Practice Sheet 2B

Algorithms & Solutions of Non-Linear Equations

Newton, Secant & False position Methods

Problem 1

1. Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use **Newton's** method to find p_2.
2. Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use **Newton's** method to find p_2. Could $p_0 = 0$ be used?

Solution:

Question	Solution
1	$p_2 = 2.6071$
2	$p_2 = -0.865684$
	If $p_0 = 0$ then p_1 cannot be computed

Problem 2

1. Let $f(x) = x^2 - 6$. With $p_0 = 3$ and $p_1 = 2$, find p_3.

   a. Use the **Secant** method.

   b. Use the method of **False Position**.

   c. which of **a** or **b** is closer to $\sqrt{6}$?

2. Let $f(x) = -x^3 - \cos x$. With $p_0 = -1$ and $p_1 = 0$, find p_3.

   a. Use the **Secant** method.

   b. Use the method of **False Position**.

Solution:

Question	Solution
1.a	2.4545
1.b	2.4444
1.c	part (a) is better
2.a	-1.2521
2.b	-0.84136

Problem 3

1. Use **Newton's** method to find solutions accurate to within 10^{-4} for the following problems.

   a. $x^3 - 2x^2 - 5 = 0, \quad [1, 4]$

   b. $x^3 + 3x^2 - 1 = 0, \quad [-3, -2]$

   c. $x - \cos x = 0, \quad [0, \pi/2]$

   d. $x - 0.8 - 0.2 \sin x = 0, \quad [0, \pi/2]$

2. Repeat Exercise 3.1 using the **Secant** Method.

3. Repeat Exercise 3.1 using the method of **False Position**.

4. Use **Newton's** method to find solutions accurate to within 10^{-5} for the following problems.

   a. $e^x + 2^{-x} + 2\cos x - 6 = 0, \quad \text{for } 1 \le x \le 2$

   b. $\ln(x-1) + \cos(x-1) = 0, \quad \text{for } 1.3 \le x \le 2$

   c. $2x \cos 2x - (x-2)^2 = 0, \quad \text{for } 2 \le x \le 3 \text{ and } 3 \le x \le 4$

   d. $(x-2)^2 - \ln x = 0, \quad \text{for } 1 \le x \le 2 \text{ and } e \le x \le 4$

   e. $e^x - 3x^2 = 0, \quad \text{for } 0 \le x \le 1 \text{ and } 3 \le x \le 5$

   f. $\sin x - e^{-x} = 0, \quad \text{for } 0 \le x \le 1 \ , \ 3 \le x \le 4 \text{ and } 6 \le x \le 7$

5. Repeat Exercise 3.4 using the **Secant** Method.

6. Repeat Exercise 3.4 using the method of **False Position**.

Solution:

Question	p_0	Solution	Question	Solution	Question	Solution
1.a	2	$p_5 = 2.6907$	**2.a**	$p_{11} = 2.6907$	**3.a**	$p_{16} = 2.6906$
1.b	-3	$p_3 = -2.8794$	**2.b**	$p_7 = -2.8794$	**3.b**	$p_6 = -2.87938$
1.c	0	$p_4 = 0.7391$	**2.c**	$p_6 = 0.7391$	**3.c**	$p_7 = 0.73908$
1.d	0	$p_3 = 0.96434$	**2.d**	$p_5 = 0.9643$	**4.d**	$p_6 = 0.96433$
4.a	1	$p_8 = 1.8294$	**5.a**	$p_7 = 1.8294$	**6.a**	$p_8 = 1.8294$
4.b	1.5	$p_4 = 1.3977$	**5.b**	$p_9 = 1.3977$	**6.b**	$p_9 = 1.3977$
4.c	2	$p_4 = 2.3707$	**5.c**	$p_6 = 2.3707$	**6.c**	$p_6 = 2.3707$
	4	$p_4 = 3.7221$		$p_7 = 3.7221$		$p_8 = 3.7221$
4.d	1	$p_4 = 1.4124$	**5.d**	$p_8 = 1.4124$	**6.d**	$p_{10} = 1.4124$
	4	$p_5 = 3.0571$		$p_7 = 3.0571$		$p_{12} = 3.0571$
4.e	1	$p_4 = 0.91001$	**5.e**	$p_6 = 0.91001$	**6.e**	$p_7 = 0.91001$
	3	$p_9 = 3.7331$		$p_{10} = 3.7331$		$p_{29} = 3.7331$
4.f	0	$p_4 = 0.58853$	**5.f**	$p_6 = 0.58853$	**6.f**	$p_9 = 0.58853$
	3	$p_3 = 3.0964$		$p_5 = 3.0964$		$p_5 = 3.0964$
	6	$p_3 = 6.285$		$p_5 = 6.285$		$p_5 = 6.285$

