Subspaces (or Linear Manifolds):

- If a subset H of a group G is itself a group under the operation of G, then H is a subgroup of G;
- A subset S of a ring R is a subring of R if S is itself a ring with operations of R;
- Let E and F be fields; if F is a subset of E and has the ~~addition~~ operations of addition and multiplication induced by E, then F is called a subfield of E, and E is called an extension field of F.
- ~~Let~~ Let W be a ~~subset~~ non-empty subset of a vector space V over a field K. W is called a subspace (or linear manifold) of V if W is itself a vector space over K with respect ~~to~~ the operations of vector addition and scalar multiplication on V.

That is, if W is subset of a vector space V, then W is a subspace of V iff:

(i) $0 \in W$

(ii) W is closed under vector addition, that is: for every $u, v \in W$, the sum $u+v \in W$.

(iii) W is closed under scalar multiplication, that is: for every $u \in W$, $k \in K$, the multiple $ku \in W$.

H- bspace $\mathbb{R}^3$

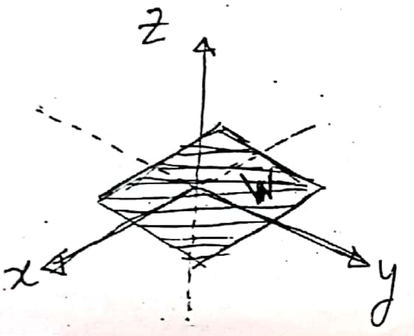
In short, W is a subspace of vector space V over the field K iff: (i) $0 \in W$, and (ii) $au + bv \in W$ for every $u, v \in W$ and $a, b \in K$.

Fact 1: Let V be a vector space. Then the set $\{0\}$ consisting of the zero vector alone, and also the entire space V are subspaces of V.

Vector space $V=$

Example 1: Let W be the xy plane in $\mathbb{R}^3$ consisting of those vectors whose third component is 0; i.e, $W = \{(a, b, 0) : a, b \in \mathbb{R}\}$

$$= \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a \text{ and } b \text{ are reals} \right\}$$



Show that W is a subspace of V.

Pf: ~~Exact~~ $V = \mathbb{R}^3$ is a vector space over the field $\mathbb{R}$.

(i) 0 vector $= (0, 0, 0)$; ~~$\in W$ since~~,
That is, zero vector is in W since, the third component of zero vector is zero;

(ii) for any vector $u \in W$ and $v \in W$ we have:
let $u = (a, b, 0)$ and $v = (c, d, 0)$ $c, d, a, b \in \mathbb{R}$.
$\Rightarrow u + v = (a, b, 0) + (c, d, 0) = (a+c, b+d, 0)$
$\Rightarrow (u+v) \in W$, since third component is zero.

(iii) Let $k \in \mathbb{R}$ be any scalar; then for any $u \in W$, $ku = k(a, b, 0) = (ka, kb, 0)$
i.e, ku is also in W since for any $k \in \mathbb{R}$, the third component of ku is also zero. ~~Q.E.D~~

$W = \{(a, b, 0) : a, b \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$ that "looks" and "acts" like $\mathbb{R}^2$ but logically different from $\mathbb{R}^2$

Note: The vector space $\mathbb{R}^2$ (over the field $\mathbb{R}$) is not a subspace of $\mathbb{R}^3$ because $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$ as because in $\mathbb{R}^3$ all vectors have three entries, the vectors in $\mathbb{R}^2$ have two.

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~~Fact-2: Let $V = M_{n,n}$ = vector space of $n\times n$ matrices with real entries, i.e., over~~

~~Then the subset W_1~~

Fact-2: Let $V = M_{n,n}$ = Vector space of $n \times n$ matrices with real entries, i.e. over the field $\mathbb{R}$.

Then the subset W_1 of (upper) triangular matrices and subset W_2 of symmetric matrices are subspaces of V since they are (i) non-empty, (ii) are closed under matrix addition and (iii) are closed under scalar multiplication.

