Probability

If an experiment is repeated n times under essentially identical conditions and the event A occurs m times, then as n gets large the ratio $\frac{m}{n}$ approaches the probability of A.

$$\boxed{P(A) = \frac{m}{n}}$$

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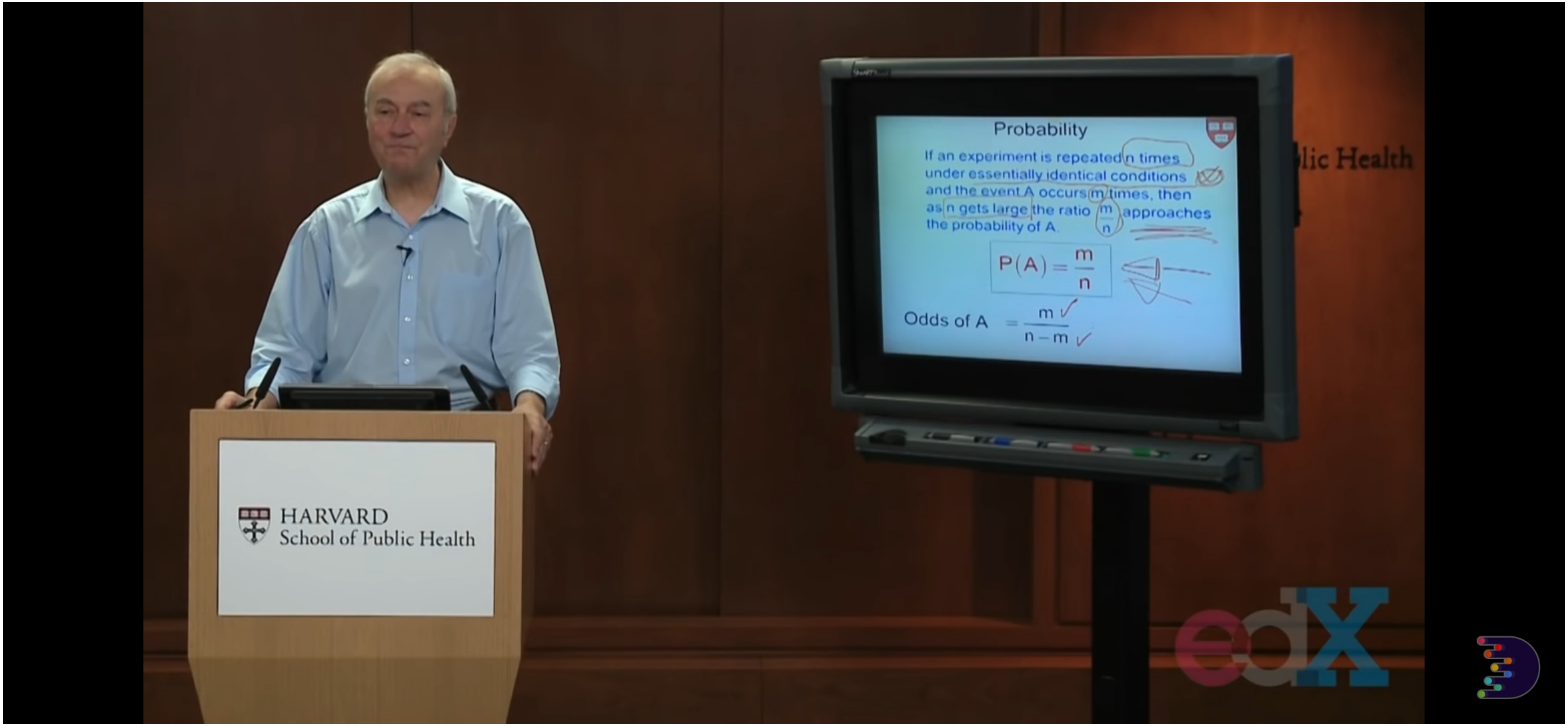
Odds of A

