UNIVERSITY OF OXFORD

Computing Choreographies

Numerical Analysis Group Internal Seminar, Mathematical Institute

Hadrien Montanelli

March 10, 2015

UNIVERSITY OF OXFORD

Overview

1. Introduction

2. Computing choreographies

3. Stability analysis

UNIVERSITY OF OXFORD

What is a choreography?

[1] Lagrange, "Essai sur le problème des trois corps", 1772.
[2] Moore, "Braids in classical dynamics", 1993.
[3] Simó, "New families of Solutions in N-body problems", 2001.

UNIVERSITY OF OXFORD

What is a choreography?

- n-body problem: motion of n bodies under the action of Newton's law of gravitation, system of n nonlinear ODEs

[1] Lagrange, "Essai sur le problème des trois corps", 1772.
[2] Moore, "Braids in classical dynamics", 1993.
[3] Simó, "New families of Solutions in N-body problems", 2001.

UNIVERSITY OF OXFORD

What is a choreography?

- n-body problem: motion of n bodies under the action of Newton's law of gravitation, system of n nonlinear ODEs
- choreographies: periodic solutions in which all of the bodies have unit masses, share a common orbit and are uniformly spread along it

[1] Lagrange, "Essai sur le problème des trois corps", 1772.
[2] Moore, "Braids in classical dynamics", 1993.
[3] Simó, "New families of Solutions in N-body problems", 2001.

UNIVERSITY OF OXFORD

What is a choreography?

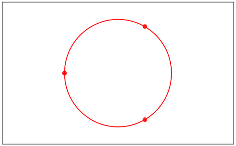
- n-body problem: motion of n bodies under the action of Newton's law of gravitation, system of n nonlinear ODEs
- choreographies: periodic solutions in which all of the bodies have unit masses, share a common orbit and are uniformly spread along it
- 3 bodies: first choreography found by Lagrange in 1772 [1], circle of radius $r \approx 0.832683$, second found more than 200 years later by Moore [2], the figure-eight

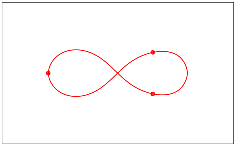
[1] Lagrange, "Essai sur le problème des trois corps", 1772.
[2] Moore, "Braids in classical dynamics", 1993.
[3] Simó, "New families of Solutions in N-body problems", 2001.

UNIVERSITY OF OXFORD

What is a choreography?

- n-body problem: motion of n bodies under the action of Newton's law of gravitation, system of n nonlinear ODEs
- choreographies: periodic solutions in which all of the bodies have unit masses, share a common orbit and are uniformly spread along it
- 3 bodies: first choreography found by Lagrange in 1772 [1], circle of radius $r \approx 0.832683$, second found more than 200 years later by Moore [2], the figure-eight



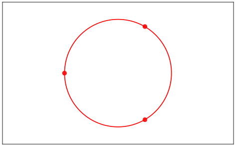


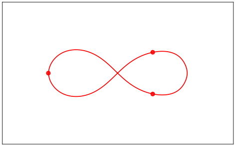
[1] Lagrange, "Essai sur le problème des trois corps", 1772.
[2] Moore, "Braids in classical dynamics", 1993.
[3] Simó, "New families of Solutions in N-body problems", 2001.

UNIVERSITY OF OXFORD

What is a choreography?

- n-body problem: motion of n bodies under the action of Newton's law of gravitation, system of n nonlinear ODEs
- choreographies: periodic solutions in which all of the bodies have unit masses, share a common orbit and are uniformly spread along it
- 3 bodies: first choreography found by Lagrange in 1772 [1], circle of radius $r \approx 0.832683$, second found more than 200 years later by Moore [2], the figure-eight





- n bodies: many new choreographies found by Simó in the early 2000s [3] using numerical optimization of the action

[1] Lagrange, "Essai sur le problème des trois corps", 1772.
[2] Moore, "Braids in classical dynamics", 1993.
[3] Simó, "New families of Solutions in N-body problems", 2001.