Fact-3: The set: ~~$P(x) = \{a_2x^2 + a_1x + a_0$ such that~~

$P_2(x) = \{a_2x^2 + a_1x + a_0$ such that $a_0, a_1, a_2 \in \mathbb{R}\}$ is subspace of the vector space ~~(over $\mathbb{R}$)~~ $P(x)$ (over $\mathbb{R}$) such that:

$P(x) = \{$ set of all polynomials with real co-efficients$\}$

In general, if $P(t)$ denotes the vector space of polynomials. Let $P_n(t)$ denote the subset of $P(t)$ that consists of all polynomials of degree $\le n$. Then $P_n(t)$ is a subspace of $P(t)$.

H.W. #1. Let U and W be subspaces of vector space V. Show that their intersection $U \cap W$ is also a subspace of V.

pf. U and W are subspaces of V $\Rightarrow$ $0 \in W$ and $0 \in U$

$\therefore$ (i) $0 \in U \cap W$

(ii) Let $u, v \in U \cap W$. Then $u, v \in U$ and $u, v \in W$.

$\Rightarrow u+v \in U$ [since U is a subspace;] and

also, $\Rightarrow u+v \in W$ [since W is a subspace;].

$\Rightarrow (u+v) \in U \cap W$.

(iii) Let k be any scalar belonged to the field over which V is a vector space.

Then $ku \in U$; [since U is a subspace]; and

also, $ku \in W$; [since W is a subspace].

$\Rightarrow ku \in U \cap W$.

Hence $U \cap W$ is a subspace $\square$

~~Note~~ Fact-4: The intersection of any number of subspaces of a vector space V is a subspace of V.

Example-2 Let $S = \{(x_1, x_2, \dots, x_6)$ such that $x_i \in \mathbb{R}$ and $x_1 = x_3 = 0\}$ in $\mathbb{R}^6$.

Show that S is a subspace of $\mathbb{R}^6$.

pf: $S = \{(0, x_2, 0, x_4, x_5, x_6) :$ ~~0, x2, 0, x4, x5~~, 0 and $x_i \in \mathbb{R}\}$

That is S contains all 6-vectors $(0, x_2, 0, x_4, x_5, x_6)$ of which the first and third components are zero.

(i) Zero vector $0 = (0,0,0,0,0,0)$ is in S because x_2, x_4, x_5, and x_6 can all be zero.

(ii) $u = (0, x_2, 0, x_4, x_5, x_6)$ and $v = (0, y_2, 0, y_4, y_5, y_6)$ and $u, v \in \mathbb{R}^6$. Then

$$u+v = (0+0,\ x_2+y_2,\ 0+0,\ x_4+y_4,\ x_5+y_5,\ x_6+y_6)$$
$$= (0,\ x_2+y_2,\ 0,\ x_4+y_4,\ x_5+y_5,\ x_6+y_6)$$

$\Rightarrow (u+v) \in S$ since the first and the third component of $(u+v)$ is zero.

(iii) $\alpha \in \mathbb{R}$, then $\alpha u = \alpha(0, x_2, 0, x_4, x_5, x_6)$
$$= (0, \alpha x_2, 0, \alpha x_4, \alpha x_5, \alpha x_6)$$

$\Rightarrow \alpha u \in S$ since the first and the third component of αu is zero.

Hence, S is a subspace of $\mathbb{R}^6$. $\square$

<u>H.W. #2</u>: Let T consists of ~~all~~ those vectors in $\mathbb{R}^4$ with all components equal. Show that T is a subspace of the vector space $\mathbb{R}^4$ (over the field $\mathbb{R}$).

<u>Pf.</u> $T = \{(x,x,x,x) : x \in \mathbb{R}\}$

(i) $(0,0,0,0)$ is zero vector and is belonged to T; [since all components are equal].

(ii) Let $u = (x, x, x, x)$ and $v = (y, y, y, y)$.

$\Rightarrow u, v \in T$; and we have:

$$u + v = (x, x, x, x) + (y, y, y, y) = (x+y, x+y, x+y, x+y)$$

$\Rightarrow (u+v) \in T$ since $(u+v)$ has all equal components.

