Root-Locus Techniques
chp ≈ 08 (Nise).

→ It is ploting system's dynamic characteristics.

Exp: If system with unity feedback is given by $G(S) = \dfrac{K}{S(S+2)}$

char. eq^n of above system

$$1+G(S)H(S)=0$$

$$\text{or} \quad 1+\frac{K}{S(S+2)}=0$$

$$S^2+2S+K=0$$

$$\frac{C(S)}{R(S)} = \frac{N(S)}{D(S)}$$

If $K=0$

$S^2+2S=0$

$\therefore S=0,\ S=-2$

If $K=1$

$S=-1,-1$

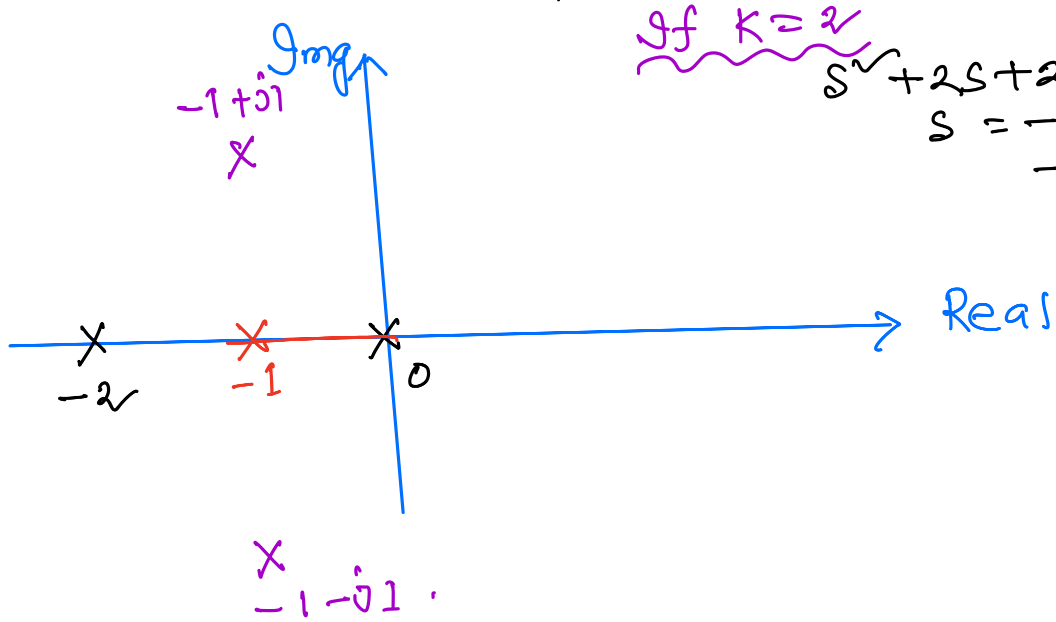
$S^2+2S+1=0$

$(S+1)^2=0$

If $K=2$

$S^2+2S+2=0$

$S=-1+j1,$
$-1-j1$



Root locus: It is the representation of the paths of the closed-loop poles as the gain is varied.

Asymptotes:

In the context of **root locus** in control theory, **asymptotes** refer to the straight lines along which the poles of the system tend to move as a system's gain parameter K approaches infinity. These asymptotes describe the direction of the roots of the characteristic equation at extreme values of K, providing insight into how the system behavior changes as the gain increases.

Key Points About Asymptotes in Root Locus:

1. **Definition**: Asymptotes are lines that represent the direction that the poles (roots of the characteristic equation) move towards as the gain K goes to infinity. The poles on the root locus may diverge to infinity, and these diverging directions are along the asymptotes.

Centriod of Asymptotes: is the point on the real axis where the asymptotes of the root locus intersect the real axis.

Loci: A locus is the collection of points that share a common geometric property.

Rules of Root locus plot:

Rule-1: The root locus is always symmetric w.r.t real axis.

$P = 0, -2$; $Z = 0$

Rule-2: Total Loci $= \max(P, Z)$; $P = 2$, $Z = 0$; $= 2$

Rule-3: Total no. of Asymptotes $= P - Z = 2 - 0 = 2$

Rule-4: Angle of Asymptotes $\theta = \frac{2x+1}{P-Z} \times 180^\circ$; $x = 0, 1, 2 \ldots$

$x = 0$, $\theta = \frac{1}{2} \times 180^\circ = 90^\circ$

$x = 1$, $\theta = 270^\circ$

Rule-5: Centroid of Asymptotes.

$P \begin{cases} 1 + j2 \\ 1 - j3 \end{cases}$, $\begin{cases} 3 - j1 \\ 3 - j1 \end{cases}$

$$= \frac{\Sigma \text{Real } P - \Sigma \text{Real } Z}{\# \text{ of } P - Z} = \frac{(-2+0) - 0}{2 - 0} = -1$$

Rule-6: Break away point

→ char. eqn $(1 + G(S)H(S) = 0)$

→ compute K = polynomial func.

→ " $\frac{dK}{dS} = 0$ and find S = break away point

Rule 7: Angle of departure

Rule 8: Intersection to Img. axis.

- → char. eqⁿ $(1 + G(S)H(S) = 0)$
- → construct Routh array
- → find K for marginarry stable system
- → place K in Aux. eqⁿ (2^{nd} order eqⁿ in Routh array)
- → s = Intersection to Imag. axis.

