Laplace Transformation

Lecture-02

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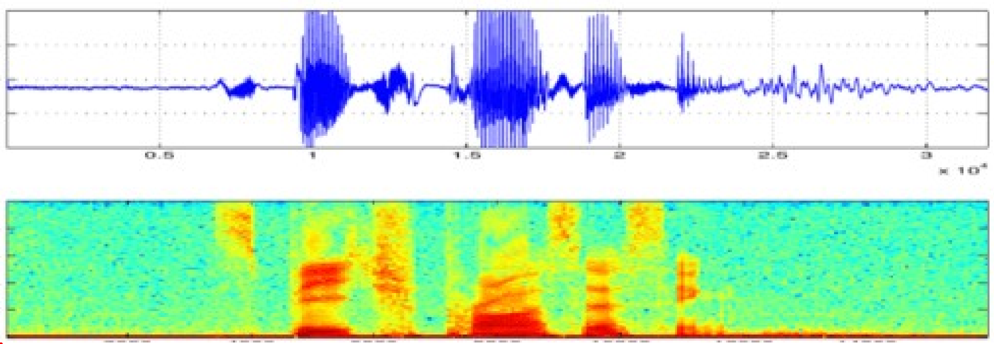

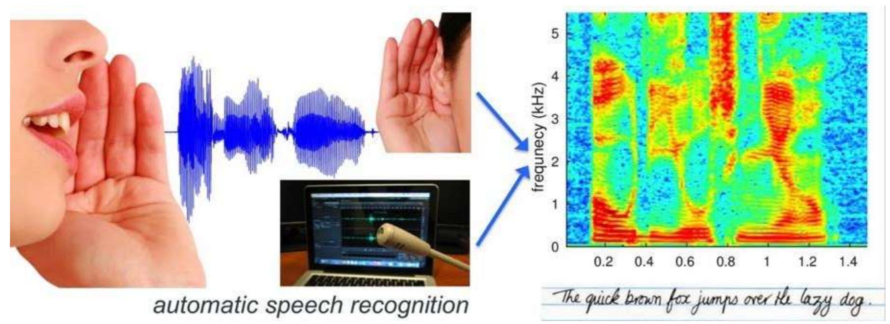

The Laplace Transform

The Laplace Transform of a function, f(t), is defined as;

$$L[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st}\,dt$$

The Inverse Laplace Transform is defined by

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{ts}\,ds$$

Time Integration:

The property is:

$$L\left[\int_0^\infty f(t)dt\right] = \int_0^\infty \left[\int_0^t f(x)dx\right] e^{-st} dt$$

Integrate by parts:

$$Let\ u = \int_0^t f(x)dx, \quad du = f(t)dt$$

and

$$dv = e^{-st} dt, \quad v = -\frac{1}{s} e^{-st}$$

Making these substitutions and carrying out the integration shows that

$$L\left[\int_0^{\infty} f(t)dt\right] = \frac{1}{s}\int_0^{\infty} f(t)e^{-st}dt$$

$$= \frac{1}{s}F(s)$$

Time Differentiation:

If the L[f(t)] = F(s), we want to show:

$$\boxed{L[\frac{df(t)}{dt}] = sF(s) - f(0)}$$

Integrate by parts:

$$u = e^{-st},\; du = -se^{-st}dt \;\; and$$

$$dv = \frac{df(t)}{dt}dt = df(t),\; so\;\; v = f(t)$$

Making the previous substitutions gives,

$$L\left[\frac{df}{dt}\right] = f(t)e^{-st}\Big|_0^{\infty} - \int_0^{\infty} f(t)\left[-se^{-st}\right]dt$$

$$= 0 - f(0) + s\int_0^{\infty} f(t)e^{-st}dt$$

$$\boxed{L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)}$$

We can extend the previous to show;

$$L\left[\frac{df(t)^2}{dt^2}\right] = s^2F(s) - sf(0) - f'(0)$$

$$L\left[\frac{df(t)^3}{dt^3}\right] = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

$$general \quad case$$

$$L\left[\frac{df(t)^n}{dt^n}\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0)$$

