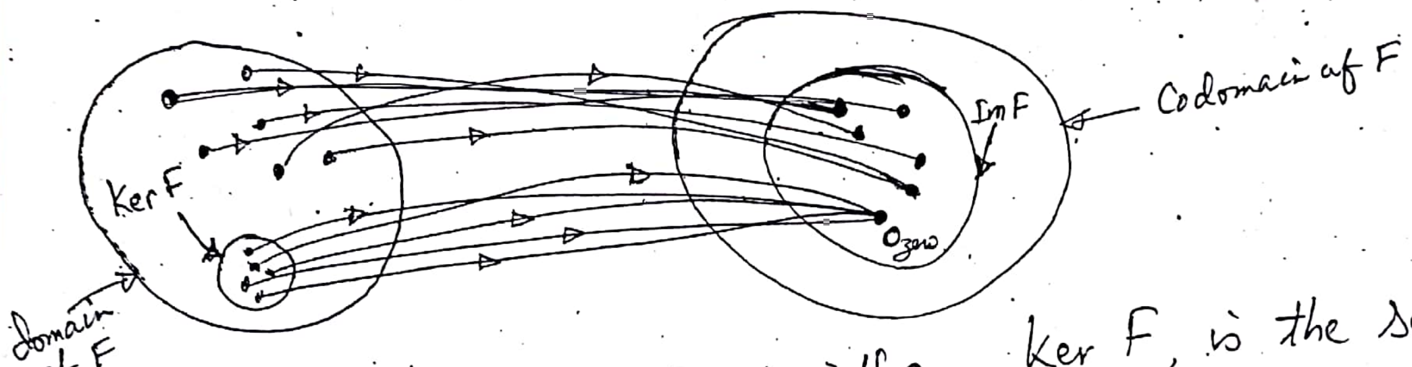
Kernel and Image (Range) of a Linear Transformation

Def^n: Let $F: V \to U$ be a linear mapping.

(i) The <u>image</u> of F, written $\operatorname{Im} F$, is the set of image points in U:

$$\operatorname{Im} F = \{u \in U : F(v) = u \text{ for some } v \in V\}$$



(ii) The <u>kernel</u> of F, written $\operatorname{Ker} F$, is the set of elements in V which map into $0 \in U$:

~~Ker F = { u ∈ U~~

$$\operatorname{Ker} F = \{v \in V : F(v) = 0\}$$

Thm: Let $F: V \to U$ be a linear transformation. Then (i) the image of F is a subspace of U and (ii) the kernel of F is a subspace of V.

pf. (i) We have to show that, (a) ~~Im~~ Ker F is non-empty, (b) ~~Im~~ Ker F is closed under vector addition; and (c) ~~Im~~ Ker F is closed under scalar multiplication.

First do Im F

(i) ~~Ker F is non-empty~~ Since F is a linear transformation, then:

(a) $F(0_V) = 0_U$, where 0_V and 0_U are the zero vectors in V and U, respectively.

Hence, Ker F is non-empty because $0_V \in \text{Ker}\, F$.

(b) Let $v_1, v_2 \in \ker F$, ~~and for any scalar~~ Then:

Then: $F(v_1) = 0$ vector and $F(v_2) = 0$ vector

Also, $F(v_1+v_2) = F(v_1) + F(v_2)$; [$\because$ F is linear transformation]

$= 0 + 0 = 0$ vector

That is, ~~$F(v_1), F(v_2) \in \text{Ker}\, F \Rightarrow F(v_1)+F(v_2) \in \ker F$~~ $v_1, v_2 \in \ker F \Rightarrow v_1 + v_2 \in \ker F$.

Hence, Ker F is closed under vector addition

(c) Let V and U be vector spaces over the field K, and let $\alpha \in K$. Then:

$F(\alpha v_1) = \alpha . F(v_1)$; [$\because$ F is a linear transformation]

$= \alpha \cdot (0_{vector})$ [$\because v_1 \in \ker F \Rightarrow F(v_1) = 0$]

$= 0$ vector

~~That is, $F(v_1) \in \ker F \Rightarrow \alpha F(v_1) \in \ker F$ for $\alpha \in K$.~~

That is, ~~$F(v_1) \in \text{Ker}\, F \Rightarrow \alpha F(v_1) \in \ker F$ for α any scalar.~~ $v_1 \in \ker F \Rightarrow \alpha v_1 \in \ker F$ for any scalar $\alpha \in K$.

Hence, Ker F is closed under scalar multiplication.

$\therefore$ Ker F is a subspace of V.

(ii) ~~Let $u_1, u_2 \in \text{Im}\, F$. Then there are vectors $v_1, v_2 \in V$ such that: $u_1 = F(v_1)$ and $u_2 = F(v_2)$~~

(ii) ⓐ Since F is a linear transformation zero vector of V is mapped into zero vector of U. Thus, $F(0_V) = 0_U \in U$ implies $0_U \in \operatorname{Im} F$. Hence $\operatorname{Im} F$ is non-empty.

do Im F first

ⓑ Let $u_1, u_2 \in \operatorname{Im} F$. ~~Then, there are vectors in~~ ~~such that~~ Then, there are vectors $v_1, v_2 \in V$ such that

$u_1 = F(v_1)$ and $u_2 = F(v_2)$; and then:

$u_1 + u_2 = F(v_1) + F(v_2) = F(v_1 + v_2)$; [$\because$ F is a linear transformation]

That is, $(u_1 + u_2)$ is image of $(v_1 + v_2)$ under F.