Rule 7: Angle of departure

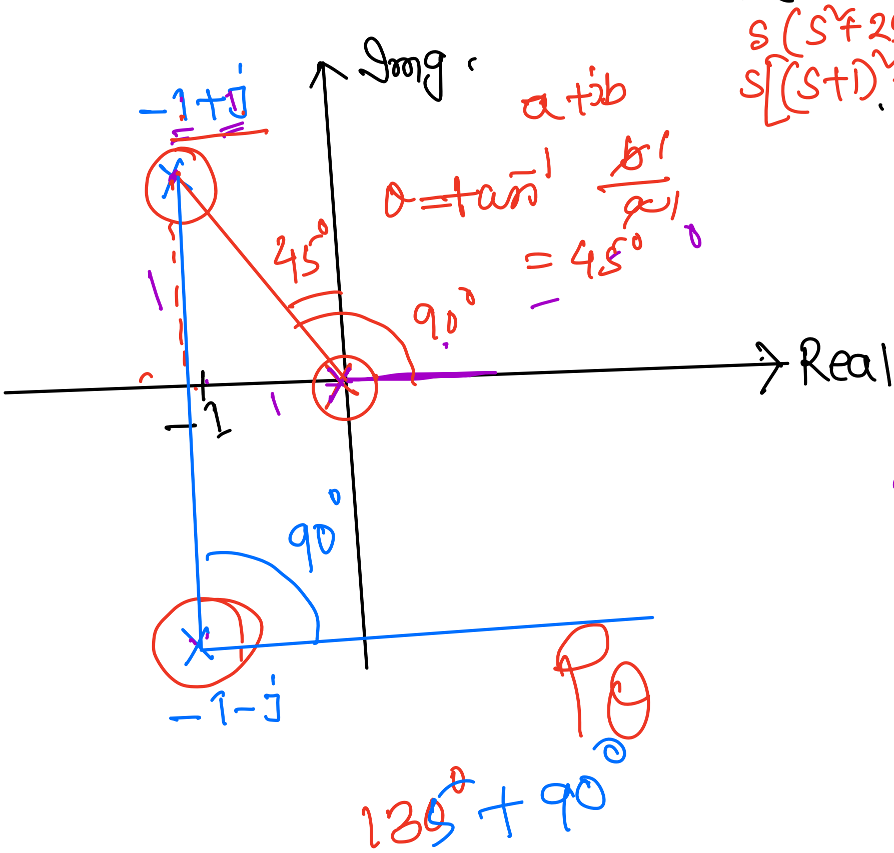
Exp: 1

$$K(S)H(S) = \frac{K}{S(S^2+2S+2)} = \frac{K}{S[(S+1)^2+1]}$$

$s(s^2+2s+1+1)$

$s[(s+1)^2+1]$

$$\boxed{\theta_d = 180^0 - (\Sigma P_\theta - \Sigma Z_\theta)}$$



$s[(s+1)^2+1] = 0$

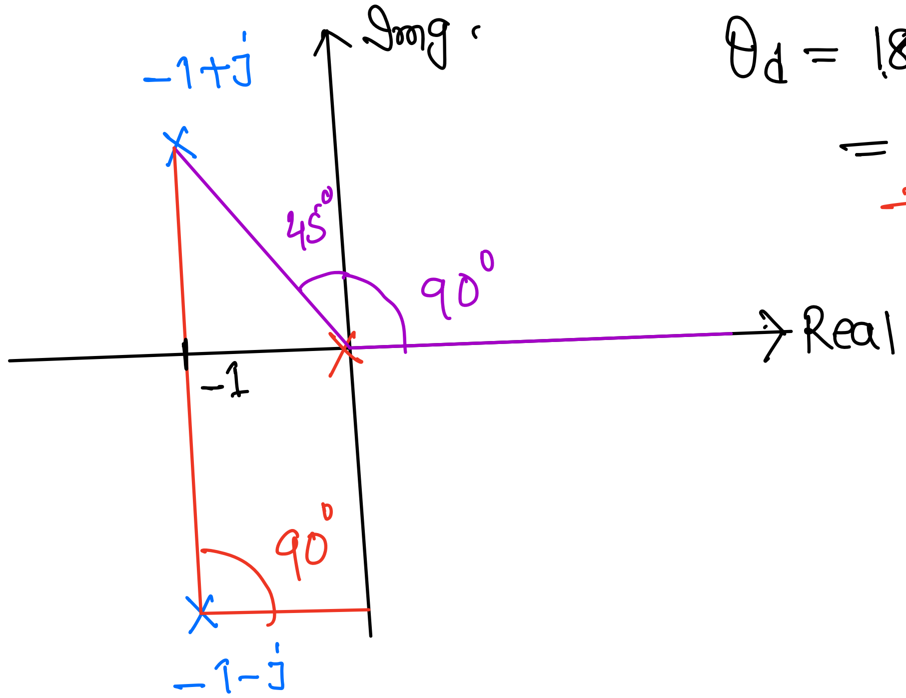
$\Rightarrow s = 0, (s+1)^2+1 = 0$

