Bases (continued)

Linear Dependence and Independence

Def^n: Let V be a vector space and $v_1, v_2, \ldots, v_n \in V$. We say that $v_1, v_2, v_3, \ldots, v_n$ are linearly dependent if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all equal to zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0 \text{ (vector)} \quad \ldots (i)$$

If no such scalars exist, ~~i.e., if (i) holds also holds~~ i.e, if $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$ implies $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$; then $v_1, v_2, \ldots, v_n$ are linearly independent. But if (i) also holds ~~for at least one of the $\alpha_i \neq 0$~~ for at least one of the $\alpha_i \neq 0$, then $v_1, v_2, \ldots, v_n$ are linearly dependent.

Independence Criterion:

Whenever

$$\alpha_1 v_1 + \alpha_1 v_2 + \ldots + \alpha_n v_n = 0$$

then

$$\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_n = 0$$

Remark 1: If ~~vector~~ 0 (vector) is an element of $\{v_1, v_2, v_3, \ldots, v_n\}$, then the set of vectors $\{v_1, v_2, v_3, \ldots, v_n\}$ must be linearly dependent. ~~Because~~, ~~So~~

check ~~Pf~~ Let $\alpha_1 \neq 0$, and $\alpha_2 = \alpha_3, \ldots, \alpha_n = 0$. Also, Let $v_1 =$ 0 vector. Then, if $\alpha_1 = 1$, we have

$$1.v_1 + 0v_2 + 0v_3 + \cdots + 0.v_{n-1} + 0.v_n$$

$$\cancel{1.v_1} = 1.0 + 0.v_2 + 0.v_3 + \cdots + 0.v_n = 0$$

i.e, even though the co-efficient of v_1 is not zero, the linear combination of $v_1, v_2, v_3, \ldots, v_n$ is zero. Hence, any set of vectors containing zero vector, is linearly dependent.

Remark 2: A set of just one vector which is not the zero vector, is, by itself, linearly independent.

~~Pf. $\alpha v = 0 \Rightarrow \alpha = 0$. □~~

~~Pf~~ Check: $\alpha v = 0$ and $v \neq 0 \Rightarrow \alpha = 0$ □

Remark 3: ~~If two vectors~~ If two of the vectors $v_1, v_2, v_3, \ldots, v_n$ are equal, or if one is a scalar multiple of the other ~~say~~ then $v_1, v_2, v_3, \ldots, v_n$ are linearly dependent

<u>Check</u>: Let $v_1 = k v_2$ and let $\alpha_1 = 1, \alpha_2 = -k$,

and $\alpha_3 = \alpha_4 = \alpha_5 = \cdots = \alpha_n = 0$.

Then $\alpha_1 v_1 + \alpha_2 v_2 + 0.v_3 + 0.v_4 + \cdots + 0.v_n$

$= 1.v_1 + (-k) v_2 + 0.v_3 + 0.v_4 + \cdots + 0.v_n$

$= v_1 - k v_2 + 0 + 0 \cdots + 0 = k v_2 - k v_2 + 0 = 0$

$\left[\therefore\ v_1 - k v_2 = 0 \Rightarrow v_1 = k v_2\right]$

~~$= v_1 - k v_2 + 0 = 0$~~

i.e, even though the co-efficients of two vectors v_1 and v_2 are not zero, their linear ~~term~~ combination is zero. Therefore, in this case are $v_1, v_2, \cdots, v_n$ linearly dependent.

Again, let, $v_1 = v_2$, and $\alpha_1 = 1, \alpha_2 = -1$ and $\alpha_3 = \alpha_4 = \cdots; \alpha_n = 0$

Then: $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n$

~~$= 1.v_1 + (-1) v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n$~~

~~$= \alpha_1 v_2 + \alpha_2 v_2 + 0.v_3 + \cdots + 0.v_n$~~

$= 1.v_1 + (-1) v_2 + 0.v_3 + 0.v_4 + \cdots + 0.v_n$

$= v_1 - v_2 + 0 + 0 \cdots + 0$

$= v_2 - v_2 + 0$

$= 0$

Again, even though the co-efficients of v_1 and v_2 are not zero, their linear combination is zero.

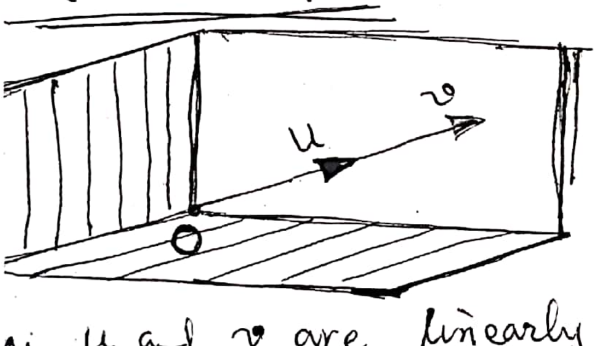
Remark 4: If the set $\{v_1, v_2, \ldots, v_m\}$ ~~Remark 6:~~

vectors is linearly independent, then any rearrangement of the vectors is also linearly independent.

