CHAPTER 4

THE LEBESGUE INTEGRAL

By definition a Riemann integrable function is bounded, and its domain is a closed interval. These two restrictions make the Riemann integral inadequate to fulfill the requirements of many scientific problems. H. Lebesgue[1] in his classical work [21] introduced a concept of an integral (called today the Lebesgue integral) based on measure theory that generalizes the Riemann integral. It has the advantage of treating at the same time both bounded and unbounded functions and allows their domains to be more general sets. Also, it gives more powerful and useful convergence theorems than the Riemann integral.

In this chapter we study the Lebesgue integral. The concepts of an upper function and its Lebesgue integral are introduced first. Consequently, the Lebesgue integrable functions are introduced as differences of two upper functions, and their properties are studied. The Lebesgue dominated convergence theorem that (under certain conditions) allows us to interchange the processes of limit and integration is proved, and various applications of this powerful result are presented. Next, it is shown that every Riemann integrable function is Lebesgue integrable, and that in this case the two integrals (the Riemann and the Lebesgue) coincide. Furthermore, the relationship between an improper Riemann integral and the Lebesgue integral is obtained. Finally, the chapter culminates with a study of product measures and iterated integrals.

For this chapter, $(X, \mathcal{S}, \mu)$ will be a fixed measure space, and unless otherwise specified, all properties of the functions will be tacitly referred to in this measure space.

21. UPPER FUNCTIONS

It was mentioned before that the step functions will be the "building blocks" for the Lebesgue integral. Recall that a function ϕ is a step function if and only if there exist a finite collection $\{A_1, \ldots, A_n\}$ of measurable sets with $\mu^*(A_i) < \infty$ for $i = 1, \ldots, n$, and real numbers $a_1, \ldots, a_n$ such that $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ holds.

[1] Henri Léon Lebesgue (1875–1941), a prominent French mathematician. The founder of the modern theory of integration.

The real number $I(\phi) = \sum_{i=1}^{n} a_i \mu^*(A_i)$ is called the Lebesgue integral of ϕ, and we have already seen (in Section 17) that it is independent of the particular representation of ϕ. From now on, the Lebesgue integral of ϕ will be denoted by its conventional symbol $\int \phi \, d\mu$, or $\int_X \phi \, d\mu$. If clarity requires the variable to be emphasized, then the notation $\int \phi(x) \, d\mu(x)$ will be used. Thus, the integral of the step function is

$$\int \phi \, d\mu = \sum_{i=1}^{n} a_i \mu^*(A_i).$$

The collection of all step functions has both the structure of an algebra and that of a function space.

Theorem 21.1. *The collection of all step functions under the pointwise operations is a function space and an algebra.*

Proof. The proof that the step functions form an algebra is straightforward. To see that this collection is also a function space, note that if the step function ϕ has the standard representation $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$, then $\phi^+ = \sum_{i=1}^{n} \max\{a_i, 0\} \chi_{A_i}$ holds. Thus, ϕ^+ is a step function, and the conclusion follows. ■

The basic properties of the Lebesgue integral for step functions were discussed in Section 17. Here we shall deal with the almost everywhere limits of increasing sequences of step functions. Such limits are used to define upper functions.

Definition 21.2. *A function $f : X \to \mathbb{R}$ is called an* **upper function** *if there exists a sequence $\{\phi_n\}$ of step functions such that*

1. $\phi_n \uparrow f$ *a.e., and*
2. $\lim \int \phi_n \, d\mu < \infty$.

Any sequence of step functions that satisfies conditions (1) and (2) of the preceding definition will be referred to as a **generating sequence** for f. Note, in particular, that by Theorem 16.6 every upper function is a measurable function. The collection of all upper functions will be denoted by $\mathcal{U}$. Clearly,

- *every step function is an upper function.*

Also, observe that an upper function *need not* be a positive function.

If f is an upper function with a generating sequence $\{\phi_n\}$, and if $\{\psi_n\}$ is another sequence of step functions such that $\psi_n \uparrow f$ a.e., then it follows from Theorem 17.5 that $\lim \int \phi_n \, d\mu = \lim \int \psi_n \, d\mu$ holds. Thus, $\{\psi_n\}$ is also a generating sequence for f, and therefore, the following definition is well justified.

Definition 21.3. *Let f be an upper function, and let $\{\phi_n\}$ be a sequence of step functions such that $\phi_n \uparrow f$ a.e. holds. Then the* **Lebesgue integral** *(or simply the* **integral***) of f is defined by*

$$\int f\,d\mu = \lim_{n\to\infty} \int \phi_n\,d\mu.$$

Again, we stress the fact that the value of the Lebesgue integral of an upper function is independent of the generating sequence of step functions. Also, it should be clear that if f is an upper function and g is another function such that $g = f$ a.e., then g is also an upper function and $\int g\,d\mu = \int f\,d\mu$ holds.

The rest of this section is devoted to the properties of upper functions.

Theorem 21.4. *For upper functions f and g, the following statements hold:*

1. *$f + g$ is an upper function and $\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$.*
2. *αf is an upper function for each $\alpha \geq 0$ and $\int(\alpha f)\,d\mu = \alpha \int f\,d\mu$.*
3. *$f \vee g$ and $f \wedge g$ are upper functions.*

Proof. Choose two generating sequences $\{\phi_n\}$ and $\{\psi_n\}$ for f and g, respectively.

(1) Clearly, $\{\phi_n + \psi_n\}$ is a sequence of step functions, and $\phi_n + \psi_n \uparrow f + g$ a.e. holds. The result now follows by observing that

$$\int (\phi_n + \psi_n)\,d\mu = \int \phi_n\,d\mu + \int \psi_n\,d\mu \uparrow \int f\,d\mu + \int g\,d\mu.$$

(2) Straightforward.

(3) Note that both $\{\phi_n \vee \psi_n\}$ and $\{\phi_n \wedge \psi_n\}$ are sequences of step functions. Now, $\phi_n \wedge \psi_n \uparrow f \wedge g$ a.e. and $\lim \int \phi_n \wedge \psi_n\,d\mu \leq \lim \int \phi_n\,d\mu < \infty$, show that $f \wedge g$ is an upper function.

To see that $f \vee g$ is an upper function, observe first that $\phi_n \vee \psi_n \uparrow f \vee g$ a.e. holds, and then use the identity

$$\phi_n \vee \psi_n = \phi_n + \psi_n - \phi_n \wedge \psi_n$$

to obtain $\int \phi_n \vee \psi_n\,d\mu = \int \phi_n\,d\mu + \int \psi_n\,d\mu - \int \phi_n \wedge \psi_n\,d\mu$. This implies

$$\int \phi_n \vee \psi_n\,d\mu = \int \phi_n\,d\mu \uparrow \int f\,d\mu + \int g\,d\mu - \int f \wedge g\,d\mu < \infty.$$

This finishes the proof of the theorem. ∎

The next theorem states that the integral is a monotone function on $\mathcal{U}$.

Theorem 21.5. *If f and g are upper functions such that $f \geq g$ a.e., then $\int f\,d\mu \geq \int g\,d\mu$ holds. In particular, if $f \in \mathcal{U}$ satisfies $f \geq 0$ a.e., then $\int f\,d\mu \geq 0$.*

Proof. Let $\{\phi_n\}$ and $\{\psi_n\}$ be generating sequences for f and g, respectively. Then $\phi_n \wedge \psi_n \uparrow g$ a.e. holds, and so $\{\phi_n \wedge \psi_n\}$ is also a generating sequence for g. By Theorem 17.3, we have $\int \phi_n\,d\mu \geq \int \phi_n \wedge \psi_n\,d\mu$ for each n. Therefore,

$$\int f\,d\mu = \lim_{n\to\infty} \int \phi_n\,d\mu \geq \lim_{n\to\infty} \int \phi_n \wedge \psi_n\,d\mu = \int g\,d\mu,$$

and the proof is finished. ■

It should be noted that if f is an upper function such that $f \geq 0$ a.e., then there exists a sequence of step functions $\{\psi_n\}$ satisfying $\psi_n \geq 0$ a.e. for each n and $\psi_n \uparrow f$ a.e. To see this, notice that if $\phi_n \uparrow f$, then $\phi_n^+ \uparrow f^+ = f$ a.e. holds.

If we take the "upper functions of $\mathcal{U}$," then we get $\mathcal{U}$ again. The details are included in the next theorem.

Theorem 21.6. *Let $f: X \to \mathbb{R}$ be a function. If there exists a sequence $\{f_n\}$ of upper functions such that $f_n \uparrow f$ a.e. and $\lim \int f_n\,d\mu < \infty$, then f is an upper function and $\int f\,d\mu = \lim \int f_n\,d\mu$.*

Proof. For each i choose a sequence $\{\phi_n^i\}$ of step functions such that $\phi_n^i \uparrow_n f_i$ a.e. holds. Now, for each n let $\psi_n = \bigvee_{i=1}^n \phi_n^i$, and note that each ψ_n is a step function such that $\psi_n \uparrow f$ a.e. holds. Also, note that $\psi_n \leq f_n$ a.e. for each n, and consequently, $\lim \int \psi_n\,d\mu \leq \lim \int f_n\,d\mu < \infty$ holds by virtue of Theorem 21.5. This shows that f is an upper function.

Now, since for each fixed i we have $\phi_n^i \leq \psi_n$ for all $n \geq i$, it follows that $\int f_i\,d\mu = \lim_{n\to\infty} \int \phi_n^i\,d\mu \leq \lim_{n\to\infty} \int \psi_n\,d\mu$ holds for all i. Therefore,

$$\lim_{n\to\infty} \int f_n\,d\mu = \lim_{n\to\infty} \int \psi_n\,d\mu = \int f\,d\mu,$$

and the proof is complete. ■

The integral satisfies an important convergence property for decreasing sequences. It is our familiar **order continuity property** of the integral.

Theorem 21.7. *If $\{f_n\}$ is a sequence of upper functions such that $f_n \downarrow 0$ a.e., then $\lim \int f_n\,d\mu = 0$ holds.*

Proof. Let $\epsilon > 0$. For each n choose a step function ϕ_n such that $0 \leq \phi_n \leq f_n$ a.e. and $\int (f_n - \phi_n)\,d\mu = \int f_n\,d\mu - \int \phi_n\,\mu < \epsilon 2^{-n}$ (remember that $-\phi_n$ is an upper function). Let $\psi_n = \bigwedge_{i=1}^{n} \phi_i$ for each n. Then $\{\psi_n\}$ is a sequence of step functions satisfying $0 \leq \psi_{n+1} \leq \psi_n$ for each n, and (in view of $\psi_n \leq f_n$ a.e. and $f_n \downarrow 0$ a.e.) we see that $\psi_n \downarrow 0$ a.e. By Theorem 17.4 we have $\lim \int \psi_n \, d\mu = 0$. Pick an integer k such that $\int \psi_n \, d\mu < \epsilon$ for all $n \geq k$. Now, the almost everywhere inequalities

$$0 \leq f_n - \psi_n = f_n - \bigwedge_{i=1}^{n} \phi_i = \bigvee_{i=1}^{n} (f_n - \phi_i) \leq \bigvee_{i=1}^{n} (f_i - \phi_i) \leq \sum_{i=1}^{n} (f_i - \phi_i)$$

imply

$$\int f_n \, d\mu - \int \psi_n \, d\mu \leq \sum_{i=1}^{n} \int (f_i - \phi_i) \, d\mu < \epsilon \left(\sum_{i=1}^{\infty} 2^{-i} \right) = \epsilon.$$

Thus,

$$0 \leq \int f_n \, d\mu < \epsilon + \int \psi_n \, d\mu < 2\epsilon$$

for all $n \geq k$, which shows that $\int f_n \, d\mu \downarrow 0$ holds. ■

Finally, we mention that $\mathcal{U}$ is not in general a vector space since it fails to be closed under multiplication by negative real numbers. An example of this type is presented in Exercise 2 of this section.

EXERCISES

1. Let L be the collection of all step functions ϕ such that there exist a finite number of sets $A_1, \ldots, A_n$ in $\mathcal{S}$ all of finite measure and real numbers $a_1, \ldots, a_n$ such that $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$. Show that L is a function space. Is L an algebra of functions?
[HINT: Use Exercise 14 of Section 12.]
2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ if $x \notin (0, 1]$, and $f(x) = \sqrt{n}$ if $x \in (\frac{1}{n+1}, \frac{1}{n}]$ for some n. Show that f is an upper function and that $-f$ is not an upper function.
[HINT: A step function is necessarily bounded.]
3. Compute $\int f \, d\lambda$ for the upper function f of the preceding exercise.
4. Verify that every continuous function $f : [a, b] \to \mathbb{R}$ is an upper function—with respect to the Lebesgue measure on $[a, b]$.
5. Let A be a measurable set, and let f be an upper function. If $\chi_A \leq f$ a.e., then show that $\mu^*(A) < \infty$.
6. Let f be an upper function, and let A be a measurable set of finite measure such that $a \leq f(x) \leq b$ holds for each $x \in A$. Then show that

a. $f\chi_A$ is an upper function, and
b. $a\mu^*(A) \le \int f\chi_A\, d\mu \le b\mu^*(A)$.

7. Let $(X, \mathcal{S}, \mu)$ be a finite measure space, and let f be a positive measurable function. Show that f is an upper function if and only if there exists a real number M such that $\int \phi\, d\mu \le M$ holds for every step function ϕ with $\phi \le f$ a.e. Also, show that if this is the case, then

$$\int f\, d\mu = \sup\left\{\int \phi\, d\mu\colon\ \phi \text{ is a step function with } \phi \le f \text{ a.e.}\right\}.$$

8. Show that every monotone function $f : [a, b] \to \mathbb{R}$ is an upper function—with respect to the Lebesgue measure on $[a, b]$.

22. INTEGRABLE FUNCTIONS

It was observed before that the collection $\mathcal{U}$ of all upper functions is not a vector space. However, if we consider the collection of all functions that can be written as an almost everywhere difference of two upper functions, then this set is a function space. The members of this collection are the Lebesgue integrable functions. The details will be explained in the following.

Definition 22.1. *A function $f : X \to \mathbb{R}$ is called* **Lebesgue integrable** *(or simply* **integrable***) if there exist two upper functions u and v such that $f = u - v$ a.e. holds. The* **Lebesgue integral** *(or simply the* **integral***) of f is defined by*

$$\int f\, d\mu = \int u\, d\mu - \int v\, d\mu.$$

It should be noted that the value of the integral is independent of the representation of f as a difference of two upper functions. Indeed, if $f = u - v = u_1 - v_1$ a.e. with u, u_1, v, and v_1 all upper functions, then $u + v_1 = u_1 + v$ a.e. holds, and by Theorem 21.4(1) we have $\int u\, d\mu + \int v_1\, d\mu = \int u_1\, d\mu + \int v\, d\mu$. Therefore, $\int u\, d\mu - \int v\, d\mu = \int u_1\, d\mu - \int v_1\, d\mu$.

An integrable function is necessarily measurable, and every upper function is Lebesgue integrable. Also, it is readily seen that if a function f is Lebesgue integrable and g is another function such that $f = g$ a.e., then g is also Lebesgue integrable and $\int g\, d\mu = \int f\, d\mu$ holds.

Historical Note: The above introduction of the Lebesgue integral is a modification of a method due to P. J. Daniell [5]. Daniell's[2] general approach to integration starts with a function space L on some nonempty set X, together with an "integral" I on L. The function $I : L \to \mathbb{R}$ is said to be an *integral* if

[2] P. J. Daniell (1889–1946), a British mathematician. He worked in functional analysis and the theory of integration.

1. $I(\alpha\phi + \beta\psi) = \alpha I(\phi) + \beta I(\psi)$ for all $\alpha, \beta \in \mathbb{R}$ and $\phi, \psi \in L$,
2. $I(\phi) \geq 0$ whenever $\phi \geq 0$, and
3. whenever $\{\phi_n\} \subseteq L$ satisfies $\phi_n(x) \downarrow 0$ for each $x \in X$, then $I(\phi_n) \downarrow 0$.

A function $u : X \to \mathbb{R}$ is called an upper function if there exists a sequence $\{\phi_n\} \subseteq L$ with $\phi_n(x) \uparrow u(x)$ for all $x \in X$ and $\lim I(\phi_n) < \infty$. As in the proof of Theorem 17.5, we can show that $\lim I(\phi_n)$ is independent of the "generating" sequence $\{\phi_n\}$. The real number $I(u) = \lim I(\phi_n)$ is the integral of u. Finally, $f : X \to \mathbb{R}$ is said to be *integrable* if there exist two upper functions u and v with $f = u - v$. The integral of f is then defined by $I(f) = I(u) - I(v)$.

Here our approach to the Lebesgue integral can be considered as a "measure theoretical Daniell method."

The set of integrable functions has all the expected nice properties.

Theorem 22.2. *The collection of all Lebesgue integrable functions is a function space.*

Proof. Let f and g be two integrable functions with representations $f = u - v$ a.e. and $g = u_1 - v_1$ a.e. Then the almost everywhere identities

$$\begin{aligned} f + g &= (u + u_1) - (v + v_1), \\ \alpha f &= \alpha u - \alpha v \ \text{ if } \alpha \geq 0, \\ \alpha f &= [(-\alpha)v] - [(-\alpha)u] \ \text{ if } \alpha < 0, \ \text{ and} \\ f^+ &= (u - v)^+ = u \vee v - v \end{aligned}$$

express the above functions as differences of two upper functions. This shows that the collection of all integrable functions is a function space. ∎

Thus, if f is integrable, then $|f|$ is also an integrable function. In particular, it follows from the last theorem that a function f is Lebesgue integrable if and only if f^+ and f^- are both integrable.

The next result describes the linearity property of the integral. Its easy proof follows directly from Definition 22.1 and is left as an exercise for the reader.

Theorem 22.3. *If f and g are two integrable functions, then*

$$\int (\alpha f + \beta g)\, d\mu = \alpha \int f\, d\mu + \beta \int g\, d\mu$$

holds for all $\alpha, \beta \in \mathbb{R}$.

Every positive integrable function is necessarily an upper function.

Theorem 22.4. *If an integrable function f satisfies $f \geq 0$ a.e., then f is an upper function.*

Proof. Choose two upper functions u and v such that $f = u - v$ a.e. holds. Since each u and v is the almost everywhere limit of a sequence of step functions, there exists a sequence $\{\psi_n\}$ of step functions such that $\psi_n \to f$ a.e. Since $f \geq 0$ a.e., it follows that $\psi_n^+ \to f$ a.e. also holds.

By Theorem 17.7, there exists a sequence $\{s_n\}$ of simple functions satisfying $0 \leq s_n \uparrow f$ a.e. Now, for each n let $\phi_n = s_n \wedge (\bigvee_{i=1}^n \psi_i^+)$. Then $\{\phi_n\}$ is a sequence of step functions such that $0 \leq \phi_n \uparrow f$ a.e. holds. To complete the proof, we show that $\{\int \phi_n \, d\mu\}$ is bounded. Indeed, from $\phi_n + v \leq f + v \leq u$ a.e. and Theorem 21.5, it follows that $\int \phi_n \, d\mu + \int v \, d\mu \leq \int u \, d\mu$, and therefore, $\int \phi_n \, d\mu \leq \int u \, d\mu - \int v \, d\mu < \infty$ holds for all n. The proof of the theorem is now complete. ∎

If f is an integrable function, then by Theorem 22.4, f^+ and f^- are both upper functions, and so, $f = f^+ - f^-$ is a decomposition of f as a difference of two positive upper functions. In particular,

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu.$$

(This formula is usually the one used by many authors to define the Lebesgue integral.)

