Mathematical Modeling-Mechanical

Lecture-04

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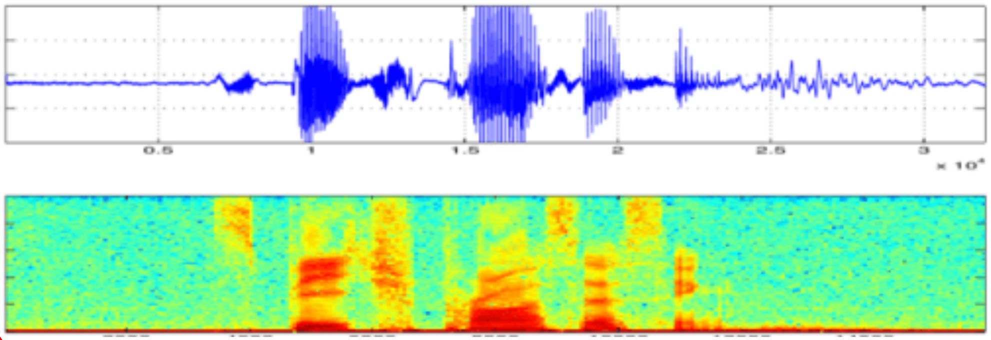
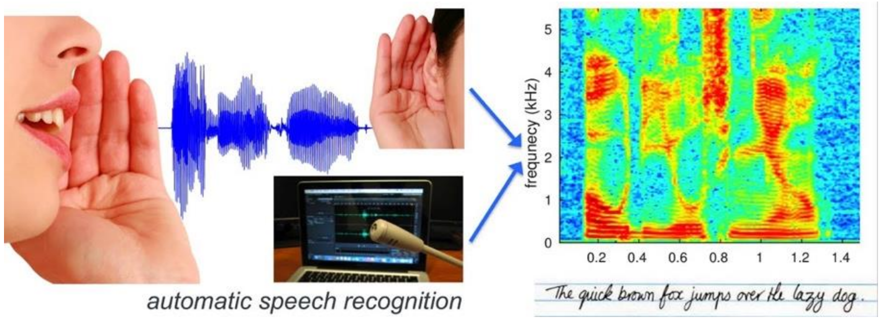




“

Some people
want it to happen,
some wish it would
happen, others
make it happen.

—Michael Jordan

Contents

Skill-Assessment Exercise 2.7

PROBLEM: If $Z_1(s)$ is the impedance of a 10 μF capacitor and $Z_2(s)$ is the impedance of a 100 kΩ resistor, find the transfer function, $G(s) = V_o(s)/V_i(s)$, if these components are used with (a) an inverting operational amplifier and (b) a noninverting amplifier as shown in Figures 2.10(c) and 2.12, respectively.

ANSWER: $G(s) = -s$ for an inverting operational amplifier; $G(s) = s + 1$ for a noninverting operational amplifier.

The complete solution is at www.wiley.com/college/nise.

In this section, we found transfer functions for multiple-loop and multiple-node electrical networks, as well as operational amplifier circuits. We developed mesh and nodal equations, noted their form, and wrote them by inspection. In the next section we begin our work with mechanical systems. We will see that many of the concepts applied to electrical networks can also be applied to mechanical systems via analogies—from basic concepts to writing the describing equations by inspection. This revelation will give you the confidence to move beyond this textbook and study systems not covered here, such as hydraulic or pneumatic systems.

2.5 Translational Mechanical System Transfer Functions

We have shown that electrical networks can be modeled by a transfer function, $G(s)$, that algebraically relates the Laplace transform of the output to the Laplace transform of the input. Now we will do the same for mechanical systems. In this section we concentrate on translational mechanical systems. In the next section we extend the concepts to rotational mechanical systems. Notice that the end product, shown in Figure 2.2, will be mathematically indistinguishable from an electrical network. Hence, an electrical network can be interfaced to a mechanical system by cascading their transfer functions, provided that one system is not loaded by the other.[6]

Three passive ele:-
K spring → capacit
Inducto
M Mass →
fv viscous damper
→ Resistor.

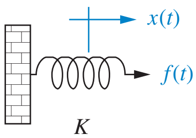
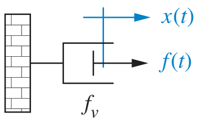
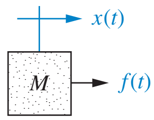
Mechanical systems parallel electrical networks to such an extent that there are analogies between electrical and mechanical components and variables. Mechanical systems, like electrical networks, have three passive, linear components. Two of them, the spring and the mass, are energy-storage elements; one of them, the viscous damper, dissipates energy. The two energy-storage elements are analogous to the two electrical energy-storage elements, the inductor and capacitor. The energy dissipator is analogous to electrical resistance. Let us take a look at these mechanical elements, which are shown in Table 2.4. In the table, K, f_v, and M are called *spring constant*, *coefficient of viscous friction*, and *mass*, respectively.

We now create analogies between electrical and mechanical systems by comparing Tables 2.3 and 2.4. Comparing the force-velocity column of Table 2.4 to the voltage-current column of Table 2.3, we see that mechanical force is analogous to electrical voltage and mechanical velocity is analogous to electrical current. Comparing the force-displacement column of Table 2.4 with the voltage-charge column of Table 2.3 leads to the analogy between the mechanical displacement and electrical charge. We also see that the spring is

f(t) force → voltage
v(t), velocity → current.
x(t) displacement → charge

[6] The concept of loading is explained further in Chapter 5.

