$\boxed{\text{solution}}$: $z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$

where, $r = |z|$

$= \sqrt{x^2+y^2}$, for $z = x+iy$;

$n = 4 \rightarrow k = 0, 1, \ldots\ldots n-1$; $\Rightarrow k = 0, 1, 2, 3$

θ = any angle of $\arg z = \operatorname{Arg} z = \arctan\left(\frac{y}{x}\right)$

Here, $z = 1-i \rightarrow r = \sqrt{(1)^2+(-1)^2}$

$= \sqrt{2}$

$= |z|$

$\theta = \arctan\left(\frac{-1}{1}\right)$

$= \arctan(-1)$

$= -\frac{\pi}{4}$

1

Thus, the required roots are :

(i) $k = 0 \rightarrow (1-i)^{1/4}$

$$= (\sqrt{2})^{1/4} \left[\cos\left(\frac{-\frac{\pi}{4}+0}{4}\right) + i\sin\left(\frac{-\frac{\pi}{4}+0}{4}\right)\right]$$

$$= \left(2^{1/2}\right)^{1/4} \left[\cos\left(-\frac{\pi}{16}\right) + i\sin\left(-\frac{\pi}{16}\right)\right]$$

$$= 2^{1/8} \left[\cos\left(\frac{\pi}{16}\right) - i\sin\left(\frac{\pi}{16}\right)\right] \sim 1.07 - 0.213\, i$$

$\square$

$$\text{(ii) } k=1 \rightarrow (1-i)^{1/4} = 2^{1/8}\left[\cos\left(\frac{-\frac{\pi}{4}+2\pi}{4}\right) + i\sin\left(\frac{-\frac{\pi}{4}+2\pi}{4}\right)\right]$$

$$= 2^{1/8}\left[\cos\left(\frac{\frac{-\pi+8\pi}{4}}{4}\right) + i\sin\left(\frac{\frac{-\pi+8\pi}{4}}{4}\right)\right]$$

$$= 2^{\frac{1}{8}}\left[\cos\left(-\frac{\pi}{16}+\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{16}+\frac{\pi}{2}\right)\right]$$

$$= 2^{\frac{1}{8}}\left[\cos\left(\frac{7\pi}{16}\right) + i\sin\left(\frac{7\pi}{16}\right)\right] \sim 0.213 + 1.07\, i$$

$$\text{(iii) } k=2 \rightarrow (1-i)^{\frac{1}{4}} = 2^{\frac{1}{8}}\left[\cos\left(\frac{-\frac{\pi}{4}+4\pi}{4}\right) + i\sin\left(\frac{-\frac{\pi}{4}+4\pi}{4}\right)\right]$$

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$$= 2^{\frac{1}{8}} \left[\cos\left(\frac{\frac{-\pi+16\pi}{4}}{4}\right) + i\sin\left(\frac{\frac{-\pi+16\pi}{4}}{4}\right)\right]$$

$$= 2^{\frac{1}{8}} \left[\cos\left(\frac{-\pi+16\pi}{16}\right) + i\sin\left(\frac{-\pi+16\pi}{16}\right)\right]$$

