Numerical Analysis

(ENME 602)

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Lecture 4

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Lecture 4

Interpolation & Polynomial Approximation

4.1 Weierstrass Approximation Theorem

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4.4 Theoretical Error Bound

4.1 Weierstrass Approximation Theorem

4.1 Weierstrass Approximation Theorem

Algebraic Polynomials

One of the most useful and well-known classes of functions mapping the set of real numbers into itself is the algebraic polynomials, the set of functions of the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_0, \ldots, a_n$ are real constants.

- One reason for their importance is that they uniformly approximate continuous functions.
- By this we mean that given any function, defined and continuous on a closed and bounded interval, there exists a polynomial that is as "close" to the given function as desired.
- This result is expressed precisely in the Weierstrass Approximation Theorem.

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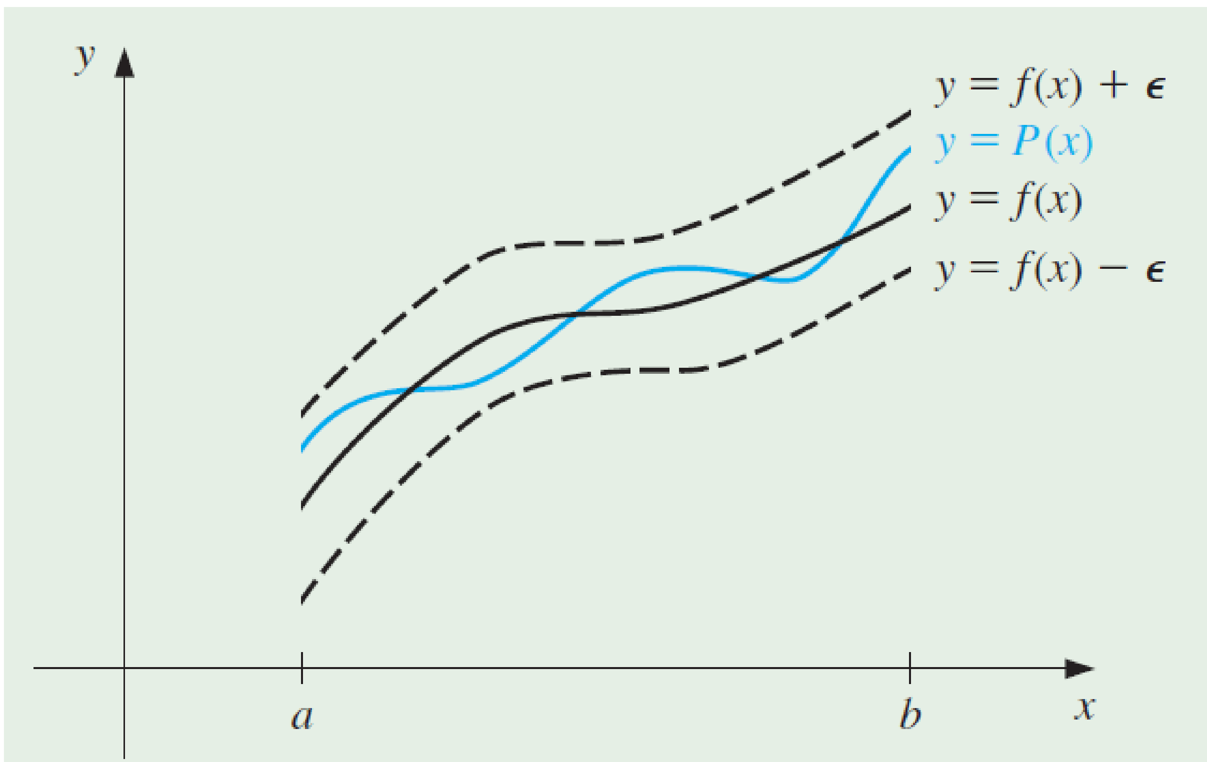
4.1 Weierstrass Approximation Theorem

Weierstrass Approximation Theorem

Suppose that f is defined and continuous on $[a, b]$. For each $\epsilon > 0$, there exists a polynomial $P(x)$, with the property that

$$|f(x) - P(x)| < \epsilon, \quad \text{for all } x \text{ in } [a, b].$$



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4.1 Weierstrass Approximation Theorem

Benefits of Algebraic Polynomials

- Another important reason for considering the class of polynomials in the approximation of functions is that the derivative and indefinite integral of a polynomial are easy to determine and are also polynomials.
- For these reasons, polynomials are often used for approximating continuous functions.

