Along two different paths

the value $f'(z)$ is same

$\therefore f(z)$ is differentiable.

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Analytic Function:

A single valued function which is differentiable at $z = z_0$ is said to be analytic at the point $z = z_0$

The point at which the function is not differentiable is ~~not~~ called a singular point of the function.

Theorem: (Cauchy - Riemann equations)

The necessary conditions for a function $f(z) = u + iv$ to analytic at all the points in a region R are.

(i) $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$

(ii) $\frac{\delta u}{\delta y} = \frac{-\delta v}{\delta x}$

proof:

Let $f(z)$ be an analytic function in a region R.

$\therefore f(z) = u + iv$

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Where u and v are the functions of x and y. Let δu and δv be the increments of u and v respectively corresponding to increments δx and δy of x and y.

$$f(z+\delta z) = (u+\delta u) + i(v+\delta v)$$

$$f(z+\delta z) - f(z) = (u+\delta u) + i(v+\delta v) - (u+iv)$$

$$\frac{f(z+\delta z)-f(z)}{\delta z} = \frac{\delta u + i\delta v}{\delta z}$$

$$\lim_{\delta z\to 0} \frac{f(z+\delta z)-f(z)}{\delta z} = \lim_{\delta z\to 0}\left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right)$$

or, $$f'(z) = \lim_{\delta z\to 0}\left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right) \quad \text{———(i)}$$

Since δz can approaches zero along any path.

Along real axis:

$Z = x+iy$, but on x axis $y=0$

$z = x$, $\delta z = \delta x$; $\delta y = 0$

putting these values in eq^n(1) we have,

$$f'(z) = \lim_{\delta x\to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} \quad \text{—(2)}$$

Along Imaginary axis:

$Z = x+iy$, ~~by~~ but on y axis $x=0$

$z = iy$, $\delta z = i\delta y$; $\delta x = 0$

putting these value in equ (i) we have,

$$f'(z) = \lim_{\delta y\to 0}\left(\frac{\delta u}{i\delta y} + i\frac{\delta v}{i\delta y}\right)$$

$$= \frac{\delta v}{\delta y} + i\frac{\delta u}{i^2\delta y}$$

$$f'(z) = \frac{\delta v}{\delta y} - i\frac{\delta u}{\delta y} \quad \text{———(3)}$$

If $f(z)$ is differentiable then the two values of $f(z)$ must be equal.

Now equation (2) and (3) we have,

$$\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} = \frac{\delta v}{\delta y} - i\frac{\delta u}{\delta y}$$

equating Real and Imaginary parts we have.

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$$

$$\frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y}$$

Known as C-R Equations.

[proved]

Exercise:

① Show that the complex variable function $f(z) = z^2$ is differentiable only at the origin.

$\Rightarrow$ $|z| = \sqrt{(x+iy)(x-iy)}$

$|z| = \sqrt{x^2 - i^2y^2}$

$|z| = \sqrt{x^2+y^2}$

Given $f(z) = |z|^2$

$= x^2+y^2$

$= u(x,y) + iv(x,y)$

Where $u(x,y) = x^2+y^2$

$v(x,y) = 0$

$\frac{\delta u}{\delta x} = 2x$; $\frac{\delta v}{\delta x} = 0$

$\frac{\delta u}{\delta y} = 2y$; $\frac{\delta v}{\delta y} = 0$

Using C-R eq^n we have,

$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ Similarly $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$

$2x = 0$ $2y = 0$

$y = 0$

$\therefore x = 0$

$\therefore$ The function differentiable only at $x=0, y=0$

$f(z)$ is dift. only at the origin.

② Show that the function $e^x(\cos y + i\sin y)$ is an analytic function, find its derivative.

