Sum of Co-prime Numbers of an Integer

forthright48 on December 8, 2018

Problem

> **Given a number N, find the sum of all numbers less than or equal to N that are co-prime with N.**

In case you forgot, a number x is co-prime with N if $gcd(x, N) = 1$.

For example, if $N = 10$, then the following numbers are co-prime with it: $[1, 3, 7, 9]$. Therefore, sum of co-prime numbers will be $1 + 3 + 7 + 9 = 20$.

Solution

Let us define a function $f(n)$, which gives us sum of all numbers less than or equal to n that are co-prime to n. Then we can calculate the value of $f(n)$ with the following formula:

$$f(n) = \frac{\phi(n)}{2} n$$

where $\phi(n)$ is Euler Phi Function.

For example, for $n = 10$, then we get the sum of co-prime as:

$$\begin{aligned} f(10) &= \frac{\phi(10)}{2} \times 10 \\ &= \frac{4}{2} \times 10 \\ &= 20 \end{aligned}$$

Proof

Prerequisite

In order to understand this section, you have to be familiar with the following topic: Euler Totient or Phi Function

The proof is very simple. We divide the proof in two sections: when $n = 2$ and when $n > 2$.

When $n = 2$

When $n = 2$, we can see that the formula works by directly inserting the values:

$$\begin{aligned} f(2) &= \frac{\phi(2)}{2} \times 2 \\ &= 1 \times 1 \\ &= 1 \end{aligned}$$

Hence $f(2) = 1$, which is correct since the only integer less than 2 co-prime with 2 is 1. So the formula works for when $n = 2$.

When $n > 2$

In order to prove the formula for the rest of the integers, we need to establish the following two facts:

1. **When $n > 2$, $\phi(n)$ is always even**
   This is easy to establish. We just need to look at the formula for $\phi(n)$. We know that $\phi(n) = n \times \frac{p_1 - 1}{p_1} \times \frac{p_2 - 1}{p_2} \ldots \times \frac{p_k - 1}{p_k}$.

   Now, every p_i divides n as they are factor of n. So we can re-write the formula as:

   $$\phi(n) = \frac{n}{p_1 \times p_2 \cdots \times p_k} \times (p_1 - 1) \times (p_2 - 1) \times \cdots \times (p_k - 1)$$

   Since $n > 2$, if n is not a power of 2 then there must be an odd prime factor, p_j, in n and $p_j - 1$ is even. Hence, $\phi(n)$ is a multiple of $p_j - 1$ or an even number when n is not a power of 2.

   When n is a power of 2, $phi(n) = \frac{n}{2} \times (2 - 1) = \frac{n}{2}$. Since $n > 2$ and a power of 2, we will always get an even number.

   Hence $\phi(n)$ is even for all values of $n > 2$.

2. **If $gcd(x, n) = 1$, then $gcd(n - x, n) = 1$**

   This is a property of gcd function. This can be proved using contradiction.

   Suppose $gcd(x, n) = 1$, but $gcd(n - x, n) = d$, where $d > 1$. Then d divides $n - x$ and n. If d divides n and $n - x$, then it must also divide x. But that's impossible since if d can divide both n and x, then $gcd(x, n) = d$. But we started with the fact that $gcd(x, n) = 1$. Contradiction! Hence, $gcd(n - x, n) \neq d$. Instead, it must be $gcd(n - x, n) = 1$.

With the two facts above established, we can now continue with our proof.

- Let $S = r_1, r_2, r_3, \ldots, r_{\phi(n)}$, where r_i is a number which is co-prime to n.
- Since r_i is co-prime to n and belongs to S, $n - r_i$ is also co-prime to n and belongs to S.
- Hence, we can form a pair: $r_i, n - r_i$ such that sum of the pair $r_i, n - r_i$ equals to n.
- Since $\phi(n)$ is even, we can form $\phi(n)/2$ such pairs.
- Each pair gives us a sum of n.
- So, if we take sum of all pairs, we get $r_1 + r_2 + r_3 + \cdots + r_{\phi(n)} = \frac{\phi(n)}{2} n$. Proved 🙂

Conclusion

The proof is very cute. Hopefully, you found it easy to understand. Unfortunately, I did not find any related problems. If you happen to know any related problems, please let me know in the comments.

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