Fast solution of stiff PDEs in 1D, 2D and 3D periodic domains and on the sphere

ICOSAHOM 2016

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Introduction

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\hat{v}_1 &= \hat{u}_n, \\
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- **Cost per time-step:** $O(2sN\log(N))$

Exponential integrators (2/5)

Method	Type	Order	Stages s	Steps q
ABNørsett4	ETD Adams-Bashforth	4	1	4
ABNørsett5	ETD Adams-Bashforth	5	1	5
ABNørsett6	ETD Adams-Bashforth	6	1	6
Friedli (VRK4)	ETD Runge-Kutta	4	4	1
Strehmel-Weiner	ETD Runge-Kutta	4	4	1
Cox-Matthews (ETDRK4)	ETD Runge-Kutta	4	4	1
Krogstad (ETDRK4-B)	ETD Runge-Kutta	4	4	1
Minchev	ETD Runge-Kutta	4	4	1
Hochbruck-Ostermann	ETD Runge-Kutta	4	5	1
Luan-Ostermann (EXPRK5S8)	ETD Runge-Kutta	5	8	1
(Mod)GenLawson41	(Mod.) Gen. Lawson	4	4	1
(Mod)GenLawson42	(Mod.) Gen. Lawson	4	4	2
(Mod)GenLawson43	(Mod.) Gen. Lawson	4	4	3
(Mod)GenLawson44	(Mod.) Gen. Lawson	5	4	4
(Mod)GenLawson45	(Mod.) Gen. Lawson	6	4	5
PEC423	Predictor-corrector	4	2	3
PECEC433	Predictor-corrector	4	3	3
PEC524	Predictor-corrector	5	2	4
PECEC534	Predictor-corrector	5	3	4
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- Predictor-Corrector methods of order ≥ 5 more accurate but often unstable

Exponential integrators (3/5)

Example 1: ETD RK schemes for KdV equation from $t = 0$ to $t = 0.005$

We plot relative error $= \frac{\|u_{\mathbf{approx}} - u_{\mathbf{exact}}\|_\infty}{\|u_{\mathbf{exact}}\|_\infty}$ vs relative time-step $= \frac{dt}{0.005}$ and time

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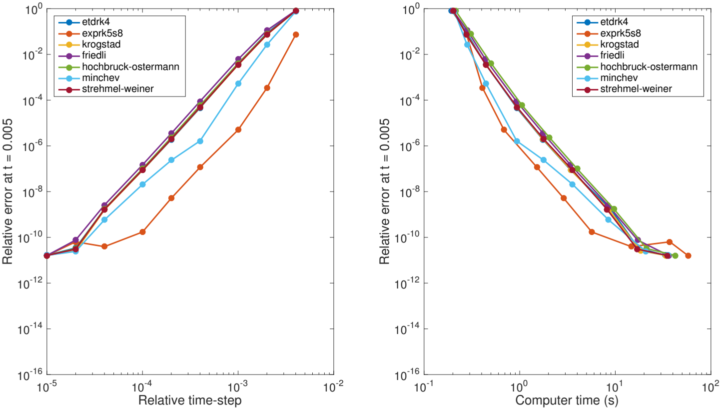

Exponential integrators (4/5)

Example 2: ETD RK schemes for KdV equation from $t = 0$ to $t = 0.015$

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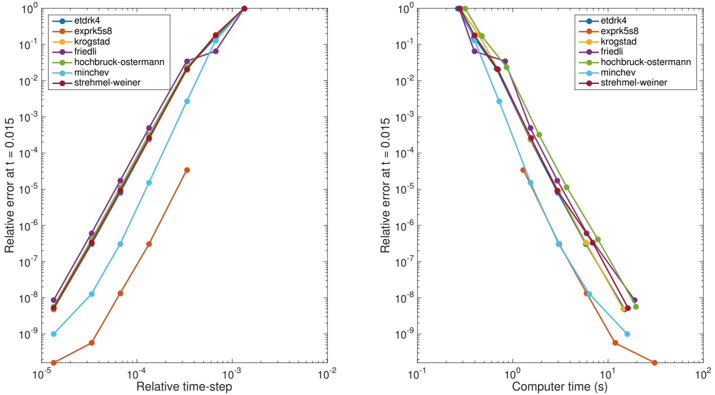

Exponential integrators (5/5)

Example 3: ETD RK schemes for Gray-Scott equations in 2D from $t = 0$ to $t = 30$

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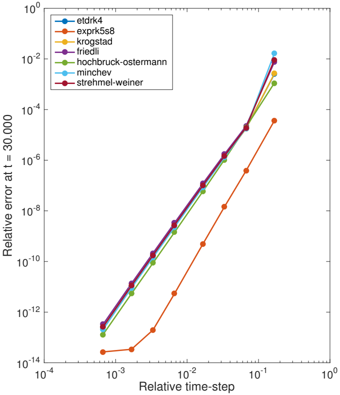