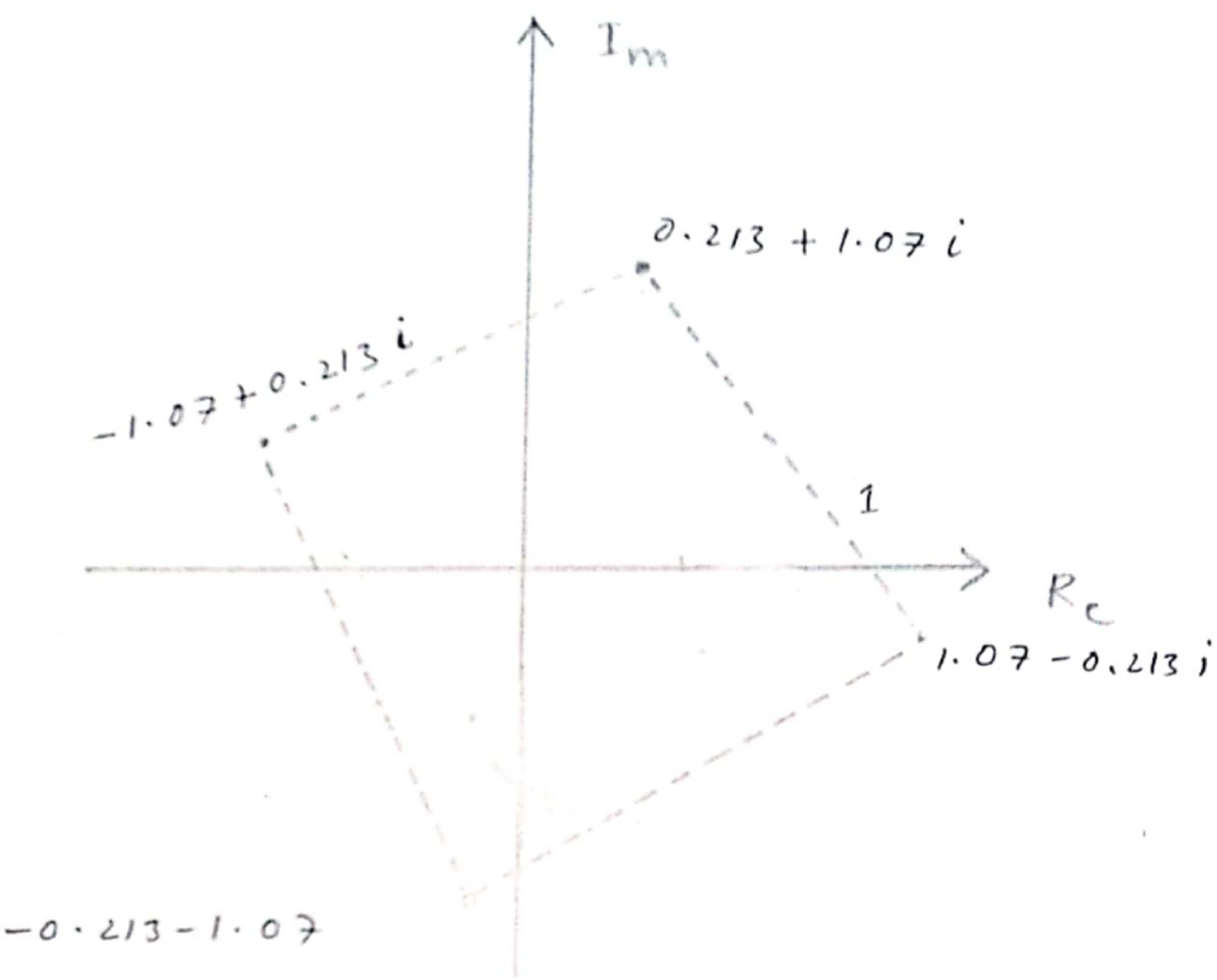
$$= 2^{\frac{1}{8}} \left[\cos\left(\frac{15\pi}{16}\right) + i\sin\left(\frac{15\pi}{16}\right) \sim 0.213 - 1.07\, i\right.$$

(iv) $K = 3 \rightarrow (1-i)^{1/4} = 2^{1/8}\left[\cos\left(\frac{-\frac{\pi}{4}+6\pi}{4}\right) + i\sin\left(\frac{-\frac{\pi}{4}+6\pi}{4}\right)\right.$

$$= 2^{\frac{1}{8}} \left[\cos\left(\frac{\frac{-\pi+24\pi}{4}}{4}\right) + i\sin\left(\frac{\frac{-\pi+24\pi}{4}}{4}\right)\right]$$

$$= 2^{\frac{1}{8}} \left[\cos\left(\frac{23\pi}{16}\right) + i\sin\left(\frac{23\pi}{16}\right)\right]$$

$$\sim -0.213 - 1.07\, i$$



Im
0.213 + 1.07 i
-1.07 + 0.213 i
1
Re
1.07 - 0.213 i
-0.213 - 1.07

$\boxed{\text{solution}}$: $f(z) = 1 - z$

$$\Rightarrow f[z(t)] = 1 - z(t)$$

$$= 1 - t + it^2$$

$z'(t) = 1 - 2it$

$$\Rightarrow f(z(t)z'(t)$$

$$= (1 - t + it^2)(1 - 2it)$$

$$= 1 - 2it - t + 2it^2 + it^2 - 2i^2t^3$$

$$= (1 - 2it - t) + 3it^2 + 2t^3$$

$$= (1 - t + 2t^3) + (3t^2 - 2t)i$$

2

$$= \int_c f(z)\,dz = \int_0^1 f(z(t)\; z'(t)\,dt$$

$$= \int_0^1 [1 - t + 2t^3] + i(3t^2 - 2t)\,dt$$

$$= \int_0^1 1\,dt - \int_0^1 t\,dt + 2\int_0^1 t^3\,dt + 3i\int_0^1 t^2\,dt - 2i\int_0^1 t\,dt$$

$$= \left[t\right]_0^1 - \frac{1}{2}\left[t^2\right]_0^1 + \frac{2}{4}\left[t^4\right]_0^1 + \frac{3i}{3}\left[t^3\right]_0^1$$

$$- \frac{2i}{i}\left[t^2\right]_0^1$$

$$= 1 - \frac{1}{2} + \frac{1}{2} + i - i$$

$$= 1$$

$\square$

3

$\boxed{\text{Solution}}$: Here c is a simple closed path, and the domain inside the c is simply connected.

But $0 \in D$, and $f'(z) = -\frac{1}{z^2}$ does not exist. Hence $f(z)$ is not analytic in the entire domain D.

Thus we can not apply Cauchy Integral Theorem.

