Model theory for metric structures

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1 Introduction

A metric structure is a many-sorted structure in which each sort is a complete metric space of finite diameter. Additionally, the structure consists of some distinguished elements as well as some functions (of several variables) (a) between sorts and (b) from sorts to bounded subsets of $\mathbb{R}$, and these functions are all required to be uniformly continuous. Examples arise throughout mathematics, especially in analysis and geometry. They include metric spaces themselves, measure algebras, asymptotic cones of finitely generated groups, and structures based on Banach spaces (where one takes the sorts to be balls), including Banach lattices, C*-algebras, etc.

The usual first-order logic does not work very well for such structures, and several good alternatives have been developed. One alternative is the logic of *positive bounded formulas with an approximate semantics* (see [23, 25, 24]). This was developed for structures from functional analysis that are based on Banach spaces; it is easily adapted to the more general metric structure setting that is considered here. Another successful alternative is the setting of *compact abstract theories* (cats; see [1, 3, 4]). A recent development is the realization that for metric structures the frameworks of positive bounded formulas and of cats are equivalent. (The full cat framework is more general.) Further, out of this discovery has come a new *continuous* version of first-order logic that is suitable for metric structures; it is equivalent to both the positive bounded and cat approaches, but has many advantages over them.

The logic for metric structures that we describe here fits into the framework of continuous logics that was studied extensively in the 1960s and then dropped (see [12]). In that work, any compact Hausdorff space X was allowed as the set of truth values for a logic. This turned out to be too general for a completely successful theory.

We take the space X of truth values to be a closed, bounded interval of real numbers, with the order topology. It is sufficient to focus on the case where X is $[0, 1]$. In [12], a wide variety of quantifiers was allowed and studied. Since our truth value set carries a natural complete linear ordering, there are two canonical quantifiers that clearly deserve special attention; these are the operations sup and inf, and it happens that these are the only quantifiers we need to consider in the setting of continuous logic and metric structures.

The continuous logic developed here is strikingly parallel to the usual first-order logic, once one enlarges the set of possible truth values from $\{0, 1\}$ to $[0, 1]$. Predicates, including the equality relation, become functions from the underlying set A of a mathematical structure into the interval $[0, 1]$. Indeed, the natural $[0, 1]$-valued counterpart of the equality predicate is a metric d on A (of diameter at most 1, for convenience). Further, the natural counterpart of the assumption that equality is a congruence relation for the predicates and operations in a mathematical structure is the requirement that the predicates and operations in a metric structure be uniformly continuous with respect to the metric d. In the $[0, 1]$-valued continuous setting, connectives are continuous functions on $[0, 1]$ and quantifiers are sup and inf.

The analogy between this continuous version of first-order logic (CFO) for metric structures and the usual first-order logic (FOL) for ordinary structures is far reaching. In suitably phrased forms, CFO satisfies the compactness theorem, Löwenheim-Skolem theorems, diagram arguments, existence of saturated and homogeneous models, characterizations of quantifier elimination, Beth's definability theorem, the omitting types theorem, fundamental results of stability theory, and appropriate analogues of essentially all results in basic model theory of first-order logic. Moreover, CFO extends FOL: indeed, each mathematical structure treated in FOL can be viewed as a metric structure by taking the underlying metric d to be discrete ($d(a, b) = 1$ for distinct a, b). All these basic results true of CFO are thus framed as generalizations of the corresponding results for FOL.

A second type of justification for focusing on this continuous logic comes from its connection to applications of model theory in analysis and geometry. These often depend on an ultraproduct construction [11, 15] or, equivalently, the nonstandard hull construction (see [25, 24] and their references). This construction is widely used in functional analysis and also arises in metric space geometry (see [19], for example). The logic of positive bounded formulas was introduced in order to provide a model theoretic framework for the use of this ultraproduct (see [24]), which it does successfully. The continuous logic for metric structures that is presented here provides an equivalent background for this ultraproduct construction and it is easier to use. Writing positive bounded formulas to express statements from analysis and geometry is difficult and often feels unnatural; this goes much more smoothly in CFO. Indeed, continuous first-order logic provides model theorists and analysts with a common language; this is due to its being closely parallel to first-order logic while also using familiar constructs from analysis (*e.g.*, sup and inf in place of $\forall$ and $\exists$).

The purpose of this article is to present the syntax and semantics of this continuous logic for metric structures, to indicate some of its key theoretical features, and to show a few of its recent application areas.

In Sections 1 through 10 we develop the syntax and semantics of continuous logic for metric structures and present its basic properties. We have tried to make this material accessible without requiring any background beyond basic undergraduate mathematics. Sections 11 and 12 discuss imaginaries and omitting types; here our presentation is somewhat more brisk and full understanding may require some prior experience with model theory. Sections 13 and 14 sketch a treatment of quantifier elimination and stability, which are needed for the applications topics later in the paper; here we omit many proofs and depend on other articles for the details. Sections 15 through 18 indicate a few areas of mathematics to which continuous logic for metric structures has already been applied; these are taken from probability theory and functional analysis, and some background in these areas is expected of the reader.

The development of continuous logic for metric structures is very much a work in progress, and there are many open problems deserving of attention. What is presented in this article reflects work done over approximately the last three years in a series of collaborations among the authors. The material presented here was taught in two graduate topics courses offered during that time: a Fall 2004 course taught in Madison by Itaï Ben Yaacov and a Spring 2005 course taught in Urbana by Ward Henson. The authors are grateful to the students in those courses for their attention and help. The authors' research was partially supported by NSF Grants: Ben Yaacov, DMS-0500172; Berenstein and Henson, DMS-0100979 and DMS-0140677; Henson, DMS-0555904.

2 Metric structures and signatures

Let (M, d) be a complete, bounded metric space[1]. A *predicate* on M is a uniformly continuous function from M^n (for some $n \geq 1$) into some bounded interval in $\mathbb{R}$. A *function* or *operation* on M is a uniformly continuous function from M^n (for some $n \geq 1$) into M. In each case n is called the *arity* of the predicate or function.

A *metric structure* $\mathcal{M}$ based on (M, d) consists of a family $(R_i \mid i \in I)$ of predicates on M, a family $(F_j \mid j \in J)$ of functions on M, and a family $(a_k \mid k \in K)$ of distinguished elements of M. When we introduce such a metric structure, we will often denote it as

$$\mathcal{M} = (M, R_i, F_j, a_k \mid i \in I, j \in J, k \in K).$$

Any of the index sets I, J, K is allowed to be empty. Indeed, they might all be empty, in which case $\mathcal{M}$ is a pure bounded metric space.

The key restrictions on metric structures are: the metric space is *complete* and *bounded*, each predicate takes its values in a *bounded interval* of reals, and the functions and predicates are *uniformly continuous*. All of these restrictions play a role in making the theory work smoothly.

Our theory also applies to *many-sorted* metric structures, and they will appear as examples. However, in this article we will not explicitly bring them into our definitions and theorems, in order to avoid distracting notation.

2.1 Examples. We give a number of examples of metric structures to indicate the wide range of possibilities.

(1) A complete, bounded metric space (M, d) with no additional structure.
(2) A structure $\mathcal{M}$ in the usual sense from first-order logic. One puts the discrete metric on the underlying set ($d(a, b) = 1$ when a, b are distinct) and a relation is considered as a predicate taking values ("truth" values) in the set $\{0, 1\}$. So, in this sense the theory developed here is a generalization of first-order model theory.
(3) If (M, d) is an unbounded complete metric space with a distinguished element a, we may view (M, d) as a many-sorted metric structure $\mathcal{M}$; for example, we could take a typical sort to be a closed ball B_n of radius n around a, equipped with the metric obtained by restricting d. The inclusion mappings $I_{mn} \colon B_m \to B_n$

1 See the appendix to this section for some relevant basic facts about metric spaces.

$(m < n)$ should be functions in $\mathcal{M}$, in order to tie together the different sorts.

(4) The unit ball B of a Banach space X over $\mathbb{R}$ or $\mathbb{C}$: as functions we may take the maps $f_{\alpha\beta}$, defined by $f_{\alpha\beta}(x,y) = \alpha x + \beta y$, for each pair of scalars satisfying $|\alpha| + |\beta| \leq 1$; the norm may be included as a predicate, and we may include the additive identity 0_X as a distinguished element. Equivalently, X can be viewed as a many-sorted structure, with a sort for each ball of positive integer radius centered at 0, as indicated in the previous paragraph.

(5) Banach lattices: this is the result of expanding the metric structure corresponding to X as a Banach space (see the previous paragraph) by adding functions such as the absolute value operation on B as well as the positive and negative part operations. In section 17 of this article we discuss the model theory of some specific Banach lattices (namely, the L^p-spaces).

(6) Banach algebras: multiplication is included as an operation; if the algebra has a multiplicative identity, it may be included as a constant.

(7) C^*-algebras: multiplication and the *-map are included as operations.

(8) Hilbert spaces with inner product may be treated like the Banach space examples above, with the addition that the inner product is included as a binary predicate. (See section 15.)

(9) If $(\Omega, \mathcal{B}, \mu)$ is a probability space, we may construct a metric structure $\mathcal{M}$ from it, based on the metric space (M,d) in which M is the measure algebra of $(\Omega, \mathcal{B}, \mu)$ (elements of $\mathcal{B}$ modulo sets of measure 0) and d is defined to be the measure of the symmetric difference. As operations on M we take the Boolean operations $\cup, \cap, ^c$, as a predicate we take the measure μ, and as distinguished elements the 0 and 1 of M. In section 16 of this article we discuss the model theory of these metric structures.

