EE 228

Signals and Systems

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Contents

1 Operations on Signals:

$$y(t) = Cx(t) \rightarrow \text{increase the amplitude} \tag{1}$$
$$y(t) = x(t-\alpha) \rightarrow \text{shift the signal x(t)} \tag{2}$$
$$y(t) = x(t/2) \rightarrow \text{expansion of x(t)} \tag{3}$$
$$y(t) = x(2t) \rightarrow \text{compression of x(t)} \tag{4}$$
$$y(t) = x(-t) \rightarrow \text{create mirror image signal about the vertical Axis} \tag{5}$$
$$\tag{6}$$

Important: for calculating the graph new equation we need use the $C_1 t_1 + D_1 = t_0$.
$t_0 = \frac{-D_0}{C_0}$

2 Symmetry (Odd and Even Functions):

Note: if the a signal is identical to its folded version, with **x(t)=x(-t)**, it is called **Even Symmetric.** If a signal and its folded version different only in sign, with **x(t)=-x(-t)**, it is called **Odd Symmetric.**

$$odd = \int_{-\alpha}^{\alpha} x_0(t)dt = 0$$
$$even = 2\int_{0}^{\alpha} x_0(t)dt$$

Important:

$$x_o * y_o \rightarrow Even$$
$$x_e + y_o \rightarrow \text{No symmetry}$$
$$x_e * y_o \rightarrow Odd$$
$$x(t) = x_e(t) + x_o(t)$$
$$x_e(t) = 0.5[x(t) + x(-t)]$$
$$x_0(t) = 0.5[x(t) - x(-t)]$$

3 Sampling (or shifting) property:

$$\int_{-\infty}^{\infty} x(t)\delta(t-\tau)dt = x(\tau)$$
$$\delta(at) = \frac{1}{|a|}\delta(t)$$

Note: Always check the boundary of the integral for $\delta(t-\tau)$ because if is not in the boundary the integral would be zero. Example:

$$\int_{-3}^{-1} t^5 \delta(3t+2)dt$$
$$\frac{1}{3} * (\frac{-2}{3})^5 \int_{-3}^{-1} \delta(t+\frac{2}{3})dt = 0 \rightarrow \frac{-2}{3} \neq -3 < t < -1$$

4 Linearity:

Check homogeneity and superposition and if we have anything like $x^2, y^2, x * x^{'}, a^y$.

5 Causality:

An LTI system is cuasal if and only if its **impulse response** is a causal function. $h(t) = 0 for t < 0$

I) Does a system respond before an input accrues

II) Does these system rely on future input.

$$y(t) = x(t+2) \rightarrow \text{It is depend of future event, so non-causal}$$
$$y(t) = x(3t) \rightarrow \text{expansion and compression}$$

6 Dynamic vs Static(instantaneous):

Dynamic system contain Energy element describe by $\frac{d}{dt} or \int dt$.
Static if the output $y(t_0)$ depends only on the instantaneous value of the input $x(t_0)$ like y(t)=Ax(t)+B. It is **static** if no derivatives are present, and every term in x and y has identical arguments.

$$y(t) \neq x(t+1) \rightarrow Dynamic$$

Because the input value for the y and x are different.

7 Time-Invariant (Shift-Invariant) Systems:

Time-scaled inputs or outputs also make a system equation time varying as with y(4t).

$$O\{x(t-t_0)\} = y(t-t_0) \rightarrow \text{(shift input by } \alpha \rightarrow \text{shift output by } \alpha)$$

To be time invariant, coefficients of the differential equation **cannot** be function of time, such as $3ty^{'}$.

8 Stability:

Stable if any "bounded" input x(t).
$|x(t)| < m_x$ for all t $m_x < \infty$ also the output is also bounded $|y(t)| < m_y, m_y < \infty$.

I) The degree of the highest derivative of x(t) must not exceed that of the highest derivative of y(t).

II) Every root of the characteristic equation must have a negative real part.

Example:

$$y^{''}(t) + 3y^{'}(t) + 2y(t) = x(t)$$

Stable because the roots of characteristic equation $s^2 + 3s + 2 = 0$ are s=-1,-2 and have a negative real part.

9 Even and Odd Functions:

$$X_e(t) = 0.5[x(t) + x(-t)]$$
$$X_o(t) = 0.5[x(t) - x(-t)]$$
$$X(t) = x_e(t) + x_o(t)$$

10 DC Average & RMS :

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$
$$P_{DC_{avg}} = \frac{1}{T} \int_{t_0}^{t_0+T} v(t)dt$$
$$P_{avg} = \frac{1}{T} \int_{t_0}^{t_0+T} v(t)^2 dt$$
$$P_{rms} = \sqrt{P_{avg}}$$

11 Convolutions:

The notation for convolutions is $x(t) \star h(t)$.

$$y(t) = x(t) \star h(t) = \int_0^t x(\lambda)h(t-\lambda)d\lambda$$
$$u(t) \star u(t) = r(t)$$
$$rect(t) \star rect(t) = tri(t)$$
$$e^{-\alpha t}u(t) \star e^{-\alpha t}u(t) = te^{-\alpha t}u(t)$$

The convolutions of any signal h(t) with impulse reproduce the h(t) signal.

$$\delta(t) \star h(t) = \delta(t)$$

12 The Natural, Forced, and Total Response:

Total Response = Natural Response + Forced Response
The roots of the characteristic equation determine only the form of the **natural** response.
The input terms (RHS) of the differential equation completely determine the forced response.
Initial conditions satisfy the total response to yield the constants in the natural response.
The **forced response** arises due to the interaction of the system with the input and thus depends on both the input and the system details.