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4.2 Inaccuracy of Taylor Polynomials

4.2 Inaccuracy of Taylor Polynomials

Interpolating with Taylor Polynomials

- The Taylor polynomials are described as one of the fundamental building blocks of numerical analysis.
- Given this prominence, you might expect that polynomial interpolation would make heavy use of these functions.
- However this is not the case.
- The Taylor polynomials agree as closely as possible with a given function at a specific point, but they concentrate their accuracy near that point.
- A good interpolation polynomial needs to provide a relatively accurate approximation over an entire interval, and Taylor polynomials do not generally do this.

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4.2 Inaccuracy of Taylor Polynomials

Example: $f(x) = e^x$

We will calculate the first six Taylor polynomials about $x_0 = 0$ for $f(x) = e^x$.

Note

Since the derivatives of $f(x)$ are all e^x, which evaluated at $x_0 = 0$ gives 1.

The Taylor polynomials are as follows:

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4.2 Inaccuracy of Taylor Polynomials

Taylor Polynomials for $f(x) = e^x$ about $x_0 = 0$

$$P_0(x) = 1$$

$$P_1(x) = 1 + x$$

$$P_2(x) = 1 + x + \frac{x^2}{2}$$

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$P_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

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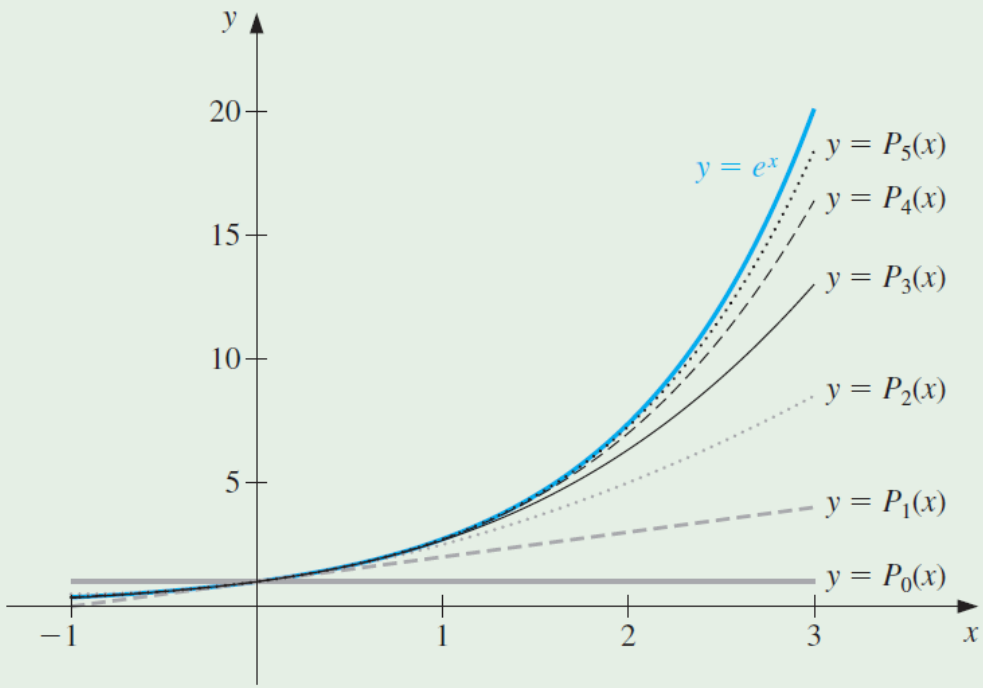
4.2 Inaccuracy of Taylor Polynomials

Taylor Polynomials for $f(x) = e^x$ about $x_0 = 0$



Notice that even for the higher-degree polynomials, the error becomes progressively worse as we move away from zero.

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4.2 Inaccuracy of Taylor Polynomials

Example: A more extreme case

- Although better approximations are obtained for $f(x) = e^x$ if higher-degree Taylor polynomials are used, this is not true for all functions.
- Consider, as an extreme example, using Taylor polynomials of various degrees for $f(x) = \frac{1}{x}$ expanded about $x_0 = 1$ to approximate $f(3) = \frac{1}{3}$.