Signatures

To each metric structure $\mathcal{M}$ we associate a *signature* L as follows. To each predicate R of $\mathcal{M}$ we associate a *predicate symbol* P and an integer $a(P)$ which is the arity of R; we denote R by $P^{\mathcal{M}}$. To each function F of $\mathcal{M}$ we associate a *function symbol* f and an integer $a(f)$ which is the arity of F; we denote F by $f^{\mathcal{M}}$. Finally, to each distinguished element a of $\mathcal{M}$ we associate a *constant symbol* c; we denote a by $c^{\mathcal{M}}$.

So, a signature L gives sets of predicate, function, and constant symbols, and associates to each predicate and function symbol its arity. In that respect, L is identical to a signature of first-order model theory. In addition, a signature for metric structures must specify more: for each predicate symbol P, it must provide a closed bounded interval I_P of real numbers and a modulus of uniform continuity[2] Δ_P. These should satisfy the requirements that $P^{\mathcal{M}}$ takes its values in I_P and that Δ_P is a modulus of uniform continuity for $P^{\mathcal{M}}$. In addition, for each function symbol f, L must provide a modulus of uniform continuity Δ_f, and this must satisfy the requirement that Δ_f is a modulus of uniform continuity for $f^{\mathcal{M}}$. Finally, L must provide a non-negative real number D_L which is a bound on the diameter of the complete metric space (M, d) on which $\mathcal{M}$ is based.[3] We sometimes denote the metric d given by $\mathcal{M}$ as $d^{\mathcal{M}}$; this would be consistent with our notation for the interpretation in $\mathcal{M}$ of the nonlogical symbols of L. However, we also find it convenient often to use the same notation "d" for the logical symbol representing the metric as well as for its interpretation in $\mathcal{M}$; this is consistent with usual mathematical practice and with the handling of the symbol $=$ in first-order logic.

When these requirements are all met and when the predicate, function, and constant symbols of L correspond exactly to the predicates, functions, and distinguished elements of which $\mathcal{M}$ consists, then we say $\mathcal{M}$ is an *L-structure*.

The key added features of a signature L in the metric structure setting are that L specifies (1) a bound on the diameter of the underlying metric space, (2) a modulus of uniform continuity for each predicate and function, and (3) a closed bounded interval of possible values for each predicate.

For simplicity, and without losing any generality, we will usually assume that our signatures L satisfy $D_L = 1$ and $I_P = [0, 1]$ for every predicate symbol P.

2.2 Remark. If $\mathcal{M}$ is an L-structure and A is a given closed subset of M^n, then $\mathcal{M}$ can be expanded by adding the predicate $x \mapsto \operatorname{dist}(x, A)$, where x ranges over M^n and dist denotes the distance function with respect to the maximum metric on the product space M^n. Note that only in very special circumstances may A itself be added to $\mathcal{M}$ as a predicate (in the form of the characteristic function χ_A of A); this could

2 See the appendix to this section for a discussion of this notion.
3 If L is many-sorted, each sort will have its own diameter bound.

be done only if χ_A were uniformly continuous, which forces A to be a positive distance from its complement in M^n.

Basic concepts such as *embedding* and *isomorphism* have natural definitions for metric structures:

2.3 Definition. Let L be a signature for metric structures and suppose $\mathcal{M}$ and $\mathcal{N}$ are L-structures.

An *embedding* from $\mathcal{M}$ into $\mathcal{N}$ is a metric space isometry

$$T\colon (M, d^{\mathcal{M}}) \to (N, d^{\mathcal{N}})$$

that commutes with the interpretations of the function and predicate symbols of L in the following sense:

Whenever f is an n-ary function symbol of L and $a_1, \dots, a_n \in M$, we have

$$f^{\mathcal{N}}(T(a_1), \dots, T(a_n)) = T(f^{\mathcal{M}}(a_1, \dots, a_n));$$

whenever c is a constant symbol c of L, we have

$$c^{\mathcal{N}} = T(c^{\mathcal{M}});$$

and whenever P is an n-ary predicate symbol of L and $a_1, \dots, a_n \in M$, we have

$$P^{\mathcal{N}}(T(a_1), \dots, T(a_n)) = P^{\mathcal{M}}(a_1, \dots, a_n).$$

An *isomorphism* is a surjective embedding. We say that $\mathcal{M}$ and $\mathcal{N}$ are *isomorphic*, and write $\mathcal{M} \cong \mathcal{N}$, if there exists an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. (Sometimes we say *isometric isomorphism* to emphasize that isomorphisms must be distance preserving.) An *automorphism* of $\mathcal{M}$ is an isomorphism between $\mathcal{M}$ and itself.

$\mathcal{M}$ is a *substructure* of $\mathcal{N}$ (and we write $\mathcal{M} \subseteq \mathcal{N}$) if $M \subseteq N$ and the inclusion map from M into N is an embedding of $\mathcal{M}$ into $\mathcal{N}$.

Appendix

In this appendix we record some basic definitions and facts about metric spaces and uniformly continuous functions; they will be needed when we develop the semantics of continuous first-order logic. Proofs of the results we state here are straightforward and will mostly be omitted.

Let (M, d) be a metric space. We say this space is *bounded* if there is a real number B such that $d(x, y) \leq B$ for all $x, y \in M$. The *diameter* of (M, d) is the smallest such number B.

Suppose (M_i, d_i) are metric spaces for $i = 1, \dots, n$ and we take M to be the product $M = M_1 \times \cdots \times M_n$. In this article we will always regard M as being equipped with the maximum metric, defined for $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ by $d(x, y) = \max\{d_i(x_i, y_i) | i = 1, \dots, n\}$.

A *modulus of uniform continuity* is any function $\Delta\colon (0, 1] \to (0, 1]$.

If (M, d) and (M', d') are metric spaces and $f\colon M \to M'$ is any function, we say that $\Delta\colon (0, 1] \to (0, 1]$ is a *modulus of uniform continuity for f* if for every $\epsilon \in (0, 1]$ and every $x, y \in M$ we have

$$\text{(UC)} \qquad d(x, y) < \Delta(\epsilon) \implies d'(f(x), f(y)) \leq \epsilon.$$

We say f is *uniformly continuous* if it has a modulus of uniform continuity.

The precise way (UC) is stated makes the property Δ *is a modulus of uniform continuity for f* a topologically robust notion. For example, if $f\colon M \to M'$ is continuous and (UC) holds for a dense set of pairs (x, y), then it holds for all (x, y). In particular, if Δ is a modulus of uniform continuity for $f\colon M \to M'$ and we extend f in the usual way to a continuous function $\bar{f}\colon \overline{M} \to \overline{M'}$ (where $\overline{M}, \overline{M'}$ are completions of M, M', resp.), then, with this definition, Δ is a modulus of uniform continuity for the extended function $\bar{f}$.

If Δ is a function from $(0, \infty)$ to $(0, \infty)$ and it satisfies (UC) for all $\epsilon \in (0, \infty)$ and all $x, y \in M$, then we will often refer to Δ as a "modulus of uniform continuity" for f. In that case, f is uniformly continuous and the restriction of the function $\min(\Delta(\epsilon), 1)$ to $\epsilon \in (0, 1]$ is a modulus of uniform continuity according to the strict meaning we have chosen to assign to this phrase, so no confusion should result.

2.4 Proposition. *Suppose $f\colon M \to M'$ and $f'\colon M' \to M''$ are functions between metrics spaces M, M', M''. Suppose Δ is a modulus of uniform continuity for f and Δ' is a modulus of uniform continuity for f'. Then the composition $f' \circ f$ is uniformly continuous; indeed, for each $r \in (0, 1)$ the function $\Delta(r\Delta'(\epsilon))$ is a modulus of uniform continuity for $f' \circ f$.*

Let M, M' be metric spaces (with metrics d, d' resp.) and let f and $(f_n \mid n \geq 1)$ be functions from M into M'. Recall that $(f_n \mid n \geq 1)$ *converges uniformly to f on M* if

$$\forall \epsilon > 0\ \exists N\ \forall n > N\ \forall x \in M\ \big(d'(f_n(x), f(x)) \leq \epsilon\big).$$

2.5 Proposition. *Let M, M', f and $(f_n \mid n \geq 1)$ be as above, and suppose $(f_n \mid n \geq 1)$ converges uniformly to f on M. If each of the functions $f_n \colon M \to M'$ is uniformly continuous, then f must also be uniformly continuous. Indeed, a modulus of uniform continuity for f can be obtained from moduli Δ_n for f_n, for each $n \geq 1$, and from a function $N \colon (0,1] \to \mathbb{N}$ that satisfies*

$$\forall \epsilon > 0 \; \forall n > N(\epsilon) \; \forall x \in M \left(d'(f_n(x), f(x)) \leq \epsilon\right).$$

Proof A modulus Δ for f may be defined as follows: given $\epsilon > 0$, take $\Delta(\epsilon) = \Delta_n(\epsilon/3)$ where $n = N(\epsilon/3) + 1$. □

2.6 Proposition. *Suppose $f, f_n \colon M \to M'$ and $f', f'_n \colon M' \to M''$ are functions $(n \geq 1)$ between metric spaces M, M', M''. If $(f_n \mid n \geq 1)$ converges uniformly to f on M and $(f'_n \mid n \geq 1)$ converges uniformly to f' on M', then $(f'_n \circ f_n \mid n \geq 1)$ converges uniformly to $f' \circ f$ on M.*

Fundamental to the continuous logic described in this article are the operations sup and inf on bounded sets of real numbers. We use these to define new functions from old, as follows. Suppose M, M' are metric spaces and $f \colon M \times M' \to \mathbb{R}$ is a bounded function. We define new functions $\sup_y f$ and $\inf_y f$ from M to $\mathbb{R}$ by

$$(\sup_y f)(x) = \sup\{f(x,y) | y \in M'\}$$

$$(\inf_y f)(x) = \inf\{f(x,y) | y \in M'\}$$

for all $x \in M$. Note that these new functions map M into the same closed bounded interval in $\mathbb{R}$ that contained the range of f. Our perspective is that $\sup_y$ and $\inf_y$ are quantifiers that bind or eliminate the variable y, analogous to the way $\forall$ and $\exists$ are used in ordinary first-order logic.