N.B. For Secant Method and Method of False Position use the endpoints of the intervals as the initial approximations (p_0 and p_1).

Problem 4

In the design of all-terrain vehicles, it is necessary to consider the failure of the vehicle when attempting to negotiate two types of obstacles. One type of failure is called hang-up failure and occurs when the vehicle attempts to cross an obstacle that causes the bottom of the vehicle to touch the ground. The other type of failure is called *nose-in failure* and occurs when the vehicle descends into a ditch and its nose touches the ground. The accompanying figure, adapted from [Bek], shows the components associated with the nose-in failure of a vehicle. In that reference it is shown that the maximum angle α that can be negotiated by a vehicle when β is the maximum angle at which hang-up failure does not occur satisfies the equation

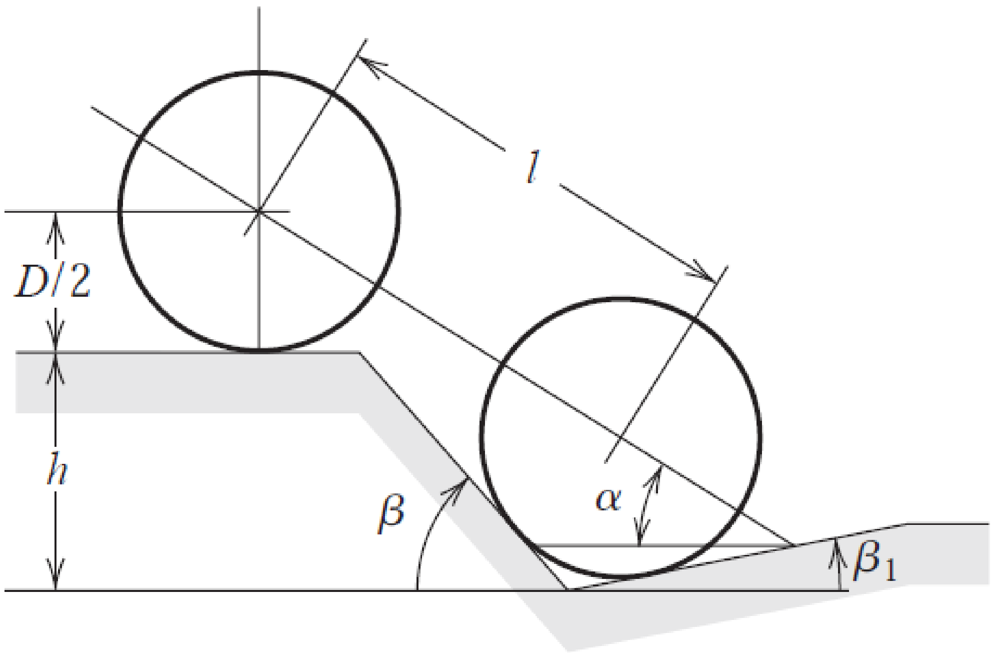
$$A \sin\alpha \cos\alpha + B \sin^2\alpha - C \cos\alpha - E \sin\alpha = 0,$$

where,

$$A = l \sin\beta_1, \quad B = l \cos\beta_1, \quad C = (h + 0.5D)\sin\beta_1 - 0.5D \tan\beta_1 \quad \text{and}$$
$$E = (h + 0.5D)\cos\beta_1 - 0.5D$$

a. It is stated that when $l = 89$ in, $h = 49$ *in*, $D = 55$ in, and $\beta_1 = 11.5°$, angle α is approximately $33°$. Verify this result.

b. Find α for the situation when l, h, and β_1 are the same as in part **a** but $D = 30$ in.