$$- \ldots - f^{(n-1)}(0)$$

Transform Pairs:

f(t)	F(s)
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$

f(t)	F(s)
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin(wt)$	$\frac{w}{s^2+w^2}$
$\cos(wt)$	$\frac{s}{s^2+w^2}$

f(t)	F(s)
$e^{-at}\sin(wt)$	$\dfrac{w}{(s+a)^2+w^2}$
$e^{-at}\cos(wt)$	$\dfrac{s+a}{(s+a)^2+w^2}$
$\sin(wt+\theta)$	$\dfrac{s\sin\theta+w\cos\theta}{s^2+w^2}$
$\cos(wt+\theta)$	$\dfrac{s\cos\theta-w\sin\theta}{s^2+w^2}$

Theorem: Initial value theorem

If the function f(t) and its first derivative are Laplace transformable and f(t) has the Laplace transform F(s), and the $\lim_{s \to \infty} sF(s)$ exists, then

$$\lim_{s \to \infty} sF(s) = \lim_{t \to 0} f(t) = f(0)$$

Initial Value Theorem

The utility of this theorem lies in not having to take the inverse of F(s) in order to find out the initial condition in the time domain. This is particularly useful in circuits and systems.

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

Example: Initial Value Theorem:

Given;

$$F(s)=\frac{(s+2)}{(s+1)^2+5^2}$$

Find f(0)

Solution:

$$f(0)=\lim_{s\to\infty} sF(s)=\lim_{s\to\infty} s\frac{(s+2)}{(s+1)^2+5^2}=\lim_{s\to\infty}\left[\frac{s^2+2s}{s^2+2s+1+25}\right]$$

$$=\lim_{s\to\infty}\frac{s^2/s^2+2s/s^2}{s^2/s^2+2s/s^2+(26/s^2)}=1$$

Final Value Theorem

If the function f(t) and its first derivative are Laplace transformable and f(t) has the Laplace transform F(s), and the $\lim_{s\to\infty} sF(s)$ exists, then

$$\boxed{\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t) = f(\infty)}$$

Final Value Theorem

Again, the utility of this theorem lies in not having to take the inverse of F(s) in order to find out the final value of f(t) in the time domain. This is particularly useful in circuits and systems.

Example: Final Value Theorem:

Given: $F(s) = \frac{(s+2)^2 - 3^2}{\left[(s+2)^2 + 3^2\right]}$ $\quad$ *note* $F^{-1}(s) = te^{-2t}\cos 3t$ $\quad$ Find $f(\infty)$.

Solution:

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} s\frac{(s+2)^2 - 3^2}{\left[(s+2)^2 + 3^2\right]} = 0$$

Nise: 2.2 Representation of Time domain ↔ Frequency domain

$$F(S) = \mathcal{L}[f(t)] = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt.$$

$$f(t) = \mathcal{L}^{-1}[F(S)] = \frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty} F(s)\, e^{st}\, ds \quad \text{[inverse laplace]}$$

$S = a + j\omega$

Some important Unilateral Laplace-Transform Pairs:

	Signal	Transform
1.	$u(t)$	$\frac{1}{S}$
2.	$\delta(t)$	$1.$
3.	$t^n u(t)$	$\frac{\lfloor n}{S^{n+1}}$
4.	$e^{-at} u(t)$	$\frac{1}{S+a}$
5.	$\sin at$ / $\sin\omega_0 t$	$\frac{a}{S^2+a^2}$ | $\frac{\omega_0}{S^2+\omega_0^2}$
6.	$\cos at$ / $\cos\omega_0 t$	$\frac{S}{S^2+a^2}$ | $\frac{S}{S^2+\omega_0^2}$