UNIVERSITY OF OXFORD

Minimization of the action (Simó)

UNIVERSITY OF OXFORD

Minimization of the action (Simó)

Periodic solutions of the n-body problem

$$z_j''(t) - \sum_{\substack{i=0 \\ i\neq j}}^{n-1} \frac{z_i(t) - z_j(t)}{\left|z_i(t) - z_j(t)\right|^3} = 0,$$

with $\quad z_j(t) = q\left(t + \frac{2\pi j}{n}\right), \quad 0 \leq j \leq n-1, \quad$ for some function $q(t)$

UNIVERSITY OF OXFORD

Minimization of the action (Simó)

Periodic solutions of the n-body problem

$$z_j''(t) - \sum_{\substack{i=0 \\ i\neq j}}^{n-1} \frac{z_i(t) - z_j(t)}{\left|z_i(t) - z_j(t)\right|^3} = 0,$$

with $\quad z_j(t) = q\left(t + \frac{2\pi j}{n}\right), \quad 0 \le j \le n-1, \quad$ for some function $q(t)$

Minima of the action

$$A = \int_0^{2\pi} L(t)\,dt, \quad L(t) = K(t) - U(t),$$

$$K(t) = \frac{n}{2}\left|q'(t)\right|^2, \quad U(t) = -\frac{n}{2}\sum_{j=1}^{n-1}\left|q(t) - q\left(t + \frac{2\pi j}{n}\right)\right|^{-1}$$

UNIVERSITY OF OXFORD

Overview

Method

UNIVERSITY OF OXFORD

Method

- ***Trigonometric interpolation:*** $q(t) \to \{c_k\} \to A$

UNIVERSITY OF OXFORD

Method

- ***Trigonometric interpolation:*** $q(t) \to \{c_k\} \to A$

  $q(t) \approx p_N(t)$ instead of $q(t) = x(t) + iy(t) \approx x_N(t) + iy_N(t)$

UNIVERSITY OF OXFORD

Method

- ***Trigonometric interpolation:*** $q(t) \to \{c_k\} \to A$

  $q(t) \approx p_N(t)$ instead of $q(t) = x(t) + iy(t) \approx x_N(t) + iy_N(t)$

- ***A closed-form expression for the gradient:*** $\{c_k\} \to \nabla A$

UNIVERSITY OF OXFORD

Method

- ***Trigonometric interpolation:*** $q(t) \to \{c_k\} \to A$

  $q(t) \approx p_N(t)$ instead of $q(t) = x(t) + iy(t) \approx x_N(t) + iy_N(t)$

- ***A closed-form expression for the gradient:*** $\{c_k\} \to \nabla A$

- ***Optimization algorithm:*** $\{c_k\} \to \{c_k^*\}$

UNIVERSITY OF OXFORD

Method

- ***Trigonometric interpolation:*** $q(t) \to \{c_k\} \to A$

  $q(t) \approx p_N(t)$ instead of $q(t) = x(t) + iy(t) \approx x_N(t) + iy_N(t)$

- ***A closed-form expression for the gradient:*** $\{c_k\} \to \nabla A$

- ***Optimization algorithm:*** $\{c_k\} \to \{c_k^*\}$

  Quasi-Newton method (BFGS) instead of Gradient methods

UNIVERSITY OF OXFORD

Trigonometric Interpolation: $q(t) \to \{c_k\} \to A$

[4] Wright, Javed, M., and Trefethen, "Extension of Chebfun to periodic functions", 2015.

UNIVERSITY OF OXFORD

Trigonometric Interpolation: $q(t) \to \{c_k\} \to A$

- Trigonometric interpolant $p_N(t)$ of $q(t)$ at N points:

$$p_N(t) = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_k e^{ikt}, \quad t \in [0, 2\pi], \quad c_k = \frac{1}{N} \sum_{j=0}^{N-1} q_j e^{-ikt_j}, \quad |k| \leq \frac{N-1}{2}$$

[4] Wright, Javed, M., and Trefethen, "Extension of Chebfun to periodic functions", 2015.