As an application of the preceding theorem, we also have the following useful result.

Theorem 22.5. *If f is an integrable function, then for every $\epsilon > 0$ the measurable set $\{x \in X : |f(x)| \geq \epsilon\}$ has finite measure.*

Proof. Fix $\epsilon > 0$, let $A = \{x \in X : |f(x)| \geq \epsilon\}$, and note that $\epsilon\chi_A \leq |f|$ holds. Now, $\frac{1}{\epsilon}|f|$ is an integrable function, in fact, by Theorem 22.4 an upper function. Let $\{\phi_n\}$ be a sequence of step functions such that $\phi_n \uparrow \frac{1}{\epsilon}|f|$ a.e. Then $\{\phi_n \wedge \chi_A\}$ is a sequence of step functions such that $\phi_n \wedge \chi_A \uparrow \chi_A$ a.e. holds. Thus, by Theorem 17.6,

$$\mu^*(A) = \lim_{n\to\infty} \int \phi_n \wedge \chi_A \, d\mu \leq \lim_{n\to\infty} \int \phi_n \, d\mu = \frac{1}{\epsilon} \int |f| \, d\mu < \infty,$$

and the proof is finished. ∎

Measurable functions "sandwiched" between integrable functions are integrable.

Theorem 22.6. *Let f be a measurable function. If there exist two integrable functions h and g such that $h \leq f \leq g$ a.e., then f is also an integrable function.*

Proof. Writing the given inequality in the form $0 \leq f - h \leq g - h$ a.e., we see that we can assume without loss of generality that $0 \leq f \leq g$ a.e. holds.

By Theorem 22.4, g is an upper function. Pick a sequence $\{\phi_n\}$ of step functions such that $0 \leq \phi_n \uparrow g$ a.e. By Theorem 17.7 there exists a sequence $\{\psi_n\}$ of simple functions such that $0 \leq \psi_n \uparrow f$ a.e. holds. But then $\{\phi_n \wedge \psi_n\}$ is a sequence of step functions such that $\phi_n \wedge \psi_n \uparrow f$ a.e., and $\int \phi_k \wedge \psi_k \, d\mu \leq \lim \int \phi_n \, d\mu = \int g \, d\mu < \infty$ for all k. Hence, $f \in \mathcal{U}$, and so, f is an integrable function. ■

More properties of the integral are included in the next theorem:

Theorem 22.7. *For integrable functions f and g we have the following:*

1. $\int |f| \, d\mu = 0$ *if and only if* $f = 0$ *a.e.*
2. *If* $f \geq g$ *a.e., then* $\int f \, d\mu \geq \int g \, d\mu$.
3. $|\int f \, d\mu| \leq \int |f| \, d\mu$.

Proof. (1) Clearly, if $f = 0$ a.e., then $\int |f| \, d\mu = 0$ holds. On the other hand, assume that $\int |f| \, d\mu = 0$. Since, by Theorem 22.4, $|f|$ is an upper function, there exists a sequence $\{\phi_n\}$ of step functions such that $0 \leq \phi_n \uparrow |f|$ a.e. holds. By Theorem 21.5, it follows that $\int \phi_n \, d\mu = 0$ for each n, and so, $\phi_n = 0$ a.e. for each n. Thus, $|f| = 0$ a.e., and so $f = 0$ a.e.

(2) Since $f - g \geq 0$ a.e., it follows from Theorem 22.4 that $f - g$ is an upper function. But then, Theorem 21.5 implies that $\int f \, d\mu - \int g \, d\mu = \int (f - g) \, d\mu \geq 0$, so that $\int f \, d\mu \geq \int g \, d\mu$ holds.

(3) The conclusion follows from (2) and the inequality $-|f| \leq f \leq |f|$. ■

The reader has probably noticed that the functions we have considered so far are real-valued. It is a custom, however, to allow a function to assume infinite values, provided that the set of all points where the function equals $-\infty$ or ∞ is a null set. The reason for this is that neither the integrability character nor the value of the integral of a function changes by altering its values on a null set. Moreover, assigning any value to the sum of two functions at the points where the form $\infty - \infty$ occurs does not affect the integrability and the value of the integral of the sum function (as long as the set of points of all such encounters has measure zero).

If one does not want to deal with functions assuming infinite values (up, of course to null sets), then one may change the infinite values to finite ones (for instance, change all the infinite values to zero) without loosing anything regarding integrability. When an extended real-valued function f is said to **define an integrable function**, it will be meant that f assumes the infinite values (or it is even undefined) on a null set, and that if finite values are assigned to these points, then f becomes an integrable function. To summarize the preceding:

- *Functions that are almost everywhere equal have the same integrability properties and can be considered as identical.*

We continue with a theorem of B. Levi[3] describing a basic monotone property of the integral.

Theorem 22.8 (Levi). *Assume that a sequence $\{f_n\}$ of integrable functions satisfies $f_n \le f_{n+1}$ a.e. for each n and $\lim \int f_n \, d\mu < \infty$. Then there exists an integrable function f such that $f_n \uparrow f$ a.e. (and hence, $\int f_n \, d\mu \uparrow \int f \, d\mu$ holds).*

Proof. Replacing $\{f_n\}$ by $\{f_n - f_1\}$ if necessary, we can assume without loss of generality that $f_n \ge 0$ a.e. holds for each n. Also, an easy argument shows that we can assume that $0 \le f_n(x) \uparrow$ holds for all $x \in X$. Let $I = \lim \int f_n \, d\mu < \infty$.

For each $x \in X$ let $g(x) = \lim f_n(x) \in \mathbb{R}^*$, and consider the set

$$E = \{x \in X : g(x) = \infty\}.$$

Clearly, $E = \bigcap_{i=1}^{\infty} [\bigcup_{n=1}^{\infty} \{x \in X : f_n(x) > i\}]$ holds, and so E is a measurable set. Next, we shall show that $\mu^*(E) = 0$.

By Theorem 22.4, each f_n is an upper function. Thus, for each i there exists a sequence $\{\phi_n^i\}$ of step functions such that $0 \le \phi_n^i \uparrow_n f_i$ a.e. holds. For each n let $\psi_n = \bigvee_{i=1}^{n} \phi_n^i$, and note that $\{\psi_n\}$ is a sequence of step functions such that $\psi_n \uparrow g$ a.e. and $\lim \int \psi_n \, d\mu = \lim \int f_n \, d\mu = I$. In particular, for each k the sequence of step functions $\{\psi_n \wedge k\chi_E\}$ satisfies $\psi_n \wedge k\chi_E \uparrow k\chi_E$ a.e. From Theorem 17.6, it follows that $\mu^*(E) < \infty$ and $k\mu^*(E) \le \lim \int \psi_n \, d\mu = I < \infty$ for each k. Hence, $\mu^*(E) = 0$.

Now, define $f : X \to \mathbb{R}$ by $f(x) = g(x)$ if $x \notin E$ and $f(x) = 0$ if $x \in E$. Then $f_n \uparrow f$ a.e. holds, and the result follows from Theorem 21.6. ∎

The series analogue of the preceding theorem is presented next.

Theorem 22.9. *Let $\{f_n\}$ be a sequence of non-negative integrable functions*

[3]Beppo Levi (1875–1961), an Italian mathematician. His main contributions were in algebraic topology, mathematical logic, and analysis.

such that $\sum_{n=1}^{\infty} \int f_n \, d\mu < \infty$. *Then* $\sum_{n=1}^{\infty} f_n$ *defines an integrable function and*

$$\int \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu.$$

Proof. For each n let $g_n = \sum_{i=1}^{n} f_i$, and note that each g_n is an integrable function such that $g_n \uparrow \sum_{i=1}^{\infty} f_i$ a.e. holds. Now, by Levi's theorem, $\sum_{n=1}^{\infty} f_n$ defines an integrable function, and

$$\sum_{n=1}^{\infty} \int f_n \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu = \int \left(\sum_{n=1}^{\infty} f_n \right) d\mu$$

holds. ■

The next result is known in the theory of integration as Fatou's[4] lemma.

Theorem 22.10 (Fatou's Lemma). *Let* $\{f_n\}$ *be a sequence of integrable functions such that* $f_n \geq 0$ *a.e. for each* n *and* $\liminf \int f_n \, d\mu < \infty$. *Then* $\liminf f_n$ *defines an integrable function, and*

$$\int \liminf f_n \, d\mu \leq \liminf \int f_n \, d\mu.$$

Proof. Without loss of generality, we can suppose that $f_n(x) \geq 0$ holds for all $x \in X$ and all n.

Given n, define $g_n(x) = \inf\{f_i(x) : i \geq n\}$ for each $x \in X$. Then each g_n is a measurable function, and $0 \leq g_n \leq f_n$ holds for all n. Thus, by Theorem 22.6, each g_n is an integrable function. Now, observe that $g_n \uparrow$ and $\lim \int g_n \, d\mu \leq \liminf \int f_n \, d\mu < \infty$ holds. Thus, by Theorem 22.8, there exists an integrable function g such that $g_n \uparrow g$ a.e. holds. It follows that $g = \liminf f_n$ a.e., and therefore, $\liminf f_n$ defines an integrable function. Moreover,

$$\int \liminf f_n \, d\mu = \int g \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu \leq \liminf \int f_n \, d\mu,$$

and the proof is finished. ■

We are now in the position to state and prove the Lebesgue dominated convergence theorem—the cornerstone of the theory of integration.

[4]Pierre Joseph Louis Fatou (1878–1929), a French mathematician. Besides his work in analysis, he also studied the motion of planets in astronomy.

Theorem 22.11 (The Lebesgue Dominated Convergence Theorem). *Let $\{f_n\}$ be a sequence of integrable functions satisfying $|f_n| \le g$ a.e. for all n and some fixed integrable function g. If $f_n \to f$ a.e., then f defines an integrable function and*

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu = \int f \, d\mu.$$

Proof. Clearly, $|f| \le g$ a.e. holds, and the integrability of f follows from Theorem 22.6. Observe next that the sequence $\{g - f_n\}$ satisfies the hypotheses of Fatou's lemma and moreover, $\liminf(g - f_n) = g - f$ a.e. Thus,

$$\begin{aligned}\int g \, d\mu - \int f \, d\mu &= \int (g - f) \, d\mu = \int \liminf(g - f_n) \, d\mu \\ &\le \liminf \int (g - f_n) \, d\mu = \int g \, d\mu - \limsup \int f_n \, d\mu,\end{aligned}$$

and hence,

$$\limsup \int f_n \, d\mu \le \int f \, d\mu.$$

Similarly, Fatou's lemma applied to the sequence $\{g + f_n\}$ yields

$$\begin{aligned}\int g \, d\mu + \int f \, d\mu &= \int (g + f) \, d\mu = \int \liminf(g + f_n) \, d\mu \\ &\le \liminf \int (g + f_n) \, d\mu = \int g \, d\mu + \liminf \int f_n \, d\mu,\end{aligned}$$

and so,

$$\int f \, d\mu \le \liminf \int f_n \, d\mu.$$

Therefore, $\lim \int f_n \, d\mu$ exists in $\mathbb{R}$, and $\lim \int f_n \, d\mu = \int f \, d\mu$ holds. ■

The next theorem characterizes the Lebesgue integrable functions in terms of some given property. It is usually employed to prove that all Lebesgue integrable functions possess a given property.

Theorem 22.12. *Let $(X, \mathcal{S}, \mu)$ be a measure space and let (P) be a property which may or may not be possessed by an integrable function. Assume that:*

1. *If f and g are integrable functions with property (P), then $f + g$ and αf for each $\alpha \in \mathbb{R}$ also have property (P).*
2. *If f is an integrable function such that for each $\epsilon > 0$ there exists an integrable function g with property (P) satisfying $\int |f - g|\, d\mu < \epsilon$, then f has property (P).*
3. *For each $A \in \mathcal{S}$ with $\mu(A) < \infty$, the characteristic function χ_A has property (P).*

Then every integrable function has property (P).

Proof. Assume first that A is a σ-set with $\mu^*(A) < \infty$. So, there exists a disjoint sequence $\{A_n\}$ of $\mathcal{S}$ such that $A = \bigcup_{n=1}^{\infty} A_n$. Put $B_n = \bigcup_{k=1}^{n} A_k$ for each n and note that $B_n \uparrow A$. From $\chi_{B_n} = \sum_{k=1}^{n} \chi_{A_k}$, (3) and (1), we see that χ_{B_n} has property (P) for each n. Since $\int |\chi_A - \chi_{B_n}|\, d\mu = \mu^*(A) - \mu(B_n) \to 0$, it follows from (2) that χ_A likewise has property (P).

Next, assume that A is an arbitrary measurable set of finite measure and let $\epsilon > 0$. Then there exists a σ-set B of finite measure such that $A \subseteq B$ and $\mu^*(B) < \mu^*(A) + \epsilon$. This implies $\int |\chi_A - \chi_B|\, d\mu = \mu^*(B) - \mu^*(A) < \epsilon$. From the above discussion and (2), we infer that χ_A satisfies property (P).

Now, from (1) we see that every step function satisfies property (P). But then, it follows from (2) that every upper function satisfies property (P). Since every integrable function is the difference of two upper functions, invoking (1) once more, we infer that indeed every integrable function satisfies property (P). ∎

It is easy to see that for every subset E of X, the collection $\mathcal{S}_E = \{E \cap A : A \in \mathcal{S}\}$ of subsets of E (called the restriction of the semiring $\mathcal{S}$ to E) is a semiring of subsets of E. If, in addition, E is a measurable subset of X, then μ^* restricted to $\mathcal{S}_E$ is also a measure. That is, $(E, \mathcal{S}_E, \mu^*)$ is a measure space for every measurable subset E of X. Also, a straightforward verification shows that the measurable subsets of $(E, \mathcal{S}_E, \mu^*)$ are precisely the subsets of E that are measurable subsets of X; see Exercise 7 of Section 15.

If E is a measurable subset of X, then a function $f : E \to \mathbb{R}$ is said to be integrable over E if f is integrable with respect to the measure space $(E, \mathcal{S}_E, \mu^*)$. Of course, the domain of f can be extended to all of X by assigning the values $f(x) = 0$ if $x \notin E$. Then f so defined is an integrable function over X, and in this case $\int_X f\, d\mu = \int_E f\, d\mu$ holds. A function $f : X \to \mathbb{R}$ is said to be **integrable over a measurable subset** E of X if the function $f\chi_E$ is integrable over X, or equivalently, if f restricted to E is integrable with respect to the measure space $(E, \mathcal{S}_E, \mu^*)$. In this case, we shall write $\int f\chi_E\, d\mu = \int_E f\, d\mu$.

The simple proof of the next result is left as an exercise for the reader.

Theorem 22.13. *Every integrable function f is integrable over every measurable subset of X. Moreover,*

$$\int_E f\,d\mu + \int_{E^c} f\,d\mu = \int_X f\,d\mu$$

holds for every measurable subset E of X.

The rest of this section deals with an observation concerning infinite Lebesgue integrals. If $\phi = \sum_{i=1}^n a_i \chi_{A_i}$ is the standard representation of a positive simple function ϕ, then the sum $\sum_{i=1}^n a_i \mu^*(A_i)$ makes sense as an extended non-negative real number. If $\sum_{i=1}^n a_i \mu^*(A_i) = \infty$, then it is a custom to write $\int \phi\,d\mu = \infty$ and say that the Lebesgue integral of ϕ is infinite.

Assume now that $f: X \to \mathbb{R}^*_+$ is a function where there exists a sequence $\{\phi_n\}$ of positive simple functions such that $\phi_n \uparrow f$ a.e. holds. Then $\lim \int \phi_n\,d\mu$ exists as an extended real number, and it can be seen easily that $\lim \int \phi_n\,d\mu$ is independent from the chosen sequence $\{\phi_n\}$. In the case that $\lim \int \phi_n\,d\mu = \infty$, we write $\int f\,d\mu = \infty$ and say that the Lebesgue integral of f is infinite—but we do not call the function integrable! See also Exercise 16 of Section 17. In this sense every positive measurable function f has a Lebesgue integral (finite or infinite) simply because, by Theorem 17.7, there exists a sequence $\{\phi_n\}$ of positive simple function such that $\phi_n \uparrow f$ a.e. holds.

Moreover, if $f: X \to \mathbb{R}^*$ defines a measurable function, then we can write $f = f^+ - f^-$ and (by the above) both integrals $\int f^+\,d\mu$ and $\int f^-\,d\mu$ exist as (non-negative) extended real numbers. If one of them is a real number, then the integral of f is defined to be the extended real number

$$\int f\,d\mu = \int f^+\,d\mu - \int f^-\,d\mu.$$

In this manner, we can assign an "integral" to a much larger class of measurable extended real-valued functions.

One advantage of the above extension of the integral is that a number of theorems can be phrased without Lebesgue integrability assumptions on the functions. For instance, Fatou's lemma can be stated as follows:

- *If $\{f_n\}$ is a sequence of measurable functions satisfying $f_n \geq 0$ a.e. for each n, then*

$$\int \liminf f_n\,d\mu \leq \liminf \int f_n\,d\mu$$

holds—where, of course, one or both sides of the inequality may be infinite.

EXERCISES

1. Show by a counterexample that the integrable functions do not form an algebra.
2. Let X be a nonempty set, and let δ be the Dirac measure on X with respect to the point a (see Example 13.4). Show that every function $f: X \to \mathbb{R}$ is integrable and that $\int f\,d\delta = f(a)$.
3. Let μ be the counting measure on $\mathbb{N}$ (see Example 13.3). Show that a function $f: \mathbb{N} \to \mathbb{R}$ is integrable if and only if $\sum_{n=1}^{\infty} |f(n)| < \infty$. Also, show that in this case $\int f\,d\mu = \sum_{n=1}^{\infty} f(n)$.
4. Show that a measurable function f is integrable if and only if $|f|$ is integrable. Give an example of a nonintegrable function whose absolute value is integrable.
5. Let f be an integrable function, and let $\{E_n\}$ be a sequence of disjoint measurable subsets of X. If $E = \bigcup_{n=1}^{\infty} E_n$, then show that

$$\int_E f\,d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\,d\mu.$$

6. Let f be an integrable function. Show that for each $\epsilon > 0$ there exists some $\delta > 0$ (depending on ϵ) such that $|\int_E f\,d\mu| < \epsilon$ holds for all measurable sets with $\mu^*(E) < \delta$. [HINT: Note that $|f| \wedge n \uparrow |f|$.]
7. Show that for every integrable function f the set $\{x \in X : f(x) \neq 0\}$ can be written as a countable union of measurable sets of finite measure—referred to as a σ-**finite set**.
8. Let $f: \mathbb{R} \to \mathbb{R}$ be integrable with respect to the Lebesgue measure. Show that the function $g: [0, \infty) \to \mathbb{R}$ defined by

$$g(t) = \sup\left\{\int |f(x+y) - f(x)|\,d\lambda(x) : |y| \leq t\right\}$$

for $t \geq 0$ is continuous at $t = 0$.
[HINT: Use Theorem 22.12.]

