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Numerical computation with periodic functions

8th International Congress on Industrial and Applied Mathematics, Beijing

Hadrien Montanelli (Oxford)

Collaborators: N.I. Gushterov, M. Javed, and L.N. Trefethen (Oxford), G.B. Wright (Boise)

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Overview

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Trigonometric Interpolation

[1] Wright, Javed, M., and Trefethen, "Extension of Chebfun to periodic functions", SISC, accepted.

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Trigonometric Interpolation

- f Lipschitz continuous and periodic function on $[0, 2\pi]$

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Trigonometric Interpolation

- f Lipschitz continuous and periodic function on $[0, 2\pi]$
- **Continuous world (true object):** f has a unique trigonometric series of the form

$$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}, \quad a_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$$

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- **Discrete world (approximation):** f has a unique trigonometric polynomial p_N that interpolates the data $f_j = f(x_j)$ at the N points $x_j = 2\pi j/N$ of the form

$$p_N(x) = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_k e^{ikx}, \quad c_k = \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-ikx_j}$$

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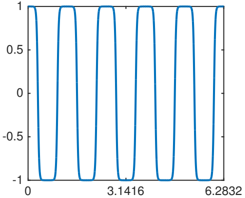
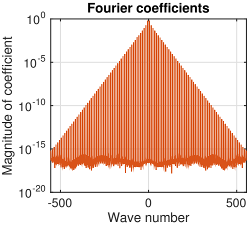
- Extension of Chebfun to periodic functions [1]:

```
f = @(x) tanh(5*cos(5*x));
p = chebfun(f,[0 2*pi],'trig');
plot(p), plotcoeffs(p)
```





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Periodic ODEs and e-value problems (1/3)

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Periodic ODEs and e-value problems (1/3)

- **Problem:** We want to solve

$$\mathcal{L}u = f, \quad \mathcal{L}u = \lambda u,$$

with periodic operator $\mathcal{L}$ on $[0, 2\pi]$ defined by

$$\mathcal{L} = u'' + a(x)u' + b(x)u = \mathcal{D}^{(2)} + \mathcal{M}[a]\mathcal{D} + \mathcal{M}[b]$$

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- **Method:** Spectral method, based on values or coefficients, with adaptive grid size
- Discretize $[0, 2\pi]$ with grid $x_j = 2\pi j/N$, f with trigonometric polynomial f_N that interpolates $f(x_j) = \texttt{ifft}(f_k)$, and look for trigonometric polynomial solution u_N

$$u_N(x) = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} u_k e^{ikx}, \quad u_k = \texttt{fft}(u(x_j))$$

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- Leads to the following systems

$$\left(D_v^{(2)} + M_v[a]D_v + M_v[b]\right)\mathbf{u} = \mathbf{f} \qquad \left(D_c^{(2)} + M_c[a]D_c + M_c[b]\right)\mathbf{u} = \mathbf{f}$$

$$\mathbf{u} = \{u(x_j)\}, \quad \mathbf{f} = \{f(x_j)\} \qquad \mathbf{u} = \{u_k\}, \quad \mathbf{f} = \{f_k\}$$

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$$\mathbf{u} = \{u(x_j)\}, \quad \mathbf{f} = \{f(x_j)\} \qquad \mathbf{u} = \{u_k\}, \quad \mathbf{f} = \{f_k\}$$

- Automatic Fréchet differentiation, and Newton's method in function space for nonlinear ODEs

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Periodic ODEs and e-value problems (2/3)

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$$0.004u'' + uu' - u = \cos(2\pi x), \quad x \in [-1, 1]$$

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```
N = chebop(-1,1);
N.op = @(x,u) .004*diff(u,2) + u.*diff(u) - u;
N.bc = 'periodic';
f = chebfun('cos(2*pi*x)','trig');
u = N\f; plot(u)
```

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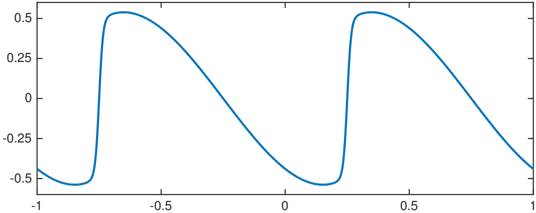

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Periodic ODEs and e-value problems (3/3)

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$$-u'' + 2q\cos(2x)u = \lambda u, \quad x \in [0, 2\pi]$$

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Periodic ODEs and e-value problems (3/3)

$$-u'' + 2q\cos(2x)u = \lambda u, \quad x \in [0, 2\pi]$$

