BASIS AND DIMENSION:

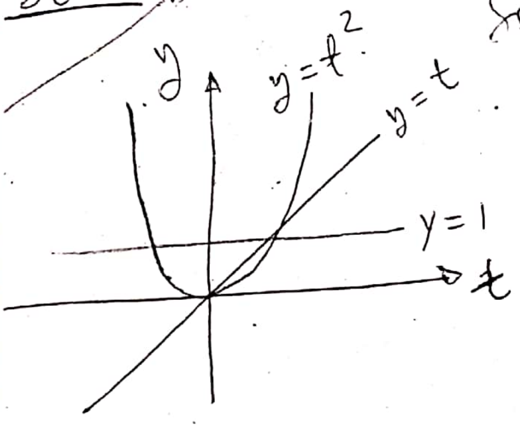
(A) **Basis**: If V is any vector space and $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ is a set of vectors in V, then $\mathcal{B}$ is called a basis for V if:

(i) $\mathcal{B}$ is linearly independent; i.e, $\mathcal{B}$ is a linearly independent set that is as large as possible; and

(ii) $\mathcal{B}$ spans $V \Leftrightarrow V = \text{Span}\{v_1, v_2, v_3, \dots, v_n\}$; i.e., $\mathcal{B}$ is a spanning set that is as small as possible.

Example Let $S = \{1, t, t^2, t^3, \dots, t^n\}$. Show that S is a basis for P_n = $\{1, 1+t, 1+t+t^2, \dots, 1+t+\dots+t^n\}$

Soln → Any $p(t) \in P_n$ can ~~be~~ written as $p(t) = \alpha_0(1) + \alpha_1 t + \dots + \alpha_n t^n$



So, it is clear that every element of P_n can be represented by the linear combination of S. That is S spans P_n. We will next, show that S is linearly independent.

Let $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n$ are scalars such that

$$(\alpha_0 \cdot 1) + (\alpha_1 \cdot t) + (\alpha_2 \cdot t^2) + \dots + (\alpha_n \cdot t^n) = 0$$

$$\Rightarrow \alpha_0 = \alpha_1 = \alpha_2 = \dots = \alpha_n = 0 \Rightarrow S \text{ is linearly independent}$$

Hence, S is a basis for P_n.

Example: Let $v_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, and $v_2 = \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix}$. Determine if $\{v_1, v_2\}$ is a basis for $\mathbb{R}^3$. Is $\{v_1, v_2\}$ a basis for $\mathbb{R}^2$? Find a subspace spanned by v_1 and v_2.

Sol^n Let

$$A = [v_1, v_2] \text{ matrix } = \begin{bmatrix} 1 & -2 \\ -2 & 7 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Columns of A do not span $\mathbb{R}^3$. Hence $\{v_1, v_2\}$ is not a basis for $\mathbb{R}^3$. Again, since v_1 and v_2 are not elements of $\mathbb{R}^2$, they can not be a basis for $\mathbb{R}^2$ either. However, we can show that v_1 and v_2 are linearly independent:

$$\alpha_1 v_1 + \alpha_2 v_2 = 0 \Rightarrow \alpha_1 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix} = 0$$

$$\Rightarrow \left.\begin{array}{ll} \alpha_1 - 2\alpha_2 = 0 & \cdots (i) \\ -2\alpha_1 + 7\alpha_2 = 0 & \cdots (ii) \\ 3\alpha_1 - 9\alpha_2 = 0 & \cdots (iii) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \alpha_1 - 2\alpha_2 = 0 \\ -2\alpha_1 + 7\alpha_2 = 0 \\ \alpha_1 - 3\alpha_2 = 0 \end{array}\right. \Rightarrow -2\alpha_2 = -3\alpha_2$$

But since $-2 \neq -3 \Rightarrow \alpha_2 = 0$, ~~then (i) gives $\alpha_1 - 0 = 0$~~

then from (i) $\alpha_1 - 2(0) = 0 \Rightarrow \alpha_1 = 0$.

$\therefore \alpha_1 = \alpha_2 = 0 \Rightarrow \{v_1, v_2\}$ are linearly independent.