UNIVERSITY OF OXFORD

Trigonometric Interpolation: $q(t) \to \{c_k\} \to A$

- Trigonometric interpolant $p_N(t)$ of $q(t)$ at N points:

$$p_N(t) = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_k e^{ikt}, \quad t \in [0, 2\pi], \quad c_k = \frac{1}{N} \sum_{j=0}^{N-1} q_j e^{-ikt_j}, \quad |k| \leq \frac{N-1}{2}$$

- Recent extension of Chebfun to periodic functions [4]

[4] Wright, Javed, M., and Trefethen, "Extension of Chebfun to periodic functions", 2015.

UNIVERSITY OF OXFORD

Trigonometric Interpolation: $q(t) \to \{c_k\} \to A$

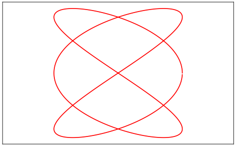
- Trigonometric interpolant $p_N(t)$ of $q(t)$ at N points:

$$p_N(t) = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_k e^{ikt}, \quad t \in [0, 2\pi], \quad c_k = \frac{1}{N}\sum_{j=0}^{N-1} q_j e^{-ikt_j}, \quad |k| \leq \frac{N-1}{2}$$

- Recent extension of Chebfun to periodic functions [4]

```
>> q = chebfun(@(t)cos(3*pi*t)+1i*sin(2*pi*t),'trig')

q =
   chebfun column (1 smooth piece)
       interval        length   endpoint values trig
[      -1,        1]        7     complex values
Epslevel = 8.437082e-16.  Vscale = 1.327490e+00.
```



[4] Wright, Javed, M., and Trefethen, "Extension of Chebfun to periodic functions", 2015.

UNIVERSITY OF OXFORD

Trigonometric Interpolation: $q(t) \to \{c_k\} \to A$

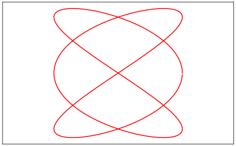
- Trigonometric interpolant $p_N(t)$ of $q(t)$ at N points:

$$p_N(t) = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_k e^{ikt}, \quad t \in [0, 2\pi], \quad c_k = \frac{1}{N}\sum_{j=0}^{N-1} q_j e^{-ikt_j}, \quad |k| \leq \frac{N-1}{2}$$

- Recent extension of Chebfun to periodic functions [4]

```
>> q = chebfun(@(t)cos(3*pi*t)+1i*sin(2*pi*t),'trig')

q =
   chebfun column (1 smooth piece)
       interval       length   endpoint values trig
[      -1,       1]       7     complex values
Epslevel = 8.437082e-16.  Vscale = 1.327490e+00.
```



- Action computed with the exponentially accurate trapezoidal rule

$$A_N = \frac{n}{2}\int_0^{2\pi} \left|p_N'(t)\right|^2 dt + \frac{n}{2}\sum_{j=1}^{n-1}\int_0^{2\pi} \left|p_N(t) - p_N\left(t + \frac{2\pi j}{n}\right)\right|^{-1} dt$$

[4] Wright, Javed, M., and Trefethen, "Extension of Chebfun to periodic functions", 2015.

UNIVERSITY OF OXFORD

Expression for the gradient: $\{c_k\} \to \nabla A$

UNIVERSITY OF OXFORD

Expression for the gradient: $\{c_k\} \to \nabla A$

- Gradient of the action with respect to the optimization variables $c_k = u_k + iv_k$ derived explicitly and used in the optimization algorithm

UNIVERSITY OF OXFORD

Expression for the gradient: $\{c_k\} \to \nabla A$

- Gradient of the action with respect to the optimization variables $c_k = u_k + iv_k$ derived explicitly and used in the optimization algorithm
- $\nabla A = (\partial A/\partial u_k, \partial A/\partial v_k)^T$