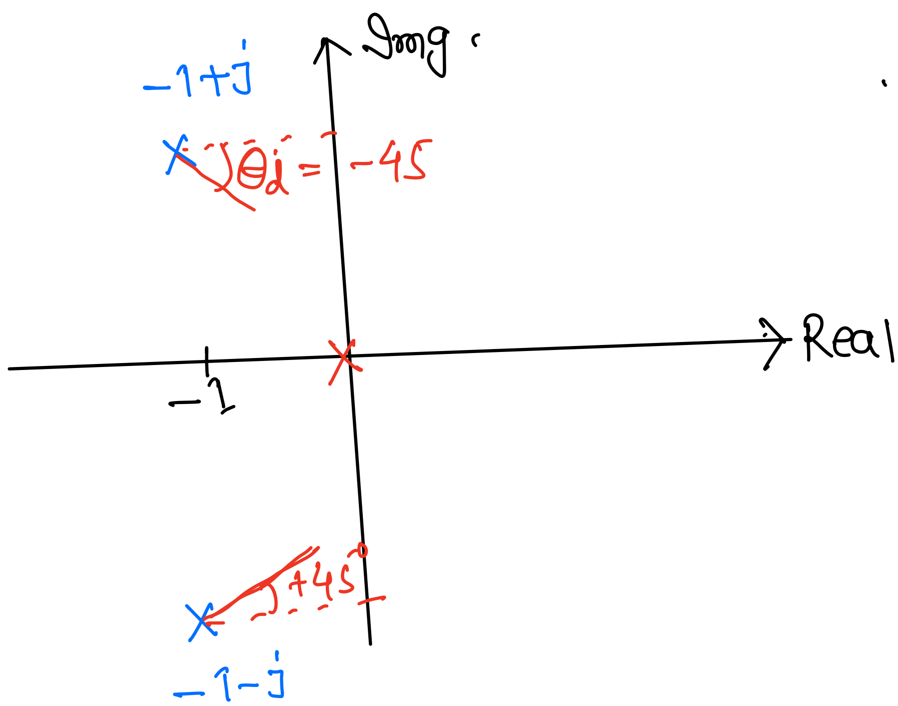
$\Rightarrow (s+1)^2 = -1$

$\Rightarrow (s+1) = \pm\sqrt{-1} = \pm j$

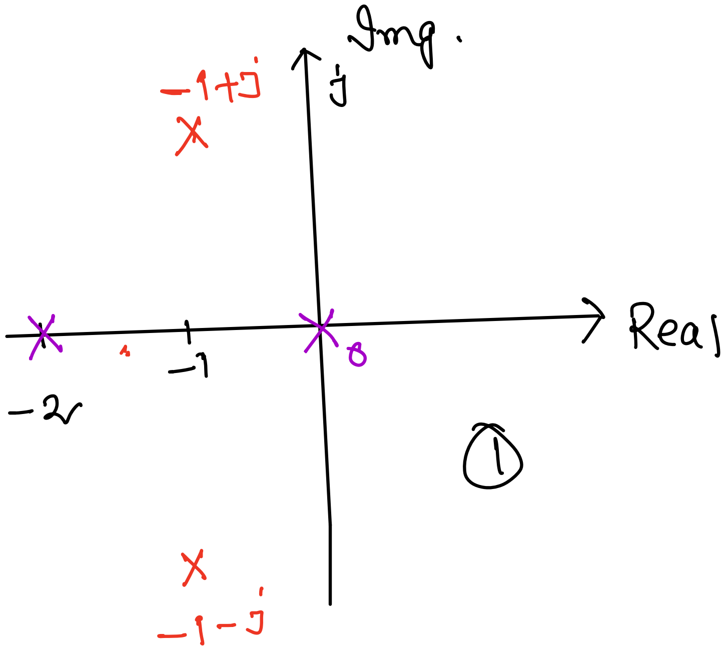
$\Rightarrow s = -1 \pm j$

$$\theta_d = 180^\circ - (135^\circ + 90 - 0)$$
$$= -45^\circ$$





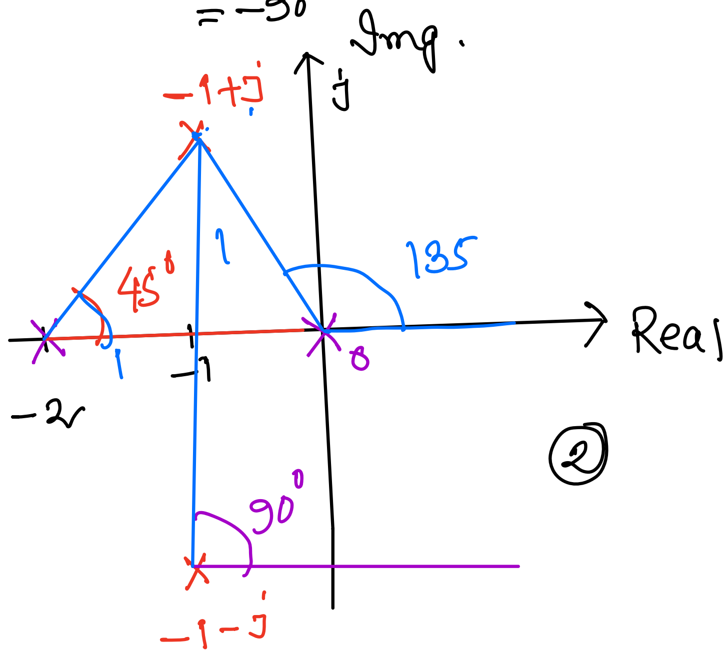
Exp: 2 $K(s)\,H(s) = \dfrac{K}{s(s+2)(s^2+2s+2)} = \dfrac{K}{s(s+2)((s+1)^2+1)}$

