# TUTORIAL 2

## ROOT FINDING USING BISECTION METHOD

Numerical Analysis (ENME 602)

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# Algorithms Pseudo Code

**Algorithm:** is an approximation procedure that describes a finite sequence of steps to be performed in a specified order.

The object of the algorithm is to implement a procedure to solve a problem or approximate a solution to the problem.

**Pseudocode:** used to describe an algorithm. It follows the rules of structured program construction.

It specifies the input, output and a stopping technique independent of the numerical technique. It can be easily translated in any programming language: e.g Matlab, Python or Java.

# Algorithms Pseudo Code

**Example:** Write an algorithm to compute:

$$\sum_{i=1}^{N} x_i = x_1 + x_2 + \ldots + x_N,$$

where $N$ and the numbers $x_1, x_2, \ldots, x_N$ are given.

**Pseudocode:**

INPUT $N, x_1, x_2, \ldots, x_N$.

OUTPUT $SUM = \sum_{i=1}^{N} x_i$.

*Step 1* Set $SUM = 0$. *(Initialize accumulator.)*

*Step 2* For $i = 1, 2, \ldots, N$ do
set $SUM = SUM + x_i$. *(Add the next term.)*

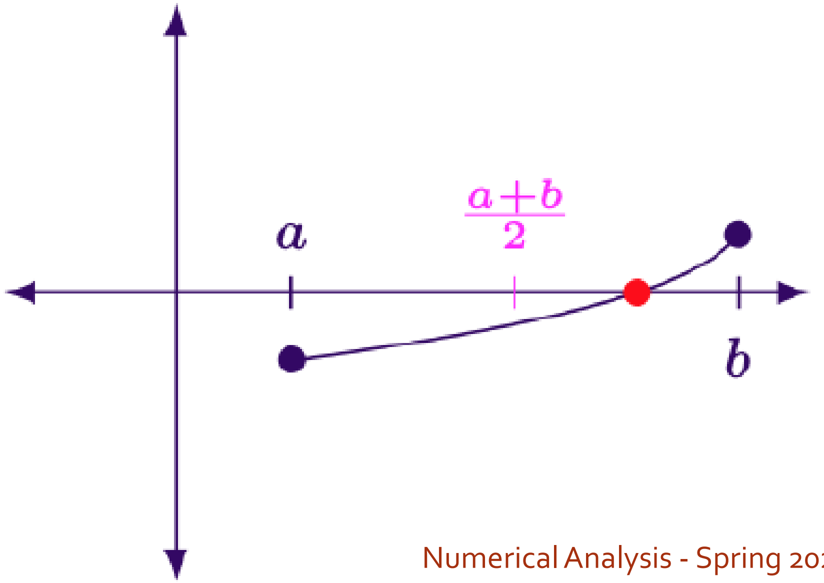
*Step 3* OUTPUT ($SUM$);
STOP.

# Root Finding Problem Statement

**The root of an equation**

The value of $x$ that satisfies $f(x) = 0$ is called the root (solution) of the equation $f(x) = 0$ or the zero of the function $f$.

Graphically, it is the x-intercept when the graph of $f$ crosses the x-axis.