$\Rightarrow$ $f(x) = e^x(\cos y + i\sin y)$

$= e^x\cos y + e^x i\sin y$

Where

$u(x,y) = e^x\cos y$

$v(x,y) = e^x\sin y$

$\frac{\delta u}{\delta x} = e^x\cos y$ $\frac{\delta v}{\delta x} = e^x\sin y$

$\frac{\delta u}{\delta y} = -e^x\sin y$ $\frac{\delta v}{\delta y} = e^x\cos y$

der

The derivative of $f(z) = e^x\cos y + ie^x\sin y$ is

$$\left.\begin{aligned}\frac{\delta u}{\delta x} &= \frac{\delta v}{\delta y}\\ \frac{\delta u}{\delta y} &= -\frac{\delta v}{\delta x}\end{aligned}\right\}$$

$f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} \Big/ \frac{\delta v}{\delta y} - i\frac{\delta u}{\delta y}$

$= e^x\cos y + ie^x\sin y$

$= e^x(\cos y + i\sin y)$

$= e^x . e^{iy}$

$= e^{x+iy}$ $= e^z$

$f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$

$f'(z) = \frac{\delta u}{\delta y}$ ...

$= \frac{\delta v}{\delta y} - i\frac{\delta u}{\delta y}$

③ Using C-R equations show that $f(z)=z^3$ is analytic.

$\Rightarrow$

$z = x+iy$

$z^3=(x+iy)^3$

$=x^3+3x^2\cdot iy+3\cdot x(iy)^2+(iy)^3$

$=x^3+i3x^2y-3xy^2-iy^3$

$=(x^3-3xy^2)+i(3x^2y-y^3)$

$\cancel{\frac{\delta u}{\delta x}=3x^2-3y^2}$

$\therefore u(x,y)+iv(x,y)$

Here,

$u(x,y)=x^3-3xy^2$

$v(x,y)=3x^2y-y^3$

$\frac{\delta u}{\delta x}=3x^2-3y^2$ $\qquad$ $\frac{\delta v}{\delta x}=6xy-0$

$\frac{\delta u}{\delta y}=-6xy$ $\qquad$ $\frac{\delta v}{\delta y}=3x^2-3y^2$

Using C-R eq^n we have,

$\frac{\delta u}{\delta x}=\frac{\delta v}{\delta y}$ $\qquad$ $\frac{\delta u}{\delta y}=-\frac{\delta v}{\delta x}$

$3x^2-3y^2=3x^2-3y^2$ $\qquad$ $-6xy=-6xy$

If $x=0$, then $y=0$

$f'(z)$ is only exists.

This function is analytic.

④ Find the points where C-R equ^s are satisfied for the function $f(z)=xy^2+ix^2y$. Where does $f'(z)$ exists? Where $f(z)$ is analytic.

$\Rightarrow$ $f(z)=xy^2+ix^2y$

$|z|=\sqrt{(x+iy)(x-iy)}$

$|z|=\sqrt{x^2-i^2y^2}$

$|z|=\sqrt{x^2+y^2}$

$|z|^2=x^2+y^2$

$u(x,y)=\cancel{x^2+y^2}\ xy^2$

$v(x,y)=x^2y$

$\frac{\delta u}{\delta x}=y^2$ $\qquad$ $\frac{\delta v}{\delta y}=x^2\ 2xy$

$\frac{\delta u}{\delta y}=2xy$ $\qquad$ $\frac{\delta v}{\delta y}=x^2$

$\frac{\delta u}{\delta x}=\frac{\delta v}{\delta y}$; $\frac{\delta u}{\delta y}=-\frac{\delta v}{\delta x}$

$y^2=x^2$ $\qquad$ $2xy=-2xy$

If $x=0$ then $y=0$

$f'(z)$ is only exists at the origin.

Then the function is analytic only at the origin

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(5) Show that the function $z|z|$ is not analytic anywhere.

