Laplace Transformation

Lecture-02

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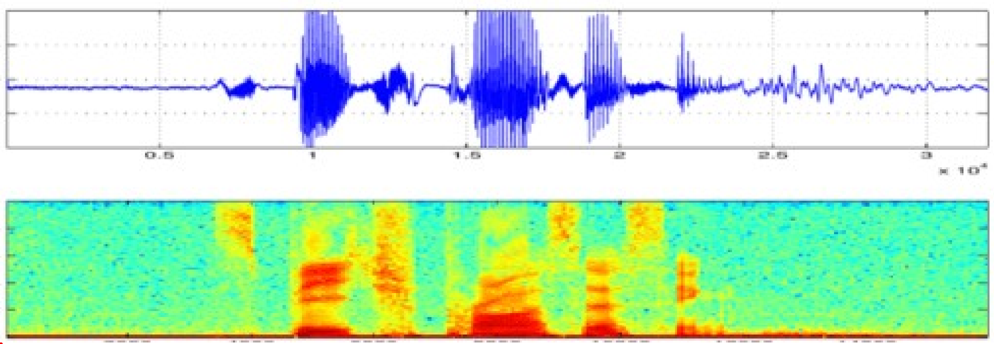

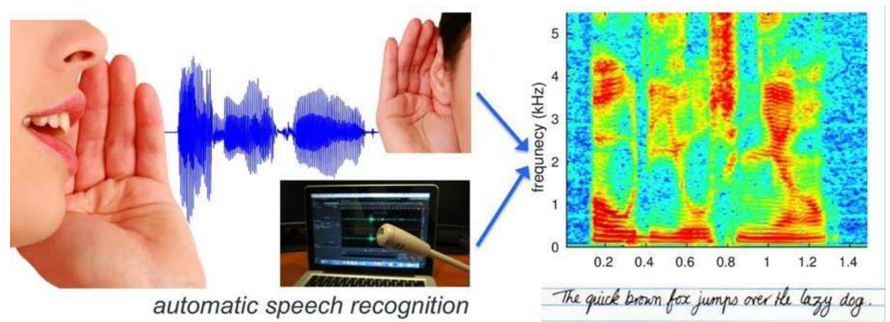

automatic speech recognition

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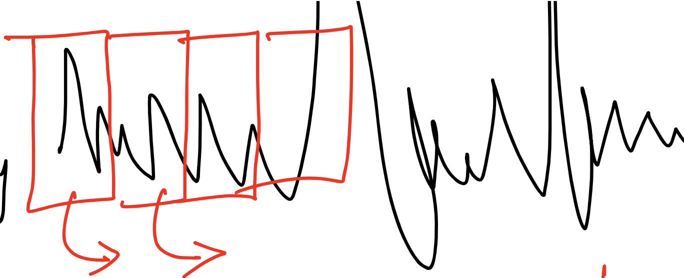
Limitation:



1. FT is suitable only for stationary signal.
For Non-Stationary signal, STFT has been used.

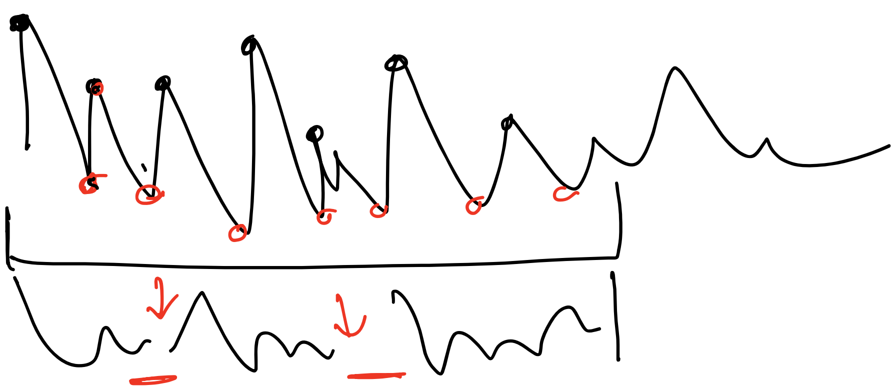
Short-time Fourier trans
STFT

2. Loss of Time Information: FT provides freq domain representation but doesn't indicate when a particular freq^n occures in time.

3. Computational complexity :- use FFT.

- **Not Suitable for Non-Stationary Signals:** Fourier Transform assumes the signal is infinite in duration and does not change over time. For non-stationary signals, **Short-Time Fourier Transform (STFT)** or **Wavelet Transform** is preferred.
- **Loss of Time Information:** FT provides frequency domain representation but does not indicate when a particular frequency occurs in time.
- **Computational Complexity:** The computation of Fourier Transform for large data sets can be time-consuming, though the **Fast Fourier Transform (FFT)** algorithm helps in reducing complexity.
- **Ideal Conditions Assumption:** In real-world scenarios, signals may not always be perfectly periodic or infinite in duration, affecting the accuracy of FT.