But we can evaluate $\oint_c f(z)\,dz = \oint_c \frac{1}{z}\,dz$ as follows: We can re-write $c = z(\theta) = x + yi = r\cos\theta + ir\sin\theta$, for $0 \le \theta \le 2\pi$ $= \cos\theta + i\sin\theta$, [$\because$ for unit circle $r = 1$, $0 \le \theta \le 2\pi$] $\Rightarrow z(\theta) = e^{i\theta}$, $z'(\theta) = ie^{i\theta}$ and we can write $\int_c \frac{1}{z}\,dz =$

$$\oint_c \frac{1}{z}\,dz = \int_0^{2\pi} f(z(\theta))\,z'(\theta)\,d\theta = \int_0^{2\pi} \frac{1}{z(\theta)}\,z'(\theta)\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{e^{i\theta}}\,ie^{i\theta}\,d\theta = i\int d\theta = i\,[\theta]_0^{2\pi} = i\,[2\pi - 0] = 2\pi i$$

$\boxed{\text{Solution}}$: $\therefore \theta = \arccos\left(\frac{1}{\sqrt{3}}\right) \cong 54.74°$

Thus, $0 \leq \theta \leq \frac{\pi}{2}$, and we use the formula:

$$\vec{u} = \left(\frac{\vec{F}\cdot\vec{r}}{\|\vec{r}\|^2}\right)\vec{r} = \frac{\{3(1)\} + \{(-2)1\} + \{6(1)\}}{\left(\sqrt{1^2+1^2+1^2}\right)^2}(\hat{i}+\hat{j}+\hat{k})$$

$$= \left(\frac{3-2+6}{3}\right)(\hat{i}+\hat{j}+\hat{k})$$

$$= \frac{7}{3}(\hat{i}+\hat{j}+\hat{k})$$

$\square$

4

$\boxed{\text{Solution}}$: Here, c is swept out by the vector

$\vec{R}(t) = 2\cos(t)\,\hat{i} + 2\sin(t)\,\hat{j} + t\hat{k}$, $t \in [0, 2]$.

Now, tangent to c, $\vec{R}'(t) = -2\sin(t)\,\hat{i} + 2\cos(t)$ $\hat{j} + \hat{k}$.

we know, the length of c ;

$l(c) = \int_a^b \| R'(t) \| \, dt$, here $a = 0$ & $b = 2$.

