Rank and Nullity of a linear Mapping:

Def^n. Let A be an $m \times n$ matrix over field K:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ where } \begin{cases} m = \text{\# of rows} \\ n = \text{\# columns} \end{cases}$$

(i) The ~~space~~ subspace spanned by $\{R_1, R_2, \cdots R_m\}$, where $R_i = (a_{i1}, a_{i2}, \cdots, a_{in})$ is called row space. Since R_i's are vectors in K^n, they span a subspace in K^n.
Similarly, the column space is also defined in K^m as because each column vector has m components.

(ii) Row Rank = Dimension of Row space in which each vector has n entries;
= maximum number of independent Row vectors.

(iii) Column Rank = Dimension of Column Space
= maximum number of independent column vectors

(iv) Rank of Matrix $A_{m\times n}$ = Row Rank of A
= Column Rank of A } [Even if $m \neq n$]

(v) Null Space = Null A = set of all solutions to the homogeneous equation $AX = 0$.

$$= \{X : X \in \mathbb{R}^n \text{ and } AX = 0\}$$

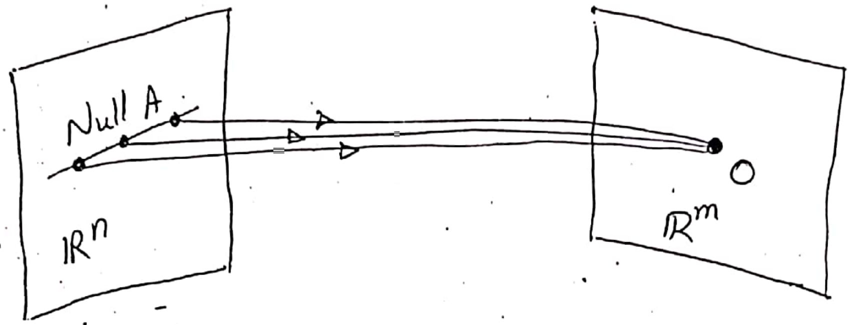
$$= \left\{ (x_1, x_2, \cdots, x_n) : \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{Vect}} = \underbrace{(0, 0, 0, \cdots, 0)}_{(m\text{-tuple})} \right.$$

Zero vector $\in \mathbb{R}^m$

= set of all $(x_1, x_2, \cdots, x_n) = X \in \mathbb{R}^n$ that are mapped into the zero vector of $\mathbb{R}^m$ ~~via~~ a via the linear transformation $F: X \to AX$

$\mathbb{R}^m$ (?)

$F: X \to AX \iff \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$



(vi) ~~Rank of~~ matrix A = number of pivot positions in A.

~~(vii) Rank A + dim Null A = # of columns of A = n~~
~~= # of pivot columns + # of non-pivot columns~~

(vii) The variables x_i corresponding to pivot columns in a matrix are called basic variables, and the other x_i's are called free ~~vatible~~ variables.

(# of basic variables) + (# of free variables)
= # of columns

(iii) Dimension of Null space of A = dim Null A
= Number of free variables in equation $AX = 0$ vector

Example : Find the dimension of the null space and column space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$\longrightarrow$

Sol^n

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \\ -3 & 6 & -1 & 1 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 5 & 10 & -10 \end{bmatrix}$$

pivot col.

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot col

Three
$\Rightarrow$ ~~4~~ free variables are x_2; x_4 and x_5

Hence, $\dim \text{Null } A = 3$

$\Rightarrow$ basic ~~vari~~ variables are two, i.e, x_1 and x_3.

$\therefore \dim \text{Col } A = 2 = \#$ of pivot columns.

Also, ~~the echelon~~ row reduced echelon form of the augmented matrix $[A \;\; 0]$ is

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

ix) Rank A + dim Null A = # of columns of $A = n$
= # of pivot columns of A + # of non-pivot columns of A.

) Let $F: V \to U$ be a linear mapping. Then $= \dim \text{Rng}(F)$
rank $F = \dim \text{Im} F = \dim R(F)$ and ~~Nullity F = di~~
Nullity $F = \dim (\text{Ker} F)$.