Condition of Fourier Transformation:

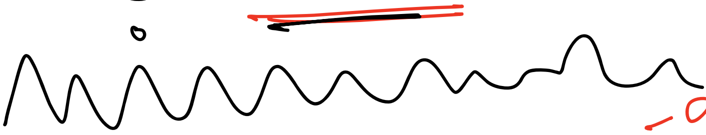


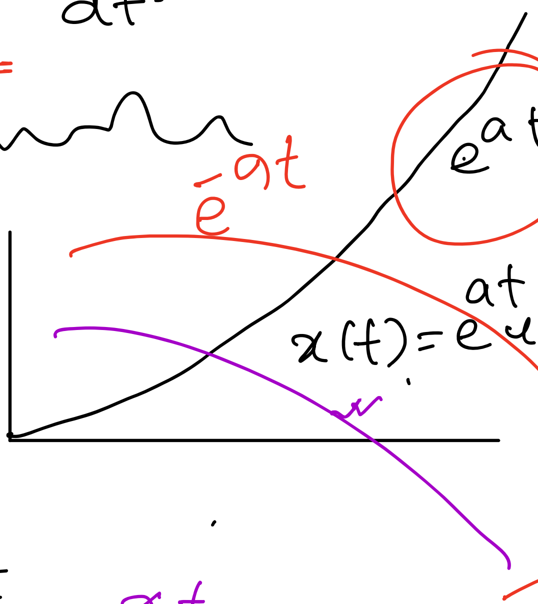
Any periodic signal $f(\theta)$ with period 2π that satisfies the Dirichlet condition, can be used to apply Fourier transformation.

Dirichlet conditions:

1) It has finite number of discontinuities in one period
2) It has finite number of maxima & minima in one period.
3) The integral $\int_{-\pi}^{\pi} |f(\theta)|\, d\theta$ is finite.

$$F(j\omega) = \int_{0}^{\infty} f(t)\, e^{-j\omega t}\, dt.$$





$$x(t) = e^{at}$$

$$x'(t) = x(t) \cdot e^{-at}.$$

$$e^{-at}\, X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, e^{-at}$$

$$e^{-at} X(j\omega) = \int_0^\infty x(t) \cdot e^{-at} \cdot e^{-j\omega t}$$

Laplace tran

$$= \int_0^\infty x(t)\, e^{-\underbrace{(a+j\omega)}_{s} t}$$

$$= \int_0^\infty x(t)\, e^{-st}\, dt \qquad s = a + j\omega$$

$$= F(s)$$

F(jω)

= Laplace transformed of x(t)

Advantages:

1) It simplifies function
2) " " operation.
3) It solves the problem of certain signals that cannot be analyzed by FT.

$$F(s) = \mathcal{L}(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}, \; s = \sigma + j\omega$$

Introduction

Laplace:

Pierre-Simon, marquis de Laplace (French; 23 March 1749 – 5 March 1827) was an influential French scholar whose work was important to the development of mathematics, statistics, physics, and astronomy.

Transformation:

It refers to conversion .

What is Laplace Transform?

The **Laplace Transform** is a powerful integral transform used to convert functions of time (usually denoted as f(t)) into functions of a complex variable (usually denoted as s).

The Laplace transform of a function f(t) is defined as:

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Here:

- t is the time-domain variable.
- s is a complex number $s=\sigma+j\omega$.
- F(s) is the Laplace transform of f(t)

Why Laplace Transform?

1. Simplifies Differential Equations:

- It converts differential equations into algebraic equations, which are easier to solve.
- For example, a second-order differential equation in time becomes a simple polynomial in s.

2. System Analysis and Control Engineering:

- Widely used in electrical, control, and mechanical engineering for analyzing linear time-invariant (LTI) systems.
- Allows engineers to study the stability, frequency response, and behavior of systems in the **s-domain** (frequency domain).

3. Initial and Final Value Theorems:

- Provides tools to quickly find the initial and steady-state behavior of a system without solving the entire differential equation.

Why Laplace Transform?

4. Handling Discontinuous and Impulse Functions:

- Easily manages step functions, impulses (Dirac delta), and other non-continuous signals, which are common in real-world systems.

5. Convolution Simplification:

- Convolution in the time domain becomes multiplication in the Laplace domain, simplifying signal processing tasks.