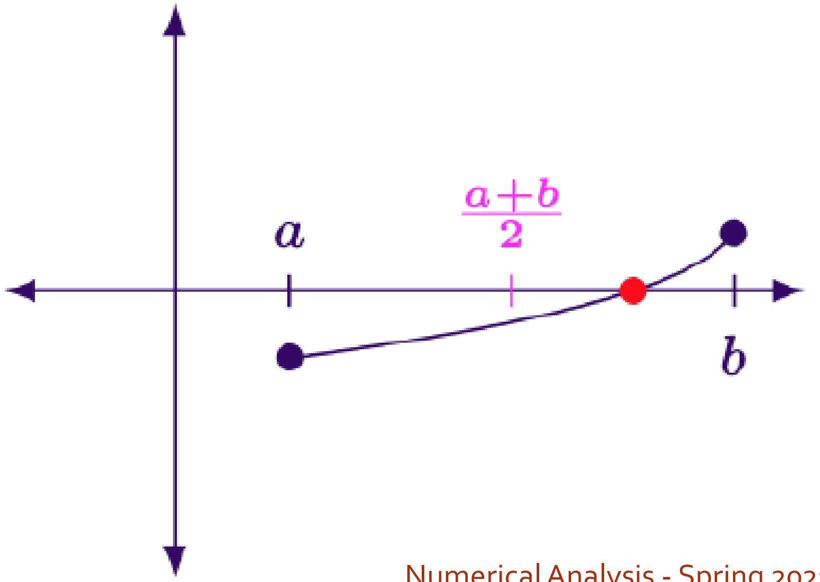
# Root Finding Problem Statement

**The root-finding problem**
is the process of approximating the root (solution) of the equation $f(x) = 0$ (to within a given tolerance).

# Bisection Method

## Idea of Bisection Method

When there is a root (red dot here), it occurs either in the right half of the interval [a,b] (like in the graph) or in the left half or at the midpoint. Take the half that contains the root!



# Bisection Method

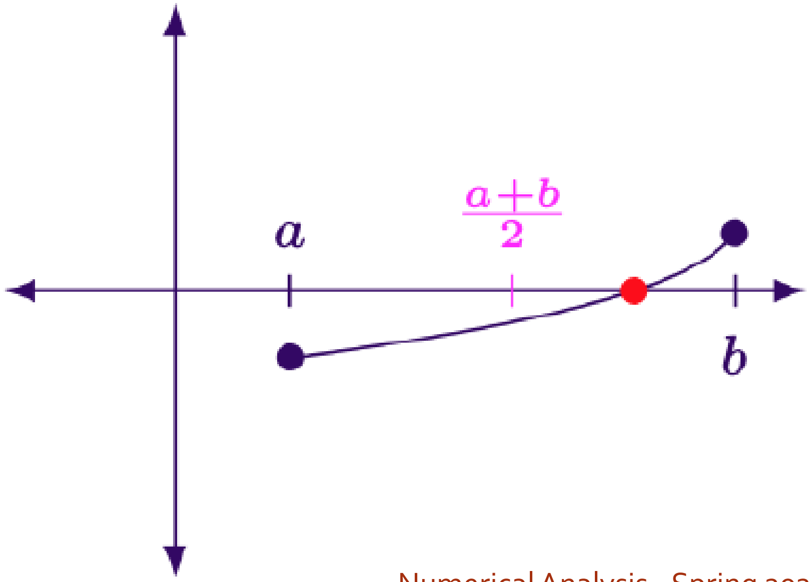
We obtain an approximation of the root by repeatedly halving the subintervals of [a,b], and "trapping" the root in smaller and smaller interval at each step.

# Bisection Method

**Q: How to know that a root exists in $[a, b]$?**

Answer: apply the Intermediate Value Theorem:

If $f$ is continuous on $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs (which is usually expressed as their product $f(a) \cdot f(b) < 0$) then there exist $p$ such that $f(p) = 0$.



