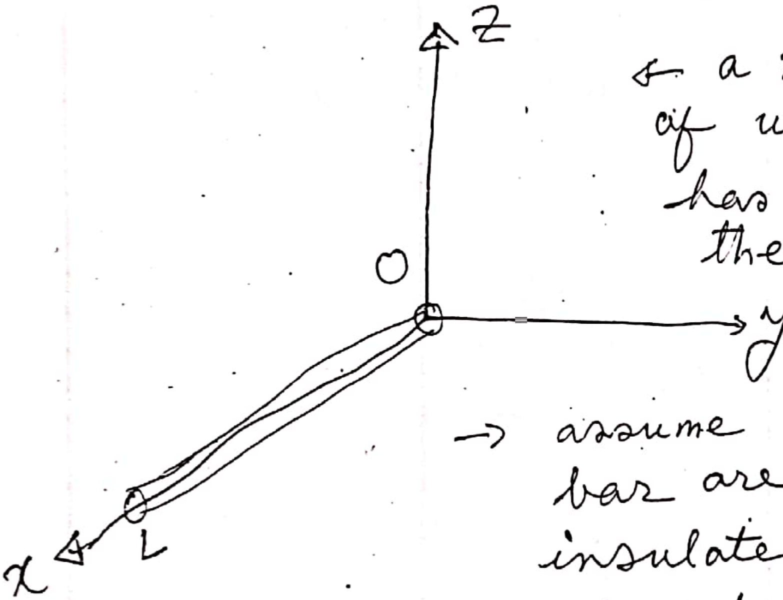
u = temperature = $u(x, t)$

THE HEAT EQUATION

→ ~~a ... is~~ a thin homogeneous bar of uniform cross-section is one which has thickness much smaller than the length;



→ a thin homogeneous bar of uniform cross section has been placed along the x-axis from 0 to L (in this figure);

→ assume that the sides of the bar are sufficiently well insulated that no heat energy is lost through them and that the bar is sufficiently thin that temperature ~~is sufficiently~~ at any given time is constant on any given cross section perpendicular to x-axis, (although of course it may differ on different cross-section);

→ to derive an eqn. for u, begin with experimentally observed fact, called Newton's Law of Cooling that the amount of heat energy per unit time passing between two parallel plates of area A, ~~distan~~

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distance d apart, the temperature T_1 and T_2, is proportional to:

$$\frac{A\,|T_1 - T_2|}{d}$$

and flows from the warmer to the cooler plate;

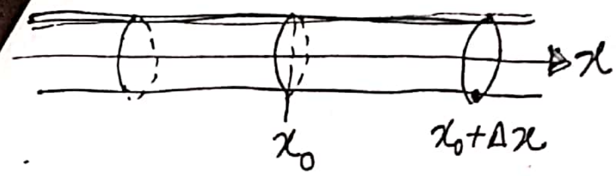
Let K be the constant of proportionality; then:

amount of heat energy per unit time flowing $= \frac{KA\,|T_1 - T_2|}{d}$ (from warmer to the cooler plate)

where, K is co-efficient of thermal conductivity and depends upon the material in the plates;

Now, by conservation of energy, the rate at which the heat flows into any portion of the bar [the "flux" term] must equal the rate at which that part of the bar absorbs heat energy (the absorption term); we shall ~~obtain~~ an equation for u by calculating the flux and absorption terms and setting them equal;

$\boxed{u = \text{temperature} = u(x,t)}$



→ For the flux term, let Δx between x_0 and $x_0+\Delta x$ is very small, then from Newton's law of cooling, the instantaneous rate of energy transfer from left to right across the section at x_0 at time t is:

$$R(x_0,t) = -\lim_{d\to 0} \frac{KA\left[u(x_0+\frac{d}{2},t)-u(x_0-\frac{d}{2},t)\right]}{d}$$

The minus sign in front of the limit is due to the fact that energy flows from left to right exactly when the temperature at the left of x_0 is greater than that to the right of x_0.

Similarly, the (instantaneous) rate of energy transfer from left to right at '$x_0+\Delta x$' and at time t is:

$$R(x_0+\Delta x,t) = -\lim_{d\to 0} \frac{KA\left[u(x_0+\Delta x+\frac{d}{2},t)-u(x_0+\Delta x-\frac{d}{2},t)\right]}{d}$$

the above two limits can be re-written as:

$$R(x_0,t) = -KA\frac{\partial u}{\partial x}(x_0,t) \; ; \; \& \; R(x_0+\Delta x,t) = -KA\frac{\partial u}{\partial x}(x_0+\Delta x,t)$$

where K and cross-sectional area A are constant.