TABLE 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring  K	$f(t) = K\int_0^t v(\tau)d\tau$ $v = \frac{1}{c}\int i\,dt$	$f(t) = Kx(t)$ $v(t) = \frac{1}{c}q(t)$	K
Viscous damper  f_v	$f(t) = f_v v(t)$ $v = Ri(t)$	$f(t) = f_v\frac{dx(t)}{dt}$ $v(t) = R\frac{dq(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$ $v = L\frac{di(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$ $v(t) = L\frac{d^2q(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

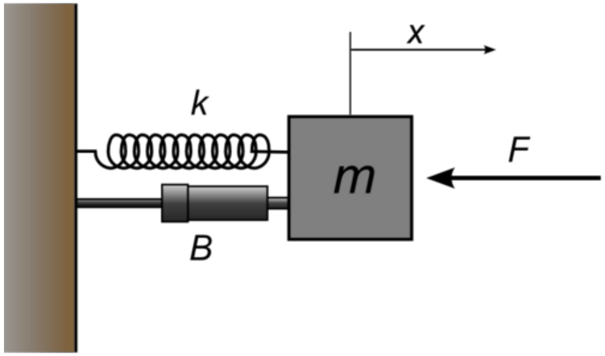
analogous to the capacitor, the viscous damper is analogous to the resistor, and the mass is analogous to the inductor. Thus, summing forces written in terms of velocity is analogous to summing voltages written in terms of current, and the resulting mechanical differential equations are analogous to mesh equations. If the forces are written in terms of displacement, the resulting mechanical equations resemble, but are not analogous to, the mesh equations. We, however, will use this model for mechanical systems so that we can write equations directly in terms of displacement.

Another analogy can be drawn by comparing the force-velocity column of Table 2.4 to the current-voltage column of Table 2.3 in reverse order. Here the analogy is between force and current and between velocity and voltage. Also, the spring is analogous to the inductor, the viscous damper is analogous to the resistor, and the mass is analogous to the capacitor. Thus, summing forces written in terms of velocity is analogous to summing currents written in terms of voltage and the resulting mechanical differential equations are analogous to nodal equations. We will discuss these analogies in more detail in Section 2.9.

We are now ready to find transfer functions for translational mechanical systems. Our first example, shown in Figure 2.15(a), is similar to the simple RLC network of Example 2.6 (see Figure 2.3). The mechanical system requires just one differential equation, called the *equation of motion*, to describe it. We will begin by assuming a positive direction of motion, for example, to the right. This assumed positive direction of motion is similar to assuming a current direction in an electrical loop. Using our assumed direction of positive motion, we first draw a free-body diagram, placing on the body all forces that act on the body either in the direction of motion or opposite to it. Next we use Newton's law to form a differential equation of motion by summing the forces and setting the sum equal to zero. Finally, assuming zero initial conditions, we take the Laplace transform of the differential equation, separate the variables, and arrive at the transfer function. An example follows.

Mechanical Network Transfer Functions

Translational: Linear Motion



Rotational: Rotational Motion

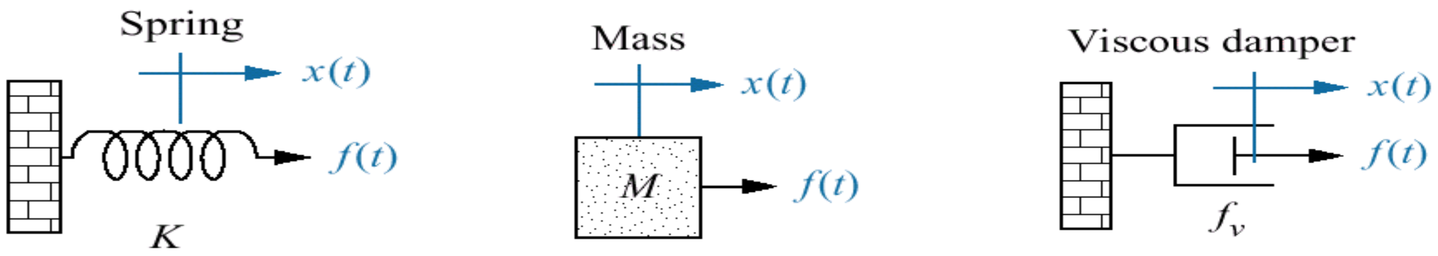
Translation

- The motion of translation is defined as a motion that takes place along or curved path.

- The variables that are used to describe translational motion are acceleration, velocity, and displacement.

Translational Mechanical System Transfer Function

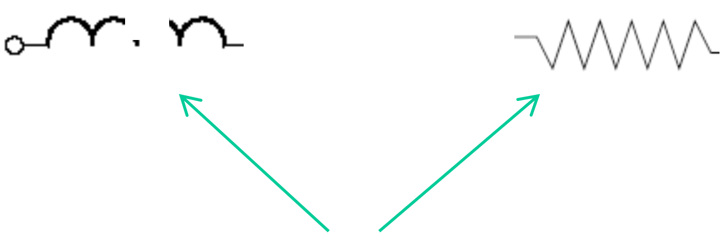
- **Electrical** we have **three passive elements:** Capacitor, Inductor, Resistor.
- **Mechanical:** Spring, Mass and Viscous damper.
- We are going to find the transfer function for a mechanical system in term of force-displacement (i.e. forces are written in terms of displacement)



Translational Spring

A translational spring is a mechanical element that can be deformed by an external force such that the deformation is directly proportional to the force applied to it.