# Bisection Method

**Q: How to select the half that contains the root?**

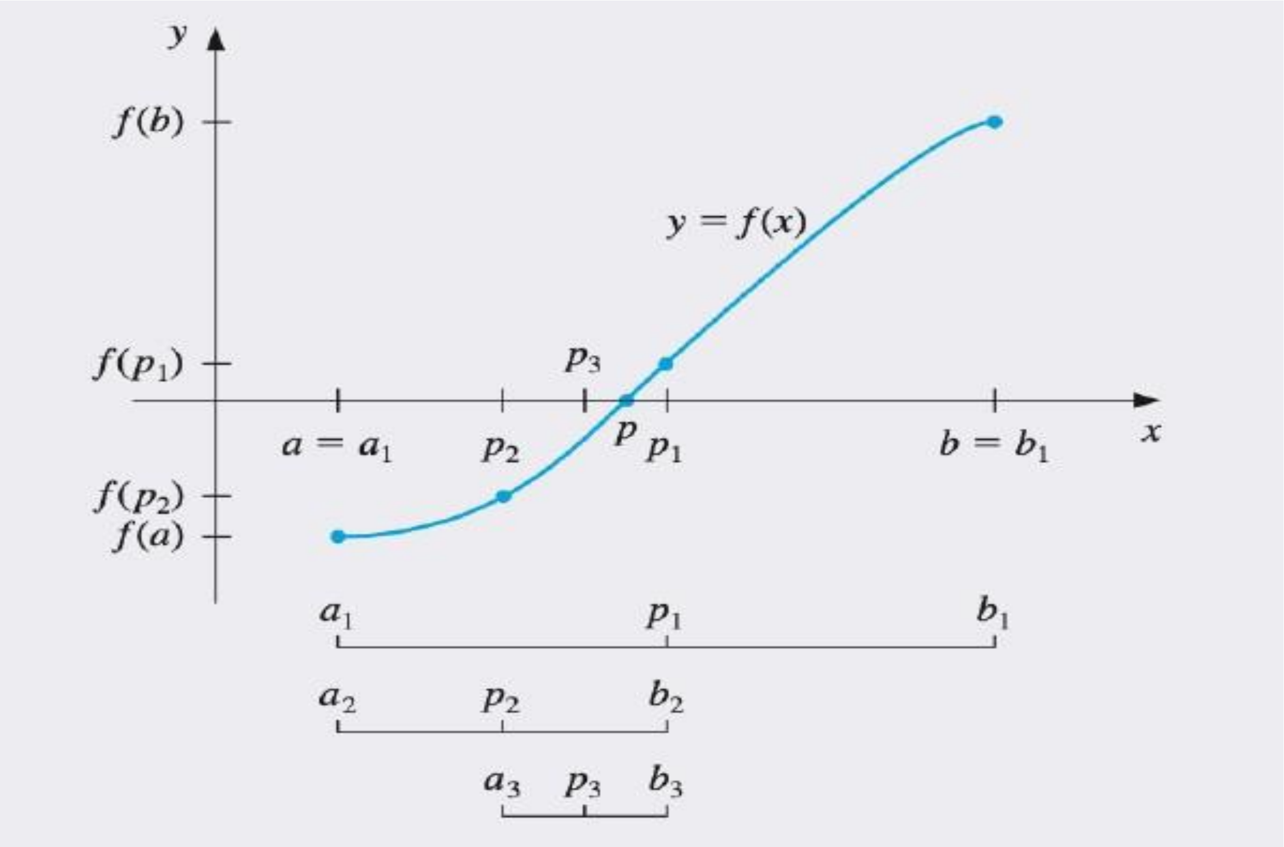
Construct the sequence of subintervals $[a_k, b_k]$ iteratively:

- Begin with first interval $[a_1, b_1] = [a, b]$ such that $f(a)f(b) < 0$ (so $f$ has a root in $[a, b]$).
- Compute the midpoint $p = \dfrac{a+b}{2}$.
- To choose left or right subinterval: check the sign of $f(p)$ and compare it to the signs of $f(a)$ and $f(b)$.

  If $f(a)f(p) < 0$ then $f$ change signs in $[a, p]$, the new interval $[a_2, b_2] = [a, p]$, otherwise, take $[p, b]$ .

- Set either $[a, p]$ or $[p, b]$ as new interval. At each iteration use $p$ as new approximation of the root.

# Bisection Method



# Bisection Method

**Q : How do we know when to stop iterations?**

If we iterate until the $n^{\text{th}}$ sub-interval length is at most $\epsilon$:

$$\frac{b-a}{2^n} \leq \epsilon \tag{1}$$

The Theorem guarantees that the absolute error is at most $\epsilon$:

$$|p - p_n| \leq \frac{b-a}{2^n} \leq \epsilon$$

To determine the number of iteration **n** necessary to reach a tolerance $\epsilon$ in $[a, b]$, solve (1) for **n**: taking ln of both sides

$$\ln\left(\frac{b-a}{2^{\mathbf{n}}}\right) < \ln(\epsilon)$$

$$\ln(b-a) - \mathbf{n}\ln 2 < \ln(\epsilon) \Longrightarrow \mathbf{n} > \frac{\ln(b-a) - \ln(\epsilon)}{\ln 2} = \frac{\ln(\frac{b-a}{\epsilon})}{\ln 2}$$

# Bisection Method

## Bisection Algorithm

To find a solution to $f(x) = 0$ given the continuous function $f$ on the interval $[a, b]$, where $f(a)$ and $f(b)$ have opposite signs:

# Bisection Method

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INPUT endpoints $a, b$; tolerance $TOL$; maximum number of iterations $N_0$.