UNIVERSITY OF OXFORD

Expression for the gradient: $\{c_k\} \to \nabla A$

- Gradient of the action with respect to the optimization variables $c_k = u_k + iv_k$ derived explicitly and used in the optimization algorithm
- $\nabla A = (\partial A/\partial u_k, \partial A/\partial v_k)^T$
- $A = A_1 + A_2$ with

$$A_1 = \frac{n}{2}\int_0^{2\pi} |q'(t)|^2 dt, \quad A_2 = \frac{n}{2}\sum_{j=1}^{n-1}\int_0^{2\pi} \left|q(t) - q\left(t + \frac{2\pi j}{n}\right)\right|^{-1} dt$$

UNIVERSITY OF OXFORD

Expression for the gradient: $\{c_k\} \to \nabla A$

- Gradient of the action with respect to the optimization variables $c_k = u_k + iv_k$ derived explicitly and used in the optimization algorithm
- $\nabla A = (\partial A/\partial u_k, \partial A/\partial v_k)^T$
- $A = A_1 + A_2$ with

$$A_1 = \frac{n}{2}\int_0^{2\pi} \left|q'(t)\right|^2 dt, \quad A_2 = \frac{n}{2}\sum_{j=1}^{n-1}\int_0^{2\pi} \left|q(t) - q\left(t + \frac{2\pi j}{n}\right)\right|^{-1} dt$$

- Parseval's identity, chain rule and patience lead to

$$\frac{\partial A_1}{\partial u_k} = 2\pi n k^2 u_k,$$

UNIVERSITY OF OXFORD

Expression for the gradient: $\{c_k\} \to \nabla A$

- Gradient of the action with respect to the optimization variables $c_k = u_k + iv_k$ derived explicitly and used in the optimization algorithm
- $\nabla A = (\partial A/\partial u_k, \partial A/\partial v_k)^T$
- $A = A_1 + A_2$ with

$$A_1 = \frac{n}{2}\int_0^{2\pi} |q'(t)|^2 dt, \quad A_2 = \frac{n}{2}\sum_{j=1}^{n-1}\int_0^{2\pi} \left|q(t) - q\left(t + \frac{2\pi j}{n}\right)\right|^{-1} dt$$

- Parseval's identity, chain rule and patience lead to

$$\frac{\partial A_1}{\partial u_k} = 2\pi n k^2 u_k, \quad \frac{\partial A_2}{\partial u_k} = -\frac{n}{2}\sum_{j=1}^{n-1}\int_0^{2\pi} \frac{\frac{\partial f_j}{\partial u_k}}{f_j^2} dt, \quad |k| \leq \frac{N-1}{2},$$

$$f_j(u_k, v_k, t)^2 = \left(\sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} a_{k,j}(t)u_k + b_{k,j}(t)v_k\right)^2 + \left(\sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} (-b_{k,j}(t))u_k + a_{k,j}(t)v_k\right)^2$$

UNIVERSITY OF OXFORD

Optimization algorithm: $\{c_k\} \to \{c_k^*\}$

[5] M., Montagnac, and Gallard, "Gradient Span Analysis Method: Application to the Multipoint Aerodynamic Shape Optimization of a Turbine Cascade", 2015.

UNIVERSITY OF OXFORD

Optimization algorithm: $\{c_k\} \rightarrow \{c_k^*\}$

- Simó used Gradient methods

[5] M., Montagnac, and Gallard, "Gradient Span Analysis Method: Application to the Multipoint Aerodynamic Shape Optimization of a Turbine Cascade", 2015.

UNIVERSITY OF OXFORD

Optimization algorithm: $\{c_k\} \to \{c_k^*\}$

- Simó used Gradient methods

- Quasi-Newton methods: numerical optimization methods based on approximations of the Hessian (or its inverse), the most popular are SR1 and BFGS [5]

[5] M., Montagnac, and Gallard, "Gradient Span Analysis Method: Application to the Multipoint Aerodynamic Shape Optimization of a Turbine Cascade", 2015.

