Numerical Analysis (5/7)
Ordinary Differential equations

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Master in Mathematics

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Outline

1. Introduction
2. Euler Methods
3. One step methods
4. Multi-step methods
5. Additional considerations
6. Summary

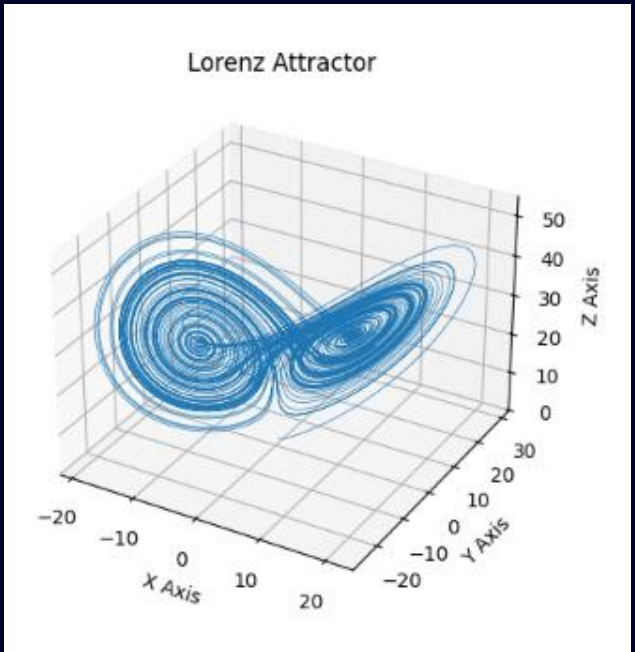
1. Introduction

A famous example – The Lorenz system

Atmospheric convection, the "Butterfly effect"

$$x'(t) = \sigma(y(t) - x(t))$$
$$y'(t) = x(t)(\rho - z) - y(t)$$
$$z'(t) = x(t)y(t) - \beta z(t)$$

Numeric test: $\rho = 28$, $\sigma = 10$, $\beta = 8/3$, $(x_0, y_0, z_0) = (1, 1, 1)$, $T = 40$, $t_0 = 0$



https://matplotlib.org/stable/gallery/mplot3d/lorenz_attractor.html

```python
import matplotlib.pyplot as plt
import numpy as np


def lorenz(xyz, *, s=10, r=28, b=2.667):
    """
    Parameters
    ----------
    xyz : array-like, shape (3,)
       Point of interest in three-dimensional space.
    s, r, b : float
       Parameters defining the Lorenz attractor.

    Returns
    -------
    xyz_dot : array, shape (3,)
       Values of the Lorenz attractor's partial derivatives at *xyz*.
    """
    x, y, z = xyz
    x_dot = s*(y - x)
    y_dot = r*x - y - x*z
    z_dot = x*y - b*z
    return np.array([x_dot, y_dot, z_dot])


dt = 0.01
num_steps = 10000

xyzs = np.empty((num_steps + 1, 3))  # Need one more for the initial values
xyzs[0] = (0., 1., 1.05)  # Set initial values
# Step through "time", calculating the partial derivatives at the current point
# and using them to estimate the next point
for i in range(num_steps):
    xyzs[i + 1] = xyzs[i] + lorenz(xyzs[i]) * dt

# Plot
ax = plt.figure().add_subplot(projection='3d')

ax.plot(*xyzs.T, lw=0.5)
ax.set_xlabel("X Axis")
ax.set_ylabel("Y Axis")
ax.set_zlabel("Z Axis")
ax.set_title("Lorenz Attractor")

plt.show()
```

Definition

- A first-order ODE is an equation of the form

$$y'(t) = f\left(t, y(t)\right), \quad \forall\ t \in I$$

where I is an interval of $\mathbb{R}$, $y : [0, +\infty[\to \mathbb{R}^N$ is a vectorial function depending on the variable t and f is a map from $I \times \mathbb{R}^N$ onto $\mathbb{R}^N$.

- An ODE of order p is an equation of the form

$$y^{(p)}(t) = f\left(t, y(t), y'(t), \ldots, y^{(p-1)}(t)\right), \quad \forall\ t \in I$$

where I is an interval of $\mathbb{R}$, $y : [0, +\infty[\to \mathbb{R}^N$ is a vectorial function with respect to t and f is an application from $I \times (\mathbb{R}^N)^p$ to $\mathbb{R}^N$.

Remark

Any ODE of order p can be written as a first-order ODE.

Indeed, by setting

$$x_1(t) = y(t),\ x_2(t) = y'(t),\ x_3(t) = y''(t), \ldots, x_p(t) = y^{(p-1)}(t)$$

the problem writes

$$x_1'(t) = x_2(t),\ x_2'(t) = x_3(t), \ldots, x_{p-1}'(t) = x_p(t)$$

and

$$x_p'(t) = f\left(t, x_1(t), x_2(t), \ldots, x_p(t)\right)$$

Remark

which can be written, by setting

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_p(t) \end{pmatrix} \quad \text{and} \quad F(t, X) = \begin{pmatrix} X_2 \\ X_3 \\ \vdots \\ f\left(t, x_1, x_2, \ldots, x_p\right) \end{pmatrix}$$

$$X'(t) = F\left(t, X(t)\right)$$

Example

Consider $y''(t) + \omega^2 y(t) = g(t)$. Define $x_1 = y, x_2 = y'$, such as

$$X'(t) = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix} = \begin{pmatrix} x_2 \\ -\omega^2 x_1 + g(t) \end{pmatrix} = F\left(t, X(t)\right)$$

Definition – Initial Value Problem

- For an interval I, $f : I \times \mathbb{R}^N \to \mathbb{R}^N$, $t_0 \in I$ and $y^0 \in \mathbb{R}^N$, solving the Cauchy problem

$$\begin{cases} y'(t) = f(t, y(t)), & \forall\, t \in I \\ y(t_0) = y^0 \end{cases}$$

means to determine all functions $y : I \to \mathbb{R}^N$ solutions to the ODE satisfying $y(t_0) = y_0$.

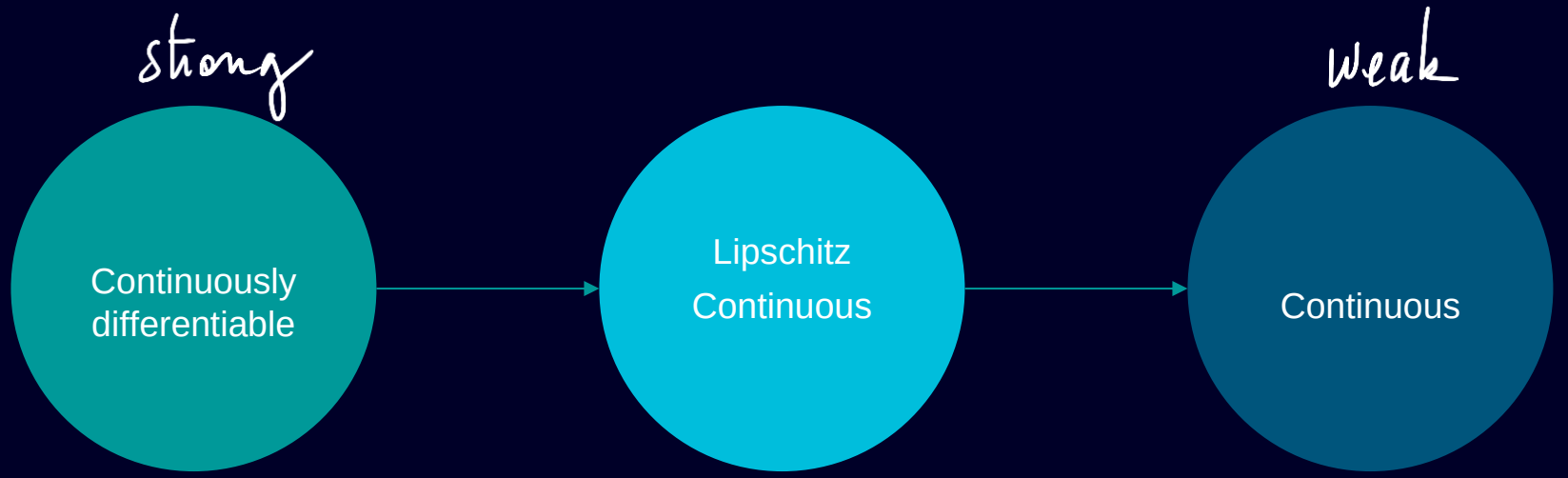
We also talk about Initial Value Problem (IVP)

Stability of first order IVP

If f is continuous in t and Lipschitz continuous in y, i.e. $|\partial_y f(t, y)| \leq L$ for $t \in I$, then the IVP has a unique solution in I. Moreover for two solutions (y_1, y_2) with different initial conditions we have

$$|y_1(t) - y_2(t)| \leq e^{L(t-t_0)} |y_1(t_0) - y_2(t_0)|$$

Take away



- Lipschitz ➔ limited in how fast in can change (bounded derivative)

$$|f(t,u) - f(t,u^*)| \leq L|u-u^*| \quad \forall u, u^* \in \mathbb{R}$$

- $f(t,u)$ continuous in t
- $f(t,u)$ Lipschitz continuous in u

} $u' = f(t,u)$ is well posed!

