Partial Differential Equations

→ A p.d.e. contains one or more partial derivative;

→ many natural phenomena ~~are described by a p.d.e. of~~ involve 'interaction of three or more variables;

→ A solution of p.d.e. is a function which satisfies the equation;

For example, $u(x,t) = \cos(2x)\, e^{-4a^2 t}$ is ~~the~~ a solution function ~~for~~ of:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

→ Some (rare) p.d.e. can be solved by inspection; consider the eqn.:

$$\frac{\partial u}{\partial x} = -4\frac{\partial u}{\partial y} \quad \dots (i) \quad (1^{st}\ \text{order})$$

→ There may be many solutions to (i); but a simple one easy to guess;

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→ if $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ were both const[illegible] we, therefore could get a solution easily; try:

$$u(x, y) = Ax + By,$$ and then:

$$\frac{\partial u}{\partial x} = A \quad \& \quad \frac{\partial u}{\partial y} = B \Rightarrow A = -4B$$

and, we can choose $B = 1$, & $A = -$

and then $u = -4x + y$ is a solution;

→ A p.d.e. ~~eqn.~~ is of order n if it contains an n^{th} partial derivative, but none higher:

(i) Laplace's eqn. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ is second order;

(ii) Heat equation. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ is also second order; &

(iii) ~~$\frac{\partial^2 y}{\partial t^2}$~~ the wave eqn.:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$ is also of second order;

→ Since, the non-linear p.d.e. is ~~too~~ very difficult to solve, we begin with linear p.d.e. Such as:

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + g(x,y) = 0 \quad \cdots (ii)$$

~~is a general~~ is the general first order p.d.e. in three variables in which u is a dependent variables and x, y are the two independent variables

→ the general linear Second order p.d.e. in three variables has the form:

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + \cancel{d(x,y)\frac{\partial u}{\partial x}}$$

$$+ d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u + g(x,y) = 0 \quad \cdots (iii)$$

→ eqns. (ii) & (iii) given above are homogeneous if $g(x,y) = 0$ for all (x,y) under consideration and are nonhomogeneous if $g(x,y) \neq 0$ for some (x,y).

* in homogeneous cases we always s[...] a non-trivial solution, that is, one which is not identically zero;

Example: Find a nontrivial solution of:

$$\frac{\partial u}{\partial x} = 0$$

Sol^n Let $u = u(y)$, then

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(u(y)) = 0.$$

$\therefore$ $u = u(y)$, that is any function of y only will work!

H. W. Prob. #'s

Find the non-trivial solutions of the followings:

HW (1) $\frac{\partial^2 u}{\partial x^2} = 0$

Sol^n $\frac{\partial^2}{\partial x^2}(u) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} u\right)$

$\Rightarrow$ ~~Let $\frac{\partial}{\partial x}(u) = f(y)$, then~~

~~Then: $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}u\right) = \frac{\partial}{\partial x} f(y)$ [=0]~~

Let $u = x + f(y)$, then:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[x + f(y)\right] = 1 + 0$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(1) = 0$$

$\therefore$ $u = x + f(y)$ is the general form of the solution function. $\square$

HW (2) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$

Sol^n: Let $u(x,y) = k(x-y)$, k constant

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left[k(x-y)\right] + \frac{\partial}{\partial y}\left[k(x-y)\right]$$

$$= \frac{\partial}{\partial x}(kx) - \frac{\partial}{\partial x}(ky) + \frac{\partial}{\partial y}(kx) - \frac{\partial}{\partial y}(ky)$$

$$= K - 0 + 0 - K = 0 \quad \square$$

$\therefore$ $u(x,y) = k(x-y)$; k constant is a solution.

checked

H.W (3) $\frac{\partial^2 u}{\partial x \partial y} = y$.

Sol^n Let $u(x,y) = \frac{1}{2} x y^2$; then

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{1}{2} x y^2\right)\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{1}{2} x (2y) + y^2 \cdot (0)\right]$$

$$= \frac{\partial}{\partial x}(xy) = x \frac{\partial}{\partial x}(y) + y \frac{\partial}{\partial x}(x)$$

$$= x \cdot (0) + y(1) = y. \quad \checkmark \square$$

H.W: #4 $\frac{\partial^4 u}{\partial x^4} - x = 0$

Sol^n: Let $u(x,y) = \frac{x^5}{120} + f(y)$, then

$$\frac{\partial u}{\partial x} = \frac{x^4}{24}; \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x^4}{24}\right) = \frac{x^3}{6}$$

$$\frac{\partial^3 u}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{x^3}{6}\right) = \frac{x^2}{2}.$$

$$\frac{\partial^4 (u)}{\partial x^4} = \frac{\partial}{\partial x}\left(\frac{x^2}{2}\right) = x$$ which matches the equation $\frac{\partial^4 u}{\partial x^4} - x = 0$ $\square$

$\Rightarrow$

Example: Given $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0$; $u(0,1) = 2$.

Find a solution satisfying the condition.

Sol^n ~~Let $u(x,y) = 2(x+y)$~~

~~Let~~ $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}$ ~~$= 0 \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$~~

~~Let $\frac{\partial u}{\partial x} = A$ & $\frac{\partial u}{\partial y} = B$~~ then

~~$\partial u = A\partial x = B\partial y$~~

~~u~~ If $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are both constant and equal we could get a solution easily; hence, we try:

$u(x,y) = Ax + Ay$, then

$\frac{\partial u}{\partial x} = A$ & $\frac{\partial u}{\partial y} = A \Rightarrow \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0$ ✓

$\therefore$ $u = A(x+y)$ is a general soln.

Now, we compute the particular Solution satisfying $u(0,1) = 2$;

$\Rightarrow 2 = A(0+1) \Rightarrow A = 2.$

Hence, $u = 2(x+y)$ is the particular solution satisfying the given conditions.

$\boxed{u(x,0) = 1}$

~~H.W. #5 Solve~~ $\cancel{\dfrac{\partial^2 u}{\partial x^2} = 0}$; $\cancel{u(1,1) = 1}$.

~~H.W. #6 Solve~~

~~Soln. If u = u(x) If u is function~~

~~Soln Let~~ $\cancel{u = Ax + Bf(y)}$, ~~then~~

$\cancel{\dfrac{\partial u}{\partial x} = A + 0 = A}$

$\cancel{\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}(A) = 0}$

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$\boxed{\text{H.W. \#5}}$ Solve $\dfrac{\partial^2 u}{\partial x^2} = 0$, $u(1,1) = 1$.

Sol^n

~~Let $u = Ax$, then $\frac{\partial u}{\partial x} = A$, $\frac{\partial^2 u}{\partial x^2} = 0$~~

Since, $u(1,1) = 1$ is only one value, we can easily solve for one unknown co-efficient.

Let $u(x,y) = A(x+y)$

then $\dfrac{\partial u}{\partial x} = A$; $\dfrac{\partial^2 u}{\partial x^2} = 0$

$\therefore$ ~~$u(x)$,~~ and $1 = A(1+1) = 2A \Rightarrow A = \frac{1}{2}$

$\therefore u(x,y) = \frac{1}{2}(x+y)$ will work!

$\therefore$ ~~$u = x$ will also work~~

$\boxed{\text{Note: } u(x,y) = u(x) = x \text{ will also work,}}$

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