Translational Spring



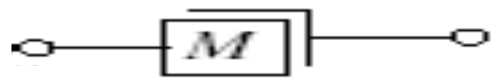
Circuit Symbols

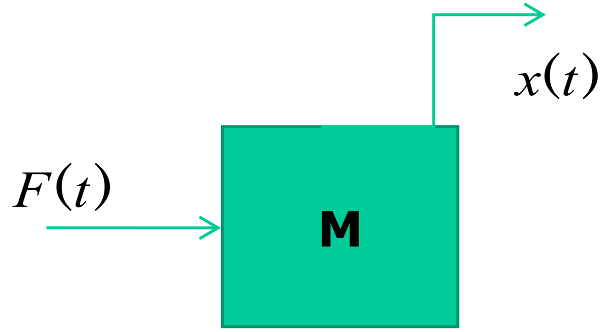
Translational Spring

Translational Mass

- Translational Mass is an inertia element.
- A mechanical system without mass does not exist.
- If a force F is applied to a mass and it is displaced to x meters then the relation b/w force and displacements is given by Newton's law.

ii) **Translational Mass**





$$F = M\ddot{x}$$

Translational Damper

- Damper opposes the rate of change of motion.
- All the materials exhibit the property of damping to some extent.
- If damping in the system is not enough then extra elements (e.g. Dashpot) are added to increase damping.

Translational Damper

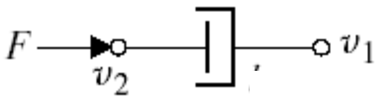


TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

[2] *Passive* means that there is no internal source of energy.

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring ($x(t)$, $f(t)$, K)	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K $K \times 1$
Viscous damper ($x(t)$, $f(t)$, f_v)	$f(t) = f_v v(t)$	$f(t) = f_v\frac{dx(t)}{dt}$	$f_v s$ $f_v \times s$
Mass ($x(t)$, M, $f(t)$)	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2 $s^2 \times M$

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

B section:

Pres:

1815, 1880, 2292

44, 46, 47, 49, 50-51, 53, 56, 7, 9.

62, 71-4, 82, 7-9, 90, 2,

C section

Pre.

1139, 0912, 1510, 1081.

93, 94, 7, 8, 101~4, 8, 10, 11, 14, 17~19, 21~23, 29, 30

31~33, 36, 38

A section

Pres.

1819, 2250, 2247, 2234, 2179, 2115, 2127

2271.

1, 2, 5, 9, 10, 11, 20, 21, 29, 27, 26, 31, 32, 38, equations are then solved for the output variable of interest in terms of the input variable from which the transfer function is evaluated. Example 2.17 demonstrates this problem-solving technique.

Example 2.17

Transfer Function—Two Degrees of Freedom

PROBLEM: Find the transfer function, $X_2(s)/F(s)$, for the system of Figure 2.17(*a*).

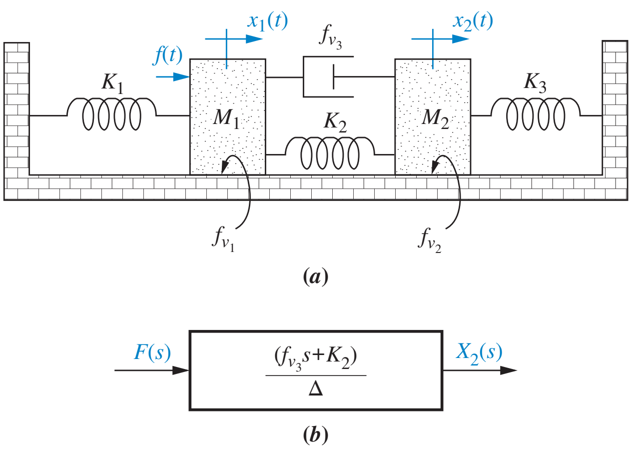


FIGURE 2.17 **a.** Two-degrees-of-freedom translational mechanical system;[8] **b.** block diagram

Virtual Experiment 2.1
Vehicle Suspension

Put theory into practice exploring the dynamics of another two-degrees-of-freedom system—a vehicle suspension system driving over a bumpy road and demonstrated with the Quanser Active Suspension System modeled in LabVIEW.

© Debra Lex

Virtual experiments are found on Learning Space.

SOLUTION: The system has two degrees of freedom, since each mass can be moved in the horizontal direction while the other is held still. Thus, two simultaneous equations of motion will be required to describe the system. The two equations come from free-body diagrams of each mass. Superposition is used to draw the free-body diagrams. For example, the forces on M_1 are due to (1) its own motion and (2) the motion of M_2 transmitted to M_1 through the system. We will consider these two sources separately.

If we hold M_2 still and move M_1 to the right, we see the forces shown in Figure 2.18(*a*). If we hold M_1 still and move M_2 to the right, we see the forces shown in Figure 2.18(*b*). The total force on M_1 is the superposition, or sum, of the forces just discussed. This result is shown in Figure 2.18(*c*). For M_2, we proceed in a similar fashion: First we move M_2 to the right while holding M_1 still; then we move M_1 to the right and hold M_2 still. For each case we evaluate the forces on M_2. The results appear in Figure 2.19.

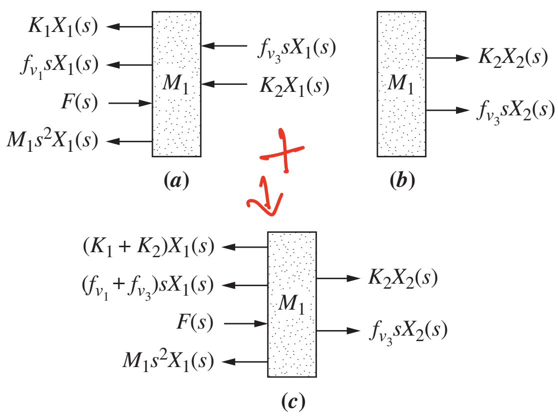


FIGURE 2.18 **a.** Forces on M_1 due only to motion of M_1; **b.** forces on M_1 due only to motion of M_2; **c.** all forces on M_1

[8] Friction shown here and throughout the book, unless otherwise indicated, is viscous friction. Thus, f_{v1} and f_{v2} are not Coulomb friction, but arise because of a viscous interface.