A simple example

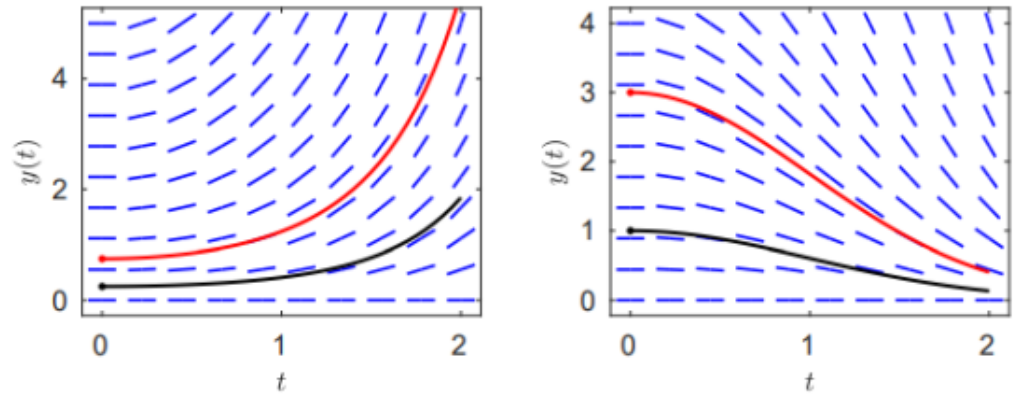
- For $a \in \mathbb{R}$, the IVP

$$\begin{cases} y'(t) = ay(t), \ \forall t > 0 \\ y(0) = y_0 \end{cases}$$

has the unique solution $y(t) = y_0 e^{at}$

- more generally, if a is a continuous function on $[0, +\infty[$, the IVP has the solution

$$y(t) = y_0 \exp\left(\int_0^t a(\sigma)\, d\sigma\right)$$



Difference in two solutions that start at nearby points for $y' = ty$ (left) and $y' = -ty$ (right)

Examples of ODEs

1. **Linear ODE**: $u' = -2u \Rightarrow$ solution is $u(t) = C_0 e^{-2t}$
2. **Separable ODE**: $u' = u^2 \Rightarrow$ solution is $u(t) = 1 / (C_0 - t)$
3. **Logistic Growth Model**: $u' = u(1 - u) \Rightarrow$ solution is $u(t) = C_0 e^t / (1 + C_0 e^t)$
4. **Exponential Growth/Decay**: $u' = -ku \Rightarrow$ solution is $u(t) = C_0 e^{-kt}$
5. **Harmonic Oscillator**: $u'' = -\omega^2 u \Rightarrow$ solution is $u(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$
6. **Pendulum**: $u'' + \frac{l}{g}\sin(u) = 0 \Rightarrow$ closed form only for small displacements $\sin(u) \approx u$ (Harmonic oscillator)
7. **Riccatti**: $u' = a(t) + b(t)u + c(t)u^2 \Rightarrow$ no closed form solution

These ODEs appear in a variety of disciplines such as physics, biology, engineering, and economics

- C_0, C_1, C_2 are the constants of integration

SIEMENS

Numerical approximation

We try to numerically solve the IVP, which means that we look for an approximate solution to

$$\begin{cases} y'(t) = f\left(t, y(t)\right), \ 0 \leq t \leq T \\ y\left(0\right) = y_0 \end{cases}$$

with $y_0 \in \mathbb{R}^N$ and $f : [0, +\infty[\times \mathbb{R}^N \to \mathbb{R}^N$

We remark that this problem is equivalent to

$$y(t) = y_0 + \int_0^t f(s, y(s))ds \quad \forall \ t \in [0, T]$$

Therefore, it is sufficient to obtain a numerical approximation to

$$\int_0^t f(s, y(s))ds$$

and, to this end we can use the ideas and methods from **numerical integration**.

Numerical approximation

For a subdivision

$$0 = t_0 < t_1 < t_2 < \ldots < t_N = T$$

our problem implies that

$$\begin{cases} y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(t, y(t))dt, & \forall\ 0 \leq n \leq N-1 \\ y(0) = y_0 \end{cases}$$

The numerical methods differ by the choice of the evaluation of the integrals

$$\int_{t_n}^{t_{n+1}} f(t, y(t))dt$$

Remark

The integrand depends on y itself, which makes the integration more complicated

2. Euler Methods

The forward Euler problem

- let us assume that our Cauchy problem admits one solution y on $[0, T]$.
- we introduce the subdivision $0 = t_0 < t_1 < \ldots < t_N = T$ and $h_n = t_{n+1} - t_n$
- Let us recall that our problems imply

$$\begin{cases} y(t_{n+1}) = y(t_n) + \displaystyle\int_{t_n}^{t_{n+1}} f(t, y(t))dt, \quad \forall\ 0 \leq n \leq N-1 \\ y(0) = y_0 \end{cases}$$

- the forward Euler method (explicit) corresponds to an approximation by the left rectangle quadrature rule

$$\int_{t_n}^{t_{n+1}} f(t, y(t))dt \simeq h_n f(t_n, y(t_n)).$$

- we then obtain, where $\tilde{y}_n$ is an approximation of $y(t_n)$,

$$\begin{cases} \tilde{y}_{n+1} = \tilde{y}_n + h_n f(t_n, \tilde{y}_n), \quad \forall\ 0 \leq n \leq N-1 \\ y_0 = \tilde{y}_0 \end{cases}$$

Convergence

It is necessary to analyze in which sense the computed value $\tilde{y}_n$ is sufficiently close to the exact value $y(t_n)$ and so we want to evaluate the discretization error

$$e_n = y(t_n) - \tilde{y}_n.$$

Definition

we say that the method is converging if

$$\max_{0 \leq n \leq N} |e_n|$$

tends towards 0 when $h \to 0$ and $\tilde{y}_0 \to y(t_0)$.

Remark

If the method is converging, by choosing h sufficiently small, and $\tilde{y}_0$ close to $y(t_0)$, we obtain a good approximation of $y(t_n)$, $n = 0, ..., N$

Convergence

Remarks

- it seems more natural to directly set $\tilde{y}_0 = y(t_0)$ in our scheme. However, in practice, if $y(t_0)$ is real-valued, it cannot be considered as exact (meaning in exact arithmetic) because of the round-off errors (on a computer). A correct analysis assumes $\tilde{y}_0 \neq y_0$
- like any computation, a stability problem arises:
  it is necessary to understand the consequences on the computation of small variations of $\tilde{y}_0$ and $f\left(t_n, \tilde{y}_n\right)$.

Consistency

We first introduce a notion called consistency of a numerical scheme:
the consistency error represents the error at the n-th step when replacing the ODE by the discrete equation

$$\varepsilon_n = y(t_{n+1}) - y(t_n) - h_n f(t_n, y(t_n)).$$

ε_n is sometimes called the local truncation error

Definition

A method is said to be consistent if

$$\lim_{h \to 0} \sum_{n=0}^{N-1} \|\varepsilon_n\| = 0.$$

Remark: consistency is a **local** notion, it supposes that the previous data are known exactly.
On the other hand, stability relates to the propagation of local errors

Stability

Definition

We say that a method is stable if there exists a constant K such that

$$\max_n \|\tilde{y}_n - \tilde{z}_n\| \leq K \left[\|\tilde{y}_0 - \tilde{z}_0\| + \sum_{n=0}^{N-1} \|\varepsilon_n\| \right]$$

for any $\tilde{z}_n$ solution to

$$\tilde{z}_{n+1} = \tilde{z}_n + h_n f\left(t_n, \tilde{z}_n\right) + \varepsilon_n, \;\; n = 0, ..., N-1.$$

This notion of stability implies that small perturbations on the initial data and all the intermediate calculations leads to small perturbations on the final result

Back to convergence

stability of the forward Euler scheme

+

consistency of the forward Euler scheme

=

convergence of the forward Euler scheme

Remark:

It can be shown that, more generally, for a one-step method, consistency and stability imply convergence.