2.7 Proposition. *Suppose M, M' are metric spaces and f is a bounded uniformly continuous function from $M \times M'$ to $\mathbb{R}$. Let Δ be a modulus of uniform continuity for f. Then $\sup_y f$ and $\inf_y f$ are bounded uniformly continuous functions from M to $\mathbb{R}$, and Δ is a modulus of uniform continuity for both of them.*

Proof Fix $\epsilon > 0$ and consider $u, v \in M$ such that $d(u,v) < \Delta(\epsilon)$. Then for every $z \in M'$ we have

$$f(v,z) \leq f(u,z) + \epsilon \leq (\sup_y f)(u) + \epsilon.$$

Taking the sup over $z \in M'$ and interchanging the role of u and v yields

$$|(\sup_y f)(u) - (\sup_y f)(v)| \leq \epsilon.$$

The function $\inf_y f$ is handled similarly. □

2.8 Proposition. *Suppose M is a metric space and $f_s \colon M \to [0,1]$ is a uniformly continuous function for each s in the index set S. Let Δ be a common modulus of uniform continuity for $(f_s \mid s \in S)$. Then $\sup_s f_s$ and $\inf_s f_s$ are uniformly continuous functions from M to $[0,1]$, and Δ is a modulus of uniform continuity for both of them.*

Proof In the previous proof, take M' to be S with the discrete metric, and define $f(x,s) = f_s(x)$. □

2.9 Proposition. *Suppose M, M' are metric spaces and let $(f_n \mid n \geq 1)$ and f all be bounded functions from $M \times M'$ into $\mathbb{R}$. If $(f_n \mid n \geq 1)$ converges uniformly to f on $M \times M'$, then $(\sup_y f_n \mid n \geq 1)$ converges uniformly to $\sup_y f$ on M and $(\inf_y f_n \mid n \geq 1)$ converges uniformly to $\inf_y f$ on M.*

Proof Similar to the proof of Proposition 2.7. □

In many situations it is natural to construct a metric space as the quotient of a pseudometric space (M_0, d_0); here we mean that M_0 is a set and $d_0 \colon M_0 \times M_0 \to \mathbb{R}$ is a pseudometric. That is,

$$\begin{aligned}
&d_0(x,x) = 0 \\
&d_0(x,y) = d_0(y,x) \geq 0 \\
&d_0(x,z) \leq d_0(x,y) + d_0(y,z)
\end{aligned}$$

for all $x, y, z \in M_0$; these are the same conditions as in the definition of a metric, except that $d_0(x,y) = 0$ is allowed even when x, y are distinct.

If (M_0, d_0) is a pseudometric space, we may define an equivalence relation E on M_0 by $E(x,y) \Leftrightarrow d_0(x,y) = 0$. It follows from the triangle inequality that d_0 is E-invariant; that is, $d_0(x,y) = d_0(x',y')$ whenever xEx' and yEy'. Let M be the quotient set M_0/E and $\pi \colon M_0 \to M$ the quotient map, so $\pi(x)$ is the E-equivalence class of x, for each $x \in M_0$. Further, define d on M by setting $d(\pi(x), \pi(y)) = d_0(x,y)$ for any $x, y \in M_0$. Then (M,d) is a metric space and π is a distance preserving function from (M_0, d_0) onto (M,d). We will refer to (M,d) as the *quotient* metric space induced by (M_0, d_0).

Suppose (M_0, d_0) and (M_0', d_0') are pseudometric spaces with quotient metric spaces (M, d) and (M', d') and quotient maps π, π', respectively. Let $f_0\colon M_0 \to M_0'$ be any function. We say that f_0 is *uniformly continuous*, with modulus of uniform continuity Δ, if

$$d_0(x, y) < \Delta(\epsilon) \quad\Longrightarrow\quad d_0'(f_0(x), f_0(y)) \le \epsilon$$

for all $x, y \in M_0$ and all $\epsilon \in (0, 1]$. In that case it is clear that $d_0(x, y) = 0$ implies $d_0'(f_0(x), f_0(y)) = 0$ for all $x, y \in M_0$. Therefore we get a well defined quotient function $f\colon M \to M'$ by setting $f(\pi(x)) = \pi'(f_0(x))$ for all $x \in M_0$. Moreover, f is uniformly continuous with modulus of uniform continuity Δ.

The following results are useful in many places for expressing certain kinds of implications in continuous logic, and for reformulating the concept of uniform continuity.

2.10 Proposition. *Let $F, G\colon X \to [0, 1]$ be arbitrary functions such that*

$$(\star) \qquad \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in X\ \ (F(x) \le \delta \Rightarrow G(x) \le \epsilon)\,.$$

Then there exists an increasing, continuous function $\alpha\colon [0, 1] \to [0, 1]$ such that $\alpha(0) = 0$ and

$$(\star\star) \qquad \forall x \in X\ \ (G(x) \le \alpha(F(x)))\,.$$

Proof Define a (possibly discontinuous) function $g\colon [0, 1] \to [0, 1]$ by

$$g(t) = \sup\{G(x) \mid F(x) \le t\}$$

for $t \in [0, 1]$. It is clear that g is increasing and that $G(x) \le g(F(x))$ holds for all $x \in X$. Moreover, statement $(\star)$ implies that $g(0) = 0$ and that $g(t)$ converges to 0 as $t \to 0$.

To complete the proof we construct an increasing, continuous function $\alpha\colon [0, 1] \to [0, 1]$ such that $\alpha(0) = 0$ and $g(t) \le \alpha(t)$ for all $t \in [0, 1]$. Let $(t_n \mid n \in \mathbb{N})$ be any decreasing sequence in $[0, 1]$ with $t_0 = 1$ and $\lim_{n\to\infty} t_n = 0$. Define $\alpha\colon [0, 1] \to [0, 1]$ by setting $\alpha(0) = 0$, $\alpha(1) = 1$, and $\alpha(t_n) = g(t_{n-1})$ for all $n \ge 1$, and by taking α to be linear on each interval of the form $[t_{n+1}, t_n]$, $n \in \mathbb{N}$. It is easy to check that α has the desired properties. For example, if $t_1 \le t \le t_0 = 1$ we have that $\alpha(t)$ is a convex combination of $g(1)$ and 1 so that $g(t) \le g(1) \le \alpha(t)$. Similarly, if $t_{n+1} \le t \le t_n$ and $n \ge 1$, we have that $\alpha(t)$ is a convex combination of $g(t_n)$ and $g(t_{n-1})$ so that $g(t) \leq g(t_n) \leq \alpha(t)$. Together with $g(0) = \alpha(0) = 0$, this shows that $g(t) \leq \alpha(t)$ for all $t \in [0, 1]$. $\square$

2.11 Remark. Note that the converse to Proposition 2.10 is also true. Indeed, if statement ($\star\star$) holds and we fix $\epsilon > 0$, then taking

$$\delta = \sup\{s \mid \alpha(s) \leq \epsilon\} > 0$$

witnesses the truth of statement ($\star$).

2.12 Remark. The proof of Proposition 2.10 can be revised to show that the continuous function α can be chosen so that it only depends on the choice of an increasing function $\Delta\colon (0, 1] \to (0, 1]$ that witnesses the truth of statement ($\star$), in the sense that

$$\forall x \in X \ (F(x) \leq \Delta(\epsilon) \Rightarrow G(x) \leq \epsilon)$$

holds for each $\epsilon \in (0, 1]$. Given such a Δ, define $g\colon [0, 1] \to [0, 1]$ by $g(t) = \inf\{s \in (0, 1] \mid \Delta(s) > t\}$. It is easy to check that $g(0) = 0$ and that g is an increasing function. Moreover, for any $\epsilon > 0$ we have from the definition that $g(t) \leq \epsilon$ for any t in $[0, \Delta(\epsilon))$; therefore $g(t)$ converges to 0 as t tends to 0. Finally, we claim that $G(x) \leq g(F(x))$ holds for any $x \in X$. Otherwise we have $x \in X$ such that $g(F(x)) < G(x)$. The definition of g yields $s \in (0, 1]$ with $s < G(x)$ and $\Delta(s) > F(x)$; this contradicts our assumptions.

Now α is constructed from g as in the proof of Proposition 2.10. This yields an increasing, continuous function $\alpha\colon [0, 1] \to [0, 1]$ with $\alpha(0) = 0$ such that whenever $F, G\colon X \to [0, 1]$ are functions satisfying

$$\forall x \in X \ (F(x) \leq \Delta(\epsilon) \Rightarrow G(x) \leq \epsilon)$$

for each $\epsilon \in (0, 1]$, then we have

$$\forall x \in X \ \ (G(x) \leq \alpha(F(x)))\,.$$

3 Formulas and their interpretations

Fix a signature L for metric structures, as described in the previous section. As indicated there (see page 6), we assume for simplicity of notation that $D_L = 1$ and that $I_P = [0, 1]$ for every predicate symbol P.