Here ⇒

$|z| =$

Given,

$f(z) = z|z|$

$= (x+iy)\sqrt{x^2+y^2}$

$= x\sqrt{x^2+y^2} + iy\sqrt{x^2+y^2}$

$= u(x,y) + iv(x,y)$

$u(x,y) = x\sqrt{x^2+y^2}$

$\frac{\delta u}{\delta x} = \sqrt{x^2+y^2} + x \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x$

$= \sqrt{x^2+y^2} + x^2(x^2+y^2)^{-\frac{1}{2}}$

$\frac{\delta u}{\delta y} = 0 + x \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y$

$= \frac{xy}{\sqrt{x^2+y^2}}$

$v(x,y) = y\sqrt{x^2+y^2}$

$\frac{\delta v}{\delta x} = 0 + y \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x$

$= \frac{xy}{\sqrt{x^2+y^2}}$

$\frac{\delta v}{\delta y} = 1 \cdot \sqrt{x^2+y^2} + y \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2y$

$= \sqrt{x^2+y^2} + \frac{y^2}{\sqrt{x^2+y^2}}$

Here,

$\frac{\delta u}{\delta x} \neq \frac{\delta v}{\delta y}$

$\frac{\delta u}{\delta y} \neq -\frac{\delta v}{\delta x}$

So the function is not analytic.

(6) Show that $2x + i(x^2 - y^2)$ is analytic or not.

$f(z) = 2x + i(x^2 - y^2)$

$= u(x,y) + iv(x,y)$

Where $u(x,y) = 2x$

$v(x,y) = x^2 - y^2$

$\frac{\delta u}{\delta x} = 2$ $\qquad \frac{\delta v}{\delta x} = 2x$

$\frac{\delta u}{\delta y} = 0$ $\qquad \frac{\delta v}{\delta y} = -2y$

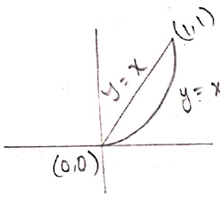
This function is not analytic.

Complex Integration:

$\int_c (Mdx + Ndy)$

$\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$

The integral values are independent upon its path.



Question 1:

Find the value of the integral $\int_c (x+y)dx + x^2y\,dy$

(a) along $y = x^2$ having $(0,0)$, $(3,9)$ and points.

(b) along $y = 3x$ between the same points Do the values depends upon path.

Answer:

Given $\int_c (x+y)dx + x^2y\,dy$

$Mdx = x+y$ $\qquad$ $Ndy = x^2y$

$\frac{\delta M}{\delta y} = 1$ $\qquad$ $\frac{\delta N}{\delta x} = 2xy$

$\frac{\delta M}{\delta y} \neq \frac{\delta N}{\delta x}$

The integral values are depends on its path.

(a) along $y = x^2$

$dy = 2x\,dx$

x varies from 0 to 3

The integral becomes

$\Rightarrow \int_0^3 (x+x^2)dx + x^2 \cdot x^2 \cdot 2x\,dx$

$\Rightarrow \left[\frac{x^2}{2} + \frac{x^3}{3} + 2 \cdot \frac{x^6}{6}\right]_0^3$

$\Rightarrow \left[\frac{3^2}{2} + \frac{3^3}{3} + \frac{3^6}{3}\right]$

$\Rightarrow \frac{9}{2} + \frac{27}{3} + \frac{729}{3}$

$\Rightarrow \frac{9}{2} + 9 + 243$

$\Rightarrow 256.5$

(b) along $y = 3x$

$dy = 3dx$

x varies from 0 to 3

The integral becomes

$\Rightarrow \int_0^3 (x + 3x)dx + x^2 \cdot 3x \cdot 3dx$

$\Rightarrow \int_0^3 (4x + 9x^3)dx$

$\Rightarrow \left[\frac{4x^2}{2} + \frac{9x^4}{4}\right]_0^3$

$\Rightarrow 2 \cdot 3^2 + \frac{9 \cdot 3^4}{4}$

$\Rightarrow 18 + 182.25$

$\Rightarrow 200.25$

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Question 2:

Evaluate $\int_{(0,0)}^{(1,1)} (3x^2+4xy+3y^2)dx+2(x^2+3xy+4y^2)dy$

ⓐ Along $y^2=x$

ⓑ Along $y=x^2$

$y=x$

Answer:

Given $\int_{(0,0)}^{(1,1)} (3x^2+4xy+3y^2)dx+2(x^2+3xy+4y^2)dy$

$Mdx=(3x^2+4xy+3y^2)$ $\quad$ $Ndy=2x^2+6xy+8y^2$

$\frac{\delta M}{\delta y}=4x+6y$ $\quad$ $\frac{\delta N}{\delta x}=4x+6y$

$\frac{\delta M}{\delta y} \cancel{\neq} = \frac{\delta N}{\delta x}$

The integral values are depends on its path.