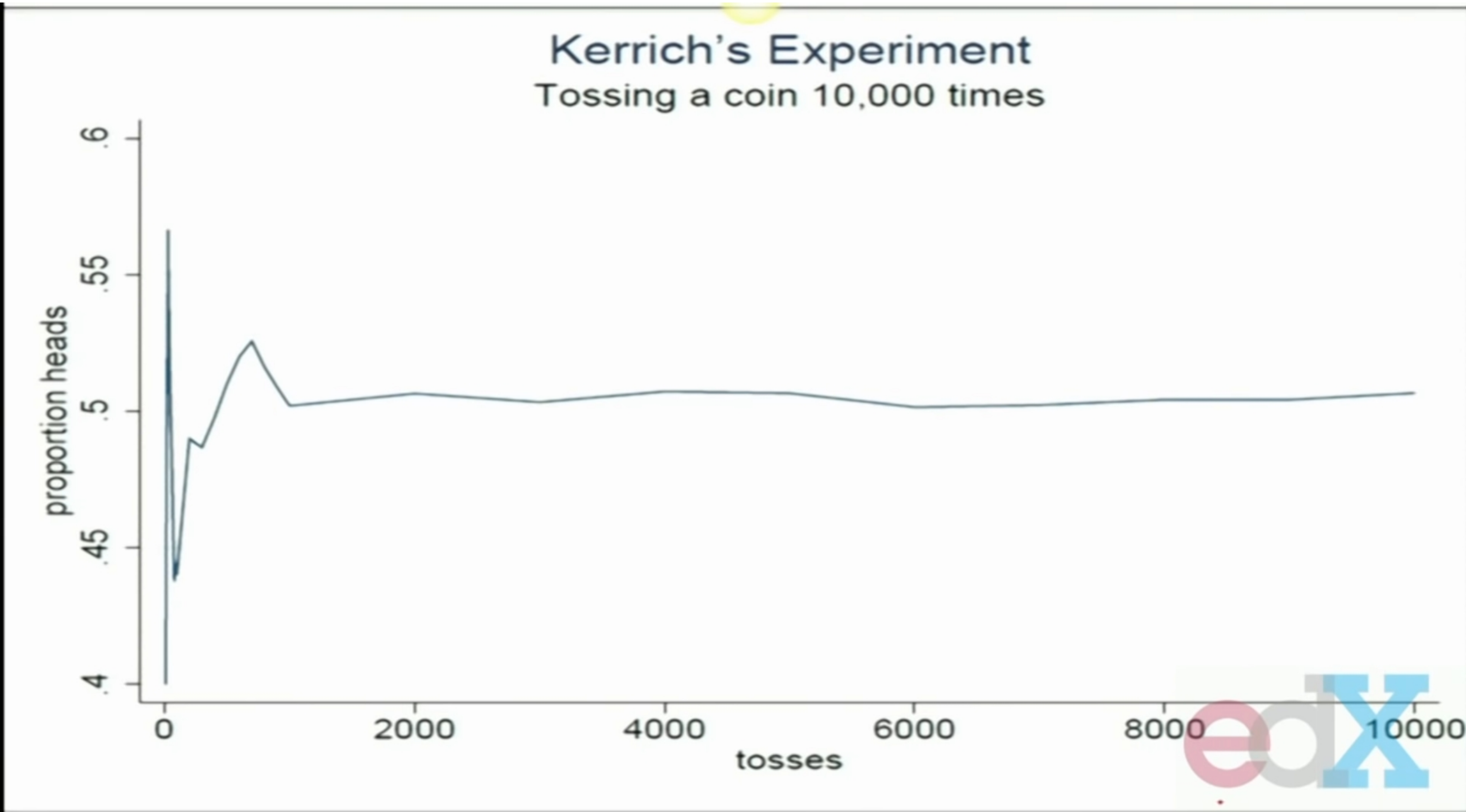

Probability
If an experiment is repeated n times
under essentially identical conditions
and the event A occurs m times, then
as n gets large the ratio m/n approaches
the probability of A.
P(A) = m/n
Odds of A = m / (n − m)
HARVARD
School of Public Health
edX

Probability

If an experiment is repeated n times under essentially identical conditions and the event A occurs m times, then as n gets large the ratio $\frac{m}{n}$ approaches the probability of A.

$$\boxed{P(A) = \frac{m}{n}}$$

$$\text{Odds of A} = \frac{m}{n-m} = \frac{P(A)}{1-P(A)}$$


Kerrich's Experiment
Tossing a coin 10,000 times
proportion heads
.4
.45
.5
.55
.6
0
2000
4000
6000
8000
10000
tosses

Something must happen:

$$\frac{n}{n} = 1$$

$$\Pr(\text{sure thing}) = 1$$

Impossible:

$$\frac{0}{n} = 0$$

$$\Pr(\text{impossible}) = 0$$

For *any* event A

$m \leq n$, so

$0 \leq Pr(A) \leq 1$

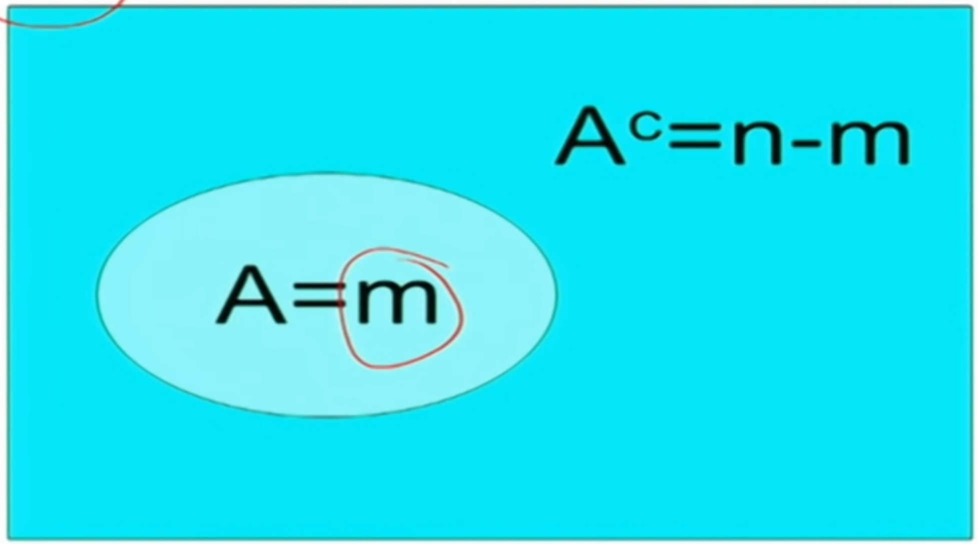
Complement

$$P(A) = \frac{m}{n}$$

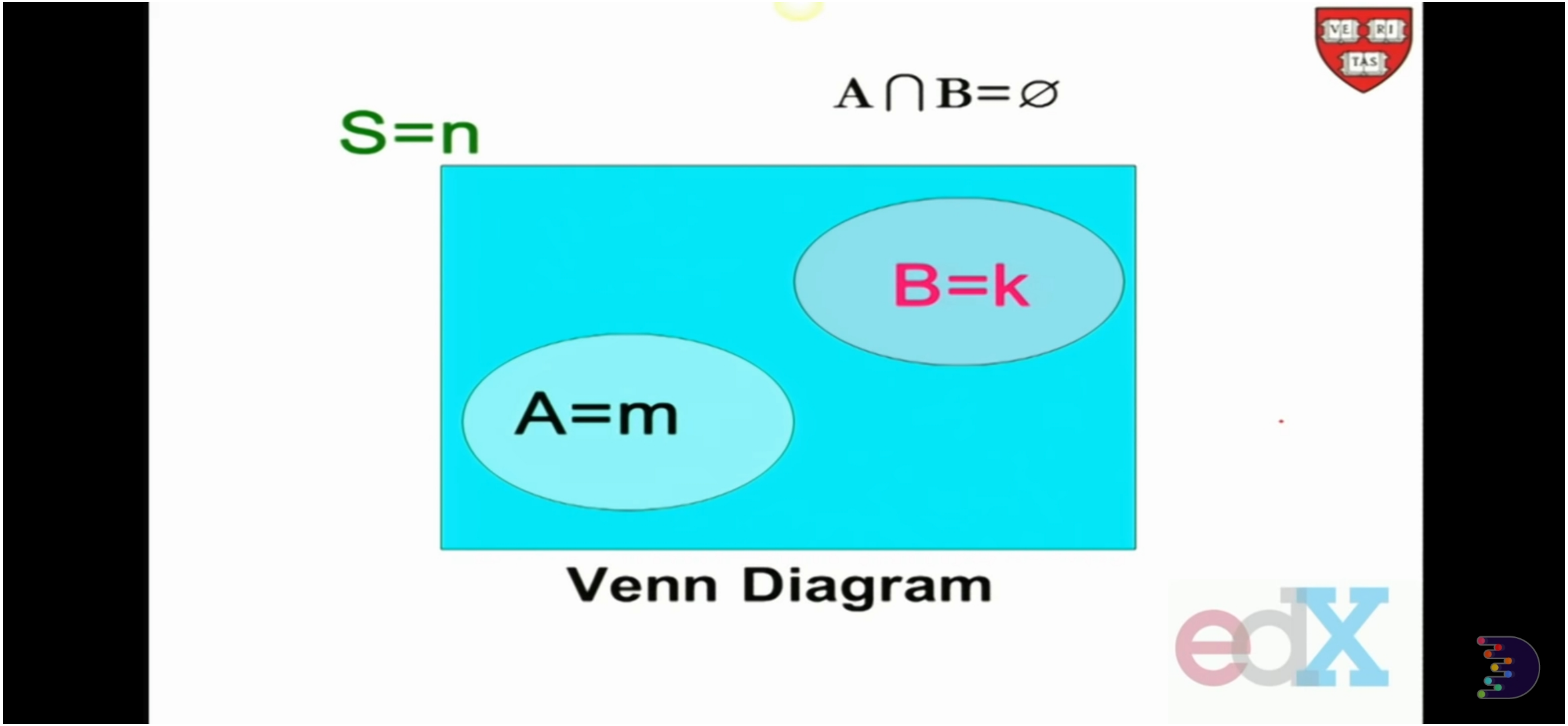
$$P(A^c) = \frac{n-m}{n} = 1 - P(A)$$

$$P(A) + P(A^c) = 1$$

$S = n$



Venn Diagram


$A \cap B = \varnothing$
S=n
B=k
A=m
Venn Diagram

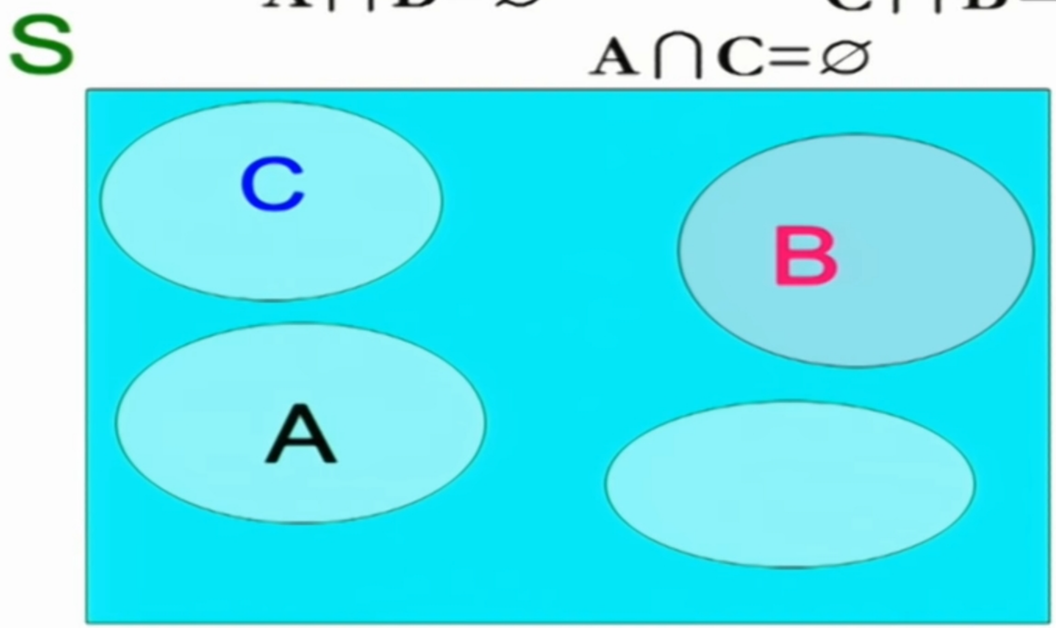
Additive Law

If the events A, B, C, ... are mutually exclusive – so at most one of them may occur at any one time – then :

$$P(A \cup B \cup C \ldots) = P(A) + P(B) + P(C) \ldots$$

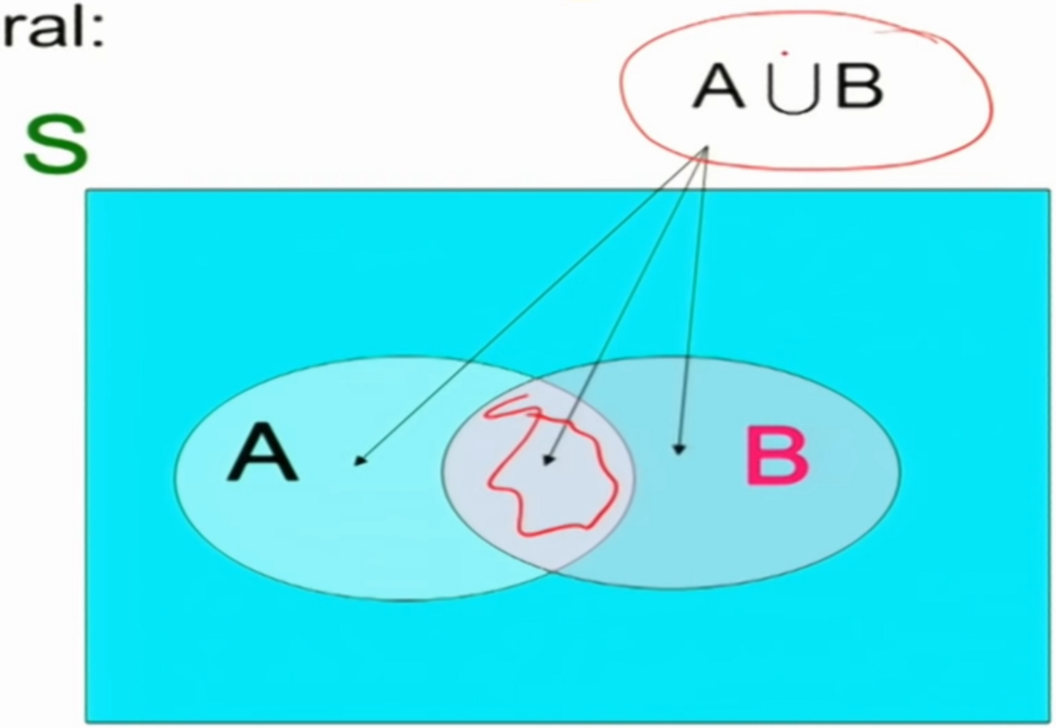
$A \cap B = \varnothing$ $\quad C \cap B = \varnothing$