Convergence of Euler's method

The forward Euler method is convergent. If f is Lipschitz and continuous one can show

$$\max_{0 \leq n \leq N} |e_n| \leq e^{LT} \left(hMT + \|e_0\| \right)$$

Example 1

Example

$$\begin{cases} y'(t) = 3y(t) - 3t & t \in [0,5] \\ y(0) = \frac{1}{3} \end{cases}$$

- the solution is $y(t) = \frac{1}{3} + t$
- now, if we consider the same problem but with the intial data $z(0) = \frac{1}{3} + \epsilon$, the solution is $z(t) = \frac{1}{3} + t + \epsilon e^{3t}$
- as a consequence, $z(5) = y(5) + \epsilon e^{15} \simeq y(5) + 3\epsilon 10^6$
- therefore, if one works with a computer with a round-off error equal to 10^{-6}, it will be impossible to approximate $y(5)$, and this, independently of the numerical method
- the problem is ill-conditioned

Example 2

Example

$$\begin{cases} y'(t) = -150\, y(t) + 50 \\ y(0) = \frac{1}{3} \end{cases}$$

- the solution is $y(t) = \frac{1}{3}$
- here, the problem is well-conditioned. Indeed, if one introduces a perturbation ϵ on the initial data we have

$$|y(t) - z(t)| \leq \epsilon e^{-150t}, \quad \forall\ t \geq 0$$

- the forward Euler method leads to

$$y_{n+1} = y_n + h_n(-150 y_n + 50)$$

that is

$$y_{n+1} - \frac{1}{3} = (1 - 150 h_n)\left(y_n - \frac{1}{3}\right)$$

Example 2

- for a constant step $h_n = h = \frac{1}{50}$, we have

$$y_{n+1} - \frac{1}{3} = (1 - 150h)^n \left(y_0 - \frac{1}{3} \right) = (-2)^n \left(y_0 - \frac{1}{3} \right)$$

- in particular

$$y_{50} - y(0) = (-2)^{50} \left(y_0 - \frac{1}{3} \right) \simeq 10^{15} \left(y_0 - \frac{1}{3} \right) !$$

- this shows that the step size is too large. On the other hand, if it is taken smaller, we will have round-off errors!
- the forward Euler scheme is a numerically unstable scheme.

Example 3

Example

$$\begin{cases} y'(t) = -\lambda y(t) & \lambda > 0 \\ y(0) = y_0 \end{cases}$$

- the solution to this problem is $y(t) = y_0 e^{-\lambda t}$
- the problem is well-conditioned. Indeed, for a small ϵ on the initial data, one gets

$$|y(t) - z(t)| \leq \epsilon e^{-\lambda t}, \quad \forall\ t \geq 0$$

- the forward Euler method applied to this problem with a constant step size h gives

$$y_{n+1} = y_n - \lambda h y_n = (1 - \lambda h) y_n$$

and so

$$y_n = (1 - \lambda h)^n y_0$$

Example 3

$$y_n = (1 - \lambda h)^n y_0$$

- even if the exact solution remains bounded

$$|y(t)| \leq |y_0| \quad \forall\ t \geq 0$$

  we see that if $|1 - \lambda h| > 1$ then the computed solution y_n will have a growing amplitude, leading to an unstable scheme
- the absolute stability condition (CFL:=Courant-Friedrichs-Lewy) writes

$$\lambda h < 2$$

- hence, the larger λ is, the smaller h must be.
- but if h is too small, then round-off errors appear !
- Stability conditions can be analyzed in the complex plane for many ODE methods

Backward Euler (implicit)

To solve the instability problem, we often use an implicit scheme like

$$\begin{cases} y_{n+1} = y_n + h_n f\left(t_{n+1}, y_{n+1}\right) & \forall\ 0 \leq n \leq N-1 \\ y_0 = \tilde{y}_0 \end{cases}$$

It comes from the approximation of

$$\int_{t_n}^{t_{n+1}} f\left(t, y\left(t\right)\right) dt$$

by the right rectangular quadrature rule

$$\int_{t_n}^{t_{n+1}} f\left(t, y\left(t\right)\right) dt \simeq h_n f(t_{n+1}, y(t_{n+1}))$$

Backward Euler (implicit)

The relation

$$y_{n+1} = y_n + h_n f\left(t_{n+1}, y_{n+1}\right)$$

defines y_{n+1} in an implicit way.

- this method is therefore more complicate to use than a forward Euler scheme
- at each iteration, this equation must be solved. Does it admit a solution? Is it unique?
- generally, the numerical solution to this equation requires the use of an iterative method (Newton, fixed point,...)
  $\rightarrow$ see Lecture on nonlinear equations
- the cost of one iteration is then higher than for the forward Euler scheme (which is explicit).
- however, the stability is greatly improved

Back to example 3

Example

$$\begin{cases} y'(t) = -\lambda y(t) & \lambda > 0 \\ y(0) = y_0 \end{cases}$$

- we have, for a constant step h, $y_{n+1} = y_n - \lambda h y_{n+1}$ that is

$$y_{n+1} = \frac{y_n}{(1+\lambda h)}$$

which also writes

$$y_n = \frac{y_0}{(1+\lambda h)^n}$$

- in particular, we have $\lambda > 0$ and for $h > 0$, $|y_n| \leq |y_0|$.
- furthermore, we can prove that this method converges as the previous one.