Symbols of L

Among the *symbols* of L are the predicate, function, and constant symbols; these will be referred to as the *nonlogical* symbols of L and the remaining ones will be called the *logical* symbols of L. Among the logical symbols is a symbol d for the metric on the underlying metric space of an L-structure; this is treated formally as equivalent to a predicate symbol of arity 2. The logical symbols also include an infinite set V_L of *variables*; usually we take V_L to be countable, but there are situations in which it is useful to permit a larger number of variables. The remaining logical symbols consist of a symbol for each continuous function $u\colon [0,1]^n \to [0,1]$ of finitely many variables $n \geq 1$ (these play the role of connectives) and the symbols sup and inf, which play the role of quantifiers in this logic.

The *cardinality* of L, denoted $\mathrm{card}(L)$, is the smallest infinite cardinal number $\geq$ the number of nonlogical symbols of L.

Terms of L

Terms are formed inductively, exactly as in first-order logic. Each variable and constant symbol is an L-term. If f is an n-ary function symbol and $t_1, \dots, t_n$ are L-terms, then $f(t_1, \dots, t_n)$ is an L-term. All L-terms are constructed in this way.

Atomic formulas of L.

The *atomic formulas* of L are the expressions of the form $P(t_1, \dots, t_n)$, in which P is an n-ary predicate symbol of L and $t_1, \dots, t_n$ are L-terms; as well as $d(t_1, t_2)$, in which t_1 and t_2 are L-terms.

Note that the logical symbol d for the metric is treated formally as a binary predicate symbol, exactly analogous to how the equality symbol $=$ is treated in first-order logic.

Formulas of L

Formulas are also constructed inductively, and the basic structure of the induction is similar to the corresponding definition in first-order logic. Continuous functions play the role of connectives and sup and inf are used formally in the way that quantifiers are used in first-order logic. The precise definition is as follows:

3.1 Definition. The class of L-formulas is the smallest class of expressions satisfying the following requirements:

(1) Atomic formulas of L are L-formulas.
(2) If $u\colon [0,1]^n \to [0,1]$ is continuous and $\varphi_1, \dots, \varphi_n$ are L-formulas, then $u(\varphi_1, \dots, \varphi_n)$ is an L-formula.
(3) If φ is an L-formula and x is a variable, then $\sup_x \varphi$ and $\inf_x \varphi$ are L-formulas.

3.2 Remark. We have chosen to take all continuous functions on $[0,1]$ as our connectives. This is both too restrictive (see section 9 in which we want to close our set of formulas under certain kinds of limits, in order to develop a good notion of *definability*) and too general (see section 6). We made this choice in order to introduce formulas as early and as directly as possible.

An L-formula is *quantifier free* if it is generated inductively from atomic formulas without using the last clause; *i.e.*, neither $\sup_x$ nor $\inf_x$ are used.

Many syntactic notions from first-order logic can be carried over word for word into this setting. We will assume that this has been done by the reader for many such concepts, including *subformula* and *syntactic substitution* of a term for a variable, or a formula for a subformula, and so forth.

Free and *bound* occurrences of variables in L-formulas are defined in a way similar to how this is done in first-order logic. Namely, an occurrence of the variable x is bound if lies within a subformula of the form $\sup_x \varphi$ or $\inf_x \varphi$, and otherwise it is free.

An *L-sentence* is an L-formula that has no free variables.

When t is a term and the variables occurring in it are among the variables $x_1, \dots, x_n$ (which we always take to be distinct in this context), we indicate this by writing t as $t(x_1, \dots, x_n)$.

Similarly, we write an L-formula as $\varphi(x_1, \dots, x_n)$ to indicate that its free variables are among $x_1, \dots, x_n$.

Prestructures

It is common in mathematics to construct a metric space as the quotient of a pseudometric space or as the completion of such a quotient, and the same is true of metric structures. For that reason we need to consider what we will call *prestructures* and to develop the semantics of continuous logic for them.

As above, we take L to be a fixed signature for metric structures. Let (M_0, d_0) be a pseudometric space, satisfying the requirement that its diameter is $\leq D_L$. (That is, $d_0(x, y) \leq D_L$ for all $x, y \in M_0$.) An *L-prestructure* $\mathcal{M}_0$ based on (M_0, d_0) is a structure consisting of the following data:

(1) for each predicate symbol P of L (of arity n) a function $P^{\mathcal{M}_0}$ from M_0^n into I_P that has Δ_P as a modulus of uniform continuity;
(2) for each function symbol f of L (of arity n) a function $f^{\mathcal{M}_0}$ from M_0^n into M_0 that has Δ_f as a modulus of uniform continuity; and
(3) for each constant symbol c of L an element $c^{\mathcal{M}_0}$ of M_0.

Given an L-prestructure $\mathcal{M}_0$, we may form its *quotient* prestructure as follows. Let (M, d) be the quotient metric space induced by (M_0, d_0) with quotient map $\pi\colon M_0 \to M$. Then

(1) for each predicate symbol P of L (of arity n) define $P^{\mathcal{M}}$ from M^n into I_P by setting $P^{\mathcal{M}}(\pi(x_1), \ldots, \pi(x_n)) = P^{\mathcal{M}_0}(x_1, \ldots, x_n)$ for each $x_1, \ldots, x_n \in M_0$;
(2) for each function symbol f of L (of arity n) define $f^{\mathcal{M}}$ from M^n into M by setting $f^{\mathcal{M}}(\pi(x_1), \ldots, \pi(x_n)) = \pi(f^{\mathcal{M}_0}(x_1, \ldots, x_n))$ for each $x_1, \ldots, x_n \in M_0$;
(3) for each constant symbol c of L define $c^{\mathcal{M}} = \pi(c^{\mathcal{M}_0})$.

It is obvious that (M, d) has the same diameter as (M_0, d_0). Also, as noted in the appendix to section 2, for each predicate symbol P and each function symbol f of L, the predicate $P^{\mathcal{M}}$ is well defined and has Δ_P as a modulus of uniform continuity and the function $f^{\mathcal{M}}$ is well defined and has Δ_f as a modulus of uniform continuity. In other words, this defines an L-prestructure (which we will denote as $\mathcal{M}$) based on the (possibly not complete) metric space (M, d).

Finally, we may define an L-structure $\mathcal{N}$ by taking a *completion* of $\mathcal{M}$. This is based on a complete metric space (N, d) that is a completion of (M, d), and its additional structure is defined in the following natural way (made possible by the fact that the predicates and functions given by $\mathcal{M}$ are uniformly continuous):

(1) for each predicate symbol P of L (of arity n) define $P^{\mathcal{N}}$ from N^n into I_P to be the unique such function that extends $P^{\mathcal{M}}$ and is continuous;

(2) for each function symbol f of L (of arity n) define $f^{\mathcal{N}}$ from N^n into N to be the unique such function that extends $f^{\mathcal{M}}$ and is continuous;
(3) for each constant symbol c of L define $c^{\mathcal{N}} = c^{\mathcal{M}}$.

It is obvious that (N, d) has the same diameter as (M, d). Also, as noted in the appendix to section 2, for each predicate symbol P and each function symbol f of L, the predicate $P^{\mathcal{N}}$ has Δ_P as a modulus of uniform continuity and the function $f^{\mathcal{N}}$ has Δ_f as a modulus of uniform continuity. In other words, $\mathcal{N}$ is an L-structure.

Semantics

Let $\mathcal{M}$ be any L-prestructure, with $(M, d^{\mathcal{M}})$ as its underlying pseudometric space, and let A be a subset of M. We extend L to a signature $L(A)$ by adding a new constant symbol $c(a)$ to L for each element a of A. We extend the interpretation given by $\mathcal{M}$ in a canonical way, by taking the interpretation of $c(a)$ to be equal to a itself for each $a \in A$. We call $c(a)$ the *name* of a in $L(A)$. Indeed, we will often write a instead of $c(a)$ when no confusion can result from doing so.

Given an $L(M)$-term $t(x_1, \ldots, x_n)$, we define, exactly as in first-order logic, the interpretation of t in $\mathcal{M}$, which is a function $t^{\mathcal{M}} \colon M^n \to M$.

We now come to the key definition in continuous logic for metric structures, in which the semantics of this logic is defined. For each $L(M)$-sentence σ, we define *the value of* σ *in* $\mathcal{M}$. This value is a real number in the interval $[0, 1]$ and it is denoted $\sigma^{\mathcal{M}}$. The definition is by induction on formulas. Note that in the definition all terms mentioned are $L(M)$-terms in which no variables occur.

3.3 Definition. (1) $\big(d(t_1, t_2)\big)^{\mathcal{M}} = d^{\mathcal{M}}(t_1^{\mathcal{M}}, t_2^{\mathcal{M}})$ for any t_1, t_2;
(2) for any n-ary predicate symbol P of L and any $t_1, \ldots, t_n$,
$$\big(P(t_1, \ldots, t_n)\big)^{\mathcal{M}} = P^{\mathcal{M}}(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}});$$
(3) for any $L(M)$-sentences $\sigma_1, \ldots, \sigma_n$ and any continuous function $u \colon [0, 1]^n \to [0, 1]$,
$$\big(u(\sigma_1, \ldots, \sigma_n)\big)^{\mathcal{M}} = u(\sigma_1^{\mathcal{M}}, \ldots, \sigma_n^{\mathcal{M}});$$
(4) for any $L(M)$-formula $\varphi(x)$,
$$\big(\sup_x \varphi(x)\big)^{\mathcal{M}}$$

is the supremum in $[0,1]$ of the set $\{\varphi(a)^{\mathcal{M}} \mid a \in M\}$;

(5) for any $L(M)$-formula $\varphi(x)$,

$$\big(\inf_x \varphi(x)\big)^{\mathcal{M}}$$

is the infimum in $[0,1]$ of the set $\{\varphi(a)^{\mathcal{M}} \mid a \in M\}$.

