Lecture sheet - 13

Q. Solve the initial value Problem by Laplace's Transform.

$$y''-10y'+9y=5t \ , \ y(0)=-1, y'(0)=2.$$

Soln: Taking Laplace transformation from both side we get,

$$L[y''-10y''+9y]=L[5t]$$

$$\Rightarrow L[y''] - L[10y'] + L[9y] = L[5t] \quad \text{———} \quad (1)$$

Let, $L[y(t)] = y(s)$

$$\Rightarrow L[y'(t)] = sy(s) - y(0)$$
$$= sy(s) - (-1)$$
$$= sy(s) + 1$$

$$L[y''(t)] = s^2y(s) - sy(0) - y'(0)$$
$$= s^2y(s) - s(-1) - 2$$
$$= s^2y(s) + s - 2$$

$$L[5t] = 5 \cdot \frac{1}{s^2} = \frac{5}{s^2}$$

Now, from (1) we get,

$$s^2y(s) + s - 2 - 10(sy(s)+1) + 9y(s) = \frac{5}{s^2}$$

$$\Rightarrow s^2y(s) + s - 2 - 10sy(s) - 10 + 9y(s) = \frac{5}{s^2}$$

$$\Rightarrow y(s)(s^2-10s+9) = \frac{5}{s^2} - s + 2 + 10$$

$\Rightarrow y(s)\,(s^2-9s-s+9) = \frac{5-s^3+12s^2}{s^2}$

$\Rightarrow y(s)\,(s-9)(s-1) = \frac{12s^2-s^3+5}{s^2}$

$\Rightarrow y = \frac{12s^2-s^3+5}{s^2(s-9)(s-1)}$

$\therefore \frac{12s^2-s^3+5}{s^2(s-9)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$

$\Rightarrow 12s^2-s^3+5 = As(s-9)(s-1) + B(s-9)(s-1) + Cs^2(s-1)$
$+ Ds^2(s-9)$

$\Rightarrow 12s^2-s^3+5 = As(s^2-10s+9) + B(s^2-10s+9)$
$+ Cs^3 - Cs^2 + Ds^3 - 9Ds^2$ ——— (2)

Equating co-efficient of s^3 from both side,

$-1 = A+C+D$ ——— (3)

Equating co-efficient of s^2 from both side,

$12 = -10A+B-C-9D$ ——— (4)

Equating co-efficient of s from both side,

$0 = 9A-10B$ ——— (5)

Taking $s=0$, from (2) we get,

$5 = 0+9B$

$\therefore 9B = 5 \quad \therefore B = \frac{5}{9}$

from (5), we get,

$$0 = 9A - 10 \cdot \frac{5}{9}$$

$$A = \frac{50}{81}$$

From ③,

$$-1 = \frac{50}{81} + C + D$$

$$\Rightarrow -1 - \frac{50}{81} - D = C$$

$$\Rightarrow \frac{-81 - 50 - 81D}{81} = C \quad \text{———} ⑥$$

$$y = \frac{12s - s^2 + 5/s}{s(s-9)(s-1)}$$

Applying Pardial Fraction we get,

$$12 = -10 \cdot \frac{50}{81} + \frac{5}{9} - \frac{-131 - 81D}{81} - 9D$$

$$12 + \frac{500}{81} - \frac{5}{9} = \frac{131 + 81D - 729D}{81}$$

$$\frac{972 + 500 - 45}{81} = \frac{131 - 648D}{81}$$

$$1472 - 45 - 131 = -648D$$

$$D = \frac{1296}{-648} = -2$$

From ⑥, $c = \frac{-131 - 81(-2)}{81} = \frac{31}{81}$

$$y(s) = \frac{12s^2 - s^3 + 5}{s^2(s-9)(s-1)}$$

$$= \frac{\frac{50}{81}}{s} + \frac{\frac{5}{9}}{s^2} + \frac{\frac{31}{81}}{(s-9)} + \frac{-2}{(s-1)}$$

Taking inverse Laplace we get,

$$y(t) = \frac{50}{81} t + \frac{5}{9} t + \frac{31}{81} e^{9t} - 2e^{t}$$

Ans

This is not the mendatory Part. I am just checking whether the solution is correct or not.

$$(y'(t)) = 0 + \frac{5}{9} + \frac{31}{81} \cdot 9e^{9t} - 2e^t$$

$$y'(0) = 0 + \frac{5}{9} + \frac{31}{81} \cdot 9 - 2$$

$$= \frac{5}{9} + \frac{31}{9} - 2$$

$$= \frac{5 + 31 - 18}{9}$$

$$= \frac{18}{9} = 2$$

$$\therefore y'(0) = 2 \text{ (Proved)}$$

Or, $y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^{t}$

$\therefore y(0) = \frac{50}{81} + \frac{5}{9}(0) + \frac{31}{81}e^{0} - 2e^{0}$

$= \frac{50}{81} + \frac{31}{81} - 2$

$= \frac{81}{81} - 2$

$= 1 - 2$

$= -1$ (Proved)