Theorem: Let V be of finite dimension and $F: V \to U$ be a linear mapping; then:

$$\dim V = \dim(\operatorname{Ker} F) + \dim(\operatorname{Im} F)$$
$$= \text{Nullity } F + \text{rank } F$$

Theorem: If $F: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there exists an $\underset{\substack{\uparrow \\ \text{row}}}{m} \times \underset{\substack{\uparrow \\ \text{column}}}{n}$ matrix A such that;

$$F(X) = AX \quad \text{for each } (x_1, x_2, \dots, x_n) = X \in \mathbb{R}^n.$$

Pf: Let $\{e_1, e_2, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$.

Then, $e_i = (0, 0, \dots, \underset{\substack{\uparrow \\ i\text{th co-ordinate}}}{1}, 0, 0, \dots, \underset{\substack{\uparrow \\ n\text{th coordinate}}}{0}), \dots, e_n = (0, 0, \dots, \underset{\substack{\uparrow \\ n\text{th coord}}}{1})$

Define,
$$T(e_i) = a_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}$$

Let $A = [a_{ij}]_{m \times n} = [a_1, a_2, \dots, a_n] = [T(e_1), T(e_2), \dots, T(e_n)]$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Now, if $X \in \mathbb{R}^n$, then
$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$$

Then applying T, we get:

$$T.X = T(x_1 e_1) + T(x_2 e_2) + \cdots + T(x_n e_n)$$
$$= x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n) ; [\because x_i \in \mathbb{R}]$$
$$= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$
$$= [a_1, a_2, \ldots, a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = AX$$

Thus, we have shows that each linear transformation T from $\mathbb{R}^n$ into $\mathbb{R}^m$ can be represented by multiplication by an $m \times n$ matrix A. We ~~refer~~ call the matrix

$$A = [a_{ij}]_{m \times n} = [a_1, a_2, \ldots, a_n]$$
$$= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

as the standard matrix of the linear transformation T.

Example: Find the standard matrix of the ~~transformation~~ linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 - x_3 \end{bmatrix}$$

Soln. $T(e_1) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1-0\\1-0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

$T(e_2) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0-1\\0-0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$

$T(e_3) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0-0\\0-1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$

Thus, we have:

$$[a_{ij}]_{2\times 3} = A = [a_{ij}]_{m\times n} = [a_1,\ a_2,\ a_3] = [T(e_1), T(e_2), T(e_3)]$$

$$= \begin{bmatrix}1 & -1 & 0\\1 & 0 & -1\end{bmatrix}$$

check: Let $(x_1, x_2, x_3) = X \in \mathbb{R}^3$, then

$$AX = \begin{bmatrix}1 & -1 & 0\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1(1) + x_2(-1) + x_3(0)\\x_1(1) + x_2(0) + x_3(-1)\end{bmatrix}$$

$$= \begin{bmatrix}x_1 - x_2\\x_1 - x_3\end{bmatrix} = TX = T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)$$ Q.E.D.

Example Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the transformation defined by:

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}x_1 - x_2\\x_1 - x_3\end{bmatrix}.$$

Find the nullity and rank of T.

Soln: If $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \ker T$, then $T(X) = 0$

$$\Rightarrow T(X) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_1 - x_2 &= 0 \quad \cdots (i) \\ x_1 - x_3 &= 0 \quad \cdots (ii) \end{aligned}\right\} \Rightarrow x_1 = x_2 = x_3.$$

$\Rightarrow$ each vector in $\ker(T)$ is a scalar multiple of the vector

$$u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

~~That~~ $\Rightarrow u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms ~~the~~ a basis for $\ker(T)$. That is, a basis of $\ker(T)$ has only one vector, $u = (1,1,1)$.

Thus, $\dim \ker(T) = 1$ and nullity of $T = 1$.

Next, if $v = \begin{bmatrix} a \\ b \end{bmatrix} \in \operatorname{Im} T \Leftrightarrow v = \begin{bmatrix} a \\ b \end{bmatrix} \in R^2(T)$,

then, $T(X) = v \Leftrightarrow T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 - x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

Therefore,

$$\begin{aligned} x_1 - x_2 &= a \\ x_1 - x_3 &= b \end{aligned}$$

Solving this system, we obtain

$$\left.\begin{aligned} x_1 &= a + x_2 \\ x_2 &= x_1 - a \\ x_3 &= x_1 - b \end{aligned}\right\}$$