The Laplace Transform

The Laplace Transform of a function, f(t), is defined as;

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

The Inverse Laplace Transform is defined by

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{ts}\,ds$$

An important point to remember:

$$f(t) \Leftrightarrow F(s)$$

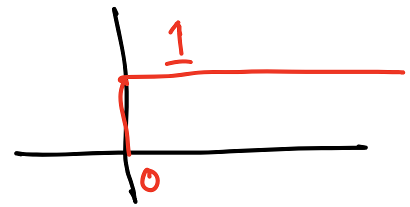
The above is a statement that f(t) and F(s) are transform pairs. What this means is that for each f(t) there is a unique F(s) and for each F(s) there is a unique f(t).

Laplace Transform of the unit step.

$$L[u(t)] = \int_0^\infty 1 e^{-st}\, dt = \frac{-1}{s} e^{-st} \Big|_0^\infty$$

$$L[u(t)] = \frac{1}{s}$$

The Laplace Transform of a unit step is: $\boxed{\frac{1}{s}}$



$u(t)$

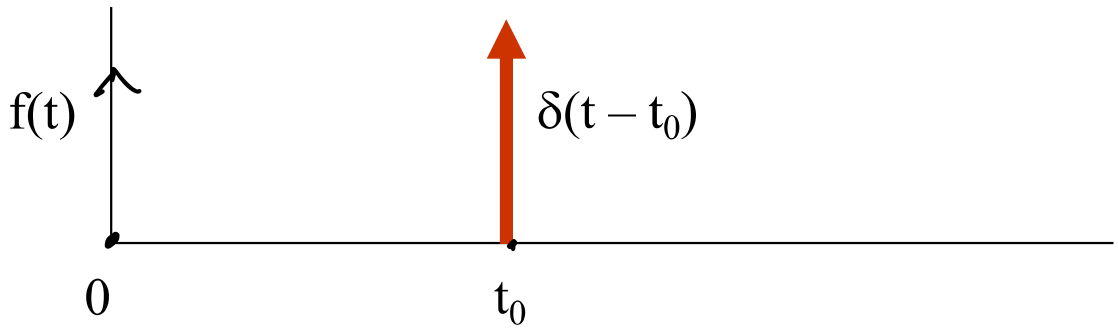
$$F(s) = \int_{-\infty \;\; 0}^{\infty} \underbrace{u(t)}_{1}\, e^{-st}\, dt = \int_0^\infty 1 \cdot e^{-st}\, dt$$

$$= -\frac{1}{s}\left[e^{-st}\right]_0^\infty = \frac{-1}{s}\left[0-1\right]$$

$$= \frac{1}{s}$$

The Laplace transform of a unit impulse:

Pictorially, the unit impulse appears as follows:



Mathematically:

$$\delta(t - t_0) = 0 \quad t \neq 0$$

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t - t_0)\,dt = 1 \qquad \varepsilon > 0$$

$$\int_{t_1}^{t_2} f(t)\delta(t - t_0)\,dt = \begin{cases} f(t_0) & t_1 < t_0 < t_2 \\ 0 & t_0 < t_1, t_0 > t_2 \end{cases}$$

The Laplace transform of a unit impulse:

In particular, if we let f(t) = δ(t) and take the Laplace

$$L[\delta(t)] = \int_0^\infty \delta(t) e^{-st} dt = e^{-0s} = 1$$

$$\int_0^\infty \underbrace{\delta(t)}_{1} e^{-st} dt = \int_0^0 1 e^{-st} = e^{0 \cdot s} = 1.$$

$\delta(t)$

$\delta(t - t_0)$

$$\mathcal{L}[e^{-at}u(t)] = \int_{-\infty}^{\infty} e^{-at}\,u(t)\cdot e^{-st}\,dt$$

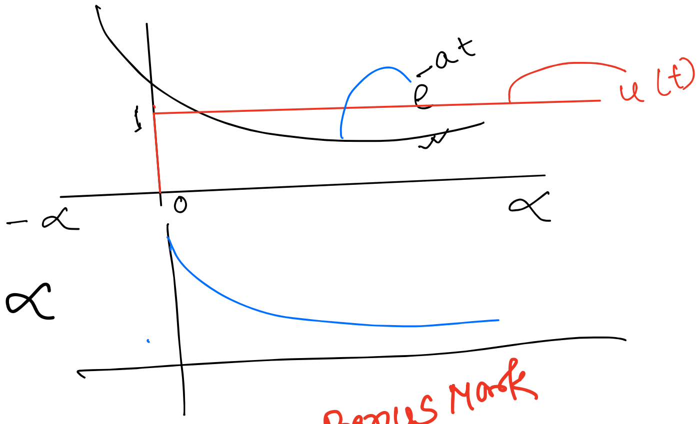
$$= \int_{0}^{\infty} e^{-at}\cdot e^{-st}\,dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t}\,dt$$

$$= \frac{-1}{s+a}\left[e^{-(s+a)t}\right]_{0}^{\infty}$$

$$= -\frac{1}{s+a}\left[\underbrace{e^{-\infty(s+a)}}_{0} - \underbrace{e^{0}}_{1}\right]$$