9. Let g be an integrable function and let $\{f_n\}$ be a sequence of integrable functions such that $|f_n| \leq g$ a.e. holds for all n. Show that if $f_n \xrightarrow{\mu} f$, then f is an integrable function and $\lim \int |f_n - f|\,d\mu = 0$ holds.
[HINT: Combine Theorem 19.4 with the Lebesgue dominated convergence theorem.]
10. Establish the following generalization of Theorem 22.9: If $\{f_n\}$ is a sequence of integrable functions such that $\sum_{n=1}^{\infty} \int |f_n|\,d\mu < \infty$, then $\sum_{n=1}^{\infty} f_n$ defines an integrable function and

$$\int \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \sum_{n=1}^{\infty} \int f_n\,d\mu.$$

[HINT: By Theorem 22.9, the series $g = \sum_{n=1}^{\infty} |f_n|$ defines an integrable function and $|\sum_{n=1}^{k} f_n| \leq g$ a.e. holds for each k. Now, use the Lebesgue dominated convergence theorem.]

11. Let f be a positive (a.e.) measurable function, and let

$$e_i = \mu^*(\{x \in X : 2^{i-1} < f(x) \leq 2^i\})$$

for each integer i. Show that f is integrable if and only if $\sum_{i=-\infty}^{\infty} 2^i e_i < \infty$.

12. Let $\{f_n\}$ be a sequence of integrable functions such that $0 \le f_{n+1} \le f_n$ a.e. holds for each n. Then show that $f_n \downarrow 0$ a.e. holds if and only if $\int f_n \, d\mu \downarrow 0$.
13. Let f be an integrable function such that $f(x) > 0$ holds for almost all x. If A is a measurable set such that $\int_A f \, d\mu = 0$, then show that $\mu^*(A) = 0$.
14. Let $(X, \mathcal{S}, \mu)$ be a finite measure space and let $f : X \to \mathbb{R}$ be an integrable function satisfying $f(x) > 0$ for almost all x. If $0 < \varepsilon \le \mu^*(X)$, then show that

$$\inf\left\{\int_E f \, d\mu : E \in \Lambda_\mu \text{ and } \mu^*(E) \ge \varepsilon\right\} > 0.$$

15. Let f be a positive integrable function. Define $\nu : \Lambda \to [0, \infty)$ by $\nu(A) = \int_A f \, d\mu$ for each $A \in \Lambda$. Show that:

    a. (X, Λ, ν) is a measure space.
    b. If Λ_ν denotes the σ-algebra of all ν-measurable subsets of X, then $\Lambda \subseteq \Lambda_\nu$. Give an example for which $\Lambda \ne \Lambda_\nu$.
    c. If $\mu^*(\{x \in X : f(x) = 0\}) = 0$, then $\Lambda = \Lambda_\nu$.
    d. If g is an integrable function with respect to the measure space (X, Λ, ν), then fg is integrable with respect to the measure space $(X, \mathcal{S}, \mu)$ and

$$\int g \, d\nu = \int gf \, d\mu.$$

16. (A Change of Variable Formula) Let I be an interval of $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be an integrable function with respect to the Lebesgue measure. For a pair of real numbers a and b with $a \ne 0$, let $J = \{(x - b)/a : x \in I\}$. Show that the function $g : J \to \mathbb{R}$ defined by $g(x) = f(ax+b)$ for $x \in J$ is integrable and that $\int_I f \, d\lambda = |a| \int_J g \, d\lambda$ holds. [HINT: Use Theorem 22.12.]
17. Let $(X, \mathcal{S}, \mu)$ be a finite measure space. For every pair of measurable functions f and g let

$$d(f, g) = \int \frac{|f - g|}{1 + |f - g|} \, d\mu.$$

    a. Show that $(\mathcal{M}, d)$ is a metric space.
    b. Show that a sequence $\{f_n\}$ of measurable functions (i.e., $\{f_n\} \subseteq \mathcal{M}$) satisfies $f_n \xrightarrow{\mu} f$ if and only if $\lim d(f_n, f) = 0$.
    c. Show that $(\mathcal{M}, d)$ is a complete metric space. That is, show that if a sequence $\{f_n\}$ of measurable functions satisfies $d(f_n, f_m) \to 0$ as $n, m \to \infty$, then there exists a measurable function f such that $\lim d(f_n, f) = 0$.

18. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. For each finite interval I let $f_I = \frac{1}{\lambda(I)} \int_I f \, d\lambda$ and $E_I = \{x \in I : f(x) > f_I\}$. Show that

$$\int_I |f - f_I| \, d\lambda = 2 \int_{E_I} (f - f_I) \, d\lambda.$$

19. Let $f : [0, \infty) \to \mathbb{R}$ be a Lebesgue integrable function such that $\int_0^t f(x) \, d\lambda(x) = 0$ for each $t \ge 0$. Show that $f(x) = 0$ holds for almost all x.

20. Let $(X, \mathcal{S}, \mu)$ be a measure space and let $f, f_1, f_2, \ldots$ be non-negative integrable functions satisfying $f_n \to f$ a.e. and $\lim \int f_n \, d\mu = \int f \, d\mu$. If E is a measurable set, then show that

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu.$$

21. If a Lebesgue integrable function $f : [0, 1] \to \mathbb{R}$ satisfies $\int_0^1 x^{2n} f(x)\, d\lambda(x) = 0$ for each $n = 0, 1, 2, \ldots$, then show that $f = 0$ a.e.
[HINT: Since the algebra generated by $\{1, x^2\}$ is uniformly dense in $C[0, 1]$, we have $\int_0^1 g(x)f(x)\, dx = 0$ for each $g \in C[0, 1]$.]

22. For each n consider the partition $\{0, 2^{-n}, 2 \cdot 2^{-n}, 3 \cdot 2^{-n}, \ldots, (2^n - 1) \cdot 2^{-n}, 1\}$ of the interval $[0, 1]$ and define the function $r_n : [0, 1] \to \mathbb{R}$ by $r_n(1) = -1$, and

$$r_n(x) = (-1)^{k-1} \text{ for } (k-1)2^{-n} \le x < k2^{-n} \text{ and each } k = 1, 2, \ldots, 2^n.$$

a. Draw the graphs of r_1 and r_2.
b. Show that if $f : [0, 1] \to \mathbb{R}$ is a Lebesgue integrable function, then

$$\lim_{n\to\infty} \int_0^1 r_n(x) f(x)\, d\lambda(x) = 0.$$

[HINT: For (b) use Theorem 22.12.]

23. Let $\{\epsilon_n\}$ be a sequence of real numbers such that $0 < \epsilon_n < 1$ for each n. Also, let us say that a sequence $\{A_n\}$ of Lebesgue measurable subsets of $[0, 1]$ is *consistent with the sequence* $\{\epsilon_n\}$ if $\lambda(A_n) = \epsilon_n$ for each n. Establish the following properties of $\{\epsilon_n\}$:

a. The sequence $\{\epsilon_n\}$ converges to zero if and only if there exists a consistent sequence $\{A_n\}$ of measurable subsets of $[0, 1]$ such that $\sum_{n=1}^{\infty} \chi_{A_n}(x) < \infty$ for almost all x.
b. The series $\sum_{n=1}^{\infty} \epsilon_n$ converges in $\mathbb{R}$ if and only if for each consistent sequence $\{A_n\}$ of measurable subsets of $[0, 1]$ we have $\sum_{n=1}^{\infty} \chi_{A_n}(x) < \infty$ for almost all x.

24. Let $(X, \mathcal{S}, \mu)$ be a finite measure space and let $f : X \to \mathbb{R}$ be a measurable function.

a. Show that if f^n is integrable for each n and $\lim \int f^n \, d\mu$ exists in $\mathbb{R}$, then $|f(x)| \le 1$ holds for almost all x.
b. If f^n is integrable for each n, then show that $\int f^n \, d\mu = c$ (a constant) for $n = 1, 2, \ldots$ if and only if $f = \chi_A$ for some measurable subset A of X.

23. THE RIEMANN INTEGRAL AS A LEBESGUE INTEGRAL

It will be shown in this section that the Lebesgue integral is a generalization of the Riemann[5] integral. We start by reviewing the definition of the Riemann integral.

[5]Georg Friedrich Bernhard Riemann (1826–1866), a German mathematician, one of the greatest mathematicians of all time. Although his life was short, his contributions left the impact of a real genius. He made path-breaking contributions to the theory of complex functions, space geometry, and mathematical physics.

For simplicity, the details will be given for functions of one variable, and at the end it will be indicated how to carry out the same results for functions of several variables. Unless otherwise specified, throughout our discussion, f will be a fixed bounded real-valued function on a closed interval $[a, b]$ of $\mathbb{R}$.

A collection of points $P = \{x_0, x_1, \dots, x_n\}$ is called a **partition** of $[a, b]$ if

$$a = x_0 < x_1 < \cdots < x_n = b$$

holds. Every partition $P = \{x_0, x_1, \dots, x_n\}$ divides $[a, b]$ into the n closed subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n].$$

The length of the largest of these subintervals is called the **mesh** of P and is denoted by $|P|$: that is, $|P| = \max\{x_i - x_{i-1} : i = 1, \dots, n\}$. A partition P is said to be **finer** than another partition Q if $Q \subseteq P$ holds. If P and Q are partitions, then $P \cup Q$ is also a partition that is finer than both P and Q.

For a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$, let

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} \quad \text{and} \quad M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

for each $i = 1, \dots, n$. Then the **lower sum** $S_*(f, P)$ of f corresponding to the partition P is defined by

$$S_*(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}),$$

and similarly, the **upper sum** $S^*(f, P)$ of f by

$$S^*(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

Clearly, $S_*(f, P) \le S^*(f, P)$ holds for every partition P of $[a, b]$.

Lemma 23.1. *If a partition P is finer than another partition Q (i.e., $Q \subseteq P$), then*

$$S_*(f, Q) \le S_*(f, P) \quad \textit{and} \quad S^*(f, P) \le S^*(f, Q).$$

Proof. We show that $S_*(f, Q) \le S_*(f, P)$ holds. The other inequality can be proven in a similar manner.

To establish the inequality, it is enough to assume that P has only one more point than Q, say t. So, let $Q = \{x_0, x_1, \dots, x_n\}$ and $P = Q \cup \{t\}$. Then there exists some k $(1 \le k \le n)$ such that $x_{k-1} < t < x_k$, and thus, $P = \{x_0, x_1, \dots, x_{k-1}, t, x_k, \dots, x_n\}$. Let $c_1 = \inf\{f(x) : x \in [x_{k-1}, t]\}$ and $c_2 = \inf\{f(x) : x \in [t, x_k]\}$. Observe that both $m_k \le c_1$ and $m_k \le c_2$ hold. Therefore,

$$\begin{aligned} S_*(f, Q) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i \ne k} m_i(x_i - x_{i-1}) + m_k(x_k - x_{k-1}) \\ &\le \sum_{i \ne k} m_i(x_i - x_{i-1}) + c_1(t - x_{k-1}) + c_2(x_k - t) \\ &= S_*(f, P) \end{aligned}$$

holds, and the proof is finished. ■

Lemma 23.2. *For every pair of partitions P and Q, we have*

$$S_*(f, P) \le S^*(f, Q).$$

Proof. From Lemma 23.1, it follows that

$$S_*(f, P) \le S_*(f, P \cup Q) \le S^*(f, P \cup Q) \le S^*(f, Q),$$

as claimed. ■

The preceding lemma states that every upper sum is an upper bound for the collection of all lower sums of f, and similarly, every lower sum is a lower bound for the collection of all upper sums.

Thus, if the **lower Riemann integral** of f is defined by

$$I_*(f) = \sup\{S_*(f, P) : P \text{ is a partition of } [a, b]\}.$$

and the **upper Riemann integral** of f by

$$I^*(f) = \inf\{S^*(f, P) : P \text{ is a partition of } [a, b]\},$$

then

$$S_*(f, P) \leq I_*(f) \leq I^*(f) \leq S^*(f, Q)$$

holds for every pair of partitions P and Q of $[a, b]$.

Definition 23.3. *A bounded function $f : [a, b] \to \mathbb{R}$ is called* **Riemann integrable** *if $I_*(f) = I^*(f)$. In this case, the common value is called the* **Riemann integral** *of f and is denoted by the classical symbol $\int_a^b f(x)\,dx$.*

Historical Note: Riemann's definition of the integral is a generalization of Eudoxus' method of exhaustion as was used by Archimedes[6] in his computation of the area of a circle. Interestingly, the value of the limit of the areas of the inscribed (or circumscribed) polygons that were employed by Archimedes to compute the area of the circle, was also called by him the integral (τὸ πᾶν). It should be historically correct to call the Riemann integral the Eudoxus–Archimedes integral or the Eudoxus–Archimedes–Riemann integral (or even the Archimedes–Riemann integral).

A characterization for the Riemann integrability of a function, known as *Riemann's criterion*, is presented next.

Theorem 23.4 (Riemann's Criterion). *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\epsilon > 0$ there exists a partition P of $[a, b]$ such that $S^*(f, P) - S_*(f, P) < \epsilon$ holds.*

Proof. Assume that f is Riemann integrable and let $\epsilon > 0$. Then let $I = \int_a^b f(x)\,dx$. Then there exist two partitions P_1 and P_2 of $[a, b]$ such that $I - S_*(f, P_1) < \epsilon$ and $S^*(f, P_2) - I < \epsilon$. Then (by Lemma 23.1), the partition $P = P_1 \cup P_2$ satisfies

$$\begin{aligned} S^*(f, P) - S_*(f, P) &\leq S^*(f, P_2) - S_*(f, P_1) \\ &= [S^*(f, P_2) - I] + [I - S_*(f, P_1)] \\ &< \epsilon + \epsilon = 2\epsilon. \end{aligned}$$

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[6]Archimedes of Syracuse (287–212 BC), a Greek mathematician and inventor. He was the most celebrated mathematician of antiquity and perhaps the best mathematician of all times. He used Eudoxus' method of exhaustion to compute areas and volumes very successfully. In his classic work, *The Measurement of the Circle*, he established that a circle has the same area as a right triangle having one leg equal to the radius of the circle and the other equal to the circumference of the circle, and that the volume of a sphere is four times the volume of a right cone with radius and height equal to the radius of the sphere.

Conversely, if the condition is satisfied, then since

$$0 \leq I^*(f) - I_*(f) \leq S^*(f, P) - S_*(f, P)$$

holds for every partition P of $[a, b]$, we have $0 \leq I^*(f) - I_*(f) < \epsilon$ for all $\epsilon > 0$. Hence, $I^*(f) - I_*(f) = 0$, or $I_*(f) = I^*(f)$, which shows that f is Riemann integrable. ■

Now, let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. A collection of points $T = \{t_1, \ldots, t_n\}$ is said to be a **selection of points** for P if $x_{i-1} \leq t_i \leq x_i$ holds for $i = 1, \ldots, n$. We write

$$R_f(P, T) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}),$$

and call it (as usual) a **Riemann sum** associated with the partition P.

The following theorem of J.-C. Darboux[7] presents some powerful approximation formulas for the Riemann integral. The theorem can be viewed as an abstract formulation of Eudoxus' exhaustion method.

Theorem 23.5 (Darboux). *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable, and let $\{P_n\}$ be a sequence of partitions of $[a, b]$ such that $\lim |P_n| = 0$. Then*

$$\lim_{n\to\infty} S_*(f, P_n) = \lim_{n\to\infty} S^*(f, P_n) = \int_a^b f(x)\,dx.$$

In particular, if a sequence of partitions $\{P_n\}$ satisfies $\lim |P_n| = 0$ and T_n is a selection of points for P_n, then

$$\lim_{n\to\infty} R_f(P_n, T_n) = \int_a^b f(x)\,dx.$$

Proof. Choose a constant $c > 0$ such that $|f(x)| < c$ holds for all $x \in [a, b]$. Let $\epsilon > 0$. By Theorem 23.4, there exists a partition $P = \{x_0, x_1, \ldots, x_m\}$ of $[a, b]$ such that $S^*(f, P) - S_*(f, P) < \epsilon$. Choose n_0 such that

$$|P_n| < \frac{\epsilon}{2cm} \quad \text{and} \quad |P_n| < \min\{x_1 - x_0, x_2 - x_1, \ldots, x_m - x_{m-1}\}$$

[7]Jean-Gastin Darboux (1842–1917), a French mathematician. He was a geometer who used his geometric intuition to solve various problems in analysis and differential equations.

for all $n \geq n_0$. Fix $n \geq n_0$, and let $P_n = \{t_0, t_1, \ldots, t_k\}$. Put

$$\begin{aligned} M_j^P &= \sup\{f(x) : x \in [x_{j-1}, x_j]\} \quad \text{for } j = 1, 2, \ldots, m, \\ M_i &= \sup\{f(x) : x \in [t_{i-1}, t_i]\} \quad \text{for } i = 1, \ldots, k. \end{aligned}$$

The definitions of m_j^P and m_i are analogous (replace the sups by infs). Then

$$\begin{aligned} 0 &\leq \int_a^b f(x)\,dx - S_*(f, P_n) \leq S^*(f, P_n) - S_*(f, P_n) \\ &= \sum_{i=1}^{k} (M_i - m_i)(t_i - t_{i-1}) = V + W, \end{aligned}$$

where V is the sum of the terms $(M_i - m_i)(t_i - t_{i-1})$ for which $[t_{i-1}, t_i]$ lies entirely in some subinterval of the partition P, and W is the sum of the remaining terms. The sums V and W are estimated separately.

We estimate V first. Note that $V = \Sigma_1 + \cdots + \Sigma_m$, where each Σ_j is the sum of the terms $(M_i - m_i)(t_i - t_{i-1})$ for which $[t_{i-1}, t_i] \subseteq [x_{j-1}, x_j]$ holds. But if $[t_{i-1}, t_i] \subseteq [x_{j-1}, x_j]$, then $M_i - m_i \leq M_i^P - m_i^P$ holds. Also, the sum of the lengths of those subintervals of the partition P_n that lie inside $[x_{j-1}, x_j]$ never exceeds $x_j - x_{j-1}$. Thus, $\Sigma_j \leq (M_j^P - m_j^P)(x_j - x_{j-1})$ holds, and hence

$$V \leq \sum_{j=1}^{m} (M_j^P - m_j^P)(x_j - x_{j-1}) = S^*(f, P) - S_*(f, P) < \epsilon.$$

Now, we estimate W. Let $(M_i - m_i)(t_i - t_{i-1})$ be a term of the sum W. Since $|P_n| < \min\{x_j - x_{j-1} : j = 1, \ldots, m\}$, there exists exactly one j with $1 \leq j \leq n$ such that $x_{j-1} < t_{i-1} < x_j < t_i < x_{j+1}$. Thus, the sum W has at most m terms, and since

$$(M_i - m_i)(t_i - t_{i-1}) \leq 2c|P_n| < 2c \cdot \frac{\epsilon}{2cm} = \frac{\epsilon}{m}$$

it follows that $W < m(\epsilon/m) = \epsilon$. Thus,

$$0 \leq \int_a^b f(x)\,dx - S_*(f, P_n) \leq V + W < \epsilon + \epsilon = 2\epsilon$$

for all $n \geq n_0$. That is, $\lim S_*(f, P_n) = \int_a^b f(x)\,dx$.