UNIVERSITY OF OXFORD

Optimization algorithm: $\{c_k\} \to \{c_k^*\}$

- Simó used Gradient methods
- Quasi-Newton methods: numerical optimization methods based on approximations of the Hessian (or its inverse), the most popular are SR1 and BFGS [5]
- Newton's method?

[5] M., Montagnac, and Gallard, "Gradient Span Analysis Method: Application to the Multipoint Aerodynamic Shape Optimization of a Turbine Cascade", 2015.

UNIVERSITY OF OXFORD

Optimization algorithm: $\{c_k\} \to \{c_k^*\}$

- Simó used Gradient methods
- Quasi-Newton methods: numerical optimization methods based on approximations of the Hessian (or its inverse), the most popular are SR1 and BFGS [5]
- Newton's method?
- Demo!

[5] M., Montagnac, and Gallard, "Gradient Span Analysis Method: Application to the Multipoint Aerodynamic Shape Optimization of a Turbine Cascade", 2015.

UNIVERSITY OF OXFORD

Overview

Method

UNIVERSITY OF OXFORD

Method

- ***First-variational equation:*** $q^*(t) \to L(t)$

UNIVERSITY OF OXFORD

Method

- ***First-variational equation:*** $q^*(t) \to L(t)$

  Coefficient matrix $L(t)$ of the first variational equation $Y' = L(t)Y$ generated by automatic differentiation

UNIVERSITY OF OXFORD

Method

- ***First-variational equation:*** $q^*(t) \to L(t)$

  Coefficient matrix $L(t)$ of the first variational equation $Y' = L(t)Y$ generated by automatic differentiation

- ***Floquet analysis:*** $L(t) \to \{\rho_i\}$

UNIVERSITY OF OXFORD

Method

- ***First-variational equation:*** $q^*(t) \to L(t)$

  Coefficient matrix $L(t)$ of the first variational equation $Y' = L(t)Y$ generated by automatic differentiation

- ***Floquet analysis:*** $L(t) \to \{\rho_i\}$

  Characteristic multipliers computed using a singular value decomposition of the differential operator which governs the equation

UNIVERSITY OF OXFORD

First-variational equation: $q^*(t) \to L(t)$

UNIVERSITY OF OXFORD

First-variational equation: $q^*(t) \to L(t)$

- Rewrite the n-body problem to a first-order problem:

$$Z'(t) = N(Z(t)), \quad t \in [0, 2\pi], \tag{1}$$

with $Z(t) = [Z_1(t), \ldots, Z_{2n}(t)]^T = [z_0(t), \ldots, z_{n-1}(t), z_0'(t), \ldots, z_{n-1}'(t)]^T$

UNIVERSITY OF OXFORD

First-variational equation: $q^*(t) \to L(t)$

- Rewrite the n-body problem to a first-order problem:

$$Z'(t) = N(Z(t)), \quad t \in [0, 2\pi], \tag{1}$$

with $Z(t) = [Z_1(t), \ldots, Z_{2n}(t)]^T = [z_0(t), \ldots, z_{n-1}(t), z_0'(t), \ldots, z_{n-1}'(t)]^T$ and

$$N(Z_j(t)) = \begin{cases} Z_{j+n}(t), \quad 1 \leq j \leq n, \\ \displaystyle\sum_{\substack{i=0 \\ i \neq j}}^{n-1} \frac{Z_{i-n}(t) - Z_{j-n}(t)}{\left|Z_{i-n}(t) - Z_{j-n}(t)\right|^3}, \quad n+1 \leq j \leq 2n \end{cases}$$