Calculations

Since

$$f(x) = x^{-1},\ f'(x) = -x^{-2},\ f''(x) = (-1)^2 2 \cdot x^{-3},$$

and, in general,

$$f^{(k)}(x) = (-1)^k k! x^{-k-1},$$

the Taylor polynomials are

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=0}^{n} (-1)^k (x-1)^k.$$

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4.2 Inaccuracy of Taylor Polynomials

Taylor Polynomials for $f(x) = \frac{1}{x}$ about $x_0 = 1$

To Approximate $f(3) = \frac{1}{3}$ by $P_n(3)$

- To approximate $f(3) = \frac{1}{3}$ by $P_n(3)$ for increasing values of n, we obtain the values shown below — rather a dramatic failure!
- When we approximate $f(3) = \frac{1}{3}$ by $P_n(3)$ for larger values of n, the approximations become increasingly inaccurate.

n	0	1	2	3	4	5	6	7
$P_n(3)$	1	-1	3	-5	11	-21	43	-85

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4.2 Inaccuracy of Taylor Polynomials

Footnotes

- For the Taylor polynomials, all the information used in the approximation is concentrated at the single number x_0, so these polynomials will generally give inaccurate approximations as we move away from x_0.
- This limits Taylor polynomial approximation to the situation in which approximations are needed only at numbers close to x_0.
- For ordinary computational purposes, it is more efficient to use methods that include information at various points.
- The primary use of Taylor polynomials in numerical analysis is not for approximation purposes, but for the derivation of numerical techniques and error estimation.

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4.3 Lagrange Polynomial

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Polynomial Interpolation

- The problem of determining a polynomial of degree one that passes through the distinct points

$$(x_0, y_0) \quad \text{and} \quad (x_1, y_1)$$

is the same as approximating a function f for which

$$f(x_0) = y_0 \quad \text{and} \quad f(x_1) = y_1$$

by means of a first-degree polynomial interpolating, or agreeing with, the values of f at the given points.

- Using this polynomial for approximation within the interval given by the endpoints is called polynomial interpolation.

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4.3 Lagrange Polynomial

Define the functions

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}.$$

Definition

The linear Lagrange interpolating polynomial though (x_0, y_0) and (x_1, y_1) is

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1).$$

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4.3 Lagrange Polynomial

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1).$$

Note that

$$L_0(x_0) = 1, \quad L_0(x_1) = 0, \quad L_1(x_0) = 0, \quad \text{and} \quad L_1(x_1) = 1,$$

which implies that

$$P(x_0) = 1 \cdot f(x_0) + 0 \cdot f(x_1) = f(x_0) = y_0$$

and

$$P(x_1) = 0 \cdot f(x_0) + 1 \cdot f(x_1) = f(x_1) = y_1.$$

So P is the unique polynomial of degree at most 1 that passes through (x_0, y_0) and (x_1, y_1).

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Example: Linear Interpolation

Determine the linear Lagrange interpolating polynomial that passes through the points $(2, 4)$ and $(5, 1)$.

Solution

In this case we have

$$L_0(x) = \frac{x-5}{2-5} = -\frac{1}{3}(x-5) \quad \text{and} \quad L_1(x) = \frac{x-2}{5-2} = \frac{1}{3}(x-2),$$

so

$$P(x) = -\frac{1}{3}(x-5)\cdot 4 + \frac{1}{3}(x-2)\cdot 1 = -\frac{4}{3}x + \frac{20}{3} + \frac{1}{3}x - \frac{2}{3} = -x + 6.$$

