Matrix

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Lecture-01: Introduction to matrix

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Definition:

A matrix is a rectangular arrangement of numbers.

In short, $\boldsymbol{M} = [\boldsymbol{a_{ij}}]$, $\boldsymbol{1 \le i \le r, 1 \le j \le c}$.

$$\boldsymbol{M} = \begin{bmatrix} 2 & 4 & -7 \\ 6 & 8 & 0 \\ -3 & 5 & 8 \end{bmatrix}$$ is a Matrix.

In general, $$\boldsymbol{M} = \begin{bmatrix} a_{11} & a_{12} \dots \dots \dots a_{1c} \\ b_{21} & b_{22} \dots \dots \dots b_{2c} \\ \dots \dots \dots & \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots & \dots \dots \dots \dots \dots \dots \\ a_{r1} & a_{r2} \dots \dots \dots a_{rc} \end{bmatrix}$$

Zero Matrix:

A matrix, $\boldsymbol{M} = [\boldsymbol{a_{ij}}]$, *where* $\boldsymbol{1 \le i \le r}$ *and* $\boldsymbol{1 \le j \le c}$, is called a **Zero Matrix** if $\boldsymbol{a_{ij} = 0}$ for all $\boldsymbol{1 \le i \le r}$ *and* $\boldsymbol{1 \le j \le c}$.

Row Matrix:

A matrix, $\boldsymbol{M} = [\boldsymbol{a_{ij}}]$, *where* $\boldsymbol{1 \le i \le r}$ *and* $\boldsymbol{1 \le j \le c}$, is called a **Row Matrix** if $\boldsymbol{r = 1}$.

Example:

$\boldsymbol{A} = [5 \quad 0 \quad -1 \quad 10]$ is a row matrix.

Column Matrix:

A matrix,$\boldsymbol{M} = [\boldsymbol{a_{ij}}]$, $where\ \mathbf{1} \leq \boldsymbol{i} \leq \boldsymbol{r}\ and\ \mathbf{1} \leq \boldsymbol{j} \leq \boldsymbol{c}$**,** is called a **Column Matrix** if $\boldsymbol{c} = \mathbf{1}$.

Example:

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ -\mathbf{4} \end{bmatrix}$$ is a column matrix.

Square Matrix:

A matrix,$\boldsymbol{M} = [\boldsymbol{a_{ij}}]$, $where\ \mathbf{1} \leq \boldsymbol{i} \leq \boldsymbol{r}\ and\ \mathbf{1} \leq \boldsymbol{j} \leq \boldsymbol{c}$**,** is called a **Square Matrix** if $\boldsymbol{r} = \boldsymbol{c}$. Then we write $\boldsymbol{M} = [\boldsymbol{a_{ij}}]$**,** where $\mathbf{1} \leq \boldsymbol{i}, \boldsymbol{j} \leq \boldsymbol{r}$**.** Here **"r"** is called the **order** of the matrix.

Example:

$$\boldsymbol{M} = \begin{bmatrix} \mathbf{2} & \mathbf{4} & -\mathbf{7} \\ \mathbf{6} & \mathbf{8} & \mathbf{0} \\ -\mathbf{3} & \mathbf{5} & \mathbf{8} \end{bmatrix}$$ is a square matrix.

Identity Matrix:

A square matrix, $\boldsymbol{M} = [\boldsymbol{a_{ij}}]$, where $\mathbf{1} \leq \boldsymbol{i}, \boldsymbol{j} \leq \boldsymbol{r}$ is called an **Identity Matrix** if $\boldsymbol{a_{ij}} = \mathbf{1}$ when $\boldsymbol{a_{ij}} =\ \ \mathbf{1}$ when $\boldsymbol{i} = \boldsymbol{j}$ and $\boldsymbol{a_{ij}} = \mathbf{0}$ otherwise.

Example:

$$\boldsymbol{M} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$ is an identity matrix.

Upper Triangular Matrix:

A square matrix, $\boldsymbol{M} = [\boldsymbol{a_{ij}}]$, where $\mathbf{1} \leq \boldsymbol{i}, \boldsymbol{j} \leq \boldsymbol{r}$ is called an **Upper Triangular Matrix** if $\boldsymbol{a_{ij}} = \mathbf{0}$ when $\boldsymbol{i} > \boldsymbol{j}$.

