Homework Problems due May 19, 2021 14:50 +03

Unit 3: Discrete random variables

Adapted from Blitzstein-Hwang Chapter 3.

> **FOR PROBLEM 1**
>
> Let X be the number of purchases that a customer will make on the online site for a certain company (in some specified time period). Suppose that the PMF of X is
>
> $$P(X = k) = e^{-\lambda}\lambda^k/k!$$
>
> for $k = 0, 1, 2, \ldots$. This distribution is called the *Poisson distribution* with parameter λ.

Problem 1a

1/1 point (graded)
(a) Find $P(X \geq 1)$ and $P(X \geq 2)$ without summing infinite series.

○ $P(X \geq 1) = 1 - \lambda e^{-\lambda},\ P(X \geq 2) = 1 - 2e^{-\lambda}$

○ $P(X \geq 1) = 1 - e^{-\lambda},\ P(X \geq 2) = 1 - \lambda e^{-\lambda}$

○ $P(X \geq 1) = e^{-\lambda},\ P(X \geq 2) = e^{-\lambda}\lambda^2/2$

● $P(X \geq 1) = 1 - e^{-\lambda},\ P(X \geq 2) = 1 - e^{-\lambda} - \lambda e^{-\lambda}$

✓

Solution:
Taking complements,

$$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-\lambda},$$
$$P(X \geq 2) = 1 - P(X \leq 1) = 1 - e^{-\lambda} - \lambda e^{-\lambda}.$$

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Answers are displayed within the problem

Problem 1b

1/1 point (graded)

(b) Suppose that the company only knows about people who have made at least one purchase on their site (a user sets up an account to make a purchase, but someone who has never made a purchase there doesn't appear in the customer database). If the company computes the number of purchases for everyone in their database, then these data are draws from the *conditional* distribution of the number of purchases, given that at least one purchase is made. Which of the following is the conditional PMF of X given $X \geq 1$? (This conditional distribution is called a *truncated Poisson distribution*.)

● $$P(X = k | X \geq 1) = \frac{e^{-\lambda}\lambda^k}{k!\,(1 - e^{-\lambda})}$$

○ $$P(X = k | X \geq 1) = \frac{\lambda^k}{k!\,(1 - \lambda e^{-\lambda})}$$

○ $$P(X = k | X \geq 1) = \frac{e^{-\lambda}\lambda^k}{k!\,(1 - \lambda e^{-\lambda})}$$

○ $$P(X = k | X \geq 1) = \frac{e^{-\lambda}}{k!\,(1 - e^{-\lambda})}$$

✓

Solution:

The conditional PMF of X given $X \geq 1$ is

$$P(X = k | X \geq 1) = \frac{P(X = k)}{P(X \geq 1)} = \frac{e^{-\lambda}\lambda^k}{k!\,(1 - e^{-\lambda})},$$

for $k = 1, 2, \ldots$.

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For Problem 2

> A book has n typos. Two proofreaders, Prue and Frida, independently read the book. Prue catches each typo with probability p_1 and misses it with probability $q_1 = 1 - p_1$, independently, and likewise for Frida, who has probabilities p_2 of catching and $q_2 = 1 - p_2$ of missing each typo. Let X_1 be the number of typos caught by Prue, X_2 be the number caught by Frida, and X be the number caught by at least one of the two proofreaders.

Problem 2a

1/1 point (graded)

(a) Find the distribution of X.

● $\text{Bin}(n, 1 - q_1 q_2)$

○ $\text{HGeom}(p_1 n, p_2 n, p_1 p_2 n)$

○ $\text{Bin}(n, p_1 \cdot p_2)$

○ $\text{HGeom}(n, n, n - 1)$

✓

Solution

By the story of the Binomial, $X \sim \text{Bin}(n, 1 - q_1 q_2)$.

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ℹ Answers are displayed within the problem

Problem 2b

1/1 point (graded)

(b) For this part only, assume that $p_1 = p_2$. Find the conditional distribution of X_1 given that $X_1 + X_2 = t$.

○ $\text{Bin}(n, t/n)$

○ $\text{HGeom}(n, t, t)$

○ $\text{Bin}(t, p_1 p_2)$

● $\text{HGeom}(n, n, t)$

✓

Solution
Let $p = p_1 = p_2$ and $T = X_1 + X_2 \sim \text{Bin}(2n, p)$. Then

$$P(X_1 = k|T = t) = \frac{P(T = t|X_1 = k)\,P(X_1 = k)}{P(T = t)} = \frac{\binom{n}{t-k}p^{t-k}q^{n-t+k}\binom{n}{k}p^k q^{n-k}}{\binom{2n}{t}p^t q^{2n-t}} = \frac{\binom{n}{t-k}\binom{n}{k}}{\binom{2n}{t}}$$

for $k \in \{0, 1, \dots, t\}$, so the conditional distribution is $\text{HGeom}(n, n, t)$.

