Practice Problems due May 28, 2021 08:50 +03

Unit 4: Continuous random variables

Adapted from Blitzstein-Hwang Chapter 5.

FOR PROBLEM 1

The *Rayleigh distribution* has PDF

$$f(x) = xe^{-x^2/2}, \quad x > 0.$$

Let X have the Rayleigh distribution.

Problem 1a

1/1 point (graded)
(a) Find $P(1 < X < 3)$.

0.595

✔ **Answer:** 0.595

0.595

Solution:
We have

$$P(1 < X < 3) = \int_1^3 xe^{-x^2/2} dx.$$

To compute this, we can make the substitution $u = -x^2/2$. Then

$$P(1 < X < 3) = F(3) - F(1) = e^{-1/2} - e^{-9/2} \approx 0.595.$$

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Answers are displayed within the problem

Problem 1b

3/3 points (graded)

(b) Find the first quartile, median, and third quartile of X; these are defined to be the values q_1, q_2, q_3 (respectively) such that $P(X \leq q_j) = j/4$ for $j = 1, 2, 3$.

0.759

✔ **Answer:** 0.759

0.759

1.177

✔ **Answer:** 1.177

1.177

1.665

✔ **Answer:** 1.665

1.665

Solution:
Integrating the PDF using the same technique as in the previous part, the CDF F is given by

$$F(x) = 1 - e^{-x^2/2}$$

for $x > 0$. We can then find q_j by setting $F(q_j) = j/4$ and solving for q_j. This gives

$$1 - e^{-q_j^2/2} = j/4,$$

which becomes

$$q_j = \sqrt{-2\log(1 - j/4)}.$$

Numerically,

$$q_1 \approx 0.759, q_2 \approx 1.177, q_3 \approx 1.665.$$

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Problem 2

1/1 point (graded)
Let $Z \sim \mathcal{N}(0, 1)$ and $X = Z^2$. Then the distribution of X is called *Chi-Square with 1 degree of freedom*. This distribution appears in many statistical methods. Find a good numerical approximation to $P(1 \leq X \leq 4)$ using facts about the Normal distribution, without querying a calculator/computer/table about values of the Normal CDF.

0.27

✔ **Answer:** 0.27

0.27

Solution:
By symmetry of the Normal,

$$P(1 \leq Z^2 \leq 4) = P(-2 \leq Z \leq -1 \text{ or } 1 \leq Z \leq 2) = 2P(1 \leq Z \leq 2) = 2(\Phi(2) - \Phi(1)).$$

By the 68-95-99.7% Rule, $P(-1 \leq Z \leq 1) \approx 0.68$. This says that 32% of the area under the Normal curve is outside of $[-1, 1]$, which by symmetry says that 16% is in $(1, \infty)$. So $\Phi(1) \approx 1 - 0.16 = 0.84$. Similarly, $P(-2 \leq Z \leq 2) \approx 0.95$ gives $\Phi(2) \approx 0.975$. So

$$P(1 \leq X \leq 4) \approx 0.27.$$

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FOR PROBLEM 3

The *Pareto distribution* with parameter $a > 0$ has PDF $f(x) = a/x^{a+1}$ for $x \geq 1$ (and 0 otherwise). This distribution is often used in statistical modeling.

Problem 3a

1/1 point (graded)
(a) Find the CDF of a Pareto r.v. with parameter a.

- (selected) $F(x) = 1 - \frac{1}{x^a}$ for $x > 1$, and $F(x) = 0$ for $x \leq 1$.
- $F(x) = 1 - \frac{1}{x^{a+1}}$ for $x > 1$, and $F(x) = 0$ for $x \leq 1$.
- $F(x) = \frac{1}{x^{a+1}}$ for $x > 1$, and $F(x) = 0$ for $x \leq 1$.
- $F(x) = 1 - \frac{a}{x^a}$ for $x > 1$, and $F(x) = 0$ for $x \leq 1$.

✔

Solution:
The CDF F is given by

$$F(x) = \int_1^x \frac{a}{t^{a+1}} dt = (-t^{-a}) \Big|_x^1 = 1 - \frac{1}{x^a}$$

for $x > 1$, and $F(x) = 0$ for $x \leq 1$. This is a valid CDF since it is increasing in x (this can be seen directly or from the fact that $F' = f$ is nonnegative), right continuous (in fact it is continuous), $F(y) \to 0$ as $x \to -\infty$, and $F(x) \to 1$ as $x \to \infty$.

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Problem 3b

1/1 point (graded)
(b) Suppose that for a simulation you want to run, you need to generate i.i.d. $\text{Pareto}(a)$ r.v.s. Suppose you have a computer that knows how to generate $\text{Unif}(0, 1)$ r.v., but does not know how to generate Pareto r.v. Let $U \sim \text{Unif}(0, 1)$. Give a function of U that has the $\text{Pareto}(a)$ distribution. (Then, to run your simulation, you can generate a lot of i.i.d. Uniform r.v.s, and then convert them into Pareto r.v.s.)

- ○ $(1 - U)^{1/a}$
- ● $\dfrac{1}{(1 - U)^{1/a}}$
- ○ $\dfrac{a}{(1 - U)^a}$
- ○ $1 - \dfrac{1}{U^a}$

✔

Solution:

By universality of the Uniform, $F^{-1}(U) \sim \text{Pareto}(a)$. The inverse of the CDF is

$$F^{-1}(u) = \frac{1}{(1-u)^{1/a}}.$$

So

$$X = \frac{1}{(1-U)^{1/a}} \sim \text{Pareto}(a).$$

To check directly that X defined in this way is Pareto(a), we can find its CDF:

$$P(X \le x) = P\left(\frac{1}{x} \le (1-U)^{1/a}\right) = P\left(U \le 1 - \frac{1}{x^a}\right) = 1 - \frac{1}{x^a},$$

for any $x \ge 1$, and $P(X \le x) = 0$ for $x < 1$. Then if we have n i.i.d. Unif$(0, 1)$ r.v.s, we can apply this transformation to each of them to obtain n i.i.d. Pareto(a) r.v.s.

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Answers are displayed within the problem

Problem 4

1/1 point (graded)

A post office has 2 clerks. Alice enters the post office while 2 other customers, Bob and Claire, are being served by the 2 clerks. She is next in line. Assume that the time a clerk spends serving a customer has the Expo(λ) distribution. What is the probability that Alice is the last of the 3 customers to be done being served?

0.5

✔ **Answer:** 0.5

0.5

Solution:

Alice begins to be served when either Bob or Claire leaves. By the memoryless property, the additional time needed to serve whichever of Bob or Claire is still there is Expo(λ). The time it takes to serve Alice is also Expo(λ), so by symmetry the probability is 1/2 that Alice is the last to be done being served.

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