(iii) Let $\alpha \in \mathbb{R}$, then $\alpha u = \alpha (x, x, x, x) = (\alpha x, \alpha x, \alpha x, \alpha x)$

$\Rightarrow \alpha u \in T$ since all components of αu are equal.

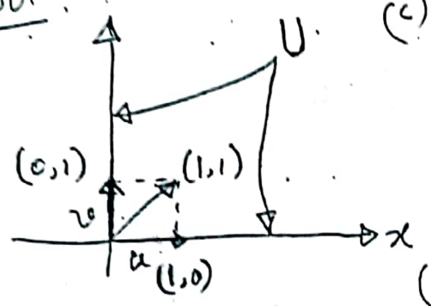
Hence, T is a subspace of $\mathbb{R}^4$. $\square$

<u>Example-3:</u> Consider the subset U of (vector space) $\mathbb{R}^2$ (over the field $\mathbb{R}$) defined by:

$$U = \{(u_1, u_2) : u_1 = 0 \text{ and/or } u_2 = 0\} = \left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ such that } u_1 = 0 \text{ and/or } u_2 = 0 \right\}.$$

Show that U is not a subspace of $\mathbb{R}^2$.

<u>Sol^n:</u>



(i) Let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in U$. Then if $v_1 = 0$ and $v_2 = 0$, $v = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (0, 0) =$ zero vector.

$\Rightarrow$ zero vector $\in U$.

(iii) Let $\alpha \in \mathbb{R}$. Then $\alpha v = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \end{bmatrix}$. If $v_1 = 0$ and/or $v_2 = 0$, then $\alpha v_1 = 0$ and/or $\alpha v_2 = 0$.

Thus $\alpha v \in U \Rightarrow U$ is closed under scalar multiplication.

(ii) Let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in U$ and $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in U$.

Then $v + w = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow (v + w) \notin U$.

Therefore, $v + w$ <u>is not</u> closed under addition.

Hence U is not a subspace of $\mathbb{R}^2$.

The vectors in $\mathbb{R}$

H.W. #3 Consider the subset U of (the vector space) $\mathbb{R}^2$ (over the field $\mathbb{R}$) defined by:

$$U = \left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ such that } u_2 = 2u_1 \right\} = \{(u_1, u_2) : u_2 = 2u_1\}.$$

Show that U is a subspace of $\mathbb{R}^2$.

Sol^n: Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in U$. Since $u_2 = 2u_1$, we can express any $u \in U$ as $u = \begin{bmatrix} u_1 \\ 2u_1 \end{bmatrix} = \begin{bmatrix} a \\ 2a \end{bmatrix}$; [since $u_1, u_2 \in \mathbb{R}$].

(i) Now if $a = 0$, then $u = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ zero vector $\in U$; i.e., U is not empty.

(ii) Let $u, v \in U$ and $u = \begin{bmatrix} a \\ 2a \end{bmatrix}$ and $v = \begin{bmatrix} b \\ 2b \end{bmatrix}$; then:

$$u + v = \begin{bmatrix} a \\ 2a \end{bmatrix} + \begin{bmatrix} b \\ 2b \end{bmatrix} = \begin{bmatrix} a+b \\ 2(a+b) \end{bmatrix} \Rightarrow (u+v) \in U.$$

(iii) Let $\alpha \in \mathbb{R}$, then $\alpha u = \alpha \begin{bmatrix} a \\ 2a \end{bmatrix} = \begin{bmatrix} \alpha a \\ \alpha 2a \end{bmatrix} = \begin{bmatrix} \alpha a \\ 2(\alpha a) \end{bmatrix}$

$$\Rightarrow \alpha u \in U.$$

That is U is non-empty, and is closed under addition and scalar multiplication. Hence U is a subspace of $\mathbb{R}^2$.

Example-4: H.W. #4 Let U be the set of all 2-vectors such that the sum of components is equal to 1; that is,

$$U = \left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ such that } u_1 + u_2 = 1 \right\} = \{(u_1, u_2) : u_1 + u_2 = 1\}.$$

Show that U is <u>not a vector space</u>.