Single step methods

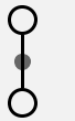
$k = \Delta t$

Forward Euler (explicit) $O(k)$

$$\frac{U^{n+1} - U^n}{k} = f(U^n) \qquad (12.2)$$

Backward Euler (implicit) $O(k)$

$$\frac{U^{n+1} - U^n}{k} = f(U^{n+1}) \qquad (12.3)$$

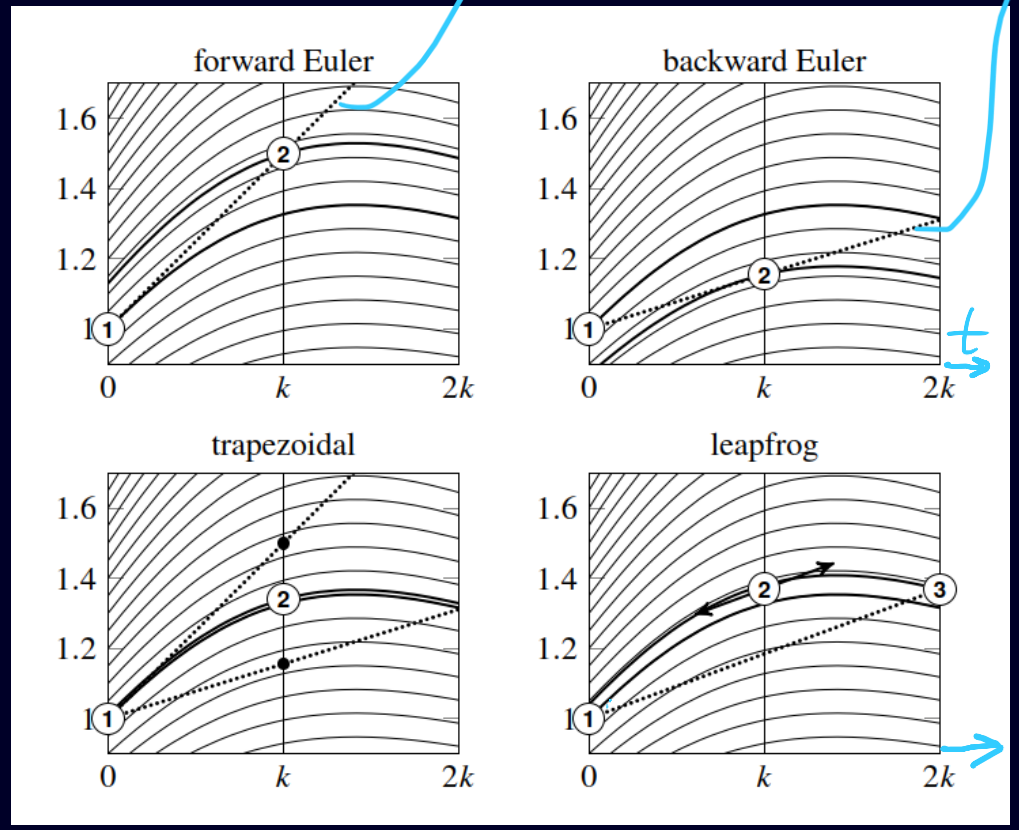


Leapfrog (explicit) $O(k^2)$

$$\frac{U^{n+1} - U^{n-1}}{2k} = f(U^n) \qquad (12.4)$$

Trapezoidal (implicit) $O(k^2)$

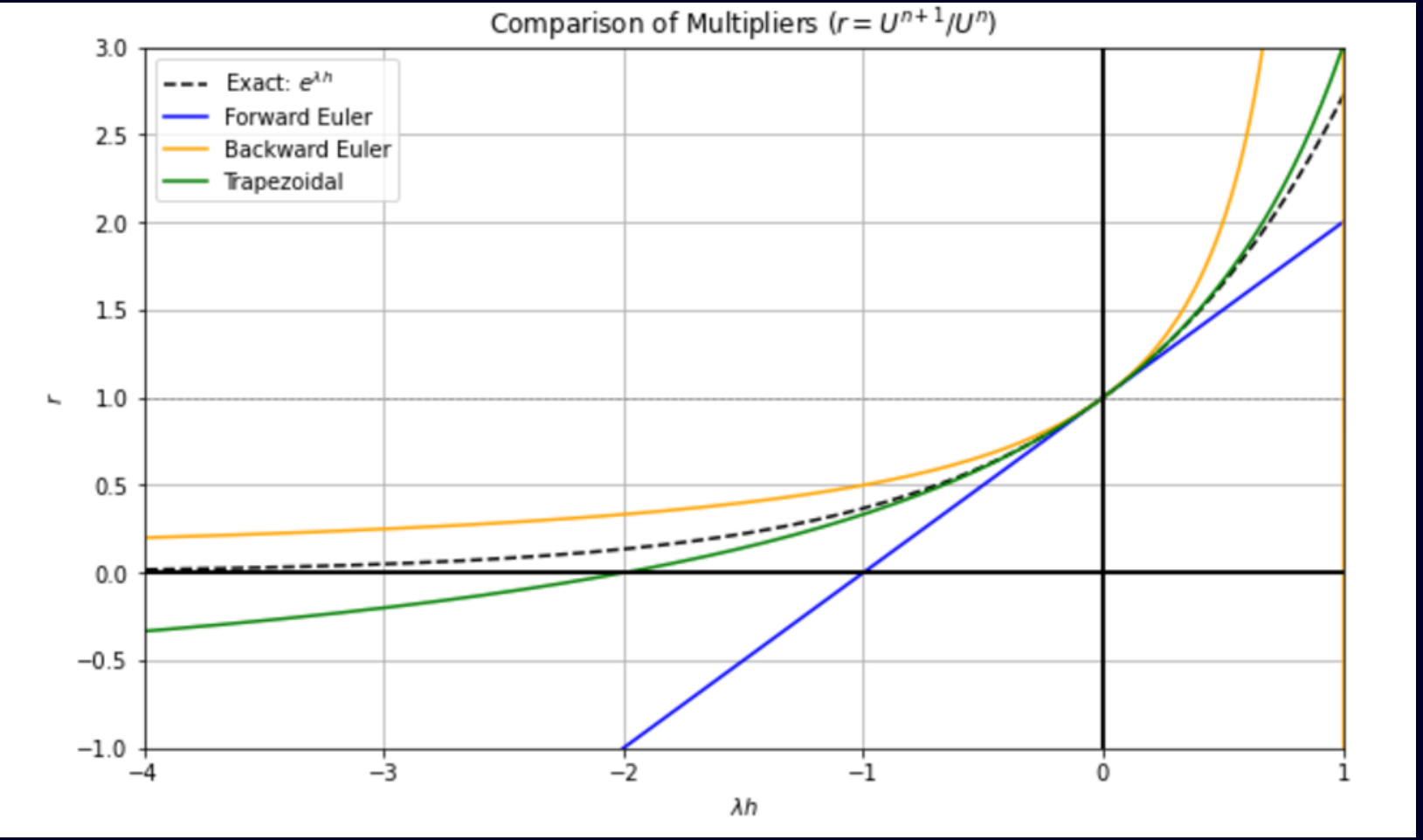
$$\frac{U^{n+1} - U^n}{k} = \frac{f(U^{n+1}) + f(U^n)}{2} \qquad (12.5)$$



Integral curves

Exercise 1

Accuracy of the different schemes for $u' = \lambda u$



General study of one-step methods

A one-step method can be written in a general way as

$$\begin{cases} y_{n+1} = y_n + h_n \Phi(t_n, y_n, h_n), & \forall\ n \in [\![0, N-1]\!] \\ y_0 = \tilde{y}_0 \end{cases}$$

- the approximation y_{n+1} of $y(t_{n+1})$ is therefore obtained uniquely from t_n, h_n and y_n the approximation of $y(t_n)$ obtained at the previous time step.
- this method can be implicit or explicit

General study of one-step methods

A one-step method can be written in a general way as

$$\begin{cases} y_{n+1} = y_n + h_n \Phi(t_n, y_n, h_n), & \forall\ n \in [\![0, N-1]\!] \\ y_0 = \tilde{y}_0 \end{cases}$$

Example

- forward Euler scheme: $y_{n+1} = y_n + h_n f(t_n, y_n)$. Here

$$\Phi(t, y, h) = f(t, y)$$

Φ is independent of h.

- backward Euler scheme : $y_{n+1} = y_n + h_n f(t_{n+1}, y_{n+1})$. Here

$$\Phi(t, y, h) = f(t + h, k)$$

with k solution to $k = y + hf(t + h, k)$

Order of accuracy

We now define the notion of order of accuracy of a one-step method

Definition

A one-step method is said to be of order p ($p > 0$), if for any solution y of $y'(t) = f(t, y(t))$ such that $y \in C^{p+1}([t_0, t_0 + T])$, there exists a real-valued parameter K which only depends on y and Φ such that

$$\sum_{n=0}^{N-1} \|\varepsilon_n\| \leq Kh^p$$

with ε_n being the local truncation error

$$\varepsilon_n = y(t_{n+1}) - y(t_n) - h_n \Phi(t_n, y(t_n), h_n)$$

Order of accuracy

Theorem

If a one-step method is stable and of order p and if $f \in C^p([t_0, t_0 + T] \times \mathbb{R}^n)$, then we have

$$\|y(t_n) - \tilde{y}_n\| \leq M \left[\|y(t_0) - \tilde{y}_0\| + Kh^p\right] \quad \forall n \in [\![0, N]\!]$$

Examples:

- the forward and backward Euler schemes are first-order
- let us consider the more general methods

Exercise 2 – logistic equation

3. Runge Kutta methods

Runge-Kutta methods

- First-order methods require too much computational time to get a given accuracy
- It is then necessary to use a high-order method: the most known are Runge-Kutta methods that consist in using high-order numerical integration rules to approximate

$$\int_{t_n}^{t_{n+1}} f(t, y(t))dt$$

which use **intermediate points** between t_n and t_{n+1}

Idea = integrate the ODE + use quadrature

$$y' = f(t, y(t)) \Rightarrow \frac{dy}{dt} = f(t, y(t)) \Rightarrow dy = f(t, y(t))\, dt$$

$$\int dy = \int f(t, y(t))\, dt$$

$$\int_{t_n}^{t_{n+1}} dy = \int_{t_n}^{t_{n+1}} f(t, y(t))\, dt$$

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t))\, dt = \sum_{i=1}^{s} b_i\, f(t_i^*, y(t_i^*))$$

weights

points

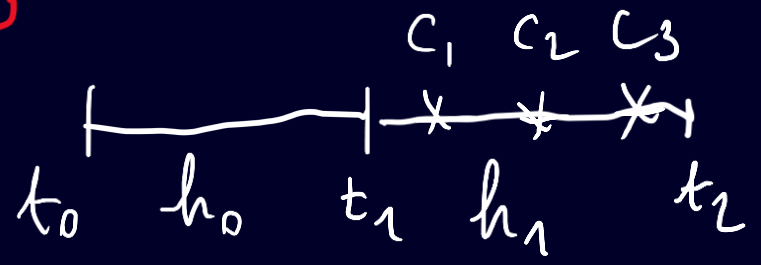
Idea = integrate the ODE + use quadrature

quadrature rule $[0,1]$
s stages (c_i, b_i)
c_i: points, b_i: weights

$$y(t_{n+1}) \simeq y(t_n) + h_n \int_0^1 f(t_n, y(t_n))\, dt$$

$$y(t_{n+1}) \simeq y(t_n) + h_n \sum_{i=1}^{s} b_i \underbrace{f(t_n^i, y(t_n^i))}_{k_i}$$

$$t_n^i = t_n + h_n c_i$$

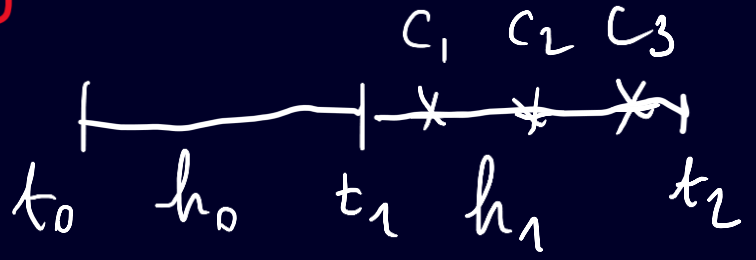


Question: how to evaluate $y(t_n^i)$ that is not known?