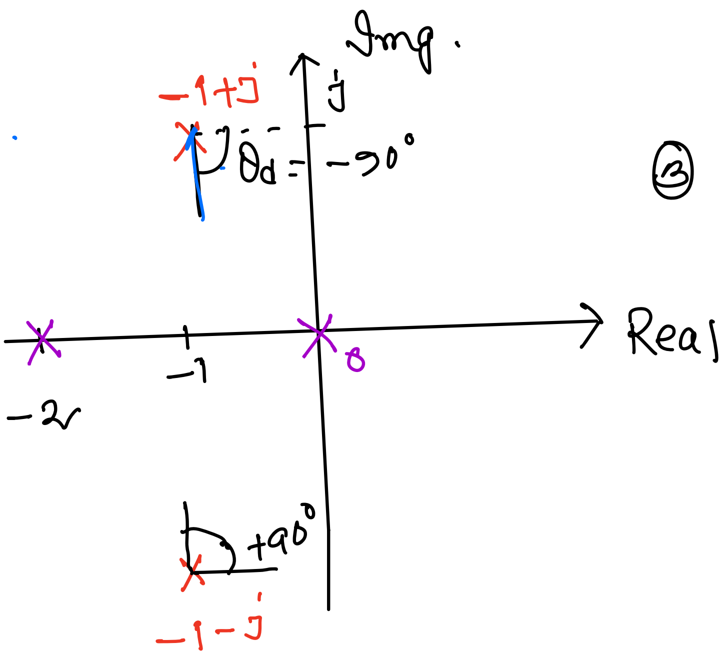


$\theta_d = 180 - (\Sigma P_\theta - \Sigma Z_\theta)$

$= 180 - (135 + 45 + 90^\circ - 0)$

$= -90^\circ$





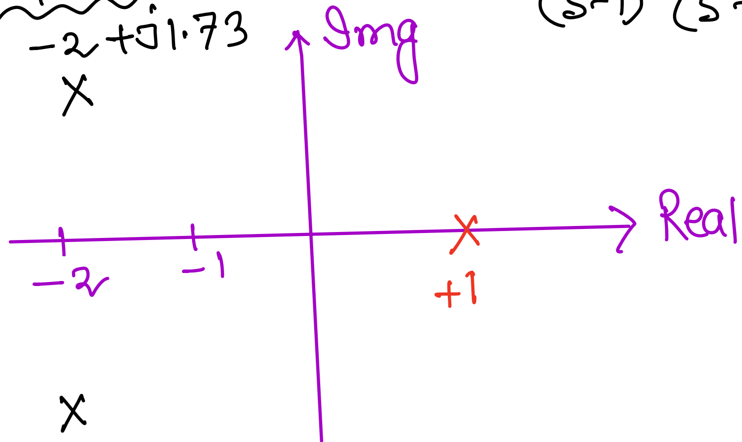
$s(s+2)\,[(s+1)^2+1] = 0$

$s = 0\,,\quad s+2 = 0\,,\quad [(s+1)^2+1] = 0$

$0,\ -2,\ -1 \pm j$

135, 90, 45

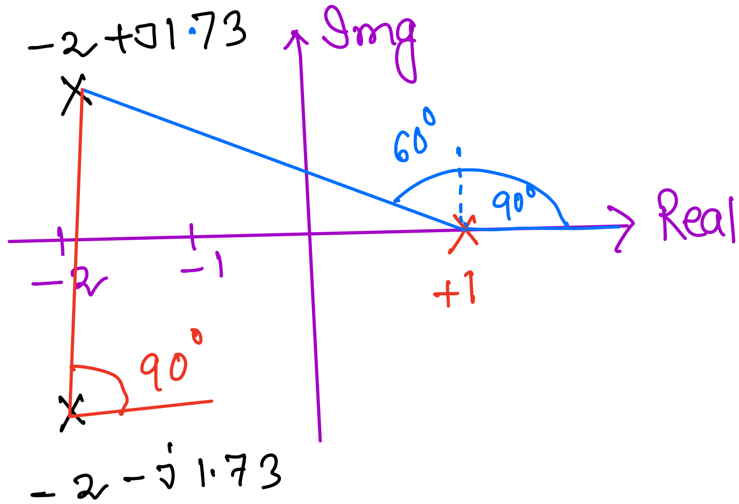
Exp 3: $K(s)\,H(s) = \dfrac{K}{(s-1)(s^2+4s+7)} = \dfrac{K}{(s-1)[(s+2)^2+3]}$



$s = 1$

$s = -2 \pm j1.73$

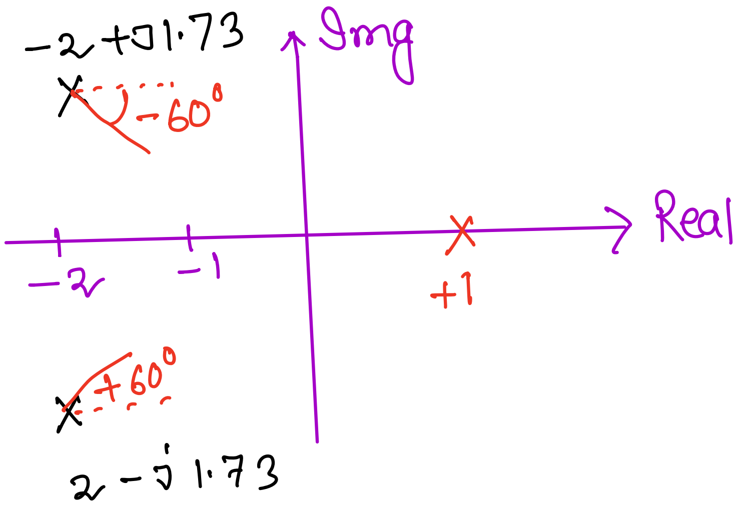
$-2 - j1.73$



$$\theta_d = 180^\circ - (150^\circ + 90^\circ)$$
$$= -60^\circ$$

$a + jb$

$$= \tan^{-1}\frac{b}{a}$$



☑ Obtain root-Locus plot for the unity feedback system with transfer function.

$$G(S) = \frac{K}{S(S+2)}$$

Poles = 0, −2

Step: 1 Obtain Loci

# of poles, P = 2

" zeros, Z = 0

∴ Total loci = Max (P, Z) = 2.

(2, 0)

Step: 2 No. of Asymptotes.

$= P - Z = 2 - 0 = 2$

Step: 3 Angle of Asymptotes.

$$\theta = \frac{2x+1}{P-Z} \times 180^0, \quad x = 0, 1$$

$$\therefore \theta = 90^0, 270^0$$

Step: 4 Centroid of Asymptotes.

$$\sigma_c = \frac{\Sigma \text{Real } P - \Sigma \text{Real } Z}{\# \text{ of } P - Z}$$

$$= \frac{(-2) - 0}{2 - 0} = -1.$$

Step: 5: Breakaway point.

→ chac. eqn:

$$1 + G(S)H(S) = 0$$

$$1 + \frac{K}{S(S+2)} \times 1 = 0$$

$$\therefore S^2 + 2S + K = 0$$

$$\therefore K = -S^2 - 2S$$

→ Calculate $\frac{dK}{dS} = 0$

$$\therefore \frac{dK}{dS} = -(2S) - 2 = 0$$

$$\therefore \boxed{S = -1}$$

Step 6: Intersection to Img. axis.

→ char. eqn.

