Fast solution of stiff PDEs in 1D, 2D and 3D

British Applied Mathematics Colloquium 2016

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UNIVERSITY OF OXFORD

Introduction

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$$u_t = \mathcal{L}u + \mathcal{N}(u), \quad t \in [0, T], \quad X \in [0, 2\pi]^d \quad (d = 1, 2, 3),$$

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$$\begin{aligned}
\hat{v}_1 &= \hat{u}_n, \\
\hat{v}_2 &= e^{C_2 h\mathbf{L}}\hat{u}_n + hA_{2,1}(h\mathbf{L})\mathbf{N}(\hat{v}_1), \\
\hat{v}_3 &= e^{C_3 h\mathbf{L}}\hat{u}_n + hA_{3,1}(h\mathbf{L})\mathbf{N}(\hat{v}_1) + hA_{3,2}(h\mathbf{L})\mathbf{N}(\hat{v}_2), \\
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Each scheme is characterized by its coefficients $A_{i,j}$, B_i and C_i which are functions of $\mathbf{L}$

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- ETDRK4 Cox & Matthews formula (2002), fourth-order accurate, is given by

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\hat{v}_1 &= \hat{u}_n, \\
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\hat{u}_{n+1} = e^{h\mathbf{L}}\hat{u}_n + h^{-2}\mathbf{L}^{-3}\Big[&(-4\mathbf{I} - h\mathbf{L} + e^{h\mathbf{L}}(4\mathbf{I} - 3h\mathbf{L} + (h\mathbf{L})^2))\mathbf{N}(\hat{v}_1) \\
+ \quad &(-4\mathbf{I} - h\mathbf{L} + e^{h\mathbf{L}}(4\mathbf{I} - 3h\mathbf{L} + (h\mathbf{L})^2))(\mathbf{N}(\hat{v}_2) + \mathbf{N}(\hat{v}_3)) \\
+ \quad &(-4\mathbf{I} - 3h\mathbf{L} - (h\mathbf{L})^2 + e^{h\mathbf{L}}(4 - h\mathbf{L}))\mathbf{N}(\hat{v}_4)\Big]
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- We have compared 30 exponential integrators of fourth and higher order and found that the ETDRK4 formula is hard to beat

Exponential integrators (3/7)

Method	Type	Order	Stages s	Steps q
ABNørsett4	ETD Adams-Bashforth	4	1	4
ABNørsett5	ETD Adams-Bashforth	5	1	5
ABNørsett6	ETD Adams-Bashforth	6	1	6
Friedli (VRK4)	ETD Runge-Kutta	4	4	1
Strehmel-Weiner	ETD Runge-Kutta	4	4	1
Cox-Matthews (ETDRK4)	ETD Runge-Kutta	4	4	1
Krogstad (ETDRK4-B)	ETD Runge-Kutta	4	4	1
Minchev	ETD Runge-Kutta	4	4	1
Hochbruck-Ostermann	ETD Runge-Kutta	4	5	1
Luan-Ostermann (EXPRK5S8)	ETD Runge-Kutta	5	8	1
(Mod)GenLawson41	(Mod.) Gen. Lawson	4	4	1
(Mod)GenLawson42	(Mod.) Gen. Lawson	4	4	2
(Mod)GenLawson43	(Mod.) Gen. Lawson	4	4	3
(Mod)GenLawson44	(Mod.) Gen. Lawson	5	4	4
(Mod)GenLawson45	(Mod.) Gen. Lawson	6	4	5
PEC423	Predictor-corrector	4	2	3
PECEC433	Predictor-corrector	4	3	3
PEC524	Predictor-corrector	5	2	4
PECEC534	Predictor-corrector	5	3	4
PEC625	Predictor-corrector	6	2	5
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Exponential integrators (4/7)

Example 1: ETD RK schemes for KdV equation from $t = 0$ to $t = 0.005$

We plot relative error $= \frac{\|u_{\textbf{approx}} - u_{\textbf{exact}}\|_\infty}{\|u_{\textbf{exact}}\|_\infty}$ vs relative time-step $= \frac{dt}{0.005}$ and time

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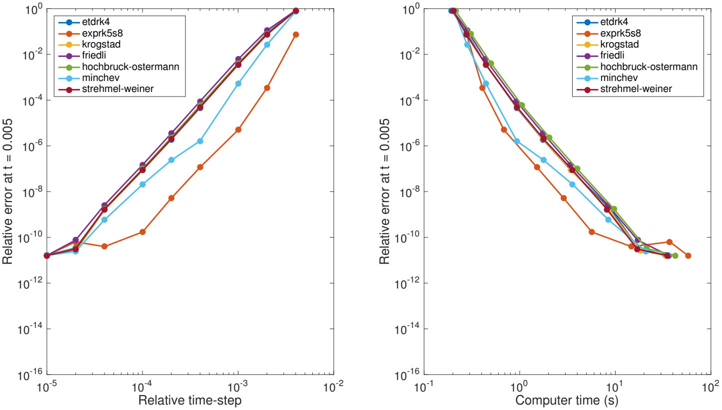

