(10 pages)

Fourier Series of Periodic functions:

→ A function defined over the whole real line (instead of $[-L, L]$) can also be expanded in a F.S. if it is periodic;

→ $f(x)$ is a periodic function of period $2L$, if $f(x+2L) = f(x)$ for all x; for example $\sin(x)$ is periodic of period 2π;

→ if $f(x)$ has period $2L$, then its graph from from L to $3L$ duplicates its graph from $-L$ to L; similarly, the graph from $3L$ to $5L$, $5L$ to $7L$, ..., $-3L$ to $-L$, $-5L$ to $-3L$, ... is identical to the graph between $-L$ to L;

→ if $f(x)$ is periodic of period $2L$, then the Fourier S. expansion on $[-L, L]$ automatically extends to the intervals $[L, 3L]$, $[3L, 5L]$, ..., $[-3L, -L]$, $[-5L, -3L]$, ... Convergence on each interval reflects convergence on $[-L, L]$, which in many cases, can be determined by Fourier Convergence Theorem discussed ~~earli~~ earlier.

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Fourier Sine and Cosine Series:

→ on an interval $[-L, L]$ the function $g(x)$ is <u>even</u> if $g(-x) = g(x)$ and ~~and~~ the graph from $-L$ to 0 (zero) looks like that from 0 to L ~~reflected~~ reflected back through the y-axis; (i.e., symmetric about the y-axis);

→ $g(x)$ is <u>odd</u> if $g(x) = -g(-x)$ and the graph from $-L$ to 0 (zero) looks like that which we would obtain by taking the graph from 0 to L and reflecting down through the ~~y-axis~~ x-axis and ~~back~~ then back across the y-axis (i.e., symmetric about the origin);

→ a function need not be either even or odd;

→ the graph $y = x^2$ is even on any interval $[-L, L]$;

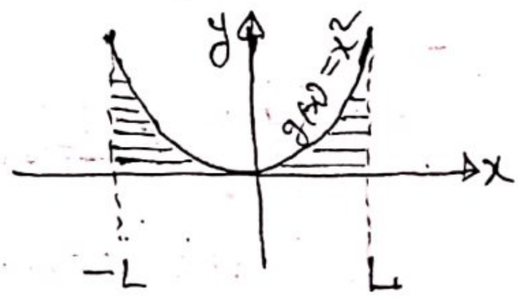
→ the graph $y = x^3$ is <u>odd</u> on any interval $[-L, L]$;

→ the graph $y = \cos(x)$ is even on $[-\pi, \pi]$;

→ the graph $y = 2x^2 + x - 1$ is neither even, nor odd on, say $[-4, 4]$;

→ if $g(x)$ is even on $[-L, L]$, then:

$$\int_{-L}^{L} g(x)\,dx = 2\int_{0}^{L} g(x)\,dx,$$ this is because the "area" under



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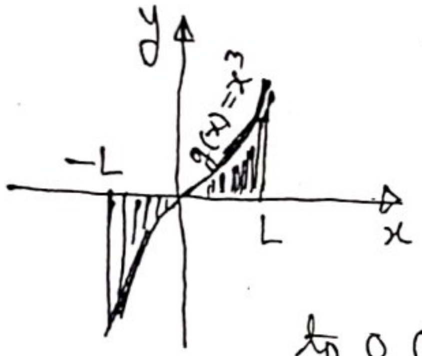
under $y = g(x)$ from $-L$ to 0 (zero) is exactly the same as that under the graph from 0 to L;



→ If $g(x)$ is odd on $[-L, L]$, then: $\int_{-L}^{L} g(x)dx = 0$; this is because the "area" determined from $-L$ to to 0 (zero) is negative to that from 0 to L;

→ a product of two even or two odd function is <u>even</u>;

→ a product of an even and an odd function is <u>odd</u>;

→ For example, if $f(x)$ is odd on $[-L, L]$, then the product $f(x) \cos\left(\frac{n\pi x}{L}\right)$ is odd and so each $a_n = 0$;

<u>Illustration</u>: The function $f(x) = x$ is odd on $[-\pi, \pi]$. The F.S. of $f(x) = x$ for $-\pi \le x \le \pi$ is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right],$$

where:

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$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos\left(\frac{n\pi x}{\pi}\right)dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin\left(\frac{n\pi x}{\pi}\right)dx \neq 0$$

$$\Rightarrow \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$= 0 + \sum_{n=1}^{\infty}\left[0 + b_n\sin\left(\frac{n\pi x}{L}\right)\right] = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx) \quad \square$$

→ If $f(x)$ is even, then $f(x)\sin\left(\frac{n\pi x}{L}\right)$ is odd, and so $b_n = 0$; and $f(x)\cos\left(\frac{n\pi x}{L}\right)$ is even, and $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$; thus, for example if $f(x) = |x|$, then $f(x) = |x|$ is even on $[-\pi, \pi]$ and the F.S. has only the Cosine ~~term~~ terms: $f(x) = |x| = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{-4}{(2n-1)^2\pi}\cos[(2n-1)x]$.

<u>In general</u>: (i) If $f(x)$ is even on $(-L, L)$, then only the cosine and constant terms $\left(\frac{a_0}{2}\right)$ appear in the F. expansion of $f(x)$ on $[-L, L]$; and

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(ii) If $f(x)$ is odd on $(-L, L)$, then only the sine term appears in the F. expansion on $[-L, L]$.

Fourier Cosine Series

The F. cosine series for $f(x)$ on $[0, L]$ is same as F. Series of $g(x)$ on $[-L, L]$, where:

$$g(x) = \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L \le x < 0 \end{cases}$$

; and the F. Cosine series of $f(x)$ on $[0, L]$ is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \text{ where, } a_0 = \frac{2}{L}\int_0^L f(x)\,dx \text{ and}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \text{ for } n = 1, 2, 3, \cdots$$

Example: Given $f(x) = e^{2x}$, $0 \le x \le 1$. Find the F. Cosine series on $[0, 1]$.

Sol^n $a_0 = \frac{2}{1}\int_0^1 e^{2x}\,dx = e^2 - 1$; and for $n = 1, 2, 3, \ldots$

$$a_n = \frac{2}{1}\int_0^1 e^{2x}\cos\left(\frac{n\pi x}{L}\right)dx = \frac{4}{4 + n^2\pi^2}\left[e^2\cos(n\pi) - 1\right]$$

(By doing integration by parts)

$\therefore$ The F. Cosine series of e^{2x} on $[0, 1]$ is:

$$\frac{e^2 - 1}{2} + \sum_{n=1}^{\infty} \frac{4}{4 + n^2\pi^2}\left[e^2\cos(n\pi) - 1\right]\cos(n\pi x)$$

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H.W. #39 Let $f(x) = \sin(x)$, $0 \le x \le \pi$. Find the f. Cosine series of $f(x)$.

Sol^n $a_0 = \frac{2}{\pi}\int_0^{\pi} \sin(x)\,dx = \frac{4}{\pi}$ and for $n = 1, 2, 3, \cdots$

~~$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\,dx =$~~ $a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos\left(\frac{n\pi x}{\pi}\right)dx$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx$$

$$= \begin{cases} 0, & \text{if } n=1 \\ \frac{1}{\pi(n-1)}\left\{1-\cos[(1-n)\pi]\right\} + \frac{1}{\pi(n+1)}\left\{1-\cos[(1+n)\pi]\right\}, & \text{if } n = 2, 3, 4, \cdots \end{cases}$$

$\because \sin(mx)\cos nx = \frac{1}{2}\sin[(m+n)x] + \frac{1}{2}\sin[(m-n)x]$

$\therefore$ The F. Cosine series of $\sin(x)$ on $[0, \pi]$ is:

$$\frac{2}{\pi} + \sum_{n=2}^{\infty}\left(\frac{1}{\pi(1-n)}\left\{1-\cos[(1-n)\pi]\right\} + \frac{1}{\pi(1+n)}\left\{1-\cos[(1+n)\pi]\right\}\right)\cos(nx).$$

Now, $\cos[(1-n)\pi] = (-1)^{1-n}$ and $\cos[(1+n)\pi] = (-1)^{1+n}$; and also, $(-1)^{1-n} = (-1)^{1+n}$; thus, the F. cosine series becomes:

$$\sin(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty}\frac{2}{\pi}\left[\frac{1-(-1)^{n+1}}{1-n^2}\right]\cos(nx)$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\sum^{\infty}\frac{\cos(2nx)}{1-4n^2} \quad \square$$