Sol^n: (i) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are elements of U, but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not an element of U, since $1+1 = 2 \neq 1$.

$\therefore$ Axiom A1 is not satisfied.

(ii) $2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$ is not an element of U; [$\because 2+0=2$]

$\therefore$ Axiom A6 is not satisfied either.

(iii) Let $\begin{bmatrix}a\\b\end{bmatrix}$ is a zero vector element of U, Then

$$\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}1+a\\0+b\end{bmatrix} = \begin{bmatrix}1+a\\b\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$\Rightarrow$ $1+a=1$ and $b=0$ $\Rightarrow$ $a=0$ and $b=0$.

That is if $\begin{bmatrix}a\\b\end{bmatrix}$ is a zero element of U, then $\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$.

But $\begin{bmatrix}0\\0\end{bmatrix}$ is not element of U, Since $0+0=0\neq 1$.

$\therefore$ Axiom A4 is not satisfied.

(iv) Since, U does not have a zero element, U has no negative of a vector.

$\therefore$ Axiom A_5 is not satisfied.

~~Therefore~~ Now, violation of just one of ten axioms disqualifies U from being a vector space. But U violates four axioms as shown above. Hence, U is not a vector space. □

H.W. #4 Consider the subset U of vector space $\mathbb{R}^2$ (over the scalar field $\mathbb{R}$) defined by:

$$U = \left\{ \begin{bmatrix}u_1\\u_2\end{bmatrix} \text{ such that } u_1+u_2=1 \right\}.$$

Show that U is not a subspace of $\mathbb{R}^2$.

Soln: ~~Sol~~ Hint: Show that U is not a vector space as example-4 above. Since, U is not vector space, it can not ~~be~~ be subspace of $\mathbb{R}^2$. □

$\boxed{\text{H-W. \#5}}$. Let V be a vector space, let $u, v \in V$ and let α be a scalar from the field over which V is a vector space. Prove that:

(i) $u+v = 0$ (zero vector) implies $v = -u$;

(ii) $0v = 0_{\text{vector}}$ [i.e, (Zero times v) = zero vector];

(iii) α times zero vector = zero vector
[i.e, $\alpha\vec{0} = \vec{0}$].

(iv) $\alpha v = \vec{0}$ implies that either $\alpha = 0$ (zero) or $v = 0$ (zero vector); and

(v) $(-1)v = -v$.

Soln. (i) Let $u+v = 0$ (zero vector). Then by axiom A4:

$$-u = -u+0 = -u+(u+v)$$

Thus, by axioms A2, A3, A4, and A5, we have:

$$-u = (-u+u)+v = 0+v = v. \quad \square$$

(ii) From axioms A8 and A10, we have:

$$v = 1v = (1+0)v = 1v+0v = v+0v$$

Then by axiom A3, we get:

$$-v+v = -v+(v+0v) = (-v+v)+0v$$

Next, by axioms A2, A4, and A5 we get:

$$0_{(\text{vector})} = 0_{(\text{vector})} + (0)(v) = 0_{(\text{zero})}v = 0v. \quad \square$$

(iii) Let $v = \alpha 0_{(\text{vector})}$; then by axioms A4 and A7 we get:

$$v+v = \alpha 0_{(\text{vector})} + \alpha 0_{(\text{vector})} = \alpha(\vec{0}+\vec{0}) = \alpha\vec{0} = v$$

Thus, $(v+v)+(-v) = v+(-v)$; and by axioms A_3 and

and axiom A_5: $v + [v+(-v)] = 0$ (vector)

$\Rightarrow \quad v + 0$(vector) $= 0$ (zero vector)

Then by axioms A2 and A4 we obtain:

$v = 0$ (vector); that is $\alpha\, 0$(vector) $= 0$(vector) □

(iv) We know $0v = 0$vector by (ii) above; then:

if $\alpha = 0$, $\alpha v = 0v = 0$(vector);

again, if $\alpha v = 0$ (vector) and $\alpha \neq 0$, then it follows from (i) above, axioms A9 and A10 that:

$$0_{(\text{vector})} = \frac{1}{\alpha}\, 0_{(\text{vector})} = \frac{1}{\alpha}(\alpha v) = \left(\frac{1}{\alpha}\alpha\right) v = 1v = v$$ □

(v) Since, $1 + (-1) = 0$, we have:

$[1+(-1)]v = 0v = 0$ (zero vector) by (ii) above.