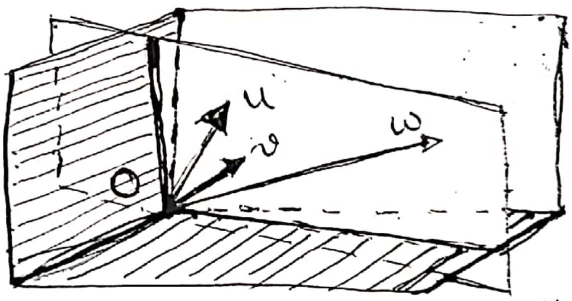
Remark 5: If a set S of vectors is linearly independent, then any subset of S is also linearly independent.

Remark 6: If S is ~~vector space~~ set of vectors. If S contains a linearly dependent subset, then S is linearly ~~dep~~ dependent.

Remark 7: In real space $\mathbb{R}^3$, linear dependence



g: u and v are linearly dependent



(b) Fig: u, v, w are linearly independent

f vectors can be described geometrically as ~~follows~~ above:

i) ~~Any~~ Any two vectors ~~are~~ u, v are linearly dependent iff they ~~are~~ lie on the same line through the origin. and

) any three vectors $u, v,$ and w are linearly dependent if and only if they lie on the same plane through the origin.

Example Show that vectors $u = (1,-1,0)$, $v = (1,3,-1)$ and $w = (5,3,-2)$ are linearly dependent.

Sol: We seek scalars α, β, γ such that

$$\alpha u + \beta v + \gamma w = 0 \text{ vectors}$$

$$\Rightarrow \alpha \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + \gamma \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha + \beta + 5\gamma \\ -\alpha + 3\beta + 3\gamma \\ 0 - \beta - 2\gamma \end{bmatrix}$$

$$\Rightarrow \alpha + \beta + 5\gamma = 0 \quad \text{---- (i)}$$

$$-\alpha + 3\beta + 3\gamma = 0 \quad \text{---- (ii)}$$

$$0 - \beta - 2\gamma = 0 \quad \text{---- (iii)}$$

From (iii) $\beta = -2\gamma$.

~~Let $\beta = 1$, then~~ Let $\gamma = 1$, then $\beta = -2$, then

from (i) $\alpha + (-2) + 5(1) = 0 \Rightarrow \alpha = -3$

$-3u - 2v + w = 0 \iff 3u + 2v - w = 0$.

Check: $3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - (1) \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = 3u + 2v - w$

$$3 + 2 - 5 = 0 = \alpha + \beta + 5\gamma$$

$$-3 + 6 - 3 = 0 = -\alpha + 3\beta + 3\gamma$$

$$0 - 2 + 2 = 0 = 0 - \beta - 2\gamma$$

Hence $u, v,$ and w are linearly dependent. $\square$

Example : Show that the vectors $u = (6,2,3,4)$, $v = (0,5,-3,1)$, and $w = (0,0,7,-2)$ are linearly independent.

Sol^n. Let α, β, γ are three unknown scalars.

We have to show that $\alpha u + \beta v + \gamma w = 0_{\text{vector}} \Rightarrow \alpha = \beta = \gamma = 0$.

We have, $\alpha u + \beta v + \gamma w = \alpha \begin{bmatrix} 6 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 5 \\ -3 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 7 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \left.\begin{array}{ll} 6\alpha + 0 + 0 = 0 & \text{---(i)} \\ 2\alpha + 5\beta + 0 = 0 & \text{---(ii)} \\ 3\alpha - 3\beta + 7\gamma = 0 & \text{---(iii)} \\ 4\alpha + \beta - 2\gamma = 0 & \text{---(iv)} \end{array}\right\} \Rightarrow$$

(i) gives $6\alpha = 0 \Rightarrow \alpha = 0$

(ii) $2\alpha + 5\beta + 0 = 0$

$\Rightarrow 2(0) + 5(\beta) = 0$

$\Rightarrow 0 + 5\beta = 0$

$\Rightarrow 5\beta = 0 \Rightarrow \beta = 0$.

Again, (iii) gives. $3\alpha - 3\beta + 7\gamma = 0 \Leftrightarrow 3(0) - 3(0) + 7\gamma = 0$

$\Leftrightarrow 0 - 0 + 7\gamma = 0 \Rightarrow 7\gamma = 0 \Rightarrow \gamma = 0$

Thus, $\alpha u + \beta v + \gamma w = 0 \Rightarrow \alpha = \beta = \gamma = 0$.