Idea = integrate the ODE + use quadrature

$$y(t_{n+1}) \simeq y(t_n) + h_n \sum_{i=1}^{s} b_i \underbrace{f(t_n^i, y(t_n^i))}_{k_i}$$

$$t_n^i = t_n + h_n c_i$$



Using the same formula!

$$y_{n+1} \simeq y_n + h_n \sum_{i=1}^{s} b_i k_i$$

$$k_i = f\left(t_n + c_i h_n, y_n + h_n \sum_{j=1}^{s} a_{ij} k_j\right)$$

Runge-Kutta methods

Let (c_j, b_j) be an elementary quadrature formula with s stages:

$$\int_0^1 g(x)dx = \sum_{j=1}^{s} b_j g(c_j)$$

Then

$$y(t_{n+1}) \simeq y(t_n) + h_n \sum_{i=1}^{s} b_i k_i$$

with $t_{n,i} = t_n + h_n c_i$, $k_i = f(t_{n,i}, y(t_{n,i}))$

Problem

How to evaluate $k_i = f(t_{n,i}, y(t_{n,i}))$ if $y(t_{n,i})$ is not known?

Runge-Kutta methods

The values $y(t_{n,i})$ are also evaluated through some numerical integration formulae by using the same points $t_{n,i}$

$$y(t_{n,i}) \simeq y(t_n) + h_n \sum_{j=1}^{s} a_{i,j} f(t_{n,j}, y(t_{n,j})) \quad \forall\, i \in [\![1, s]\!]$$

The Runge-Kutta methods consists in replacing $\simeq$ by $=$
$y(t_{n,j})$ are given at other intermediates points where it can be evaluated !

Runge-Kutta methods

Remarks

- if the matrix $(a_{i,j})$ is strictly lower triangular, then the RK method define explicitly the values of $y_{n,j}$, otherwise implicitly.
- the method is a one-step method. Indeed, this scheme can be written as

$$y_{n+1} = y_n + h_n \Phi(t_n, y_n, h_n)$$

where $\Phi(.,.,.)$ is the function defined by the equations

$$\Phi(t_n, y_n, h_n) = \sum_{i=1}^{s} b_i k_i, \;\; k_i = f(t_n + c_i h_n, y_n + h_n \sum_{j=1}^{s} a_{i,j} k_j) \quad \forall\, i \in [\![1, s]\!]$$

Examples of Runge Kutta method

$$y_{n+1} = y_n + h_n \int_0^1 f(t_n, y_n)\, dt$$

- Explicit midpoint method: we take the midpoint formula

$$y(t_{n+1}) \simeq y(t_n) + h_n f\left(t_n + \frac{h_n}{2}, y(t_n + \frac{h_n}{2})\right)$$

and we replace the unknown value $y(t_n + \frac{h_n}{2})$ by the Euler method

$$y(t_n + \frac{h_n}{2}) \simeq y(t_n) + \frac{h_n}{2} f(t_n, y(t_n))$$

This provides

$$y_{n+1} = y_n + h_n f\left(t_n + \frac{h_n}{2}, y_n + \frac{h_n}{2} f(t_n, y_n)\right)$$

Examples of Runge Kutta method

$$y_{n+1} \simeq y_n + \int_0^1 f(t_n, y_n)\, dt$$

- Trapezoidal method: we take the trapezoidal quadrature formula

$$y(t_{n+1}) \simeq y(t_n) + \frac{h_n}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right)$$

which is the implicit trapezoidal method. If we replace the unknown value $y(t_{n+1})$ by Euler approximation, we obtain the explicit trapezoidal method

$$\begin{aligned} y_{n+1} &= y_n + \frac{h_n}{2}\left[f(t_n, y_n) + f\left(t_{n+1}, y_n + h_n f(t_n, y_n)\right)\right] \\ &= y_n + \frac{h_n}{2}\left[k_1 + k_2\right], \\ \text{with} \quad k_1 &= f(t_n, y_n), \quad k_2 = f(t_n + h_n, y_n + h_n k_1) \end{aligned}$$

↳ nested rule.

Butcher Tableau

$$y_{n+1} = y_n + h_n \sum_{i=1}^{s} b_i k_i$$

$$k_i = f\left(t_n + c_i,\ y_n + h_n \sum_j a_{ij} k_j\right)$$

A Runge-Kutta method is completely known when we have: s, the coefficients $a_{i,j}$, b_j and c_j. Usually, we use the following Butcher Tableaux

$$\begin{array}{c|cccc} c_1 & a_{1,1} & a_{1,2} & \dots & a_{1,s} \\ c_2 & a_{2,1} & a_{2,2} & \dots & a_{2,s} \\ \vdots & \vdots & \vdots & & \vdots \\ c_s & a_{s,1} & a_{s,2} & \dots & a_{s,s} \\ \hline & b_1 & b_2 & \dots & b_s \end{array}$$

points

coupled weights

main weights

Example

Explicit Euler: $\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$, Explicit midpoint: $\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}$,

Runge Kutta 4 (RK4)

Simpson's rule $b_i = \left(\frac{1}{6} \quad \frac{2}{3} \quad \frac{1}{6}\right)$, $c_i = \left(0, \frac{1}{2}, 1\right)$

- Example for RK4: based on Simpson's rule integration and explicit midpoint rule

0	0	0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	0	0
1	0	0	1	0
	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

$$k_{n,1} = f(t_n, y_n) \qquad k_{n,2} = f\left(t_n + \frac{h_n}{2}, y_n + \frac{h_n}{2} k_{n,1}\right)$$
$$k_{n,3} = f\left(t_n + \frac{h_n}{2}, y_n + \frac{h_n}{2} k_{n,2}\right) \qquad k_{n,4} = f(t_{n+1}, y_n + h_n k_{n,3})$$

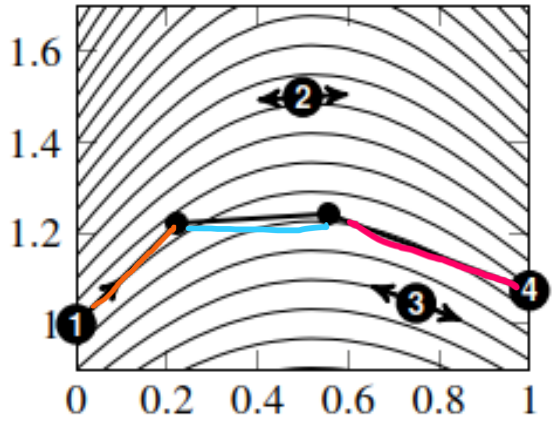
$$y_{n+1} = y_n + \frac{h_n}{6}\left[k_{n,1} + 2k_{n,2} + 2k_{n,3} + k_{n,4}\right]$$

Remarks on Runge Kutta

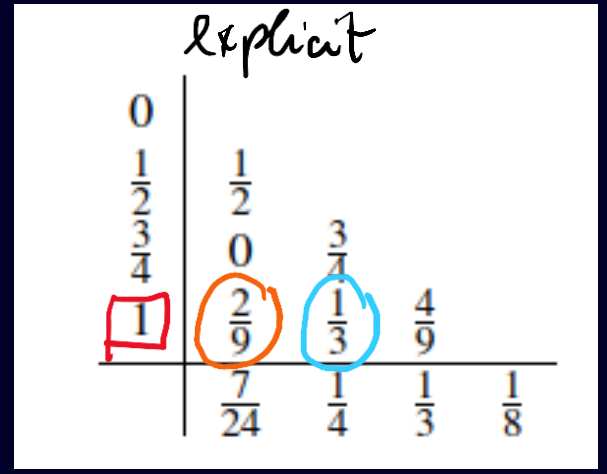
Under some regularity assumptions on f, it can be proved that the Runge-Kutta methods are stable. Being stable and consistent, they are convergent.