Thus, by axiom A8, we get:

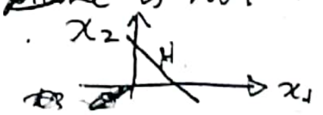
$1v + (-1)v = 0$ or by using axiom A10, we get

$v + (-1)v = 0$, and then by (i) above, we get.

$(-1)v = -v$. □

H.W #6 Prove that a ~~plane~~ line (H) in $\mathbb{R}^2$ not through the origin is not a subspace of $\mathbb{R}^2$.

pf. The ~~plane~~ line does not contain the zero vector of $\mathbb{R}^2$. Therefore, such a ~~plane~~ line is not a subspace of $\mathbb{R}^2$. □



H.W. # 7. Let $\mathbb{Q}$ = Field of Rational Numbers. Show that the set $M_2(\mathbb{Q})$ of 2×2 matrices with entries from $\mathbb{Q}$ is a vector space over $\mathbb{Q}$.

Soln: (i) Sum of two 2×2 matrices of $M_2(\mathbb{Q})$ is again a 2×2 matrix with entries from $\mathbb{Q} \Rightarrow$ axiom A1 is satisfied.

(ii) Commutative law also holds satisfying A2.

(iii) Let $\begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}$, $\begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix}$, and $\begin{bmatrix} c_1 & c_3 \\ c_2 & c_4 \end{bmatrix}$ are belonged to $M_2(\mathbb{Q})$. Then:

$$\left\{\begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix}\right\} + \begin{bmatrix} c_1 & c_3 \\ c_2 & c_4 \end{bmatrix}$$

$$= \begin{bmatrix} a_1+b_1 & a_3+b_3 \\ a_2+b_2 & a_4+b_4 \end{bmatrix} + \begin{bmatrix} c_1 & c_3 \\ c_2 & c_4 \end{bmatrix} = \begin{bmatrix} a_1+b_1+c_1 & a_3+b_3+c_3 \\ a_2+b_2+c_2 & a_4+b_4+c_4 \end{bmatrix}$$

Again;

$$\begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} + \left\{\begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix} + \begin{bmatrix} c_1 & c_3 \\ c_2 & c_4 \end{bmatrix}\right\} = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} + \begin{bmatrix} b_1+c_1 & b_3+c_3 \\ b_2+c_2 & b_4+c_4 \end{bmatrix}$$

$$= \begin{bmatrix} a_1+b_1+c_1 & a_3+b_3+c_3 \\ a_2+b_2+c_2 & a_4+b_4+c_4 \end{bmatrix}$$

$\therefore$ Associativity law also holds satisfying axiom A3.

(iv) The zero vector $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is in $M_2(Q)$ as because 0 is $\in Q$. This satisfies axiom A4.

(V.) The negative of any non-zero vector of $M_2(Q)$ exists as:

The negative of $\begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} = -\begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_3 \\ -a_2 & -a_4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} + \begin{bmatrix} -a_1 & -a_3 \\ -a_2 & -a_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is zero vector.

$\therefore$ The axiom A5 is also satisfied.

(vi) Let $\alpha \in Q$, then

$\alpha \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} = \begin{bmatrix} \alpha a_1 & \alpha a_3 \\ \alpha b_2 & \alpha a_4 \end{bmatrix}$ is an element of $M_2(Q)$.

This satisfies axiom A6.

(vii) $\alpha \left\{ \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix} \right\} = \alpha \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} + \alpha \begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix}$

$\therefore$ The axiom A7 is also satisfied.

(viii) Let β is also an element of Q; then.

