Fourier Series Solution of the Heat Equation:

Example: Consider the boundary value problem of determining the temperature distribution $u(x,t)$ in a thin, homogeneous bar of length L, given the initial temperature throughout the bar and temperature at both ends at all times. The boundary value problem is:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad ; \quad (0<x<L, \; t>0)$$

Boundary Conditions (temperatures at both ends):

$$u(0,t) = 0 = u(L,t), \quad t>0$$

Initial ~~condition~~ value or condition (Initial temperature):

$$u(x,0) = f(x), \quad (0<x<L)$$

Solve the problem to find the solution function $u(x,t)$.

Sol^n Let $u(x,t) = X(x)\,T(t)$

Then, $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left[\dfrac{\partial}{\partial x}(XT)\right] = \dfrac{\partial}{\partial x}(X'T) = TX''$

and $\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial t}[XT) = XT'$, and we can ~~write~~ re-write the given p.d.e. as:

$$XT' = a^2 X'' T$$

$u_n(x,t) =$ [illegible]

$$\Rightarrow \frac{T'}{a^2T} = \frac{X''}{X}$$

Since, ~~t and~~ left side depends only on time (t) and the right side only on ~~time~~ position x, and t and x are independent we can write:

$$\frac{T'}{a^2T} = \frac{X''}{X} = \lambda$$

$$\Rightarrow X'' - \lambda X = 0 \quad \cdots (i) \quad \& \quad T' - a^2 T\lambda = 0 \quad \cdots (ii)$$

Now given $u(0,t) = 0 = u(L,t)$ we get two boundary conditions; B.C.-1:

$u(0,t) = X(0)T(t) = 0$ which implies two cases:

Case-I: $T(t) = 0$ is trivial case, and is not take

Case-II: $X(0) = 0$ is non-trivial case and must be chosen

Boundary Condition-2:

$u(L,t) = X(L)T(t) = 0$ which implies two cases:

Case-I: $T(t) = 0$ is trivial case and is not chosen

Case-II: $X(L) = 0$ is non-trivial case and must be taken

Thus, we get: $X(0) = X(L) = 0$; then from eqn. (i) we get:

$$(0 < x < L)$$

$$X'' - \lambda X = 0,$$

$$X(0) = 0, \quad X(L) = 0$$

(Is a Sturm-Liouville or Eigenvalue problem) and therefore, λ must be real, not complex) because to be for physically realistic, solution must be real valued)

Case-I $\lambda = 0 \Rightarrow X'' = 0, \Rightarrow X = Ax + B \Rightarrow X(0) = 0 + B$

$\Rightarrow B = 0$ & $[\because X(0) = 0]$ and $X(L) = AL + 0 \Rightarrow$

$0 = AL$ $[\because X(L) = 0]$ $\Rightarrow A = 0$; Then

we get trivial solution $X(x) = 0$ and we

$u(x,t) = X(x)T(t) = 0 \cdot T(t) = 0$ is also trivial solution of the given p.d.e., and can not be allowed. i.e., $\lambda = 0$ can not be chosen; &

Case-II $\lambda > 0$, and let $\lambda = a^2$ (for convenience) where $a > 0$. Then $X'' - a^2 X = 0$ has characteristic eqn:

$$\beta^2 - a^2 = 0 \Rightarrow \beta_1 = a \quad \& \quad \beta_2 = -a$$

with solutions: $X_1 = e^{\beta_1 x}$, $X_2 = e^{\beta_2 x}$

$\Rightarrow X_1 = e^{ax}$ and $X_2 = e^{-ax}$ and we general soln.

$$X = AX_1 + BX_2 = Ae^{ax} + Be^{-ax}$$

$\Rightarrow X(0) = Ae^0 + Be^0 = A + B = 0$ $[\because X(0) = 0]$

$\Rightarrow A = -B \Rightarrow X(L) = Ae^{aL} - Ae^{-aL} = A(e^{aL} - e^{-aL})$

But $X(L) = 0 \Rightarrow 0 = A(e^{aL} - e^{-aL}) \Rightarrow 0$

Thus $A = 0 = B \Rightarrow X(x) = Ae^{ax} + Be^{-ax} = 0 + 0 = 0$

$u(x,t) = 0$ ...

and then $u(x,t) = X(x)T(t) = 0$. $T(t) = 0$ gives ~~no positive no~~ trivial case also. Hence, no positive value for λ i.e. $\lambda > 0$ is not also allowed.

Case - III: Let $\lambda < 0$ say $\lambda = -a^2$, with $a > 0$.

Then $X'' + a^2 X = 0$ has characteristic equ.:

$$\beta^2 + a^2 = 0 \quad \Rightarrow \beta_1 = +ia \;\&\; \beta_2 = -ia$$

$$\Rightarrow X = A\cancel{X_1} + BX_2 = Ae^{iax} + Be^{-iax}$$

$$\Rightarrow X = A\cos(ax) + B\sin(ax)$$

Now, from $X(0) = 0 \Rightarrow 0 = A\cos(0) + B\sin(0)$

$\Rightarrow A = 0$ and from ~~X(L)~~ $X(L) = 0$ we get.

$0 = A\cos(aL) + B\sin(aL) = 0\cos(aL) + B\sin(aL)$

$\Rightarrow B\sin(aL) = 0$, but $B = 0$ gives trivial case only which ~~can~~ is not chosen. We have;

Therefore,

$$\sin(aL) = 0, \quad \text{where } a > 0.$$

$\therefore aL = n\pi$ for $n = 1, 2, 3, \dots$

$$\Rightarrow \lambda = -a^2 = -\frac{n^2\pi^2}{L^2} \Rightarrow \lambda_1 = -\frac{\pi^2}{L^2},\ \lambda_2 = -\frac{4\pi^2}{L^2},$$

$$\lambda_3 = -\frac{9\pi^2}{L^2}, \dots \text{ so on.}$$

Thus we get:

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)$$

That is for each positive integer n, we have a different Solution for λ and X. The λ_n's are called <u>eigenvalues</u> of the boundary-value problem; and $B_n \sin\left(\frac{n\pi x}{L}\right)$ are called the corresponding eigenfunctions.