3.4 Definition. Given an $L(M)$-formula $\varphi(x_1,\dots,x_n)$ we let $\varphi^{\mathcal{M}}$ denote the function from M^n to $[0,1]$ defined by

$$\varphi^{\mathcal{M}}(a_1,\dots,a_n) = \big(\varphi(a_1,\dots,a_n)\big)^{\mathcal{M}}.$$

A key fact about formulas in continuous logic is that they define uniformly continuous functions. Indeed, the modulus of uniform continuity for the predicate does not depend on $\mathcal{M}$ but only on the data given by the signature L.

3.5 Theorem. *Let $t(x_1,\dots,x_n)$ be an L-term and $\varphi(x_1,\dots,x_n)$ an L-formula. Then there exist functions Δ_t and Δ_φ from $(0,1]$ to $(0,1]$ such that for any L-prestructure $\mathcal{M}$, Δ_t is a modulus of uniform continuity for the function $t^{\mathcal{M}}\colon M^n \to M$ and Δ_φ is a modulus of uniform continuity for the predicate $\varphi^{\mathcal{M}}\colon M^n \to [0,1]$.*

Proof The proof is by induction on terms and then induction on formulas. The basic tools concerning uniform continuity needed for the induction steps in the proof are given in the appendix to section 2. □

3.6 Remark. The previous result is the counterpart in this logic of the Perturbation Lemma in the logic of positive bounded formulas with the approximate semantics. See [24, Proposition 5.15].

3.7 Theorem. *Let $\mathcal{M}_0$ be an L-prestructure with underlying pseudometric space (M_0,d_0); let $\mathcal{M}$ be its quotient L-structure with quotient map $\pi\colon M_0 \to M$ and let $\mathcal{N}$ be the L-structure that results from completing $\mathcal{M}$ (as explained on page 15). Let $t(x_1,\dots,x_n)$ be any L-term and $\varphi(x_1,\dots,x_n)$ be any L-formula. Then:*

(1) $t^{\mathcal{M}}(\pi(a_1),\dots,\pi(a_n)) = t^{\mathcal{M}_0}(a_1,\dots,a_n)$ for all $a_1,\dots,a_n \in M_0$;

(2) $t^{\mathcal{N}}(b_1,\dots,b_n) = t^{\mathcal{M}}(b_1,\dots,b_n)$ for all $b_1,\dots,b_n \in M$.

(3) $\varphi^{\mathcal{M}}(\pi(a_1),\dots,\pi(a_n)) = \varphi^{\mathcal{M}_0}(a_1,\dots,a_n)$ for all $a_1,\dots,a_n \in M_0$;

(4) $\varphi^{\mathcal{N}}(b_1,\dots,b_n) = \varphi^{\mathcal{M}}(b_1,\dots,b_n)$ for all $b_1,\dots,b_n \in M$.

Proof The proofs are by induction on terms and then induction on formulas. In handling the quantifier cases in (3) the key is that the quotient map π is surjective. For the quantifier cases in (4), the key is that the functions $\varphi^{\mathcal{N}}$ are continuous and that $\mathcal{M}$ is dense in $\mathcal{N}$. $\square$

3.8 Caution. Note that we only use words such as *structure* when the underlying metric space is *complete*. In some constructions this means that we must take a metric space completion at the end. Theorem 3.7 shows that this preserves all properties expressible in continuous logic.

Logical equivalence

3.9 Definition. Two L-formulas $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are said to be *logically equivalent* if

$$\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = \psi^{\mathcal{M}}(a_1, \ldots, a_n)$$

for every L-structure $\mathcal{M}$ and every $a_1, \ldots, a_n \in M$.

If $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are L-formulas, we can extend the preceding definition by taking the *logical distance* between φ and ψ to be the supremum of all numbers

$$|\varphi^{\mathcal{M}}(a_1, \ldots, a_n) - \psi^{\mathcal{M}}(a_1, \ldots, a_n)|$$

where $\mathcal{M}$ is any L-structure and $a_1, \ldots, a_n \in M$. This defines a pseudometric on the set of all formulas with free variables among $x_1, \ldots, x_n$, and two formulas are logically equivalent if and only if the logical distance between them is 0.

3.10 Remark. Note that by Theorem 3.7, we could use L-prestructures in place of L-structures in the preceding Definition without changing the meaning of the concepts defined.

3.11 Remark. (Size of the space of L-formulas)
Some readers may be concerned that the set of L-formulas is be too large, because we allow all continuous functions as connectives. What matters, however, is the size of a set of L-formulas that is dense in the set of all L-formulas with respect to the logical distance defined in the previous paragraph. By Weierstrass's Theorem, there is a countable set of functions from $[0, 1]^n$ to $[0, 1]$ that is dense in the set of all continuous functions, with respect to the sup-distance between such functions. (The

sup-distance between f and g is the supremum of $|f(x) - g(x)|$ as x ranges over the common domain of f, g.) If we only use such connectives in building L-formulas, then (a) the total number of formulas that are constructed is $\mathrm{card}(L)$, and (b) any L-formula can be approximated arbitrarily closely in logical distance by a formula constructed using the restricted connectives. Thus the density character of the set of L-formulas with respect to logical distance is always $\leq \mathrm{card}(L)$. (We explore this topic in more detail in section 6.)

Conditions of L

An L-*condition* E is a formal expression of the form $\varphi = 0$, where φ is an L-formula. We call E *closed* if φ is a sentence. If $x_1, \dots, x_n$ are distinct variables, we write an L-condition as $E(x_1, \dots, x_n)$ to indicate that it has the form $\varphi(x_1, \dots, x_n) = 0$ (in other words, that the free variables of E are among $x_1, \dots, x_n$).

If E is the $L(M)$-condition $\varphi(x_1, \dots, x_n) = 0$ and $a_1, \dots, a_n$ are in M, we say E is *true of* $a_1, \dots, a_n$ *in* $\mathcal{M}$ and write $\mathcal{M} \models E[a_1, \dots, a_n]$ if $\varphi^{\mathcal{M}}(a_1, \dots, a_n) = 0$.

3.12 Definition. Let E_i be the L-condition $\varphi_i(x_1, \dots, x_n) = 0$, for $i = 1, 2$. We say that E_1 and E_2 are *logically equivalent* if for every L-structure $\mathcal{M}$ and every $a_1, \dots, a_n$ we have

$$\mathcal{M} \models E_1[a_1, \dots, a_n] \quad \text{iff} \quad \mathcal{M} \models E_2[a_1, \dots, a_n].$$

3.13 Remark. When φ and ψ are formulas, it is convenient to introduce the expression $\varphi = \psi$ as an abbreviation for the condition $|\varphi - \psi| = 0$. (Note that $u\colon [0,1]^2 \to [0,1]$ defined by $u(t_1, t_2) = |t_1 - t_2|$ is a connective.) Since each real number $r \in [0,1]$ is a connective (thought of as a constant function), expressions of the form $\varphi = r$ will thereby be regarded as conditions for any L-formula φ and $r \in [0,1]$. Note that the interpretation of $\varphi = \psi$ is semantically correct; namely for any L-structure $\mathcal{M}$ and elements a of M, $|\varphi - \psi|^{\mathcal{M}}(a) = 0$ if and only if $\varphi^{\mathcal{M}}(a) = \psi^{\mathcal{M}}(a)$.

Similarly, we introduce the expressions $\varphi \leq \psi$ and $\psi \geq \varphi$ as abbreviations for certain conditions. Let $\dot{-}\colon [0,1]^2 \to [0,1]$ be the connective defined by $\dot{-}(t_1, t_2) = \max(t_1 - t_2, 0) = t_1 - t_2$ if $t_1 \geq t_2$ and 0 otherwise. Usually we write $t_1 \dot{-} t_2$ in place of $\dot{-}(t_1, t_2)$. We take $\varphi \leq \psi$ and $\psi \geq \varphi$ to be abbreviations for the condition $\varphi \dot{-} \psi = 0$. (See section 6, where this connective plays a central role.) In $[0,1]$-valued logic, the condition $\varphi \le \psi$ can be seen as family of implications, from the condition $\psi \le r$ to the condition $\varphi \le r$ for each $r \in [0,1]$.

4 Model theoretic concepts

Fix a signature L for metric structures. In this section we introduce several of the most fundamental model theoretic concepts and discuss some of their basic properties.

4.1 Definition. A *theory* in L is a set of closed L-conditions. If T is a theory in L and $\mathcal{M}$ is an L-structure, we say that $\mathcal{M}$ *is a model* of T and write $\mathcal{M} \models T$ if $\mathcal{M} \models E$ for every condition E in T. We write $\mathrm{Mod}_L(T)$ for the collection of all L-structures that are models of T. (If L is clear from the context, we write simply $\mathrm{Mod}(T)$.)

If $\mathcal{M}$ is an L-structure, the *theory of* $\mathcal{M}$, denoted $\mathrm{Th}(\mathcal{M})$, is the set of closed L-conditions that are true in $\mathcal{M}$. If T is a theory of this form, it will be called *complete*.

If T is an L-theory and E is a closed L-condition, we say E *is a logical consequence of* T and write $T \models E$ if $\mathcal{M} \models E$ holds for every model $\mathcal{M}$ of T.