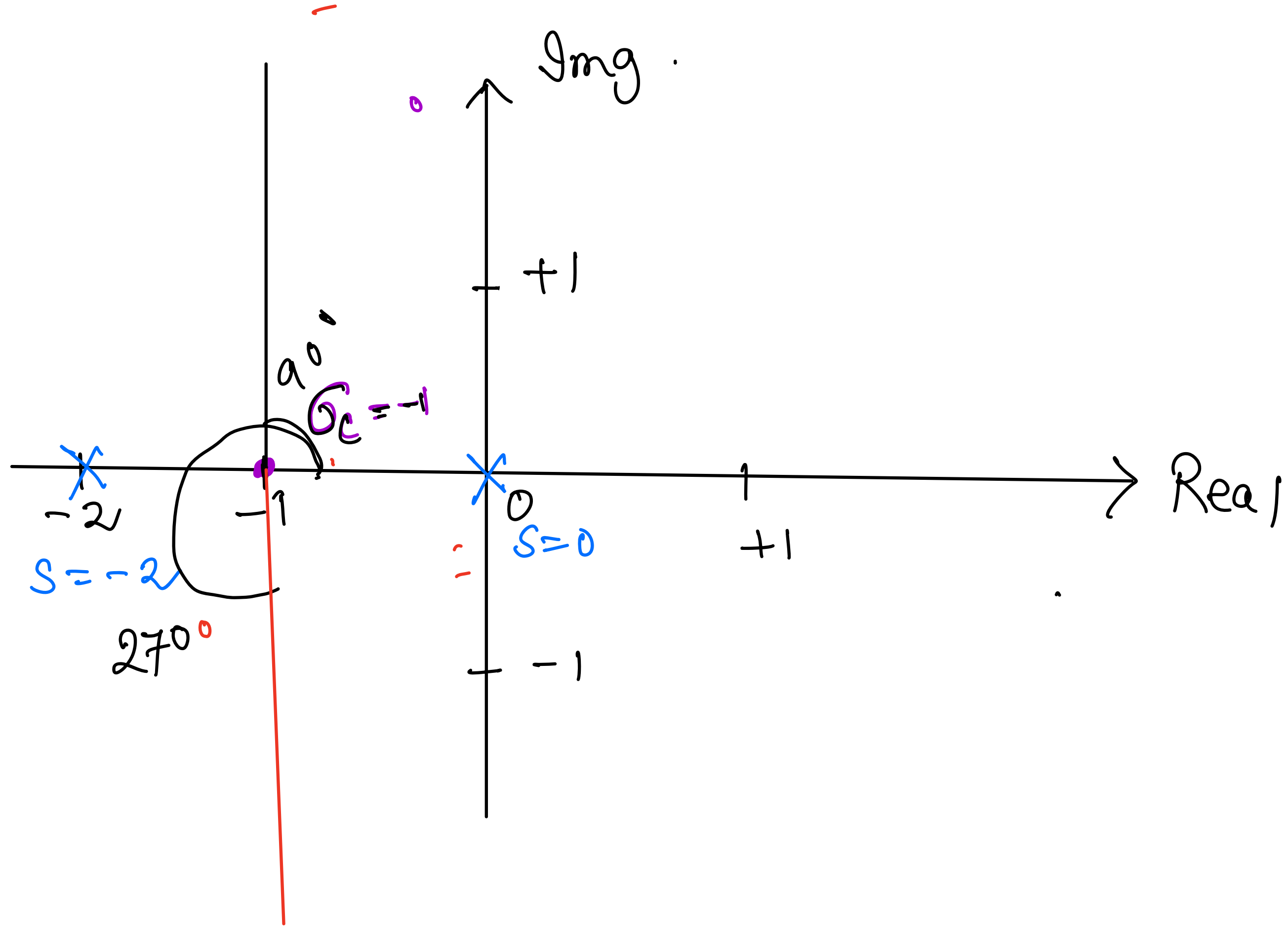
$1 + G(s)H(s) = 0$. ∴ $s^2 + 2s + K = 0$

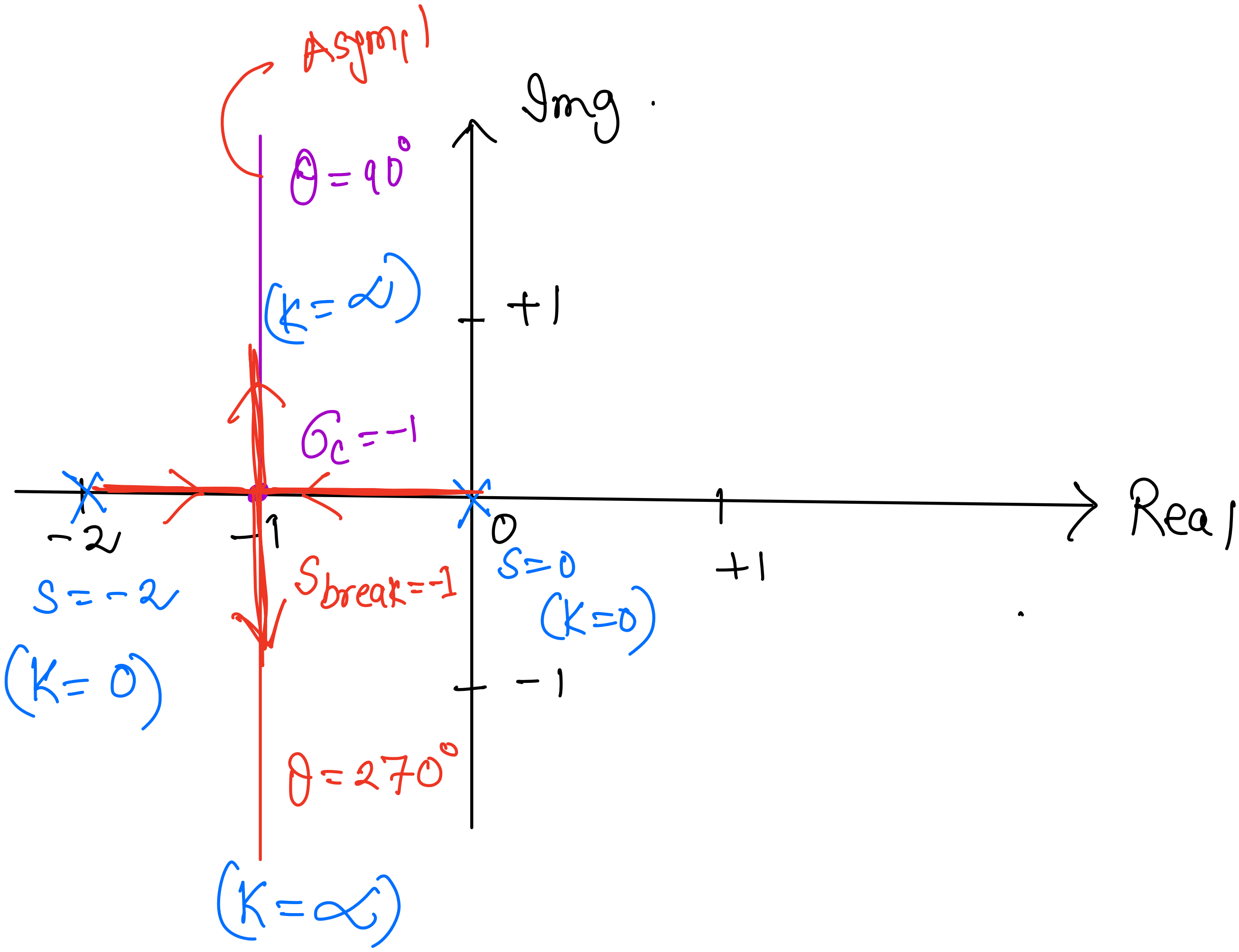
→ Routh array

s^2	1	K
s^1	2	0
s^0	K	

For stability, $K > 0$. So, no cross point.

