TUTORIAL 5

DIVIDED DIFFERENCES

Numerical Analysis (ENME 602)

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Divided Differences

$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$

- Determining a_0 is easy: $a_0 = P_n(x_0) = f_0$
- To determine a_1 we compute

$$P_n(x_1) = a_0 + a_1(x - x_0)$$
$$f_1 = f_0 + a_1(x_1 - x_0)$$

- Solving for a_1 we have

$$a_1 = \frac{f_1 - f_0}{x_1 - x_0}$$

Divided Differences

- **Definition**: If the (k−1)st divided differences

  $f[x_i,\ldots,x_{i+k-1}]$ and $f[x_{i+1},\ldots,x_{i+k}]$

  are given, the kth divided difference relative to $x_i,\ldots,x_{i+k}$ is given by

$$f[x_i,\ldots,x_{i+k}] = \frac{f[x_{i+1},\ldots,x_{i+k}] - f[x_i,\ldots,x_{i+k-1}]}{x_{i+k} - x_i}$$

Divided Differences

- The divided differences are computed in table:

x	f(x)	Ist DD	IInd DD	IIIrd DD	IVth DD
x_0	f_0				
x_1	f_1	$f[x_0,x_1]$			
x_2	f_2	$f[x_1,x_2]$	$f[x_0,x_1,x_2]$		
x_3	f_3	$f[x_2,x_3]$	$f[x_1,x_2,x_3]$	$f[x_0,x_1,x_2,x_3]$	
x_4	f_4	$f[x_3,x_4]$	$f[x_2,x_3,x_4]$	$f[x_1,x_2,x_3,x_4]$	$f[x_0,x_1,x_2,x_3,x_4]$
…	….	….	…	….	

Divided Differences Interpolation

Newton's Forward Formula

- So the nth interpolating polynomial becomes:
  $P_n(x)=f[x_0]+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2](x-x_0)(x-x_1)+\ldots+f[x_0,\ldots,x_n](x-x_0)\ldots(x-x_{n-1})$
- **Definition**: This formula is called Newton's interpolatory forward divided difference formula.

Divided Differences Interpolation

Newton’s Backward Formula

- If the interpolating nodes are reordered as

  $$x_n, x_{n-1}, \dots x_1, x_0$$

  a formula similar to the Newton’s forward divided difference formula can be established.
- $$P_n(x) = f[x_n] + f[x_n, x_{n-1}](x - x_n) + \dots + f[x_n, \dots, x_0](x - x_n)\dots(x - x_1)$$
- **Definition**: This formula is called Newton’s backward divided difference formula.

Error of Interpolation using Divided Differences

- The nth degree polynomial generated by the Newton's divided difference formula is the **exact same polynomial** generated by Lagrange formula. Thus, the error is the same:

$$E_n(x,f)=\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)\ldots(x-x_n)$$

- Recall also that

$$E_n(x,f)=f(x)-P_n(x)$$

Error of Interpolation using Divided Differences

- Often f(x) is NOT known, and the nth derivative of f(x) is also not known. Therefore, it is hard to bound the error.
- We saw that

$$f[x_0,\ldots,x_n]=\frac{f^{(n)}(\xi)}{n!}$$

- Thus, the nth divided difference is an estimate of the nth derivative of f.

Error of Interpolation using Divided Differences

- This means that the error is approximated by the value of the next term to be added:

$$E_n(x,f) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)\ldots(x-x_n)$$

$$\approx f[x_0,\ldots,x_n,x_{n+1}](x-x_0)\ldots(x-x_n)$$

- $E_n(x,f) \approx$ the value of the next term that would be added to $P_n(x)$.

Interpolation of Equally-Spaced Points

- **Definition**: The points $x_0, x_1, \ldots, x_n$ are called equally spaced if

  $$x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h \text{ (step)}.$$

Interpolation of Equally-Spaced Points

Forward Differences

- The (k+1)st forward difference $\Delta^{k+1} f(x_i)$ is defined as follows:

$$\Delta^{k+1} f(x_i) = \Delta^k f(x_{i+1}) - \Delta^k f(x_i)$$

- In general,

$$f[x_i, \ldots, x_{i+k}] = \frac{\Delta^k f(x_i)}{k!\,h^k}$$

Interpolation of Equally-Spaced Points

$$s = \frac{x - x_0}{h}$$

- Newton's forward difference formula is given by:

$$P_n(s) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \ldots$$

$$\ldots + \frac{s(s-1)\ldots(s-n+1)}{n!}\Delta^n f(x_0)$$

Interpolation of Equally-Spaced Points

Backward Differences

- As before, we can rearrange the points and define backward differences:

$$x_n \quad x_{n-1} \; \ldots \; x_1 \quad x_0$$

Interpolation of Equally-Spaced Points

- **Definition**: The kth backward difference at the point x_i is defined as follows:

$$\nabla^k f(x_i) = \nabla^{k-1} f(x_i) - \nabla^{k-1} f(x_{i-1})$$

- **Definition**: Newton's backward difference formula is given by

$$P_n(s) = f(x_n) + s\nabla f(x_n) + \frac{s(s+1)}{2!}\nabla^2 f(x_n) + \ldots$$

$$\ldots + \frac{s(s+1)\ldots(s+n-1)}{n!}\nabla^n f(x_n)$$

where $s=(x-x_n)/h$.

Problem 1

Use the Newton forward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.

a. $f(8.4)$ if $f(8.1) = 16.94410$, $f(8.3) = 17.56492$, $f(8.6) = 18.50515$, $f(8.7) = 18.82091$

Problem 3

Use the Newton backward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specified value using each of the polynomials.

$a.$ $f(-\frac{1}{3})$ if $f(-0.75) = -0.07181250$, $f(-0.5) = -0.024750$, $f(-0.25) = 0.33493750$, $f(0) = 1.1010$

Problem 4

The following data are given for a polynomial $P(x)$ of unknown degree.

x	0	1	2	3
$P(x)$	4	9	15	18

Determine the coefficient of x^3 in $P(x)$ if all forth-order forward differences are 1.

Problem 6

a. Approximate $f\,(0.05)$ using the following data and the Newton forward-difference formula:

x	0.0	0.2	0.4	0.6	0.8
$P(x)$	1.00000	1.22140	1.49182	1.82212	2.22554

b. Use the Newton backward-difference formula to approximate $f\,(0.65)$.

Thank you!