4.2 Caution. Note that we only use words such as *model* when the underlying metric space is *complete*. Theorem 3.7 shows that whenever T is an L-theory and $\mathcal{M}_0$ is an L-prestructure such that $\varphi^{\mathcal{M}_0} = 0$ for every condition $\varphi = 0$ in T, then the completion of the canonical quotient of $\mathcal{M}_0$ is indeed a *model* of T.

4.3 Definition. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are L-structures.

(1) We say that $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent*, and write $\mathcal{M} \equiv \mathcal{N}$, if $\sigma^{\mathcal{M}} = \sigma^{\mathcal{N}}$ for all L-sentences σ. Equivalently, this holds if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.

(2) If $\mathcal{M} \subseteq \mathcal{N}$ we say that $\mathcal{M}$ is an *elementary substructure of* $\mathcal{N}$, and write $\mathcal{M} \preceq \mathcal{N}$, if whenever $\varphi(x_1, \dots, x_n)$ is an L-formula and $a_1, \dots, a_n$ are elements of M, we have

$$\varphi^{\mathcal{M}}(a_1, \dots, a_n) = \varphi^{\mathcal{N}}(a_1, \dots, a_n).$$

In this case, we also say that $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$.

(3) A function F from a subset of M into N is an *elementary map* from $\mathcal{M}$ into $\mathcal{N}$ if whenever $\varphi(x_1,\dots,x_n)$ is an L-formula and $a_1,\dots,a_n$ are elements of the domain of F, we have

$$\varphi^{\mathcal{M}}(a_1,\dots,a_n) = \varphi^{\mathcal{N}}(F(a_1),\dots,F(a_n)).$$

(4) An *elementary embedding* of $\mathcal{M}$ into $\mathcal{N}$ is a function from all of M into N that is an elementary map from $\mathcal{M}$ into $\mathcal{N}$.

4.4 Remark. (1) Every elementary map from one metric structure into another is distance preserving.
(2) The collection of elementary maps is closed under composition and formation of the inverse.
(3) Every isomorphism between metric structures is an elementary embedding.

In the following result we refer to sets $\mathcal{S}$ of L-formulas that are *dense with respect to logical distance*. (See page 18.) That is, such a set has the following property: for any L-formula $\varphi(x_1,\dots,x_n)$ and any $\epsilon > 0$ there is $\psi(x_1,\dots,x_n)$ in $\mathcal{S}$ such that for any L-structure $\mathcal{M}$ and any $a_1,\dots,a_n \in M$

$$|\varphi^{\mathcal{M}}(a_1,\dots,a_n) - \psi^{\mathcal{M}}(a_1,\dots,a_n)| \le \epsilon.$$

4.5 Proposition. *(Tarski-Vaught Test for $\preceq$) Let $\mathcal{S}$ be any set of L-formulas that is dense with respect to logical distance. Suppose $\mathcal{M}$, $\mathcal{N}$ are L-structures with $\mathcal{M} \subseteq \mathcal{N}$. The following statements are equivalent:*

(1) $\mathcal{M} \preceq \mathcal{N}$;
(2) *For every L-formula $\varphi(x_1,\dots,x_n,y)$ in $\mathcal{S}$ and $a_1,\dots,a_n \in M$,*

$$\inf\{\varphi^{\mathcal{N}}(a_1,\dots,a_n,b) \mid b \in N\} = \inf\{\varphi^{\mathcal{N}}(a_1,\dots,a_n,c) \mid c \in M\}$$

Proof If (1) holds, then we may conclude (2) for the set of all L-formulas directly from the meaning of $\preceq$. Indeed, if $\varphi(x_1,\dots,x_n,y)$ is any L-formula and $a_1,\dots,a_n \in A$, then from (1) we have

$$\inf\{\varphi^{\mathcal{N}}(a_1,\dots,a_n,b) \mid b \in N\} = \big(\inf_y \varphi(a_1,\dots,a_n,y)\big)^{\mathcal{N}} =$$

$$\big(\inf_y \varphi(a_1,\dots,a_n,y)\big)^{\mathcal{M}} = \inf\{\varphi^{\mathcal{M}}(a_1,\dots,a_n,c) \mid c \in M\} =$$

$$\inf\{\varphi^{\mathcal{N}}(a_1,\dots,a_n,c) \mid c \in M\}.$$

For the converse, suppose (2) holds for a set $\mathcal{S}$ that is dense in the set of all L-formulas with respect to logical distance. First we will prove that (2) holds for the set of all L-formulas. Let $\varphi(x_1, \ldots, x_n, y)$ be any L-formula. Given $\epsilon > 0$, let $\psi(x_1, \ldots, x_n, y)$ be an element of $\mathcal{S}$ that approximates $\varphi(x_1, \ldots, x_n, y)$ to within ϵ in logical distance. Let $a_1, \ldots, a_n$ be elements of M. Then we have

$$\begin{aligned}
&\inf\{\varphi^{\mathcal{N}}(a_1, \ldots, a_n, b) \mid b \in N\} \\
&\leq \inf\{\psi^{\mathcal{N}}(a_1, \ldots, a_n, b) \mid b \in N\} + \epsilon \\
&= \inf\{\psi^{\mathcal{N}}(a_1, \ldots, a_n, c) \mid c \in M\} + \epsilon \\
&\leq \inf\{\varphi^{\mathcal{N}}(a_1, \ldots, a_n, c) \mid c \in M\} + 2\epsilon.
\end{aligned}$$

Letting ϵ tend to 0 and recalling $M \subseteq N$ we obtain the desired equality for $\varphi(x_1, \ldots, x_n, y)$.

Now assume that (2) holds for the set of all L-formulas. One proves the equivalence

$$\psi^{\mathcal{M}}(a_1, \ldots, a_n) = \psi^{\mathcal{N}}(a_1, \ldots, a_n)$$

(for all $a_1, \ldots, a_n$ in M) by induction on the complexity of ψ, using (2) to cover the case when ψ begins with sup or inf. $\square$

5 Ultraproducts and compactness

First we discuss ultrafilter limits in topology. Let X be a topological space and let $(x_i)_{i \in I}$ be a family of elements of X. If D is an ultrafilter on I and $x \in X$, we write

$$\lim_{i,D} x_i = x$$

and say *x is the D-limit of $(x_i)_{i \in I}$* if for every neighborhood U of x, the set $\{ i \in I \mid x_i \in U \}$ is in the ultrafilter D. A basic fact from general topology is that X is a compact Hausdorff space if and only if for every family $(x_i)_{i \in I}$ in X and every ultrafilter D on I the D-limit of $(x_i)_{i \in I}$ exists and is unique.

The following lemmas are needed below when we connect ultrafilter limits and the semantics of continuous logic.

5.1 Lemma. *Suppose X, X' are topological spaces and $F \colon X \to X'$ is continuous. For any family $(x_i)_{i \in I}$ from X and any ultrafilter D on I, we have that*

$$\lim_{i,D} x_i = x \quad \Longrightarrow \quad \lim_{i,D} F(x_i) = F(x)$$

where the ultrafilter limits are taken in X and X' respectively.

Proof Let U be an open neighborhood of $F(x)$ in X'. Since F is continuous, $F^{-1}(U)$ is open in X, and it contains x. If x is the D-limit of $(x_i)_{i\in I}$, there exists $A \in D$ such that for all $i \in A$ we have $x_i \in F^{-1}(U)$ and hence $F(x_i) \in U$. □

5.2 Lemma. *Let X be a closed, bounded interval in $\mathbb{R}$. Let S be any set and let $(F_i \mid i \in I)$ be a family of functions from S into X. Then, for any ultrafilter D on I*

$$\sup_x \big(\lim_{i,D} F_i(x)\big) \quad \leq \quad \lim_{i,D} \big(\sup_x F_i(x)\big), \textit{ and}$$

$$\inf_x \big(\lim_{i,D} F_i(x)\big) \quad \geq \quad \lim_{i,D} \big(\inf_x F_i(x)\big).$$

where in both cases, $\sup_x$ *and* $\inf_x$ *are taken over $x \in S$. Moreover, for each $\epsilon > 0$ there exist $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ in S such that*

$$\lim_{i,D} F_i(x_i) + \epsilon \quad \geq \quad \lim_{i,D} \big(\sup_x F_i(x)\big), \textit{ and}$$

$$\lim_{i,D} F_i(y_i) - \epsilon \quad \leq \quad \lim_{i,D} \big(\inf_x F_i(x)\big).$$

Proof We prove the statements involving sup; the inf statements are proved similarly (or by replacing each F_i by its negative).

Let $r_i = \sup_x F_i(x)$ for each $i \in I$ and let $r = \lim_{i,D} r_i$. For each $\epsilon > 0$, let $A(\epsilon) \in D$ be such that $r - \epsilon < r_i < r + \epsilon$ for every $i \in A(\epsilon)$.

First we show $\sup_x \lim_{i,D} F_i(x) \leq r$. For each $i \in A(\epsilon)$ and $x \in S$ we have $F_i(x) \leq r_i < r + \epsilon$. Hence the D-limit of $(F_i(x))_{i\in I}$ is $\leq r + \epsilon$. Letting ϵ tend to 0 gives the desired inequality.

