Nonlinear (systems of) equations

Numerical analysis - University of Luxembourg

Exercises

Introduction

Exercise 1.

Consider the system of non-linear equations in two dimensions

$$x_1^2 - x_2 + \gamma = 0$$
$$-x_1 + x_2^2 + \gamma = 0$$

Give a geometrical interpretation to the solutions of this non-linear system, by plotting the iso-contours of the two functions for different values of γ.

- define a grid $[-2, 2] \times [2, 2]$ using np.meshgrid
- plot the isolines using plt.contour (you can also experiment plt.contourf - filled contours)
- note that you can specify the zero-valued isoline with "levels = [0]"
- How many solutions do you visually find for $\gamma = 0.5$, $\gamma = 0.25$, $\gamma = -0.5$ and $\gamma = -1$ respectively?
- finally, experiment 3D rendering

1D Root-finding

Exercise 2. *Bisection method*
Implement the bisection method and test you methods on the following functions

- $x^2 = 4\sin(x)$
- $x^3 - 2x - 5 = 0$
- $x\sin(x) = 1$
- $e^{-x} = x$
- $x^3 - 3x^2 + 3x - 1 = 0$

What termination criterion should you use? What convergence rate is achieved? Compare your result (solution and convergence rate) with those for a library routine for solving nonlinear equations (scipy.optimize).

Exercise 3. *Fixed point method*
For the equation

$$f(x) = x^2 - x - 2 = 0$$

Each of the following functions yields an equivalent fixed-point problem

$$g_1(x) = \left(x^2 - 2\right) \tag{1}$$
$$g_2(x) = \sqrt{x+2} \tag{2}$$
$$g_3(x) = 1 + 2/x \tag{3}$$
$$g_4(x) = \left(x^2 + 2\right)/(2x-1). \tag{4}$$

Analyze the convergence properties of each of the corresponding fixed-point iteration schemes for the root $x = 2$ by considering $|g_i'(2)|$.
Confirm your analysis by implementing each of the schemes and verifying its convergence (or lack thereof) and approximate convergence rate

Exercise 4. *Secant and Newton's method*
Consider again exercise 2, but now solve it using Secant and Newton's method.

Exercise 5. *Newton fractal*
Consider the polynomial function

$$f : x \mapsto x^3 - 1$$

What are the roots of this polynomial ?
Report on a 1D plot the behaviour of Newton's method for many different starting points (convergence towards a root or divergence - use a different color for each behaviour and each root). Do the same exercise but for many starting points in the *complex plane*. Plot the so-called basin of attraction of the system, that is report for many different starting points the behaviour of the method. Use a different color for each root.

Replot the basin of attraction with the secant method. For the Newton or secant method you should obtain a fractal pattern. Try to explain and interpret what you observe.

Root-finding in higher dimensions

Exercise 6. *2D zero finding*
Find the zeros of the following systems

$$\begin{aligned} xy + y^2 &= 1 \\ xy^3 + x^2y^2 &= -1 \end{aligned}$$

and

$$\begin{aligned} x^2 &= y - x\cos(\pi x) \\ yx &= 1/x - e^{-y} \end{aligned}$$

using Newton's method in 2D. You will need to compute the Jacobian matrix.

Exercise 7. *3D zero finding*
Let F be defined by

$$F : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} xy^2 + 2y^2z + 10y^2 + x + 2z + 10 \\ xz^2 - 2yz^2 - 10z^2 + 2x - 4y - 20 \\ 5y^3 + 4z^3 + 4y^2z + 5yz^2 - 5y^2 - 5z^2 \end{pmatrix}$$

The goal of this exercise is to find a zero of F by using three methods.

- Let ϕ such that

$$\phi(x,y,z) = f_1(x,y,z)^2 + f_2(x,y,z)^2 + f_3(x,y,z)^2$$

  where f_1, f_2, f_3 are the functions that define the components of F. By using `scipy.optimize.minimize`, find a minimum of ϕ. Is it a zero of F ? What happens if you change the initial value ?

- Code the Newton method for F.

- Code the Broyden method for F.

- Run the two methods, interpret your results and conclude