)$ exists, then

$$f[x_0,x_1]=f'(\xi)$$

for some ξ between x_0 and x_1.

- The following Theorem generalizes this:

- **Theorem** Suppose f has n continuous derivatives and x0,x1,…,xn are distinct numbers in [a,b]. Then ξ in (a,b) exists with

$$f[x_0,\ldots,x_n] = \frac{f^{(n)}(\xi)}{n!}$$

5.3 Error of Interpolation using Divided Differences

Error Estimation when $f(x)$ is Unknown: Next Term Rule

- Often f(x) is NOT known, and the nth derivative of f(x) is also not known. Therefore, it is hard to bound the error.
- We saw that

$$f[x_0,\ldots,x_n]=\frac{f^{(n)}(\xi)}{n!}$$

- Thus, the nth divided difference is an estimate of the nth derivative of f.

5.3 Error of Interpolation using Divided Differences

Error Estimation when $f(x)$ is Unknown: Next Term Rule

- This means that the error is approximated by the value of the next term to be added:

$$E_n(x,f) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)\ldots(x-x_n)$$

$$\approx f[x_0,\ldots,x_n,x_{n+1}](x-x_0)\ldots(x-x_n)$$

- $E_n(x,f) \approx$ the value of the next term that would be added to $P_n(x)$.

5.3 Error of Interpolation using Divided Differences

Example 5: Next Term Rule

- For the function

$$f(x) = x^2 e^{-\frac{x}{2}}$$

  - Construct the divided difference table for the points
    $x_0=1.1 \quad x_1=2 \quad x_2=3.5 \quad x_3=5 \quad x_4=7.1$
  - Find the Newton's forward divided difference polynomial of degree 1.
  - Use the next term rule to estimate the error of the interpolate for f(1.75).

$$|E_1(x,f)| = \frac{|f''(\xi(x))|}{2!}|(x-1.1)(x-2)|$$
$$\le \frac{0.3679}{2} 0.2025 = 0.03725$$

Example 5: Next Term Rule

- The divided difference table is:

x	f(x)	Ist DD	IInd DD
1.1	0.6981		
2	1.4715	0.8593	
3.5	2.1287	0.4381	-0.1755

- $P_1(x) = 0.6981 + 0.8593(x-1.1)$
- $P_2(x) = P_1(x) - 0.1755(x-1.1)(x-2)$

- The next term rule gives:
- $E_1(1.75, f) \approx -0.17755(1.75-1.1)(1.75-2) = 0.02852$

5.4 Interpolation of Equally-Spaced Points

5.4 Interpolation of Equally-Spaced Points

- **Definition**: The points $x_0, x_1, \ldots, x_n$ are called equally spaced if

  $$x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h \text{ (step)}.$$

- **Example**: $x_0 = 1 \quad x_1 = 1.5 \quad x_2 = 2 \quad x_3 = 2.5$
- If the data are equally spaced getting the interpolation polynomial is simpler.
- When we compute the divided differences we will always divide by the same number.
- In this case it is more convenient to define ordinary differences.

5.4 Interpolation of Equally-Spaced Points

- **Definition**: The first forward difference $\Delta f(x_i)$ is defined as

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i)$$

- Then,

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{\Delta f(x_i)}{h}$$

- **Example**: Let $f(x)=\ln(x)$. The first forward difference at the points $x_0=1$ $x_1=2$ is
  $\Delta f(x_0)=f(2)-f(1)=\ln(2)-\ln(1)=\ln(2)=0.69315$

5.4 Interpolation of Equally-Spaced Points

- The second forward difference $\Delta^2 f(x_i)$ is defined as follows:

$$\Delta^2 f(x_i) = \Delta f(x_{i+1}) - \Delta f(x_i)$$

- Consequently the second divided difference expressed in terms of the ordinary difference is:

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_{i+1}, x_i]}{x_{i+2} - x_i} =$$

$$= \frac{1}{2h}\left[\frac{\Delta f(x_{i+1})}{h} - \frac{\Delta f(x_i)}{h}\right] = \frac{\Delta^2 f(x_i)}{2h^2}$$

5.4 Interpolation of Equally-Spaced Points

- The (k+1)st forward difference $\Delta^{k+1} f(x_i)$ is defined as follows:

$$\Delta^{k+1} f(x_i) = \Delta^k f(x_{i+1}) - \Delta^k f(x_i)$$

- In general,

$$f[x_i, \ldots, x_{i+k}] = \frac{\Delta^k f(x_i)}{k!\,h^k}$$

- Computing ordinary differences is the same as computing divided differences – in a table.

