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Computing planar and spherical choreographies

17th Biennial Oxford/Cambridge Applied Mathematics Meeting

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Collaborator: Nikola I. Gushterov (Oxford)

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Overview

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Minimization of the action (Simó)

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- Based on the principle of least action applied to the n-body problem

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Choreographies of the n-body problem

$$z_j''(t) - \sum_{\substack{i=0 \\ i \neq j}}^{n-1} \frac{z_i(t) - z_j(t)}{\left|z_i(t) - z_j(t)\right|^3} = 0, \quad 0 \leq j \leq n-1,$$

with $z_j(t) = q\left(t + \frac{2\pi j}{n}\right)$ for some 2π-periodic function $q(t) : [0, 2\pi] \to \mathbb{C}$

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Minima of the action

$$A = \int_0^{2\pi} \left(K(t) - U(t)\right) dt,$$

$$K(t) = \frac{n}{2}\left|q'(t)\right|^2, \quad U(t) = -n \sum_{j=1}^{n-1} \left|q(t) - q\left(t + \frac{2\pi j}{n}\right)\right|^{-1}$$

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Overview

Method

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- ***Trigonometric interpolation:*** $q(t) \to p_N(t) \to A_N$

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$q(t) \approx p_N(t)$ instead of $q(t) = x(t) + iy(t) \approx x_N(t) + iy_N(t)$

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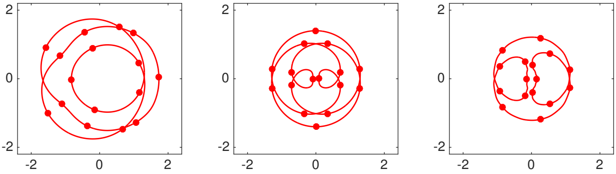

Trigonometric Interpolation: $q(t) \to p_N(t) \to A_N$

[1] Wright, Javed, Montanelli, and Trefethen, "Extension of Chebfun to periodic functions", 2015.

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Trigonometric Interpolation: $q(t) \to p_N(t) \to A_N$

- Trigonometric interpolant $p_N(t)$ of $q(t)$ at N points:

$$p_N(t) = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_k e^{ikt}, \quad t \in [0, 2\pi], \quad c_k = \frac{1}{N} \sum_{j=0}^{N-1} q_j e^{-ikt_j}, \quad |k| \le \frac{N-1}{2}$$

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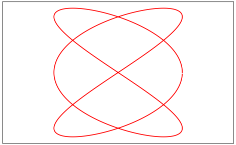
- Recent extension of Chebfun to periodic functions [1]

```
>> q = chebfun(@(t)cos(3*pi*t)+1i*sin(2*pi*t),'trig')
q =
   chebfun column (1 smooth piece)
       interval          length   endpoint values trig
[      -1,         1]         7    complex values
```



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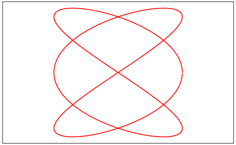

- Action computed with the exponentially accurate trapezoidal rule

$$A_N = \frac{n}{2} \int_0^{2\pi} |p_N'(t)|^2 dt + n \sum_{j=1}^{n-1} \int_0^{2\pi} \left| p_N(t) - p_N\left(t + \frac{2\pi j}{n}\right) \right|^{-1} dt$$

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Numerical results

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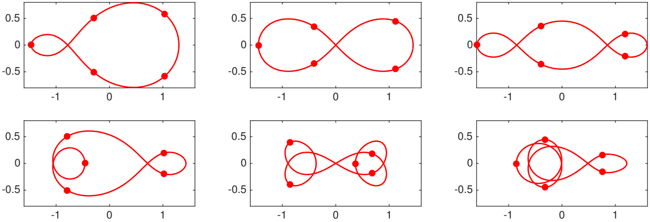
Numerical results



Action	68.8516	71.3312	77.1588	88.4397	109.6366	119.3191
Computer time (s)	0.79	0.49	0.44	0.70	0.98	0.86
2-norm of the gradient	1.26e-02	1.39e-02	8.87e-03	9.68e-03	1.18e-02	1.28e-02
Smallest coefficient	4.71e-06	6.45e-08	2.26e-06	3.33e-06	2.08e-05	2.75e-05
∞-norm of the residual	9.31e-02	1.09e-03	1.30e-02	4.43e-02	2.83e-01	6.56e-01