Let $x_3 = t$ ~~$= x_1 - b$~~ Then, $t = x_1 -$

$\Rightarrow x_1 = b + t$ ~~$x_2 = x_1 - a$~~

$x_2 = x_1 - a = b + t - a$

Therefore, every $v = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ is the image $T(X)$ of a vector:

$$X = \begin{bmatrix} b+t \\ b-a+t \\ t \end{bmatrix} \text{ in } \mathbb{R}^3 \text{ for all } t \in \mathbb{R}.$$

~~Thus, $R(T) = \mathbb{R}^2$ and $\dim \mathbb{R}^2 = 2$~~

Thus, $\mathrm{Im}(T) = R(T) = \mathbb{R}^2$ and $\dim \mathbb{R}^2 = 2$.

Therefore, the rank of $T = 2$; and we have

nullity of T + rank of $T = 1 + 2 = 3 = \dim \mathbb{R}^3$

[where, $\mathbb{R}^3$ is the domain of T]. $\square$

H.W. Prob-#33 Let $T: \mathbb{R}^2 \to \mathbb{R}^1$ be a linear transformation for which:

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 2 \text{ and } T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = -3.$$

(a) Find $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$, and (b) Find $T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)$.

Soln: (a) Since $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$, we can express:

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \alpha_1\begin{bmatrix}1\\0\end{bmatrix} + \alpha_2\begin{bmatrix}1\\1\end{bmatrix}$$

where α_1 and α_2 are scalars. Then:

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}\alpha_1+\alpha_2\\\alpha_2\end{bmatrix} \Rightarrow \left\{\begin{array}{ll} x_1 = \alpha_1+\alpha_2 & \cdots(i) \\ x_2 = \alpha_2 & \cdots(ii)\end{array}\right.$$

From (i) and (ii) we get:

$\alpha_2 = x_2 \Rightarrow \alpha_1 = x_1 - \alpha_2 = x_1 - x_2$. Thus:

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \alpha_1\begin{bmatrix}1\\0\end{bmatrix} + \alpha_2\begin{bmatrix}1\\1\end{bmatrix} = (x_1-x_2)\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}1\\1\end{bmatrix}$$

Now, applying T, we get:

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = T\left((x_1-x_2)\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}1\\1\end{bmatrix}\right)$$

$$= T\left((x_1-x_2)\begin{bmatrix}1\\0\end{bmatrix}\right) + T\left(x_2\begin{bmatrix}1\\1\end{bmatrix}\right)$$

$$= (x_1-x_2)\,T\begin{pmatrix}1\\0\end{pmatrix} + x_2\,T\begin{pmatrix}1\\1\end{pmatrix}; \quad [\because x_1, x_2 \text{ are scalars}]$$

$$= (x_1-x_2)(2) + x_2(-3) = 2x_1 - 5x_2$$

(b) $T\left(\begin{bmatrix}2\\4\end{bmatrix}\right) = ?$

From (a) above, we have:

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = 2x_1 - 5x_2$$

$$\Rightarrow T\left(\begin{bmatrix}2\\4\end{bmatrix}\right) = 2(2) - 5(4) = 4-20 = -16 \;\square$$

H.W. Prob. #34 Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a matrix transformation such that:

$$T(\hat{i}) = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}; \quad T(\hat{j}) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\end{bmatrix};$$

and $T(\hat{k}) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\-1\end{bmatrix}$. (a) Find the matrix A such that $T(X) = AX$ and (b) find $T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$

Sol: (a) $T(X) = AX$ and $A = [a_{ij}]_{2\times3} = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\end{bmatrix}$

Let $X = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, then $T(X) = AX = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$,

$$= \begin{bmatrix}a_{11}x_1 + a_{12}x_2 + a_{13}x_3\\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3\end{bmatrix}$$

Now, from $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}$, we have:

we have:

$$\left.\begin{aligned} a_{11}\cdot(1) + a_{12}\cdot(0) + a_{13}\cdot(0) &= 1 \\ a_{21}\cdot(1) + a_{22}\cdot(0) + a_{23}\cdot(0) &= 1 \end{aligned}\right\} \Rightarrow a_{11} = 1 \text{ and } a_{21} = 1$$