UNIVERSITY OF OXFORD

First-variational equation: $q^*(t) \to L(t)$

- Rewrite the n-body problem to a first-order problem:

$$Z'(t) = N(Z(t)), \quad t \in [0, 2\pi], \tag{1}$$

with $Z(t) = [Z_1(t), \ldots, Z_{2n}(t)]^T = [z_0(t), \ldots, z_{n-1}(t), z_0'(t), \ldots, z_{n-1}'(t)]^T$ and

$$N(Z_j(t)) = \begin{cases} Z_{j+n}(t), & 1 \le j \le n, \\ \displaystyle\sum_{\substack{i=0 \\ i \neq j}}^{n-1} \frac{Z_{i-n}(t) - Z_{j-n}(t)}{\left|Z_{i-n}(t) - Z_{j-n}(t)\right|^3}, & n+1 \le j \le 2n \end{cases}$$

- $q^*(t)$ choreography $\Leftrightarrow$ $\Phi(t)$ solution of (1) with

$$\Phi(t) = [z_0(t), \ldots, z_{n-1}(t), z_0'(t), \ldots, z_{n-1}'(t)]^T, \quad z_j(t) = q^*(t + \frac{2\pi j}{n})$$

UNIVERSITY OF OXFORD

First-variational equation: $q^*(t) \to L(t)$

- Rewrite the n-body problem to a first-order problem:

$$Z'(t) = N(Z(t)), \quad t \in [0, 2\pi], \tag{1}$$

with $Z(t) = [Z_1(t), \dots, Z_{2n}(t)]^T = [z_0(t), \dots, z_{n-1}(t), z_0'(t), \dots, z_{n-1}'(t)]^T$ and

$$N(Z_j(t)) = \begin{cases} Z_{j+n}(t), & 1 \le j \le n, \\ \displaystyle\sum_{\substack{i=0 \\ i \neq j}}^{n-1} \frac{Z_{i-n}(t) - Z_{j-n}(t)}{\left|Z_{i-n}(t) - Z_{j-n}(t)\right|^3}, & n+1 \le j \le 2n \end{cases}$$

- $q^*(t)$ choreography $\Leftrightarrow$ $\Phi(t)$ solution of (1) with

$$\Phi(t) = [z_0(t), \dots, z_{n-1}(t), z_0'(t), \dots, z_{n-1}'(t)]^T, \quad z_j(t) = q^*(t + \frac{2\pi j}{n})$$

- The stability of $\Phi(t)$ is determined by the first-variational equation

$$Y'(t) = L(t)Y(t), \quad L_{i,j}(t) = \frac{\partial N(Z_i(t))}{\partial Z_j(t)}(\Phi(t)), \quad 1 \le i,j \le 2n, \quad L(t+2\pi) = L(t)$$

UNIVERSITY OF OXFORD

First-variational equation: $q^*(t) \to L(t)$

- Rewrite the n-body problem to a first-order problem:

$$Z'(t) = N(Z(t)), \quad t \in [0, 2\pi], \tag{1}$$

with $Z(t) = [Z_1(t), \dots, Z_{2n}(t)]^T = [z_0(t), \dots, z_{n-1}(t), z_0'(t), \dots, z_{n-1}'(t)]^T$ and

$$N(Z_j(t)) = \begin{cases} Z_{j+n}(t), & 1 \leq j \leq n, \\ \displaystyle\sum_{\substack{i=0 \\ i \neq j}}^{n-1} \frac{Z_{i-n}(t) - Z_{j-n}(t)}{\left|Z_{i-n}(t) - Z_{j-n}(t)\right|^3}, & n+1 \leq j \leq 2n \end{cases}$$

- $q^*(t)$ choreography $\Leftrightarrow$ $\Phi(t)$ solution of (1) with

$$\Phi(t) = [z_0(t), \dots, z_{n-1}(t), z_0'(t), \dots, z_{n-1}'(t)]^T, \quad z_j(t) = q^*(t + \frac{2\pi j}{n})$$

- The stability of $\Phi(t)$ is determined by the first-variational equation

$$Y'(t) = L(t)Y(t), \quad L_{i,j}(t) = \frac{\partial N(Z_i(t))}{\partial Z_j(t)}(\Phi(t)), \quad 1 \leq i, j \leq 2n, \quad L(t + 2\pi) = L(t)$$