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Overview

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Spherical n-body problem

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Spherical n-body problem

Equations of the motion

$$X_j''(t) - \sum_{\substack{i=0 \\ i\neq j}}^{n-1} \frac{R^3 X_i(t) - R(X_i(t)\cdot X_j(t))X_j(t)}{\left[R^4 - (X_i(t)\cdot X_j(t))^2\right]^{3/2}} + R^{-2}\left\|X_j'(t)\right\|^2 X_j(t) = 0,$$

with $X_j(t) \in \mathbb{S}^2_R = \{X \in \mathbb{R}^3, \|X\| = R\} \subset \mathbb{R}^3$, $0 \leq j \leq n-1$

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Potential

$$U(t) = -\frac{1}{R}\sum_{j=0}^{n-1}\sum_{\substack{i=0 \\ i\neq j}}^{n-1} \cot\frac{d_{GC}(X_i(t), X_j(t))}{R},$$

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with

$$d_{GC}(X_i(t), X_j(t)) = R \arccos\frac{X_i(t)\cdot X_j(t)}{R^2}$$

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Spherical choreographies

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Choreographies of the spherical n-body problem

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$$A = \int_0^{2\pi} \left(K(t) - U(t)\right) dt,$$

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Computing spherical choreographies

[2] Montanelli and Gushterov, "Computing planar and spherical choreogprahies", 2015.

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Computing spherical choreographies

- Our method [2] is based on stereographic projection

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$\mathbb{S}^2_R$	$\mathbb{C}$
$X = (x_1, x_2, x_3)^T$	$z = P_R(X) = \dfrac{Rx_1 + iRx_2}{R - x_3}$

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$d(X, Y) = \lVert X - Y \rVert$	$d(z, \xi) = \dfrac{2R^2\lvert z - \xi\rvert}{\sqrt{(R^2 + \lvert z\rvert^2)(R^2 + \lvert\xi\rvert^2)}}$

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$d(X, Y) = \\|X - Y\\|$	$d(z, \xi) = \dfrac{2R^2\|z - \xi\|}{\sqrt{(R^2 + \|z\|^2)(R^2 + \|\xi\|^2)}}$
$d_{GC}(X, Y) = R \arccos(\dfrac{X \cdot Y}{R^2})$	$d_{GC}(z, \xi) = 2R \arcsin \dfrac{d(z, \xi)}{2R}$

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$d_{GC}(X, Y) = R \arccos(\dfrac{X \cdot Y}{R^2})$	$d_{GC}(z, \xi) = 2R \arcsin \dfrac{d(z, \xi)}{2R}$

$$A = \frac{n}{2}\int_0^{2\pi} \left(\frac{2R^2|q'(t)|}{R^2 + |q(t)|^2}\right)^2 dt + \frac{n}{R}\sum_{j=1}^{n-1}\int_0^{2\pi} \frac{2R^2 - d(q(t), z_j(t))^2}{d(q(t), z_j(t))\sqrt{4R^2 - d(q(t), z_j(t))^2}}\, dt$$

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$X = (x_1, x_2, x_3)^T$	$z = P_R(X) = \dfrac{Rx_1 + iRx_2}{R - x_3}$
$Q(t),\ X_j(t) = Q\left(t + \frac{2\pi j}{n}\right)$	$q(t) = P_R(Q(t)),\ z_j(t) = P_R(X_j(t))$
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- Trigonometric interpolation, closed-form of the gradient, quasi-Newton methods

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Numerical results (1/2)

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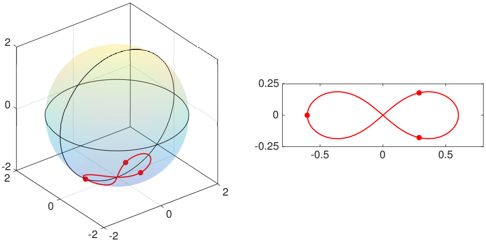

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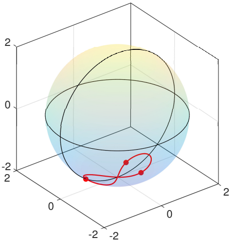