From $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\end{bmatrix}$ we have:

$$\left.\begin{aligned} a_{11}\cdot(0) + a_{12}\cdot(1) + a_{13}\cdot(0) &= 2 \\ \text{and } a_{21}\cdot(0) + a_{22}\cdot(1) + a_{23}\cdot(0) &= 0 \end{aligned}\right\} \Rightarrow a_{12} = 2, \quad a_{22} = 0$$

From $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\-1\end{bmatrix}$ we have:

$$\left.\begin{aligned} a_{11}\cdot(0) + a_{12}\cdot(0) + a_{13}\cdot(1) &= 3 \\ a_{21}\cdot(0) + a_{22}\cdot(0) + a_{23}\cdot(1) &= -1 \end{aligned}\right\} \Rightarrow a_{13} = 3 \text{ and } a_{23} = -1$$

Hence, we obtain:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix}$$

(b) Let $X = \begin{bmatrix}1\\1\\1\end{bmatrix}$, then:

$$Ax = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}6\\0\end{bmatrix} = T(X)$$

Thus $T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}6\\0\end{bmatrix}$. $\square$

Linear Functionals and the Dual Sp

Let V be a vector space over a field K. Then a linear mapping $\phi : V \to K$ is termed a linear functional (or linear form). ~~If for every $v \in V$~~ ~~In other words~~ (or scalar valued funct

Illustrations

Examples (a) Let $\pi_i : K^n \to K$ be the ith projection linear mapping, i.e.,

$$\pi_i(a_1, a_2, \cdots, a_n) = a_i \Rightarrow \pi_1(a_1, a_2, \cdots, a_n) = a_1$$

$$\pi_2(a_1, a_2, \cdots, a_n) = a_2$$

$$\pi_3(a_1, a_2, \cdots, a_n) = a_3$$

$$\vdots$$

$$\pi_n(a_1, a_2, \cdots, a_n) = a_n.$$

Then π_i is a linear functional on K^n.

(b) Let V be the vector space of polynomials in t over $\mathbb{R}$. Let $J : V \to \mathbb{R}$ be the integral operator defined by:

$$J(p(t)) = \int_0^1 p(t)\,dt$$ is linear functional on V.

(c) Let V be the vector space of n-square matrices over K. Let $T : V \to K$ be the trace mapping:

$$T(A) = a_{11} + a_{22} + \cdots + a_{nn} \text{ where } V \ni A = [a_{ij}].$$

Then T is a linear functional on V.

Remark 1: Let V and U be vector spaces over a field K. Then the collection of all linear mappings from V into U forms a vector space over K and is usually denoted by $Hom(V, U)$ where "Hom" means homomorphism. If V and U are of finite dimensional, then:

$$\dim Hom(V, U) = mn; \quad \left[\text{where, } \dim V = m \text{ and } \dim U = n\right]$$

Dual Space: Let V be a vector ~~space~~ over the field K, then ~~the set of~~ K can viewed as a one-dimensional vector space over itself, even though V can be of one or multiple dimensions. The set of all linear maps of V into K is also a vector space over the field K, ~~called~~ and such set is denoted and defined by:

$$V^* = \mathcal{L}(V, K)$$

~~The vect~~ Such vector space V^* is known as the dual space, and the elements of V^* are called functionals. ~~Also dim V^* = m.1)~~
Also, $\dim V^* = m\cdot(1) = m$, $\left[\text{where, } m = \dim V; \text{ and } \dim K = 1\right]$

Remark 2: ~~For~~ Let V is a vector space of over the field K. ~~The V^*~~ Then

The dual space V^* is the set of all linear functionals from V into K. Also, since V^* is itself a vector space over K, then V^{**} is the set of all linear functionals from V^* into K, and V^{**} is called the <u>Second Dual Space</u>, and so on. That is, every vector space has a dual space (of linear functionals) and every dual space has a second dual space.

~~Senses → intellect~~ → intuition

<u>Remark 3</u>: Let V, U, and W are vector spaces over the same field K and let: be linear mapping $F: V \to U$ and $G: U \to W$

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[rr, bend left, "G \circ F"] & U \arrow[r, "G"] & W \end{tikzcd}$$

The the composition function $G \circ F$ is the mapping from V into W defined by:

~~$(G \circ F)(u)$~~ $(G \circ F)(v) = G(F(v))$, where, $v \in V$.

Whenever F and G are linear, $G \circ F$ is also linear.