```
q=2;
L = chebop([0 2*pi]);
L.op = @(x,u) -diff(u,2) + 2*q*cos(2*x).*u;
L.bc = 'periodic';
[V,D] = eigs(L,5); plot(V)
```

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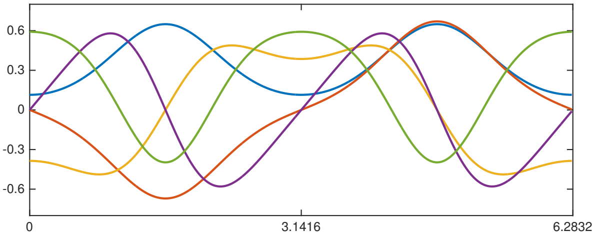

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Linear PDEs (1/2)

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- **Problem:** We want to solve

$$u_t(x,t) = \mathcal{L}u(x,t),$$

for some linear periodic operator $\mathcal{L}$ on $[0,2\pi] \times [0,T]$ and initial condition $u(x,0)$

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- When the problem is well-posed, solution given by the operator exponential

$$u(x,t) = e^{\mathcal{L}t}u(x,0)$$

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- E.g., advection equation

$$u_t = \mathcal{L}u, \quad \mathcal{L} = a(x)\frac{\partial}{\partial x} = \mathcal{M}[a]\mathcal{D}$$

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$$u_t = \mathcal{L}u, \quad \mathcal{L} = a(x)\frac{\partial}{\partial x} = \mathcal{M}[a]\mathcal{D}$$

$$\mathbf{u(t)} = e^{M_{\boldsymbol{v}}[a]D_{\boldsymbol{v}}t}\mathbf{u(0)}$$

$$\mathbf{u(t)} = \{u(x_j,t)\}, \quad \mathbf{u(0)} = \{u(x_j,0)\}$$

$$\mathbf{u(t)} = e^{M_{\boldsymbol{c}}[a]D_{\boldsymbol{c}}t}\mathbf{u(0)}$$

$$\mathbf{u(t)} = \{u_k(t)\}, \quad \mathbf{u(0)} = \{u_k(0)\}$$

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Linear PDEs (2/2)

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$$u_t = -(\frac{1}{5} + \sin^2(x-1))u_x, \quad x \in [0, 2\pi], \; t \in [0, 20]$$

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```
c = chebfun(@(x) -(1/5 + sin(x-1).^2),[0 2*pi]);
L = chebop(@(x,u) c.*diff(u),[0 2*pi]);
L.bc = 'periodic';
u0 = chebfun(@(x) exp(-100*(x-1).^2),[0 2*pi]);
u = expm(L,0:.5:20,u0), waterfall(u,0:.5:20)
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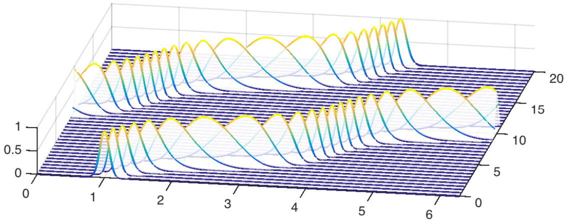

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Overview

1 Extension of Chebfun to periodic functions

2 Application 1: Nonlinear stiff PDEs

3 Application 2: Choreographies

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Problem of interest

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- **Problem:** We want to solve nonlinear stiff PDEs of the form

$$u_t = \mathcal{L}u + \mathcal{N}(t, u(x,t)),$$

for some periodic op. $\mathcal{L}$ and $\mathcal{N}$ on $[0, 2\pi] \times [0, T]$ and initial condition $u(x, 0)$

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- **Method:** Discretization with a spectral method in space with M points leads to

$$u'(t) = Lu(t) + N(t, u(t)),$$

with $u(t) = \{u_k(t)\}$, i.e., a system of M ODEs

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- If N is a constant,

$$u(t) = e^{tL}u_0 + L^{-1}\left[e^{tL} - I\right]N$$

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- For general N, one need to find a way to capture the nonlinearity in a time-stepping method

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Exponential integrators

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- **Method:** From the value $u_n = u(t_n)$ at time t_n, the numerical approximation u_{n+1} at time $t_{n+1} = t_n + h$ is given by the formula