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Problem 3

1/1 point (graded)
People are arriving at a party one at a time. While waiting for more people to arrive they entertain themselves by comparing their birthdays. Let X be the number of people needed to obtain a birthday match, i.e., before person X arrives there are no two people with the same birthday, but when person X arrives there is a match. For example, $X = 10$ would mean that the first nine people to arrive all have different birthdays, but the tenth person to arrive matches one of the first nine. Find $P(X = 3 \text{ or } X = 4)$.

0.0136 ✓ **Answer:** 0.0136

0.0136

Solution
We will make the usual assumptions as in the birthday problem (e.g., exclude February 29). The support of X is $\{2, 3, \dots, 366\}$ since if there are 365 people there and no match, then every day of the year is accounted for and the 366th person will create a match. Let's start with a couple simple cases and then generalize:

$$P(X = 2) = \frac{1}{365},$$

since the second person has a $1/365$ chance of having the same birthday as the first,

$$P(X = 3) = \frac{364}{365} \cdot \frac{2}{365},$$

since $X = 3$ means that the second person didn't match the first but the third person matched one of the first two. In general, for $2 \le k \le 366$ we have

$$\begin{aligned} P(X = k) &= P(X > k - 1 \text{ and } X = k) \\ &= \frac{365 \cdot 364 \cdots (365 - k + 2)}{365^{k-1}} \cdot \frac{k - 1}{365} \\ &= \frac{(k - 1) \cdot 364 \cdot 363 \cdots (365 - k + 2)}{365^{k-1}}. \end{aligned}$$

Therefore,

$$P(X = 3 \text{ or } X = 4) = P(X = 3) + P(X = 4) \approx 0.0136.$$

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ℹ Answers are displayed within the problem

> **FOR PROBLEM 4**
>
> Let X be the number of Heads in 10 fair coin tosses.

Problem 4a

1/1 point (graded)

(a) Find the conditional PMF of X, given that the first two tosses both land Heads.

○ $\frac{1}{1024}\binom{10}{k-2}$, for $k = 2, 3, \dots, 10$

● $\frac{1}{256}\binom{8}{k-2}$, for $k = 2, 3, \dots, 10$

○ $\frac{1}{128}\binom{8}{k}$, for $k = 2, 3, \dots, 10$

○ $\frac{1}{1013}\binom{10}{k}$, for $k = 2, 3, \dots, 10$

✓

Solution

Let X_2 and X_8 be the number of Heads in the first 2 and last 8 tosses, respectively. Then the conditional PMF of X given $X_2 = 2$ is

$$\begin{aligned} P(X = k|X_2 = 2) &= P(X_2 + X_8 = k|X_2 = 2) \\ &= P(X_8 = k - 2|X_2 = 2) \\ &= P(X_8 = k - 2) \\ &= \binom{8}{k-2}\left(\frac{1}{2}\right)^{k-2}\left(\frac{1}{2}\right)^{8-(k-2)} \\ &= \frac{1}{256}\binom{8}{k-2}, \end{aligned}$$

for $k = 2, 3, \dots, 10$.

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Answers are displayed within the problem

Problem 4b

1/1 point (graded)

(b) Find the conditional PMF of X, given that at least two tosses land Heads.

○ $\frac{1}{1024}\binom{10}{k-2}$, for $k=2,3,\dots,10$

○ $\frac{1}{256}\binom{8}{k-2}$, for $k=2,3,\dots,10$

○ $\frac{1}{128}\binom{8}{k}$, for $k=2,3,\dots,10$

● $\frac{1}{1013}\binom{10}{k}$, for $k=2,3,\dots,10$

✓

Solution

The conditional PMF of X given $X \geq 2$ is

$$\begin{aligned}
P(X=k|X\geq 2) &= \frac{P(X=k, X\geq 2)}{P(X\geq 2)} \\
&= \frac{P(X=k)}{1-P(X=0)-P(X=1)} \\
&= \frac{\binom{10}{k}\left(\frac{1}{2}\right)^{10}}{1-\left(\frac{1}{2}\right)^{10}-10\left(\frac{1}{2}\right)^{10}} \\
&= \frac{1}{1013}\binom{10}{k},
\end{aligned}$$

for $k=2,3,\dots,10.$

Submit

You have used 1 of 2 attempts

Answers are displayed within the problem