TD — Linear Algebra for Machine Learning

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Exercise 1 — Determinant, Eigenvalues, Eigenvectors, and Diagonalization

Consider the matrix

$$A = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} & \frac{1}{2} \\ -1 & 1 & 2 \end{pmatrix}.$$

1. Compute the determinant of A and verify if it is invertible.
2. Find all eigenvalues of A by solving the characteristic polynomial.
3. For each eigenvalue, determine a corresponding eigenvector.
4. Check whether A is diagonalizable, and if yes, find matrices P and D such that $A = PDP^{-1}$.

Exercice 2

Consider the shear matrix

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

1. Show that S has a single eigenvalue $\lambda = 1$ with only one independent eigenvector. Conclude that S is *not* diagonalizable.
2. Compute $S^\top S$ and its eigenvalues $\{\lambda_1, \lambda_2\}$; set the singular values $\sigma_i = \sqrt{\lambda_i}$.
3. Find the right singular vectors v_i as eigenvectors of $S^\top S$ and the left singular vectors $u_i = \frac{Sv_i}{\sigma_i}$.
4. Conclude the SVD $S = U\Sigma V^\top$ with $U = [u_1\ u_2]$, $V = [v_1\ v_2]$, $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2)$.

Exercise 3 — PCA on a tiny 2D dataset

Given the four points (rows are samples)

$$X = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 3 & 1 \\ 1 & 3 \end{pmatrix} \in \mathbb{R}^{4\times 2}.$$

1. Center the data: compute the column mean μ and $Z = X - \mathbf{1}\mu^\top$.
2. Compute the sample covariance matrix $C = \frac{1}{n-1} Z^\top Z$.
3. Compute the eigenvalues/eigenvectors of C (principal components).
4. Order components by decreasing eigenvalue, and give the explained variance ratio for $k = 1$ and $k = 2$.
5. Project the centered data on the first principal component: $T_1 = Z v_1$.