They span the subspace of $\mathbb{R}^3 = \text{Span}(v_1, v_2)$

That is, $\{v_1, v_2\}$ is a basis for $\text{span}(v_1, v_2)$, means $\{v_1, v_2\}$ is a basis for the subspace of $\mathbb{R}^3$ spanned by ~~v_1, v_2~~ v_1 and v_2. $\square$

Example: Let $v_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, $v_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}$

~~Find a basis for the subspace (W) spanned by {v1, v2, v3 and v4.~~ Find a basis for the subspace W spanned by $\{v_1, v_2, v_3, v_4\}$.

Soln Let $A = \text{matrix } [v_1 \; v_2 \; v_3 \; v_4] = \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix}$.

$$\Rightarrow A \sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 20 & 4 & -20 \\ 0 & -25 & -5 & 25 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 5 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & -3 & 4 \\ 6 & 2 & -1 \\ 2 & -2 & 3 \\ -4 & -8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 0 & 0 \\ 0 & 4 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot pivot

Since, columns of v_1 and v_2 are pivot columns, They form a basis ~~of Col A~~ $= W = \text{span}(v_1, v_2, v_3, v_4)$.

That is $\{v_1, v_2\}$ is a basis for W.

or, $\{v_1, v_3\}$ is also a basis for W.

Example. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\}$

Then every vector in H is a linear combination of v_1 and v_2 because:

$$\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};$$

Is $\{v_1, v_2\}$ a basis for H?

Soln. Let $s=0$, then $\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \neq v_1$ and $v_2 \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Let $s=1$, then $\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \neq v_1$ and $v_2 \neq \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

$\therefore$ v_1 and $v_2 \notin H$. Therefore, $\{v_1, v_2\}$ can not be a basis for H. In fact, $\{v_1, v_2\}$ is a basis for the plane of all vectors of the form $(\alpha_1, \alpha_2, 0)$. But H is only a line ~~set~~ of all vectors of the form $(\alpha, \alpha, 0)$. That is H is line ~~on~~ ~~and~~ on the plane formed by the first and second coordinate axes; and every point (ordered tripples) i.e. every vector in H maintains equal distance from the first and second coordinate axes. □

H.W. #19 Consider $\mathcal{B} = \{e_1, e_2, e_3, \ldots, e_n\}$, where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Show that $\mathcal{B}$ is a basis for $\mathbb{R}^n$.

Sol^n: Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \in \mathbb{R}$. We will show that $\alpha_1 e_1 + \alpha_2 e_2 + \ldots \alpha_n e_n = 0$ (vector) $\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$ to prove that $e_1, e_2, e_3, \ldots, e_n$ are linearly independent.

$$\Leftrightarrow \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n = 0_{\text{vector}} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \ldots + \alpha_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 0 + \ldots + 0 + 0 \\ 0 + \alpha_2 + 0 + \ldots + 0 \\ \vdots \\ 0 + 0 + \ldots + 0 + \alpha_n \end{bmatrix} \Leftrightarrow \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \end{bmatrix}$$

$$\Rightarrow \alpha_1 = 0, \quad \alpha_2 = 0, \ldots, \alpha_{n-1} = 0, \quad \alpha_n = 0$$

$$\Leftrightarrow \alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_{n-1} = \alpha_n = 0$$

Thus, $\mathcal{B} = \{e_1, e_2, e_3, \cdots, e_{n-1}, e_n\}$ is linearly independent.

Now, we need to show that any $v \in \mathbb{R}^n$ can be expressed as a linear combination of $e_1, e_2, \cdots, e_n$.

Let any vector $v \in \mathbb{R}^n$. Then:

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}; \text{ But } v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + v_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} v_1 + 0 + \cdots + 0 \\ 0 + v_2 + \cdots + 0 \\ \vdots \\ 0 + 0 + 0 + \cdots + v_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$\Leftrightarrow v = v_1 e_1 + v_2 e_2 + v_3 e_3 + \cdots + v_n e_n$. Since ~~That is,~~

any vector v can be ~~or~~ expressed as a suitable linear combination of $e_1, e_2, \cdots, e_n$, we conclude that $\mathcal{B} = \{e_1, e_2, \cdots, e_n\}$ spans $\mathbb{R}^n$.

