Homework Problems due May 10, 2021 20:50 +03

Unit 2: Conditioning

Adapted from Blitzstein-Hwang Chapter 2.

FOR PROBLEM 1

Fred is answering a multiple-choice problem on an exam, and has to choose one of n options (exactly one of which is correct). Let K be the event that he knows the answer, and R be the event that he gets the problem right (either through knowledge or through luck). Suppose that if he knows the right answer he will definitely get the problem right, but if he does not know then he will guess completely randomly. Let $P(K) = p$.

Problem 1a

1/1 point (graded)

(a) Find $P(K|R)$ (in terms of p and n).

- ( ) $$\frac{np}{np+1}$$

- ( ) $$\frac{np}{p+n(1-p)}$$

- ( ) $$\frac{p}{p+n(1-p)}$$

- (•) $$\frac{p}{p+(1-p)/n}$$

Solution

By Bayes' rule and the law of total probability,

$$P(K|R) = \frac{P(R|K)\,P(K)}{P(R|K)\,P(K) + P(R|K^c)\,P(K^c)} = \frac{p}{p + (1-p)/n}.$$

Submit You have used 1 of 2 attempts

Answers are displayed within the problem

Problem 1b

1/1 point (graded)

(b) When (if ever) does $P(K|R)$ equal p?

- Never
- In the limit as $n \to \infty$
- In each of the following extreme cases: $n = 1$, $p = 0$, $p = 1$ (selected)
- If $p = 1/2$

Solution

For the extreme case $p = 0$, we have $P(K|R) = 0 = p$. So assume $p > 0$. By the result of (a), $P(K|R) \geq p$ is equivalent to $p + (1-p)/n \leq 1$, which is a true statement since $p + (1-p)/n \leq p + 1 - p = 1$.

This makes sense intuitively since getting the question right should increase our confidence that Fred knows the answer. Equality holds if and only if one of the extreme cases $n = 1$, $p = 0$, or $p = 1$ holds. If $n = 1$, it's not really a multiple-choice problem, and Fred getting the problem right is completely uninformative; if $p = 0$ or $p = 1$, then whether Fred knows the answer is a foregone conclusion, and no evidence will make us more (or less) sure that Fred knows the answer.

Submit You have used 1 of 2 attempts

Answers are displayed within the problem

Problem 2

1/1 point (graded)

A hat contains 100 coins, where 99 are fair but one is double-headed (always landing Heads). A coin is chosen uniformly at random. The chosen coin is flipped 7 times, and it lands Heads all 7 times. Given this information, what is the probability that the chosen coin is double-headed? (Of course, another approach here would be to *look at both sides of the coin*—but this is a metaphorical coin.)

0.564

✔ **Answer:** 0.564

0.564

Solution

Let A be the event that the chosen coin lands Heads all 7 times, and B be the event that the chosen coin is double-headed. Then

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A|B)\,P(B) + P(A|B^c)\,P(B^c)} = \frac{0.01}{0.01 + (1/2)^7 \cdot 0.99} = \frac{128}{227} \approx 0.564.$$

Submit

You have used 2 of 5 attempts

Answers are displayed within the problem

For Problem 3

Jimmy decides to take a series of n tests, to diagnose whether he has a certain disease (any individual test is not perfectly reliable, so he hopes to reduce his uncertainty by taking multiple tests). Let D be the event that he has the disease, $p = P(D)$ be the prior probability that he has the disease, and $q = 1 - p$. Let T_j be the event that he tests positive on the jth test.

Problem 3a

1/1 point (graded)

(a) Assume for this part that the test results are conditionally independent given Jimmy's disease status. Let $a = P(T_j|D)$ and $b = P(T_j|D^c)$, where a and b don't depend on j. Find the posterior probability that Jimmy has the disease, given that he tests positive on exactly k out of the n tests.