- $L(t)$ computed by automatic differentiation in Chebfun

UNIVERSITY OF OXFORD

Floquet analysis: $L(t) \to \{\rho_i\}$

UNIVERSITY OF OXFORD

Floquet analysis: $L(t) \to \{\rho_i\}$

- First, need to compute the principal fundamental matrix solution $\Psi(t)$ of

$$Y'(t) = L(t)Y(t), \quad L(t+2\pi) = L(t), \tag{2}$$

  i.e., need to solve $2n$ times a system of $2n$ linear equations, computationally expensive

UNIVERSITY OF OXFORD

Floquet analysis: $L(t) \to \{\rho_i\}$

- First, need to compute the principal fundamental matrix solution $\Psi(t)$ of

$$Y'(t) = L(t)Y(t), \quad L(t+2\pi) = L(t), \tag{2}$$

  i.e., need to solve $2n$ times a system of $2n$ linear equations, computationally expensive
- Then, characteristic multipliers $\{\rho_i\}$ = eigenvalues of $\Psi(2\pi)$ and stable iff $\max|\rho_i| \le 1$

UNIVERSITY OF OXFORD

Floquet analysis: $L(t) \to \{\rho_i\}$

- First, need to compute the principal fundamental matrix solution $\Psi(t)$ of

$$Y'(t) = L(t)Y(t), \quad L(t+2\pi) = L(t), \tag{2}$$

  i.e., need to solve $2n$ times a system of $2n$ linear equations, computationally expensive
- Then, characteristic multipliers $\{\rho_i\}$ = eigenvalues of $\Psi(2\pi)$ and stable iff $\max|\rho_i| \leq 1$
- Other approach: a matrix solution $\Delta(t)$ of (2) is a set of functions which maps the nullspace of the operator $Y(t) \mapsto Y'(t) - L(t)Y(t)$, can be calculated in Chebfun using a (continuous) singular value decomposition of the operator $Y(t)$

UNIVERSITY OF OXFORD

Floquet analysis: $L(t) \to \{\rho_i\}$

- First, need to compute the principal fundamental matrix solution $\Psi(t)$ of

$$Y'(t) = L(t)Y(t), \quad L(t+2\pi) = L(t), \tag{2}$$

  i.e., need to solve $2n$ times a system of $2n$ linear equations, computationally expensive
- Then, characteristic multipliers $\{\rho_i\}$ = eigenvalues of $\Psi(2\pi)$ and stable iff $\max|\rho_i| \leq 1$
- Other approach: a matrix solution $\Delta(t)$ of (2) is a set of functions which maps the nullspace of the operator $Y(t) \mapsto Y'(t) - L(t)Y(t)$, can be calculated in Chebfun using a (continuous) singular value decomposition of the operator $Y(t)$
- Once we have the matrix solution $\Delta(t)$,

$$\Psi(2\pi) = \Delta(2\pi)\Delta^{-1}(0)$$

UNIVERSITY OF OXFORD

Floquet analysis: $L(t) \to \{\rho_i\}$

- First, need to compute the principal fundamental matrix solution $\Psi(t)$ of

$$Y'(t) = L(t)Y(t), \quad L(t+2\pi) = L(t), \tag{2}$$

  i.e., need to solve $2n$ times a system of $2n$ linear equations, computationally expensive
- Then, characteristic multipliers $\{\rho_i\}$ = eigenvalues of $\Psi(2\pi)$ and stable iff $\max|\rho_i| \leq 1$
- Other approach: a matrix solution $\Delta(t)$ of (2) is a set of functions which maps the nullspace of the operator $Y(t) \mapsto Y'(t) - L(t)Y(t)$, can be calculated in Chebfun using a (continuous) singular value decomposition of the operator $Y(t)$
- Once we have the matrix solution $\Delta(t)$,

$$\Psi(2\pi) = \Delta(2\pi)\Delta^{-1}(0)$$

	$n = 2$	3	4	5
Circle	1.00000	85.0197	221.540	365.304