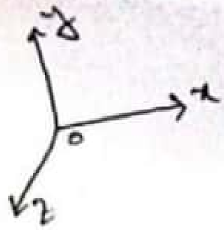
MATH 3

Complex Number System

- $a+ib$
  - Real Number
    - a
      - Rational Number $\frac{p}{q}; q \neq 0$
        - integer
          - Negative
          - zero
          - Positive
            - Natural Number
            - Prime Number
        - fraction
          - proper fraction $\frac{N}{D} = N<D = \frac{1}{3}$
          - improper fraction $N>D = \frac{3}{2}$
          - Mix fraction $1\frac{1}{2}, 2\frac{3}{2}$
      - irrational Number $\pi, \sqrt{2}, 2\pi$
  - imaginary Number
    - ib where $i = \sqrt{-1}$

$p \to$ Numerator

$q \to$ Denominator



Let , $z_1 = a + ib$

$z_2 = c + id$

Addition:

$z_1 + z_2$

$\Rightarrow (a+ib) + (c+id)$

$\Rightarrow (a+c) + i(b+d)$

Another complex Number.

Substraction:

$z_1 - z_2$

$\Rightarrow (a+ib) - (c+id)$

$\Rightarrow (a-c) + i(b-d)$

Multipication

$z_1 * z_2$

$\Rightarrow (a+ib) * (c+id)$

$\Rightarrow ac + iad + ibc + i^2 bd \quad [i^2 = -1]$

$\Rightarrow (ac - bd) + i(ad + bc)$

Division

$\frac{z_1}{z_2}$

$\Rightarrow \frac{a+ib}{c+id}$

$\Rightarrow \frac{(a+ib)(c-id)}{(c+id)(c-id)}$

$\Rightarrow \frac{ac - iad + ibc - i^2 bd}{c^2 - i^2 d^2}$

$\Rightarrow \frac{(ac+bd) + i(bc - ad)}{c^2 + d^2}$

$\Rightarrow \frac{ac+bd}{c^2+d^2} + i\left(\frac{bc-ad}{c^2+d^2}\right)$

$\therefore z = x + iy$

Proper form of a Complex Number

~~x+ia~~

$z = x+iy$

$x = r\cos\theta$

$y = r\sin\theta$

$\theta = \tan^{-1}\frac{y}{x}$

$z = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$

Complex variable function

The complex variable function is $w = f(z)$

$u+iv = f(x+iy)$

Each of the real number u and v depends on the real variables x and y and it follows that $f(z)$ can be expressed in terms of a pair of real valued functions of the real variable x and y.

i.e. $f(z) = u(x,y) + iv(x,y)$

The Polar form is

$u+iv = f(re^{i\theta})$

$\therefore f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$.

Limit:

Let $f(z)$ be a single valued function defined at all points in some neighbourhood of a point z_0. Then the limit of $f(z)$ as z ~~app~~ approaches to z_0 is w_0.

$\lim_{z\to z_0} f(z) = w_0$

Continuity:-

$f(z)$ is said to be continuous at $z=z_0$

if $\lim_{z\to z_0} f(z) = f(z_0)$

Differentiability:-

Let $f(z)$ be a single valued function of the variable z. then, $f(z) = \lim_{\delta z\to 0} \frac{f(z+\delta z)-f(z)}{\delta z}$

provided that the limit exists and is independent of the path $\delta z \to 0$.

Exercises

① Express $\frac{(1+i)(2+i)}{(3+i)}$ in the form of $a+ib$

② " $\frac{(6+i)(2-i)}{(4+3i)(1-2i)}$ "

③ " $\frac{1+2i}{1-3i}$ in the form of $r\cos\theta + ir\sin\theta$

④ " $\left(\frac{2+i}{3-i}\right)^2$ in polar form.

⑤ $w = f(z) = z^2+3z$
find u and v and calculate the value of f at $z=1+3i$.

⑥ $w = f(z) = 2iz + 6\bar{z}$
find u and v and the value of f at $z = \frac{1}{2}+4i$.

⑦ Show
$f(z) = 4x+y+i(-x+4y)$ is differentiable or not.