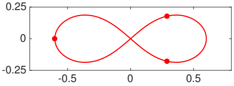

Action	22.036690
Number of coefficients	45
Computer time (s)	0.64
2-norm of the gradient	1.14e-02
Smallest coefficient	8.57e-07
∞-norm of the residual	1.07e-03

Numerical results (2/2)

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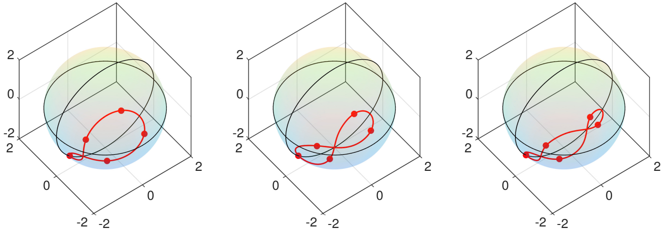

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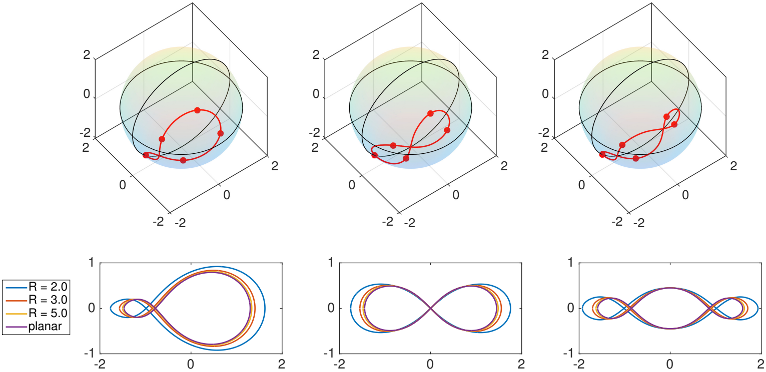

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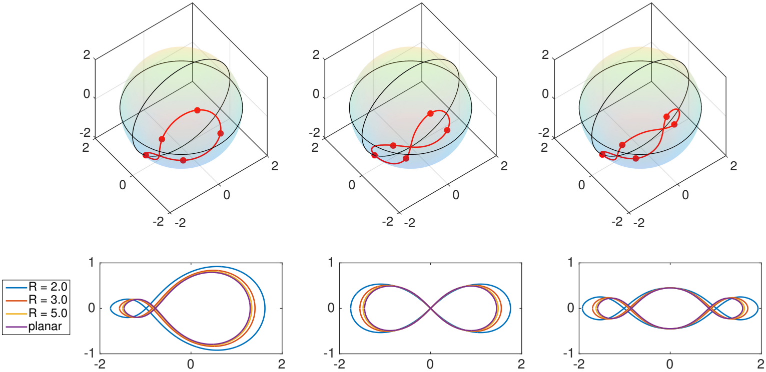
Numerical results (2/2)



	$R = 2$	3	5	10	20
Left	3.23e-01	1.05e-01	3.38e-02	8.08e-03	1.99e-03
Middle	3.06e-01	1.04e-01	3.39e-02	8.16e-03	2.02e-03
Right	3.35e-01	1.12e-01	3.64e-02	8.77e-03	1.97e-03

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Conclusion and future work

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- Choreographies are very special periodic solutions of the n-body problem characterized by a single orbit

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- Choreographies are very special periodic solutions of the *n*-body problem characterized by a single orbit
- They also exist on a sphere in a cotangent potential, and as in the plane, they can be computed using trigonometric interpolation and minimization of the action

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- Choreographies are very special periodic solutions of the n-body problem characterized by a single orbit
- They also exist on a sphere in a cotangent potential, and as in the plane, they can be computed using trigonometric interpolation and minimization of the action
- Stability properties of spherical choreographies have not been discussed

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- They also exist on a sphere in a cotangent potential, and as in the plane, they can be computed using trigonometric interpolation and minimization of the action
- Stability properties of spherical choreographies have not been discussed
- Apply these ideas to the n-vortex problem

$$z_j'(t) = \frac{i}{2\pi} \sum_{\substack{k=0 \\ k\neq j}}^{n-1} \Gamma_k \frac{z_j(t) - z_k(t)}{|z_j(t) - z_k(t)|^2}, \quad 0 \leq j \leq n-1$$