TUTORIAL 4

INTERPOLATION & POLYNOMIAL APPROXIMATION

Numerical Analysis (ENME 602)

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Table of Contents

- Lagrange Polynomial Interpolation

- Theoretical Error Bound.

Lagrange Polynomial Interpolation

- Why approximating functions to **Algebraic Polynomials**?
  - Derivative and indefinite integral are easy to determine.
  - Data Fitting for unknown functions.

Lagrange Polynomial Interpolation

- **Algebraic Polynomials:** the set of functions in the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a non-negative integer and $a_0 \ldots a_n$ are real constants

Given any function $f(x)$ defined and continuous on a closed and bounded interval, their exist a polynomial $P_n(x)$ that is “close” to the given function as “desired”

Lagrange Polynomial Interpolation

Lagrange Polynomial Interpolation

- **Lagrange Polynomial:** Degree n Construction
  - Consider the construction of polynomial of degree at most n that passes through the $n+1$ points $(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_n, f(x_n))$ which called "nodes"

$$P_n(x) = \sum_{k=0}^{n} L_{n,k}(x) f(x_k)$$

$$L_{n,k}(x) = \prod_{i=0,\ i\neq k}^{n} \frac{(x - x_i)}{(x_k - x_i)} = \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots((x - x_n))}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots((x_k - x_n))}$$

Problem 1

1. For the given functions $f(x)$, let $x_0 = 1$, $x_1 = 1.25$ and $x_2 = 1.6$. Construct interpolation polynomials of degree at most one and two to approximate $f(1.4)$, and find the absolute error.

   b. $f(x) = \sqrt[3]{x-1}$

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$P_1(x)$ $\quad$ $P_2(x)$

b. $f(x) = \sqrt[3]{x-1}$

$(1, 0)$

$(1.25, 0.62996)$

$(1.6, 0.84343)$

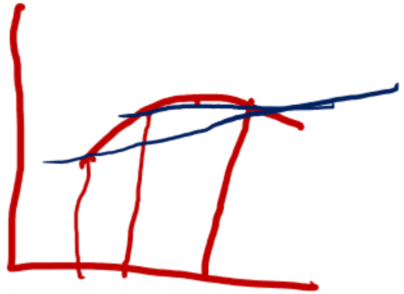
$f(1.4) = 0.7368$

$$P_1(x) = 0.62996 \times \frac{x - 1.6}{1.25 - 1.6} + 0.84343 \frac{x - 1.25}{1.6 - 1.25}$$

$$\boxed{= 0.6099x - 0.1325}$$

$$P_1(1.4) = 0.6099(1.4) - 0.1325 = 0.72136$$

$$\epsilon = |f(1.4) - P_1(1.4)| = 0.$$



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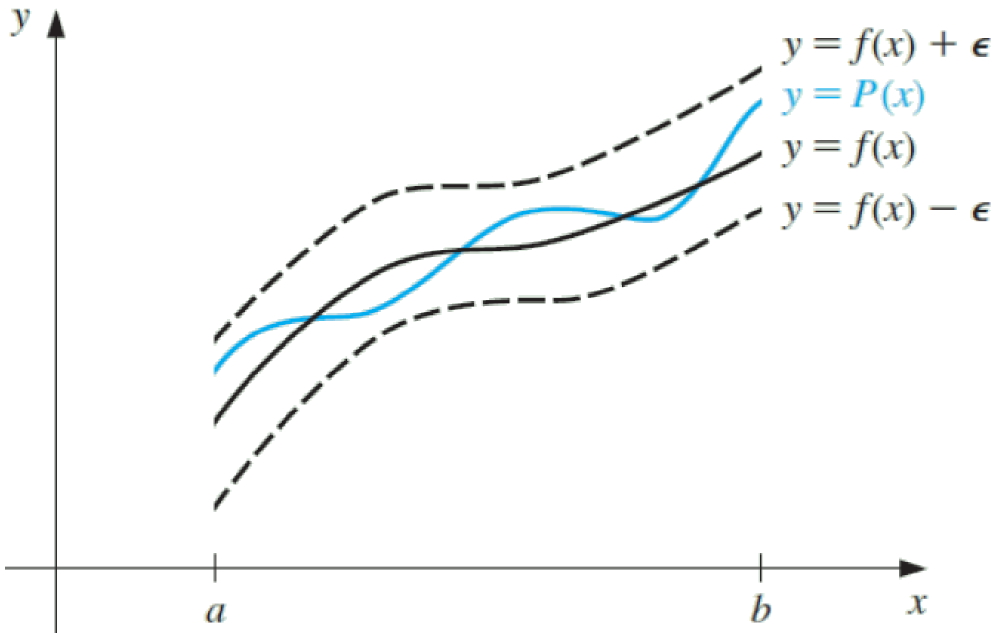
Theoretical Error Bound