$A \cap C = \varnothing$

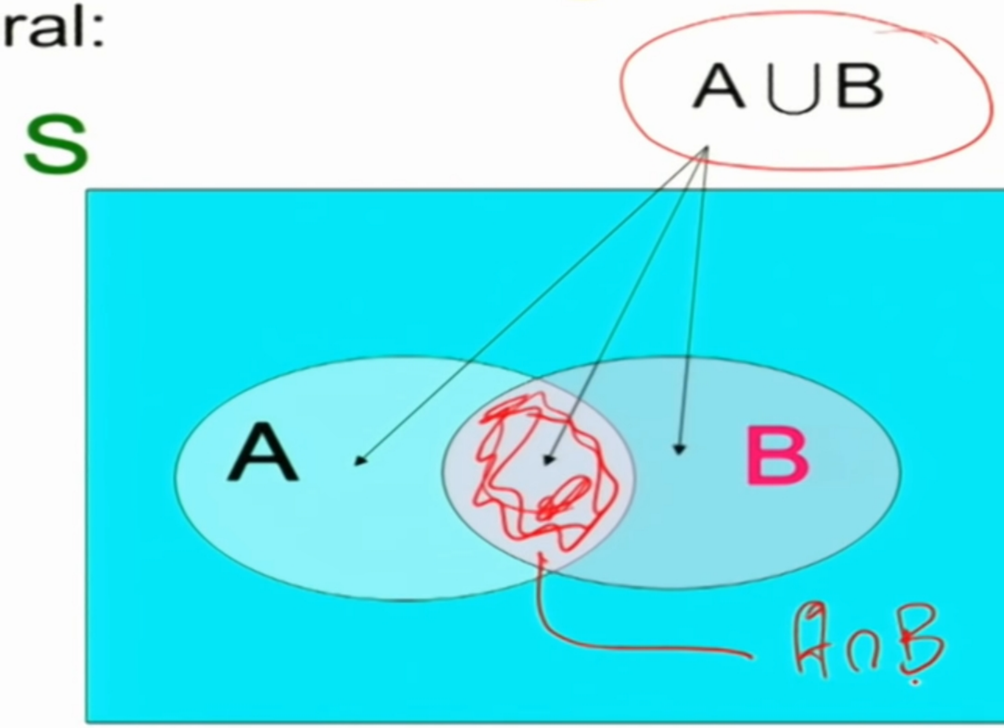


Venn Diagram

In general:



In general:



So $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Conditional Probability

Notation: $P(B \mid A)$

is the probability of B *given,* or knowing that the event A has happened.

Conditional Probability

Notation: P(B | A)   P(B)

is the probability of B *given,* or knowing that the event A has happened.

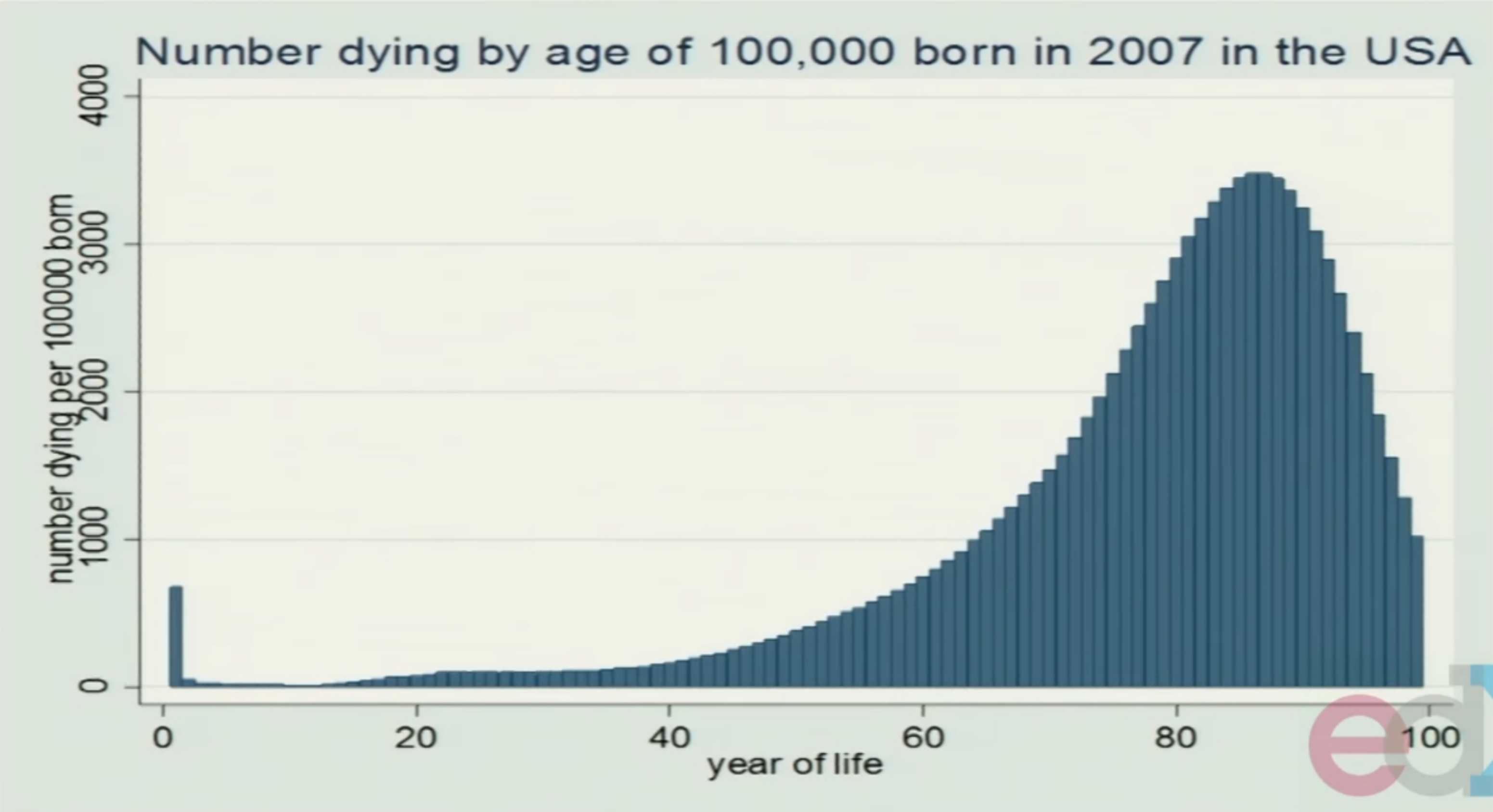
B="A person in US will live to be 70"