No K for marginally stable.




Asym 1
Img
θ = 90°
(K = ∞)
+1
σc = −1
Real
−2
−1
0
+1
S = −2
(K = 0)
Sbreak = −1
S = 0
(K = 0)
−1
θ = 270°
(K = ∞)

☑ Obtain root-Locus plot for a feedback control system with open loop transfer function.

$$G(s)\,H(s) = \frac{K(s+2)(s+3)}{(s+1)(s-1)}$$

where k varies from 0 to ∞

soln: $Z = -2, -3$

$P = -1, 1$.

Step1: No. of loci = max (P, Z) = 2

Step2: No. of Asymptotes = P−Z = 2−2 = 0

Step3: Break away point.

− Char. eqn: $1 + G(s)\,H(s) = 0$

or, $1 + \dfrac{K(s+2)(s+3)}{(s+1)(s-1)} = 0$

$\therefore\ K = -\dfrac{s^2-1}{s^2+5s+6}$

− find $\dfrac{dK}{ds} = 0$

$\Rightarrow \dfrac{(s^2+5s+6)(2s) - (s^2-1)(2s+5)}{(s^2+5s+6)^2} = 0$

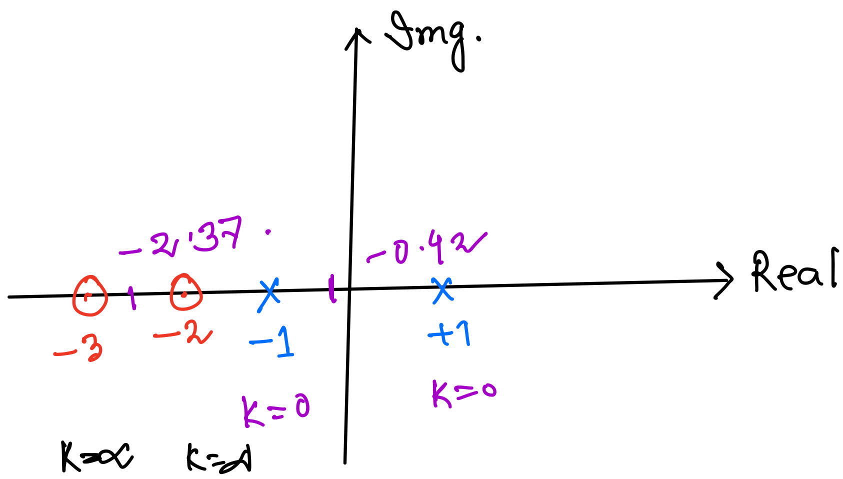
$\Rightarrow$ ....

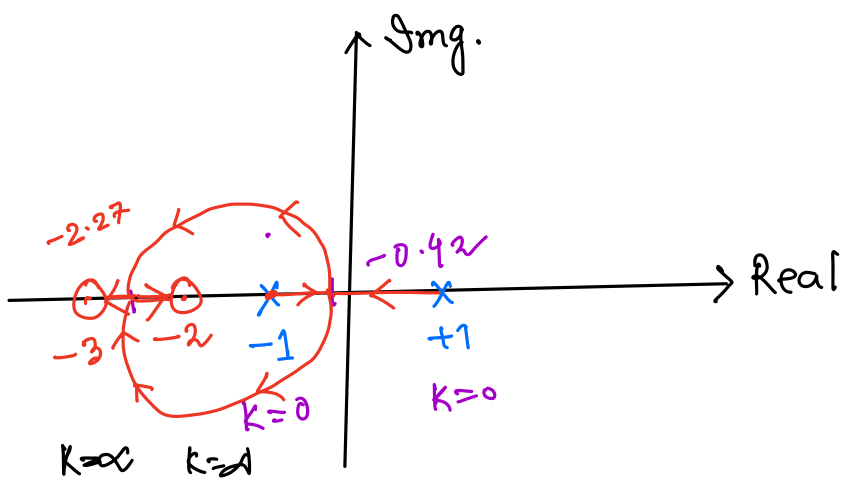
$\Rightarrow 5s^2 + 14s + 5 = 0$

$\Rightarrow s = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= -0.42, -2.37$.

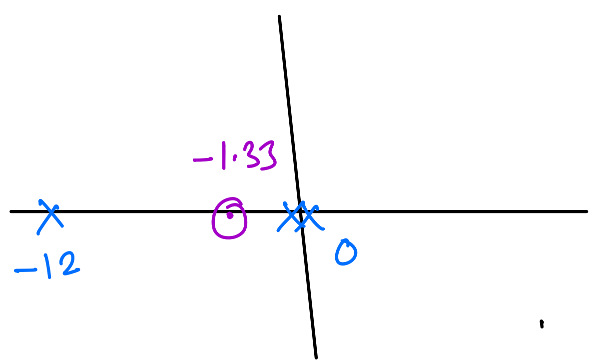
$\dfrac{d}{dx}(A/B) = \dfrac{B\dfrac{dA}{dx} - A\dfrac{dB}{dx}}{B^2}$





Exp-3 A unity feedback system has an open loop T.F. of $G(S) = \frac{K(S+4/3)}{S^2(S+12)}$

soln: $z_1 = -4/3$, $P_1, P_2 = 0$, $P_3 = -12$



Step1:

No. of Loci = Max (P, Z)

= Max (3, 1)