Example:

$$\boldsymbol{U_1} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{4} & \mathbf{5} \\ \mathbf{0} & \mathbf{0} & \mathbf{6} \end{bmatrix} \ and\ \boldsymbol{U_2} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{4} \end{bmatrix}$$ are two upper triangular matrices.

Lower Triangular Matrix:

A square matrix, $\boldsymbol{M} = [\boldsymbol{a_{ij}}]$, where $\mathbf{1} \leq \boldsymbol{i}, \boldsymbol{j} \leq \boldsymbol{r}$ is called an **Lower Triangular Matrix** if $\boldsymbol{a_{ij}} = \mathbf{0}$ when $\boldsymbol{i} < \boldsymbol{j}$.

Example:

$$\boldsymbol{L_1} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{2} & \mathbf{3} & \mathbf{0} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \end{bmatrix} \ and\ \boldsymbol{L_2} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{3} & \mathbf{0} & \mathbf{5} \end{bmatrix}$$ are two lower triangular matrices.

Diagonal Matrix:

A square matrix, $\boldsymbol{M} = [\boldsymbol{a_{ij}}]$, where $\mathbf{1} \leq \boldsymbol{i}, \boldsymbol{j} \leq \boldsymbol{r}$ is called an **Diagonal Matrix** if $\boldsymbol{a_{ij}} = \mathbf{0}$ when $\boldsymbol{i} \neq \boldsymbol{j}$.

Example:

$$\boldsymbol{D} = \begin{bmatrix} \mathbf{5} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{7} \end{bmatrix}$$ is a diagonal matrix.

Scalar Matrix:

A square matrix, $\boldsymbol{M} = [\boldsymbol{a_{ij}}]$, where $\mathbf{1} \leq \boldsymbol{i}, \boldsymbol{j} \leq \boldsymbol{r}$ is called an **Scalar Matrix** if $\boldsymbol{a_{ij}} = \boldsymbol{\lambda}$ when $\boldsymbol{i} = \boldsymbol{j}$ and $\boldsymbol{a_{ij}} = \mathbf{0}$ otherwise. Here $\boldsymbol{\lambda}$ is a constant.

Example:

$$\boldsymbol{S} = \begin{bmatrix} \mathbf{5} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{5} \end{bmatrix} \text{ is a scalar matrix.}$$

Lecture-02: Basic Operations in Matrix

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Addition:

Let $\boldsymbol{A} = [\boldsymbol{a_{ij}}]$ and $\boldsymbol{B} = [\boldsymbol{b_{ij}}]$ be two matrices where $\mathbf{1} \leq \boldsymbol{i} \leq \boldsymbol{r}\ \boldsymbol{and}\ \mathbf{1} \leq \boldsymbol{j} \leq \boldsymbol{c}$. Then $\boldsymbol{C} = [\boldsymbol{c_{ij}}]$, where $\mathbf{1} \leq \boldsymbol{i} \leq \boldsymbol{r}\ \boldsymbol{and}\ \mathbf{1} \leq \boldsymbol{j} \leq \boldsymbol{c}$, is the sum of $\boldsymbol{A}$ and $\boldsymbol{B}$, that is, $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$ if and only if $\boldsymbol{c_{ij}} = \boldsymbol{a_{ij}} + \boldsymbol{b_{ij}}$ for all $\mathbf{1} \leq \boldsymbol{i} \leq \boldsymbol{r}\ \boldsymbol{and}\ \mathbf{1} \leq \boldsymbol{j} \leq \boldsymbol{c}$.

- Note that if the dimensions of $\boldsymbol{A}$ and $\boldsymbol{B}$ are not same, then their sum is **undefined**.

Example:

$$\begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{3} & \mathbf{7} \end{bmatrix} = undefined$$

$$\text{And } \begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{0} & \mathbf{3} \end{bmatrix} = \begin{bmatrix} \mathbf{3} & \mathbf{6} \\ \mathbf{4} & \mathbf{3} \end{bmatrix}$$

Subtraction:

Let $\boldsymbol{A} = [\boldsymbol{a_{ij}}]$ and $\boldsymbol{B} = [\boldsymbol{b_{ij}}]$ be two matrices where $\boldsymbol{1 \le i \le r \; and \; 1 \le j \le c}$. Then $\boldsymbol{C} = [\boldsymbol{c_{ij}}]$, where $\boldsymbol{1 \le i \le r \; and \; 1 \le j \le c}$, is the difference of $\boldsymbol{A}$ and $\boldsymbol{B}$, that is, $\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{B}$ if and only if $\boldsymbol{c_{ij}} = \boldsymbol{a_{ij}} - \boldsymbol{b_{ij}}$ for all $\boldsymbol{1 \le i \le r \; and \; 1 \le j \le c}$.