$$\| \vec{R}(t) \| = \sqrt{\{-2\sin(t)\}^2 + \{2\cos(t)\}^2 + (1)^2}$$

$$= \sqrt{4(\sin^2 t + \cos^2 t) + 1}$$

$$= \sqrt{4(1) + 1}$$

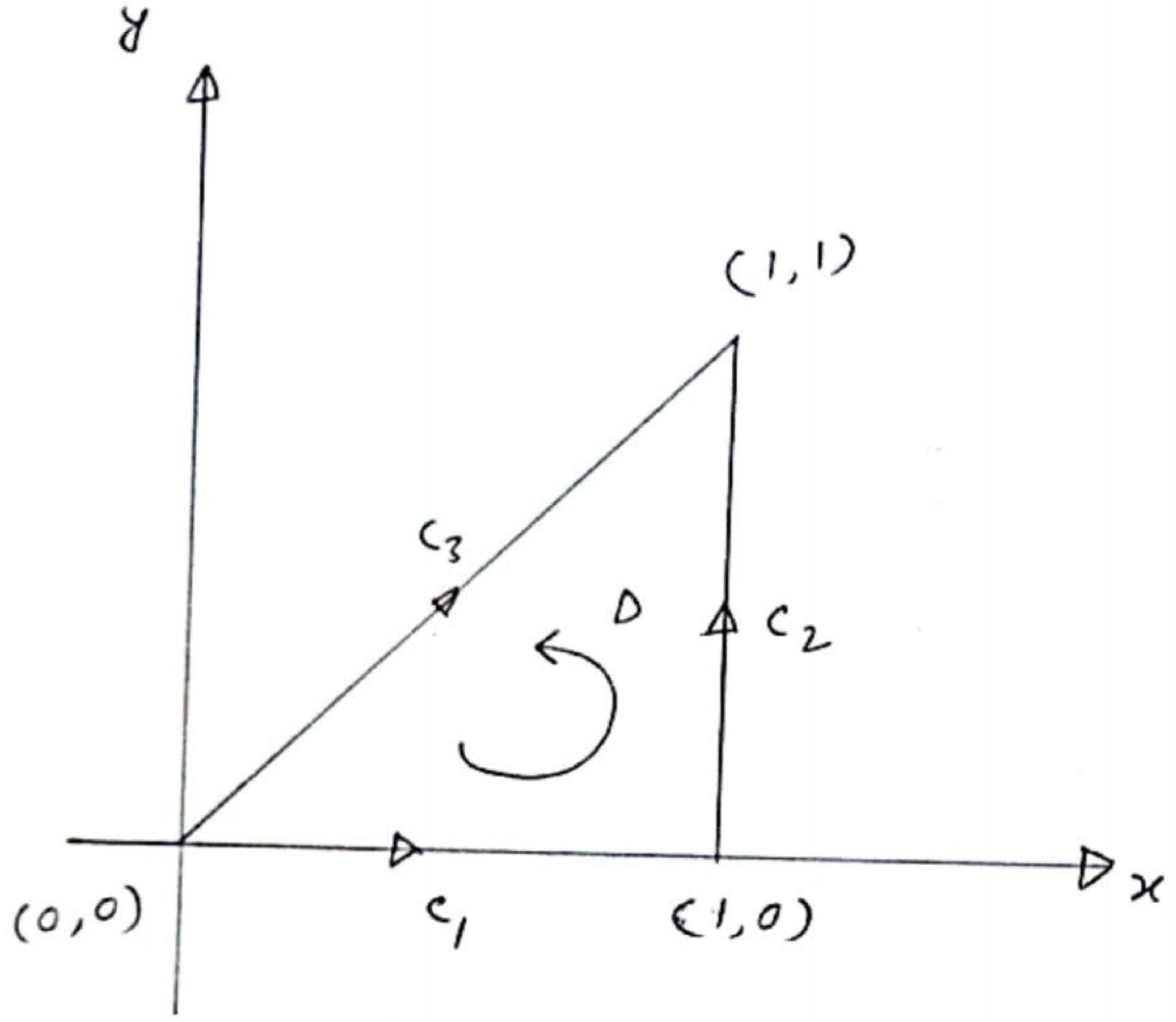
$$= \sqrt{5}.$$

5

$$\therefore l(c) = \int_0^2 \sqrt{5}\, dt = \sqrt{5} \int_0^2 dt = \sqrt{5}\, [t]_0^2$$

$$= 2\sqrt{5} \text{ unit}.$$

$\boxed{\text{Solution}}$:



Here, c consists of three smooth pieces :

c_1 , c_2 & c_3 as shown : we may parametrize

these as :

$c_1 : x = x , y = 0 ; x : 0 \to 1 ,$

$c_2 : x = 1 , y = y ; y : 0 \to 1 .$

$c_3 : x = y : x : 1 \to 0 .$

i.e $x(x) = x , y(x) = x ; x : 1 \to 0$

6

$= \int [F_1\, dx + F_2\, dy] = \int_{C_1} (F_1\, dx + F_2\, dy + \int_{C_2} (F_1\, dx +$

$F_2\, dy) + \int_{C_3} (F_1\, dx + F_2\, dy)$

$\int_{C_2} [F_1\, dx + F_2\, dy]$; $x(x) = x$, $y(x) = 0$, $x : 0 \to 1$

$dx = dx$, $dy = d(0) = 0$ (x is parameter)

$F_1\, (x(x), y(x)) = y(x) = 0$, $F_2\, (x(x), y(x) = 2(x)(0) = 0$

$\int_{C_1} F_1\, dx + F_2\, dy = \int_0^1 0\, dx + \int_0^1 0\, d(0) = 0$

For, $\int_{C_2} F_1\, dx + F_2\, dy$ we can use y as the parameter

For $\int_{C_2} F_1\, dx + F_2\, dy$ we get :

$x = x(y) = 1$, $x(y) = y$; $y : 0 \to 1$

$dx = d(1) = 0$, $dy = dy$, $F_1(x(y), y(y)) = y$, and

$F_2(x(y), y(y)) = 2(1)y = 2y$

$\therefore \int_{C_2} F_1\,dx + F_2\,dy = \int_0^1 y\,d(1) + 2y\,dy = \int_0^1 (0 + 2y\,dy)$

$= 2\int_0^1 y\,dy = 2\left[\frac{y^2}{2}\right]_0^1 = [y^2]_0^1 = 1^2 - 0^2$

$= 1$

Now, For $\int_C F_1\,dx + F_2\,dy$ we get :

$F_1(x(x), y(x)) = x$, $F_2(x(x), y(x)) = 2(x)(x)$

$dx = dx \qquad dy(x) = dx \qquad x : 1 \to 0$.

$\therefore \int_1^0 x\,dx = 2x^2\,dx = \left[\frac{x^2}{2}\right]_1^0 + \left[2\,\frac{x^3}{3}\right]_1^0$

$= \frac{1}{2}[x^2]_1^0 + \frac{2}{3}[x^3]_1^0 = \frac{1}{2}[0^2 - 1^2] + \frac{2}{3}$

$[0^3 - 1^3]$

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$$= -\frac{1}{2} - \frac{2}{3} = \frac{-3-4}{6} = \frac{-7}{6}$$ and thus, we get :

$$\int_{C_1} \vec{F} + \int_{C_2} \vec{F} + \int_{C_3} \vec{F} = 0 + 1 + \left(\frac{-7}{6}\right) = \frac{6-7}{6} = -\frac{1}{6}$$

$\square$

Thus we have found that :

$$\int_C F_1\, dx + F_2\, dy = \iint_D \left(\frac{\partial F_2}{dx} - \frac{\partial F_1}{\partial y}\right) dx\, dy$$

$\square$

Solution : Gauss's Divergence Formula is :