Sol^n

① Given $z = \dfrac{(1+i)(2+i)}{(3+i)}$

$$= \frac{2+i+2i+i^2}{3+i}$$

$$= \frac{(2-1)+i(1+2)}{3+i} \quad [i^2 = -1]$$

$$= \frac{(1+3i)(3-i)}{(3+i)(3-i)}$$

$$= \frac{3-i+9-3i^2}{3^2-i^2}$$

$$= \frac{6+8i}{10}$$

$$= \frac{6}{10} + \frac{8}{10}i$$

$$= \frac{3}{5} + i\frac{4}{5}$$

$$= a+ib$$

where, $a = \frac{3}{5}$, $b = \frac{4}{5}$

③ $z = \dfrac{1+2i}{1-3i}$

$$= \frac{(1+2i)(1+3i)}{(1-3i)(1+3i)}$$

$$= \frac{1+3i+2i+6i^2}{1^2-9i^2}$$

$$= \frac{1+3i+2i-6}{1+9}$$

$$= \frac{-5+5i}{10}$$

$$= \frac{-5}{10} + \frac{5i}{10}$$

$$= -\frac{1}{2} + i\frac{1}{2}$$

$r\cos\theta = -\frac{1}{2}$

$r\sin\theta = \frac{1}{2}$

$r^2\cos^2\theta + r^2\sin^2\theta = \left(-1/2\right)^2 + \left(1/2\right)^2$

$r^2(\cos^2\theta + \sin^2\theta) = \frac{2}{4}$

$$r^2 = \frac{1}{2}$$

$$r = \frac{1}{\sqrt{2}}$$

$\Rightarrow r\cos\theta = -\frac{1}{2}$

$\frac{1}{\sqrt{2}}\cos\theta = -\frac{1}{2}$

$\cos\theta = -\frac{1}{\sqrt{2}} = \cos\frac{3\pi}{4}$

$\Rightarrow r\sin\theta = \frac{1}{2}$

$\frac{1}{\sqrt{2}}\sin\theta = \frac{1}{2}$

$\sin\theta = \frac{1}{\sqrt{2}} = \sin\frac{3\pi}{4}$

$\therefore z = r\cos\theta + ir\sin\theta$

$= \frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

(7.) Given

$$f(z) = 4x + y + i(-x + 4y)$$

where,

$$u = 4x + y$$

$$v(x, y) = -x + 4y$$

$$f(z + \delta z) = 4(x + \delta x) + (y + \delta y) - i(x + \delta x) + i4(y + \delta y).$$

$$= 4x + 4\delta x + y + \delta y - ix - i\delta x + i4y + i4\delta y.$$

$f(z + \delta z) - f(z)$

$$\Rightarrow \cancel{4x} + 4\delta x + \cancel{y} + \delta y - \cancel{ix} - i\delta x + \cancel{i4y} + i4\delta y - \cancel{4x} - \cancel{y} + \cancel{ix} - \cancel{4iy}$$

$$\frac{f(z + \delta z) - f(z)}{\delta z} = \frac{4\delta x + \delta y - i\delta x + 4i\delta y}{\delta z}$$

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \frac{4\delta x + \delta y - i\delta x + 4i\delta y}{\delta x + i\delta y}$$

Along real axis, $\delta y = 0$

$$\frac{\delta f}{\delta z} = \frac{4\delta x - i\delta x}{\delta x} = \frac{\delta x(4 - i)}{\delta x} = 4 - i$$

Along imaginary axis, $\delta x = 0$

$$\frac{\delta f}{\delta z} = \frac{\delta y + 4i\delta y}{i\delta y} = \frac{1 + 4i}{i} = \frac{i + 4i^2}{i^2} = -(i - 4) = 4 - i$$

Along different Paths the value $f'(z)$ is same $f(z)$ is differentiable.