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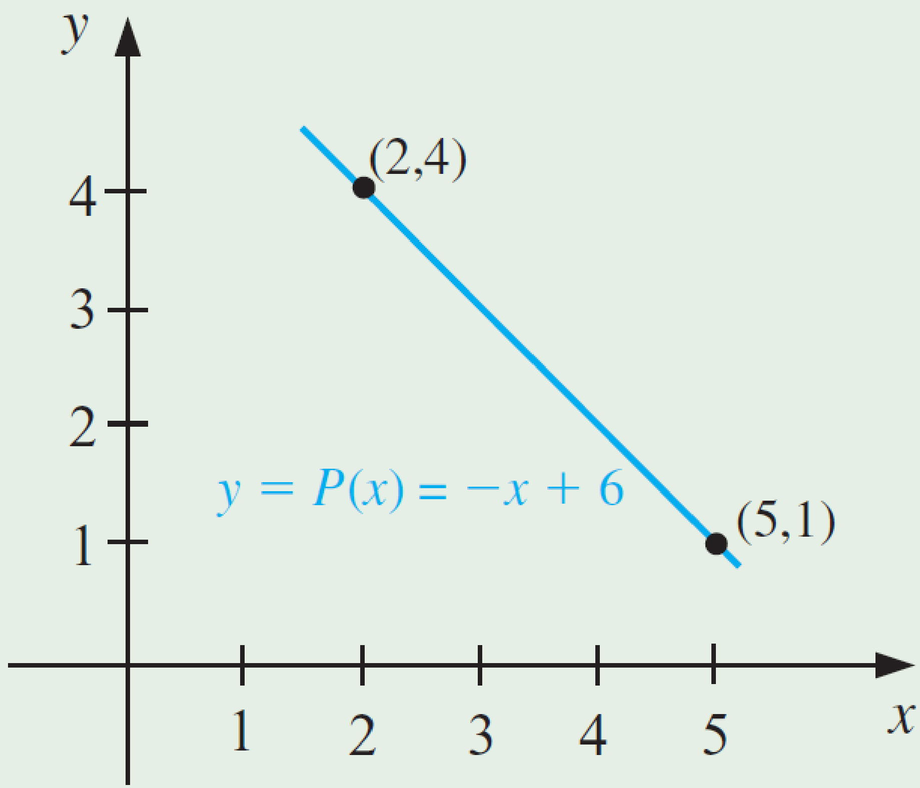
4.3 Lagrange Polynomial



The linear Lagrange interpolating polynomial that passes through the points $(2, 4)$ and $(5, 1)$.

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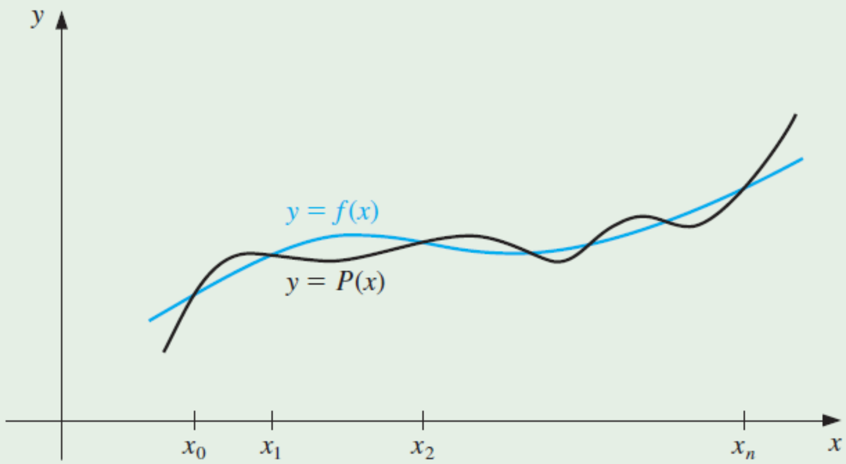
The Lagrange Polynomial: Degree n Construction



To generalize the concept of linear interpolation, consider the construction of a polynomial of degree at most n that passes through the $n+1$ points

$$(x_0, f(x_0)),\ (x_1, f(x_1)), \ldots, (x_n, f(x_n)).$$

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Constructing the Degree n Polynomial