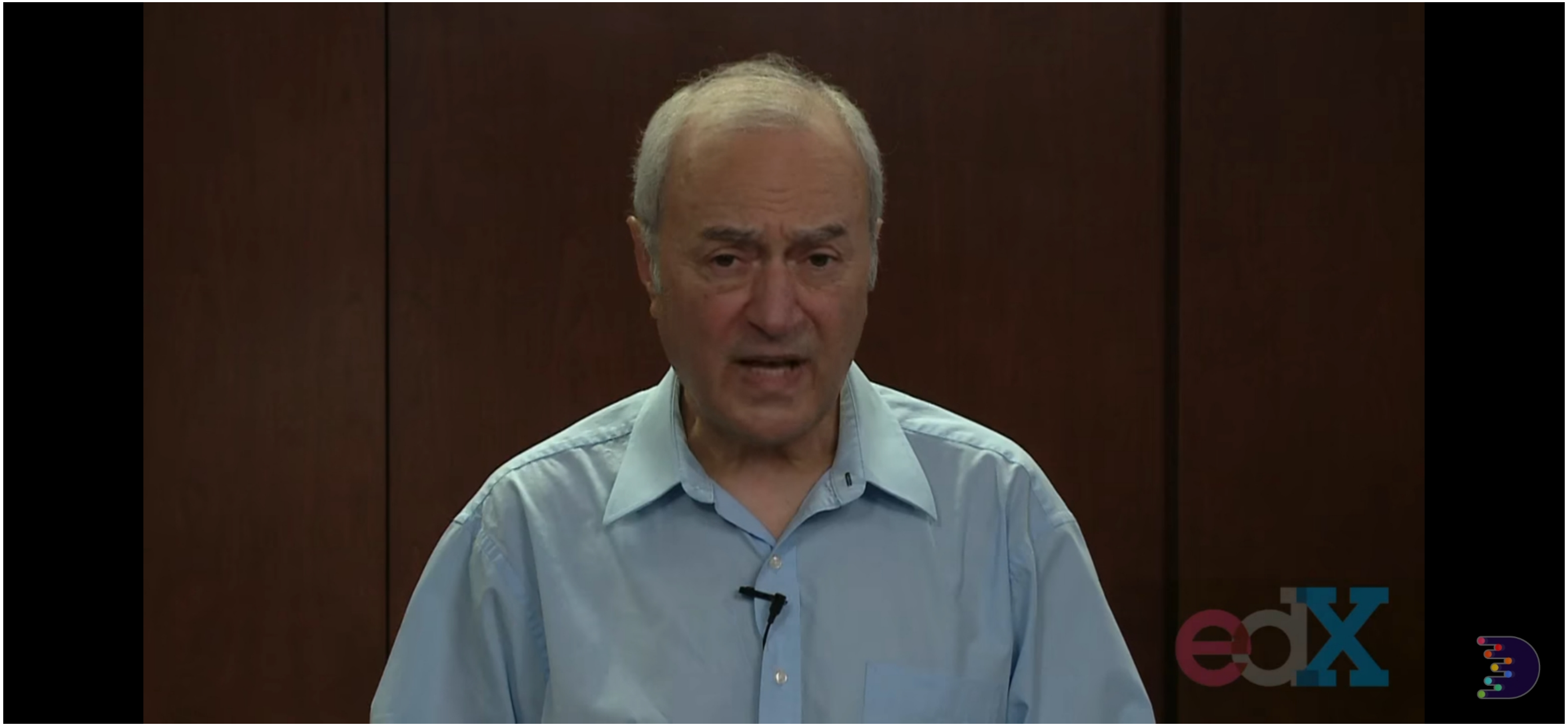
A="A person is alive at age 65"

Then

B | A="A 65 year old person will be alive at 70"