ⓐ along $y^2=x$

$dx=2y\,dy$

y varies from 0 to 1

The integral becomes

$\Rightarrow \int_0^1 \{3(y^2)^2+4.y^2.y+3y^2\}\overset{2ydy}{\cancel{dx}}+2\{(y^2)^2+3.y^2.y+4y^2\}dy$

$\Rightarrow \int_0^1 (3y^4+4y^3+3y^2)2y\,dy+2(y^4+3y^3+4y^2)dy$

$\Rightarrow \int_0^1 (6y^5+8y^4+6y^3)dy+(2y^4+6y^3+8y^2)dy$

$=\int_0^1 (6y^5+8y^4+6y^3+2y^4+6y^3+8y^2)dy$

$=\int_0^1 (6y^5+10y^4+12y^3+8y^2)dy$

$=\left[\cancel{6}.\frac{y^6}{\cancel{6}}+10.\frac{y^5}{5}+12.\frac{y^4}{4}+8.\frac{y^3}{3}\right]_0^1$

$=\left[y^6+2y^5+3y^4+\frac{8y^3}{3}\right]_0^1$

$=1^6+2.1^5+3.1^4+\frac{8.1^3}{3}$

$=1+2+3+\frac{8}{3}$

$=\frac{18+8}{3}$

$=8.67$

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(b) along $y=x^2$

$dy = 2x\,dx$

x varies from 0 to 1.

The integral becomes =

$$\int_0^1 \{3x^2+4.x.x^2+3.(x^2)^2\}dx+2\{x^2+3.x.x^2+4.(x^2)^2\}dy$$

$$=\int_0^1 (3x^2+4x^3+3x^4)dx+2(x^2+3x^3+4x^4)\cancel{dy}\,2x\,dx$$

$$=\int_0^1 (3x^2+4x^3+3x^4)+(2x^2+6x^3+8x^4)2x\,dx$$

$$=\int_0^1 (3x^2+4x^3+3x^4+4x^3+12x^4+16x^5)dx$$

$$=\int_0^1 (3x^2+8x^3+15x^4+16x^5)dx$$

$$=\left[3.\frac{x^3}{3}+8.\frac{x^4}{4}+15.\frac{x^5}{5}+16.\frac{x^6}{6}\right]_0^1$$

$$=\left[x^3+2x^4+3x^5+\frac{8x^6}{3}\right]_0^1$$

$$=1^3+2.1^4+3.1^5+\frac{8.1^6}{3}$$

$$=6+\frac{8}{3}$$

$$=\frac{18+8}{3}$$

$$=\frac{26}{3}$$

(a) along $y=x$

$dy=dx$

x varies from 0 to 1

The integral becomes ⇒

$$\int_0^1 \{3x^2+4.x.x+3(x^2)\}.dx+2\{x^2+3.x.x+4.(x^2)\}dx$$

$$=\int_0^1 (3x^2+4x^2+3x^2)dx+(2x^2+6x^2+8x^2)dx$$

$$=\int_0^1 (10x^2+16x^2)dx$$

$$=\int_0^1 26x^2dx$$

$$=\left[26.\frac{x^3}{3}\right]_0^1$$

$$=\frac{26}{3}.$$

5.03.19

Harmonic function:

$$\frac{\delta u^2}{\delta x^2} + \frac{\delta u^2}{\delta y^2} = 0 \text{ (laplace eq}^n\text{)}$$

Any function which Satisfies the laplace's equation is known as harmonic function.

<u>Theorem</u>: If $f(z) = u+iv$ is an analytic function, then u and v both harmonic functions.