$\therefore$ u, v, w are linearly independent. $\square$

Lemma: Suppose two or more vectors $v_1, v_2, \ldots, v_m$ are linearly dependent. Then one of the vectors is a linear combination of the preceding vectors, that is, there exists a $m \geq k > 1$ such that:

$$v_k = a_1 v_1 + a_2 v_2 + \cdots + a_{k-2} v_{k-2} + a_{k-1} v_{k-1}.$$

→ for Lemma

Illustration: A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:

1. All non-zero rows are above any rows of all zeros.
2. Each leading entry of row is in column to the right of leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form (or reduced row echelon form).

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is only non-zero entry in its column.

Consider the following matrix in echelon form:

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 4 & -4 & 4 & -4 & 4 \\ 0 & 0 & 0 & 0 & 7 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 6 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Viewing the non-zero rows from the bottom up, R_4, R_3, R_2, R_1, no row is a linear combination of the previous rows. ~~Thus~~, the rows are linearly independent (by the ~~g~~ above ~~Lem~~ Lemma).

Example: ~~Show that~~ Let $P_2 = \{$polynomials $P(x)$ of degree $\leq 2\}$.

Let $\vec{P_1}(x)$, $\vec{P_2}(x)$, & $\vec{P_3}(x) \in P_2$ such that: $\vec{P_1}(x) = P_1(x) = 1$, $\vec{P_2}(x) = P_2(x) = 1+x$, $\vec{P_3}(x) = P_3(x) = 1+x+x$?

Show that $\vec{P_1}, \vec{P_2}, \& \vec{P_3}$ are linearly independent.

Sol^n: Let there exist $\alpha_1, \alpha_2, \alpha_3$ (scalars) such that: $\alpha_1 \vec{P_1} + \alpha_2 \vec{P_2} + \alpha_3 \vec{P_3} = \vec{0}$ (vector); then:

$$\alpha_1 P_1(x) + \alpha_2 P_2(x) + \alpha_3 P_3(x) = 0$$

$$\Rightarrow \alpha_1 (1) + \alpha_2 (1+x) + \alpha_3 (1+x+x^2) = 0$$

$$\Rightarrow \underbrace{(\alpha_1 + \alpha_2 + \alpha_3)} + \underbrace{(\alpha_2 + \alpha_3)} x + \alpha_3 x^2 = 0$$

$$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = 0 \quad \text{----} (i)$$

$$\alpha_2 + \alpha_3 = 0 \quad \text{----} (ii)$$

$$\alpha_3 = 0 \quad \text{----} (iii)$$

By (iii) in (ii), we get:

$\alpha_2 + 0 = 0 \iff \alpha_2 = 0$, then from (i):

$\alpha_1 + \alpha_2 + \alpha_3 = 0 \iff \alpha + (0) + 0 = 0 \Rightarrow \alpha_1 = 0$

$\therefore \alpha_1 = \alpha_2 = \alpha_3 = 0$.

Thus, $\{1, 1+x, 1+x+x^2\}$ is a linearly independent set of vectors. $\square$

CW-# 18 Show that functions $f(x) = e^x$ and $g(x) = e^{2x}$ [such that $x \in (-\infty, \infty)$] are ~~st~~ linearly independent.

Sol^n: Let there exist scalars a, b such that $ae^x + be^{2x} = 0$ for $x \in (-\infty, \infty)$; Then

$$\frac{d}{dx}(ae^x + be^x) = \frac{d}{dx}(0) \iff ae^x + 2be^{2x} = 0$$

Now, we have:

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$$ae^{x} + be^{2x} = 0 \quad \text{-----} \quad (i)$$

$$ae^{x} + 2be^{2x} = 0 \quad \text{-----} \quad (ii)$$

(ii) minus (i) gives $be^{2x} = 0 \Rightarrow b = 0$.

Then from (i) $ae^{x} + be^{2x} = 0 \Leftrightarrow ae^{x} + 0(e^{2x}) = 0$

$\Rightarrow ae^{x} = 0 \quad \Rightarrow \quad a = 0$; $\therefore \; a = b = 0$.

$\therefore$ The set of vectors $\{f(x), g(x)\} = \{e^{x}, e^{2x}\}$ is linearly independent.

H.W. # ~~Let~~ $\{v_1, v_2, v_3, \ldots, v_n\}$ be a set of n vectors in $\mathbb{R}^n$ and ~~let~~ A be an $n \times n$ matrix such that $v_1, v_2, \ldots, v_n$ form the column of A. We know that A ~~will~~ be

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