= 3

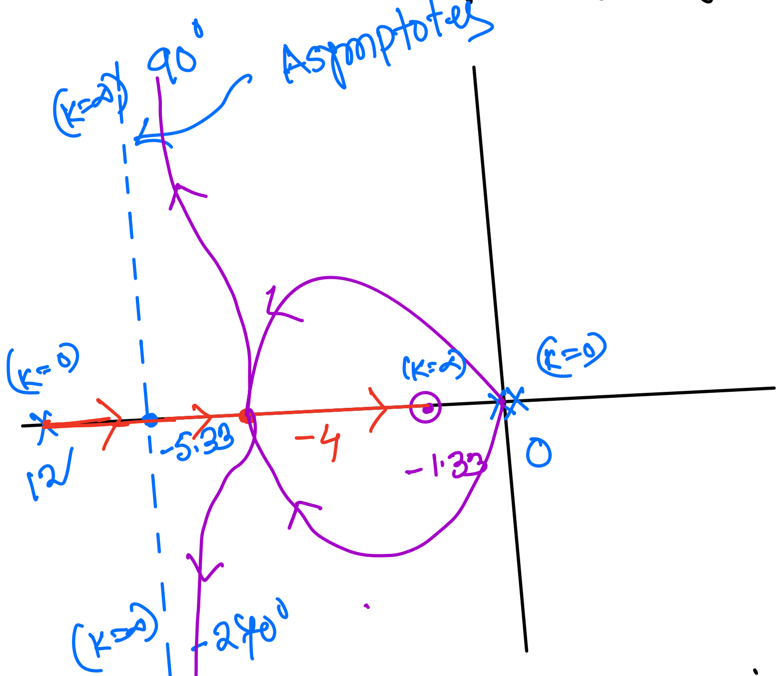
Step 2 : No. of Asymptotes $= P - Z = 3 - 1 = 2$

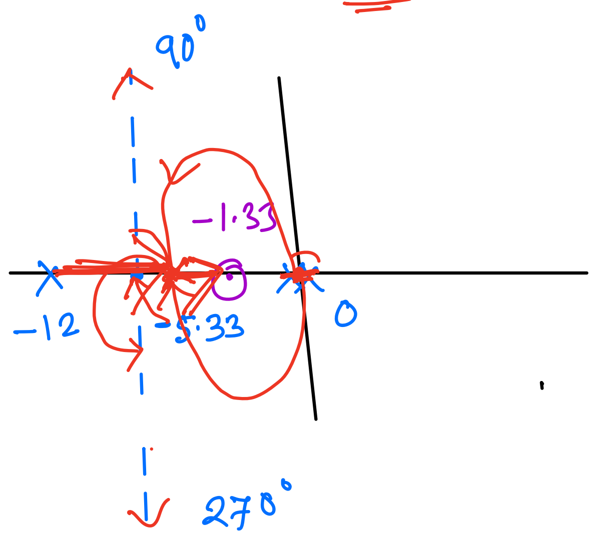
" 3 : Centroid " " $= \dfrac{\Sigma \text{ real of poles} - \Sigma \text{ real of zeros}}{P - Z}$

$$= \frac{(0+0-12) - (-1.33)}{3-1} = -5.33$$

Step 4: Angles of Asymptotes $= \dfrac{2K+1}{P-Z} \times 180^0, \; K = 0, 1, 2 \ldots$

$$= 90^0, \; 270^0$$





Step 5: Break away point

chr. eqn $1 + K G(S) H(S) = 0$

$$K = -\frac{S^3 + 12S^2}{S + 4/3}$$

$\rightarrow \dfrac{dK}{dS} = 0$.

$$\frac{(S+4/3)(3S^2+24S) - (S^3+12S^2)}{(S+4/3)^2} = 0$$

$\therefore \; 2S^3 + 16S^2 + 32S = 0$

$\therefore \; S = 0, -4, -4$

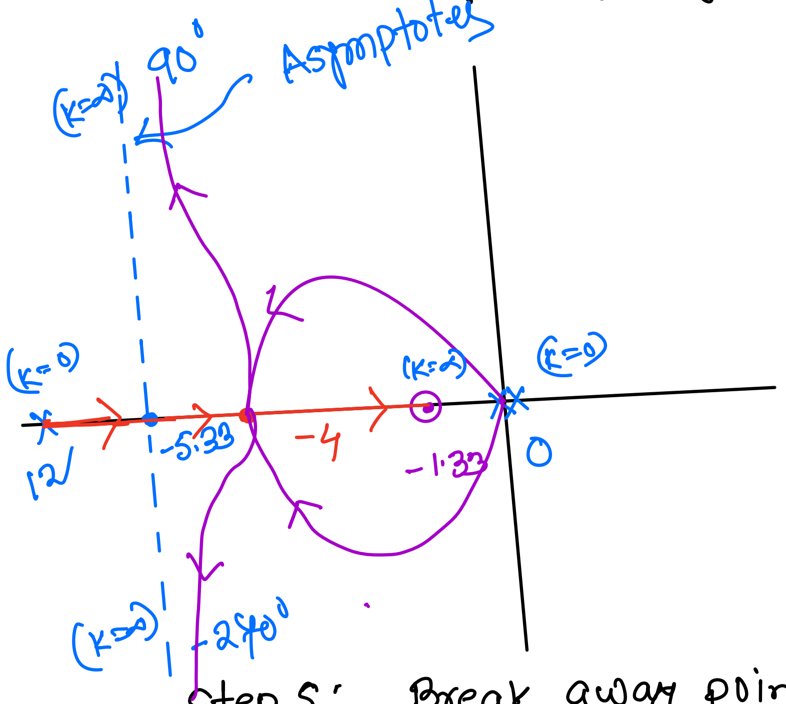
Step 2: No. of Asymptotes $= P - Z = 3 - 1 = 2$