- ( ) $$\frac{pa^k}{pa^k + qb^k}$$
- ( ) $$\frac{pa^k b^{n-k}}{pa^k b^{n-k} + q(1-a)^k(1-b)^{n-k}}$$
- (•) $$\frac{pa^k(1-a)^{n-k}}{pa^k(1-a)^{n-k} + qb^k(1-b)^{n-k}}$$
- ( ) $$\binom{n}{k} a^k (1-a)^{n-k}$$

✔

Solution

Let X be the number of positive test results. By Bayes' rule and LOTP,

$$P(D|X = k) = \frac{P(X = k|D)P(D)}{P(X = k)} = \frac{pa^k(1-a)^{n-k}}{pa^k(1-a)^{n-k} + qb^k(1-b)^{n-k}}.$$

Submit

You have used 1 of 2 attempts

Answers are displayed within the problem

Problem 3b

1/1 point (graded)

(b) Now suppose that Jimmy tests positive on all n tests. However, some people have a certain gene that makes them *always* test positive. Let G be the event that Jimmy has the gene. Assume that $P(G) = 1/2$ and that D and G are independent. If Jimmy does *not* have the gene, then the test results are conditionally independent given his disease status. Let $a_0 = P(T_j|D, G^c)$ and $b_0 = P(T_j|D^c, G^c)$, where a_0 and b_0 don't depend on j. Find the posterior probability that Jimmy has the disease, given that he tests positive all n times.

- ( ) $$\frac{p(1+a_0)}{p(1+a_0)+q(1+b_0)}$$
- (•) $$\frac{p(1+a_0^n)}{p(1+a_0^n)+q(1+b_0^n)}$$
- ( ) $$\frac{pa_0^n}{pa_0^n+qb_0^n}$$
- ( ) $$\frac{pa_0^n+q}{pa_0^n+qb_0^n+1}$$

✓

Solution

Let T be the event that Jimmy tests positive on all n tests. Then

$$P(D|T) = \frac{P(T|D)\,P(D)}{P(T)} = \frac{pP(T|D)}{pP(T|D)+qP(T|D^c)}.$$

Conditioning on whether or not he has the gene, we have

$$P(T|D) = P(T|D,G)\,P(G|D) + P(T|D,G^c)\,P(G^c|D) = \frac{1}{2} + \frac{a_0^n}{2},$$
$$P(T|D^c) = P(T|D^c,G)\,P(G|D^c) + P(T|D^c,G^c)\,P(G^c|D^c) = \frac{1}{2} + \frac{b_0^n}{2}.$$

Thus,

$$P(D|T) = \frac{p(1+a_0^n)}{p(1+a_0^n)+q(1+b_0^n)}.$$

Submit You have used 1 of 2 attempts

Answers are displayed within the problem

Problem 4

1/1 point (graded)

An *exit poll* in an election is a survey taken of voters just after they have voted. One major use of exit polls has been so that news organizations can try to figure out as soon as possible who won the election, before the votes are officially counted. This has been notoriously inaccurate in various elections, sometimes because of *selection bias*: the sample of people who are invited to and agree to participate in the survey may not be similar enough to the overall population of voters.

Consider an election with two candidates, Candidate A and Candidate B. Every voter is invited to participate in an exit poll, where they are asked whom they voted for; some accept and some refuse. For a randomly selected voter, let A be the event that they voted for A, and W be the event that they are willing to participate in the exit poll. Suppose that $P(W|A) = 0.7$ but $P(W|A^c) = 0.3$. In the exit poll, 60% of the respondents say they voted for A (assume that they are all honest), suggesting a comfortable victory for A. Find $P(A)$, the true proportion of people who voted for A.

0.391

Answer: 0.391

0.391

Solution

We have $P(A|W) = 0.6$ since 60% of the respondents voted for A. Let $p = P(A)$. Then

$$0.6 = P(A|W) = \frac{P(W|A)\,P(A)}{P(W|A)\,P(A) + P(W|A^c)\,P(A^c)} = \frac{0.7p}{0.7p + 0.3\,(1-p)}.$$

Solving for p, we obtain

$$P(A) = \frac{9}{23} \approx 0.391.$$

So actually A received fewer than half of the votes!

Submit You have used 1 of 5 attempts

Answers are displayed within the problem