(5.) $w = f(z) = z^2 + 3z$

$z = 1 + 3i$

$\therefore f(z) = z^2 + 3z$

$= (x+iy)^2 + 3(x+iy) \quad [\because z = x+iy]$

$= x^2 + 2ixy + i^2y^2 + 3x + 3iy \quad [\because i^2 = -1]$

$= x^2 + 2ixy - y^2 + 3x + 3iy$

$= (x^2 + 3x - y^2) + i(2xy + 3y)$

$u(x,y) = x^2 + 3x - y^2$

$v(x,y) = 2xy + 3y$

Given, $z = 1 + 3i$ | $x = 1$, $y = 3$

$\therefore u(x,y) = x^2 + 3x - y^2 = (1)^2 + 3.1 - (3)^2$

$= 1 + 3 - 9 = 4 - 9 = 5$

$\therefore v(x,y) = 2x - 6y = 2 \cdot 1 \cdot 3 + 3 \cdot 3$

$= 6 + 9 = 15$

$\therefore f(x) = u(x,y) + iv(x,y) = -5 + i15$

(6.) $w = f(z) = 2iz + 6\bar{z}$

$\therefore f(z) = 2iz + 6\bar{z}$

$= 2i(x+iy) + 6(x-iy) \quad [\because z = x+iy,\ \bar{z} = x-iy]$

$= 2ix + 2i^2y + 6x - 6iy$

$= (6x - 2y) + i(2x - 6y)$

$u(x,y) = 6x - 2y$

$v(x,y) = 2x - 6y$

Given, $z = \frac{1}{2} + 4i$ | $x = \frac{1}{2}$, $y = 4$

$\therefore u(x,y) = 6x - 2y = 6 \cdot \frac{1}{2} - 2 \cdot 4$

$= 3 - 8 = -5$

$v(x,y) = 2x - 6y$

$= 2 \cdot \frac{1}{2} - 6 \cdot 4$

$= 1 - 24$

$= -23$

$\therefore f(z) = u(x,y) + iv(x,y)$

$= -5 + i(-23)$

$= -5 - 23i$

(3) $\dfrac{(6+i)(2-i)}{(4+3i)(1-3i)}$

$$= \frac{12-6i+2i-i^2}{4-12i+3i-9i^2}$$

$$= \frac{(13-4i)}{(13-9i)}$$

$$= \frac{(13-4i)(13+9i)}{(13-9i)(13+9i)}$$

$$= \frac{169-52i+117i-36i^2}{169-81i^2}$$

$$= \frac{205+65i}{250}$$

$$= \frac{205}{250} + \frac{65}{250} i$$

$$= \frac{41}{50} + \frac{13}{50} i = a+ib$$

where, $a = \frac{41}{50}$ & $b = \frac{13}{50}$

(4.) $\left(\dfrac{2+i}{3-i}\right)^2$

$$= \frac{(2+i)^2}{(3-i)^2}$$

$$= \frac{4+4i+i^2}{9-6i+i^2}$$

$$= \frac{3+4i}{8-6i}$$

$$= \frac{(3+4i)(8+6i)}{(8-6i)(8+6i)}$$

$$= \frac{24+18i+32i+24i^2}{64-48i+48i-36i^2}$$

$$= \frac{50i}{100}$$

$$= \frac{0}{100} + \frac{50}{100} i$$

$$= 0 + \frac{1}{2} i$$

$r\cos\theta = 0$

$r\sin\theta = \frac{1}{2}$

$r^2\cos^2\theta + r^2\sin^2\theta = 0^2 + \left(\frac{1}{2}\right)^2$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = \frac{1}{4}$

$\Rightarrow r^2 = \frac{1}{4}$

$\Rightarrow r = \frac{1}{2}$

$r\cos\theta = 0$

$\Rightarrow \frac{1}{2}\cos\theta = 0$

$\Rightarrow \cos\theta = 0 = \cos\frac{\pi}{2}$

$r\sin\theta = \frac{1}{2}$

$\Rightarrow \frac{1}{2}\sin\theta = \frac{1}{2}$

$\Rightarrow \sin\theta = 1$

$\Rightarrow \sin\theta = \sin\frac{\pi}{2}$

$\therefore z = r\cos\theta + i r\sin\theta$

$= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\frac{1}{2}$

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