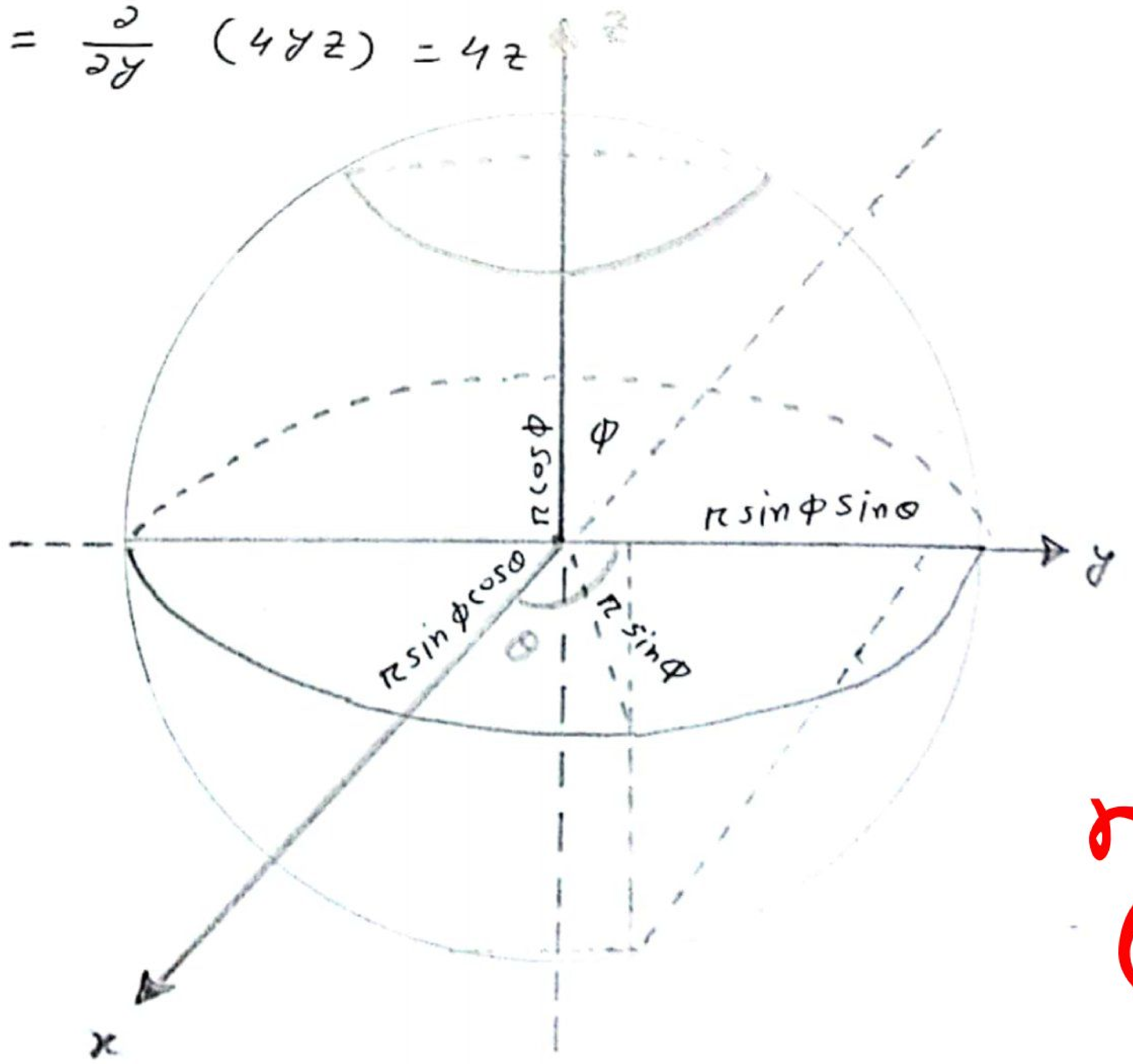
$$\iint_S \vec{F} = \iiint_V \vec{\nabla} \cdot \vec{F} \, dV$$

First , let us compute the

R.H.S

$$\vec{\nabla} \cdot \vec{F} = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot 4yz \hat{j}$$

$$= \frac{\partial}{\partial y} (4yz) = 4z$$



7

Hence in cartesian coordinates:

$$\iiint_V \vec{\nabla} \cdot \vec{F} \, dV = \iiint_V 4z \, dx\,dy\,dz$$

since, limits are given is spherical coordinates on S enclosing the sphere with radius R; we transform to spherical coordinates as: $4z = 4r\cos\phi$, $dx\,dy\,dz = r^2 \sin\phi \, dr\,d\theta\,d\phi$ with $0 \le r \le R$, $0 \le \theta \le 2\pi$, and $0 \le \phi \le \pi$; and we get:

$$\iiint_V 4z \, dx\,dy\,dz = \int_0^{\pi} \int_0^{2\pi} \int_0^{R} (4r\cos\phi) r^2 \sin\phi \, dr\,d\theta\,d\phi$$