Density function

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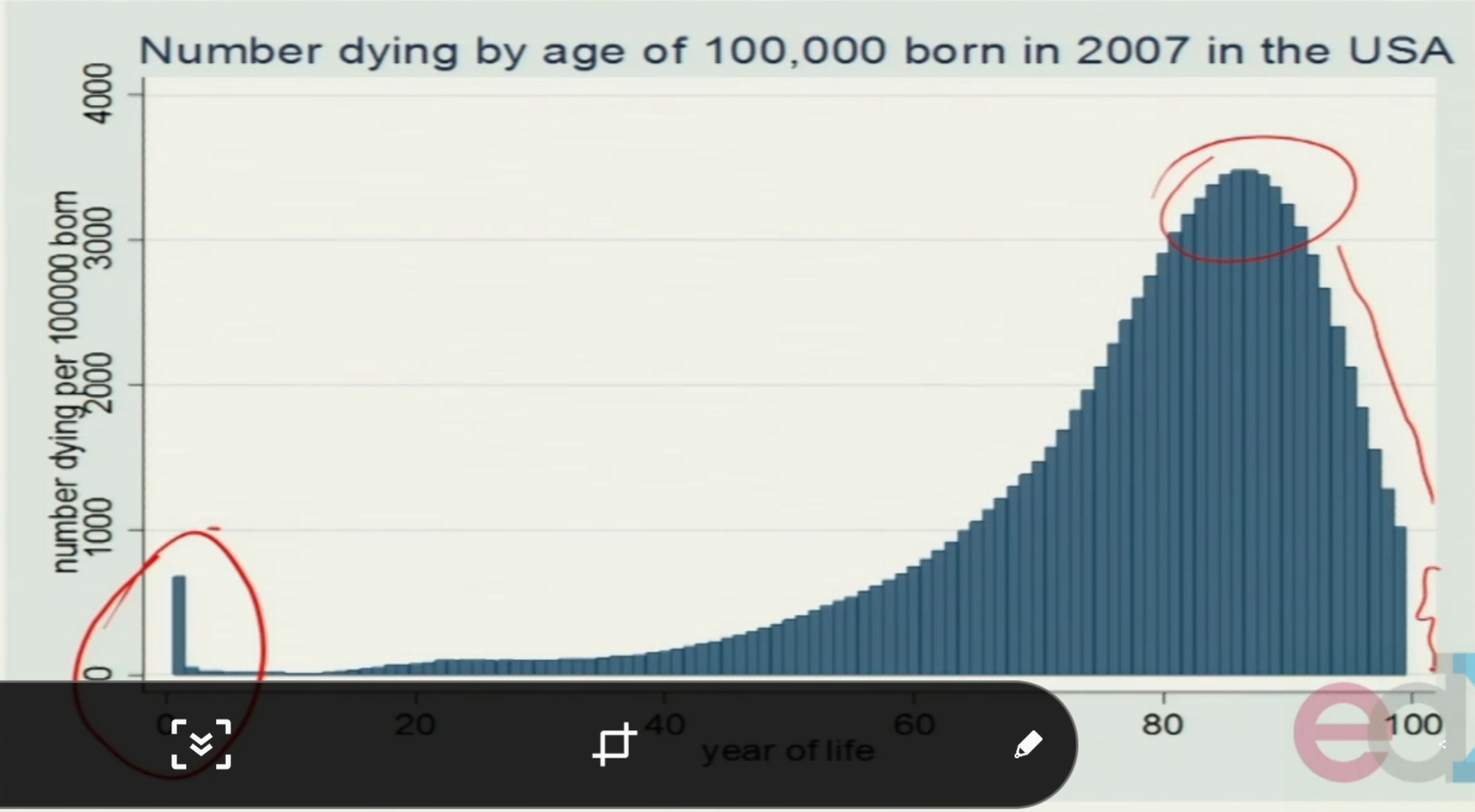
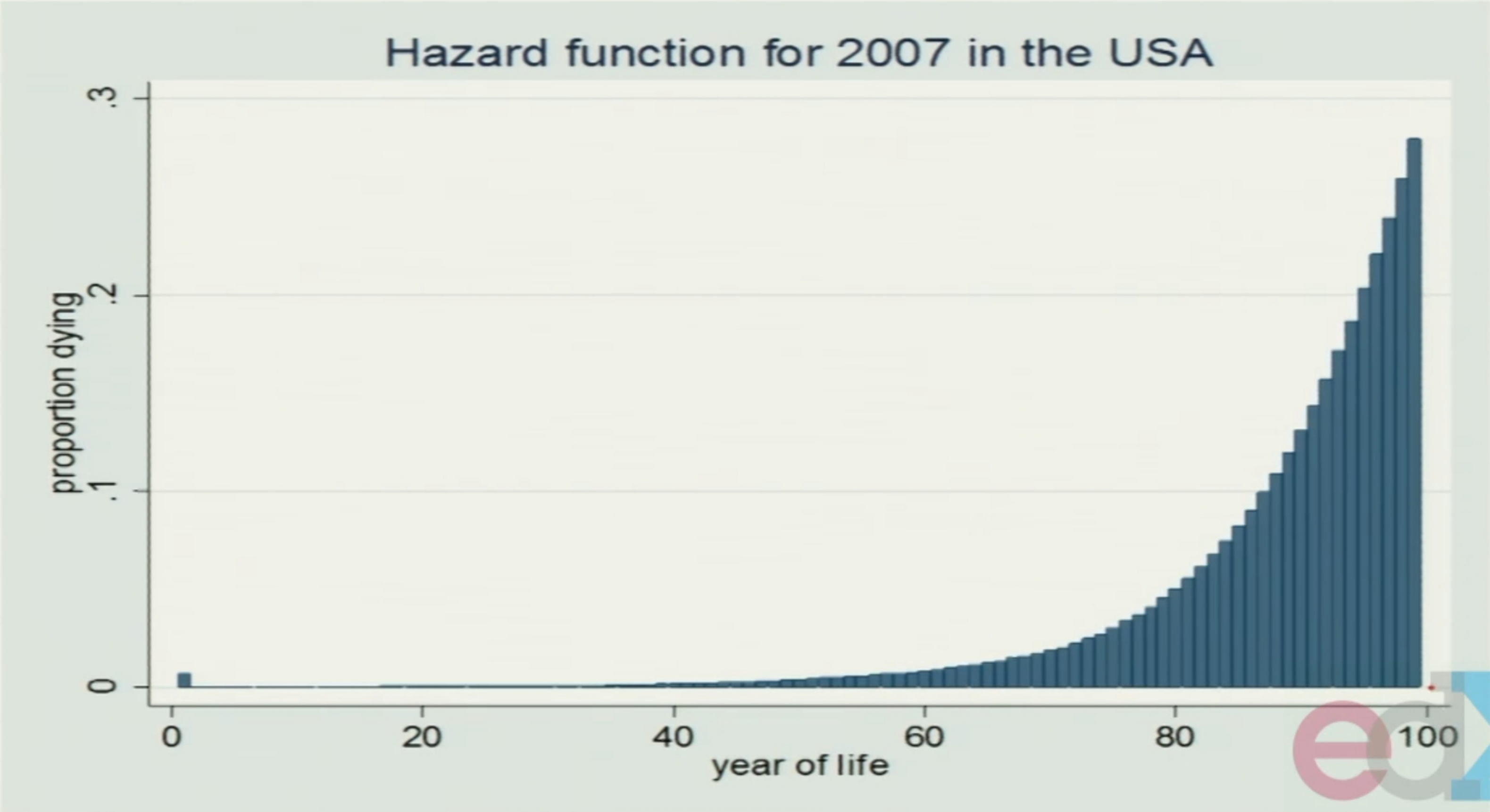
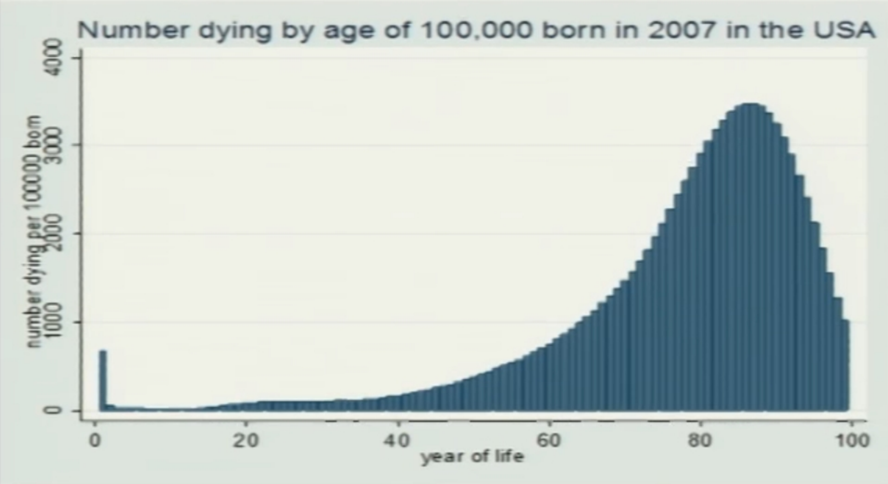
Density function



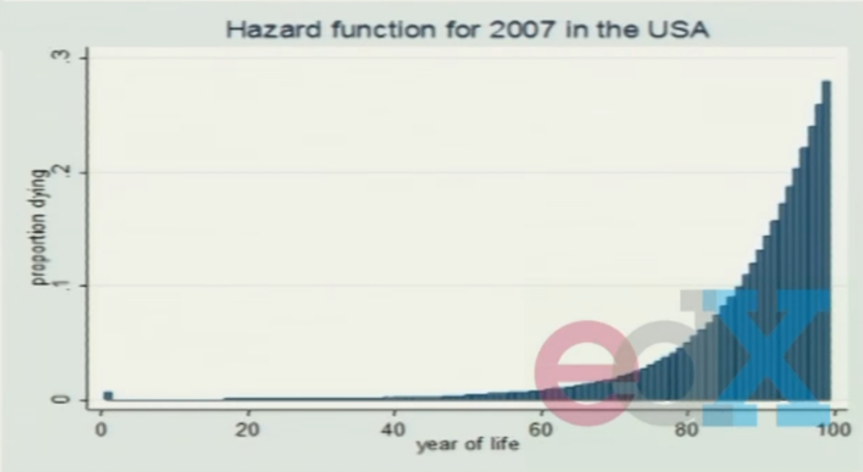
Proportion who die in their next year



Density function



Proportion who die
in their next year (hazard)