<u>proof</u>: Let $f(z) = u+iv$, be an analytic function, then we have,

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \quad \text{—— (i)}$$

$$\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} \quad \text{—— (ii)}$$

Now differentiating (i) w.r.to x we have,

$$\frac{\delta^2 u}{\delta x^2} = \frac{\delta^2 v}{\delta x \delta y} \quad \text{—— (iii)}$$

differentiating (ii) w.r.to y we have,

$$\frac{\delta^2 u}{\delta y^2} = -\frac{\delta^2 v}{\delta x \delta y}$$

Adding eq^n (iii) & (iv) we have,

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = \frac{\delta^2 v}{\delta x \delta y} - \frac{\delta^2 v}{\delta x \delta y} = 0$$

$\therefore u$ is a Harmonic Function.

Similarly we can prove, $\frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 0$

(proved)

Harmonic Conjugate Function:

If a Harmonic Function satisfies the C-R eq^n or if it is analytic then the function is called harmonic conjugate function.

<u>Exercises:</u>

1. Prove that $u = x^2 - y^2$ and $v = \frac{y}{x^2+y^2}$ are harmonic functions of (x,y) but are not Harmonic conjugate.

$\Rightarrow$ Given $u = x^2 - y^2$

$$\frac{\delta u}{\delta x} = 2x$$

$$\frac{\delta^2 u}{\delta x^2} = 2$$

$$\frac{\delta u}{\delta y} = -2y$$

$$\frac{\delta^2 u}{\delta y^2} = -2$$

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 2-2 = 0$$

$\therefore u$ is harmonic

$$v = \frac{y}{x^2+y^2}$$

$$\frac{\delta v}{\delta x} = \frac{(x^2+y^2)\frac{\delta}{\delta x}y - y\cdot\frac{\delta}{\delta x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{0 - y\cdot 2x}{(x^2+y^2)^2}$$

$$\frac{\delta^2 v}{\delta x^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)^2\frac{\delta}{\delta x}(-2xy) - (2xy)\frac{\delta}{\delta x}(x^2+y^2)^2}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)(-2y) + 2xy\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$$

$$= \frac{-2y(x^2+y^2)+8x^2y}{(x^2+y^2)^3}$$

$$= \frac{-2y^3+6x^2y}{(x^2+y^2)^3}$$

$$\frac{\delta v}{\delta y} = \frac{(x^2+y^2)\frac{\delta}{\delta y}(y) - y\cdot\frac{\delta}{\delta y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-y\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\delta^2 v}{\delta y^2} = \frac{(x^2+y^2)^2\frac{\delta}{\delta y}(x^2-y^2)-(x^2-y^2)\frac{\delta}{\delta y}(x^2+y^2)^2}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)^2\cdot(-2y)-(x^2-y^2)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)(-2y)-(x^2-y^2)4y}{(x^2+y^2)^3}$$

$$= \frac{-2x^2y-2y^3-4x^2y+4y^3}{(x^2+y^2)^3}$$

$$= \frac{-6x^2y+2y^3}{(x^2+y^2)^3}$$

Now,

$$\frac{\delta^2 v}{\delta x^2}+\frac{\delta^2 v}{\delta y^2}$$

$$= \frac{-2y^3+6x^2y}{(x^2+y^2)^3}+\frac{-6x^2y+2y^3}{(x^2+y^2)^3}$$

$$= 0$$

$\therefore v$ is also Harmonic.

Here,

$$\frac{\delta u}{\delta x} \neq \frac{\delta v}{\delta y}$$

$$\frac{\delta u}{\delta y} \neq -\frac{\delta v}{\delta x}$$

$\therefore u$ and v are not harmonic conjugate.