$(\alpha+\beta) \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} = \alpha \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} + \beta \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}$

$\therefore$ A8 is also satisfied.

(ix) $\alpha \left(\beta \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} \right) = \alpha \begin{bmatrix} \beta a_1 & \beta a_3 \\ \beta a_2 & \beta a_4 \end{bmatrix} = \begin{bmatrix} \alpha\beta a_1 & \alpha\beta a_3 \\ \alpha\beta a_2 & \alpha\beta a_4 \end{bmatrix}$

Similarly, $(\alpha\beta) \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} = \begin{bmatrix} \alpha\beta a_1 & \alpha\beta a_3 \\ \alpha\beta a_2 & \alpha\beta a_4 \end{bmatrix}$

$\therefore$ Axiom A9 is also satisfied.

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~~(x) The identity element~~

(x) The unit scalar $1 \in \mathbb{Q}$ and

$$1 \cdot \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}.$$

$\therefore$ The axiom A10 is also satisfied.

$\therefore$ $M_2(\mathbb{Q})$ is a vector space over $\mathbb{Q}$. $\square$

H.W. #8. Let K^n is a vector space over the field K. Let A is vector in K^n. Let W be a set of all elements B in K^n such that $B \cdot A = 0$ (dot product), i.e., such that B is perpendicular to A. Show that W is a subspace of K^n.

Sol^n: (i) 0(zero vector)$\cdot A = \vec{0} \cdot \vec{A} = 0$. Therefore, zero vector is in W.

(ii) Let B, C are both perpendicular to A.

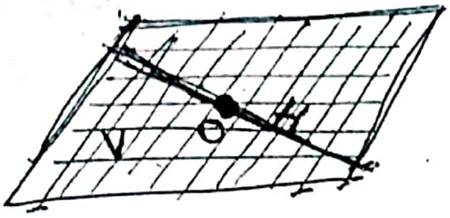
Then: $(B+C) \cdot A = (B \cdot A) + (C \cdot A) = 0 + 0 = 0$

$\Rightarrow (B+C)$ is also perpendicular to A; and, therefore $(B+C) \in W$.

(iii) Let $x \in K$. Then: $(xB) \cdot A = x(A \cdot B) = x(0) = 0$

$\Rightarrow xB$ is also perpendicular to A; and, Therefore, is belonged to W. $\therefore$ W is a subspace of K^n.

$u, v \in V$ and $a, b \in K$.



In general, every subspace is a vector ~~spa~~ space; and every vector space is a subspace of itself and possibly other larger spaces. ~~When~~ The term subspace is used when at least two vector spaces are in mind, with one inside the other, and the phrase "subspace of V" identifies V as the larger space. In ~~this~~ this figure H is a subspace of V and H is a line containing the ~~origin~~ O (zero vector).

Example. Given v_1 and v_2 in vector space V. ~~Let H = Span{H, H}~~ Let $H = \text{Span}\{v_1, v_2\}$. Show that H is a subspace of V.

Sol^n: The term "linear combination" refers to any sum of scalar multiples of vectors, and $\text{Span}\{v_1, v_2, \dots, v_n\}$ denotes the set of all vectors that can be written as linear combinations of $v_1, v_2, \dots, v_n$.

Now, $0(\text{vector}) = 0v_1 + 0v_2 \Rightarrow 0\text{vector} \in H$

Let $u, w \in H$ and a_1, a_2, a_3, a_4 are scalars such that

$u = a_1v_1 + a_2v_2$ and $w = a_3v_1 + a_4v_2$, then:

$u + w = (a_1v_1 + a_2v_2) + (a_3v_1 + a_4v_2) = (a_1+a_3)v_1 + (a_2+a_4)v_2 \Rightarrow u + w \in H$

Let b is any scalar, then for any $u \in H$ we have

$bu = b(a_1v_1 + a_2v_2) = (ba_1)v_1 + (ba_2)v_2 \Rightarrow bu \in H$.

That is, (i) H contains zero vector (ii) H is closed under vector addition and (iii) H is closed under scalar multiplication. Therefore, H is a subspace of V. □