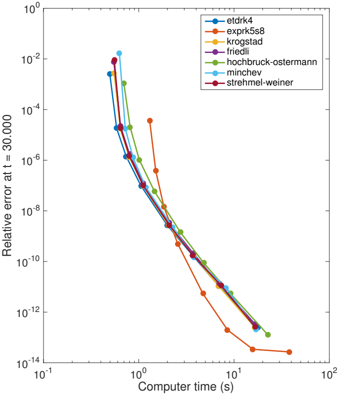

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- **Define your own problem:**

```
dom = [0 32*pi]; tspan = [0 100];
S = spinop(dom, tspan);
S.linearPart = @(u) -diff(u,2)-diff(u,4);
S.nonlinearPart = @(u) -.5*diff(u.^2);
S.init = chebfun(@(x) cos(x/16).*(1 + sin(x/16)), dom, 'trig');
u = spin(S);
```

Stiff PDEs on the sphere (1/2)

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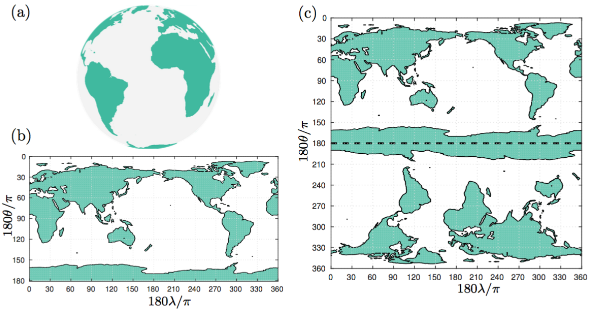

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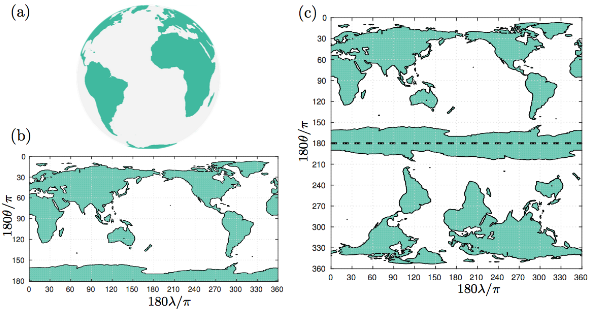

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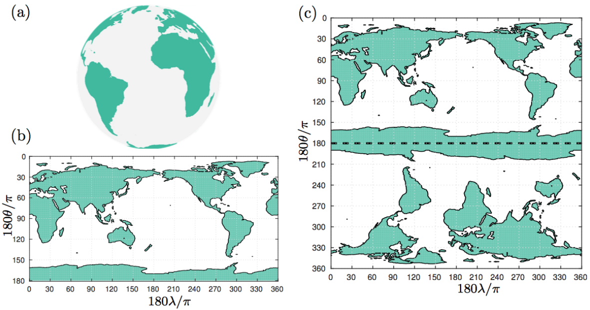

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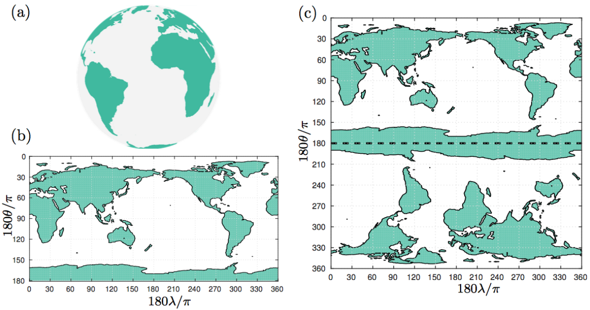

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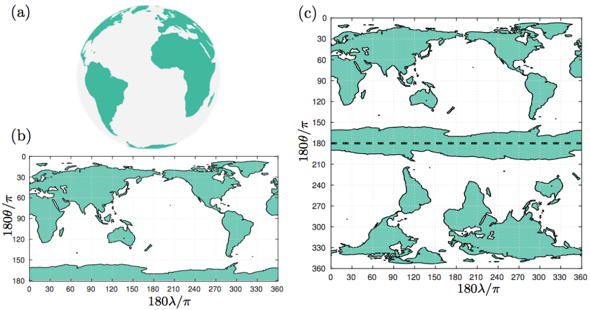

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```
u = spherefun(@(lam,tt) 1./(1 + (cos(lam).*sin(tt)).^2) + cos(tt).^2);
```

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- If you work with stiff PDEs, you definitely want to try `spin`
- Fourier spectral methods and exponential integrators can be used for solving PDEs on the sphere with the Double Fourier Sphere method (in coefficient space) and diagonalization by block