- Note that if the dimensions of $\boldsymbol{A}$ and $\boldsymbol{B}$ are not same, then their difference is **undefined**.

Example:

$$\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 4 \\ 0 & 3 & 7 \end{bmatrix} = undefined$$

And $\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & -3 \end{bmatrix}$

Scalar Multiplication:

Let $\boldsymbol{A} = [\boldsymbol{a_{ij}}]$ where $\boldsymbol{1 \le i \le r \; and \; 1 \le j \le c}$. Then $\boldsymbol{C} = [\boldsymbol{c_{ij}}]$, where $\boldsymbol{1 \le i \le r \; and \; 1 \le j \le c}$, is a scalar multiple of $\boldsymbol{A}$, that is, $\boldsymbol{C} = \boldsymbol{\alpha}.\boldsymbol{A}$ if an only if $\boldsymbol{c_{ij}} = \boldsymbol{\alpha}.\boldsymbol{a_{ij}}$ for all $\boldsymbol{1 \le i \le r \; and \; 1 \le j \le c}$.

Example:

$$\frac{1}{2} \times \begin{bmatrix} 2 & 4 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

Matrix Multiplication:

Let $\boldsymbol{A} = [\boldsymbol{a_{ij}}]$ where $\boldsymbol{1 \le i \le r \; and \; 1 \le j \le c'}$

and $\boldsymbol{B} = [\boldsymbol{b_{ij}}]$ where $\boldsymbol{1 \le i \le c' \; and \; 1 \le j \le c}$ be two matrices.

Then $\boldsymbol{C} = [\boldsymbol{c_{ij}}]$ where $\boldsymbol{1 \le i \le r \; and \; 1 \le j \le c}$ is the matrix product of $\boldsymbol{A}$ and $\boldsymbol{B}$, that is, $\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$ if and only if $\boldsymbol{c_{ij}} = \sum_{k=1}^{n} \boldsymbol{a_{ik}} \times \boldsymbol{b_{kj}}$.

For example, $\boldsymbol{c_{11}} = \boldsymbol{a_{11}b_{11}} + \boldsymbol{a_{12}b_{21}} + \boldsymbol{a_{13}b_{31}} + \cdots + \boldsymbol{a_{1n}b_{n1}} = \sum_{k=1}^{n} \boldsymbol{a_{1k}}.\boldsymbol{b_{k1}}$

Example:

$$\begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} = undefined$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [14]$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Here, $\boldsymbol{AB \neq BA}$.

So, **Matrix Multiplication** does not follow **Communicative Law of Multiplication**.

Transpose of A Matrix:

Let $= [\boldsymbol{a_{ij}}]$ **,** where $\boldsymbol{1 \leq i \leq r}$ *and* $\boldsymbol{1 \leq j \leq c}$**.** Then $\boldsymbol{C} = [\boldsymbol{c_{ij}}]$**,** where $\boldsymbol{1 \leq i \leq r}$ *and* $\boldsymbol{1 \leq j \leq c}$, is the transpose matrix of $\boldsymbol{A}$, denoted by $\boldsymbol{C = A^T}$ if an only if $\boldsymbol{c_{ij} = a_{ji}}$ for all $\boldsymbol{1 \leq i \leq r}$ *and* $\boldsymbol{1 \leq j \leq c}$.

Example:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 5 & 7 & 0 \\ 5 & 0 & 4 & 1 \\ 7 & 4 & 9 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} then\ \boldsymbol{A^T = A}.$$

Symmetric Matrix:

A square matrix, $\boldsymbol{M} = [\boldsymbol{a_{ij}}]$**,** where $\boldsymbol{1 \leq i, j \leq r}$**,** is called a **Symmetric Matrix** if $\boldsymbol{a_{ij} = a_{ji}}$ for all $\boldsymbol{1 \leq i, j \leq r}$.