$$= \int_0^{\pi} \int_0^{2\pi} \cos\phi \sin\phi \left[r^4\right]_0^R d\theta\,d\phi = R^4 \int_0^{\pi} \int_0^{2\pi} \cos\phi \sin\phi \, d\theta\,d\phi$$

$$= R^4 \int_0^{\pi} \cos\phi \sin\varphi \, [\theta]_0^{2\pi} \, d\phi = 2\pi R^4 \int_0^{\pi} \cos\varphi \sin\Phi \, d\varphi =$$

$$2\pi R^4 \int_0^{\pi} \frac{1}{2} \sin(2\varphi) \theta$$

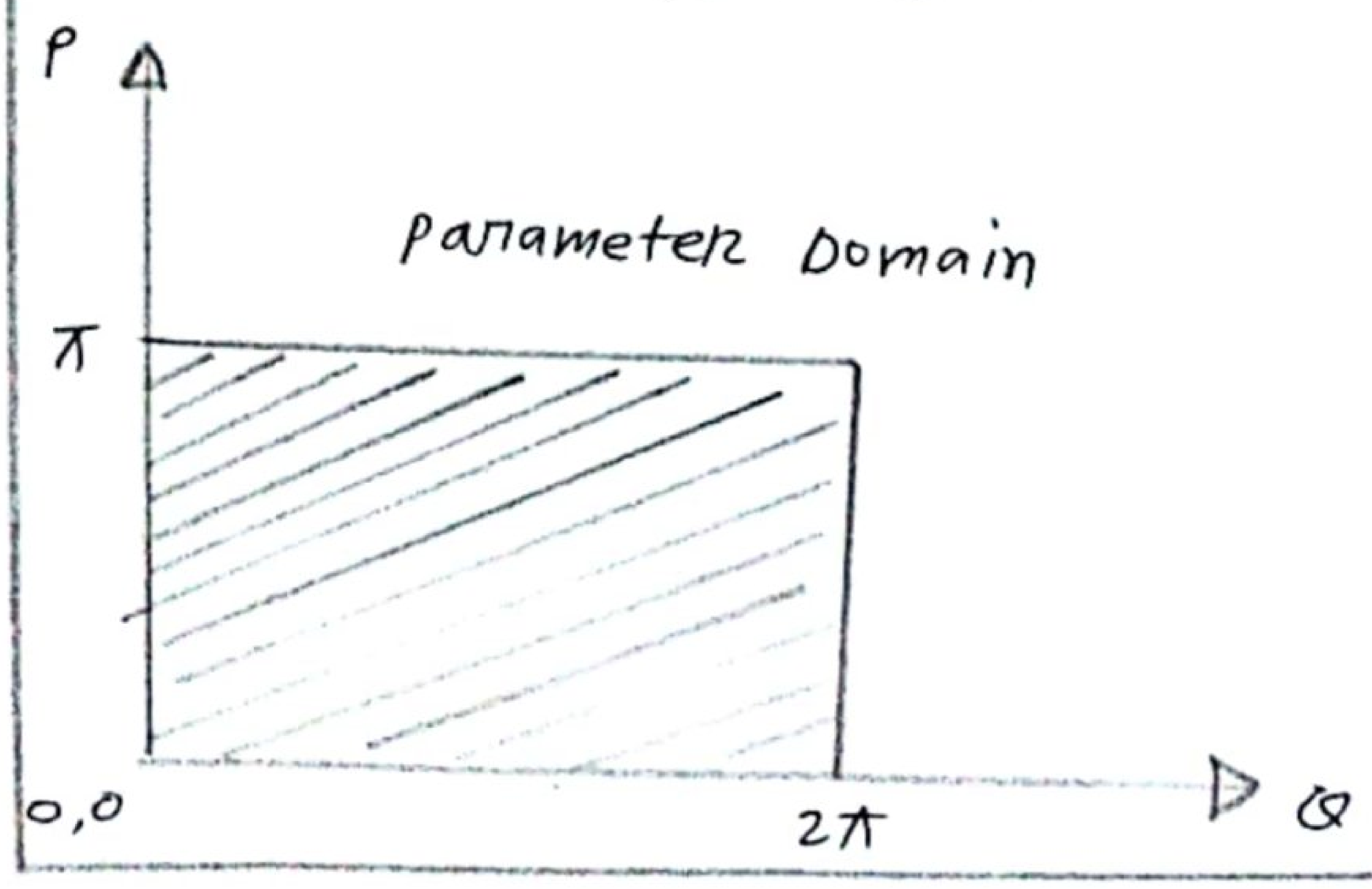
$$= \pi R^4 \int_0^{\pi} \sin(2\varphi) \, d\varphi = -\pi R^4 \left(\frac{1}{2}\right) [\cos(2\varphi)]_0^{\pi} =$$

$$-\frac{\pi}{2} R^4 [1-1] = 0$$

Now, we compute L.H.S :

$$\iint_S \vec{F} = \iint_D [\vec{F}(x,(\theta,\varphi), y(\theta,\varphi), z(\theta,\Phi)).$$

