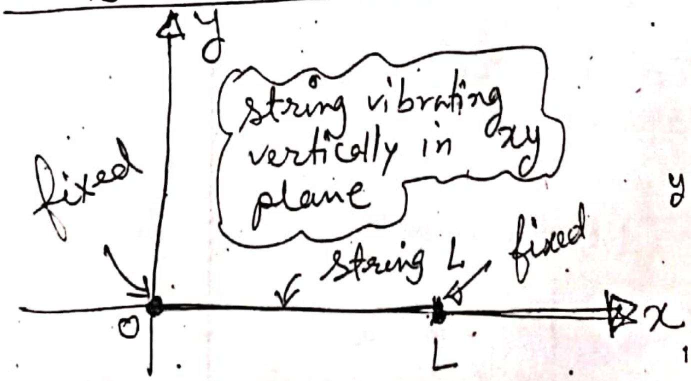
Lecture: PDE II

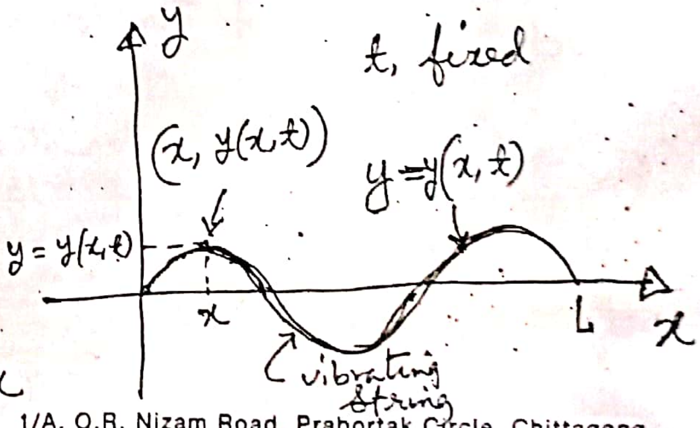
P-1

Derivation of the Wave & Heat Eqns...

- will derive several p.d.e. governing wave motion & heat conduction;
- through derivation we can get lessons about the role of boundary and initial conditions;
- [usually a p.d.e. is in terms of a function of time and one or more space variables
- in most physically motivated problems, one has data given at some time, say, at $t=0$ and this constitutes the initial conditions;
- one also often has conditions specified at different extremes of the space variables, giving boundary conditions;]
- a boundary value problem consists of a p.d.e., together with boundary conditions; often, initial conditions are also present;

The Wave Equation:



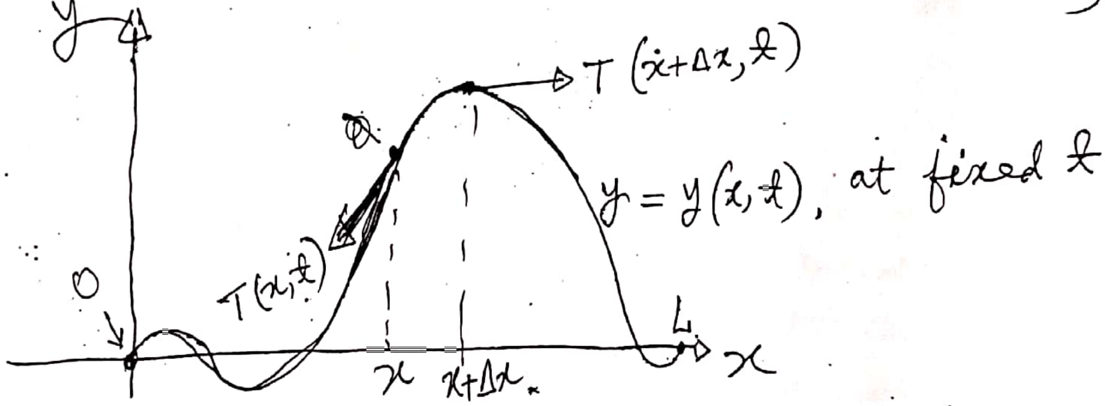


→ ~~at t, and~~ at time t, and horizontal coordinate x,
let $y(x,t)$ be the vertical displacement of string.

→ thus, the graph of $y = y(x,t)$ at any time shows
the shape of the string at that time;

→ we want to know $y(x,t)$ for $0 \le x \le L$ at time $t > 0$.

→ for simple case p.d.e., neglect damping force (due to air resistance) and weight of the string, and assume that the tension $T(x,t)$ in the string always acts tangentially; and let ρ be the mass per unit length (and ρ is constant).