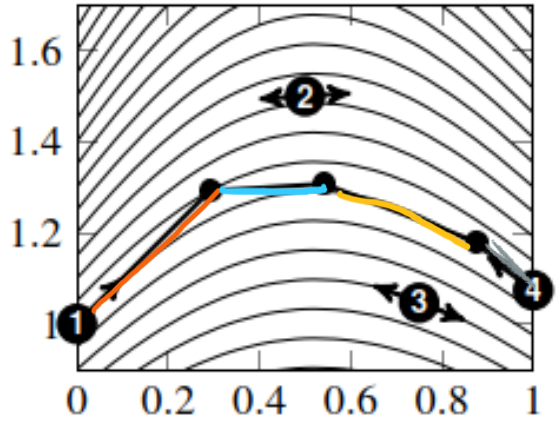
- A RK method with s stages is of order s
- RK methods are costly, they require many function evaluations
- varying step size RK methods can be derived, such as RK23 and RK45
- Implicit RK schemes are costly, we privilege explicit RK in practice

Butcher Tableau - interpretation

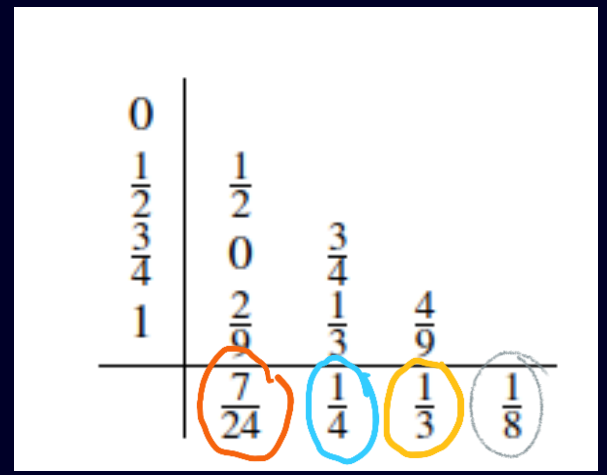


The fourth row says we now try to find the solution at $t = 1$ by first traveling $\frac{2}{9}$ with a slope given by ❶, then a distance $\frac{1}{3}$ with a slope given by ❷, and then a distance $\frac{4}{9}$ with slope from ❸.

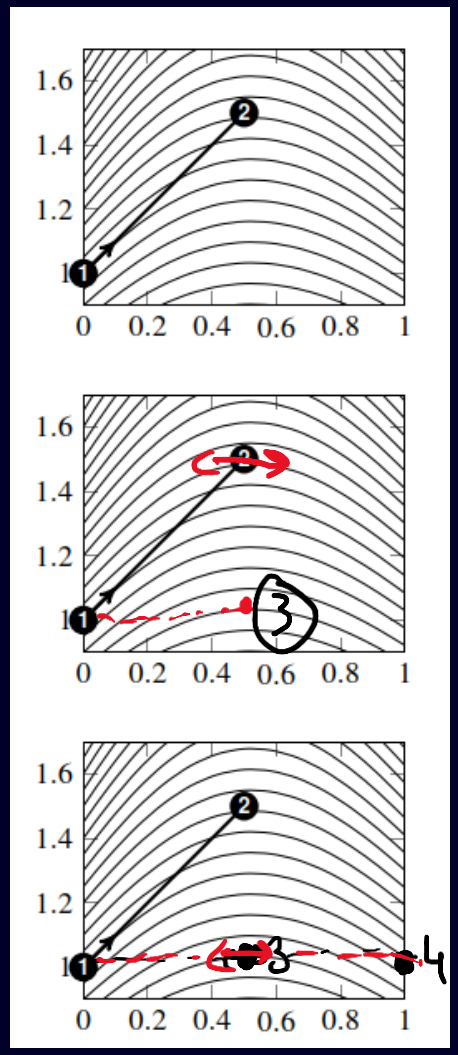




The final row tells us to first travel $\frac{7}{24}$ with a slope from ❶, then $\frac{1}{4}$ with a slope from ❷, then $\frac{1}{3}$ with a slope from ❸, and finally $\frac{1}{8}$ with a slope from ❹. The solution provides a third-order correction over the solution from ❹



RK4 - interpretation



we start with second row to find (2), we travel $\frac{1}{2}$ with the slope from (1)

third row, we travel $\frac{1}{2}$ with the slope we just found at (2) and nothing from (1)

fourth row, we travel 1 with the slope we found at (3) and nothing from (1) and (2)

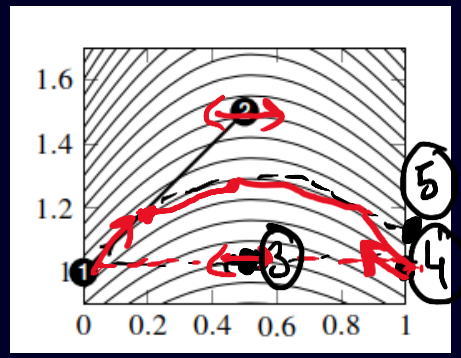
	1	2	3	4
0	0	0	0	0
→ $\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	0	0
1	0	0	1	0
	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

0	0	0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
→ $\frac{1}{2}$	0	$\frac{1}{2}$	0	0
1	0	0	1	0
	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

0	0	0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	0	0
→ 1	0	0	1	0
	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

RK4 - interpretation

5



Last now, we combine the slopes from ①, ②, ③, ④ to obtain the result at $t = 1$

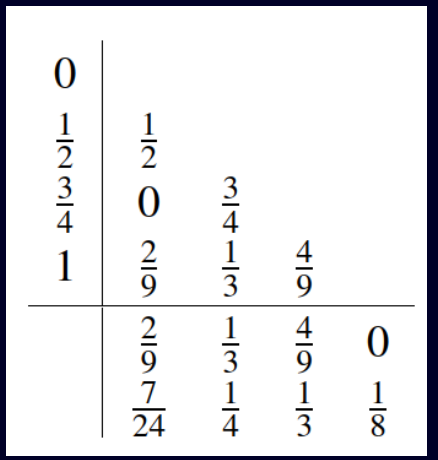
	①	②	③	④
0	0	0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	0	0
1	0	0	1	0
→	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

We use 4 different slopes defined recursively!

Adaptive step size

- One advantage of RK methods is that step size can be varied
- With an error estimator this allows to use smaller steps to minimize errors and maintain stability
- Often, a RK method embeds a lower-order method inside a higher-order method,
  - sharing the same Butcher tableau!
- For example, the Bogacki–Shampine method (Python "RK23“, Matlab ode23, Julia’s BS3) combines a 2nd and 3rd order Runge–Kutta method using the same quadrature points
- The notation RKmn is occasionally used
  - m is the order to obtain the solution
  - n is the order of the method to obtain the error estimate

Bogacki–Shampine



RK2, RK4 and RK23 (explicit)

$$
\begin{aligned}
K_1 &= f(t_n, U^n) \\
K_2 &= f\left(t_n + \tfrac{1}{2}k, U^n + \tfrac{1}{2}kK_1\right) \\
U^{n+1} &= U^n + kK_2
\end{aligned}
$$

$$
\begin{array}{c|cc}
0 & & \\
\frac{1}{2} & \frac{1}{2} & \\
\hline
 & 0 & 1
\end{array}
$$

We travel h/2 with slope from 1

We travel h with slope from 2

$$
\begin{aligned}
K_1 &= f(t_n, U^n) \\
K_2 &= f\left(t_n + \tfrac{1}{2}k, U^n + \tfrac{1}{2}kK_1\right) \\
K_3 &= f\left(t_n + \tfrac{1}{2}k, U^n + \tfrac{1}{2}kK_2\right) \\
K_4 &= f(t_n + k, U^n + kK_3) \\
U^{n+1} &= U^n + \tfrac{1}{6}k(K_1 + 2K_2 + 2K_3 + K_4).
\end{aligned}
$$

$$
\begin{array}{c|ccc}
0 & & & \\
\frac{1}{2} & \frac{1}{2} & & \\
\frac{1}{2} & 0 & \frac{1}{2} & \\
1 & 0 & 0 & 1 \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}
$$