② Similarly prove for $u = x^2-y^2$
$v = 2xy$

Given $u = x^2-y^2$

$$\frac{\delta u}{\delta x} = 2x$$

$$\frac{\delta^2 u}{\delta x^2} = 2$$

$$\frac{\delta u}{\delta y} = -2y$$

$$\frac{\delta^2 u}{\delta y^2} = -2$$

$$\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2} = 2-2 = 0$$

u is harmonic

Again,

$$\frac{\delta v}{\delta x} = 2y$$

$$\frac{\delta^2 v}{\delta x^2} = 0$$

$$\frac{\delta v}{\delta y} = 2x$$

$$\frac{\delta^2 v}{\delta y^2} = 0$$

$$\therefore \frac{\delta^2 v}{\delta x^2}+\frac{\delta^2 v}{\delta y^2} = 0$$

$\therefore v$ is also Harmonic.

$2 \cdot \frac{1}{\sqrt{x}}$

Here,

$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$

u and v are harmonic Conjugate.

③ Similarly prove for $u = x^3 - 3xy^2$
$v = 3x^2y - y^3$

Given $u = x^3 - 3xy^2$

$\frac{\delta u}{\delta x} = 3x^2 - 3y^2$

$\frac{\delta^2 u}{\delta x^2} = 6x$

$\frac{\delta u}{\delta y} = -6xy$

$\frac{\delta^2 u}{\delta y^2} = -6x$

$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 6x - 6x$
$= 0$

u is harmonic function.

Again $v = 3x^2y - y^3$

$\therefore \frac{\delta v}{\delta x} = \cancel{3y^2 - 0}\ 6xy$

$\frac{\delta^2 v}{\delta x^2} = \cancel{0}\ 6y$

$\frac{\delta v}{\delta y} = 3x^2 - 3y^2$

$\frac{\delta^2 v}{\delta y^2} = \cancel{6x - 6y} - 6y$

$\frac{\delta^2 v}{\delta x^2} \cancel{= 6y} + \frac{\delta^2 v}{\delta y^2} = 6y - 6y$
$= 0$

v is also harmonic

$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$

$\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$

u and v are harmonic conjugate.

Date: 6.03.19

Cauchy's Integral Theorem

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a closed curve C, then

$$\int_C f(z)\,dz = 0$$

Proof:

Let the region enclosed by the curve C be R and let,

$f(z) = u + iv$; $z = x + iy$; $dz = dx + idy$

$\int_C f(z)\,dz = \int_C f(u+iv)(dx + idy)$

$= \int_C (udx + iudy + ivdx + i^2 vdy)$

$= \int_C (udx - vdy) + i(udy + vdx)$

Now Applying Green's theorem,

$\int_C f(z)\,dz = \iint_R \left(-\frac{\delta v}{\delta x} - \frac{\delta u}{\delta y}\right) dx\,dy + i \iint_R \left(\frac{\delta v}{\delta y} - \frac{\delta u}{\delta x}\right) dx\,dy$

Replacing, $-\frac{\delta v}{\delta x} = \frac{\delta u}{\delta y}$

$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$

we get,

$$\int_c f(z)dz = \iint_R \left(-\frac{\delta u}{\delta y} - \frac{\delta u}{\delta y}\right) dxdy + i\iint_c \left(\frac{\delta u}{\delta x} - \frac{\delta y}{\delta x}\right) dxdy$$

$= 0$ (proved)

Cauch's ~~Ingr~~ Integral Formula:

If $f(z)$ is analytic within and on a closed curve c and if 'a' is any point within c, then

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz$$

proof:

Consider the function $\frac{f(z)}{z-a}$, which is analytic at all points ~~the~~ within c except $z=a$

with the point 'a' as centre and radius r draw a small circle c_1 lying entirely within c.