Solution:

(a) Using Newton's Method with $\alpha_0 = 35°$, $\alpha \approx 33.972$ *VERIFIED.*

(b) Using Newton's Method with $\alpha_0 = 35°$; $\alpha \approx 33.2$.

Fixed Point method

Problem 5

1. Use algebraic manipulation to show that each of the following functions has a fixed point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

   a. $g_1(x) = (3 + x - 2x^2)^{1/4}$

   b. $g_2(x) = \left(\frac{x + 3 - x^4}{2}\right)^{1/2}$

   c. $g_3(x) = \left(\frac{x + 3}{x^2 + 2}\right)^{1/2}$

   d. $g_4(x) = \frac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$

2. **a.** Perform four iterations, if possible, on each of the functions g defined in Exercise 1. Let $p_0 = 1$ and $p_{n+1} = g(p_n)$, for $n = 0, 1, 2, 3$.

   b. Which function do you think gives the best approximation to the solution ?

Solution:

Question	Solution
1.a	$x = (3 + x - 2 * x^2)^{1/4} \Leftrightarrow x^4 = 3 + x - 2 * x^2 \Leftrightarrow f(x) = 0$
1.b	$x = (\frac{x+3-x^4}{2})^{1/2} \Leftrightarrow 2x^2 = x + 3 - x^4 \Leftrightarrow f(x) = 0$
1.c	$x = (\frac{x+3}{x^2+2})^{1/2} \Leftrightarrow x^2(x^2 + 2) = x + 3 \Leftrightarrow f(x) = 0$
2.a	$x = (\frac{3x^4+2x^2+3}{4x^3+4x-1}) \Leftrightarrow 4x^4 + 4x^2 - x = 3x^4 + 2x^2 + 3 \Leftrightarrow f(x) = 0$
2.a.a	$p_4 = 1.1078$
2.a.b	$p_4 = 0.98751$
2.a.c	$p_4 = 1.1236$
2.a.d	$p_4 = 1.1241$
2.b	Part (d) gives the best answer since $\lvert p_4 - p_3 \rvert$ is smallest for (d)

Problem 6

1. Use a **fixed-point** iteration method to determine a solution accurate to within 10^{-2} for $x^4 - 3x^2 - 3 = 0$ on $[1, 2]$. Use $p_0 = 1$.

2. Use a **fixed-point** iteration method to find an approximation to $\sqrt{3}$ that is accurate to within 10^{-4}.

Solution:

Question	Solution
1	$g(x) = (3x^2 + 3)^{1/4}$ and $p_0 = 1$; $p_6 = 1.9433$
2	$g(x) = 0.5(x + {}^3/_x)$ and $p_0 = 1$; $p_4 = 1.7321$

Problem 7

An object falling vertically through the air is subjected to viscous resistance as well as to the force of gravity. Assume that an object with mass m is dropped from a height s_0 and that the height of the object after t seconds is

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2 g}{k^2}(1 - e^{-kt/m}),$$

where $g = 32.17\ ft/s^2$ and k represents the coefficient of air resistance in lb–s/ft. Suppose $s_0 = 300\ ft$, $m = 0.25\ lb$, and $k = 0.1\ lb{-}s/ft$. Find, to within $0.01\ s$, the time it takes this quarter-pounder to hit the ground.

Solution:

With $g(t) = 6.2302 - 2.5e^{-0.4t}$ and $p_0 = 4$; $p_3 = 6.0035$