Some Property:—

1. Homogenous property:— $f_1(t) + f_2(t) \rightarrow F_1(S) + F_2(S)$

$a f_1(t) + b f_2(t) \rightarrow a F_1(S) + b F_2(S)$

2. ~~Time~~ Shifting property:—

— S domain :— $e^{-at} f(t) \rightarrow F(S+a)$; $e^{at} f(t) \rightarrow F(S-a)$

t

— t " :— $f(t-a) \rightarrow F(S)\, e^{-aS}$

$$\mathcal{L}^{-1}\{F(S+a)\} \longrightarrow e^{-at} f(t)$$

Explanation:-

$$\mathcal{L}[f(t-a)U(t-a)] = \int_0^{\alpha} f(t-a)\ \exp(-st)\ dt.$$

$$= \int_a^{\alpha} f(t-a)\ \exp(-st)\ dt.$$

Let, $t-a=\tau$

$$\therefore \mathcal{L}[f(t-a)\ u(t-a)] = \int_0^{\alpha} f(\tau)\ \exp(-sa-\tau s)\ d\tau$$

$$= e^{-sa}\int_0^{\alpha} f(\tau)\ e^{-\tau s}\ d\tau$$

$$= e^{-sa}\ F(s).$$

Explanation:-

$$\mathcal{L}[f(t)e^{at}u(t)] = \int_0^{\alpha} f(t)\ \exp\{-(s-a)t\}dt$$

3. Scaling property:

$$f(at) \leftrightarrow \frac{1}{a}\ F(s/a)$$

4. Differentiation Property:

$$\frac{df(t)}{dt} \leftrightarrow sF(s) - f(0)$$

5. ~~Inteo~~

$$\frac{d^n f(t)}{dt^n} \leftrightarrow s^n F(s) - s^{n-1} f(0) - \cdots$$

$$- s^{n-2} f'(0^-) - \cdots s^{n-n} f$$

$$\frac{d^n f(t)}{dt^n} \leftrightarrow s^n F(s) - \sum s^{n-k} f^{(k-1)}(0+)$$

$$\leftrightarrow s^n F(s) - s^{n-1} f(0^+) - s^{n-2} f'(0^+) - s^{n-3} f''(0^+) - \cdots$$

5. Integral Property:

$$\int_0^t f(\tau)\,d\tau \leftrightarrow \frac{F(s)}{s}$$

6. Initial value theorem:

$$f(0^+) = \lim_{s\to\infty} sF(s).$$

7. Final value theorem:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

☑ If $F(s) = \frac{N(s)}{D(s)}$

Condition : 1) Order of s in N(s) < Order of s in D(s)

2) The root of D(s) should be real & distinct

Partial Fraction:

- $\frac{s+1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$
- $\frac{s+1}{(s+2)^2(s+3)} = \frac{A}{s+3} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$
- $\frac{s+1}{(s+2)(s^2+3)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+3}.$

Let's Practice

Ex: 2.1: Find the Laplace transform of $f(t) = t\,e^{-5t}$

$$t \longleftrightarrow \frac{1}{s^2}$$

$$e^{-at} \longleftrightarrow \frac{1}{s+a}$$

$$t\,e^{-at} \longleftrightarrow \frac{1}{(s+a)^2}$$

$$t\,e^{-5t} \longleftrightarrow \frac{1}{(s+5)^2}$$

$$t\,e^{+5t} \longleftrightarrow \frac{1}{(s-5)^2}$$

$$t^2 e^{-5t} \longleftrightarrow \frac{\lfloor 2}{(s+5)^3}$$

Ex: 2.1: Find the inverse Laplace transform of $F(s) = \frac{10}{s(s+2)(s+3)^2}$

$$F(s) = \frac{10}{s(s+2)(s+3)^2}$$

$$\frac{10}{s(s+2)(s+3)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+3)} + \frac{D}{(s+3)^2}$$

$$10 = A(s+2)(s+3)^2 + Bs(s+3)^2 + C \cdots + \cdots$$

$$\frac{10}{s(s+2)(s+3)^2} = \frac{5}{9s} - \frac{5}{s+2} + \frac{40}{9}\cdot\frac{1}{s+3} + \frac{10}{3}\cdot\frac{1}{(s+3)^2}$$

Inverse L.T.