- We first construct, for each $k = 0, 1, \ldots, n$, a function $L_{n,k}(x)$ with the property that $L_{n,k}(x_i) = 0$ when $i \neq k$ and $L_{n,k}(x_k) = 1$.
- To satisfy $L_{n,k}(x_i) = 0$ for each $i \neq k$ requires that the numerator of $L_{n,k}(x)$ contain the term

$$(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n).$$

- To satisfy $L_{n,k}(x_k) = 1$, the denominator of $L_{n,k}(x)$ must be this same term but evaluated at $x = x_k$.
- Thus

$$L_{n,k}(x) = \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}.$$

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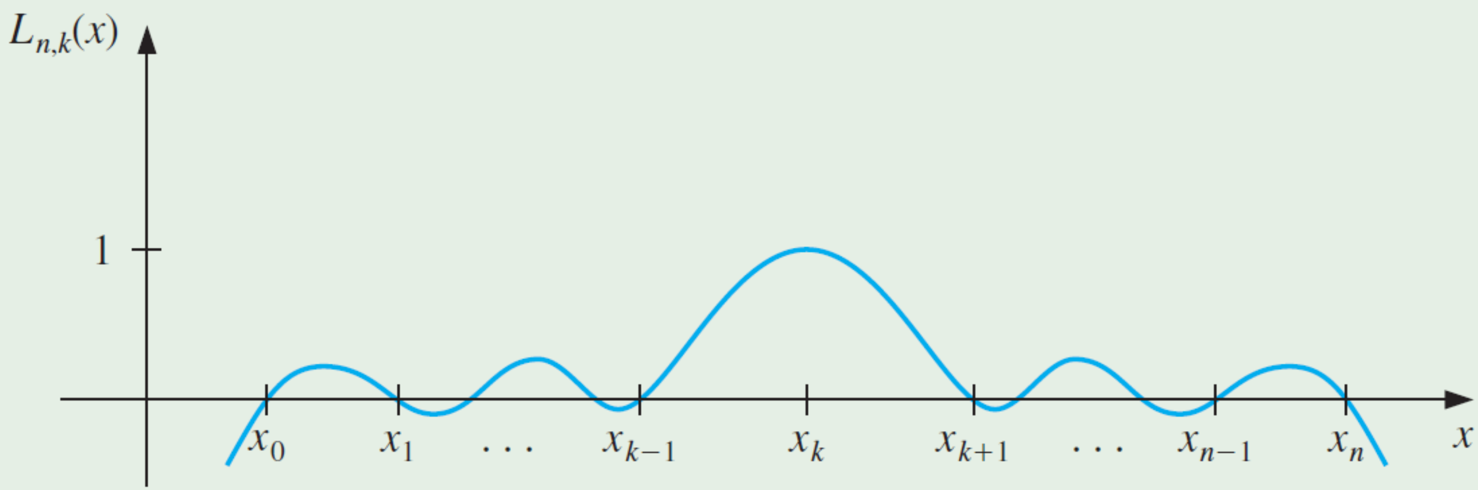
4.3 Lagrange Polynomial

The Lagrange Polynomial: The General Case

$$L_{n,k}(x) = \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}.$$



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4.3 Lagrange Polynomial

Theorem: *n*-th Lagrange interpolating polynomial

If $x_0, x_1, \ldots, x_n$ are $n+1$ distinct numbers and f is a function whose values are given at these numbers, then a unique polynomial $P(x)$ of degree at most n exists with

$$f(x_k) = P(x_k), \quad \text{for each } k = 0, 1, \ldots, n.$$

This polynomial is given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

where, for each $k = 0, 1, \ldots, n$, $L_{n,k}(x)$ is defined as follows:

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4.3 Lagrange Polynomial

The Lagrange Polynomial: The General Case

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

Definition of $L_{n,k}(x)$

$$\begin{aligned} L_{n,k}(x) &= \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)} \\ &= \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)} \end{aligned}$$

We will write $L_{n,k}(x)$ simply as $L_k(x)$ when there is no confusion as to its degree.

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4.3 Lagrange Polynomial

Example: $f(x) = \frac{1}{x}$

(a) Use the numbers (called nodes) $x_0 = 2$, $x_1 = 2.75$ and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = \frac{1}{x}$.

(b) Use this polynomial to approximate $f(3) = \frac{1}{3}$.