Bogacki–Shampine

$$
\begin{array}{c|ccc}
0 & & & \\
\frac{1}{2} & \frac{1}{2} & & \\
\frac{3}{4} & 0 & \frac{3}{4} & \\
1 & \frac{2}{9} & \frac{1}{3} & \frac{4}{9} \\
\hline
 & \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & 0 \\
 & \frac{7}{24} & \frac{1}{4} & \frac{1}{3} & \frac{1}{8}
\end{array}
$$

Exercise 3 - pendulum

4. Multi-step methods

Adams-Bashforth methods with r + 1 steps

- we assume that we know the approximate values y_n of $y(t_n)$ and $f_n, f_{n-1}, ..., f_{n-r}$ of $f(t, y(t))$ respectively at points $t_n, t_{n-1}, ..., t_{n-r}$.
- the polynomial P_n is chosen as the polynomial of degree less or equal to r such that

$$P_n(t_{n-i}) = f_{n-i} \qquad \forall i = 0, ..., r.$$

- the approximation of $y(t_{n+1})$ is then defined by

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} P_n(t)dt$$

Newton formula (incrementally build polynomials) - interpolation

Newton's Divided Difference Formula

Newton's method constructs the interpolating polynomial incrementally, leveraging the concept of **divided differences**. The polynomial is expressed as:

$$P(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

where:

- $f[x_0] = f(x_0)$ (the value of the function at x_0),
- $f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$ (first divided difference),
- $f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$ (second divided difference), and so on.

This formulation is particularly useful because it allows for the **incremental addition of terms** if new points are introduced, without having to recompute the entire polynomial.

Adams-Bashforth methods with r + 1 steps

- if one represents the polynomial P_n by the Newton formula

$$P_n(t) = \sum_{i=0}^{r} f[t_n, t_{n-1}, \ldots, t_{n-i}] \prod_{j=0}^{i-1} (t - t_{n-j})$$

the methods becomes

$$y_{n+1} = y_n + \sum_{i=0}^{r} f[t_n, t_{n-1}, \ldots, t_{n-i}] \left(\int_{t_n}^{t_{n+1}} \prod_{j=0}^{i-1} (t - t_{n-j}) dt \right)$$

The Adams-Bashforth methods with r + 1 steps

- in the case of a constant step h i.e. $t_j = t_0 + jh$, the divided differences can be written as

$$f[t_n, t_{n-1}, \ldots, t_{n-i}] = \frac{\Delta^i f_n}{i! h^i}$$

where

$$\Delta^i f_k = \begin{cases} f_k & \text{if } i = 0 \\ \Delta^{i-1} f_k - \Delta^{i-1} f_{k-1} & \text{if } i \geq 1 \end{cases}$$

are the backward finite differences

- the formula then becomes

$$y_{n+1} = y_n + \sum_{i=0}^{r} \frac{\Delta^i f_n}{i! h^i} \left(\int_{t_n}^{t_{n+1}} \prod_{j=0}^{i-1} (t - t_{n-j}) dt \right)$$

The Adams-Bashforth methods with r + 1 steps

- now, by setting $t = t_n + sh$, $s \in [0, 1]$, we have

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \prod_{j=0}^{i-1} (t - t_{n-j}) dt &= h^{i+1} \int_0^1 \prod_{j=0}^{i-1} (j - s) ds \\ &= h^{i+1} i! \int_0^1 \binom{s+i-1}{i} ds \end{aligned}$$

where $\binom{s}{k}$ is the binomial coefficient generalized to non integer values

$$\binom{s}{k} = \frac{s\,(s-1)\ldots(s-k+1)}{1.2\ldots k}.$$

The Adams-Bashforth methods with r + 1 steps

- Hence

$$y_{n+1} = y_n + h\sum_{i=0}^{r} \gamma_i \Delta^i f_n \quad \text{with} \quad \gamma_i = \int_0^1 \binom{s+i-1}{i} ds$$

- we show that the γ_i satisfy the relation

$$\gamma_0 = 1,\ 1 = \frac{\gamma_0}{i+1} + \frac{\gamma_1}{i} + \ldots + \frac{\gamma_{i-1}}{2} + \gamma_i$$

  which leads to their recursive computation.

- it is important to notice that they do not depend on r, which is useful when one wants to make the order r vary in a same computation.
- one then gets

$$\gamma_0 = 1,\ \gamma_1 = \frac{1}{2},\ \gamma_2 = \frac{5}{12},\ \gamma_3 = \frac{3}{8},\ \gamma_4 = \frac{251}{720},\ \gamma_5 = \frac{95}{288}.$$

The Adams-Bashforth methods with r + 1 steps

- In practice, we prefer to explicitly write the relation as a function of the values of f_{n-i}, leading to

$$y_{n+1} = y_n + h \sum_{i=0}^{r} b_{i,r} f_{n-i}.$$

- from the finite difference formula, we can check that

$$b_{r,r} = (-1)^r \gamma_r, \; b_{i,r} = b_{i,r-1} + (-1)^i \binom{r}{i} \gamma_r, \; 0 \leq i \leq r.$$

The Adams-Bashforth methods with r + 1 steps

- one gets the following tableaux:

	$b_{0,r}$	$b_{1,r}$	$b_{2,r}$	$b_{3,r}$	$b_{4,r}$	$b_{5,r}$	$b_{6,r}$	γ_r
$r=0$	1							1
$r=1$	$\frac{3}{2}$	$-\frac{1}{2}$						$\frac{1}{2}$
$r=2$	$\frac{23}{12}$	$-\frac{4}{3}$	$\frac{5}{12}$					$\frac{5}{12}$
$r=3$	$\frac{55}{24}$	$-\frac{59}{24}$	$\frac{37}{24}$	$-\frac{3}{8}$				$\frac{3}{8}$
$r=4$	$\frac{1901}{720}$	$-\frac{1387}{360}$	$\frac{109}{30}$	$-\frac{637}{360}$	$\frac{251}{720}$			$\frac{251}{720}$
$r=5$	$\frac{4277}{1440}$	$-\frac{7923}{1440}$	$\frac{4991}{720}$	$-\frac{3649}{720}$	$\frac{959}{480}$	$-\frac{95}{288}$		$\frac{95}{288}$
$r=6$	$\frac{199441}{60840}$	$-\frac{18817}{2520}$	$\frac{238783}{20160}$	$-\frac{10979}{945}$	$\frac{139313}{20160}$	$-\frac{5783}{2520}$	$\frac{19807}{60840}$	$\frac{19807}{60840}$

The Adams-Bashforth methods with r + 1 steps

- for $r = 0$

$$y_{n+1} = y_n + hf_n \quad \text{(Euler)}$$

- for $r = 1$

$$y_{n+1} = y_n + \frac{h}{2}(3f_n - f_{n-1})$$

- for $r = 2$

$$y_{n+1} = y_n + \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2})$$

- for $r = 3$

$$y_{n+1} = y_n + \frac{h}{124}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})$$

The Adams-Bashforth methods with r + 1 steps

- this last method (Adams-Bashforth with 4 steps) is usually used.
- if one wishes to apply it to the resolution of our Cauchy problem, we have to know the four initial approximations y_0, y_1, y_2 and y_3. Next, we can use the recursive formula to compute y_4, y_5, ...
- Adams computed the Taylor series of the exact solution around the initial value to determine the initial approximations that are not known
- clearly, we can also get them by using a one-step method
- this method is of order 4 and is stable under the natural smoothness assumptions on f

The Adams-Bashforth methods with r + 1 steps

- however, the stability constant are often very large which implies some numerical instabilities analogous to the one that have been underlined in the case of the forward Euler scheme
- to overcome this drawback, one uses some implicit methods

The Adams-Moulton methods with r + 1 steps

- one gets the tableau

	$b^*_{-1,r}$	$b^*_{0,r}$	$b^*_{1,r}$	$b^*_{2,r}$	$b^*_{3,r}$	$b^*_{4,r}$	$b^*_{5,r}$	$b^*_{6,r}$	$b^*_{7,r}$
$r=0$	$\frac{1}{2}$	$\frac{1}{2}$							1
$r=1$	$\frac{5}{12}$	$\frac{2}{3}$	$-\frac{1}{12}$						$-\frac{1}{2}$
$r=2$	$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$					$-\frac{1}{12}$
$r=3$	$\frac{251}{720}$	$\frac{323}{360}$	$-\frac{11}{30}$	$\frac{53}{360}$	$-\frac{19}{720}$				$-\frac{1}{24}$
$r=4$	$\frac{95}{288}$	$\frac{1427}{1440}$	$-\frac{133}{240}$	$\frac{241}{720}$	$-\frac{173}{1440}$	$\frac{3}{160}$			$-\frac{19}{720}$
$r=5$	$\frac{19087}{60480}$	$\frac{2713}{2520}$	$-\frac{15487}{20160}$	$\frac{586}{945}$	$-\frac{6737}{20160}$	$\frac{263}{2520}$	$-\frac{863}{60480}$		$-\frac{3}{160}$
$r=6$	$\frac{36799}{120960}$	$\frac{139849}{120960}$	$-\frac{121797}{120960}$	$\frac{123133}{120960}$	$-\frac{88545}{120960}$	$\frac{41499}{120960}$	$-\frac{11351}{120960}$	$\frac{275}{24192}$	$-\frac{863}{60480}$

