Supervised learning

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Context

Setup Consider data $\mathcal{X} \subseteq \mathbb{R}^d$, labels $\mathcal{Y} \subseteq \mathbb{R}$ and a classifier $f : \mathcal{X} \mapsto \mathcal{Y}$. We approximate f by $\hat{f}$ using n labelled data $(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_n, y_n) \in \mathcal{X}_0 \times \mathcal{Y}$, $\mathcal{X}_0 \subset \mathcal{X}$.

Bias-variance tradeoff For $y = f(\boldsymbol{x}) + \epsilon$, $\epsilon \sim \mathrm{N}(0, \sigma^2)$,

$$\mathrm{E}([y - \hat{f}(\boldsymbol{x})]^2) = [\mathrm{Bias}(\hat{f}(\boldsymbol{x}))]^2 + \mathrm{var}(\hat{f}(\boldsymbol{x})) + \sigma^2,$$
$$\mathrm{Bias}(\hat{f}(\boldsymbol{x})) = \mathrm{E}(\hat{f}(\boldsymbol{x})) - f(\boldsymbol{x}),$$
$$\mathrm{var}(\hat{f}(\boldsymbol{x})) = \mathrm{E}([\hat{f}(\boldsymbol{x}) - \mathrm{E}(\hat{f}(\boldsymbol{x}))]^2).$$

1 Naive Bayes $\mathcal{Y} = \{0, 1\}$

$$\hat{f}(\boldsymbol{x}) = \arg\max_{y \in \mathcal{Y}} \left(\mathrm{P}(Y = y) \prod_{j=1}^{d} \mathrm{P}(X_j = x_j | Y = y) \right)$$

Training $\mathcal{O}(nd)$

- Compute the class prior $\mathrm{P}(Y = y)$ from $\mathcal{X}_0$.
- Choose a model for each $\mathrm{P}(X_j = x_j | Y = y)$.
- Use the closed-form for the MLE to find the parameters for each $\mathrm{P}(X_j = x_j | Y = y)$.

Classifying $\mathcal{O}(d)$ Evaluate $\hat{f}(\boldsymbol{x})$.

2 k-NNs $\mathcal{Y} = \{0, 1\}$

$$\hat{f}(\boldsymbol{x}) = \mathrm{label}\left(\arg\min_{1 \leq i \leq n} d_{\boldsymbol{W}}(\boldsymbol{x}, \boldsymbol{x}_i) \right) \quad (k = 1)$$

Training $\mathcal{O}(nd^2 s + n\log(n)d)$

- Find distance $d_{\boldsymbol{W}}(\boldsymbol{x}_i, \boldsymbol{x}_j) = (\boldsymbol{x}_i - \boldsymbol{x}_j)^T \boldsymbol{W} (\boldsymbol{x}_i - \boldsymbol{x}_j)$, with s steps of BFGS, that best separates the data.
- Choose k. Find medians to produce k-d trees.

Classifying $\mathcal{O}(\log(n)d)$ Evaluate $\hat{f}(\boldsymbol{x})$.

3 Perceptron $\mathcal{Y} = \{-1, 1\}$

$$\hat{f}(\boldsymbol{x}) = \mathrm{sign}\left(\boldsymbol{w}^T \boldsymbol{x} + w_0\right)$$

Training $\mathcal{O}(ndP)$, with $P = \max_{1 \leq i \leq n} \|\boldsymbol{x}_i\|^2/\gamma^2$ (margin γ)

- Initialize $\boldsymbol{w} = \boldsymbol{0}$ and $w_0 = 0$.
- For $k = 1, 2, \ldots,$
  - if there is $\boldsymbol{x}_i \in \mathcal{X}_0$ such that
  $$\mathrm{sign}\left(\boldsymbol{w}^T \boldsymbol{x}_i + w_0\right) \neq y_i,$$
  set $\boldsymbol{w} = \boldsymbol{w} + y_i \boldsymbol{x}_i$ and $w_0 = w_0 + y_i$.

Classifying $\mathcal{O}(d)$ Evaluate $\hat{f}(\boldsymbol{x})$.

4 Kernel perceptron $\mathcal{Y} = \{-1, 1\}$

$$\hat{f}(\boldsymbol{x}) = \mathrm{sign}\left(\sum_{i=1}^{n} \alpha_i y_i K(\boldsymbol{x}_i, \boldsymbol{x}) \right)$$

Training $\mathcal{O}(ndP)$, with $P = \max_{1 \leq i \leq n} \|\boldsymbol{x}_i\|^2/\gamma^2$ (margin γ)

- Choose K. (Perceptron is $K(\boldsymbol{x}, \boldsymbol{y}) = (1 + \boldsymbol{x}^T \boldsymbol{y})$.)
- Initialize $\boldsymbol{\alpha} = \boldsymbol{0}$.
- For $k = 1, 2, \ldots,$
  - set $\alpha_i = \alpha_i + 1$ if there is $\boldsymbol{x}_i \in \mathcal{X}_0$ such that
  $$\mathrm{sign}\left(\sum_{j=1}^{n} \alpha_j y_j K(\boldsymbol{x}_j, \boldsymbol{x}_i) \right) \neq y_i.$$

Classifying $\mathcal{O}(nd)$ Evaluate $\hat{f}(\boldsymbol{x})$.