~~Hence~~ Thus, $u_1, u_2 \in \operatorname{Im} F \Rightarrow (u_1 + u_2) \in \operatorname{Im} F$.

Hence $\operatorname{Im} F$ is closed under vector addition.

ⓒ For any scalar $\alpha \in K$, we have:

$\alpha u_1 = \alpha F(v_1) = F(\alpha v_1)$; [$\because$ F is a linear transformation]

That is αu_1 is image of αv_1 under F.

Thus, $u_1 \in \operatorname{Im} F \Rightarrow \alpha u_1 \in \operatorname{Im} F$ for $\alpha \in K$.

Hence $\operatorname{Im} F$ is closed under scalar multiplication.

$\therefore$ $\operatorname{Im} F$ is a subspace of co-domain U. $\square$

Illustrations:

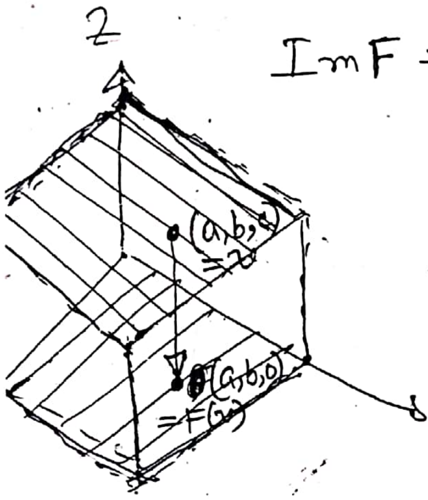
(a) Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ be (linear) projection mapping into the xy plane. That is,

$$F(x,y,z) = (x,y,0). \text{ Then:}$$

$$\text{Im}F = \{(a,b,0) : a,b \in \mathbb{R}\} = \text{entire } xy \text{ plane.}$$

$$\text{Ker}F = \{(0,0,c) : c \in \mathbb{R}\} = z \text{ axis.}$$

~~Since, $F(x,y,z) = (x,y,0) \Rightarrow F(0,0,c)$)~~



[Since $F(x,y,z) = (x,y,0)$ implies that $F(0,0,c) = (0,0,0)$ for any $c \in \mathbb{R}$].

(b) Let V be the vector space of polynomials over field $\mathbb{R}$ and let $T: V \to V$ be the third derivative operator (linear transformation, $V \to V$), that is,

$$T[f(t)] = \frac{d^3(f)}{dt^3}; \quad \text{then:}$$

$$\text{Ker}T = \{\text{polynomials of degree} \le 2\} \text{ and}$$

$\text{Im}T = V$; [$\because$ every polynomial $f(t)$ in V is the third derivative of some polynomial].

(c) Consider an arbitrary 4×3 matrix A over a field K:

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix};$$

which we view as a linear mapping. $A: K^3 \to K^4$. We know, the standard basis $\{e_1, e_2, e_3\}$ of K^3 spans K^3, ~~and so their values Ae_1, Ae_2, Ae_3 under~~ $\to$

The values of ~~Ae_1, Ae_2, Ae_3~~ of e_1, e_2, e_3 under A is Ae_1, Ae_2, Ae_3; and $\{Ae_1, Ae_2, Ae_3\}$ span the image of A. But the vectors Ae_1, Ae_2, and Ae_3 are columns of A as follows:

$$Ae_1 = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix}$$

$$Ae_2 = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}$$

$$Ae_3 = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{pmatrix}$$

Thus, image A (, a 4×3 matrix) under A is precisely the column space of A. On the other hand, the kernel of A consists of all vectors v for which $Av = 0$. This means that the kernel of A is the solution space of the homogeneous system $AX = 0$.

<u>Remark</u>: In general, if A is any $m\times n$ matrix which viewed as a linear mapping:

$$A: K^n \to K^m \text{ and}$$

$E = \{e_1, e_2, \ldots, e_n\}$ is the usual basis ~~for~~ of K^n, then –

$Ae_1, Ae_2, \ldots, Ae_n$ are the columns of A, and

$\text{Ker}\, A = \text{nullsp}\, A$ and $\text{Im}\, A = \text{Colsp}\, A$.

<u>Notes</u>: Let A be an arbitrary $m \times n$ matrix ~~of~~ over field K such as:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{nn} \end{pmatrix}$$, then the rows

~~of A are: $R_i = \{(a_{i1}, a_{i2}, \ldots, a_{in}\}$ for $i = 1, 2, \ldots, m$~~

of A are: $R_i = \{(a_{i1}, a_{i2}, \ldots, a_{in})\}$ for $i = 1, 2, \ldots, m$ may be viewed as vectors in K^n and hence they span a subspace of K^n called the <u>row space</u> of A and denoted by rowsp A. That is,

$$\text{row sp}\, A = \text{Span}\,(R_1, R_2, R_3, \ldots, R_m) = \text{is a subspace of } K^n.$$

The columns of A are: $\{C_j\}$ as:

$$C_j = \left\{ \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \right\} = \{(a_{1j}, a_{2j}, \ldots, a_{mj})\} \text{ for } j = 1, 2, \ldots, n$$

may be viewed as vectors in K^m and hence they span a subspace of K^m called the column space of A and denoted by ColspA. That is,

$$\text{Colsp}A = \text{span}(C_1, C_2, \cdots, C_n) = \text{rowsp}A^T$$

$$\because A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

~~Rank and Nullity~~