$\vec{N}(\theta,\varphi)] \, d\theta \, d\varphi$ where D is the parameter domain, $D = \{(\theta,\varphi) : 0 \le \theta \le 2\pi, 0 \le \varphi \le R\}$



Here $\vec{F}(x(\theta,\phi), y(\theta,\phi), z(\theta,\phi)) = 4yz\hat{j}$

$= 4r\sin\phi \sin\theta \cos\phi \,\hat{j} = 4r^2 \sin\theta \sin\phi \cos\phi \,\hat{j}$

Now, $\vec{N}(\theta,\phi) = \frac{\partial(y,z)}{\partial(\theta,\phi)}\hat{i} + \frac{\partial(z,x)}{\partial(\theta,\phi)}\hat{j} + \frac{\partial(x,y)}{\partial(\theta,\phi)}\hat{k}$

[in Jacobian notation]

$= \left(\frac{\partial y}{\partial z}\frac{\partial z}{\partial \phi} - \frac{\partial z}{\partial \theta}\frac{\partial y}{\partial \phi}\right)\hat{i} + \frac{\partial z}{\partial \theta}\frac{\partial x}{\partial \phi} - \frac{\partial x}{\partial \theta}\frac{\partial z}{\partial \phi}\Big)\hat{j}$

$+ \left(\frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \phi} - \frac{\partial y}{\partial \theta}\frac{\partial x}{\partial \phi}\right)\hat{k}$; where, we get :

$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(r\sin\phi\cos\theta) = -r\sin\phi\sin\theta$, $\frac{\partial x}{\partial \phi} = \frac{\partial}{\partial \phi}$

$(r\sin\phi\cos\theta) = r\cos\phi\cos\theta$

$\frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(r\sin\phi\sin\theta) = r\sin\phi\cos\theta$; $\frac{\partial y}{\partial \phi} = \frac{\partial}{\partial \phi}$

$(r\sin\phi\sin\theta) = r\cos\phi\sin\theta$

$\frac{\partial z}{\partial \theta} = \frac{\partial}{\partial \theta}(r\cos\phi) = 0$, $\frac{\partial z}{\partial \phi} = \frac{\partial}{\partial \phi}(r\cos\phi) = -r\sin\phi$

$\therefore \vec{N}(\theta,\phi) = [(r\sin\phi\cos\theta)(-r\sin\phi) - 0\,(r\cos\phi$

$\sin\theta)]\hat{i} + [0\,(r\cos\phi\sin\theta)$

$-(-r\sin\phi\sin\theta)(-r\sin\phi)]\hat{j} + [(-r\sin\phi\sin\theta)$

$(r\cos\phi\sin\theta) - r\sin\phi\cos\theta\,[r\cos\phi\cos\theta)]\hat{k}$

$= -r^2\sin^2\phi\cos\theta\, i - r^2\sin^2\phi\sin\theta\,\hat{j} + (-r^2\sin\phi$

$\sin^2\theta - r^2\sin\phi\cos\phi\cos^2\theta)\hat{k})$

$= -r^2\sin^2\phi\cos\theta\,\hat{i} - r^2\sin^2\phi\sin\theta\,\hat{j} - r^2\sin\phi\cos\phi$

$(\sin^2\theta + \cos^2\theta)\hat{k}$

$= -r^2\sin\phi\,[\cos\theta\sin\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\phi\,\hat{k}]$

$\therefore F\,(x(\theta,\phi), y(\theta,\phi), z(\theta,\phi))$.

$\vec{N}(\theta,\phi) = [4r^2\sin\theta\sin\phi\cos\phi\,\hat{j}]$.

$[-r^2\sin\phi\,\{\cos\theta\sin\theta\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\phi\,\hat{k}\}]$

$= (4r^2\sin\theta\sin\phi\cos\phi)(-r^2\sin\theta\sin^2\phi) = -4r^4\sin$

$\sin^3\theta \cos\phi$

$\therefore \iint_S \vec{F} = \int_0^{2\pi} \int_0^{\pi} (-4r^4 \sin^2\theta \sin^3\phi \cos\phi)$

$d\phi\, d\theta = -4r^4 \int_0^{2\pi} \sin^2\theta \left[\int_0^{\pi} \sin^3\phi \cos\phi \, d\phi\right] d\theta$

$= 0$

Thus L.H.S = R.H.S $\square$

$\boxed{\text{solution}}$: The distribution is modrately skewed as because its skewness $= 0.3$. Hence, we can use the following formula :

$$\text{Mean} - \text{Mode} = 3\,(\text{Mean} - \text{Median})$$

$$\rightarrow \text{Mean} - 50 = 3\,(\text{Mean} - 55)$$

$$\rightarrow 3\,\text{Mean} - \text{Mean} = 3\,(55) - 50 \rightarrow 2\,\text{Mean} = 115.$$

$$\therefore \text{Mean} = \frac{115}{2} = 57.5$$

□

Also, Given $Sk(P_1) = 0.3 = \dfrac{\text{Mean} - \text{Mode}}{\text{Standard Deviation}}$

$$\Rightarrow \text{Standard deviation} = \frac{57.5 - 50}{0.3} = \frac{7.5}{0.3} = 25$$

$$Sk(P_2) = \frac{3\,(\text{Mean} - \text{Median})}{\text{Standard Deviation}} = \frac{3\,(57.5 - 55)}{25}$$

$$= \frac{3\,(2.5)}{25} = \frac{7.5}{25} = 0.3$$ □

8

Thus, for this distribution $S_K(P_1) = S_K(P_2)$. $\square$

[That is the degree of skewness from both formulas is 0.3 and the skewness is positive].