The Adams-Moulton methods with r + 1 steps

- for $r = 0$

$$y_{n+1} = y_n + \frac{h}{2}(f_{n+1} + f_n)$$

- for $r = 1$

$$y_{n+1} = y_n + \frac{h}{12}(5f_{n+1} + 8f_n - f_{n-1})$$

- for $r = 2$

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2})$$

The Adams-Moulton methods with r + 1 steps

- this last method (3-steps Adams-Moulton method) is the most commonly used method.
- we show that under some smoothness assumptions this method is of order 4 and stable
- the stability coefficients are much better (smaller) than for the explicit fourth-order Adams-Bashforth method
- of course, we must pay the price since we implicitly define y_{n+1} through $f_{n+1} = f\left(t_{n+1}, y_{n+1}\right).$
- a nonlinear system must then be solved.
- to this end, we can consider the following predictor-corrector method

Predictor-Corrector methods

- To solve the equation

$$y_{n+1} = y_n + \frac{h}{24}(9f(t_{n+1}, y_{n+1}) + 19f_n - 5f_{n-1} + f_{n-2})$$

  we can use a successive approximation method (i.e. fixed-point) consisting in building the sequence $\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_p$ defined by

$$\begin{cases} \tilde{y}_{p+1} = y_n + \frac{h}{24}\left(9f\left(t_{n+1}, \tilde{y}_p\right) + 19f_n - 5f_{n-1} + f_{n-2}\right) \\ \tilde{y}_0 \text{ to choose.} \end{cases}$$

- one can iterate until convergence (in general $\tilde{y}_p$ converges towards y_{n+1} when p tends to infinity)

Predictor-Corrector methods

- most of the time, one only iterates a few times, even sometimes 1 or 2.
- in addition, the initial value $\tilde{y}_0$ is often obtained through one step of an explicit method of the same order
- then, we have a predictor-corrector method : the evaluation of $\tilde{y}_0$ corresponds to a prediction; this value is then next corrected through one or two iterations of a fixed point algorithm.

Predictor-Corrector methods

- finally, the following scheme is often used

$$\begin{cases} \text{Predictor: fourth-order Adams-Bashforth method} \\ \tilde{y}_0 = y_n + \frac{h}{24}\left(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right) \\ \text{Corrector: one or two iterations of the Adams-Moulton method} \\ \text{of order 4} \\ \tilde{y}_{p+1} = y_n + \frac{h}{24}\left(9f\left(t_{n+1}, \tilde{y}_p\right) + 19f_n - 5f_{n-1} + f_{n-2}\right), \ p = 0, 1. \end{cases}$$

- we show that this method is also of order 4
- its stability is clearly better than for the Adams-Bashforth scheme
- the solution to the nonlinear system related to the Adams-Moulton formula is finally done explicitly.

Backward differentiation formula

A last category of multi-step method consists of evaluating f at the end of the current step (t_{n+s}, y_{n+s}), and driving an interpolating polynomial for y with the points $(t_{n+s}, \dots, t_n)$.
We start from $y'(t_{n+s}) = f(t_{n+s}, y(t_{n+s}))$ and use the approximation

$$p'_{n,s}(t_{n+s}) = f(t_{n+s}, y_{n+s})$$

By doing so we end up with BDF schemes of order s

$$\sum_{k=0}^{s} a_k y_{n+k} = h\beta f(t_{n+s}, y_{n+s})$$

Example

BDF1: $y_{n+1} - y_n = hf(t_{n+1}, y_{n+1})$
BDF2: $y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2h}{3}f(t_{n+2}, y_{n+2})$

Backward differentiation formula

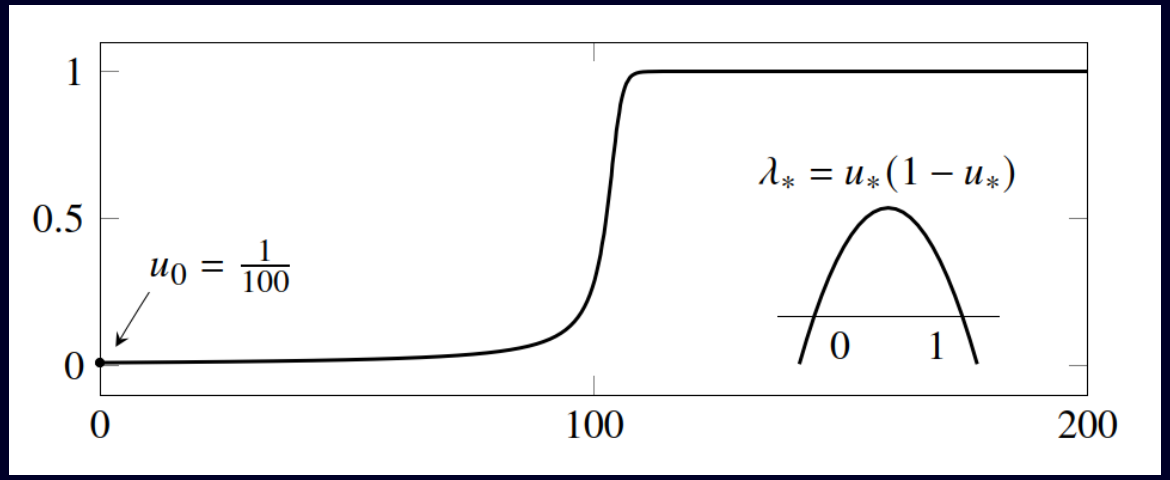
Remarks

- BDF schemes with s stages are of order s
- BDF schemes are implicit
- they are popular for stiff problems because of their stability property
- methods with $s > 6$ cannot be used

5. Additional considerations

Stiff equations

- A solution is said to be stiff if the dynamics of the solution react on at least two vastly different time scales
  - Example: diffusion equations are stiff
- The solution evolves quickly in the presence of a large spatial gradient and then slows to a crawl once it smooths out.
- Advanced schemes (adaptive and implicit) are required for such problems



Non-linear problems

- **Explicit** is no-brainer, you simply evaluate the function at previous time step (like if linear)

$$u^{n+1} = u^n + h \cdot f(u^n)$$

  - **No iterative solver** like Newton-Raphson is used because explicit methods avoid solving systems of equations.
  - However, stability imposes some restrictions on the time step (CFL condition in hyperbolic problems)

- **Implicit**
  - To solve the resulting non-linear system, you perform **inner iterations**: Start with an initial guess (usually the solution from the previous step).
  - Apply a non-linear solver (e.g., Fixed point or Newton-Raphson), which involves solving linearized systems iteratively until convergence.
  - Update the solution and repeat until the residual meets the tolerance.

Implicit or Explicit?

- **Stiffness**: Implicit schemes are preferred for stiff problems as explicit schemes can require impractically small time steps.
- **Computational cost**: Implicit schemes are more expensive per step due to the inner iterations, but fewer time steps may offset this cost.
- **Accuracy**: Both schemes can achieve high accuracy if the time step is small, but implicit methods often allow larger steps without losing stability.
- **Inner iterations**: In implicit schemes, the non-linear solver (e.g., Newton-Raphson) introduces another layer of convergence:
  - Each non-linear iteration linearizes the problem around the current guess (Fixed point or Newton).
  - Inner convergence depends on the problem's non-linearity and the initial guess.

Exercise 4 – 2 mass & spring
Exercise 5 – spaceship

6. Summary

Summary

We have seen some numerical methods to solve IVP

1. One-step methods of different orders (Euler, RK)
2. Multi-step methods (AM, AB, BDF)
3. Predictor-corrector methods

They are various differences between the schemes

- The methods can be explicit or implicit, and of different orders of accuracy
- Implicit methods are more stable than explicit methods, but are also more costly
- Usually an adaptive step size solver is necessary
- Writing an efficient ODE solver requires a good knowledge and experience of these methods