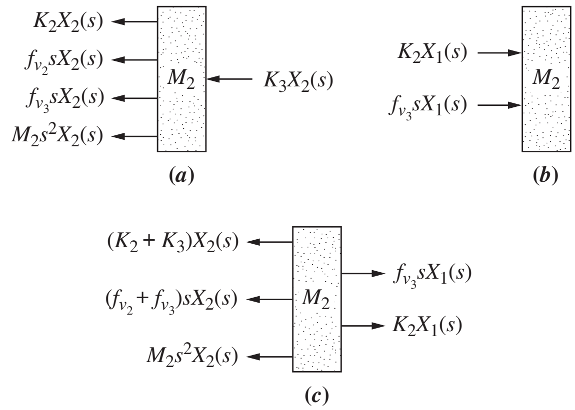


FIGURE 2.19 **a.** Forces on M_2 due only to motion of M_2; **b.** forces on M_2 due only to motion of M_1; **c.** all forces on M_2

The Laplace transform of the equations of motion can now be written from Figures 2.18(*c*) and 2.19(*c*) as

$$[M_1s^2(F_{v_1}+f_{v_3})s+(K_1+K_2)]X_1(s)-(f_{v_3}s+K_2)X_2(s)=F(s) \tag{2.118a}$$

$$-(f_{v_3}s+K_2)X_1(s)+[M_2s^2+(f_{v_2}+f_{v_3})s+(K_2+K_3)]X_2(s)=0 \tag{2.118b}$$

From this, the transfer function, $X_2(s)/F(s)$, is

$$\frac{X_2(s)}{F(s)}=G(s)=\frac{(f_{v_3}s+K_2)}{\Delta} \tag{2.119}$$

as shown in Figure 2.17(*b*) where

$$\Delta=\begin{vmatrix}[M_1s^2+(f_{v_1}+f_{v_3})s+(K_1+K_2)] & -(f_{v_3}s+K_2)\\ -(f_{v_3}s+K_2) & [M_2s^2+(f_{v_2}+f_{v_3})s+(K_2+K_3)]\end{vmatrix}$$

Notice again, in Eq. (2.118), that the form of the equations is similar to electrical mesh equations:

$$\left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } x_1\end{array}\right]X_1(s)-\left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_2\end{array}\right]X_2(s)=\left[\begin{array}{c}\text{Sum of}\\ \text{applied forces}\\ \text{at } x_1\end{array}\right] \tag{2.120a}$$

$$-\left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_2\end{array}\right]X_1(s)+\left[\begin{array}{c}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } x_2\end{array}\right]X_2(s)=\left[\begin{array}{c}\text{Sum of}\\ \text{applied forces}\\ \text{at } x_2\end{array}\right] \tag{2.120b}$$

The pattern shown in Eq. (2.120) should now be familiar to us. Let us use the concept to write the equations of motion of a three-degrees-of-freedom mechanical network by inspection, without drawing the free-body diagram.

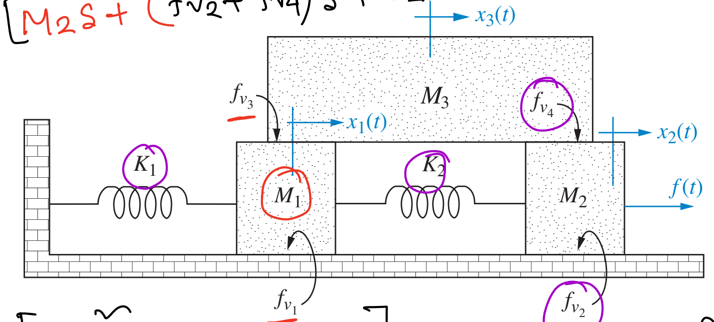
Example 2.18

Equations of Motion by Inspection

PROBLEM: Write, but do not solve, the equations of motion for the mechanical network of Figure 2.20.

$M_1 : [M_1 s^2 + (f_{v_1} + f_{v_3}) s + (K_1 + K_2)] x_1 - K_2 x_2 - s f_{v_3} x_3 = 0$

$M_2 : [M_2 s^2 + (f_{v_2} + f_{v_4}) s + K_2] x_2 - K_2 x_1 - s f_{v_4} x_3 = F(s).$



$M_3 : [M_3 s^2 + s(f_{v_3} + f_{v_4})] x_3 - s f_{v_4} x_2 - s f_{v_3} x_3 = 0$

FIGURE 2.20 Three-degrees-of-freedom translational mechanical system

SOLUTION: The system has three degrees of freedom, since each of the three masses can be moved independently while the others are held still. The form of the equations will be similar to electrical mesh equations. For M_1,

$$\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } x_1\end{bmatrix} X_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_2\end{bmatrix} X_2(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_3\end{bmatrix} X_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied forces}\\ \text{at } x_1\end{bmatrix} \tag{2.121}$$

Similarly, for M_2 and M_3, respectively,

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_2\end{bmatrix} X_1(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } x_2\end{bmatrix} X_2(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_2 \text{ and } x_3\end{bmatrix} X_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied forces}\\ \text{at } x_2\end{bmatrix} \tag{2.122}$$

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_3\end{bmatrix} X_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_2 \text{ and } x_3\end{bmatrix} X_2(s)$$

$$+\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } x_3\end{bmatrix} X_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied forces}\\ \text{at } x_3\end{bmatrix} \tag{2.123}$$

M_1 has two springs, two viscous dampers, and mass associated with its motion. There is one spring between M_1 and M_2 and one viscous damper between M_1 and M_3. Thus, using Eq. (2.121),

$$[M_1s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)]X_1(s) - K_2X_2(s) - f_{v_3}sX_3(s) = 0 \tag{2.124}$$

Similarly, using Eq. (2.122) for M_2,

$$-K_2X_1(s) + [M_2s^2 + (f_{v_2} + f_{v_4})s + K_2]X_2(s) - f_{v_4}sX_3(s) = F(s) \tag{2.125}$$

and using Eq. (2.123) for M_3,

$$-f_{v_3}sX_1(s) - f_{v_4}sX_2(s) + [M_3s^2 + (f_{v_3} + f_{v_4})s]X_3(s) = 0 \tag{2.126}$$

Equations (2.124) through (2.126) are the equations of motion. We can solve them for any displacement, $X_1(s)$, $X_2(s)$, or $X_3(s)$, or transfer function.