Similarly, $0 \leq S^*(f, P_n) - \int_a^b f(x)\,dx \leq V + W < 2\epsilon$ holds for all $n \geq n_0$, and so, $\lim S^*(f, P_n) = \int_a^b f(x)\,dx$. The last part follows immediately from the inequalities

$$S_*(f, P_n) \le R_f(P_n, T_n) \le S^*(f, P_n).$$

The proof of the theorem is now complete. ∎

We are now ready to establish that the Lebesgue integral is a generalization of the Riemann integral. Here, "f is Lebesgue integrable" means that f is integrable with respect to the Lebesgue measure.

Theorem 23.6. *Every Riemann integrable function $f : [a, b] \to \mathbb{R}$ is Lebesgue integrable, and in this case the two integrals coincide. That is,*

$$\int f \, d\lambda = \int_a^b f(x) \, dx.$$

Proof. For each n let $P_n = \{x_0, x_1, \dots, x_{2^n}\}$ be the partition that divides $[a, b]$ into 2^n subintervals all of the same length $(b-a)2^{-n}$; that is, $x_i = a + i(b-a)2^{-n}$. Let

$$\phi_n = \sum_{i=1}^{2^n} m_i \chi_{[x_{i-1}, x_i)} \quad \text{and} \quad \psi_n = \sum_{i=1}^{2^n} M_i \chi_{[x_{i-1}, x_i)},$$

where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ and $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$. Clearly, $\{\phi_n\}$ and $\{\psi_n\}$ are two sequences of step functions satisfying the propetries $\phi_n(x) \uparrow \le f(x) \le \psi_n(x) \downarrow$ for each $x \in [a, b)$.

Now, if $\phi_n(x) \uparrow g(x)$ and $\psi_n(x) \downarrow h(x)$, then by Theorem 22.6, both functions g and h are Lebesgue integrable and $g(x) \le f(x) \le h(x)$ holds for all $x \in [a, b)$. Also, by definition, $\int \phi_n \, d\lambda = S_*(f, P_n)$ and $\int \psi_n \, d\lambda = S^*(f, P_n)$. Therefore, since $\psi_n(x) - \phi_n(x) \downarrow h(x) - g(x) \ge 0$, it follows that

$$\begin{aligned} 0 \le \int (h - g) \, d\lambda &= \lim_{n\to\infty} \int (\psi_n - \phi_n) \, d\lambda = \lim_{n\to\infty} \int \psi_n \, d\lambda - \lim_{n\to\infty} \int \phi_n \, d\lambda \\ &= \lim_{n\to\infty} S^*(f, P_n) - \lim_{n\to\infty} S_*(f, P_n) = 0, \end{aligned}$$

where the last equality holds true by virtue of Theorem 23.5. This implies $h - g = 0$ a.e., and hence, $h = g = f$ a.e. holds. In particular, $\phi_n \uparrow f$ a.e. and $\psi_n \downarrow f$ a.e. hold, which show that f is Lebesgue integrable—in fact, an upper function. Finally,

$$\int f \, d\lambda = \lim_{n\to\infty} \int \phi_n \, d\lambda = \lim_{n\to\infty} S_*(f, P_n) = \int_a^b f(x) \, dx.$$

and the proof of the theorem is finished. ∎

The next theorem, due to H. Lebesgue and G. Vitali, characterizes the Riemann integrable functions in terms of their discontinuities. (The almost everywhere relations are considered with respect to the Lebesgue measure.)

Theorem 23.7 (Lebesgue–Vitali). *A bounded function $f:[a,b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous almost everywhere.*

Proof. For each n, let P_n, ϕ_n, and ψ_n be as they were introduced in the proof of Theorem 23.6.

Assume first that f is Riemann integrable. Then a glance at the proof of Theorem 23.6 guarantees the existence of a (Lebesgue) null subset A of $[a,b]$ such that $\phi_n(x) \uparrow f(x)$ and $\psi_n(x) \downarrow f(x)$ for all $x \notin A$. Clearly, $D = A \cup (\bigcup_{n=1}^{\infty} P_n)$ has Lebesgue measure zero, and we claim that f is continuous on $[a,b] \setminus D$.

To see this, let $s \in [a,b] \setminus D$ and $\epsilon > 0$. Pick some n with $f(s) - \phi_n(s) < \epsilon$ and $\psi_n(s) - f(s) < \epsilon$. Then there exists some subinterval $[x_{i-1}, x_i]$ of the partition P_n such that $s \in (x_{i-1}, x_i)$. Clearly, $\phi_n(s) = m_i$ and $\psi_n(s) = M_i$. Therefore, if $x \in (x_{i-1}, x_i)$, then

$$-\epsilon < m_i - f(s) \leq f(x) - f(s) \leq M_i - f(s) < \epsilon.$$

Since (x_{i-1}, x_i) is a neighborhood of s, the last inequality shows that f is continuous at s. This establishes that f is continuous almost everywhere.

For the converse, assume that f is continuous almost everywhere. Let $s \neq b$ be a point of continuity of f. If $\epsilon > 0$ is given, then choose some $\delta > 0$ such that

$$f(s) - \epsilon < f(x) < f(s) + \epsilon \qquad (\star)$$

for all $x \in [a,b]$ with $|x - s| < \delta$. Pick some k so that $|P_k| < \delta$. Then for some subinterval $[x_{i-1}, x_i]$ of P_k, we must have $s \in [x_{i-1}, x_i)$. In particular, $|x - s| < \delta$ must hold for all $x \in [x_{i-1}, x_i]$, and so, from $(\star)$ we get

$$f(s) - \epsilon \leq m_i = \phi_k(s) < f(s) + \epsilon.$$

Since $\phi_n(s) \uparrow$, it easily follows that $\phi_n(s) \uparrow f(s)$. Similarly, $\psi_n(s) \downarrow f(s)$ holds.

Since f is continuous almost everywhere, we conclude that $\phi_n \uparrow f$ a.e. and $\psi_n \downarrow f$ a.e. both hold. This shows that f is Lebesgue integrable. Moreover, by the Lebesgue dominated convergence theorem we have

$$S_*(f, P_n) = \int \phi_n \, d\lambda \uparrow \int f \, d\lambda \quad \text{and} \quad S^*(f, P_n) = \int \psi_n \, d\lambda \downarrow \int f \, d\lambda.$$

Thus, $\lim[S^*(f, P_n) - S_*(f, P_n)] = 0$, and so, by Theorem 23.3, the function f is Riemann integrable. The proof of the theorem is now complete. ■

An immediate consequence of Theorem 23.7 is the following:

Theorem 23.8. *The collection of all Riemann integrable functions on a closed interval is a function space and an algebra of functions.*

It is easy to present examples of bounded Lebesgue integrable functions that are not Riemann integrable. Here is an example:

Example 23.9. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 0$ if x is a rational number and $f(x) = 1$ if x is irrational (in other words, f is the characteristic function of the irrationals of $[0, 1]$). Then f is discontinuous at every point of $[0, 1]$, and thus, by Theorem 23.7, f is not Riemann integrable. On the other hand, $f = \mathbf{1}$ a.e. holds (since the set of rational numbers has Lebesgue measure zero), and so, f is Lebesgue integrable. Also, note that $\int f\,d\lambda = 1$ holds. ■

It follows from Theorem 23.7 that if a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then f restricted to any closed subinterval of $[a, b]$ is also Riemann integrable there. Moreover, by the same theorem, if two functions f and g are Riemann integrable on $[a, c]$ and $[c, b]$, then the function $h : [a, b] \to \mathbb{R}$, defined by $h(x) = f(x)$ if $x \in [a, c]$, and $h(x) = g(x)$ if $x \in (c, b]$, is Riemann integrable.

Clearly, by Theorem 23.7, every continuous function on a closed interval is Riemann integrable. To compute the Riemann (and hence, the Lebesgue) integral of a continuous function, one usually uses the fundamental theorem of calculus, one form of which is stated next. Since any "reasonable" calculus book contains a proof of this important result, its proof is omitted. (See also Exercise 6 at the end of the section.) The fundamental theorem of calculus is due to I. Newton[8] and independently to G. Leibniz.[9]

Theorem 23.10 (The Fundamental Theorem of Calculus). *For a continuous function $f : [a, b] \to \mathbb{R}$ we have the following:*

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[8]Isaac Newton (1642–1727), a great British mathematician, physicist, astronomer, and philosopher. He discovered the law of gravity and was one of the founders of calculus. His pioneering original contributions to mathematics and science revolutionized the modern scientific approach.

[9]Gottfried Wilhelm Leibniz (1646–1716), a prominent German mathematician and philosopher. He was a person with "universal" scientific interests. Besides his philosophical and metaphysical contributions, he contributed substantially to mathematics, mathematical logic, and physics. Together with Newton, he is considered the founder of calculus.

1. *If $A:[a,b] \to \mathbb{R}$ is an* **area function** *of f (i.e., $A(x) = \int_c^x f(t)\,dt$ holds for all $x \in [a,b]$ and some fixed $c \in [a,b]$), then A is an antiderivative of f. That is, $A'(x) = f(x)$ holds for each $x \in [a,b]$.*
2. *If $F:[a,b] \to \mathbb{R}$ is an antiderivative of f, i.e., $F'(x) = f(x)$ holds for each $x \in [a,b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

In a conventional way, the integral $\int_b^a f(x)\,dx$ is defined to be $-\int_a^b f(x)\,dx$; that is, $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$. Also, $\int_a^a f(x)\,dx$ is defined to be zero. By doing so, the useful identity

$$\int_c^d f(x)\,dx = \int_c^e f(x)\,dx + \int_e^d f(x)\,dx$$

holds regardless of the ordering between the points c, d, and e of $[a,b]$.

We now indicate how to extend the above results to functions of several variables. In the general case, the interval $[a,b]$ is replaced by a **cell** $J = [a_1,b_1] \times \cdots \times [a_n,b_n]$, and its Lebesgue measure is $\lambda(J) = \prod_{i=1}^n (b_i - a_i)$. A **partition** P of J is a set of points of the form $P = P_1 \times \cdots \times P_n$, where P_i is a partition of $[a_i,b_i]$ for each $i = 1,\dots,n$. Clearly, any partition P divides J into a finite number of subcells.

Now, if $f: J \to \mathbb{R}$ is a bounded function and the partition P divides J into the subcells $J_1,\dots,J_k$, then we define again the numbers

$$\begin{aligned} m_i &= \inf\{f(x_1,\dots,x_n) : (x_1,\dots,x_n) \in J_i\}, \\ M_i &= \sup\{f(x_1,\dots,x_n) : (x_1,\dots,x_n) \in J_i\}. \end{aligned}$$

The lower and upper sums corresponding to the partition P are defined as before by the formulas

$$S_*(f,P) = \sum_{i=1}^k m_i\lambda(J_i) \quad \text{and} \quad S^*(f,P) = \sum_{i=1}^k M_i\lambda(J_i),$$

respectively.

The **lower Riemann integral** of f is defined (as before) by

$$I_*(f) = \sup\{S_*(f,P) : P \text{ is a partition of } J\},$$

and the **upper Riemann integral** of f by.

$$I^*(f) = \inf\{S^*(f, P) : P \text{ is a partition of } J\}.$$

As in the one-dimensional case,

$$-\infty < I_*(f) \leq I^*(f) < \infty$$

holds. The function f is called **Riemann integrable** if $I_*(f) = I^*(f)$. This common number is called the **Riemann integral** of f and is denoted by

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} f(x_1, x_2, \ldots, x_n)\, dx_1 dx_2 \cdots dx_n.$$

All the results given in this section are valid in this general setting. Their proofs parallel the ones presented here, and for this reason we leave them as an exercise for the reader.

EXERCISES

1. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Show that f is Riemann integrable on every closed subinterval of $[a, b]$. Also, show that
$$\int_c^d f(x)\, dx = \int_c^e f(x)\, dx + \int_e^d f(x)\, dx$$
holds for every three points c, d, and e of $[a, b]$.
2. Let $f : [a. b] \to \mathbb{R}$ be Riemann integrable. Then show that
$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{i(b-a)}{n}\right).$$
3. Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a, b]$ such that $\{f_n\}$ converges uniformly to a function f. Show that f is Riemann integrable and that
$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$
4. For each n. let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = \frac{nx^{n-1}}{1+x}$ for all $x \in [0, 1]$. Then show that $\lim \int_0^1 f_n(x)\, dx = \frac{1}{2}$.
[HINT: Use integration by parts.]
5. Let $f : [a. b] \to \mathbb{R}$ be an increasing function. Show that f is Riemann integrable.
[HINT: Verify that f satisfies Riemann's criterion.]
6. (*The Fundamental Theorem of Calculus*) If $f : [a, b] \to \mathbb{R}$ is a Riemann integrable function, define its **area function** $A : [a, b] \to \mathbb{R}$ by $A(x) = \int_a^x f(t)\, dt$ for each $x \in [a, b]$. Show that

a. A is a uniformly continuous function.
b. If f is continuous at some point c of $[a, b]$, then A is differentiable at c and $A'(c) = f(c)$ holds.
c. Give an example of a Riemann integrable function f whose area function A is differentiable and satisfies $A' \neq f$.

[HINT: For part (c) use the function defined in Exercise 7 of Section 9.]

7. **(Arzelà)** Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a, b]$ such that $\lim f_n(x) = f(x)$ holds for each $x \in [a, b]$ and f is Riemann integrable. Also, assume that there exists a constant M such that $|f_n(x)| \leq M$ holds for all $x \in [a, b]$ and all n. Show that

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

8. Determine the lower and upper Riemann integrals for the function of Example 23.9.
9. Let C be the Cantor set (see Example 6.15). Show that χ_C is Riemann integrable over $[0, 1]$, and that $\int_0^1 \chi_C\,dx = 0$.
10. Let $0 < \epsilon < 1$, and consider the ϵ-Cantor set C_ϵ of $[0, 1]$. Show that χ_{C_ϵ} is not Riemann integrable over $[0, 1]$. Also, determine $I_*(\chi_{C_\epsilon})$ and $I^*(\chi_{C_\epsilon})$.
[HINT: Show that the set of all discontinuities of χ_{C_ϵ} is C_ϵ.]
11. Give a proof of the Riemann integrability of a continuous function based upon its uniform continuity (Theorem 7.7).
12. Establish the familiar change of variable formula for the Riemann integral of continuous functions: *If* $[a, b] \xrightarrow{g} [c, d] \xrightarrow{f} \mathbb{R}$ *are continuous functions with* g *continuously differentiable (i.e.,* g *has a continuous derivative), then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

13. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x\to\infty} f(x) = \delta$. Show that $\lim_{n\to\infty} \int_0^a f(nx)\,dx = a\delta$ for each $a > 0$.
14. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that $f(x + 1) = f(x)$ for all $x \geq 0$. If $g : [0, 1] \to \mathbb{R}$ is an arbitrary continuous function, then show that

$$\lim_{n\to\infty} \int_0^1 g(x)f(nx)\,dx = \left(\int_0^1 g(x)\,dx\right) \cdot \left(\int_0^1 f(x)\,dx\right).$$

15. Let $f : [0, 1] \to [0, \infty)$ be Riemann integrable on every closed subinterval of $(0, 1]$. Show that f is Lebesgue integrable over $[0, 1]$ if and only if $\lim_{\epsilon\downarrow 0} \int_\epsilon^1 f(x)\,dx$ exists in $\mathbb{R}$. Also, show that if this is the case, then $\int f\,d\lambda = \lim_{\epsilon\downarrow 0} \int_\epsilon^1 f(x)\,dx$.
16. As an application of the preceding exercise, show that the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = x^p$ if $x \in (0, 1]$ and $f(0) = 0$ is Lebesgue integrable if and only if $p > -1$. Also, show that if f is Lebesgue integrable, then

$$\int f\,d\lambda = \frac{1}{1 + p}.$$

17. Let $f : [0, 1] \to \mathbb{R}$ be a function and define $g : [0, 1] \to \mathbb{R}$ by $g(x) = e^{f(x)}$.

a. Show that if f is measurable (or Borel measurable), then so is g.
b. If f is Lebesgue integrable, is then g necessarily Lebesgue integrable?
c. Give an example of an essentially unbounded function f which is continuous on $(0, 1]$ such that f^n is Lebesgue integrable for each $n = 1, 2, \ldots$. (A function f is "essentially unbounded," if for each $M > 0$ the set $\{x \in [0, 1] : f(x)| > M\}$ has positive measure.)

[HINT: For (b) consider the function $f(x) = x^{-\frac{1}{2}}$.]

18. Let $f : [0, 1] \to \mathbb{R}$ be Lebesgue integrable. Assume that f is differentiable at $x = 0$ and $f(0) = 0$. Show that the function $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = x^{-\frac{3}{2}} f(x)$ for $x \in (0, 1]$ and $g(0) = 0$ is Lebesgue integrable.

19. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function. Show that the Riemann integral of f can be computed with two iterated integrations. That is, show that

$$\int_a^b \int_c^d f(x, y)\,dx dy = \int_a^b \left[\int_c^d f(x, y)\,dy\right] dx = \int_c^d \left[\int_a^b f(x, y)\,dx\right] dy.$$

Generalize this to a continuous function of n variables.

20. Assume that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are two continuous functions such that $f(x) \le g(x)$ for $x \in [a, b]$. Let $A = \{(x, y) \in \mathbb{R}^2 : x \in [a, b] \text{ and } f(x) \le y \le g(x)\}$.

a. Show that A is a closed set—and hence, a measurable subset of $\mathbb{R}^2$.
b. If $h : A \to \mathbb{R}$ is a continuous function, then show that h is Lebesgue integrable over A and that

$$\int_A h\,d\lambda = \int_a^b \left[\int_{f(x)}^{g(x)} h(x, y)\,dy\right] dx.$$

21. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function—with one-sided derivatives at the end-points. If the derivative f' is uniformly bounded on $[a, b]$, then show that f' is Lebesgue integrable and that

$$\int_{[a,b]} f'\,d\lambda = f(b) - f(a).$$

22. Let $f, g : [a, b] \to \mathbb{R}$ be two Lebesgue integrable functions satisfying

$$\int_a^x f(t)\,d\lambda(t) \le \int_a^x g(t)\,d\lambda(t)$$

for each $x \in [a, b]$. If $\phi : [a, b] \to \mathbb{R}$ is a non-negative decreasing function, then show that the functions ϕf and ϕg are both Lebesgue integrable over $[a, b]$ and that they satisfy

$$\int_a^x \phi(t) f(t)\,d\lambda(t) \le \int_a^x \phi(t) g(t)\,d\lambda(t)$$

for all $x \in [a, b]$.
[HINT: Prove it first for a decreasing function of the form $\phi = \sum_{i=1}^k c_i \chi_{[a_{i-1}, a_i)}$, where $a_0 < a_1 < \cdots < a_k$ is a partition of $[a, b]$, and then use the fact that there

exists a sequence $\{\phi_n\}$ of such step functions satisfying $\phi_n(t) \uparrow \phi(t)$ for almost all t in $[a, b]$; see Exercise 8 of Section 21.]