$$= \frac{5}{9}u(t) - 5e^{-2t} + \frac{40}{9}\cdot e^{-3t} + \frac{10}{3}\cdot e^{-3t}\,t$$

$$+\frac{10}{3}\cdot\frac{1}{(s+3)^3}$$

$$= \frac{\cancel{10}^{5}}{3}\cdot\frac{1}{\cancel{2}}\times\underbrace{\frac{\lfloor 2}{(s+3)^3}}$$

$$= \frac{5}{3}e^{-3t}t^{2}$$

$$+\frac{10}{3}\cdot\frac{1}{(s+3)^4}$$

$$= \frac{10}{3}\times\frac{1}{6}\times\frac{\lfloor 3}{(s+3)^4}$$

$$= \frac{5}{9}\times e^{-3t}t^{3}.$$

Sec : A (01 ~ 39, 68)

Absent : 6, 7, 14, 16, 18, 19, 22, 24, 25, 28, 30,
32, 33, 37.

Recor : Present : 1579, 1678, 2250, 1976, 2234
2255, 2115, 2252, 2127,
2175, 1819, 2247, 1940.

Sec : C (93 ~ 139)

Pres : 93, 94, 97, 99, 101, 3, 4, 14, 17, 19,
24, 31, 36,
1696, 1105, 1069, 1025, 1081
1701331020 1510.

Sec B : (44 ~ 92) 65–7

Abs : 45, 48, 52, 54, 58, 60, 61, 63,
76, 79, 81, 83, 85, 91,
Pres : 2292, 1135, 2314, 1880, 1815.

Let's Practice

Ex: 2.1: Find the Laplace transform of $f(t) = t e^{-5t}$

$$\mathcal{L}[\overset{f(t)}{t}] = \frac{1}{s^{1+1}} = \frac{1}{s^2} \quad \left[t^n u(t) \leftrightarrow \frac{\lfloor n}{s^{n+1}} \right.$$

Using Time shifting in s domain property: $\therefore t^n e^{-at} = \frac{1}{(s+a)^{n+1}}$

$$\mathcal{L}[t e^{-5t}] = \frac{1}{(s+5)^2}$$

$$e^{-at} f(t) \longleftrightarrow F(s+a).$$

Ex: 2.1: Find the inverse Laplace transform of $F(s) = \frac{10}{s(s+2)(s+3)^2}$

Let,

$$\frac{10}{s(s+2)(s+3)^2} = \frac{A}{s} + \frac{B}{(s+2)} + \frac{C}{(s+3)} + \frac{D}{(s+3)^2}$$

or, $10 = A(s+2)(s+3)^2 + B(s+3)^2 s + C \cdot s(s+2)(s+3) + D s (s+2)$

Putting, $s = 0$	$s = -2$	$s = -3$	
$10 = A \cdot 18$	$10 = B(-2)$	$10 = D(-3)(-1)$	$C = \frac{40}{9}$
$A = 5/9$	$\therefore B = -5$	$\therefore D = \frac{10}{3}$	

$$\therefore \quad \frac{10}{s(s+2)(s+3)^2} = \frac{5}{9s} - \frac{5}{s+2} + \frac{40}{9}\cdot\frac{1}{s+3} + \frac{10}{3}\cdot\frac{1}{(s+3)^2}$$

$$\therefore \mathcal{L}^{-1}\left[F(s)\right] = \frac{5}{9}u(t) - 5e^{-2t} + \frac{40}{9}e^{-3t} + \frac{10}{3}te^{-3}$$

Ans

Conclusion:

It converts differential equations into algebraic equations in 's' domain so that you can solve them in 's' domain and then take their inverse to obtain solution in time domain.

In electrical engineering dynamic analysis of circuits and systems in scalar or vector form uses Laplace transform and its application extensively.

Subjects like Control system, Network theory, System theory, Power system analysis and simulation etc will be impossible to follow without the use of Laplace transform.

Mathematical Problem

Class : Exercise 2.5 Nise.

Assignment

Exercise: 2.1, 2.2, 2.3, 2.4

Example: 2.4, 2.5,

Thank you!
Question ?