$\left(\because 1+(-1)^{n+1} = \begin{cases} 0, & n \text{ odd} \\ 2, & n \text{ even} \end{cases}\right)$

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<u>F. Sine Series</u>: The F. Sine series for $f(x)$ on $[0, L]$ is the same as F. Series of $h(x)$ on $[-L, L]$ where:

$$h(x) = \begin{cases} f(x), & 0 \le x \le L \\ -f(-x), & -L \le x < 0 \end{cases}; \text{ and the}$$

F. ~~Series~~ Sine Series for $f(x)$ on $[0, L]$ is:

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \text{ where}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \text{ for } n = 1, 2, 3, \cdots$$

<u>Example</u>: Given $f(x) = e^{2x}$, $0 \le x \le 1$. Find the F. sine series for $f(x)$ on $[0, 1]$.

<u>Sol^n</u> $b_n = \frac{2}{1}\int_0^1 e^{2x}\sin(n\pi x)dx = \frac{2n\pi}{4+n^2\pi^2}\left[1 - e^2\cos(n\pi)\right]$

Thus, the F. ~~series~~ sine series of e^{2x} on $[0,1]$ is:

$$\sum_{n=1}^{\infty} \frac{2n\pi}{4+n^2\pi^2}\left[1 - e^2\cos(n\pi)\right]\sin(n\pi x).$$

<u>H. W.</u> Given $f(x) = \sin(x)$, $0 \le x \le \pi$. Find the F. sine series of $f(x)$.

<u>Sol^n</u> $b_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\sin(nx)dx$

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$$= \begin{cases} 1, & \text{if } n=1 \\ 0, & \text{if } n = 2, 3, 4, \ldots \end{cases}$$

$\therefore$ The required F. sine series becomes:

$$1\left(\sin\left[\frac{(1)\pi x}{\pi}\right]\right) + (0) \cdot \sin\left(\frac{2\pi x}{\pi}\right) + \cdots$$

$$= \sin(x) + 0 + 0 + \cdots = \sin(x) \quad \square$$

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Lecture -24

P-9 1

Convergence of Fourier Cosine Series:

Let f(x) be sectionally continuous on [0, L]; then:

(i) If $0 < x_0 < L$ and both right and left derivatives exist at x_0 the Fourier cosine series of f(x) converges to:

$$\frac{1}{2}\left[\lim_{x\uparrow x_0} f(x) + \lim_{x\downarrow x_0} f(x)\right]$$

(ii) If f(x) is continuous at x_0, then ~~the~~ Fourier Cosine series of f(x) converges to $f(x_0)$ (at x_0); and

(iii) At 0, if $f_R'(0)$ exists, then the series converges to $f(0^+)$; at $x = L$, if $f_L'(L)$ exists, then ~~the~~ Fourier cosine series of f(x) converges to $f(L^-)$: ~~this is because~~

For $x = 0$

$$\frac{1}{2}\left[\lim_{h\to 0^+} f(0+h) + \lim_{h\to 0^+} f(0-h)\right]$$

$$= \frac{1}{2}\left[\lim_{h\to 0^+} f(h) + \lim_{h\to 0^+} f(-h)\right]$$

$$= \frac{1}{2}\left[f(0^+) + f(-0^+)\right] = \frac{1}{2}\left[f(0^+) + f(0^+)\right] = f(0^+)$$

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Similarly: ~~For~~ $x = L$

$$\frac{1}{2}\left[\lim_{h\to 0^+} f(L+h) + \lim_{h\to 0^+} f(L-h)\right]$$

$$= \frac{1}{2}\left[\lim_{h\to 0^+}\right.$$

Convergence of Fourier Sine Series:

Let $f(x)$ be sectionally continuous on $[0, L]$. Then:

(i) If $0 < x_0 < L$ and both right and left derivatives exist at x_0, then at $x = x_0$ the Fourier sine series of $f(x)$ converges to the average of the left and right limits at x_0.

(ii) If $f(x)$ is continuous at x_0, then Fourier sine series converges to $f(x_0)$; and

(iii) At 0 and L, ~~the~~ Fourier Sine Series converges to 0 (zero);

$$\therefore \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = 0 \text{ if } \begin{cases} x = 0 \text{ and} \\ x = L \end{cases}$$

for $n = 1, 2, 3, \ldots$

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