For the other sup statement, fix $\epsilon > 0$ and for each $i \in I$ choose $x_i \in S$ so that $r_i \leq F_i(x_i) + \epsilon/2$. Then for $i \in A(\epsilon/2)$ we have $r \leq F_i(x_i) + \epsilon$. Taking the D-limit gives the desired inequality. □

Ultraproducts of metric spaces

Let $((M_i, d_i) \mid i \in I)$ be a family of bounded metric spaces, all having diameter $\leq K$. Let D be an ultrafilter on I. Define a function d on the Cartesian product $\prod_{i\in I} M_i$ by

$$d(x, y) = \lim_{i,D} d_i(x_i, y_i)$$

where $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$. This D-limit is taken in the interval $[0, K]$. It is easy to check that d is a pseudometric on $\prod_{i\in I} M_i$.

For $x, y \in \prod_{i\in I} M_i$, define $x \sim_D y$ to mean that $d(x, y) = 0$. Then $\sim_D$ is an equivalence relation, so we may define

$$\Big(\prod_{i\in I} M_i\Big)_D = \Big(\prod_{i\in I} M_i\Big)\Big/\sim_D.$$

The pseudometric d on $\prod_{i\in I} M_i$ induces a metric on this quotient space, and we also denote this metric by d.

The space $(\prod_{i\in I} M_i)_D$ with the induced metric d is called the *D-ultraproduct of* $((M_i, d_i) \mid i \in I)$. We denote the equivalence class of $(x_i)_{i\in I} \in \prod_{i\in I} M_i$ under $\sim_D$ by $((x_i)_{i\in I})_D$.

If $(M_i, d_i) = (M, d)$ for every $i \in I$, the space $(\prod_{i\in I} M_i)_D$ is called the *D-ultrapower of* M and it is denoted $(M)_D$. In this situation, the map $T\colon M \to (M)_D$ defined by $T(x) = ((x_i)_{i\in I})_D$, where $x_i = x$ for every $i \in I$, is an isometric embedding. It is called the *diagonal embedding* of M into $(M)_D$.

A particular case of importance is the D-ultrapower of a compact metric space (M, d). In that case the diagonal embedding of M into $(M)_D$ is surjective. Indeed, if $(x_i)_{i\in I} \in M^I$ and x is the D-limit of the family $(x_i)_{i\in I}$, which exists since (M, d) is compact, then it is easy to show that $((x_i)_{i\in I})_D = T(x)$. In particular, any ultrapower of a closed bounded interval may be canonically identified with the interval itself.

Since we require that structures are based on *complete* metric spaces, it is useful to note that every ultraproduct of such spaces is itself complete.

5.3 Proposition. *Let $((M_i, d_i) \mid i \in I)$ be a family of complete, bounded metric spaces, all having diameter $\leq K$. Let D be an ultrafilter on I and let (M, d) be the D-ultraproduct of $((M_i, d_i) \mid i \in I)$. The metric space (M, d) is complete.*

Proof Let $(x^k)_{k\geq 1}$ be a Cauchy sequence in (M, d). Without loss of generality we may assume that $d(x^k, x^{k+1}) < 2^{-k}$ holds for all $k \geq 1$; that is, to prove (M, d) complete it suffices to show that all such Cauchy sequences have a limit. For each $k \geq 1$ let x^k be represented by the family $(x_i^k)_{i\in I}$. For each $m \geq 1$ let A_m be the set of all $i \in I$ such that $d_i(x_i^k, x_i^{k+1}) < 2^{-k}$ holds for all $k = 1, \ldots, m$. Then the sets $(A_m)_{m\geq 1}$ form a decreasing chain and all of them are in D.

We define a family $(y_i)_{i\in I}$ that will represent the limit of the sequence

$(x^k)_{k\geq 1}$ in (M,d). If $i \notin A_1$, then we take y_i to be an arbitrary element of M_i. If for some $m \geq 1$ we have $i \in A_m \setminus A_{m+1}$, then we set $y_i = x_i^{m+1}$. If $i \in A_m$ holds for all $m \geq 1$, then $(x_i^m)_{m\geq 1}$ is a Cauchy sequence in the complete metric space (M_i, d_i) and we take y_i to be its limit.

An easy calculation shows that for each $m \geq 1$ and each $i \in A_m$ we have $d_i(x_i^m, y_i) \leq 2^{-m+1}$. It follows that $((y_i)_{i\in I})_D$ is the limit in the ultraproduct (M,d) of the sequence $(x^k)_{k\geq 1}$. $\square$

Ultraproducts of functions

Suppose $((M_i, d_i) \mid i \in I)$ and $((M_i', d_i') \mid i \in I)$ are families of metric spaces, all of diameter $\leq K$. Fix $n \geq 1$ and suppose $f_i\colon M_i^n \to M_i'$ is a uniformly continuous function for each $i \in I$. Moreover, suppose the single function $\Delta\colon (0,1] \to (0,1]$ is a modulus of uniform continuity for all of the functions f_i. Given an ultrafilter D on I, we define a function

$$\Big(\prod_{i\in I} f_i\Big)_D \colon \Big(\prod_{i\in I} M_i\Big)_D^n \to \Big(\prod_{i\in I} M_i'\Big)_D$$

as follows. If for each $k = 1,\dots,n$ we have $(x_i^k)_{i\in I} \in \prod_{i\in I} M_i$, we define

$$\Big(\prod_{i\in I} f_i\Big)_D \big(((x_i^1)_{i\in I})_D, \dots, ((x_i^n)_{i\in I})_D\big) = \Big(\Big(f_i(x_i^1,\dots,x_i^n)\Big)_{i\in I}\Big)_D.$$

We claim that this defines a uniformly continuous function that also has Δ as its modulus of uniform continuity. For simplicity of notation, suppose $n = 1$. Fix $\epsilon > 0$. Suppose the distance between $\big((x_i)_{i\in I}\big)_D$ and $\big((y_i)_{i\in I}\big)_D$ in the ultraproduct $\Big(\prod_{i\in I} M_i\Big)_D$ is $< \Delta(\epsilon)$. There must exist $A \in D$ such that for all $i \in A$ we have $d_i(x_i, y_i) < \Delta(\epsilon)$. Since Δ is a modulus of uniform continuity for all of the functions f_i, it follows that $d_i'(f_i(x_i), f_i(y_i)) \leq \epsilon$ for all $i \in A$. Hence the distance in the ultraproduct $\Big(\prod_{i\in I} M_i'\Big)_D$ between $\big((f(x_i))_{i\in I}\big)_D$ and $\big((f(y_i))_{i\in I}\big)_D$ must be $\leq \epsilon$. This shows that $\Big(\prod_{i\in I} f_i\Big)_D$ is well defined and that it has Δ as a modulus of uniform continuity. (Note that the precise form of our definition of "modulus of uniform continuity" played a role in this argument.)

Ultraproducts of L-structures

Let $(\mathcal{M}_i \mid i \in I)$ be a family of L-structures and let D be an ultrafilter on I. Suppose the underlying metric space of $\mathcal{M}_i$ is (M_i, d_i). Since there is a uniform bound on the diameters of these metric spaces, we may form their D-ultraproduct. For each function symbol f of L, the functions $f^{\mathcal{M}_i}$ all have the same modulus of uniform continuity Δ_f. Therefore the D-ultraproduct of this family of functions is well defined. The same is true if we consider a predicate symbol P of L. Moreover, the functions $P^{\mathcal{M}_i}$ all have their values in $[0,1]$, whose D-ultrapower can be identified with $[0,1]$ itself; thus the D-ultraproduct of $(P^{\mathcal{M}_i} \mid i \in I)$ can be regarded as a $[0,1]$-valued function on M.

Therefore we may define the *D-ultraproduct of the family* $(\mathcal{M}_i \mid i \in I)$ *of L-structures* to be the L-structure $\mathcal{M}$ that is specified as follows:

The underlying metric space of $\mathcal{M}$ is given by the ultraproduct of metric spaces

$$M = \Big(\prod_{i\in I} M_i\Big)_D.$$

For each predicate symbol P of L, the interpretation of P in $\mathcal{M}$ is given by the ultraproduct of functions

$$P^{\mathcal{M}} = \Big(\prod_{i\in I} P^{\mathcal{M}_i}\Big)_D$$

which maps M^n to $[0,1]$. For each function symbol f of L, the interpretation of f in $\mathcal{M}$ is given by the ultraproduct of functions

$$f^{\mathcal{M}} = \Big(\prod_{i\in I} f^{\mathcal{M}_i}\Big)_D$$

which maps M^n to M. For each constant symbol c of L, the interpretation of c in $\mathcal{M}$ is given by

$$c^{\mathcal{M}} = \big((c^{\mathcal{M}_i})_{i\in I}\big)_D.$$

The discussion above shows that this defines $\mathcal{M}$ to be a well-defined L-structure. We call $\mathcal{M}$ the *D-ultraproduct of the family* $(\mathcal{M}_i \mid i \in I)$ and denote it by

$$\mathcal{M} = \Big(\prod_{i\in I} \mathcal{M}_i\Big)_D.$$

If all of the L-structures $\mathcal{M}_i$ are equal to the same structure $\mathcal{M}_0$, then $\mathcal{M}$ is called the *D-ultrapower of* $\mathcal{M}_0$ and is denoted by

$$\big(\mathcal{M}_0\big)_D.$$

This ultraproduct construction finds many applications in functional analysis (see [24] and its references) and in metric space geometry (see

[19]). Its usefulness is partly explained by the following theorem, which is the analogue in this setting of the well known result in first-order logic proved by J. Łos. This is sometimes known as the *Fundamental Theorem of Ultraproducts*.

