Practice Problems due May 2, 2021 02:50 +03

Unit 1: Probability and Counting

Adapted from Blitzstein-Hwang Chapter 1.

PRACTICE PROBLEMS

Recall from the syllabus that the **practice** problems are provided to help you practice with the concepts before tackling the homework problems. They are graded on completion, not correctness. You have many attempts to put in the correct answer and improve your grade. Use the show answer feature within the problems to see a detailed solution.

NOTE

This course includes numerical entry problems. Review the sample ungraded problem in Unit 0 if you want to practice entering numerical input.

Problem 1a

1/1 point (graded)

(a) How many ways are there to split a dozen people into 3 teams, where one team has 2 people, and the other two teams have 5 people each?

8316

✔ **Answer:** 8316

8316

Solution:
Pick any 2 of the 12 people to make the 2 person team, and then any 5 of the remaining 10 for the first team of 5, and then the remaining 5 are on the other team of 5; this overcounts by a factor of 2 though, since there is no designated “first" team of 5. So the number of possibilities is $\binom{12}{2}\binom{10}{5}/2 = 8316.$
Alternatively, politely ask the 12 people to line up, and then let the first 2 be the team of 2, the next 5 be a team of 5, and the last 5 be a team of 5. There are $12!$ ways for them to line up, but it does not matter which order they line up in *within* each group, nor does the order of the 2 teams of 5 matter, so the number of possibilities is $\frac{12!}{2!5!5!\cdot 2} = 8316.$

Submit You have used 4 of 99 attempts

Answers are displayed within the problem

Problem 1b

1/1 point (graded)
(b) How many ways are there to split a dozen people into 3 teams, where each team has 4 people?

5775 **Answer:** 5775

5775

Solution:
By either of the approaches above, there are $\frac{12!}{4!4!4!}$ ways to divide the people into a Team A, a Team B, and a Team C, if we care about which team is which (this is called a *multinomial coefficient*). Since here it doesn't matter which team is which, this overcounts by a factor of $3!$, so the number of possibilities is $\frac{12!}{4!4!4!3!} = 5775.$

Submit You have used 1 of 99 attempts

Answers are displayed within the problem

Problem 2

1/1 point (graded)

Consider the following identity. For all positive integers n and k with $n \geq k$,

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

This can be demonstrated either algebraically or via a story proof. To prove the identity algebraically, we can write

$$\begin{aligned}\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-k+1)!} \\ &= \frac{(n-k+1)\,n! + (k)\,n!}{k!\,(n-k+1)!} \\ &= \frac{n!\,(n+1)}{k!\,(n-k+1)!} \\ &= \binom{n+1}{k}.\end{aligned}$$

Which of the following is a story proof of the identity?

- [ ] Consider a group of $n-k+1$ peacocks and k toucans. The right-hand side is the number of ways to choose k of these birds, with order not mattering. The left-hand side counts the same thing in a different way, breaking it into two cases: all k toucans are chosen (first term) or at least one toucan is not chosen (second term).
- [x] Consider $n+1$ people, with one of them pre-designated as the “president" of the group. The right-hand side is the number of ways to choose k out of these $n+1$ people, with order not mattering. The left-hand side counts the same thing in a different way, by considering two disjoint cases: the president is not in the chosen subset (first term) or is in the chosen subset (second term).

- ○ Consider a group of $n-1$ peacocks and 2 toucans. The right-hand side is the number of ways to choose k of these birds, with order not mattering. The left-hand side counts the same thing in a different way, breaking it into two cases: no toucans are chosen (first term) or at least one toucan is chosen (second term).

- ○ Consider $n+1$ people, with one of them pre-designated as the “president" of the group. The right-hand side is the number of committees of size k that can be formed from the people. The left-hand side is the number of committees of size k or $k-1$ that can be formed, if the president is not allowed to be on the committee.

✔

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Answers are displayed within the problem

For Problem 3

For each part, decide whether the blank should be filled in with equal to, less than, or greater than.

Problem 3a

1/1 point (graded)

A die is a cube whose 6 sides are labeled with the integers from 1 to 6. The die is *fair* if all 6 sides are equally likely to come up on top when the die is rolled. The plural form of "die" is "dice".

(a) $P(\text{the total after rolling 4 fair dice is 21})$ [>] ✔

Answer: > $P(\text{the total after rolling 4 fair dice is 22})$

Solution:
$\boxed{>}$. All *ordered* outcomes are equally likely here. So for example with two dice, obtaining a total of 11 is more likely than obtaining a total of 12 since there are two ways to get a 6 and a 5, and only one way to get two 6's. To get a 21, the outcome must be a permutation of $(6,6,6,3)$ (4 possibilities), $(6,5,5,5)$ (4 possibilities), or $(6,6,5,4)$ ($4!/2 = 12$ possibilities). To get a 22, the outcome must be a permutation of $(6,6,6,4)$ (4 possibilities), or $(6,6,5,5)$ ($4!/2^2 = 6$ possibilities). So getting a 21 is more likely; in fact, it is exactly twice as likely as getting a 22.

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You have used 1 of 99 attempts

Answers are displayed within the problem

Problem 3b

1/1 point (graded)
(b) A *palindrome* is an expression such as "A man, a plan, a canal: Panama" that reads the same backwards as forwards, ignoring spaces, capitalization, and punctuation. Assume for this problem that all words of the specified length are equally likely, that there are no spaces or punctuation, and that the alphabet consists of the lowercase letters a,b,...,z.

$P(\text{a random 2-letter word is a palindrome})$ [=]

Answer: = $P(\text{a random 3-letter word is a palindrome})$

Solution:
$\boxed{=}$. The probabilities are equal, since for both 2-letter and 3-letter words, being a palindrome means that the first and last letter are the same. In particular, both sides equal $1/26$.

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You have used 1 of 99 attempts

Answers are displayed within the problem

Problem 4

1/1 point (graded)

Elk dwell in a certain forest. There are N elk, of which a simple random sample of size n are captured and tagged ("simple random sample" means that all $\binom{N}{n}$ sets of n elk are equally likely). The captured elk are returned to the population, and then a new sample is drawn, this time with size m. This is an important method that is widely used in ecology, known as *capture-recapture*. What is the probability that exactly k of the m elk in the new sample were previously tagged? (Assume that an elk that was captured before doesn't become more or less likely to be captured again.)

- [ ] $$\binom{n}{k}\left(\frac{m}{N}\right)^k\left(\frac{N-m}{N}\right)^{n-k}$$

- [ ] $$\frac{\binom{n}{m}\cdot\binom{N-n}{k}}{\binom{N}{m}}$$

- [ ] $$\binom{m}{k}\left(\frac{n}{N}\right)^k\left(\frac{N-n}{N}\right)^{m-k}$$

- [x] $$\frac{\binom{n}{k}\cdot\binom{N-n}{m-k}}{\binom{N}{m}}$$

✓

Solution:

We can use the naive definition here since we're assuming all samples of size m are equally likely. To have exactly k be tagged elk, we need to choose k of the n tagged elk, and then $m-k$ from the $N-n$ untagged elk. So the probability is

$$\frac{\binom{n}{k} \cdot \binom{N-n}{m-k}}{\binom{N}{m}},$$

for k such that $0 \leq k \leq n$ and $0 \leq m-k \leq N-n$, and the probability is 0 for all other values of k (for example, if $k > n$ the probability is 0 since then there aren't even k tagged elk in the entire population!). This is known as a *Hypergeometric* probability; we will encounter these again later in the course.

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You have used 1 of 99 attempts

Answers are displayed within the problem