Now, Let us solve for T from eqn. (ii)

$T' - a^2\lambda T = 0 \iff T' + \frac{n^2\pi^2a^2}{L^2}T = 0$ is a first order linear o.d.e. of the form:

$y' + P(x)y = q(x)$ with solution

$$y = e^{-\int p dx}\int q(x)\, e^{\int p dx}dx + ce^{-\int p dx}.$$

$$\Rightarrow T = e^{-\int \frac{n^2\pi^2a^2}{L^2}dt}\int (0).(\,)dt$$

$$\Rightarrow \cancel{T = e^{\int -\frac{n^2\pi^2a^2}{L^2}}} + Ce^{-\int \frac{n^2\pi^2a^2}{L^2}dt} = C_0 \exp\left(-\frac{n^2\pi^2a^2}{L^2}t\right)$$

$$\therefore T_n = C_n \exp\left(\frac{-n^2\pi^2a^2}{L^2}t\right) \text{ for } n = 1, 2, 3, \cdots$$

Thus, we get the solution function of the given boundary value problem as:

$$u_n(x, t) = D_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(\frac{-n^2\pi^2a^2t}{L}\right)$$

But to satisfy initial value $u(x,0) = \dots$

Let $f(x) = u(x,0) = u_n(x, \cancel{t})$

$$= D_1 \sin\left(\frac{\pi x}{L}\right) \exp\left(-\frac{\pi^2 a^2 t}{L^2}\right) + D_2 \sin\left(\frac{2\pi x}{L}\right) \exp\left(-\frac{4\pi^2 a^2 t}{L^2}\right)$$

$$+ \dots$$

$$= \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(\frac{-n^2\pi^2 a^2 t}{L^2}\right)$$ and for

$t = 0$, we get

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right)$$ which is

a Fourier sine series for $f(x)$ on $[0, L)$.

(Here $f(x)$ is at least sectionally continuous as because ~~rod~~ the bar is real)

$$\therefore D_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$ is chosen

(from defn of Fourier sine series)

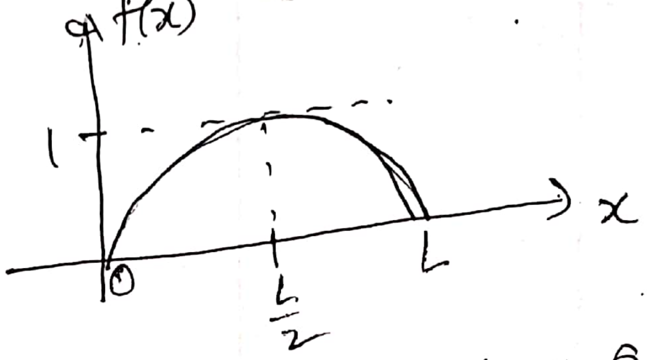
$\therefore$ The solution becomes:

$$\cancel{u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{L}\int_0^L f(\gamma) \sin\left(\frac{n\pi\gamma}{L}\right) d\gamma\right]}$$

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{L}\int_0^L f(\gamma) \sin\left(\frac{n\pi\gamma}{L}\right) d\gamma \sin\left(\frac{n\pi x}{L}\right) \exp\left(\frac{-n^2\pi^2 a^2 t}{L^2}\right)\right]$$

$\square$

(H.W.) Let $f(x) = \frac{4x}{L}\left(-\frac{x}{L}+1\right)$ ~~given~~ specified for $0 \leq x \leq L$ (for the above example):



Find $u(x,t)$ (by calculating ~~B~~ D_n).

<u>Sol^n:</u> $D_n = \frac{2}{L}\int_0^L \left[-\frac{4x^2}{L^2}\sin\left(\frac{n\pi x}{L}\right) + \frac{4x}{L}\sin\left(\frac{n\pi x}{L}\right)\right]dx$

$$= -\frac{16}{n^3\pi^3}\left[\cos(n\pi) - 1\right] = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{32}{n^3\pi^3}, & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{32}{n^3\pi^3}\sin\left(\frac{n\pi x}{L}\right)\exp\left(\frac{-n^2\pi^2a^2t}{L^2}\right)$$

for $n = 1, 3, 5, 7, \ldots$

$$= \sum_{n=1}^{\infty} \frac{32}{(2n-1)^3\pi^3}\sin\left[\frac{(2n-1)\pi x}{L}\right]\exp\left[-\frac{(2n-1)^2\pi^2a^2t}{L^2}\right]$$

$\square$

H.W. Solve Boundary Value Heat Equation:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad (0<x<L, \ t>0)$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, \quad t>0 \ \text{(insulated ends)}$$

$$u(x,0) = f(x), \quad (0<x<L) \ \text{(initial temp.)}$$

H.W. Solve the Boundary value wave equation:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \quad (0<x<L, \ t>0)$$

$$y(0,t) = y(L,t) = 0, \quad (t>0)$$

$$y(x,0) = 0, \quad \frac{\partial y}{\partial t}(x,0) = g(x).$$

H.W. Solve the Boundary Value Wave problem:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}; \quad (0<x<L, \ t>0)$$

$$y(0,t) = 0, \quad y(L,t) = 0$$

$$y(x,0) = f(x), \quad \frac{\partial y}{\partial t}(x,0) = g(x).$$