$$u_{n+1} = e^{hL}u_n + h\sum_{i=1}^{s} B_i(hL)N(t_n + c_i h, v_i),$$

with internal stages $v_1 = u_n$ and, for $2 \leq i \leq s$,

$$v_i = e^{c_i hL}u_n + h\sum_{j=1}^{i-1} A_{i,j}(hL)N(t_n + c_j h, v_j)$$

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- Coefficients $A_{i,j}$, B_i and c_i given via Butcher tableau

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- Coefficients $A_{i,j}$, B_i and c_i given via Butcher tableau
- In Chebfun: available soon, use 4th- and 5th-order time-stepping schemes, easy to add another scheme, essentially just give the Butcher tableau

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Example 1: Kuramoto–Sivashinsky

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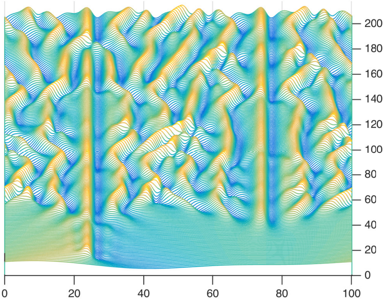

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$$u_t = -u_{xxx} - uu_x, \quad x \in [-\pi, \pi],\ t \in [0, .01]$$

$$u(x,0) = 3A^2 \operatorname{sech}^2\left(\frac{A(x+2)}{2}\right) + 3B^2 \operatorname{sech}^2\left(\frac{B(x+1)}{2}\right)$$

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```
S = spin('kdv',{[-pi pi], [0 .01]}); A = 25; B = 16;
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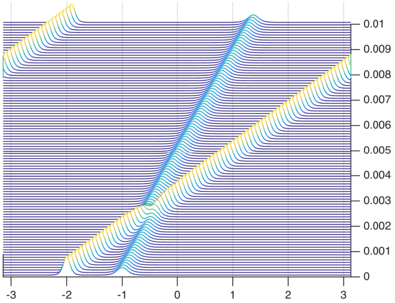

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Overview

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Minimization of the action (Simó)

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- Based on the principle of least action applied to the n-body problem

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Choreographies of the n-body problem

$$z_j''(t) - \sum_{\substack{i=0 \\ i \neq j}}^{n-1} \frac{z_i(t) - z_j(t)}{\left|z_i(t) - z_j(t)\right|^3} = 0, \quad 0 \leq j \leq n-1,$$

with $z_j(t) = q\left(t + \frac{2\pi j}{n}\right)$ for some 2π-periodic function $q(t) : [0, 2\pi] \to \mathbb{C}$

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Minima of the action

$$A = \int_0^{2\pi} \left(K(t) - U(t)\right) dt,$$

$$K(t) = \frac{n}{2}\left|q'(t)\right|^2, \quad U(t) = -n\sum_{j=1}^{n-1} \left|q(t) - q\left(t + \frac{2\pi j}{n}\right)\right|^{-1}$$

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Method

[2] M. and Gushterov, "Computing planar and spherical choreogprahies", SIADS, submitted.

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Method

- ***Trigonometric interpolation:*** $q(t) \to p_N(t) \to A_N$

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- Can be generalized to the sphere using stereographic projection [2]

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Numerical results

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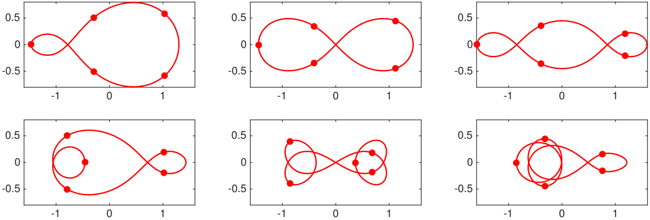

Action	68.8516	71.3312	77.1588	88.4397	109.6366	119.3191
Computer time (s)	0.79	0.49	0.44	0.70	0.98	0.86
2-norm of the gradient	1.26e-02	1.39e-02	8.87e-03	9.68e-03	1.18e-02	1.28e-02
Smallest coefficient	4.71e-06	6.45e-08	2.26e-06	3.33e-06	2.08e-05	2.75e-05
∞-norm of the residual	9.31e-02	1.09e-03	1.30e-02	4.43e-02	2.83e-01	6.56e-01

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Conclusion and future work

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```
L = chebop([0 2*pi]);
L.op = @(t,u) diff(u) + cos(2*t).*u;
L.bc = 'periodic';
L'
```