5 SVMs $\mathcal{Y} = \{-1, 1\}$

$$\hat{f}(\boldsymbol{x}) = \mathrm{sign}\left(\boldsymbol{w}^T \boldsymbol{x} + w_0\right)$$

Training Minimize, with s steps of PGD,

$$J(\boldsymbol{w}) = \frac{1}{2}\|\boldsymbol{w}\|^2,$$

such that $y_i(\boldsymbol{w}^T \boldsymbol{x}_i + w_0) \geq 1$, $1 \leq i \leq n$.

Classifying $\mathcal{O}(d)$ Evaluate $\hat{f}(\boldsymbol{x})$.

6 Kernel SVMs $\mathcal{Y} = \{-1, 1\}$

$$\hat{f}(\boldsymbol{x}) = \mathrm{sign}\left(\sum_{i=1}^{n} \alpha_i y_i K(\boldsymbol{x}_i, \boldsymbol{x}) + w_0 \right)$$

Training

- Choose K. (SVMs is $K(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^T \boldsymbol{y}$.)
- Maximize, with s steps of SQP,
$$J(\boldsymbol{\alpha}) = \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j K(\boldsymbol{x}_j, \boldsymbol{x}_i),$$
$$\text{such that } \sum_{i=1}^{n} \alpha_i y_i = 0 \text{ and } \alpha_i \geq 0,\ 1 \leq i \leq n.$$
- For i with $\alpha_i \neq 0$, $w_0 = y_i - \sum_{j=1}^{n} \alpha_j y_j K(\boldsymbol{x}_j, \boldsymbol{x}_i)$.

Classifying $\mathcal{O}(nd)$ Evaluate $\hat{f}(\boldsymbol{x})$.

7 Logistic regression $\mathcal{Y} = [0, 1]$

$$\hat{f}(\boldsymbol{x}) = \frac{1}{1 + e^{-(\boldsymbol{w}^T\boldsymbol{x} + w_0)}}$$

Training $\mathcal{O}(nd^3 s)$ Minimize the likelihood with s steps of Newton's method.

Classifying $\mathcal{O}(d)$ Evaluate $\hat{f}(\boldsymbol{x})$.

8 Neural networks $\mathcal{Y} = [0, 1]$

$$\hat{f}(\boldsymbol{x}) = \sigma\left(\boldsymbol{w}^{(2)T}\sigma\left(\boldsymbol{W}^{(1)T}\boldsymbol{x} + \boldsymbol{w}_0^{(1)}\right) + w_0^{(2)}\right) \quad \text{(shallow)}$$

Training $\mathcal{O}(ndNs)$ Minimize, with s steps of GD,

$$J(\boldsymbol{w}_0^{(1)}, \boldsymbol{W}^{(1)}, w_0^{(2)}, \boldsymbol{w}^{(2)}) = \frac{1}{n}\sum_{i=1}^{n} L\left(\hat{f}(\boldsymbol{x}_i), y_i\right).$$

Note that $\boldsymbol{w}_0^{(1)}$ is $N \times 1$, $\boldsymbol{W}^{(1)}$ is $d \times N$ and $\boldsymbol{w}^{(2)}$ is $N \times 1$.

Classifying $\mathcal{O}(dN)$ Evaluate $\hat{f}(\boldsymbol{x})$.

9 Decision trees

$$\hat{f}(\boldsymbol{x}) = \mathrm{label}\left(C(\boldsymbol{x})\right)$$

Training Find feature and threshold that maximize

$$u(C) - (p_L u(C_L) + p_R u(C_R),\ u(C) = -\sum_{y\in\mathcal{Y}} p_y \log p_y.$$

Classifying $\mathcal{O}(\log(n)d)$ Evaluate $\hat{f}(\boldsymbol{x})$.

10 Random forests

$$\hat{f}(\boldsymbol{x}) = \frac{1}{T}\sum_{t=1}^{T} \mathrm{label}\left(C_t(\boldsymbol{x})\right)$$

Training Sample T times, with replacement, $m < n$ training examples from $\mathcal{X}_0$ to construct T decision trees.

Classifying $\mathcal{O}(\log(n)dT)$ Evaluate $\hat{f}(\boldsymbol{x})$.