5.4 Interpolation of Equally-Spaced Points

Example 6

- Compute the ordinary differences table for

$$f(x) = 2x^3$$

  for the points:
  $x_0=0,\ x_1=0.5,\ x_2=1,\ x_3=1.5,\ x_4=2,\ x_5=2.5$
- Compute the divided differences table for the same function and the same points.
- Compare the two tables.

Table of Ordinary Differences

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	0				
0.5	0.25	0.25			
1	2	1.75	1.5		
1.5	6.75	4.75	3.0	1.5	
2	16	9.25	4.5	1.5	0
2.5	31.25	15.25	6.0	1.5	0
3	54	22.75	7.5	1.5	0

5.4 Interpolation of Equally-Spaced Points

Example 6

- Compute the ordinary differences table for

$$f(x) = 2x^3$$

  for the points:
  $x_0=0,\ x_1=0.5,\ x_2=1,\ x_3=1.5,\ x_4=2,\ x_5=2.5$
- Compute the divided differences table for the same function and the same points.
- Compare the two tables.

Table of Divided Differences

x	f(x)	Ist DD	IInd DD	IIIrd DD	IVth DD
0	0				
0.5	0.25	0.5			
1	2	3.5	3		
1.5	6.75	9.5	6	2	
2	16	18.5	9	2	0
2.5	31.25	30.5	12	2	0
3	54	45.5	15	2	0

5.4 Interpolation of Equally-Spaced Points

Example 6

Table of Ordinary Differences

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	0				
0.5	0.25	0.25			
1	2	1.75	1.5		
1.5	6.75	4.75	3.0	1.5	
2	16	9.25	4.5	1.5	0
2.5	31.25	15.25	6.0	1.5	0
3	54	22.75	7.5	1.5	0

Table of Divided Differences

x	f(x)	Ist DD	IInd DD	IIIrd DD	IVth DD
0	0				
0.5	0.25	0.5			
1	2	3.5	3		
1.5	6.75	9.5	6	2	
2	16	18.5	9	2	0
2.5	31.25	30.5	12	2	0
3	54	45.5	15	2	0

$$f[x_i, \ldots, x_{i+k}] = \frac{\Delta^k f(x_i)}{k!\,h^k}$$

5.4 Interpolation of Equally-Spaced Points

- An interpolation polynomial of degree n can be written in terms of ordinary differences.
- The independent variable in this polynomial is typically not x but s:

$$s = \frac{x - x_0}{h}$$

- Newton's forward difference formula is given by:

$$P_n(s) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \ldots$$

$$\ldots + \frac{s(s-1)\ldots(s-n+1)}{n!}\Delta^n f(x_0)$$

5.4 Interpolation of Equally-Spaced Points

Example 7

- Given the table of x_i and $f(x_i)$:

x	0	0.2	0.4	0.6	0.8	1.0	1.2
f(x)	0	0.203	0.423	0.684	1.03	1.557	2.572

- Compute the forward differences to order four.
- Find f(0.73) from a cubic interpolating polynomial.

5.4 Interpolation of Equally-Spaced Points

Example 7

- We complete the table

x	f(x)	Ist diff	IInd diff	IIIrd diff	IVth diff
0	0				
0.2	0.203	0.203			
0.4	0.423	0.22	0.017		
0.6	0.684	0.261	0.041	0.024	
0.8	1.03	0.346	0.085	0.044	0.2
1.0	1.557	0.527	0.181	0.096	0.052
1.2	2.572	1.015	0.488	0.307	0.211