Now, $\frac{f(z)}{z-a}$ is analytic in the region between c and c_1, Hence by Cauchy's Integral theorem for ~~all~~ multiple connected region we have,

$$\int_c \frac{f(z)}{z-a}dz = \int_{c_1}\frac{f(z)}{z-a}dz.$$

$$= \int_{c_1}\frac{f(z)-f(a)+f(a)}{z-a}dz$$

$$= \int_{c_1}\frac{f(z)-f(a)}{z-a}dz + \int_{c_1}\frac{f(a)}{z-a}dz$$

for any point on c_1 we have,

$$\int_{c_1}\frac{f(z)-f(a)}{z-a}dz$$

Here, $z-a = re^{i\theta}$

$z = a+re^{i\theta}$

$dz = ire^{i\theta}d\theta$

θ varies from 0 to 2π

$$= \int_0^{2\pi}\frac{f(a+re^{i\theta})-f(a)}{re^{i\theta}}\, ire^{i\theta}d\theta$$

$$\int_{c_1}\frac{f(z)-f(a)}{z-a}dz = \int_0^{2\pi}\left(f(a)+f(re^{i\theta})-f(a)\right)id\theta$$

if r tends to zero.

$$\int_c \frac{f(z)-f(a)}{z-a}\,dz = 0$$

Now,

$$\int_{C_1} \frac{f(a)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a)}{re^{i\theta}}\, ire^{i\theta}\, d\theta$$

$$= if(a) \cdot \int_0^{2\pi} d\theta$$

$$= if(a)\,[\theta]_0^{2\pi}$$

$$= 2\pi i f(a)$$

Now putting these values in eqn(i) we have.

$$\int_c \frac{f(z)}{z-a}\,dz = 0 + 2\pi i f(a)$$

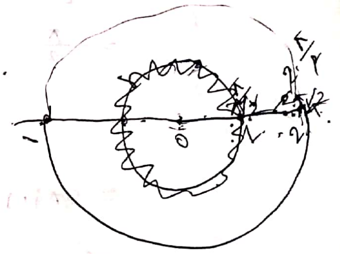
$$\therefore f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}\,dz$$

[Proved]

Exercise:

① Evaluate the complex Integral $\int_c \tan z\, dz$. Where C is $|z|=2$

$\Rightarrow$ Given $\int_c \tan z\, dz$



Which we can write,

$$\int_c \frac{\sin z}{\cos z}\,dz$$

c is $|z|=2$

$|z-0|=2$

Here given closed curve C having centre at $z=0$ and radius $=2$

Here we can find poles by equating the denominator is 0.

$\cos z = 0$:

$z = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}$

The integrand has two poles within the given circle i.e. $z=-\frac{\pi}{2}$ and $z=\frac{\pi}{2}$.

Now Applying Cauchy's Integral formula we have,

$$\int_c \frac{\sin z}{\cos z}\,dz = \int_{C_1} \frac{\sin z}{\cos z}\,dz + \int_{C_2} \frac{\sin z}{\cos z}\,dz$$

$= 2\pi i \, [\sin z]_z$

$= -\frac{\pi}{2} + 2\pi i \, [\sin z]_z$

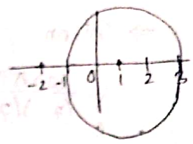
$= -\frac{\pi}{2}$

$= 2\pi i \cdot 1 + 2\pi i \left[-\sin \frac{\pi}{2}\right]$

$= 0$

(2) Evaluate $\int_C \frac{e^z}{z-1}\,dz$ where $|z-1| = 2$

$\Rightarrow$ Given $\int_C \frac{e^z}{z-1}\,dz$



which we can write,

$$\int_C \frac{e^z}{z-1}\,dz$$

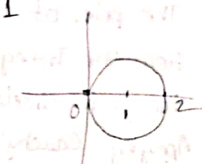
Here centre of circle is 1 and radius is 2

putting denomitor is equal to zero we have.

$z - 1 = 0$

$z = 1$

The poles of the given function is 1



$$\int_C \frac{e^z}{z-1}\,dz = \int_{C_1} \frac{\frac{e^z}{z+1}}{z-1}\,dz$$

$= 2\pi i \left[\cancel{\frac{e^z}{z+1}}\right]_{z=1}$ $[f(z)]_{z=1}$

$z - 1 = 0$

$z = 1$

$= 2\pi i \left[\frac{e^z}{2}\right]$ $[e^z]_{z=1}$

Which in the

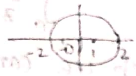
$= i\pi e^z$

Ans: $2\pi i \, e^1$

③ Evaluate $\int \frac{\cos\pi z^2}{(z-1)(z-2)}$ where $|z|=2$

⇒ Given

$$\int_c \frac{\cos\pi z^2}{(z-1)(z-2)} \; ; \; |z|=2$$



Here centre of circle is 0 radius is 2.

we can write,

putting denominator is equal to zero we have,

$$(z-1)(z-2)=0$$

$$z=1,2$$

The poles of the given function is 1 and 2.