Remark: If $\boldsymbol{M}$ is a symmetric matrix, then $\boldsymbol{M = M^T}$.

Skew-symmetric Matrix:

A square matrix, $\boldsymbol{M} = [\boldsymbol{a_{ij}}]$, where $\mathbf{1} \leq \boldsymbol{i}, \boldsymbol{j} \leq \boldsymbol{r}$, is called a **Symmetric Matrix** if $\boldsymbol{a_{ij}} = -\boldsymbol{a_{ji}}$ for all $\mathbf{1} \leq \boldsymbol{i}, \boldsymbol{j} \leq \boldsymbol{r}$.

Remark: If $\boldsymbol{M}$ is a skew-symmetric matrix, then $\boldsymbol{M} = -\boldsymbol{M^T}$.

Some Properties of Matrix Transpose:

1. $(\boldsymbol{A} + \boldsymbol{B})^{\boldsymbol{T}} = \boldsymbol{A^T} + \boldsymbol{B^T}$
2. $(\boldsymbol{A} - \boldsymbol{B})^{\boldsymbol{T}} = \boldsymbol{A^T} - \boldsymbol{B^T}$
3. $(\boldsymbol{\alpha A})^{\boldsymbol{T}} = \boldsymbol{\alpha A^T}$
4. $(\boldsymbol{AB})^{\boldsymbol{T}} = \boldsymbol{B^T A^T}$
5. $(\boldsymbol{A^T})^{\boldsymbol{T}} = \boldsymbol{A}$

Class Work:

For any square matrix, A, do the following operations and comment about the result matrices:

1. $\boldsymbol{A} + \boldsymbol{A^T}$
2. $\boldsymbol{A} - \boldsymbol{A^T}$
3. $\boldsymbol{A} \times \boldsymbol{A^T}$

Comments: $\boldsymbol{A} + \boldsymbol{A^T}$ = Symmetric, $\boldsymbol{A} - \boldsymbol{A^T}$ = Skew-symmetric and

$\boldsymbol{A} \times \boldsymbol{A^T}$ = symmetric.

Theorem:

For any square matrix, $\boldsymbol{A}$ –

(i) $\boldsymbol{A} + \boldsymbol{A^T}$ is symmetric;

(ii) $\boldsymbol{A} - \boldsymbol{A^T}$ is skew-symmetric and

(iii) $\boldsymbol{A} \times \boldsymbol{A^T}$ is symmetric.

Home Work:

Prove the above theorem.

#Proof of the previous theorem:

(i) Let $\boldsymbol{X = A + A\hat{}T}$

We have

$$\boldsymbol{X^T = (A + A^T)^T = A^T + (A)^T = A^T + A = A + A^T = X}$$

As $\boldsymbol{X = X^T}$, $\boldsymbol{X}$ is symmetric.

(ii) Let $\boldsymbol{X = A - A\hat{}T}$

We have

$$\boldsymbol{X^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) = -X}$$

As $\boldsymbol{X = -X^T}$, $\boldsymbol{X}$ is skew-symmetric.

(iii) Let $\boldsymbol{X = A.A^T}$

We have

$$\boldsymbol{X^T = (A.A^T)^T = (A^T)^T.A^T = A.A^T = X}$$

As $\boldsymbol{X = X^T}$, $\boldsymbol{X}$ is symmetric.

[Proved]

#Problem:

Let $\boldsymbol{A}$ be any square matrix. Find $\boldsymbol{B}$ and $\boldsymbol{C}$ such that $\boldsymbol{A = B + C}$ where $\boldsymbol{B}$ is symmetric and $\boldsymbol{C}$ is skew-symmetric.

Solution:

Theory

Let $\boldsymbol{A = [a_{ij}], B = [b_{ij}]}$ & $\boldsymbol{C = [c_{ij}]}$ be three square matrices such that $\boldsymbol{1 \le i, j \le r}$.

If $\boldsymbol{A = B + C}$ where $\boldsymbol{B}$ is symmetric and $\boldsymbol{C}$ is skew-symmetric, then

$$\boldsymbol{b_{ij} = \frac{a_{ij} + a_{ji}}{2} \;\; and \;\; c_{ij} = \frac{a_{ij} - a_{ji}}{2}}$$

For all $\boldsymbol{1 \le i, j \le r}$.