Skill-Assessment Exercise 2.8

M1

PROBLEM: Find the transfer function, $G(s)=X_2(s)/F(s)$, for the translational mechanical system shown in Figure 2.21.

$[M_1 s^2 + s(f_{v_1} + f_{v_2} + f_{v_3}) + K]x_1 - \{[s(f_{v_1} + f_{v_2} + f_{v_3}] + K\} x_2 = F(s).$

$(s^2 + 3s + 1)x_1 - (3s+1)x_2 = F(s)$

1

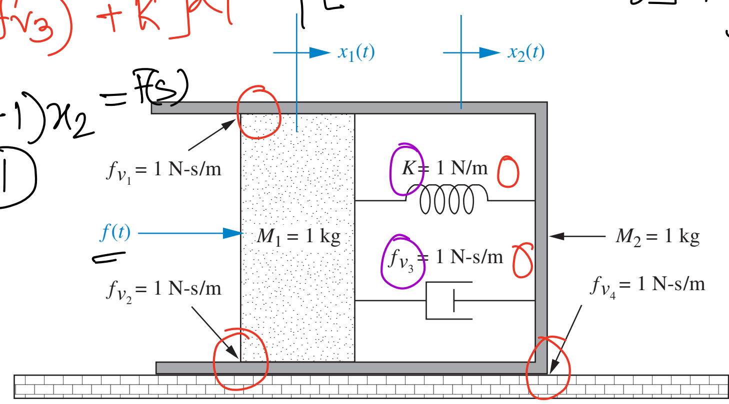


FIGURE 2.21 Translational mechanical system for Skill-Assessment Exercise 2.8

ANSWER: $G(s) = \dfrac{3s + 1}{s(s^3 + 7s^2 + 5s + 1)}$

The complete solution is at www.wiley.com/college/nise.

M2: $[M_2 s^2 + s(f_{v_1} + f_{v_2} + f_{v_3} + f_{v_4}) + K]x_1 - [(f_{v_1} + f_{v_2} + f_{v_3})s + K]x_2 = 0$

$$(s^2+4s+1)x_1 - (3s+1)x_2 = 0 \quad \text{—— (2)}$$

$$x_2 = \frac{\begin{vmatrix} s^2+3s+1 & 0 \\ s^2+4s+1 & F(s) \end{vmatrix}}{\begin{vmatrix} s^2+3s+1 & -(3s+1) \\ s^2+4s+1 & -(3s+1) \end{vmatrix}}$$

$$= \frac{F(s)\cdot(s^2+3s+1)}{}$$

Rotational Mechanical System

- The rotational motion can be defined as motion about a fixed axis.
- The extension of Newton's Law of motion for rotational motion states that the algebraic sum of moments or torque about a fixed axis is equal to the product of the inertia and the angular acceleration about the axis where,

J = Inertia

T = Torque

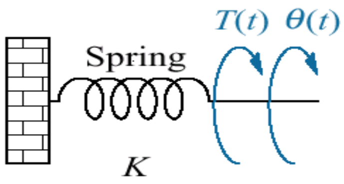
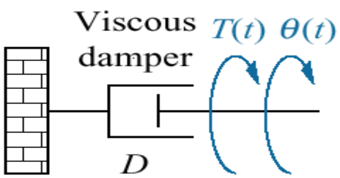
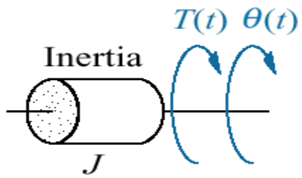
θ = Angular Displacement

ω = Angular Velocity

where Newton's second law for rotational system are,

$$\sum Torque(T) = J\alpha, \quad where: \alpha = \ angular \ \ acceleration$$

Modelling of Rotational Mechanical System

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)\,d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

2.6 Rotational Mechanical System Transfer Functions

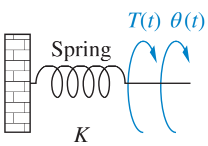
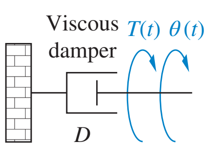
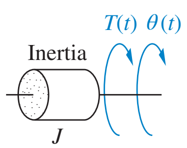
Having covered electrical and translational mechanical systems, we now move on to consider rotational mechanical systems. Rotational mechanical systems are handled the same way as translational mechanical systems, except that torque replaces force and angular displacement replaces translational displacement. The mechanical components for rotational systems are the same as those for translational systems, except that the components undergo rotation instead of translation. Table 2.5 shows the components along with the relationships between torque and angular velocity, as well as angular displacement. Notice that the symbols for the components look the same as translational symbols, but they are undergoing rotation and not translation.

Also notice that the term associated with the mass is replaced by inertia. The values of K, D, and J are called *spring constant*, *coefficient of viscous friction*, and *moment of inertia*, respectively. The impedances of the mechanical components are also summarized in the last column of Table 2.5. The values can be found by taking the Laplace transform, assuming zero initial conditions, of the torque-angular displacement column of Table 2.5.