Newton's 2nd Law to the segment between x and $x+\Delta x$:

Net force Due to Tension = (Segment mass × segment acceleration)

For small Δx, consideration of the vertical components (of the forces on both sides):

$$T_v(x+\Delta x, t)\sin(\theta+\Delta\theta) - T_v(x,t)\sin\theta = \rho\,\Delta x\,\frac{\partial^2 y}{\partial t^2}(\bar{x}, t)$$

where, $\bar{x}$ = x-coordinate of the center of mass of the string segment ~~(of length L)~~; ?

$$\Rightarrow \frac{T_v(x+\Delta x, t)\sin(\theta+\Delta\theta) - T_v(x,t)\sin\theta}{\Delta x} = \rho\frac{\partial^2 y}{\partial t^2}(\bar{x}, t) \quad \cdots (i)$$

As a convenience, let the vertical component of tension

$T_v(x+\Delta x, t)\sin(\theta+\Delta\theta) =$ ~~T(x,t)~~ $v(x+\Delta x, t)$ and ~~T(x,t)~~
$T_v(x,t)\sin\theta = v(x,t)$ respectively, then (i) becomes:

$$\frac{v(x+\Delta x, t) - v(x,t)}{\Delta x} = \rho\frac{\partial^2 y}{\partial t^2}(\bar{x}, t) \text{ ; and as } \Delta x \to 0,$$

we get:

$$\frac{\partial v}{\partial x} = \rho\frac{\partial^2 y}{\partial t^2} \quad \cdots (a)$$

$[\because x < \bar{x} < x+\Delta x$, and hence, $\bar{x} \to x$ as $\Delta x \to 0]$

~~For~~ As because the horizontal component of tension is zero, we get:

$$T_H(x+\Delta x, t)\cos(\theta+\Delta\theta) - T_H(x,t)\cos\theta = 0$$

for convenience we write for horizontal component

$T_H(x+\Delta x, t)\cos(\theta+\Delta\theta) = h(x+\Delta x, t)$, $T_H(x,t)\cos\theta = h(x,t)$ and

we get:

$$h(x+\Delta x, t) - h(x,t) = 0 \Rightarrow h(x,t) = h(x+\Delta x, t)$$

That is, $h(x,t)$ is independent of x; But

$$v = h\tan\theta = h\frac{\partial y}{\partial x}; \quad \left[\because \theta = \frac{\Delta y}{\Delta x} \to \frac{\partial y}{\partial x}\right]$$

$\cdots$ (b)

→ Substituting this v from (b) into (a), we get:

$$\frac{\partial}{\partial x}\left(h\frac{\partial y}{\partial x}\right) = \rho\frac{\partial^2 y}{\partial t^2} \Leftrightarrow h\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2}$$

which can be re-written as:

$$\frac{\partial^2 y}{\partial x^2} = a^2\frac{\partial^2 y}{\partial t^2}, \quad \text{where } a^2 = \frac{h}{\rho}$$

is the 1-dimensional wave equation (i.e., one independent space variable, x).

→ To uniquely determine a solution, we must know both the initial position and initial velocity of the string and thus, we must be given the initial conditions:

$y(x,0) = f(x)$, for $0 \le x \le L$; (initial position)

and $\frac{\partial y}{\partial t}(x,0) = g(x)$, for $0 \le x \le L$; (initial velocity)

Further, the data should reflect the fact that the string is fixed at both ends; and we have the boundary conditions:

$y(0,t) = y(L,t) = 0$ for $t \ge 0$;

Thus, we have found the boundary value ~~prob~~ problem for vibrating string:

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}; & 0 \le x \le L,\ t \ge 0 \\ y(0,t) = y(L,t) = 0, & t \ge 0 \\ y(x,0) = f(x),\ \frac{\partial y}{\partial t}(x,0) = g(x); & 0 \le x \le L \end{cases}$$

→ we learn to solve later

Remarks: (i) If we want an external force acting parallel to y-axis, say, having magnitude F units per unit ~~leg~~ length, the wave eqn. becomes:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} + \frac{F}{\rho} \;;\quad [\because \rho = \text{mass/unit length}]$$

(ii) If F were just the weight of the string, then the wave equation become

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} - g \;;\quad \left[\because \frac{F}{\rho} = \frac{\rho g}{\rho} = g\right]$$

(iii) If F were a damping force, say, proportional to velocity of the string ~~having~~ and having constant of proportionality (damping constant) α, then we would have:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} - \alpha \frac{\partial y}{\partial t}$$

~~(iv) If two dimensional vibration In two dimension~~

(iv) If we are looking at vibrations in a stretched membrane or drum (in two dimensions) (with no forcing), then the wave eqn. becomes:

$$\frac{\partial^2 z}{\partial t^2} = a^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

where, the membrane is initially stretched over a frame in xy plane, and vertical ~~displace~~ displacement (z) is measured as a function of x, y, and t; i.e, $z = z(x, y, t)$.

(v) If we add a force term to vibrating ~~memb~~ membrane, we get:

$$\cancel{\frac{\partial^2 z}{\partial t^2} = a\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)}$$

$$\frac{\partial^2 z}{\partial t^2} = a^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + \frac{F}{\rho}$$