Thus $\mathcal{B}$ is the linearly independent spanning set of vectors for $\mathbb{R}^n$. Hence, $\mathcal{B}$ is a basis for $\mathbb{R}^n$. $\square$

$\boxed{\text{H-W-\# 20}}$ Let $M_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,

$M_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $M_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Show that the set $\mathcal{B} = \{M_{11}, M_{12}, M_{21}, M_{22}\}$ is a basis for the vector space $R_{2\times 2}$ of 2×2 real matrices.

Sol

<u>Sol'n</u>: First we will prove that $\mathcal{B} = \{M_{11}, M_{12}, M_{21}, M_{22}\}$ is linearly independent:

Let $\alpha_1 M_{11} + \alpha_2 M_{12} + \alpha_3 M_{21} + \alpha_4 M_{22} = O_{2\times 2}$; for $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$.

$$\Leftrightarrow \alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \alpha_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \alpha_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} \alpha_1 + 0 + 0 + 0 & 0 + \alpha_2 + 0 + 0 \\ 0 + 0 + \alpha_3 + 0 & 0 + 0 + 0 + \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 = 0,\ \alpha_4 = 0 \Leftrightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

$\therefore$ We conclude that $\mathcal{B}$ is linearly independent.

Next, we will prove that any element of $\mathbb{R}_{2\times 2}$ can be expressed as a linear combination of $M_{11}, M_{12}, M_{21}, M_{22}$.

Let A be any element of $\mathbb{R}_{2\times 2}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\text{Then } \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a M_{11} + b M_{12} + c M_{21} + d M_{22}$$

$\therefore$ $\mathcal{B}$ spans $\mathbb{R}_{2\times 2}$. ~~Hence~~ Hence, $\mathcal{B}$ is the linearly independent spanning set of vectors for $\mathbb{R}_{2\times 2}$. Thus, $\mathcal{B}$ is a basis for $\mathbb{R}$

H.W.-#21 Show that $\mathcal{B} = \{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space P_n = {set of all polynomials of degree $\le n$.}

Soln: First we will show that $\{1, x, x^2, \ldots, x^n\}$ is a linearly independent set of vectors.

Let $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ such that: $\alpha_0(1) + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n = 0$; Hence since, $1 \ne 0$, and x can be ~~any~~ of any value. It, thus implies that for $x \ne 0$, $\alpha_0 = \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Therefore, $\mathcal{B}$ is linearly independent. Next, we will show that any element of P_n can be expressed as linear combination of the elements of $\mathcal{B}$. Let $p(x) \in P_n$, then

$$p(x) = \alpha_0 1 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n$$

where $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ ~~ad~~ suitable for the validity of equality. ~~Hence $\mathcal{B}$ = 1, x,~~

Hence, $\mathcal{B} = \{1, x, x^2, \ldots, x^n\}$ spans P_n.

That is, $\mathcal{B}$ is a linearly independent ~~span~~ ~~set~~ spanning set of P_n. Thus $\mathcal{B}$ is a basis for P_n $\square$

Dimension: If V is any vector space and $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, then V is called a finite-dimensional vector space and is said to have dimension n, denoted by:

i.e. $\dim V = n$.

That is, the number of vectors in a basis is the dimension (of the vector space). If the vector space $V = \{0 \text{ vector}\}$, i.e., V is the zero vector space containing zero vector only, then its dimension is defined to be zero. On the other hand, if the vector space V is not spanned by a finite set, i.e., if a basis of V ~~has~~ does not have finite number of vectors, then V is said to be infinite-dimensional.

Examples: (i) $\dim \mathbb{R}^n = n$, because a basis for $\mathbb{R}^n = \{e_1, e_2, \cdots, e_n\}$
$= \{(1,0,0,\cdots,0), (0,1,0,\cdots,0), \ldots, (0,0,0,\cdots,1)\}$

(ii) $\dim R_{2\times 2} = 4$, because, a basis for $R_{2\times 2}$ is:

$$\{M_{11}, M_{12}, M_{21}, M_{22}\} = \left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}.$$

(iii) $\dim P_n = n+1$, because a basis for P_n is $\{1, x, x^2, x^3, \ldots, x^n\}$ has $n+1$ elements.

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