24. APPLICATIONS OF THE LEBESGUE INTEGRAL

If a function $f : [a, \infty) \to \mathbb{R}$ (where, of course, $a \in \mathbb{R}$) is Riemann integrable on every closed subinterval of $[a, \infty)$, then its **improper Riemann integral** is defined by

$$\int_a^\infty f(x)\,dx = \lim_{r\to\infty} \int_a^r f(x)\,dx,$$

provided that the limit on the right-hand side exists in $\mathbb{R}$. The existence of the prior limit is also expressed by saying that the (improper Riemann) integral $\int_a^\infty f(x)\,dx$ exists. Similarly, if $f : (-\infty, a] \to \mathbb{R}$ is Riemann integrable on every closed subinterval of $(-\infty, a]$, then $\int_{-\infty}^a f(x)\,dx$ is defined by

$$\int_{-\infty}^a f(x)\,dx = \lim_{r\to-\infty} \int_r^a f(x)\,dx$$

whenever the limit exists in $\mathbb{R}$. It should be clear that if $\int_a^\infty f(x)\,dx$ exists, then $\int_b^\infty f(x)\,dx$ also exists for each $b > a$ and

$$\int_a^\infty f(x)\,dx = \int_a^b f(x)\,dx + \int_b^\infty f(x)\,dx.$$

Theorem 24.1. *Assume that $f : [a, \infty) \to \mathbb{R}$ is Riemann integrable on every closed subinterval of $[a, \infty)$. Then $\int_a^\infty f(x)\,dx$ exists if and only if for every $\epsilon > 0$ there exists some $M > 0$ (depending on ϵ) such that $|\int_s^t f(x)\,dx| < \epsilon$ for all $s, t \geq M$.*

Proof. Assume that $I = \int_a^\infty f(x)\,dx$ exists. Pick a real number $M > 0$ such that $|I - \int_a^r f(x)\,dx| < \epsilon$ holds for all $r \geq M$. If $s, t \geq M$, then

$$\begin{aligned}\left|\int_s^t f(x)\,dx\right| &= \left|\int_a^t f(x)\,dx - \int_a^s f(x)\,dx\right| \\ &\leq \left|I - \int_a^t f(x)\,dx\right| + \left|I - \int_a^s f(x)\,dx\right| < 2\epsilon.\end{aligned}$$

Conversely, assume that the condition is satisfied. If $\{a_n\}$ is a sequence of $[a, \infty)$ such that $\lim a_n = \infty$, then it is easy to see that the sequence $\{\int_a^{a_n} f(x)\,dx\}$ is

Cauchy. Thus, $A = \lim \int_a^{a_n} f(x)\,dx$ exists in $\mathbb{R}$. Now, let $\{b_n\}$ be another sequence of $[a, \infty)$ with $\lim b_n = \infty$; let $B = \lim \int_a^{b_n} f(x)\,dx$. From the inequality

$$|A - B| \leq \left|A - \int_a^{a_n} f(x)\,dx\right| + \left|\int_{a_n}^{b_n} f(x)\,dx\right| + \left|B - \int_a^{b_n} f(x)\,dx\right|,$$

it is easy to see that $|A - B| < \epsilon$ holds for all $\epsilon > 0$. Thus, $A = B$, and so the limit is independent of the chosen sequence. This shows that $\int_a^\infty f(x)\,dx$ exists, and the proof is finished. ■

From the preceding theorem, and the inequality $|\int_s^t f(x)\,dx| \leq \int_s^t |f(x)|\,dx$ for $s < t$, we have the following:

Lemma 24.2. *If a function $f : [a, \infty) \to \mathbb{R}$ is Riemann integrable on every closed subinterval of $[a, \infty)$ and $\int_a^\infty |f(x)|\,dx$ exists then $\int_a^\infty f(x)\,dx$ also exists and $|\int_a^\infty f(x)\,dx| \leq \int_a^\infty |f(x)|\,dx$.*

The converse of this lemma is false. That is, there are functions f whose improper Riemann integrals $\int_a^\infty f(x)\,dx$ exist but for which $\int_a^\infty |f(x)|\,dx$ fails to exist. A well-known example is provided by the function $f(x) = \frac{\sin x}{x}$ (with $f(0) = 1$) over $[0, \infty)$. We shall see later that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$ but $\int_0^\infty \frac{|\sin x|}{x}\,dx$ does not exist. To see the latter, note that

$$\int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\,dx \geq \frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi} |\sin x|\,dx = \frac{2}{k\pi}$$

holds for each k. Therefore,

$$\int_0^{n\pi} \frac{|\sin x|}{x}\,dx = \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\,dx \geq \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k},$$

which shows that $\int_0^\infty \frac{|\sin x|}{x}\,dx$ does not exist in $\mathbb{R}$.

If the improper Riemann integral of a function exists, then it is natural to ask whether the function is, in fact, Lebesgue integrable. In general, this is not the case. However, if the improper Riemann integral of the absolute value of the function exists, then the function is Lebesgue integrable. The details follow.

Theorem 24.3. *Let $f : [a, \infty) \to \mathbb{R}$ be Riemann integrable on every closed subinterval of $[a, \infty)$. Then f is Lebesgue integrable if and only if the improper*

Riemann integral $\int_a^\infty |f(x)|\,dx$ *exists. Moreover, in this case*

$$\int f\,d\lambda = \int_a^\infty f(x)\,dx.$$

Proof. Assume that f is Lebesgue integrable over $[a,\infty)$. Then f^+ is also Lebesgue integrable over $[a,\infty)$. Let $\{r_n\}$ be a sequence of $[a,\infty)$ such that $\lim r_n = \infty$. For each n, let $f_n(x) = f^+(x)$ if $x \in [a, r_n]$ and $f_n(x) = 0$ if $x > r_n$. Then $\lim f_n(x) = f^+(x)$ and $0 \le f_n(x) \le f^+(x)$ hold for all $x \in [a,\infty)$. Moreover, by Theorem 23.6, $\{f_n\}$ is a sequence of Lebesgue integrable functions such that $\int f_n\,d\lambda = \int_a^{r_n} f^+(x)\,dx$. Thus, by the Lebesgue dominated convergence theorem

$$\int f^+\,d\lambda = \lim_{n\to\infty}\int f_n\,d\lambda = \lim_{n\to\infty}\int_a^{r_n} f^+(x)\,dx.$$

This shows that $\int_a^\infty f^+(x)\,dx$ exists and $\int_a^\infty f^+(x)\,dx = \int f^+\,d\lambda$. Similarly, $\int_a^\infty f^-(x)\,dx$ exists and $\int_a^\infty f^-(x)\,dx = \int f^-\,d\lambda$. But then, it follows from $f = f^+ - f^-$ and $|f| = f^+ + f^-$ that both improper Riemann integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty |f(x)|\,dx$ exist. Moreover,

$$\int_a^\infty f(x)\,dx = \int f\,d\lambda \quad\text{and}\quad \int_a^\infty |f(x)|\,dx = \int |f|\,d\lambda.$$

For the converse, assume that the improper Riemann integral $\int_a^\infty |f(x)|\,dx$ exists. Clearly, $\lim \int_a^{a+n} |f(x)|\,dx = \int_a^\infty |f(x)|\,dx$. Define $f_n(x) = |f(x)|$ if $x \in [a, a+n]$ and $f_n(x) = 0$ if $x > a+n$ for each n. Note that $0 \le f_n(x) \uparrow |f(x)|$ holds for all $x \ge a$. Since f_n is Riemann integrable on $[a, a+n]$, f_n is an upper function on $[a,\infty)$ satisfying $\int f_n\,d\lambda = \int_a^{a+n} f_n(x)\,dx = \int_a^{a+n} |f(x)|\,dx$. Thus, by Theorem 21.6, $|f|$ is an upper function (and hence, Lebesgue integrable) such that $\int |f|\,d\lambda = \int_a^\infty |f(x)|\,dx$. The Lebesgue integrability of f now follows from Theorem 22.6 by observing that f is a measurable function, since it is a measurable function on every closed subinterval of $[a,\infty)$. The proof of the theorem is now complete. ■

The next result deals with interchanging the processes of limit and integration.

Theorem 24.4. *Let* $(X, \mathcal{S}, \mu)$ *be a measure space, let* J *be a subinterval of* $\mathbb{R}$, *and let* $f : X \times J \to \mathbb{R}$ *be a function such that* $f(\cdot, t)$ *is a measurable function for each* $t \in J$. *Assume also that there exists an integrable function* g

such that for each $t \in J$ we have $|f(x,t)| \le g(x)$ for almost all x. If for some accumulation point t_0 (including possibly $\pm\infty$) of J there exists a function h such that $\lim_{t\to t_0} f(x,t) = h(x)$ exists in $\mathbb{R}$ for almost all x, then h defines an integrable function, and

$$\lim_{t\to t_0} \int f(x,t)\,d\mu(x) = \int \lim_{t\to t_0} f(x,t)\,d\mu(x) = \int h\,d\mu.$$

Proof. Assume that the function $f: X \times J \to \mathbb{R}$ satisfies the hypotheses of the theorem. Let $\{t_n\}$ be a sequence of J such that $\lim t_n = t_0$. Put $h_n(x) = f(x, t_n)$ for $x \in X$, and note that $|h_n| \le g$ a.e. holds for each n and $h_n \to h$ a.e. By Theorem 22.6, each h_n is integrable. Moreover, by the Lebesgue dominated convergence theorem, h defines an integrable function and

$$\lim_{n\to\infty} \int f(x,t_n)\,d\mu(x) = \lim_{n\to\infty} \int h_n\,d\mu = \int h\,d\mu$$

holds, from which our conclusion follows. ■

If $f: X \times (a,b) \to \mathbb{R}$ is a function and $t_0 \in (a,b)$, then its difference quotient function at t_0 is defined by

$$D_{t_0}(x,t) = \frac{f(x,t) - f(x,t_0)}{t - t_0}$$

for all $x \in X$ and all $t \in (a,b)$ with $t \ne t_0$. To make the function D_{t_0} defined everywhere, we let $D_{t_0}(x,t_0) = 0$ for all $x \in X$.

As usual, $\lim_{t\to t_0} D_{t_0}(x,t)$, whenever the limit exists, is called the **partial derivative** of f with respect to t at the point (x, t_0) and is denoted by $\frac{\partial f}{\partial t}(x, t_0)$. That is,

$$\frac{\partial f}{\partial t}(x,t_0) = \lim_{t\to t_0} D_{t_0}(x,t) = \lim_{t\to t_0} \frac{f(x,t) - f(x,t_0)}{t - t_0}.$$

The next result deals with differentiation of a function defined by an integral.

Theorem 24.5. *Let $(X, \mathcal{S}, \mu)$ be a measure space and let $f: X \times (a,b) \to \mathbb{R}$ be a function such that $f(\cdot, t)$ is Lebesgue integrable for every $t \in (a,b)$. Assume that for some $t_0 \in (a,b)$ the partial derivative $\frac{\partial f}{\partial t}(x,t_0)$ exists for almost all x. Suppose also that there exists an integrable function g and a neighborhood V of t_0 such that for each $t \in V$ we have $|D_{t_0}(x,t)| \le g(x)$ for almost all $x \in X$. Then*

1. *$\frac{\partial f}{\partial t}(\cdot, t_0)$ defines an integrable function, and*

2. *the function* $F:(a,b)\to \mathbb{R}$ *defined by* $F(t)=\int f(x,t)\,d\mu(x)$ *is differentiable at* t_0 *and*

$$F'(t_0)=\int \frac{\partial f}{\partial t}(x,t_0)\,d\mu(x).$$

Proof. Put $\frac{\partial f}{\partial t}(x,t_0)=0$ at each point x where the partial derivative does not exist. Then $\lim_{t\to t_0} D_{t_0}(x,t)=\frac{\partial f}{\partial t}(x,t_0)$ holds for almost all x. Thus, by Theorem 24.4, $\frac{\partial f}{\partial t}(\cdot,t_0)$ defines an integrable function and

$$\begin{aligned}\frac{F(t)-F(t_0)}{t-t_0} &= \int \frac{f(x,t)-f(x,t_0)}{t-t_0}\,d\mu(x)\\ &= \int D_{t_0}(x,t)\,d\mu(x) \longrightarrow \int \frac{\partial f}{\partial t}(x,t_0)\,d\mu(x)\end{aligned}$$

as $t\to t_0$. This shows that F is differentiable at t_0 and that $F'(t_0)=\int \frac{\partial f}{\partial t}(x,t_0)\,d\mu(x)$ holds. ■

There is a criterion for testing the boundedness of the difference quotient function $D_{t_0}(x,t)$ by an integrable function that is very useful for applications. It requires the existence of a neighborhood V of t_0 satisfying the following two properties:

1. the partial derivative $\frac{\partial f}{\partial t}(x,t)$ exists for all x and all $t\in V$, and
2. there exists a non-negative integrable function g such that for each $t\in V$, we have $|\frac{\partial f}{\partial t}(x,t)|\le g(x)$ for almost all x.

Indeed, if the previous two conditions hold, then by the mean value theorem, it easily follows that for each $t\in V$ we have $|D_{t_0}(x,t)|\le g(x)$ for almost all x.

Next, as applications of the last two theorems we shall compute a number of classical improper Riemann integrals.

Theorem 24.6 (Euler).[10] *We have* $\int_0^\infty e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$.

Proof. The existence of the improper Riemann integral follows from the inequalities $0\le e^{-x^2}\le e^{-x}$ for $x\ge 1$ and $\int_1^\infty e^{-x}\,dx=e^{-1}$. Also, since $e^{-x^2}\ge 0$ for all x, the improper Riemann integral is a Lebesgue integral over $[0,\infty)$.

[10] Leonhard Euler (1707–1783), a great Swiss mathematician. He was one of the most prolific writers throughout the history of science. He is considered (together with Gauss and Riemann) as one of the three greatest mathematicians of modern times.

For the computation of the integral, consider the real-valued functions on $\mathbb{R}$ defined by

$$f(t) = \left(\int_0^t e^{-x^2}\,dx\right)^2 \quad \text{and} \quad g(t) = \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,dx.$$

The derivatives of the previous functions will be determined separately. For f we use the fundamental theorem of calculus and the chain rule. Thus,

$$f'(t) = 2e^{-t^2}\int_0^t e^{-x^2}\,dx.$$

For g observe that

$$\left|\frac{\partial}{\partial t}\left(\frac{e^{-t^2(1+x^2)}}{1+x^2}\right)\right| = \left|-2te^{-t^2(1+x^2)}\right| \le 2|t| \le M$$

holds for every $x \in [0, 1]$ and every $t \in \mathbb{R}$ in any bounded neighborhood of some fixed point t_0. The constant M depends, of course, upon the choice of the neighborhood of t_0. By Theorem 24.5, and the fact that the Lebesgue integrals are Riemann integrals, we get

$$g'(t) = \int_0^1 \frac{\partial}{\partial t}\left(\frac{e^{-t^2(1+x^2)}}{1+x^2}\right)dx = -2e^{-t^2}\int_0^1 te^{-x^2t^2}\,dx$$

for all $t \in \mathbb{R}$. Substituting $u = xt$ (for $t \neq 0$), we obtain

$$g'(t) = -2e^{-t^2}\int_0^t e^{-u^2}\,du = -2e^{-t^2}\int_0^t e^{-x^2}\,dx$$

for each $t \in \mathbb{R}$.

Thus, $f'(t)+g'(t) = 0$ holds for each $t \in \mathbb{R}$, and so, $f(t)+g(t) = c$ (a constant) holds for all t. In particular,

$$c = f(0) + g(0) = \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4},$$

and so $f(t) + g(t) = \frac{\pi}{4}$ for each t. Now, observe that for each x and t we have

$$\left|\frac{e^{-t^2(1+x^2)}}{1+x^2}\right| \le \frac{1}{1+x^2},$$

and clearly

$$\lim_{t\to\infty} \frac{e^{-t^2(1+x^2)}}{1+x^2} = 0.$$

Thus, by Theorem 24.4, $\lim_{t\to\infty} g(t) = 0$. Consequently,

$$\frac{\pi}{4} = \lim_{t\to\infty} f(t) + \lim_{t\to\infty} g(t) = \left(\int_0^\infty e^{-x^2}\,dx\right)^2,$$

from which it follows that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$. ■

Theorem 24.7. *For each $t \in \mathbb{R}$ we have*

$$\int_0^\infty e^{-x^2}\cos(2xt)\,dx = \frac{\sqrt{\pi}}{2}e^{-t^2}.$$

Proof. Let $F(t) = \int_0^\infty e^{-x^2}\cos(2xt)\,dx$ for all $t \in \mathbb{R}$. Since $|e^{-x^2}\cos(2xt)| \le e^{-x^2}$ holds for all x and t, it follows that the improper Riemann integral $F(t)$ exists and, moreover, is a Lebesgue integral over $[0, \infty)$. Now,

$$\left|\frac{\partial}{\partial t}\left[e^{-x^2}\cos(2xt)\right]\right| = \left|-2xe^{-x^2}\sin(2xt)\right| \le 2xe^{-x^2} = g(x)$$

holds for all $x \ge 0$ and t. Hence, since the function g is positive on $[0, \infty)$ and the improper Riemann integral $\int_0^\infty g(x)\,dx$ exists (its value is 1), g is Lebesgue integrable over $[0, \infty)$. Therefore, by Theorem 24.5 (and the remarks after it), it follows that

$$F'(t) = \int_0^\infty \frac{\partial}{\partial t}\left[e^{-x^2}\cos(2xt)\right]dx = -2\int_0^\infty xe^{-x^2}\sin(2xt)\,dx.$$

Integrating by parts, we obtain

$$\begin{aligned} -2\int_0^\infty xe^{-x^2}\sin(2xt)\,dx &= e^{-x^2}\sin(2xt)\Big|_0^\infty - 2t\int_0^\infty e^{-x^2}\cos(2xt)\,dx \\ &= -2t\int_0^\infty e^{-x^2}\cos(2xt)\,dx. \end{aligned}$$

Thus, $F'(t) = -2tF(t)$ holds for all t. Solving the differential equation, we get $F(t) = F(0)e^{-t^2}$. By Theorem 24.6, $F(0) = \frac{\sqrt{\pi}}{2}$, and so, $F(t) = \frac{\sqrt{\pi}}{2}e^{-t^2}$, as claimed. ■

For the next result, the value of $\frac{\sin x}{x}$ at zero will be assumed to be 1.

Theorem 24.8. *If $t \geq 0$, then*

$$\int_0^\infty \frac{\sin x}{x} e^{-xt}\, dx = \frac{\pi}{2} - \arctan t$$

holds.

Proof. Fix some $t_0 \geq 0$. The cases $t_0 > 0$ and $t_0 = 0$ will be treated separately.