- 0.73

5.4 Interpolation of Equally-Spaced Points

Example 7

- Since 0.73 falls between 0.6 and 0.8 and we need 4 point to obtain a cubic polynomial, we use the closest points to 0.73:

x_0	x_1	x_2	x_3
0.4	0.6	0.8	1

- We take the appropriate subtable:

x	f(x)	Ist diff	IInd diff	IIIrd diff
0.4	0.423			
0.6	0.684	0.261		
0.8	1.03	0.346	0.085	
1.0	1.557	0.527	0.181	0.096

5.4 Interpolation of Equally-Spaced Points

Example 7

- We obtain the polynomial:

$$P_3(s) = 0.423 + 0.261s + 0.085\frac{s(s-1)}{2} + $$
$$+0.096\frac{s(s-1)(s-2)}{6}$$

- Since x=0.73, then

  $s=(x-x_0)/h=(0.73-0.4)/0.2=1.65$

  $P(1.65) = 0.893$

- Note: The function f(x)=tan(x). So f(0.73)=0.895. Thus the actual error of the approximation is 0.002.

5.4 Interpolation of Equally-Spaced Points

Backward Differences

- As before, we can rearrange the points and define backward differences:

  $x_n \quad x_{n-1} \quad \ldots \quad x_1 \quad x_0$

- **Definition**: The first backward difference at x_i is defined as follows:

$$\nabla f(x_i) = f(x_i) - f(x_{i-1})$$

- **Note:**

$$\nabla f(x_i) = \Delta f(x_{i-1})$$

5.4 Interpolation of Equally-Spaced Points

Backward Differences

- **Definition**: The kth backward difference at the point x_i is defined as follows:

$$\nabla^k f(x_i) = \nabla^{k-1} f(x_i) - \nabla^{k-1} f(x_{i-1})$$

- **Definition**: Newton's backward difference formula is given by

$$P_n(s) = f(x_n) + s\nabla f(x_n) + \frac{s(s+1)}{2!}\nabla^2 f(x_n) + \ldots$$

$$\ldots + \frac{s(s+1)\ldots(s+n-1)}{n!}\nabla^n f(x_n)$$

where $s=(x-x_n)/h$.

5.4 Interpolation of Equally-Spaced Points

Example 8

- Given the data:

x	−0.75	−0.5	−0.25	0
f(x)	−0.0718125	−0.02475	0.3349375	1.101

- Construct the forward difference table.
- Use Newton's forward difference formula to construct the interpolating polynomial of degree 3.
- Use Newton's backward difference formula to construct the interpolating polynomial of degree 3.
- Use either polynomial to approximate $f(-1/3)$.

5.4 Interpolation of Equally-Spaced Points

Example 8

- We construct the forward difference table:

x	f(x)	Ist diff	IInd diff	IIIrd diff
−0.75	−0.0718125			
−0.5	−0.02475	0.0470625		
−0.25	0.3349375	0.3596875	0.312625	
0	1.101	0.7660625	0.406375	0.09375

- The forward difference polynomial is

$$P_3(s) = -0.0718125 + 0.0470625s + 0.312625\frac{s(s-1)}{2!} + $$

$$+0.09375\frac{s(s-1)(s-2)}{3!}$$

5.4 Interpolation of Equally-Spaced Points

Example 8

- The backward difference table is exactly the same as the forward difference table

x	f(x)	Ist diff	IInd diff	IIIrd diff
−0.75	−0.0718125			
−0.5	−0.02475	0.0470625		
−0.25	0.3349375	0.3596875	0.312625	
0	1.101	0.7660625	0.406375	0.09375

- The backward difference polynomial is:

$$P_3(s) = 1.101 + 0.7660625s + 0.406375\frac{s(s+1)}{2!} + $$
$$+0.09375\frac{s(s+1)(s+2)}{3!}$$

$$P_n(s) = f(x_n) + s\nabla f(x_n) + \frac{s(s+1)}{2!}\nabla^2 f(x_n) + \ldots$$
$$\ldots + \frac{s(s+1)\ldots(s+n-1)}{n!}\nabla^n f(x_n)$$

Example 8

- We have to use either polynomial to estimate $f(-1/3)$.
- If we use the backward polynomial,

$$s=(x-x_n)/h=x/h = -4/3$$

- We compute $P_3(-4/3)\approx 0.1745185$

Thank You

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