Here the Integrand has two poles both are within the given circle $|z|=2$

Applying Cauchy's Integral formula,

$$\int \frac{\cos\pi z^2}{(z-1)(z-2)} = \int_{c_1} \frac{\left(\frac{\cos\pi z^2}{z-1}\right) f(z)}{z-2}\,dz + \int_{c_2} \frac{\left(\frac{\cos\pi z^2}{z-2}\right) f(z)}{z-1}\,dz$$

$$= 2\pi i\left[\frac{\cos\pi z^2}{z-1}\right]_{z=2} + 2\pi i\left[\frac{\cos\pi z}{z-2}\right]_{z=1}$$

$$= 2\pi i\left[\frac{\cos 4\pi}{1}\right] + 2\pi i\left[\frac{\cos\pi}{-1}\right]$$

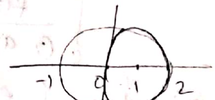
$$= 2\pi i \times 1 + 2\pi i\left(\frac{-1}{-1}\right)$$

$$= \cancel{8}\, 4\pi i$$

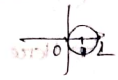
④ Evaluate $\int_c \frac{3z^2+2}{z^2-1}$ where $|z-1|=1$

⇒ Given,

$$\cancel{\int_c \cos\pi z^2}$$



$$\int_c \frac{3z^2+2}{z^2-1} \; ; \; |z-1|=1$$



Here centre of circle is 1 and radius is 1

we can write,

putting denominator is equal to zero we have,

$$z^2-1=0$$

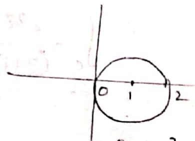
$$z=\pm 1$$

z=1 is within the given circle

$z = -1, +1$

$$\int_c \frac{3z^2+2}{z^2-1} = \int_{c_1} \frac{\left(\frac{3z^2+2}{z+1}\right) f(z)}{z-1}\,dz$$

$$= 2\pi i\left[\frac{3z^2+2}{z+1}\right]_{z=1}$$



0, 1, 2

$$= 2\pi i\,\frac{3+1}{2}$$

$$= 2\pi i\,\frac{4}{2}$$

$$= 4\pi i$$

Cauchy's Integral formula for derivative.

$$\int_c \frac{f(z)}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^n(a).$$

(15) Evaluate $\int_c \frac{e^{2z}}{(z+1)^4} dz$ where $|z|=2$

$\Rightarrow$ Here Centre is $z=-1$

$f(z)=e^{2z}$

$f'(z)=2e^{2z}$

$f''(z)=4e^{2z}$

$f'''(z)=8e^{2z}$

Here $z=-1$

$f'''(-1)=8e^{-2}$

$\therefore \int_c \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} f'''(-1)$

$= \frac{2\pi i}{6} 8e^{-2}$

$= \frac{8}{3}\pi i e^{-2}$

(16) Evaluate $\int_c \frac{e^{-z}}{(z+2)^5}$ where $|z|=2$

$\Rightarrow$ Here the centre is -2

$f(z)=e^{-z}$

$f'(z)=-e^{-z}$

$f''(z)=e^{-z}$

$f'''(z)=-e^{-z}$

$f''''(z)=e^{-z}$

Here $z=-2$

$f''''(2)=e^{2}$

$\int_c \frac{e^{-z}}{(z+2)^5} dz = \frac{2\pi i}{4!} f''''(-2)$

$= \frac{2\pi i}{24} e^2$

$= \frac{e^2\pi i}{12}$