Case I. Assume $t_0 > 0$.

For each fixed $t > 0$, note that $|e^{-xt}\frac{\sin x}{x}| \leq e^{-xt}$ holds for all $x \geq 0$. Thus, the improper Riemann integral exists as a Lebesgue integral over $[0, \infty)$. Let $F(t) = \int_0^\infty e^{-xt}\frac{\sin x}{x}\, dx$ for $t > 0$. Then F satisfies the following properties:

$$\lim_{t\to\infty} F(t) = 0, \tag{$\star$}$$

and

$$F'(t) = -\frac{1}{1+t^2} \quad \text{for each } t > 0. \tag{$\star\star$}$$

To see ($\star$), note that if $g(x) = 1$ for $x \in [0, 1]$ and $g(x) = e^{-x}$ for $x > 1$, then g is Lebesgue integrable over $[0, \infty)$ for each $t > 0$ and $|e^{-xt}\frac{\sin x}{x}| \leq g(x)$ holds for all $x \geq 0$ and all $t \geq 1$. On the other hand, $\lim_{t\to\infty} e^{-xt}\frac{\sin x}{x} = 0$ holds for all $x > 0$, and so, by Theorem 24.4, $\lim_{t\to\infty} F(t) = 0$.

To establish ($\star\star$), note first that $\frac{\partial}{\partial t}[e^{-xt}\frac{\sin x}{x}] = -e^{-xt}\sin x$ holds for all $x \geq 0$, and that for each fixed $a > 0$, the inequality $|e^{-xt}\frac{\sin x}{x}| \leq e^{-ax}$ holds for each $t \geq a$ and all $x \geq 0$. By Theorem 24.5, $F'(t) = -\int_0^\infty e^{-xt}\sin x\, dx$ for all $t > a$ (and all $a > 0$), where the last equality holds since the improper Riemann integral is a Lebesgue integral. Thus, $F'(t) = -\int_0^\infty e^{-xt}\sin x\, dx$ holds for all $t > 0$. Since from elementary calculus we have

$$\int_0^r e^{-xt}\sin x\, dx = -\frac{e^{-rt}(t\sin r + \cos r)}{1+t^2} + \frac{1}{1+t^2},$$

by letting $r \to \infty$, we get $F'(t) = -\frac{1}{1+t^2}$ for $t > 0$.

Integrating ($\star\star$) from t to t_0 yields

$$F(t_0) - F(t) = -\int_t^{t_0} \frac{dx}{1+x^2} = \arctan t - \arctan t_0,$$

and by letting $t \to \infty$ it follows that $F(t_0) = \frac{\pi}{2} - \arctan t_0$.

Case II. Assume $t_0 = 0$.

In this case, $\frac{\sin x}{x}$ is not Lebesgue integrable over $[0, \infty)$. However, the improper Riemann integral $\int_0^\infty \frac{\sin x}{x}\,dx$ exists. Indeed, if $0 < s < t$, then an integration by parts yields

$$\int_s^t \frac{\sin x}{x}\,dx = -\frac{\cos x}{x}\Big|_s^t - \int_s^t \frac{\cos x}{x^2}\,dx = \frac{\cos s}{s} - \frac{\cos t}{t} - \int_s^t \frac{\cos x}{x^2}\,dx,$$

and thus,

$$\left|\int_s^t \frac{\sin x}{x}\,dx\right| \le \frac{1}{s} + \frac{1}{t} + \int_s^t \frac{dx}{x^2} = \frac{2}{s}.$$

By Theorem 24.1, $\int_0^\infty \frac{\sin x}{x}\,dx$ exists. In particular, note that $\lim_{n\to\infty} \int_n^{n+1} \frac{\sin x}{x}\,dx = 0$.

Now, for each n define $f_n(t) = \int_0^n e^{-xt}\frac{\sin x}{x}\,dx$ for $t \ge 0$, and note that the relation $|f_n(n)| \le \int_0^n e^{-xn}\,dx = \frac{1-e^{-n^2}}{n} \le \frac{1}{n}$ implies $\lim f_n(n) = 0$. By Theorem 24.5, we have

$$f_n'(t) = -\int_0^n e^{-xt}\sin x\,dx = \frac{e^{-nt}(t\sin n + \cos n) - 1}{1+t^2},$$

and so, $\lim_{n\to\infty} f_n'(t) = -\frac{1}{1+t^2}$ holds for all $t > 0$. Also,

$$|f_n'(t)| \le \frac{1 + (1+t)e^{-t}}{1+t^2}$$

holds for each $t > 0$, and the dominating function for the sequence $\{f_n'\}$ is Lebesgue integrable over $[0, \infty)$.

Let $g_n = f_n'\chi_{[0,n]}$, and note that $\{g_n\}$ is a sequence of Lebesgue integrable functions over $[0, \infty)$. Also, since $|g_n| \le |f_n'|$ and $\lim_{n\to\infty} g_n(t) = -\frac{1}{1+t^2}$ for $t > 0$, the Lebesgue dominated convergence theorem yields

$$\lim_{n\to\infty}\int_0^n f_n'(t)\,dt = \lim_{n\to\infty}\int_0^\infty g_n(t)\,dt = -\int_0^\infty \frac{dt}{1+t^2} = -\frac{\pi}{2}.$$

Since $\int_0^n f_n'(t)\,dt = f_n(n) - f_n(0)$ and $\lim f_n(n) = 0$, we get $\lim f_n(0) = \frac{\pi}{2}$. Finally, letting $n \to \infty$ in the identity

$$\int_0^{n+1}\frac{\sin x}{x}\,dx = \int_0^n \frac{\sin x}{x}\,dx + \int_n^{n+1}\frac{\sin x}{x}\,dx = f_n(0) + \int_n^{n+1}\frac{\sin x}{x}\,dx,$$

we easily get $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$. ∎

EXERCISES

1. Show that

$$\int_0^\infty x^{2n} e^{-x^2}\,dx = \frac{(2n)!}{2^{2n} n!} \cdot \frac{\sqrt{\pi}}{2}$$

holds for $n = 0, 1, 2, \ldots$.

2. Show that $\int_0^\infty e^{-tx^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{t}}$ for each $t > 0$.
3. Show that $f(x) = \frac{\ln x}{x^2}$ is Lebesgue integrable over $[1, \infty)$ and that $\int f\,d\lambda = 1$.
4. Show that

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\,dx = 1.$$

5. Let $f\colon [0, \infty) \to (0, \infty)$ be a continuous, decreasing, and Lebesgue integrable function. Show that

$$\lim_{x\to\infty} \frac{1}{f(x)} \int_x^\infty f(s)\,ds = 0 \quad \text{if and only if} \quad \lim_{x\to\infty} \frac{f(x+t)}{f(x)} = 0 \quad \text{for each } t > 0.$$

6. Show that the improper Riemann integrals, $\int_0^\infty \cos(x^2)\,dx$ and $\int_0^\infty \sin(x^2)\,dx$ (which are known as the **Fresnel**[11] **integrals**) both exist. Also, show that $\cos(x^2)$ and $\sin(x^2)$ are not Lebesgue integrable over $[0, \infty)$.
7. Show that $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$.
8. Let $(X, \mathcal{S}, \mu)$ be a measure space, T a metric space, and $f\colon X \times T \to \mathbb{R}$ a function. Assume that $f(\cdot, t)$ is a measurable function for each $t \in T$ and $f(x, \cdot)$ is a continuous function for each $x \in X$. Assume also that there exists an integrable function g such that for each $t \in T$ we have $|f(x, t)| \le g(x)$ for almost all $x \in X$. Show that the function $F\colon T \to \mathbb{R}$, defined by

$$F(t) = \int_X f(x, t)\,d\mu(x),$$

is a continuous function.

9. Show that

$$\int_0^\infty \frac{e^{-x} - e^{-xt}}{x}\,dx = \ln t$$

holds for each $t > 0$.

10. For each $t > 0$, let $F(t) = \int_0^\infty \frac{e^{-xt}}{1+x^2}\,dx$.
    a. Show that the integral exists as an improper Riemann integral and as a Lebesgue integral.
    b. Show that F has a second-order derivative and that $F''(t) + F(t) = \frac{1}{t}$ holds for each $t > 0$.

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[11] Augustin Jean Fresnel (1788–1827) was a French physicist who worked extensively in the field of optics. Using the integrals that bear his name, he was the first to demonstrate the wave conception of light.

11. Show that the improper Riemann integral, $\int_0^{\frac{\pi}{2}} \ln(t\cos x)\,dx$, exists for each $t > 0$ and that it is also a Lebesgue integral. Also, show that

$$\int_0^{\frac{\pi}{2}} \ln(t\cos x)\,dx = \frac{\pi}{2}\ln\left(\frac{t}{2}\right)$$

holds for all $t > 0$.

12. Show that for each $t \geq 0$, the improper Riemann integral $\int_0^{\infty} \frac{\sin xt}{x(1+x^2)}\,dx$ exists as a Lebesgue integral and that

$$\int_0^{\infty} \frac{\sin xt}{x(1+x^2)}\,dx = \frac{\pi}{2}(1-e^{-t}).$$

13. The **Gamma function** for $t > 0$ is defined by an integral as follows:

$$\Gamma(t) = \int_0^{\infty} x^{t-1}e^{-x}\,dx.$$

a. Show that the integral

$$\int_0^{\infty} x^{t-1}e^{-x}\,dx = \lim_{\substack{r\to\infty \\ \epsilon\to 0+}} \int_{\epsilon}^{r} x^{t-1}e^{-x}\,dx$$

exists as an improper Riemann integral (and hence, as a Lebesgue integral).

b. Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

c. Show that $\Gamma(t+1) = t\Gamma(t)$ holds for all $t > 0$, and use this conclusion to establish $\Gamma(n+1) = n!$ for $n = 1, 2, \ldots$.

d. Show that Γ is differentiable at every $t > 0$ and that

$$\Gamma'(t) = \int_0^{\infty} x^{t-1}e^{-x}\ln x\,dx$$

holds.

e. Show that Γ has derivatives of all order and that

$$\Gamma^{(n)}(t) = \int_0^{\infty} x^{t-1}e^{-x}(\ln x)^n\,dx$$

holds for $n = 1, 2, \ldots$ and all $t > 0$.

14. Let $f : [0, 1] \to \mathbb{R}$ be a Lebesgue integrable function and define the real-valued function $F : [0, 1] \to \mathbb{R}$ by $F(t) = \int_0^1 f(x)\sin(xt)\,d\lambda(x)$.

a. Show that the integral defining F exists and that F is a uniformly continuous function.

b. Show that F has derivatives of all orders and that

$$F^{(2n)}(t) = (-1)^n \int_0^1 x^{2n} f(x)\sin(xt)\,d\lambda(x)$$

and

$$F^{(2n-1)}(t) = (-1)^n \int_0^1 x^{2n-1} f(x) \cos(xt)\, d\lambda(x)$$

for $n = 1, 2, \dots$ and each $t \in [0, 1]$.

c. Show that $F = 0$ (i.e., $F(t) = 0$ for all $t \in [0, 1]$) if and only if $f = 0$ a.e.

25. APPROXIMATING INTEGRABLE FUNCTIONS

The following type of approximation problem is commonly encountered in analysis.

- *Given an integrable function f, a family of $\mathcal{F}$ of integrable functions, and $\epsilon > 0$, determine whether there exists a function g in the collection $\mathcal{F}$ such that $\int |f - g|\, d\mu < \epsilon$.*

By the definition of an integrable function, the following result is immediate.

Theorem 25.1. *Let f be an integrable function and $\epsilon > 0$. Then there exists a step function ϕ such that $\int |f - \phi|\, d\mu < \epsilon$.*

Let us denote by L the collection of all step functions ϕ for which there exist sets $A_1, \dots, A_n \in \mathcal{S}$ all of finite measure, and real numbers $a_1, \dots, a_n$ (all depending on ϕ) such that $\phi = \sum_{i=1}^n a_i \chi_{A_i}$ holds. Then L is a function space; see Exercise 1 of Section 21.

Theorem 25.2. *Let f be an integrable function, and let $\epsilon > 0$. Then there exists a function $\phi \in L$ such that $\int |f - \phi|\, d\mu < \epsilon$.*

Proof. Let $\mathcal{F}$ denote the collection of all integrable functions f such that for each $\epsilon > 0$, there exists some function $\phi \in L$ such that $\int |f - \phi|\, d\mu < \epsilon$. It should be clear that $\mathcal{F}$ is a vector space such that $\chi_A \in \mathcal{F}$ holds true for each $A \in \mathcal{S}$. Now, a glance at Theorem 22.12 guarantees that $\mathcal{F}$ consists of all integrable functions. ■

The next result deals with the approximation of integrable functions by continuous functions. Recall that if X is a topological space and $f : X \to \mathbb{R}$ is a function, then the closure of the set $\{x \in X : f(x) \neq 0\}$ is called the **support** of f (denoted by $\operatorname{Supp} f$). If the support of f happens to be a compact set, then f is said to have **compact support**.

Theorem 25.3. *Let X be a Hausdorff locally compact topological space, and let μ be a regular Borel measure on X. Assume that f is an integrable function* *with respect to the measure space* $(X, \mathcal{B}, \mu)$. *Then given* $\epsilon > 0$, *there exists a continuous function* $g: X \to \mathbb{R}$ *with compact support such that* $\int |f - g|\, d\mu < \epsilon$.

Proof. By Theorem 25.2, we can assume without loss of generality that f is the characteristic function of some Borel set of finite measure. So, assume $f = \chi_A$ for some Borel set A with $\mu(A) < \infty$.

Since μ is a regular Borel measure, there exist a compact set K such that $K \subseteq A$ and $\mu(A) - \mu(K) < \epsilon$ (see Lemma 18.5) and an open set V such that $A \subseteq V$ and $\mu(V) - \mu(A) < \epsilon$. By Theorem 10.8, there exists a continuous function $g: X \to [0, 1]$ (and hence, g is Borel measurable) with compact support such that $g(x) = 1$ for each $x \in K$ and $\operatorname{Supp} g \subseteq V$. Clearly, g is integrable and $|\chi_A - g| \le \chi_V - \chi_K$ holds. Therefore,

$$\int |\chi_A - g|\, d\mu \le \int (\chi_V - \chi_K)\, d\mu = \mu(V) - \mu(K) < 2\epsilon,$$

and the proof of the theorem is finished. ■

The following theorem describes an important property of the Lebesgue integrable functions on $\mathbb{R}$. It is usually referred to as the **Riemann–Lebesgue Lemma.**

Theorem 25.4 (Riemann–Lebesgue). *If* $f: \mathbb{R} \to \mathbb{R}$ *is Lebesgue integrable, then*

$$\lim_{n\to\infty} \int f(x)\cos(nx)\, d\lambda(x) = \lim_{n\to\infty} \int f(x)\sin(nx)\, d\lambda(x) = 0.$$

Proof. Note first that the inequality $|f(x)\cos(nx)| \le |f(x)|$ for all x, combined with Theorem 22.6, shows that the function $f(x)\cos(nx)$ is a Lebesgue integrable function for each n. Also, by Theorem 25.2, in order to establish the theorem, it suffices to consider functions of the form $f = \chi_{[a,b)}$. Thus, let $f = \chi_{[a,b)}$. In this case, the Lebesgue integrals are Riemann integrals, and

$$\left|\int f(x)\cos(nx)\, d\lambda(x)\right| = \left|\int_a^b \cos(nx)\, dx\right| = \frac{1}{n}\,|\sin(nb) - \sin(na)| \le \frac{2}{n} \to 0$$

as $n \to \infty$. Similarly, $\lim \int f(x)\sin(nx)\, d\lambda(x) = 0$. ■

A sequence of integrable functions $\{f_n\}$ is said to **converge in the mean** to some function f if $\lim \int |f_n - f|\, d\mu = 0$ holds.

Theorem 25.5. *Let $\{f_n\}$ be a sequence of integrable functions. If f is an integrable function such that $\lim \int |f_n - f|\, d\mu = 0$, then there exists a subsequence $\{f_{k_n}\}$ of $\{f_n\}$ such that $f_{k_n} \to f$ a.e. holds.*

Proof. Let $\epsilon > 0$. For each n let $E_n = \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$, and note that each E_n is a measurable set of finite measure (see Theorem 22.5). Now, since $\epsilon\chi_{E_n} \leq |f_n - f|$ holds for each n, it follows that $\epsilon\mu^*(E_n) \leq \int |f_n - f|\, d\mu$ also holds for each n. Thus, $\lim \mu^*(E_n) = 0$, and so, $f_n \xrightarrow{\mu} f$. The conclusion now follows immediately from Theorem 19.4. ∎

A sequence of integrable functions that converges in the mean to some function need not converge pointwise to that function. An example of this situation is provided by the sequence $\{f_n\}$ of Example 19.6.

EXERCISES

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Show that

$$\lim_{t\to\infty} \int f(x)\cos(xt)\, d\lambda(x) = \lim_{t\to\infty} \int f(x)\sin(xt)\, d\lambda(x) = 0.$$

2. A function $f : \mathcal{O} \to \mathbb{R}$ (where $\mathcal{O}$ is a nonempty open subset of $\mathbb{R}^n$) is said to be a C^∞**-function** if f has continuous partial derivatives of all orders.
   a. Consider the function $\rho : \mathbb{R} \to \mathbb{R}$ defined by $\rho(x) = \exp[\frac{1}{x^2-1}]$ if $|x| < 1$ and $\rho(x) = 0$ if $|x| \geq 1$. Show that ρ is a C^∞-function such that $\operatorname{Supp}\rho = [-1, 1]$. (Induction and L'Hôpital's[12] rule are needed here.)
   b. For $\epsilon > 0$ and $a \in \mathbb{R}$ show that the function $f(x) = \rho(\frac{x-a}{\epsilon})$ is also a C^∞-function with $\operatorname{Supp} f = [a - \epsilon, a + \epsilon]$.

3. Let $[a, b]$ be an interval, $\epsilon > 0$ such that $a + \epsilon < b - \epsilon$, and ρ as in the previous exercise. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = \int_a^b \rho(\frac{t-x}{\epsilon})\, dt$ for all $x \in \mathbb{R}$. Then show that
   a. $\operatorname{Supp} h \subseteq [a - \epsilon, b + \epsilon]$,
   b. $h(x) = c$ (a constant function) for all $x \in [a + \epsilon, b - \epsilon]$,
   c. h is a C^∞-function and $h^{(n)}(x) = \int_a^b \frac{\partial^n}{\partial x^n}\rho(\frac{t-x}{\epsilon})\, dt$ holds for all $x \in \mathbb{R}$, and
   d. the C^∞-function $f = \frac{1}{c}h$ satisfies $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}$, $f(x) = 1$ for all $x \in [a + \epsilon, b - \epsilon]$, and $\int |\chi_{[a,b)} - f|\, d\lambda < 4\epsilon$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function with respect to the Lebesgue measure, and let $\epsilon > 0$. Show that there exists a C^∞-function g such that $\int |f - g|\, d\lambda < \epsilon$.
   [HINT: Use Theorem 25.2 and the preceding exercise.]