The concept of degrees of freedom carries over to rotational systems, except that we test a point of motion by *rotating* it while holding still all other points of motion. The number of points of motion that can be rotated while all others are held still equals the number of equations of motion required to describe the system.

Writing the equations of motion for rotational systems is similar to writing them for translational systems; the only difference is that the free-body diagram consists of torques rather than forces. We obtain these torques using superposition. First, we rotate a body while holding all other points still and place on its free-body diagram all torques due to the body's own motion. Then, holding the body still, we rotate adjacent points of motion one at a time and add the torques due to the adjacent motion to the free-body diagram. The process is repeated for each point of motion. For each free-body diagram, these torques are summed and set equal to zero to form the equations of motion.

TABLE 2.5 Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\dfrac{d\theta(t)}{dt}$	Ds
	$T(t) = J\dfrac{d\omega(t)}{dt}$	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$	Js^2

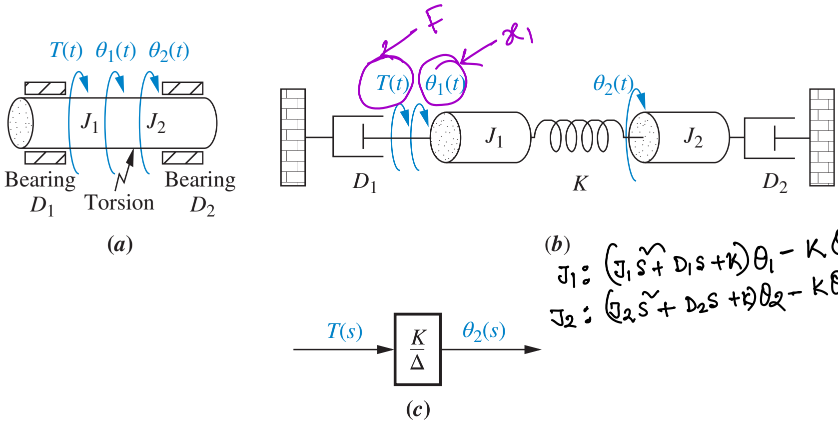
Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad (radians), $\omega(t)$ – rad/s (radians/second), K – N-m/rad (newton-meters/radian), D – N-m-s/rad (newton-meters-seconds/radian). J – kg-m^2(kilograms-meters^2 – newton-meters-seconds^2/radian).

Two examples will demonstrate the solution of rotational systems. The first one uses free-body diagrams; the second uses the concept of impedances to write the equations of motion by inspection.

Example 2.19

Transfer Function—Two Equations of Motion

PROBLEM: Find the transfer function, $\theta_2(s)/T(s)$, for the rotational system shown in Figure 2.22(a). The rod is supported by bearings at either end and is undergoing torsion. A torque is applied at the left, and the displacement is measured at the right.



J_1: $(J_1 s^2 + D_1 s + K)\theta_1 - K\theta_2 = T(s)$
J_2: $(J_2 s^2 + D_2 s + K)\theta_2 - K\theta_1 = 0$

FIGURE 2.22 **a.** Physical system; **b.** schematic; **c.** block diagram

SOLUTION: First, obtain the schematic from the physical system. Even though torsion occurs throughout the rod in Figure 2.22(a),[9] we approximate the system by assuming that the torsion acts like a spring concentrated at one particular point in the rod, with an inertia J_1 to the left and an inertia J_2 to the right.[10] We also assume that the damping inside the flexible shaft is negligible. The schematic is shown in Figure 2.22(b). There are two degrees of freedom, since each inertia can be rotated while the other is held still. Hence, it will take two simultaneous equations to solve the system.

Next, draw a free-body diagram of J_1, using superposition. Figure 2.23(a) shows the torques on J_1 if J_2 is held still and J_1 rotated. Figure 2.23(b) shows the torques on J_1 if J_1 is held still and J_2 rotated. Finally, the sum of Figures 2.23(a) and 2.23(b) is shown in Figure 2.23(c), the final free-body diagram for J_1. The same process is repeated in Figure 2.24 for J_2.

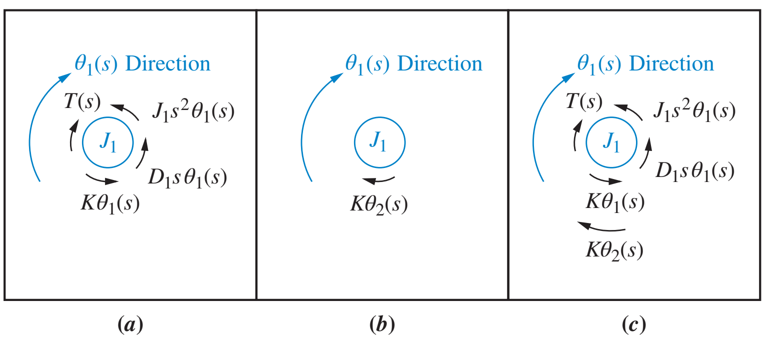


FIGURE 2.23 **a.** Torques on J_1 due only to the motion of J_1; **b.** torques on J_1 due only to the motion of J_2; **c.** final free-body diagram for J_1

[9] In this case the parameter is referred to as a *distributed* parameter.
[10] The parameter is now referred to as a *lumped* parameter.