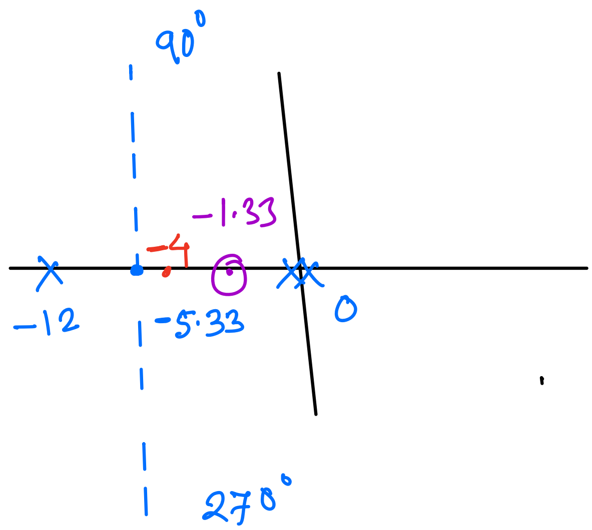
Step 3: Centroid $= \dfrac{\sum \text{real of poles} - \sum \text{real of zeros}}{P - Z}$

$$= \frac{(0 + 0 - 12) - (-1.33)}{3 - 1} = -5.33$$

Step 4: Angles of Asymptotes $= \dfrac{2K+1}{P-Z} \times 180^\circ$, $K = 0, 1, 2 \ldots$

$= 90^\circ, 270^\circ$





Step 5: Break away point

chr. eqn $1 + K G(S) H(S) = 0$

$$K = -\frac{S^3 + 12S^2}{S + 4/3}$$

$\rightarrow \dfrac{dK}{dS} = 0$.

$$\frac{(S + 4/3)(3S^2 + 24S) - (S^3 + 12S^2)}{(S + 4/3)^2} = 0$$

$\therefore 2S^3 + 16S^2 + 32S = 0$

$\therefore S = 0, -4, -4$

Exp: 4 $G(S)\,H(S) = \dfrac{K}{S(S+1)(S+2)(S+3)}$

$\rightarrow P = 0, -1, -2, -3$

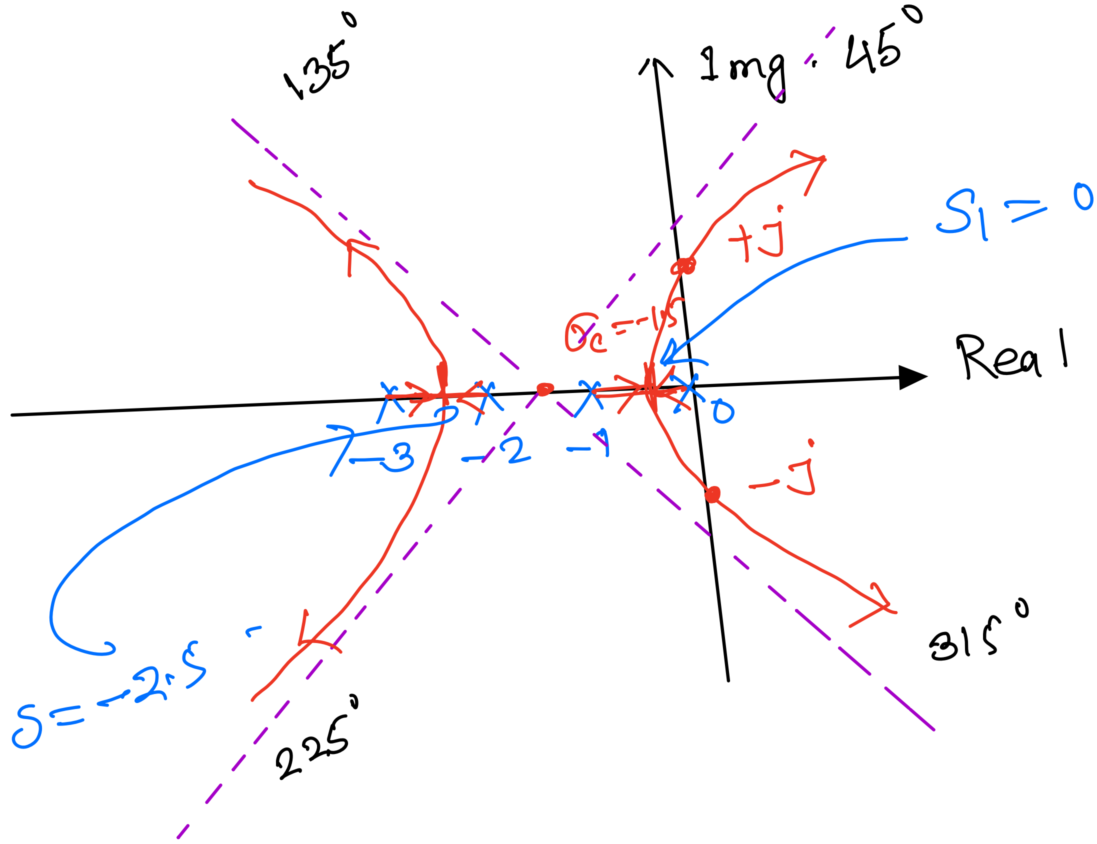
Step 1: No. of Loci = Max $(P, Z) = 4$

Step 2: " " Asyp. $= P - Z = 4$

Step 3: centroid $= -1.5$

Step 4: Angles, $\theta = \dfrac{(2K+1)\,180^{\circ}}{P-Z}$, $K = 0, 1, 2, 3$

$= 45^{\circ}, 135^{\circ}, 225^{\circ}, 315^{\circ}$



Step 5: Break away :

char. eq^n: $1 + K(S)\,G(S) = 0$

$\therefore K =$

$\dfrac{dK}{dS} = 0$

$\therefore\ 4s^3 + 18s^2 + 22s + 6$

$\therefore\ s_1 \approx -0.4, \quad s_2 = -2.6$

Step 6 Intersection to long axis.

char eq^n $= s^4 + 6s^3 + 11s^2 + 6s + K = 0$

s^4	1	11	K
s^3	6	6	
s^2	10	K	
s^1	$\frac{60-6K}{10}$	0	
s^0	K	0	

$\therefore\ \frac{60-6K}{10} > 0$

or $60 > 6K$

$\therefore\ K < 10$

→ Aux eq^n

$10s^2 + K = 0$

$s^2 = -K/10$

If $K \approx 10$,

$s^2 = -10/10 = -1$

$\therefore\ s = \pm j$