OUTPUT approximate solution $p$ or message of failure.

*Step 1* Set $i = 1$;
$FA = f(a)$.

*Step 2* While $i \le N_0$ do Steps 3–6.

*Step 3* Set $p = a + (b - a)/2$; (*Compute $p_i$.*)
$FP = f(p)$.

*Step 4* If $FP = 0$ or $(b - a)/2 < TOL$ then
OUTPUT ($p$); (*Procedure completed successfully.*)
STOP.

*Step 5* Set $i = i + 1$.

*Step 6* If $FA \cdot FP > 0$ then set $a = p$; (*Compute $a_i, b_i$.*)
$FA = FP$
else set $b = p$. (*FA is unchanged.*)

*Step 7* OUTPUT ('Method failed after $N_0$ iterations, $N_0 =$', $N_0$);
(*The procedure was unsuccessful.*)
STOP.

# Problem 1

2. Let $f(x) = 3(x+1)\left(x-\frac{1}{2}\right)(x-1)$. Use the Bisection method on the following interval to find $p_3$:

   [ -2, 1.5 ]

# Problem 1

3. Use the Bisection method to find solutions to within $10^{-2}$ for $x^3 - 7x^2 + 14x - 6 = 0$ on the following interval:

   [ 0, 1 ]

# Problem 2

1. Sketch the graphs of $y = x$ and $y = \sin(x)$.

# Problem 2

2. Use the Bisection method to find an approximation to within $10^{-5}$ to the first positive value of x with $x = 2\sin(x)$.

# Problem 2

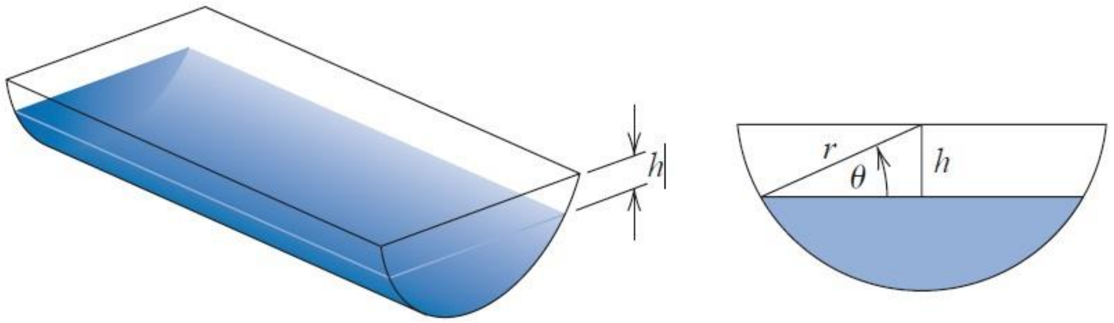
4. Use Theorem 2.1 to find a bound for the number of iterations needed to achieve an approximation with accuracy $10^{-3}$ to the solution of $x^3 + x - 4 = 0$ lying in the interval [ 1, 4]. Find an approximation to the root with this degree of accuracy.

# Problem 3

A trough of length L has a cross-section in the shape of a semicircle with radius r. (see the accompanying figure). When filled with water to within a distance h of the top, the volume V of the water is:

$$V = L\left[\,0.5\pi r^2 - r^2 \arcsin\left(\frac{h}{r}\right) - h(r^2 - h^2)^{\frac{1}{2}}\,\right]$$

Suppose L = 10 ft, r = 1 ft and V = 10.4 ft$^3$. Find the depth of water in the trough to within 0.01 ft.



# Thank you!