Exponential integrators (5/7)

Example 2: ETD RK schemes for KdV equation from $t = 0$ to $t = 0.015$

Exponential integrators (5/7)

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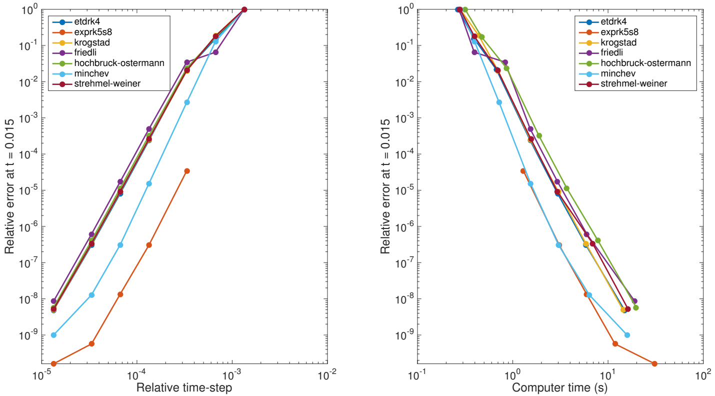
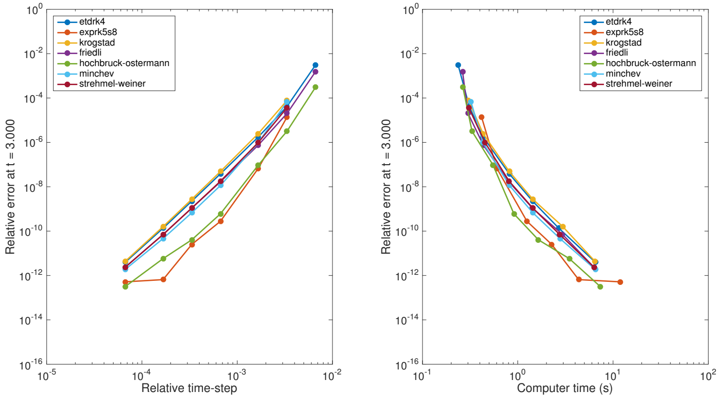


Exponential integrators (6/7)

Example 3: ETD RK schemes for Cahn-Hilliard equation from $t = 0$ to $t = 3$

Exponential integrators (6/7)

Example 3: ETD RK schemes for Cahn-Hilliard equation from $t = 0$ to $t = 3$

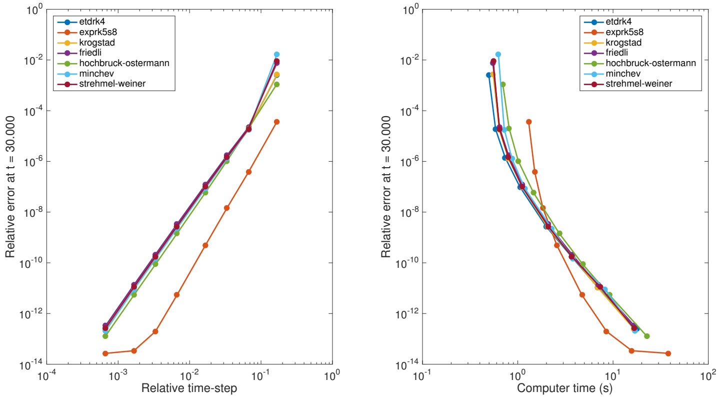


Exponential integrators (7/7)

Example 4: ETD RK schemes for Gray-Scott equations in 2D from $t = 0$ to $t = 30$

Exponential integrators (7/7)

Example 4: ETD RK schemes for Gray-Scott equations in 2D from $t = 0$ to $t = 30$



In Chebfun: spin, spin2 and spin3

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- **Preloaded demos:** Nine demos in 1D, four in 2D, four in 3D

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u = spin('kdv');

u = spin2('gs2');

u = spin3('sh3');
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u = spin('kdv');

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```

- **Define your own problem:**

```
dom = [0 32*pi]; tspan = [0 100];
S = spinop(dom, tspan);
S.linearPart = @(u) -diff(u,2)-diff(u,4);
S.nonlinearPart = @(u) -.5*diff(u.^2);
S.init = chebfun(@(x) cos(x/16).*(1 + sin(x/16)), dom, 'trig');
u = spin(S);
```

Conclusion

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- Future work includes stiff PDEs on the sphere with Grady Wright (Boise State University), Alex Townsend (MIT) and Paul Matthews (University of Nottingham)