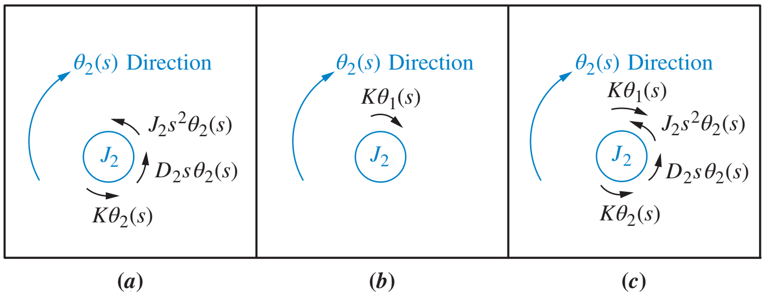


FIGURE 2.24 **a.** Torques on J_2 due only to the motion of J_2; **b.** torques on J_2 due only to the motion of J_1; **c.** final free-body diagram for J_2

Summing torques, respectively, from Figures 2.23(*c*) and 2.24(*c*) we obtain the equations of motion,

$$(J_1s^2 + D_1s + K)\theta_1(s) \qquad\qquad - K\theta_2(s) = T(s) \tag{2.127a}$$

$$-K\theta_1(s) + (J_2s^2 + D_2s + K)\theta_2(s) = 0 \tag{2.127b}$$

from which the required transfer function is found to be

$$\frac{\theta_2(s)}{T(s)} = \frac{K}{\Delta} \tag{2.128}$$

as shown in Figure 2.22(*c*), where

$$\Delta = \begin{vmatrix} (J_1s^2 + D_1s + K) & -K \\ -K & (J_2s^2 + D_2s + K) \end{vmatrix}$$

Notice that Eqs. (2.127) have that now well-known form

$$\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_1\end{bmatrix}\theta_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_2\end{bmatrix}\theta_2(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_1\end{bmatrix} \tag{2.129a}$$

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_2\end{bmatrix}\theta_1(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_2\end{bmatrix}\theta_2(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_2\end{bmatrix} \tag{2.129b}$$

TryIt 2.9

Use the following MATLAB and Symbolic Math Toolbox statements to help you get Eq. (2.128).

```
syms s J1 D1 K T J2 D2...
  theta1 theta2
A=[(J1*s^2+D1*s+K) -K
   -K (J2*s^2+D2*s +K)];
B=[theta1
   theta2];
C=[T
   0];
B=inv(A)*C;
theta2=B(2);
'theta2'
pretty(theta2)
```

Example 2.20

Equations of Motion by Inspection

PROBLEM: Write, but do not solve, the Laplace transform of the equations of motion for the system shown in Figure 2.25.

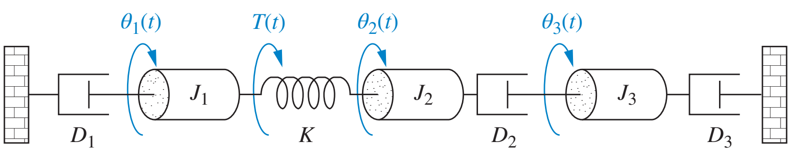
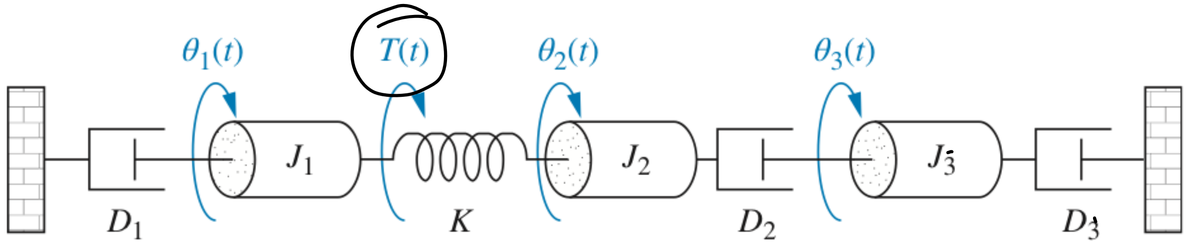


FIGURE 2.25 Three-degrees-of-freedom rotational system



J_1:
$$(J_1 s^2 + D_1 s + K)\theta_1 - K\theta_2 = T(s)$$

J_2:
$$(J_2 s^2 + D_2 s + K)\theta_2 - K\theta_1 - D_2 s\,\theta_3 = 0$$

J_3:
$$[J_3 s^2 + (D_2 + D_3)s]\theta_3 - D_2 s\,\theta_2 = 0$$

SOLUTION: The equations will take on the following form, similar to electrical mesh equations:

$$\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_1\end{bmatrix}\theta_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_2\end{bmatrix}\theta_2(s)$$
$$- \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_3\end{bmatrix}\theta_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_1\end{bmatrix} \tag{2.130a}$$

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_2\end{bmatrix}\theta_1(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_2\end{bmatrix}\theta_2(s)$$
$$- \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_2 \text{ and } \theta_3\end{bmatrix}\theta_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_2\end{bmatrix} \tag{2.130b}$$

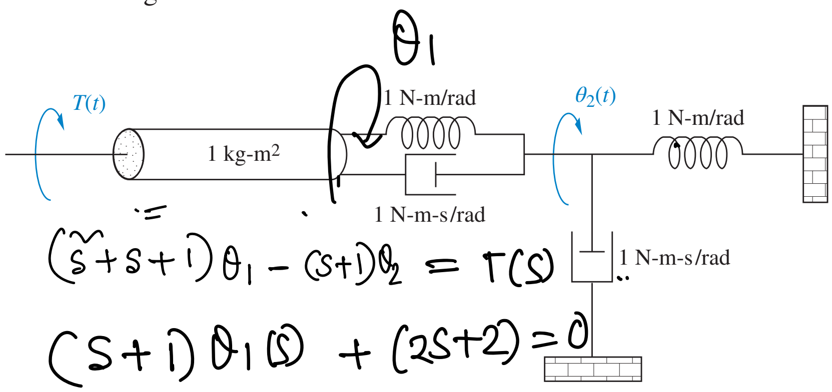
$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_3\end{bmatrix}\theta_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_2 \text{ and } \theta_3\end{bmatrix}\theta_2(s)$$
$$+ \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_3\end{bmatrix}\theta_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_3\end{bmatrix} \tag{2.130c}$$