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[12]Guillaume-François-Antoine de L'Hôpital (1661–1704), a French mathematician. He is mainly remembered for the familiar rule of computing the limit of a fraction whose numerator and denominator tend simultaneously to zero (or to $\pm\infty$).

5. The purpose of this exercise is to establish the following general result: *If* $f:\mathbb{R}^n \to \mathbb{R}$ *is an integrable function (with respect to the Lebesgue measure) and* $\epsilon > 0$, *then there exists a* C^∞*-function* g *such that* $\int |f-g|\,d\lambda < \epsilon$.

   a. Let $a_i < b_i$ for $i = 1, \ldots, n$, and put $I = \prod_{i=1}^n (a_i, b_i)$. Choose $\epsilon > 0$ such that $a_i + \epsilon < b_i - \epsilon$ for each i. Use Exercise 3 to select for each i a C^∞-function $f_i:\mathbb{R} \to \mathbb{R}$ such that $0 \le f_i(x) \le 1$ for all x, $f_i(x) = 1$ if $x \in [a+\epsilon, b_i - \epsilon]$, and $\operatorname{Supp} f_i \subseteq [a_i - \epsilon, b_i + \epsilon]$. Now define $h:\mathbb{R}^n \to \mathbb{R}$ by $h(x_1, \ldots, x_n) = \prod_{i=1}^n f_i(x_i)$. Then show that h is a C^∞-function on $\mathbb{R}^n$ and that
$$\int |\chi_I - h|\,d\lambda \le 2\left[\prod_{i=1}^n (b_i - a_i + 2\epsilon) - \prod_{i=1}^n (b_i - a_i)\right].$$

   b. Let $f:\mathbb{R}^n \to \mathbb{R}$ be Lebesgue integrable, and let $\epsilon > 0$. Then use part (a) to show that there exists a C^∞-function g with compact support such that
$$\int |f-g|\,d\lambda < \epsilon.$$

6. Let μ be a regular Borel measure on $\mathbb{R}^n$, $f:\mathbb{R}^n \to \mathbb{R}$ a μ-integrable function, and $\epsilon > 0$. Show that there exists a C^∞-function $g:\mathbb{R}^n \to \mathbb{R}$ such that $\int |f-g|\,d\mu < \epsilon$.
7. Let $f:[a,b] \to \mathbb{R}$ be a Lebesgue integrable function, and let $\epsilon > 0$. Show that there exists a polynomial p such that $\int |f-p|\,d\lambda < \epsilon$, where the integral is considered, of course, over $[a,b]$.
   [HINT: Use the Stone–Weierstrass approximation theorem.]

26. PRODUCT MEASURES AND ITERATED INTEGRALS

Throughout this section, $(X, \mathcal{S}, \mu)$ and (Y, Σ, ν) will be two fixed measure spaces. The **product semiring** $\mathcal{S} \otimes \Sigma$ of subsets of $X \times Y$ is defined by
$$\mathcal{S} \otimes \Sigma = \{A \times B : A \in \mathcal{S} \text{ and } B \in \Sigma\}.$$

The members of $\mathcal{S} \otimes \Sigma$ are called **rectangles**.

The above collection $\mathcal{S} \otimes \Sigma$ is indeed a semiring of subsets of $X \times Y$. This follows immediately from the identities

1. $(A \times B) \cap (A_1 \times B_1) = (A \cap A_1) \times (B \cap B_1)$, and
2. $(A \times B) \setminus (A_1 \times B_1) = [(A \setminus A_1) \times B] \cup [(A \cap A_1) \times (B \setminus B_1)]$,

and the fact that $A \setminus A_1$ and $B \setminus B_1$ can be written as finite disjoint unions of members of $\mathcal{S}$ and Σ, respectively. The product semiring was discussed in some detail in Section 20.

Now, define the set function $\mu \times \nu : \mathcal{S} \otimes \Sigma \to [0, \infty]$ by $\mu \times \nu(A \times B) = \mu(A) \cdot \nu(B)$ for each $A \times B \in \mathcal{S} \otimes \Sigma$ (keep in mind that $0 \cdot \infty = 0$). This set function is a measure on the product semiring $\mathcal{S} \otimes \Sigma$, called the **product measure** of μ and ν. The details follow.

Theorem 26.1. *The set function $\mu \times \nu : \mathcal{S} \otimes \Sigma \to [0, \infty]$ defined by*

$$\mu \times \nu(A \times B) = \mu(A) \cdot \nu(B)$$

for each $A \times B \in \mathcal{S} \otimes \Sigma$ is a measure. (This measure can be thought of as the generalization of the familiar formula "Length $\times$ Width" for computing the area of a rectangle in elementary geometry!)

Proof. Clearly, $\mu \times \nu(\emptyset) = 0$. For the σ-additivity of $\mu \times \nu$, let $A \times B \in \mathcal{S} \otimes \Sigma$ and let $\{A_n \times B_n\}$ be a sequence of mutually disjoint sets of $\mathcal{S} \otimes \Sigma$ such that $A \times B = \bigcup_{n=1}^{\infty} A_n \times B_n$. It must be established that

$$\mu(A) \cdot \nu(B) = \sum_{n=1}^{\infty} \mu(A_n) \cdot \nu(B_n). \tag{$\star$}$$

Obviously, ($\star$) holds true if either A or B has measure zero. Thus, we can assume that $\mu(A) > 0$ and $\nu(B) > 0$.

Since $\chi_{A \times B} = \sum_{n=1}^{\infty} \chi_{A_n \times B_n}$, we see that

$$\chi_A(x) \cdot \chi_B(y) = \sum_{n=1}^{\infty} \chi_{A_n}(x) \cdot \chi_{B_n}(y)$$

holds true for all x and y. For each fixed $y \in B$ let $K = \{i \in \mathbb{N} : y \in B_i\}$ and note that $\chi_A(x) = \sum_{i \in K} \chi_{A_i}(x)$ for each $x \in X$. Since the collection $\{A_i : i \in K\}$ is pairwise disjoint (why?), the latter implies $\mu(A) = \sum_{i \in K} \mu(A_i)$. Therefore,

$$\mu(A) \cdot \chi_B(y) = \sum_{i \in K} \mu(A_i) \chi_{B_i}(y) = \sum_{n=1}^{\infty} \mu(A_n) \cdot \chi_{B_n}(y) \tag{$\star\star$}$$

holds for all $y \in Y$. Since a term with $\mu(A_n) = 0$ does not alter the sum in ($\star$) or ($\star\star$), we can assume that $\mu(A_n) > 0$ for all n.

Now, if both A and B have finite measures, then by using Theorem 22.9 and integrating ($\star\star$) term by term, we see that ($\star$) is valid. On the other hand, if either A or B has infinite measure, then $\sum_{n=1}^{\infty} \mu(A_n) \cdot \nu(B_n) = \infty$ must hold. Indeed, if the last sum is finite, then, by Theorem 22.9, $\mu(A)\chi_B(y)$ defines an integrable function, which is impossible. Thus, in this case ($\star$) holds with both sides infinite, and the proof is finished. ■

The next few results will unravel the basic properties of the product measure $\mu \times \nu$. As usual $(\mu \times \nu)^*$ denotes the outer measure generated by the measure space $(X \times Y, \mathcal{S} \otimes \Sigma, \mu \times \nu)$ on $X \times Y$.

Theorem 26.2. *If $A \subseteq X$ and $B \subseteq Y$ are measurable sets of finite measure, then*

$$(\mu \times \nu)^*(A \times B) = \mu^* \times \nu^*(A \times B) = \mu^*(A) \cdot \nu^*(B).$$

Proof. Clearly, $\mathcal{S} \otimes \Sigma \subseteq \Lambda_\mu \otimes \Lambda_\nu$ holds. Now, let $\{A_n \times B_n\}$ be a sequence of $\mathcal{S} \otimes \Sigma$ such that $A \times B \subseteq \bigcup_{n=1}^\infty A_n \times B_n$. Since, by Theorem 26.1, $\mu^* \times \nu^*$ is a measure on the semiring $\Lambda_\mu \otimes \Lambda_\nu$ and $A \times B \in \Lambda_\mu \otimes \Lambda_\nu$, it follows from Theorem 13.8 that

$$\mu^* \times \nu^*(A \times B) \le \sum_{n=1}^\infty \mu^* \times \nu^*(A_n \times B_n) = \sum_{n=1}^\infty \mu \times \nu(A_n \times B_n),$$

and so,

$$\mu^* \times \nu^*(A \times B) \le (\mu \times \nu)^*(A \times B).$$

On the other hand, if $\epsilon > 0$ is given, choose two sequences $\{A_n\} \subseteq \mathcal{S}$ and $\{B_n\} \subseteq \Sigma$, with $A \subseteq \bigcup_{n=1}^\infty A_n$, $B \subseteq \bigcup_{n=1}^\infty B_n$, such that $\sum_{n=1}^\infty \mu(A_n) < \mu^*(A) + \epsilon$ and $\sum_{n=1}^\infty \nu(B_n) < \nu^*(B) + \epsilon$. Then, $A \times B \subseteq \bigcup_{n=1}^\infty \bigcup_{m=1}^\infty A_n \times B_m$ holds, and so

$$\begin{aligned}(\mu \times \nu)^*(A \times B) &\le \sum_{n=1}^\infty \sum_{m=1}^\infty \mu \times \nu(A_n \times B_m) = \sum_{n=1}^\infty \sum_{m=1}^\infty \mu(A_n) \cdot \nu(B_m) \\ &= \left[\sum_{n=1}^\infty \mu(A_n)\right] \cdot \left[\sum_{m=1}^\infty \nu(B_m)\right] < [\mu^*(A) + \epsilon] \cdot [\nu^*(B) + \epsilon]\end{aligned}$$

for all $\epsilon > 0$. That is,

$$(\mu \times \nu)^*(A \times B) \le \mu^*(A) \cdot \nu^*(B) = \mu^* \times \nu^*(A \times B).$$

Therefore, $(\mu \times \nu)^*(A \times B) = \mu^* \times \nu^*(A \times B)$ holds, as required. ∎

It is expected that the members of $\Lambda_\mu \otimes \Lambda_\nu$ are $\mu \times \nu$-measurable subsets of $X \times Y$; that is, $\Lambda_\mu \otimes \Lambda_\nu \subseteq \Lambda_{\mu \times \nu}$ holds. The next theorem shows that this is actually the case.

Theorem 26.3. *If A is a μ-measurable subset of X and B a ν-measurable subset of Y, then $A \times B$ is a $\mu \times \nu$-measurable subset of $X \times Y$.*

Proof. Assume $A \in \Lambda_\mu$, $B \in \Lambda_\nu$, and fix $C \times D \in \mathcal{S} \otimes \Sigma$ with $\mu \times \nu(C \times D) = \mu(C) \cdot \nu(D) < \infty$. In order to establish the $\mu \times \nu$-measurability of $A \times B$, by

Theorem 15.2, it suffices to show that

$$(\mu \times \nu)^*((C \times D) \cap (A \times B)) + (\mu \times \nu)^*((C \times D) \cap (A \times B)^c) \leq \mu \times \nu(C \times D).$$

If $\mu \times \nu(C \times D) = 0$, then the previous inequality is obvious (both sides are zero). So, we can assume $\mu(C) < \infty$ and $\nu(D) < \infty$. Clearly,

$$\begin{aligned}(C \times D) \cap (A \times B) &= (C \cap A) \times (D \cap B), \text{ and}\\ (C \times D) \cap (A \times B)^c &= [(C \cap A^c) \times (D \cap B)] \cup [(C \cap A) \times (D \cap B^c)]\\ &\quad \cup [(C \cap A^c) \times (D \cap B^c)]\end{aligned}$$

hold with every member of the previous union having finite measure.

Now, the subadditivity of $(\mu \times \nu)^*$, combined with Theorem 26.2, yields

$$\begin{aligned}&(\mu \times \nu)^*((C \times D) \cap (A \times B)) + (\mu \times \nu)^*((C \times D) \cap (A \times B)^c)\\ &\leq \mu^*(C \cap A) \cdot \nu^*(D \cap B) + \mu^*(C \cap A^c) \cdot \nu^*(D \cap B)\\ &\qquad + \mu^*(C \cap A) \cdot \nu^*(D \cap B^c) + \mu^*(C \cap A^c) \cdot \nu^*(D \cap B^c)\\ &= [\mu^*(C \cap A) + \mu^*(C \cap A^c)] \cdot [\nu^*(D \cap B) + \nu^*(D \cap B^c)]\\ &= \mu(C) \cdot \nu(D) = \mu \times \nu(C \times D),\end{aligned}$$

as required. ■

In general, it is not true that the measure $\mu^* \times \nu^*$ is the only extension of $\mu \times \nu$ from $\mathcal{S} \otimes \Sigma$ to a measure on $\Lambda_\mu \otimes \Lambda_\nu$. However, if both $(X, \mathcal{S}, \mu)$ and (Y, Σ, ν) are σ-finite measure spaces, then $(X \times Y, \mathcal{S} \otimes \Sigma, \mu \times \nu)$ is likewise a σ-finite measure space, and therefore, by Theorem 15.10, $\mu^* \times \nu^*$ is the only extension of $\mu \times \nu$ to a measure on the semiring $\Lambda_\mu \otimes \Lambda_\nu$. Moreover, in view of $\Lambda_\mu \otimes \Lambda_\nu \subseteq \Lambda_{\mu \times \nu}$ and the fact that $(\mu \times \nu)^*$ is a measure on $\Lambda_{\mu \times \nu}$, it follows in this case that $(\mu \times \nu)^* = \mu^* \times \nu^*$ holds on $\Lambda_\mu \otimes \Lambda_\nu$.

We now turn our attention to measurability properties of arbitrary subsets of $X \times Y$. Let us recall a few definitions from Section 20. If A is a subset of $X \times Y$ and $x \in X$, then the **x-section** A_x of A is the subset of Y defined by

$$A_x = \{y \in Y : (x, y) \in A\}.$$

Similarly, if $y \in Y$, then the **y-section** A^y of A is the subset of X defined by

$$A^y = \{x \in X : (x, y) \in A\}.$$

The geometrical meanings of the x- and y-sections are shown in Figure 4.1. Regarding sections of sets, we have the following identities.

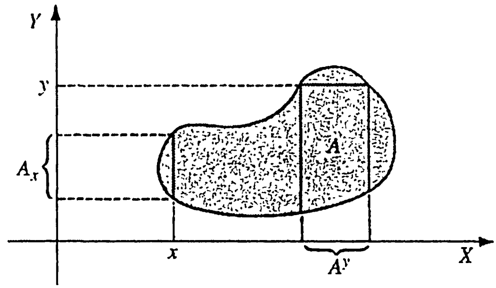

FIGURE 4.1.

a. $(\bigcup_{i\in I} A_i)_x = \bigcup_{i\in I}(A_i)_x$ and $(\bigcup_{i\in I} A_i)^y = \bigcup_{i\in I}(A_i)^y$;
b. $(\bigcap_{i\in I} A_i)_x = \bigcap_{i\in I}(A_i)_x$ and $(\bigcap_{i\in I} A_i)^y = \bigcap_{i\in I}(A_i)^y$;
c. $(A \setminus B)_x = A_x \setminus B_x$ and $(A \setminus B)^y = A^y \setminus B^y$.

The proofs of the previous identities are straightforward, and they are left as an exercise for the reader.

The next theorem demonstrates the relationship between the $\mu \times \nu$-measurable subsets of $X \times Y$ and the measurable subsets of X and Y, and it is a key result for this section.

Recall that an extended real-valued function f that is undefined on a set of measure zero is said to **define an integrable function** if there exists an integrable function g such that $f = g$ almost everywhere. That is, if arbitrary values are assigned to f on the points where it is undefined or attains an infinite value, then f becomes an integrable function. (The value of the integral does not depend, of course, upon the choices of these values.)

Theorem 26.4. *Let E be a $\mu \times \nu$-measurable subset of $X \times Y$ satisfying $(\mu \times \nu)^*(E) < \infty$. Then for μ-almost all x the set E_x is a ν-measurable subset of Y, and the function $x \mapsto \nu^*(E_x)$ defines an integrable function on X such that*

$$(\mu \times \nu)^*(E) = \int_X \nu^*(E_x)\, d\mu(x).$$

Similarly, for ν-almost all y, the set E^y is a μ-measurable subset of X, and the function $y \mapsto \mu^*(E^y)$ defines an integrable function on Y such that

$$(\mu \times \nu)^*(E) = \int_Y \mu^*(E^y)\, d\nu(y).$$

Proof. By the symmetry of the situation, it suffices to establish the first formula. The proof goes by steps.

Step I. Assume $E = A \times B \in \mathcal{S} \otimes \Sigma$.

Clearly, $E_x = B$ if $x \in A$ and $E_x = \emptyset$ if $x \notin A$. Thus, E_x is a ν-measurable subset of Y for each $x \in X$, and

$$\nu(E_x) = \nu(B)\chi_A(x) \qquad (\star)$$

holds for all $x \in X$. Since $(\mu \times \nu)^*(E) = (\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B) < \infty$, two possibilities arise:

a. *Both A and B have finite measure.*

In this case, $(\star)$ shows that $x \mapsto \nu^*(E_x)$ is an integrable function (actually, it is a step function) satisfying

$$\int_X \nu^*(E_x)\,d\mu(x) = \int \nu(B)\chi_A\,d\mu = \mu(A) \cdot \nu(B) = (\mu \times \nu)^*(E).$$

b. *Either A or B has infinite measure.*

In this case, the other set must have measure zero, and so, $(\star)$ guarantees $\nu(E_x) = 0$ for μ-almost all x. Thus, $x \mapsto \nu^*(E_x)$ defines the zero function, and hence

$$\int_X \nu^*(E_x)\,d\mu(x) = 0 = (\mu \times \nu)^*(E).$$

Step II. Assume that E is a σ-set of $\mathcal{S} \otimes \Sigma$.

Choose a disjoint sequence $\{E_n\}$ of $\mathcal{S} \otimes \Sigma$ such that $E = \bigcup_{n=1}^{\infty} E_n$. In view of $E_x = \bigcup_{n=1}^{\infty} (E_n)_x$ and the preceding step, it follows that E_x is a measurable subset of Y for each $x \in X$.

Now, define $f(x) = \nu^*(E_x)$ and $f_n(x) = \sum_{i=1}^{n} \nu((E_i)_x)$ for each $x \in X$ and all n. By Step I, each f_n defines an integrable function and

$$\int f_n\,d\mu = \sum_{i=1}^{n} \int_X \nu((E_i)_x)\,d\mu(x) = \sum_{i=1}^{n} \mu \times \nu(E_i) \uparrow (\mu \times \nu)^*(E) < \infty.$$

Since $\{(E_n)_x\}$ is a disjoint sequence of Σ, we have $\nu^*(E_x) = \sum_{n=1}^{\infty} \nu((E_n)_x)$, and so, $f_n(x) \uparrow f(x)$ holds for each $x \in X$. Thus, by Levi's theorem (Theorem 22.8), f defines an integrable function and

$$\int_X \nu^*(E_x)\,d\mu(x) = \int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu = \sum_{i=1}^{\infty} \mu \times \nu(E_i) = (\mu \times \nu)^*(E).$$

Step III. Assume that E is a countable intersection of σ-sets of finite measure.

