Practice Problems due May 19, 2021 14:50 +03

Unit 3: Discrete random variables

Adapted from Blitzstein-Hwang Chapter 3.

Problem 1

1/1 point (graded)

Benford's law states that in a very large variety of real-life data sets, the first digit approximately follows a particular distribution with about a 30% chance of a 1, an 18% chance of a 2, and in general

$$P(D=j)=\log_{10}\left(\frac{j+1}{j}\right),\ \text{for } j\in\{1,2,3,\ldots,9\},$$

where D is the first digit of a randomly chosen element. Check whether this is a valid PMF (**using properties of logs, not with a calculator**).

- [x] This is a valid PMF
- [ ] This is not a valid PMF

✓

Solution:

The function $P(D=j)$ is nonnegative and the sum over all values is

$$\sum_{j=1}^{9}\log_{10}\frac{j+1}{j}=\sum_{j=1}^{9}\left(\log_{10}(j+1)-\log_{10}(j)\right).$$

All terms cancel except $\log_{10}10-\log_{10}1=1$ (this is a *telescoping series*). Since the values add to 1 and are nonnegative, $P(D=j)$ is a PMF.

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You have used 1 of 99 attempts

Answers are displayed within the problem

Problem 2

1/1 point (graded)

In a chess tournament, 10 games are being played, independently. Each game ends in a win for one player with probability 0.4 and ends in a draw (tie) with probability 0.6. Find the probability that exactly 5 games end in a draw.

0.20

Answer: 0.20

0.20

Solution:

Let G be the number of games ending in a draw. Then $G \sim \mathrm{Bin}\,(10, 0.6)$, so the PMF is

$$P\,(G = g) = \binom{10}{g} 0.6^g \cdot 0.4^{n-g} \text{ for } g = 0, 1, \ldots, n$$

$$P\,(G = 5) = \binom{10}{5} 0.6^5 \cdot 0.4^{10-5} \approx 0.20$$

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For Problem 3

There are two coins, one with probability p_1 of Heads and the other with probability p_2 of Heads. One of the coins is randomly chosen (with equal probabilities for the two coins). It is then flipped $n \geq 2$ times. Let X be the number of times it lands Heads.

Problem 3a

1/1 point (graded)

(a) Which one of the following is the PMF of X?

- ( ) $P(X = k) = \binom{n}{k}\left(\frac{p_1+p_2}{2}\right)^{n-k}\left(1 - \frac{p_1+p_2}{2}\right)^{k}$
- ( ) $P(X = k) = \binom{n}{k}\left(\frac{p_1+p_2}{2}\right)^{k}\left(1 - \frac{p_1+p_2}{2}\right)^{n-k}$
- (•) $P(X = k) = \frac{1}{2}\binom{n}{k}p_1^k(1-p_1)^{n-k} + \frac{1}{2}\binom{n}{k}p_2^k(1-p_2)^{n-k}$
- ( ) $P(X = k) = \binom{n}{k}\left(p_1^k(1-p_1)^{n-k} + p_2^k(1-p_2)^{n-k}\right)$

✓

Solution:

By LOTP, conditioning on which coin is chosen, we have

$$P(X = k) = \frac{1}{2}\binom{n}{k}p_1^k(1-p_1)^{n-k} + \frac{1}{2}\binom{n}{k}p_2^k(1-p_2)^{n-k},$$

for $k = 0, 1, \ldots, n$.

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Answers are displayed within the problem

Problem 3b

1/1 point (graded)

(b) Is the distribution of X Binomial if $p_1 = p_2$?

- [x] Yes
- [ ] No

✔

Solution:
Yes. For $p_1 = p_2$, the above expression reduces to the $\text{Bin}(n, p_1)$ PMF.

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You have used 1 of 99 attempts

Answers are displayed within the problem

Problem 3c

1/1 point (graded)
(c) Is the distribution of X Binomial if $p_1 \neq p_2$?

- [ ] Yes
- [x] No

✔

Solution:
No. A mixture of two Binomials is *not* Binomial (except in the degenerate case $p_1 = p_2$). Marginally, each toss has probability $(p_1 + p_2)/2$ of landing Heads, but the tosses are *not* independent since earlier tosses give information about which coin was chosen, which in turn gives information about later tosses.
Let n be large, and imagine repeating the entire experiment many times (each repetition consists of choosing a random coin and flipping it n times). We would expect to see *either* approximately np_1 Heads about half the time, and approximately np_2 Heads about half the time. In contrast, with a $\text{Bin}(n, p)$ distribution we would expect to see approximately np Heads; no fixed choice of p can create the behavior described above.

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Answers are displayed within the problem

Problem 4

1/1 point (graded)

There are n eggs, each of which hatches a chick with probability p (independently). Each of these chicks survives with probability r, independently. Let H be the number of eggs that hatch and X be the number of hatchlings that survive. Find the distribution of H and the distribution of X.

- ( ) $$H \sim \mathrm{Bin}(n, p), X \sim \mathrm{Bin}(n, r)$$
- ( ) $$H \sim \mathrm{Bin}(n, r), X \sim \mathrm{Bin}(n, p+r)$$
- (•) $$H \sim \mathrm{Bin}(n, p), X \sim \mathrm{Bin}(n, pr)$$
- ( ) $$H \sim \mathrm{Bin}(n, p), X \sim \mathrm{Bin}(n, p+r)$$

✔

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✔ Correct (1/1 point)