5.4 Theorem. *Let $(\mathcal{M}_i \mid i \in I)$ be a family of L-structures. Let D be any ultrafilter on I and let $\mathcal{M}$ be the D-ultraproduct of $(\mathcal{M}_i \mid i \in I)$. Let $\varphi(x_1, \dots, x_n)$ be an L-formula. If $a_k = \big((a_i^k)_{i \in I}\big)_D$ are elements of $\mathcal{M}$ for $k = 1, \dots, n$, then*

$$\varphi^{\mathcal{M}}(a_1, \dots, a_n) = \lim_{i,D} \varphi^{\mathcal{M}_i}(a_i^1, \dots, a_i^n).$$

Proof The proof is by induction on the complexity of φ. Basic facts about ultrafilter limits (discussed at beginning of this section) are used in the proof. □

5.5 Corollary. *If $\mathcal{M}$ is an L-structure and $T \colon M \to (M)_D$ is the diagonal embedding, then T is an elementary embedding of $\mathcal{M}$ into $(\mathcal{M})_D$.*

Proof From Theorem 5.4. □

5.6 Corollary. *If $\mathcal{M}$ and $\mathcal{N}$ are L-structures and they have isomorphic ultrapowers, then $\mathcal{M} \equiv \mathcal{N}$.*

Proof Immediate from the preceding result. □

The converse of the preceding corollary is also true, in a strong form:

5.7 Theorem. *If $\mathcal{M}$ and $\mathcal{N}$ are L-structures and $\mathcal{M} \equiv \mathcal{N}$, then there exists an ultrafilter D such that $(\mathcal{M})_D$ is isomorphic to $(\mathcal{N})_D$.*

The preceding result is an extension of the Keisler-Shelah Theorem from ordinary model theory. (See [36] and Chapter 6 in [13].) A detailed proof of the analogous result for normed space structures and the approximate logic of positive bounded formulas is given in [24, Chapter 10], and that argument can be readily adapted to continuous logic for metric structures.

There are characterizations of elementary equivalence that are slightly more complex to state than Theorem 5.7 but are much easier to prove, such as the following: $\mathcal{M} \equiv \mathcal{N}$ if and only if $\mathcal{M}$ and $\mathcal{N}$ have isomorphic elementary extensions that are each constructed as the union of an infinite sequence of successive ultrapowers. Theorem 5.7 has the positive feature that it connects continuous logic in a direct way to application areas in which the ultrapower construction is important, such as the theory of Banach spaces and other areas of functional analysis. Indeed, the result shows that the mathematical properties a metric structure and its ultrapowers share in common are exactly those that can be expressed by sentences of the continuous analogue of first-order logic. Theorem 5.7 also yields a characterization of axiomatizable classes of metric structures (5.14 below) whose statement is simpler than would otherwise be the case.

Compactness theorem

5.8 Theorem. *Let T be an L-theory and $\mathcal{C}$ a class of L-structures. Assume that T is finitely satisfiable in $\mathcal{C}$. Then there exists an ultraproduct of structures from $\mathcal{C}$ that is a model of T.*

Proof Let Λ be the set of finite subsets of T. Let $\lambda \in \Lambda$, and write $\lambda = \{E_1, \dots, E_n\}$. By assumption there is an L-structure $\mathcal{M}_\lambda$ in $\mathcal{C}$ such that $\mathcal{M}_\lambda \models E_j$ for all $j = 1, \dots, n$.

For each $E \in T$, let $S(E)$ be the set of all $\lambda \in \Lambda$ such that $E \in \lambda$. Note that the collection of sets $\{S(E) \mid E \in T\}$ has the finite intersection property. Hence there is an ultrafilter D on Λ that contains this collection.

Let

$$\mathcal{M} = \Big(\prod_{\lambda\in\Lambda} \mathcal{M}_\lambda\Big)_D.$$

Note that if $\lambda \in S(E)$, then $\mathcal{M}_\lambda \models E$. It follows from Theorem 5.4 that $\mathcal{M} \models E$ for every $E \in T$. In other words, the ultraproduct $\mathcal{M}$ of structures from $\mathcal{C}$ is a model of T. $\square$

In many applications it is useful to note that the Compactness Theorem remains true even if the finite satisfiability hypothesis is weakened to an approximate version. This is an immediate consequence of basic properties of the semantics for continuous logic.

5.9 Definition. For any set Σ of L-conditions, Σ^+ is the set of all conditions $\varphi \leq 1/n$ such that $\varphi = 0$ is an element of Σ and $n \geq 1$.

5.10 Corollary. *Let T be an L-theory and $\mathcal{C}$ a class of L-structures. Assume that T^+ is finitely satisfiable in $\mathcal{C}$. Then there exists an ultraproduct of structures from $\mathcal{C}$ that is a model of T.*

Proof This follows immediately from Theorem 5.8, because T and T^+ obviously have the same models. □

The next result is a version of the Compactness Theorem for formulas. In it we allow an arbitrary family $(x_j \mid j \in J)$ of possible free variables.

5.11 Definition. Let T be an L-theory and $\Sigma(x_j \mid j \in J)$ a set of L-conditions. We say that Σ is *consistent with* T if for every finite subset F of Σ there exists a model $\mathcal{M}$ of T and elements a of M such that for every condition E in F we have $\mathcal{M} \models E[a]$. (Here a is a finite tuple suitable for the free variables in members of F.)

5.12 Corollary. *Let T be an L-theory and $\Sigma(x_j \mid j \in J)$ a set of L-conditions, and assume that Σ^+ is consistent with T. Then there is a model $\mathcal{M}$ of T and elements $(a_j \mid j \in J)$ of M such that*

$$\mathcal{M} \models E[a_j \mid j \in J]$$

for every L-condition E in Σ.

Proof Let $(c_j \mid j \in J)$ be new constants and consider the signature $L(\{c_j \mid j \in J\})$. This corollary is proved by applying the Compactness Theorem to the set $T \cup \Sigma^+(c_j \mid j \in J)$ of closed $L(\{c_j \mid j \in J\})$-conditions. As noted in the proof of the previous result, anything satisfying Σ^+ will also satisfy Σ. □

Axiomatizability of classes of structures

5.13 Definition. Suppose that $\mathcal{C}$ is a class of L-structures. We say that $\mathcal{C}$ is *axiomatizable* if there exists a set T of closed L-conditions such that $\mathcal{C} = \mathrm{Mod}_L(T)$. When this holds for T, we say that T is a set of *axioms* for $\mathcal{C}$ in L.

In this section we characterize axiomatizability in continuous logic using ultraproducts. The ideas are patterned after a well known characterization of axiomatizability in first-order logic due to Keisler [31]. (See Corollary 6.1.16 in [13].)

5.14 Proposition. *Suppose that $\mathcal{C}$ is a class of L-structures. The following statements are equivalent:*

(1) *$\mathcal{C}$ is axiomatizable in L;*
(2) *$\mathcal{C}$ is closed under isomorphisms and ultraproducts, and its complement, $\{\mathcal{M} \mid \mathcal{M}$ is an L-structure not in $\mathcal{C}\}$, is closed under ultrapowers.*

Proof (1)⇒(2) follows from the Fundamental Theorem of Ultraproducts.

To prove (2)⇒(1), we let T be the set of closed L-conditions that are satisfied by every structure in $\mathcal{C}$. We claim that T is a set of axioms for $\mathcal{C}$. To prove this, suppose $\mathcal{M}$ is an L-structure such that $\mathcal{M} \models T$.

We claim that $\mathrm{Th}(\mathcal{M})^+$ is finitely satisfiable in $\mathcal{C}$. If not, there exist L-sentences $\sigma_1, \dots, \sigma_n$ and $\epsilon > 0$ such that $\sigma_j^{\mathcal{M}} = 0$ for all $j = 1, \dots, n$, but such that for any $\mathcal{N} \in \mathcal{C}$, we have $\sigma_j^{\mathcal{N}} \geq \epsilon$ for some $j = 1, \dots, n$. This means that the condition $\max(\sigma_1, \dots, \sigma_n) \geq \epsilon$ is in T but is not satisfied in $\mathcal{M}$, which is a contradiction.

So $\mathrm{Th}(\mathcal{M})^+$ is finitely satisfiable in $\mathcal{C}$. By the Compactness Theorem this yields an ultraproduct $\mathcal{M}'$ of structures from $\mathcal{C}$ such that M' is a model of $\mathrm{Th}(\mathcal{M})^+$. One sees easily that this implies $\mathcal{M}' \equiv \mathcal{M}$. Theorem 5.7, the extension of the Keisler-Shelah theorem to this continuous logic, yields an ultrafilter D such that $(\mathcal{M}')_D$ and $(\mathcal{M})_D$ are isomorphic. Statement (2) implies that $\mathcal{M}$ is in $\mathcal{C}$. □

5.15 Remark. The proof of Proposition 5.14 contains the following useful elementary result: let $\mathcal{C}$ be a class of L-structures and let T be the set of all closed L-conditions E such that $\mathcal{M} \models E$ holds for all $\mathcal{M} \in \mathcal{C}$. Then, every model of T is elementarily equivalent to some ultraproduct of structures from $\mathcal{C}$.

6 Connectives

Recall that in our definition of *formulas* for continuous logic, we took a *connective* to be a continuous function from $[0,1]^n$ to $[0,1]$, for some $n \geq 1$. This choice is somewhat arbitrary; from one point of view it is too general, and from another it is too restrictive. We begin to discuss these issues in this section.

Here we continue to limit ourselves to *finitary* connectives, and our intention is to limit the connectives we use when building formulas. We consider a restricted set of formulas to be adequate if every formula can be "uniformly approximated" by formulas from the restricted set.