Part (a): Solution

We first determine the coefficient polynomials $L_0(x)$, $L_1(x)$, and $L_2(x)$:

$$L_0(x) = \frac{(x-2.75)(x-4)}{(2-2.5)(2-4)} = \frac{2}{3}(x-2.75)(x-4)$$

$$L_1(x) = \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)} = -\frac{16}{15}(x-2)(x-4)$$

$$L_2(x) = \frac{(x-2)(x-2.75)}{(4-2)(4-2.5)} = \frac{2}{5}(x-2)(x-2.75)$$

Also, since $f(x) = \frac{1}{x}$:

$$f(x_0) = f(2) = 1/2, \qquad f(x_1) = f(2.75) = 4/11, \qquad f(x_2) = f(4) = 1/4$$

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4.3 Lagrange Polynomial

Example: $f(x) = \frac{1}{x}$

(a) Use the numbers (called nodes) $x_0 = 2$, $x_1 = 2.75$ and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = \frac{1}{x}$.

(b) Use this polynomial to approximate $f(3) = \frac{1}{3}$.

Part (a): Solution (Cont'd)

Therefore, we obtain

$$
\begin{aligned}
P(x) &= \sum_{k=0}^{2} f(x_k)L_k(x) \\
&= \frac{1}{3}(x - 2.75)(x - 4) - \frac{64}{165}(x - 2)(x - 4) + \frac{1}{10}(x - 2)(x - 2.75) \\
&= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}.
\end{aligned}
$$

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$$P(x) = \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}$$

(b) Use this polynomial to approximate $f(3) = \frac{1}{3}$.

Part (b): Solution

An approximation to $f(3) = \frac{1}{3}$ is

$$f(3) \approx P(3) = \frac{9}{22} - \frac{105}{88} + \frac{49}{44} = \frac{29}{88} \approx 0.32955.$$

Earlier, we we found that no Taylor polynomial expanded about $x_0 = 1$ could be used to reasonably approximate $f(x) = 1/x$ at $x = 3$.

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4.4 Theoretical Error Bound

4.4 Theoretical Error Bound

Theorem

Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \ldots, x_n$, and hence in (a, b), exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

where $P(x)$ is the interpolating polynomial given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

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Example: Second Lagrange Polynomial for $f(x) = \frac{1}{x}$

In an earlier example, ‣ Original Example we found the second Lagrange polynomial for $f(x) = \frac{1}{x}$ on $[2, 4]$ using the nodes $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$. Determine the error form for this polynomial, and the maximum error when the polynomial is used to approximate $f(x)$ for $x \in [2, 4]$.

Note

We will make use of the theoretical result ‣ Theorem written in the form

$$|f(x) - P(x)| \leq \max_{[2,4]} \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right| \cdot \max_{[2,4]} \left| \prod_{i=0}^{n} (x - x_i) \right|$$

with $n = 2$

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4.4 Theoretical Error Bound

Solution (1/3)

Because $f(x) = x^{-1}$, we have

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad \text{and} \quad f'''(x) = -\frac{6}{x^4}$$

As a consequence, the second Lagrange polynomial has the error form

$$\frac{f'''(\xi(x))}{3!}(x - x_0)(x - x_1)(x - x_2) = -\frac{1}{\xi(x)^4}(x - 2)(x - 2.75)(x - 4)$$

for $\xi(x)$ in $(2, 4)$. The maximum value of $\frac{1}{\xi(x)^4}$ on the interval is $\frac{1}{2^4} = 1/16$.

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Solution (2/3)

We now need to determine the maximum value on $[2, 4]$ of the absolute value of the polynomial

$$g(x) = (x-2)(x-2.75)(x-4) = x^3 - \frac{35}{4}x^2 + \frac{49}{2}x - 22$$

Because

$$g'(x) = 3x^2 - \frac{35}{2}x + \frac{49}{2} = \frac{1}{2}(3x-7)(2x-7),$$

the critical points occur at

$$x = \frac{7}{3} \text{ with } g\left(\frac{7}{3}\right) = \frac{25}{108} \qquad \text{and} \qquad x = \frac{7}{2} \text{ with } g\left(\frac{7}{2}\right) = -\frac{9}{16}$$

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4.4 Theoretical Error Bound

Solution (3/3)

Hence, the maximum error is

$$\max_{[2,4]} \left| \frac{f'''(\xi(x))}{3!} \right| \cdot \max_{[2,4]} |(x - x_0)(x - x_1)(x - x_2)|$$

$$\leq \frac{1}{3!} \cdot \frac{1}{16} \cdot \frac{9}{16}$$

$$= \frac{3}{512}$$

$$\approx 0.00586$$

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Thank You

"Newton Divided Differences and Hermite Interpolation"