Hence,

$$\begin{aligned}
(J_1s^2 + D_1s + K)\theta_1(s) \quad & -K\theta_2(s) & -0\theta_3(s) &= T(s)\\
-K\theta_1(s) \quad & +(J_2s^2 + D_2s + K)\theta_2(s) & -D_2s\theta_3(s) &= 0\\
-0\theta_1(s) \quad & -D_2s\theta_2(s) & +(J_3s^2 + D_3s + D_2s)\theta_3(s) &= 0
\end{aligned} \tag{2.131a,b,c}$$

Skill-Assessment Exercise 2.9

PROBLEM: Find the transfer function, $G(s) = \theta_2(s)/T(s)$, for the rotational mechanical system shown in Figure 2.26.



J_1: $(s^2+s+1)\theta_1 - (s+1)\theta_2 = T(s)$

J_2: $(s+1)\theta_1(s) + (2s+2) = 0$

FIGURE 2.26 Rotational mechanical system for Skill-Assessment Exercise 2.9

ANSWER: $G(s) = \dfrac{1}{2s^2 + s + 1}$

The complete solution is at www.wiley.com/college/nise.

2.7 Transfer Functions for Systems with Gears

Now that we are able to find the transfer function for rotational systems, we realize that these systems, especially those driven by motors, are rarely seen without associated gear trains driving the load. This section covers this important topic.

Gears provide mechanical advantage to rotational systems. Anyone who has ridden a 10-speed bicycle knows the effect of gearing. Going uphill, you shift to provide more torque and less speed. On the straightaway, you shift to obtain more speed and less torque. Thus, gears allow you to match the drive system and the load—a trade-off between speed and torque.

For many applications, gears exhibit *backlash*, which occurs because of the loose fit between two meshed gears. The drive gear rotates through a small angle before making contact with the meshed gear. The result is that the angular rotation of the output gear does not occur until a small angular rotation of the input gear has occurred. In this section, we idealize the behavior of gears and assume that there is no backlash.

The linearized interaction between two gears is depicted in Figure 2.27. An input gear with radius r_1 and N_1 teeth is rotated through angle $\theta_1(t)$ due to a torque, $T_1(t)$. An output gear with radius r_2 and N_2 teeth responds by rotating through angle $\theta_2(t)$ and delivering a torque, $T_2(t)$. Let us now find the relationship between the rotation of Gear 1, $\theta_1(t)$, and Gear 2, $\theta_2(t)$.

From Figure 2.27, as the gears turn, the distance traveled along each gear's circumference is the same. Thus,

$$r_1\theta_1 = r_2\theta_2 \tag{2.132}$$

or

$$\boxed{\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}} \tag{2.133}$$

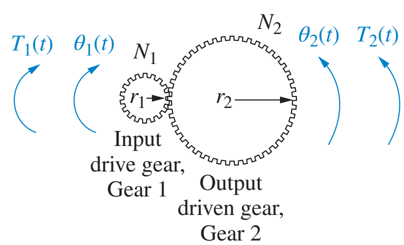


FIGURE 2.27 A gear system

Related Problems (Ref. Nise)

Exam: 2.17, 2.18,2.19, 2.20

Exercise: 2.9

Application:

In automobile suspension

An electrical system be developed for mechanical system

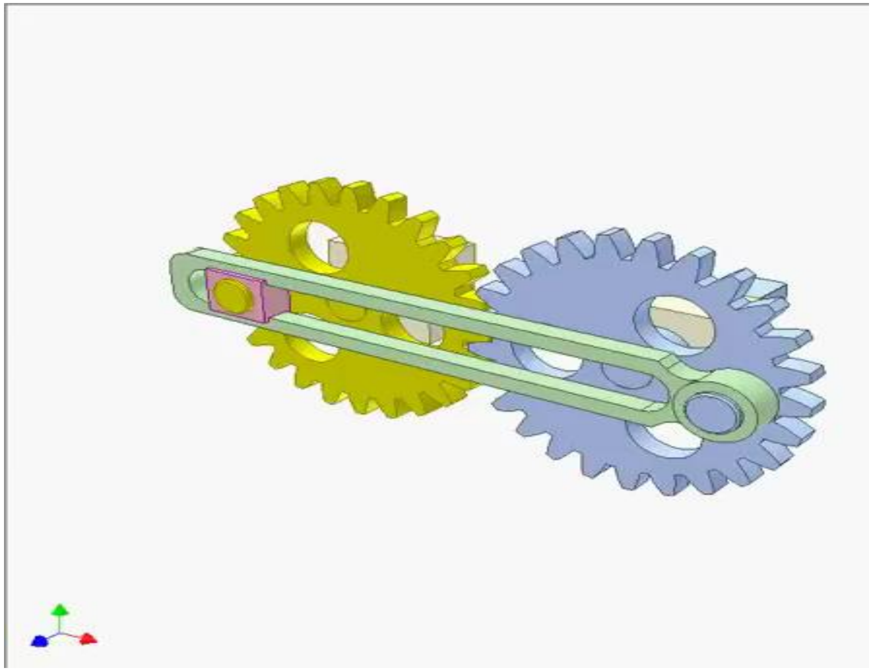
Electric motor

Electrical energy required to rotate the motor

Electric Generator

Mechanical energy is required to generates electricity

Application:



Conclusion:

From this discussion topic we can conclude that , an electrical network can be interfaced to a mechanical system by cascading their transfer functions, provided that one system is not loaded by the other.

Thank you!
Question ?