Then net <u>rate</u> F at which heat energy <u>flows</u> into the portion between x_0 and $x_0+\Delta x$ is then:

$$F = R(x_0,t)-R(x_0+\Delta x,t) = KA\left[\frac{\partial u}{\partial x}(x_0+\Delta x,t)-\frac{\partial u}{\partial x}(x_0,t)\right]$$

u = temperature = $u(x, t)$

The amount of heat energy entering this portion of the bar in time Δt is then $F\Delta t$, or

$$Q_1 = F\Delta t = kA\left[\frac{\partial u}{\partial x}(x_0+\Delta x, t) - \frac{\partial u}{\partial x}(x_0, t)\right]\Delta t;$$ is the

<u>flux</u> term.

← For the absorption term, the average change Δu in temperature over the time Δt is directly proportional to the flux $F\Delta t$ and inversely proportional to the mass Δm. Thus, for some constant s (called the specific heat of the bar),

$$\Delta u = \frac{F\Delta t}{s\Delta m} = \frac{F\Delta t}{s\rho A\Delta x}$$

where ρ = density of conducting media, and therefore ~~$\Delta m = A\rho\Delta$~~ $\Delta m = A\Delta x\rho$.

Now, the average temperature change Δu is equal to the actual temperature change at some point $\bar{x}$ between x_0 and $x_0+\Delta x$, (where $\bar{x}$ is the center of mass) then:

$$\Delta u = u(\bar{x}, t+\Delta t) - u(\bar{x}, t) \doteq \frac{F\Delta t}{s\rho A\Delta x}$$ and then:

$$Q_2 = F\Delta t = s\rho A\left[u(\bar{x}, t+\Delta t) - u(\bar{x}, t)\right]\Delta x$$ where the right side is the absorption term and

we know: $Q_1 = Q_2 = F\Delta t$ which means:

$$kA\left[\frac{\partial u}{\partial x}(x_0+\Delta x, t) - \frac{\partial u}{\partial x}(x_0, t)\right]\Delta t = s\rho A\left[u(\bar{x}, t+\Delta t) - u(\bar{x}, t)\right]\Delta x$$

Now, dividing the above equation by $A\Delta x\Delta t$, we get:

$$k\left[\frac{\frac{\partial u}{\partial x}(x_0+\Delta x, t) - \frac{\partial u}{\partial x}(x_0, t)}{\Delta x}\right] = s\rho\left[\frac{u(\bar{x}, t+\Delta t) - u(\bar{x}, t)}{\Delta t}\right]$$

$u = \text{temp.} = u(x,t)$

t $\Delta x \to 0$ and $\Delta t \to 0$, and noting that $\bar{x} \to x_0$ as $\Delta x \to 0$; then:

$$k\frac{\partial^2 u}{\partial x^2} = s\rho\frac{\partial u}{\partial t}$$ and is often written in the form:

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$$

where $a^2 = \dfrac{k}{s\rho}$ is called the thermal diffusivity of the bar.

Now, to determine $u(x,t)$ for all $t \geq 0$ and $0 \leq x \leq L$, we need boundary conditions ~~data at the ends of~~ ends (i.e., the data at the ends of the bar) and initial value (i.e., temperature throughout the bar at time $t = 0$). For example, we may have the following boundary value problem:

(a)

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}; \quad (0 < x < L,\ t > 0)$$

boundary conditions: $u(0,t) = u(L,t) = T_1$, $(t > 0)$

initial ~~for~~ values (temp.): $u(x,t) = f(x)$, $(0 < x < L)$

This problem specifies that the ends of the bar are kept at constant temperature T_1 and that temperature at time $t = 0$ at point x is $f(x)$.

For another example, we may have insulation conditions as:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \; ; \; (0 < x < L, \; t > 0)$$

insulated boundary ends: $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0, \; (t > 0)$

initial temp.: $u(x,0) = f(x)$

In the above problem, boundary conditions specify no heat flow across the ends of the bar; hence the name insulation conditions.

In two dimensions, the heat equation is:

(c) $$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right);$$ and in 3-D the heat equations becomes:

(d) $$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$