$$= \frac{-1}{s+a}(-1) = \frac{1}{s+a}$$



Bonus Mark

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Building transform pairs:

$$L[e^{-at}u(t)] = \int_0^{\infty} e^{-at}e^{-st}\,dt = \int_0^{\infty e} e^{-(s+a)t}\,dt$$

$$L[e^{-at}u(t)] = \frac{-e^{-st}}{(s+a)}\Big|_0^{\infty} = \frac{1}{s+a}$$

$$\boxed{e^{-at}u(t) \quad \Leftrightarrow \quad \frac{1}{s+a}}$$

$$L(e^{-at}u(t))$$

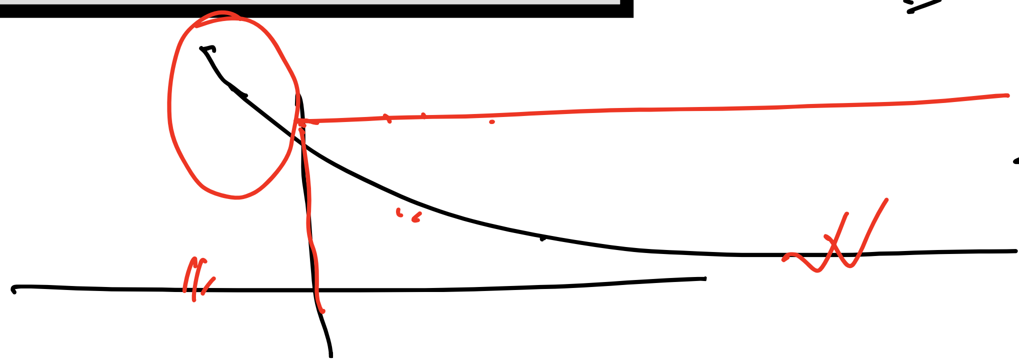
$$= \int_{-\infty}^{\infty} \underbrace{e^{-at}u(t)}\,.\,e^{-st}\,dt$$

$$= \int_0^{\infty} e^{-(s+a)t}\,dt$$

$$= \frac{1}{-(s+a)}\left[e^{-(s+a)t}\right]_0^{\infty}$$

$$= -\frac{1}{s+a}\,.\,[0-1]$$

$$= \frac{1}{s+a}$$



$$\boxed{\;}\quad L[f(t-a)u(t-a)].$$

$$= \int_{-\alpha}^{\alpha} f(t-a)\,u(t-a)\,e^{-st}\,dt$$

$$= \int_{t=a}^{t=\alpha} f(t-a)\,e^{-st}\,dt.$$

dt, $t-a = x \Rightarrow t = x+a$

$dt = dx$

$$= \int_{x=0}^{\alpha} f(x)\,e^{-s(x+a)}\,dx$$

$t=a$, $x = t-a = a-a =$

$t=\alpha$. $x = \alpha$

$$= \int_{x=0}^{\alpha} f(x)\cdot e^{-xs}\cdot e^{-sa}$$

$$= e^{-as}\underbrace{\int_{x=0}^{\alpha} f(x)\,e^{-xs}}_{\mathcal{L}(f(x)) = F(s)}$$

$$= e^{-as}\cdot F(s)$$

$$\mathcal{L}\left[e^{-2t}u(t)\right] = \frac{1}{s+2}.$$

$$\mathcal{L}\left[e^{-2(t-3)}u(t-3)\right] = \frac{e^{-3s}}{s+2}$$

Time Shift

$$L[f(t-a)u(t-a)] = \int_a^\infty f(t-a)e^{-st}$$

Let $x = t-a$, *then* $dx = dt$ *and* $t = x+a$

As $t \to a,\ x \to 0$ *and as* $t \to \infty, x \to \infty$. *So,*

$$\int_0^\infty f(x)e^{-s(x+a)}dx = e^{-as}\int_0^\infty f(x)e^{-sx}dx$$

$$L[f(t-a)u(t-a)] = e^{-as}F(s)$$

$L[f(t-a)u(t-a)]$

$= \int_{-\infty}^{\infty} f(t-a)u(t-a)e^{-st}dt$

$= \int_a^\infty f(t-a)e^{-st}dt$

Let, $t-a = x$

$t = a,\ x = 0$

$t = \infty,\ x = \infty$

$= \int_0^\infty f(x)e^{-(x+a)s}dx$

$= e^{-as}\int_0^\infty f(x)e^{-xs}.dx$

$= e^{-as}F(s).$

Exap

$\mathcal{L}[e^{-2t}u(t)] = \frac{1}{s+2}$

$\mathcal{L}[e^{-2t+3}u(t-3/2)] = \frac{e^{-3s/2}}{s+2}$

Thank you!
Question ?