Theorem 3.3:

Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in c^{n+1}[a, b]$. Then, for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \ldots, x_n$, and hence in $[a, b]$, thus the error bound ($\boldsymbol{\varepsilon}$) for the function $f(x)$ is defined as:



Weierstrass Approximation Theorem

$$\epsilon = |f(x) - p(x)| \leq \max_{[a,b]} \left| \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \right| \times \max_{[a,b]} \left| \prod_{i=0}^{n} (x - x_i) \right|$$

Problem 1

2. Use Theorem 3.3 to find an error bound for the approximations in Exercise **1**.

b. $f(x) = \sqrt[3]{x-1}$

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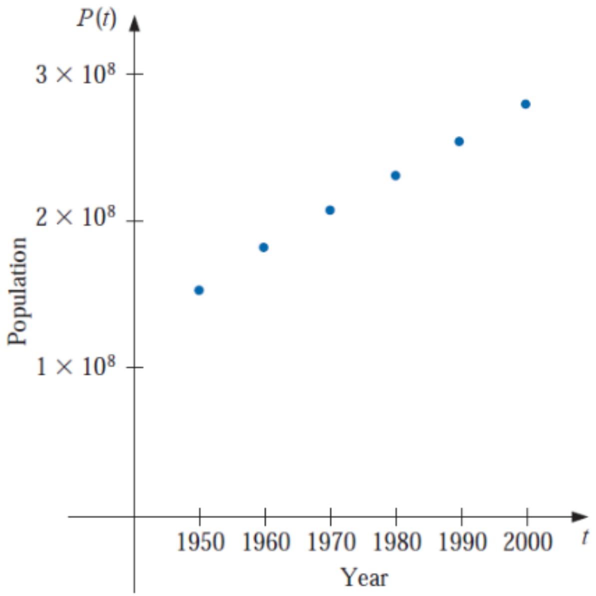
Problem 3

A census of the population of the United States is taken every 10 years. The following table lists the population, in thousands of people, from 1950 to 2000, and the data are also represented in the figure.

1. Use Lagrange interpolation to approximate the population in the years 1940, 1975, and 2020.

2. The population in 1940 was approximately 132,165,000. How accurate do you think your 1975 and 2020 figures are?



Year	1950	1960	1970	1980	1990	2000
Population (in thousands)	151,326	179,323	203,302	226,542	249,633	281,422

Problem 4

It is suspected that the high amounts of tannin in mature oak leaves inhibit the growth of the winter moth (Operophtera bromata L., Geometridae) larvae that extensively damage these trees in certain years. The following table lists the average weight of two samples of larvae at times in the first 28 days after birth. The first sample was reared on young oak leaves, whereas the second sample was reared on mature leaves from the same tree.

1. Use Lagrange interpolation to approximate the average weight curve for each sample.
2. Find an approximate maximum average weight for each sample by determining the maximum of the interpolating polynomial.

Day	0	6	10	13	17	20	28
Sample 1 average weight (mg)	6.67	17.33	42.67	37.33	30.10	29.31	28.74
Sample 2 average weight (mg)	6.67	16.11	18.89	15.00	10.56	9.44	8.89

Thank you!