$\boxed{\text{solution}}$: $P(x) = \binom{N}{x} p^x q^{N-x}$;

Here, $x = 2$, $N = 6$ $p = \frac{1}{2}$, $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$$\therefore P(2) = \frac{N!}{x!\,(N-x)!} \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right)^{N-x}$$

$$= \frac{6!}{2!\,(6-2)!} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{6-2}$$

$$= \frac{15}{64} \quad \square$$

9

$\boxed{\text{solution}}$: $z_1 = \frac{x_1 - \mu}{\sigma} = \frac{1-5}{2} = -2$ & $z_2 = \frac{x_2 - \mu}{\sigma}$

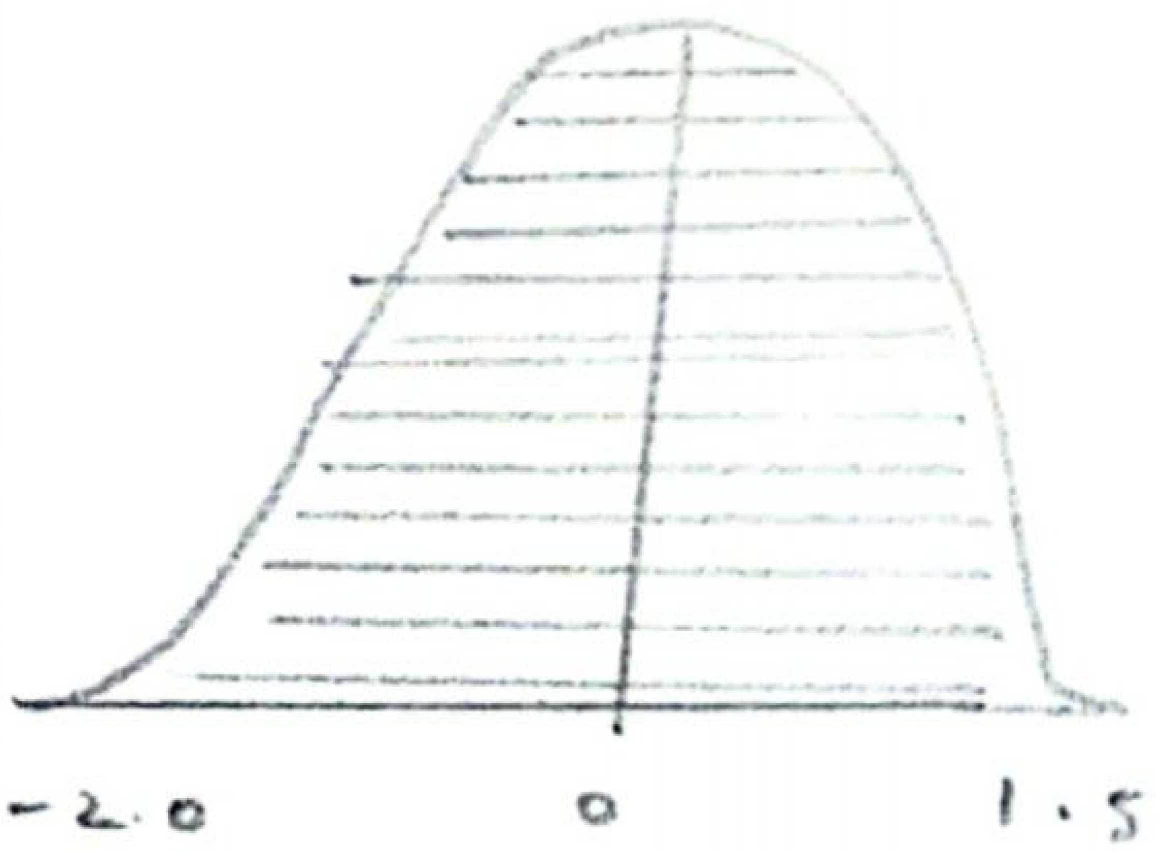
$= \frac{8-5}{2}$

$= 1.5$

$p(x_1 < x < x_2) = p(1 < x < 8)$

$= p(z_1 < z < z_2) =$

$p(-2 < z < 1.5)$



$= p(-\infty < z < 1.5) - p(-\infty < z < -2.0)$

$= \phi(1.5) - \phi(-2.0)$

$= \phi(1.5) - \{1 - \phi(2.0)$

$= 0.9332 - (1 - 0.9773)$

$= 0.9105.$

□

10