Md Sadman Hafiz
2018831057

Matrix Chain Rule Assignment

Q1: Given $f(z) = \log_e(1+z)$ where $z = x^T x$, $x \in \mathbb{R}^d$

~~what~~ if $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$ then $x^T = [x_1, x_2 \dots x_d]$

Now, $x^T x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_d \end{bmatrix} \times [x_1, x_2 \dots x_d]$

$$= [x_1^2 + x_2^2 + \dots x_d^2]$$

Applying chain rule:

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{d}{dz}\left(\log_e(1+z)\right) \cdot \frac{d}{dx}(x^T x)$$

$$= \frac{1}{1+z}\left(\frac{d}{dz} z\right) \cdot \frac{d}{dx}\left(x_1^2 + x_2^2 \dots x_d^2\right)$$

$$= \frac{1}{1+z}\left(2x_1 + 2x_2 \dots + 2x_d\right)$$

$$= \frac{2}{1+z} \sum_{i=1}^{d} x_i$$

(Ans)

2. $f(z) = e^{-z/2}$, where $z = g(y), g(y) = y^T S^{-1} y$,
$y = h(x), h(x) = x - \mu$

We know,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

now, $$\frac{df}{dz} = \frac{d}{dz}\left(e^{-z/2}\right) = -\frac{e^{-z/2}}{2}$$

$$\frac{dz}{dy} = \frac{d}{dy}\left(y^T S^{-1} y\right)$$

$$= \lim_{h\to 0} \frac{g(y + h) - g(y)}{h}$$

$$= \lim_{h\to 0} \frac{(y^T + h) S^{-1} (y + h) - y^T S^{-1} y}{h}$$

$$= \lim_{h\to 0} \frac{y^T S^{-1} y + y^T S^{-1} h + h S^{-1} y + h^2 S^{-1} - y^T S^{-1} y}{h}$$

$$= \lim_{h\to 0} \frac{h(y^T S^{-1} + S^{-1} y + h S^{-1})}{h}$$

$$= \lim_{h\to 0} \left(y^T S^{-1} + S^{-1} y + h S^{-1}\right)$$

$$= y^T S^{-1} + S^{-1} y$$

Now,

$$\frac{dy}{dx} = \frac{d(x-\mu)}{dx} = 1$$

$$\therefore \frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$= -\frac{e^{-z/2}}{2} \cdot (y^T \Sigma' + \Sigma' y) \cdot 1$$

$$= -\frac{e^{-z/2}}{2} \cdot \frac{1}{\Sigma} (y^T + y)$$

(Ans.)