Choose a sequence $\{E_n\}$ of σ-sets such that $E = \bigcap_{n=1}^{\infty} E_n$, $(\mu \times \nu)^*(E_1) < \infty$, and $E_{n+1} \subseteq E_n$ for all n.

For each n, let $g_n(x) = 0$ if $\nu^*((E_n)_x) = \infty$ and $g_n(x) = \nu^*((E_n)_x$ if $\nu^*((E_n)_x < \infty$. By Step II, each g_n is an integrable function over X such that $\int g_n \, d\mu = (\mu \times \nu)^*(E_n)$ holds. In view of $E_x = \bigcap_{n=1}^{\infty}(E_n)_x$, it follows that E_x is a ν-measurable set for each $x \in X$. Also, since $\nu^*((E_1)_x) < \infty$ holds for μ-almost all x, it follows from Theorem 15.4 that $g_n(x) = \nu^*((E_n)_x) \downarrow \nu^*(E_x)$ holds for μ-almost all x. Thus, $x \mapsto \nu^*(E_x)$ defines an integrable function and

$$\int_X \nu^*(E_x)\, d\mu(x) = \lim_{n\to\infty} \int g_n \, d\mu = \lim_{n\to\infty} (\mu \times \nu)^*(E_n) = (\mu \times \nu)^*(E),$$

where the last equality holds again by virtue of Theorem 15.4.

Step IV: Assume that E is a null set, i.e., $(\mu \times \nu)^*(E) = 0$.

Arguing as in the proof of Theorem 15.11, we see that there exists a measurable set G, which is a countable intersection of σ-sets of finite measure, such that $E \subseteq G$ and $(\mu \times \nu)^*(G) = 0$.

By Step III, $\int_X \nu^*(G_x)\, d\mu(x) = (\mu \times \nu)^*(G) = 0$, and so, by Theorem 22.7(1), $\nu^*(G_x) = 0$ holds for μ-almost all x. In view of $E_x \subseteq G_x$ for all x, we must have $\nu^*(E_x) = 0$ for μ-almost all x. Therefore, E_x is ν-measurable for μ-almost all x and $x \mapsto \nu^*(E_x)$ defines the zero function. Thus,

$$\int_X \nu^*(E_x)\, d\mu(x) = 0 = (\mu \times \nu)^*(E).$$

Step V. The general case.

Choose a $\mu \times \nu$-measurable set F that is a countable intersection of σ-sets all of finite measure such that $E \subseteq F$ and $(\mu \times \nu)^*(F) = (\mu \times \nu)^*(E)$. Let $G = F \setminus E$. Then G is a null set, and thus, by Step IV, $\nu^*(G_x) = 0$ holds for μ-almost all x. Therefore, E_x is ν-measurable and $\nu^*(E_x) = \nu^*(F_x)$ holds for μ-almost all x. By Step III, $x \mapsto \nu^*(F_x)$ defines an integrable function, and so, $x \mapsto \nu^*(E_x)$ defines an integrable function and

$$(\mu \times \nu)^*(E) = (\mu \times \nu)^*(F) = \int_X \nu^*(F_x)\, d\mu(x) = \int_X \nu^*(E_x)\, d\mu(x)$$

holds. The proof of the theorem is now complete. ■

Now, let $f : X \times Y \to \mathbb{R}$ be a function. Then for each fixed $x \in X$, the symbol f_x will denote the function $f_x : Y \to \mathbb{R}$ defined by $f_x(y) = f(x, y)$ for all $y \in Y$. Similarly, for each $y \in Y$ the notation f^y denotes the function $f^y : X \to \mathbb{R}$ defined by $f^y(x) = f(x, y)$ for all $x \in X$.

Definition 26.5. *Let $f : X \times Y \to \mathbb{R}$ be a function. Then the* **iterated integral** *$\int\int f\, d\mu d\nu$ is said to exist if*

1. *f^y is an integrable function over X for ν-almost all y, and*
2. *the function*

$$g(y) = \int f^y\, d\mu = \int_X f(x, y)\, d\mu(x)$$

defines an integrable function over Y.

The value of the iterated integral $\int\int f\, d\mu d\nu$ is computed by starting with the innermost integration and by continuing with the second as follows:

$$\int\int f\, d\mu d\nu = \int_Y \left[\int_X f(x, y)\, d\mu(x) \right] d\nu(y),$$

holding y fixed while the integration over X is performed.

The meaning of the iterated integral $\int\int f\, d\nu d\mu$ is analogous. That is,

$$\int\int f\, d\nu d\mu = \int_X \left[\int_Y f(x, y)\, d\nu(y) \right] d\mu(x).$$

If E is a $\mu \times \nu$-measurable subset of $X \times Y$ with $(\mu \times \nu)^*(E) < \infty$, then it is readily seen from Theorem 26.4 that both iterated integrals $\int\int \chi_E\, d\mu d\nu$ and $\int\int \chi_E\, d\nu d\mu$ exist, and that

$$\int\int \chi_E\, d\mu d\nu = \int\int \chi_E\, d\nu d\mu = \int \chi_E\, d(\mu \times \nu) = (\mu \times \nu)^*(E).$$

Since every $\mu \times \nu$-step function is a linear combination of characteristic functions of $\mu \times \nu$-measurable sets of finite measure, it follows from the prior observation that if ϕ is a $\mu \times \nu$-step function, then both iterated integrals $\int\int \phi\, d\mu d\nu$ and $\int\int \phi\, d\nu d\mu$ exist and, moreover,

$$\int\int \phi\, d\mu d\nu = \int\int \phi\, d\nu d\mu = \int \phi\, d(\mu \times \nu).$$

The previous identities regarding iterated integrals are special cases of a more general result known as Fubini's[13] theorem.

[13] Guido Fubini (1879–1943), an Italian mathematician. He made important contributions in analysis, geometry, and mathematical physics.

Theorem 26.6 (Fubini). *Let $f : X \times Y \to \mathbb{R}$ be a $\mu \times \nu$-integrable function. Then both iterated integrals exist and*

$$\int f\, d(\mu \times \nu) = \iint f\, d\mu d\nu = \iint f\, d\nu d\mu.$$

Proof. Without loss of generality we can assume that $f(x, y) \geq 0$ holds for all x and y. Choose a sequence $\{\phi_n\}$ of step functions such that $0 \leq \phi_n(x, y) \uparrow f(x, y)$ holds for all x and y. Thus,

$$\int_X \left[\int_Y \phi_n(x, y)\, d\nu(y)\right] d\mu(x) = \int \phi_n\, d(\mu \times \nu) \uparrow \int f\, d(\mu \times \nu) < \infty. \qquad (\star)$$

By Theorem 26.4, for each n the function

$$g_n(x) = \int (\phi_n)_x\, d\nu = \int_Y \phi_n(x, y)\, d\nu(y)$$

defines an integrable function over X; and clearly, $g_n(x) \uparrow$ holds for μ-almost all x. But then, by Levi's Theorem 22.8, there exists a μ-integrable function $g : X \to \mathbb{R}$ such that $g_n(x) \uparrow g(x)$ μ-a.e. holds. That is, there exists a μ-null subset A of X such that $\int (\phi_n)_x\, d\nu \uparrow g(x) < \infty$ holds for all $x \notin A$. Since $(\phi_n)_x \uparrow f_x$ holds for each x, it follows that f_x is ν-integrable for all $x \notin A$ and

$$g_n(x) = \int (\phi_n)_x\, d\nu = \int_Y \phi_n(x, y)\, d\nu(y) \uparrow \int_Y f_x\, d\nu$$

holds for all $x \notin A$.

Now, $(\star)$ combined with Theorem 21.6 implies that the function $x \mapsto \int_Y f_x\, d\nu$ defines an integrable function such that

$$\int f\, d(\mu \times \nu) = \int_X \left(\int_Y f_x\, d\nu\right) d\mu = \iint f\, d\nu d\mu.$$

Similarly, $\int f\, d(\mu \times \nu) = \iint f\, d\mu d\nu$, and the proof is complete. ■

The existence of the iterated integrals is by no means enough to ensure that the function is integrable over the product space. For instance, let $X = Y = [0, 1]$, $\mu = \nu = \lambda$ (the Lebesgue measure), and $f(x, y) = (x^2 - y^2)/(x^2 + y^2)^2$ if

$(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Then, it is easy to see that

$$\iint f \, d\mu d\nu = -\frac{\pi}{4} \quad \text{and} \quad \iint f \, d\nu d\mu = \frac{\pi}{4}.$$

Fubini's theorem shows, of course, that f is not integrable over $[0, 1] \times [0, 1]$.

However, there is a converse to Fubini's theorem according to which the existence of one of the iterated integrals is sufficient for the integrability of the function over the product space. The theorem is known as Tonelli's[14] theorem, and this result is frequently used in applications.

Theorem 26.7 (Tonelli). *Let $(X, \mathcal{S}, \mu)$ and (Y, Σ, ν) be two σ-finite measure spaces, and let $f : X \times Y \to \mathbb{R}$ be a $\mu \times \nu$-measurable function. If one of the iterated integrals $\iint |f| \, d\mu d\nu$ or $\iint |f| \, d\nu d\mu$ exists, then the function f is $\mu \times \nu$-integrable—and hence, the other iterated integral exists and*

$$\int f \, d(\mu \times \nu) = \iint f \, d\mu d\nu = \iint f \, d\nu d\mu.$$

Proof. We can assume without loss of generality that $f(x, y) \geq 0$ holds for all x and y. Since $(X, \mathcal{S}, \mu)$ and (Y, Σ, ν) are σ-finite measure spaces, it is easy to see that the product measure space is also a σ-finite measure space. Choose a sequence $\{A_n\}$ of $\mu \times \nu$-measurable sets such that $(\mu \times \nu)^*(A_n) < \infty$ for each n and $A_n \uparrow X \times Y$. By Theorem 17.7, there exists a sequence $\{\psi_n\}$ of $\mu \times \nu$-simple functions such that $0 \leq \psi_n(x, y) \uparrow f(x, y)$ holds for all x and y. Let $\phi_n = \psi_n \cdot \chi_{A_n}$ for each n. Then $\{\phi_n\}$ is a sequence of $\mu \times \nu$-step functions such that $0 \leq \phi_n(x, y) \uparrow f(x, y)$ holds for all x and y.

Now, assume that $\iint f \, d\mu d\nu$ exists. This means that for ν-almost all y, the integral $\int f(x, y) \, d\mu(x)$ exists and defines a ν-integrable function. From $\phi_n(x, y) \uparrow f(x, y)$ it follows that $\int \phi_n(x, y) \, d\mu(x) \uparrow \int f(x, y) \, d\mu(x)$ holds for ν-almost all y. But then, by applying the Lebesgue dominated convergence theorem, we get

$$\int \phi_n d(\mu \times \nu) = \int_Y \left[\int_X \phi_n(x, y) \, d\mu(x) \right] d\nu(y) \uparrow \iint f \, d\mu d\nu < \infty.$$

This shows that f is a $\mu \times \nu$-upper function and that $\int f \, d(\mu \times \nu) = \iint f \, d\mu d\nu$ holds. The rest of the proof now follows immediately from Fubini's theorem. ■

The Fubini and Tonelli theorems are usually referred to as "the method of computing a double integral by changing the order of integration."

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[14]Leonida Tonelli (1885–1946), an Italian mathematician. He contributed to measure theory, the theory of integration, and to calculus of variations.

In general, it is a difficult problem to determine whether or not a given function $f : X \times Y \to \mathbb{R}$ is $\mu \times \nu$-measurable. However, in a number of applications the $\mu \times \nu$-measurability of f can be established from topological considerations. For instance, if $X = Y = \mathbb{R}$ and $\mu = \nu =$ the Lebesgue measure, then it should be clear that the product measure $\mu \times \nu$ on $\mathbb{R}^2$ is precisely the Lebesgue measure on $\mathbb{R}^2$. Therefore, every continuous real-valued function on $\mathbb{R}^2$ is necessarily $\mu \times \nu$-measurable. For more about the joint measurability of functions, see also Section 20.

EXERCISES

1. Let $(X, \mathcal{S}, \mu)$ and (Y, Σ, ν) be two measure spaces, and let $A \times B \in \Lambda_\mu \otimes \Lambda_\nu$.
   a. Show that $\mu^*(A) \cdot \nu^*(B) \leq (\mu \times \nu)^*(A \times B)$.
   b. Show that if $\mu^*(A) \cdot \nu^*(B) \neq 0$, then $(\mu \times \nu)^*(A \times B) = \mu^*(A) \cdot \nu^*(B)$.
   c. Give an example for which $(\mu \times \nu)^*(A \times B) \neq \mu^*(A) \cdot \nu^*(B)$.
2. Let $(X, \mathcal{S}, \mu)$ and (Y, Σ, ν) be two σ-finite measure spaces. Then show that
$$(\mu \times \nu)^*(A \times B) = \mu^*(A) \cdot \nu^*(B)$$
holds for each $A \times B \in \Lambda_\mu \otimes \Lambda_\nu$.
3. Let $(X, \mathcal{S}, \mu)$ and (Y, Σ, ν) be two measure spaces. Assume that A and B are subsets of X and Y, respectively, such that $0 < \mu^*(A) < \infty$, and $0 < \nu^*(B) < \infty$. Then show that $A \times B$ is $\mu \times \nu$-measurable if and only if both A and B are measurable in their corresponding spaces. Is the above conclusion true if either A or B has measure zero?
4. Let $(X, \mathcal{S}, \mu)$ and (Y, Σ, ν) be two σ-finite measure spaces, and let $f : X \times Y \to \mathbb{R}$ be a $\mu \times \nu$-measurable function. Show that for μ-almost all x, the function f_x is a ν-measurable function. Similarly, show that for ν-almost all y, the function f^y is μ-measurable.
5. Show that if $f(x, y) = (x^2 - y^2)/(x^2 + y^2)^2$, with $f(0, 0) = 0$, then
$$\int_0^1 \left[\int_0^1 f(x, y)\,dx\right]dy = -\frac{\pi}{4} \quad \text{and} \quad \int_0^1 \left[\int_0^1 f(x, y)\,dy\right]dx = \frac{\pi}{4}.$$
6. Let $X = Y = \mathbb{N}$, $\mathcal{S} = \Sigma =$ the collection of all subsets of $\mathbb{N}$, and $\mu = \nu =$ the counting measure. Give an interpretation of Fubini's theorem in this case.
7. Establish the following result, known as **Cavalieri's[15] principle**. Let $(X, \mathcal{S}, \mu)$ and (Y, Σ, ν) be two measure spaces, and let E and F be two $\mu \times \nu$-measurable subsets of $X \times Y$ of finite measure. If $\nu^*(E_x) = \nu^*(F_x)$ holds for μ-almost all x, then
$$(\mu \times \nu)^*(E) = (\mu \times \nu)^*(F).$$
8. For this exercise, λ denotes the Lebesgue measure on $\mathbb{R}$. Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space, and let $f : X \to \mathbb{R}$ be a measurable function such that $f(x) \geq 0$

[15]Bonaventura Cavalieri (1589–1647), an Italian mathematician. He was a geometer who worked on problems regarding volumes of solids and wrote several monographs on this subject.

holds for all $x \in X$. Then show that

a. The set $A = \{(x, y) \in X \times \mathbb{R} : 0 \le y \le f(x)\}$, called the **ordinate set** of f, is a $\mu \times \lambda$-measurable subset of $X \times \mathbb{R}$.
b. The set $B = \{(x, y) \in X \times \mathbb{R} : 0 \le y < f(x)\}$ is a $\mu \times \lambda$-measurable subset of $X \times \mathbb{R}$ and $(\mu \times \lambda)^*(A) = (\mu \times \lambda)^*(B)$ holds.
c. The graph of f, i.e., the set $G = \{(x, f(x)) : x \in X\}$, is a $\mu \times \lambda$-measurable subset of $X \times \mathbb{R}$.
d. If f is μ-integrable, then $(\mu \times \lambda)^*(A) = \int f\, d\mu$ holds.
e. If f is μ-integrable, then $(\mu \times \lambda)^*(G) = 0$ holds.

9. Let $g : X \to \mathbb{R}$ be a μ-integrable function, and let $h : Y \to \mathbb{R}$ be a ν-integrable function. Define $f : X \times Y \to \mathbb{R}$ by $f(x, y) = g(x)h(y)$ for each x and y. Show that f is $\mu \times \nu$-integrable and that

$$\int f d(\mu \times \nu) = \left(\int_X g\, d\mu\right) \cdot \left(\int_Y h\, d\nu\right).$$

10. Use Tonelli's theorem to verify that

$$\int_\epsilon^r \frac{\sin x}{x}\, dx = \int_0^\infty \left(\int_\epsilon^r e^{-xy} \sin x\, dx\right) dy$$

holds for each $0 < \epsilon < r$. By letting $\epsilon \to 0^+$ and $r \to \infty$ (and justifying your steps) give another proof of the formula

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

11. Show that if $f(x, y) = ye^{-(1+x^2)y^2}$ for each x and y, then

$$\int_0^\infty \left[\int_0^\infty f(x, y)\, dx\right] dy = \int_0^\infty \left[\int_0^\infty f(x, y)\, dy\right] dx.$$

Use the previous equality to give an alternate proof of the formula

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

12. Show that

$$\int_0^\infty \left(\int_0^r e^{-xy^2} \sin x\, dx\right) dy = \int_0^r \left(\int_0^\infty e^{-xy^2} \sin x\, dy\right) dx$$

holds for all $r > 0$. By letting $r \to \infty$ show that

$$\int_0^\infty \frac{\sin x}{\sqrt{x}}\, dx = \frac{\sqrt{2\pi}}{2}.$$

In a similar manner show that $\int_0^\infty \frac{\cos x}{\sqrt{x}}\, dx = \sqrt{2\pi}/2$.

13. Using the conclusions of the preceding exercise (and an appropriate change of variable), show that the values of the **Fresnel integrals** (see Exercise 6 of Section 24) are

$$\int_0^\infty \sin(x^2)\, dx = \int_0^\infty \cos(x^2)\, dx = \frac{\sqrt{2\pi}}{4}.$$

14. Let $X = Y = [0, 1]$, $\mu =$ the Lebesgue measure on $[0, 1]$, and $\nu =$ the counting measure on $[0, 1]$. Consider the "diagonal" $\Delta = \{(x, x) : x \in X\}$ of $X \times Y$. Show that:

a. Δ is a $\mu \times \nu$-measurable subset of $X \times Y$, and hence, χ_Δ is a non-negative $\mu \times \nu$-measurable function.

b. Both iterated integrals $\int\int \chi_\Delta \, d\mu d\nu$ and $\int\int \chi_\Delta \, d\nu d\mu$ exist.

c. The function χ_Δ is not $\mu \times \nu$-integrable. Why doesn't this contradict Tonelli's theorem?

15. Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable. Then show that the functions $f(x + y)$ and $f(x - y)$ are both $\lambda \times \lambda$-measurable.
[HINT: Consider first $f = \chi_V$ for some open set V.]