Table 4.1 Form of the Natural Response for Analog LTI Systems

Entry	Root of Characteristic Equation	Form of Natural Response
1	Real and distinct: r	Ke^{rt}
2	Complex conjugate: $\beta \pm j\omega$	$e^{\beta t}[K_1 \cos(\omega t) + K_2 \sin(\omega t)]$
3	Real, repeated: r^{p+1}	$e^{rt}(K_0 + K_1 t + K_2 t^2 + \cdots + K_p t^p)$
4	Complex, repeated: $(\beta \pm j\omega)^{p+1}$	$e^{\beta t} \cos(\omega t)(A_0 + A_1 t + A_2 t^2 + \cdots + A_p t^p)$ $+ e^{\beta t} \sin(\omega t)(B_0 + B_1 t + B_2 t^2 + \cdots + B_p t^p)$

Table 4.2 Form of the Forced Response for Analog LTI Systems

Note: If the right-hand side (RHS) is $e^{\alpha t}$, where α is also a root of the characteristic equation repeated r times, the forced response form must be multiplied by t^r.

Entry	Forcing Function (RHS)	Form of Forced Response
1	C_0 (constant)	C_1 (another constant)
2	$e^{\alpha t}$ (see note above)	$Ce^{\alpha t}$
3	$\cos(\omega t + \beta)$	$C_1 \cos(\omega t) + C_2 \sin(\omega t)$ or $C \cos(\omega t + \theta)$
4	$e^{\alpha t} \cos(\omega t + \beta)$ (see note above)	$e^{\alpha t}[C_1 \cos(\omega t) + C_2 \sin(\omega t)]$
5	t	$C_0 + C_1 t$
6	t^p	$C_0 + C_1 t + C_2 t^2 + \cdots + C_p t^p$
7	$te^{\alpha t}$ (see note above)	$e^{\alpha t}(C_0 + C_1 t)$
8	$t^p e^{\alpha t}$ (see note above)	$e^{\alpha t}(C_0 + C_1 t + C_2 t^2 + \cdots + C_p t^p)$
9	$t \cos(\omega t + \beta)$	$(C_1 + C_2 t)\cos(\omega t) + (C_3 + C_4 t)\sin(\omega t)$

13 Laplace Transform

$$\mathscr{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)dt$$

14 LPF:

$$y(s) = X(s) * \frac{Z_c}{Z_R + Z_c}$$
$$H(s) = \frac{\frac{1}{\tau}}{s + \frac{1}{\tau}}$$
$$\frac{d}{dQ} \to y(s)[s + \frac{1}{\tau}] = \frac{1}{\tau} X(s)$$
$$\mathscr{L} \to y(t)' + \frac{1}{\tau} y(t) = \frac{x(t)}{\tau}$$
$$h(t) = \frac{1}{\tau} e^{\frac{-t}{\tau}} u(t)$$
$$y_{step} = \int_0^t h(t)dt = -1 \int_0^t \frac{1}{\tau} e^{\frac{-t}{\tau}} dt = -(e^{\frac{-t}{\tau}} - 1)$$

15 BIBO (bounded-input,bounded output):

1. **Proper:** The highest derivative of input never exceed the highest derivative of output $\frac{P(s)}{Q(s)}$ degree of P never exceed the degree of Q.
2. Poles should be inside of left hand side.
3. h(t) must absolutely integrable.

Example:

$$H(s) = \frac{s+1}{s^2+4} \text{not stable because of s=} \pm j$$
$$H(s) = \frac{s^3+2}{s^2+3s+2} \text{unstable because of H(s) is not proper}$$
$$H(s) = \frac{s+2}{(s+1)(s-3)} \text{unstable because RHP s=3}$$

Marginally stable: if order of higher derivative of input exceeds the higher derivative of output.

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega^2 = f(t)$$
$$s^2 + 2\alpha s + \omega_0 = 0$$
$$\zeta = \frac{\alpha}{\omega}$$
$$\zeta_{settle} = \frac{u}{\zeta\omega_n}$$
$$\%overshoot = 100e^{\frac{-2\pi}{\sqrt{1-\zeta^2}}}$$
$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$
$$x_c = K_1e^{s_1t} + K_2e^{s_2t} \to \zeta > 1 \text{ overdamped}$$
$$x_c = K_1e^{s_1t} + K_2te^{s_1t} \to \zeta = 1 \text{ critically damped}$$
$$x_c = K_1e^{-\alpha t}cos(\omega_nt) + K_2e^{-\alpha t}sin(\omega_nt) \to \zeta < 1 \text{ underdapmped}$$
$$\omega_n = \sqrt{\omega_0^2 - \alpha^2}$$
$$\alpha = \frac{1}{2RC}$$
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

16 Initial value and final value Theorem:

The initial value theorem predicts the initial value x(0+) of the signal x(t) from its strictly proper transform X(s) as

$$x(0+) = \lim_{s\to\infty}[sX(s)]$$

The final value theorem predicts the value of the time signal x(t) as $t \to \infty$ from its transform X(s) and reads

$$x(\infty) = \lim_{s\to 0}[sX(s)]$$

17 Fourrier Transform:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$
$$inverse_{transform} = x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}dt$$
$$e^{i\theta} = cos(\theta) + isin(\theta)$$
$$e^{-i\theta} = cos(\theta)isin(\theta)$$
$$sin(\theta) = \frac{1}{2}[e^{i\theta} - e^{-i\theta}]$$
$$cos(\theta) = \frac{1}{2}[e^{i\theta} + e^{-i\theta}]$$