Formula:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

B = "A person will be alive at 70"

A = "A person will be alive at age 65"

B | A = "A 65 year old person will be alive at 70"

$A \cap B$ = "A person will reach 65 & 70"

= "A person reaches 70"

Life table (segment) for the total population: United States, 2007

100,000

Age	Probability of dying between ages x to $x+1$	Number surviving to age x	Number dying between ages x to $x+1$
	q_x	l_x	d_x
65–66	0.013600	83,587	1,137
66–67	0.014722	82,451	1,214
67–68	0.015959	81,237	1,296
68–69	0.017288	79,940	1,382
69–70	0.018755	78,558	1,473
70–71	0.020424	77,085	1,574

National Vital Statistics Reports, Vol. 59, No. 9, September 28, 2011

Formula: $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$

e.g. 2007 lifetable: born: 100,000

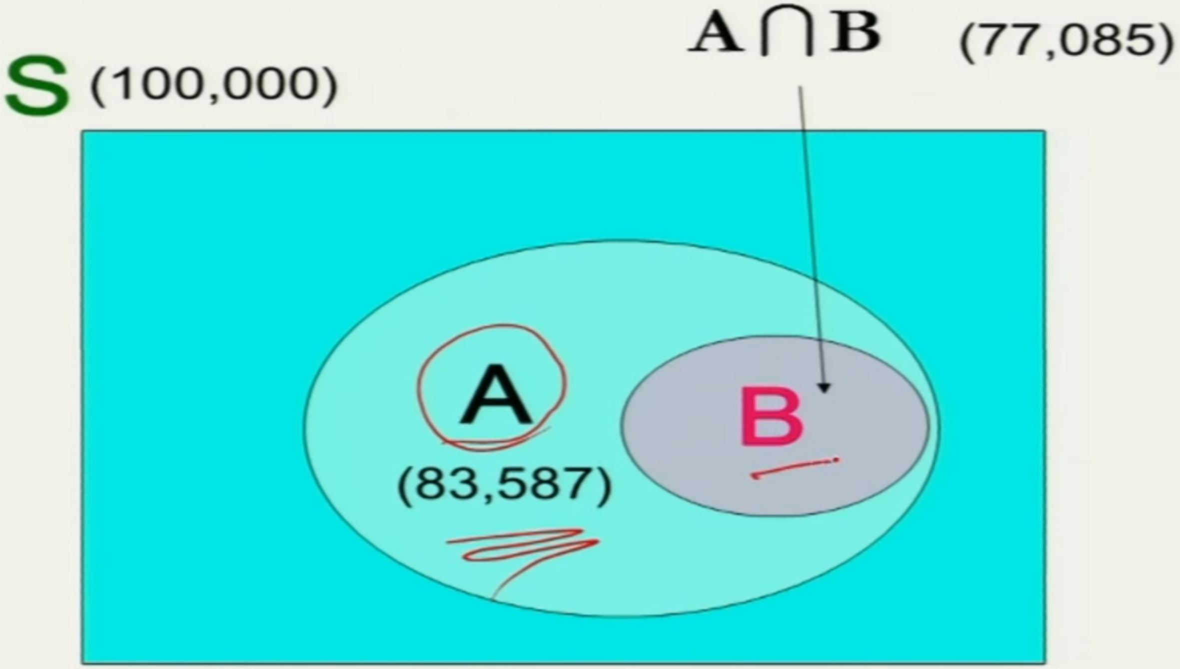
$\boxed{65 : 83,587}$ $\boxed{70 : 77,085}$

$$P(A \cap B) = \frac{77,085}{100,000} = 0.77$$

$$P(A) = \frac{83,587}{100,000}$$

$$P(B|A) = \boxed{\frac{77,085/100,000}{83,587/100,000}} = \frac{77,085}{83,587} = 0.92$$

Venn Diagram



$$P(B) = 0.77 \quad \text{versus} \quad P(B \mid A) = \frac{77{,}085}{83{,}587} = 0.92$$

From the formula for conditional probability,

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

we get the multiplicative law

$$P(A \cap B) = P(A)\, P(B \mid A)$$

Note:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Independence

A and B are said to be independent if:

$$P(A \cap B) = P(A)\,P(B)$$

and since in general

$$P(A \cap B) = P(A)\,P(B \mid A)$$

So independence implies

$$P(B \mid A) = P(B)$$

Similarly

$$P(A \mid B) = P(A)$$

Mendelian Segregation

Gregor Mendel
1823-1884

Sperm

Egg	A 1/2	a 1/2
A 1/2		
a 1/2		

Probabilities

Gregor Mendel
1823-1884

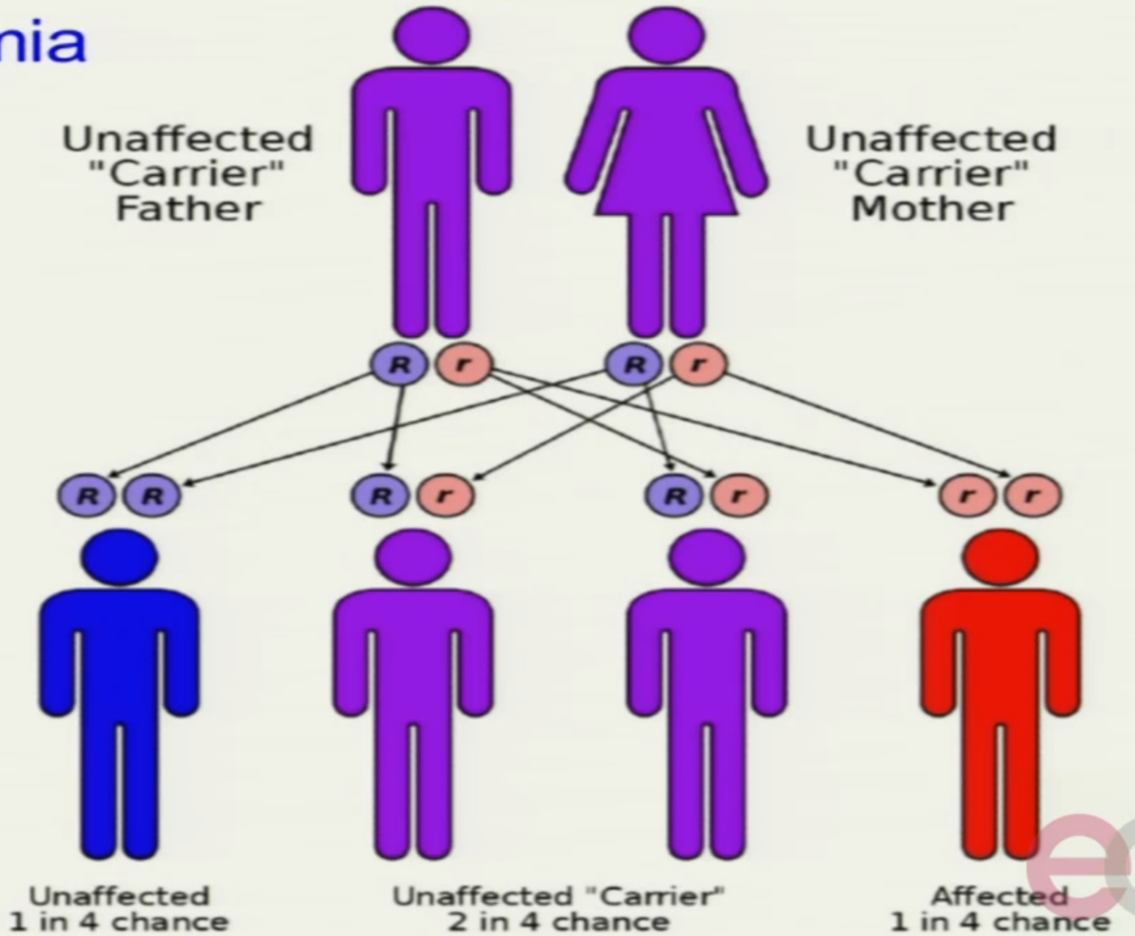
Mendelian Segregation

(A,a) $\longleftrightarrow$ (A,a)

$\downarrow$

(A,A)	1/4
(A,a)	1/2
(a,a)	1/4

Thalassemia



Clarification aid:

IF A and B are mutually exclusive then
(Additive Law)

$$P(A \cup B) = P(A) + P(B)$$

IF A and B are independent then
(Multiplicative Law)

$$P(A \cap B) = P(A) \times P(B)$$

Sally Clark and Roy Meadow

Sally Clark was a British solicitor

Had a son in September of 1996.

He died in December of 1996.

Had a son in November of 1997.

He died in February of 1998.

She is accused and tried for murder.

Found guilty.

"Expert" witness, a pediatrician, Roy Meadow claims that the chance of two SIDS deaths in a family is "one in 73 million", and that carried the day.

Meadow's Law:

one cot death is a tragedy, two cot deaths is suspicious and, until the contrary is proved, three cot deaths is murder.

The CESDI study looked at 472,823 live births.

363 deaths were identified as SIDS.

P(SIDS) = 363/472,823 = 1/1300

Meadow testimony

(i) P(SIDS) = 1/8543

(ii) P(2 SIDS) = $(1/8543)^2$ = 1/ 73 million.

Fleming P, Bacon C, Blair P, Berry PJ (eds). *Sudden Unexpected Deaths in Infancy, The CESDI Studies 1993-1996*. London: The Stationery Office, 2000.

In the CESDI report, they carried out a case-control study:

Among the 323 SIDS families studied, there were 5 previous SIDS,

P(prev. SIDS in 323) = 5/323 ≈ 0.0155

Among the 1288 control families, there were 2 previous SIDS.

P(prev. SIDS in 1288) = 2/1288 ≈ 0.00156

Ray Hill, Multiple sudden infant deaths – coincidence or beyond coincidence?
Paediatric and Perinatal Epidemiology 2004, **18,** 320–326

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× 10

Ray Hill, Multiple sudden infant deaths – coincidence or beyond coincidence? *Paediatric and Perinatal Epidemiology 2004,* **18,** 320–326

Return to the multiplication rule and now look at P(A | B):

$$P(A \mid B) = \frac{P(B \cap A)}{P(B)}$$

$$= \frac{P(A)P(B \mid A)}{P(B)},$$

assuming P(B)>0.

Return to the multiplication rule and now look at $P(A \mid B)$:

$$\boxed{P(A \mid B)} = \frac{P(B \cap A)}{P(B)}$$

$$= \frac{P(A)\boxed{P(B \mid A)}}{P(B)},$$

assuming $P(B)>0$.

This is known as Bayes' Theorem

$$P(B) = P(B \cap A) + P(B \cap A^C)$$

$$= P(A)\,P(B \mid A) + P(A^C)\,P(B \mid A^C)$$

Diagnostic tests

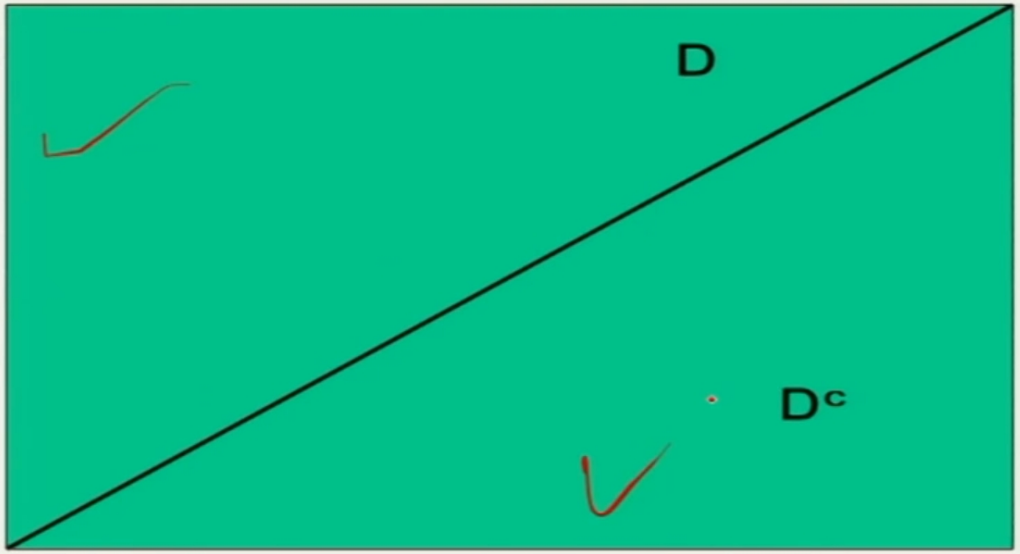
$A = D$ = "have disease"

$A^C = D^c$ = "do *not* have disease"

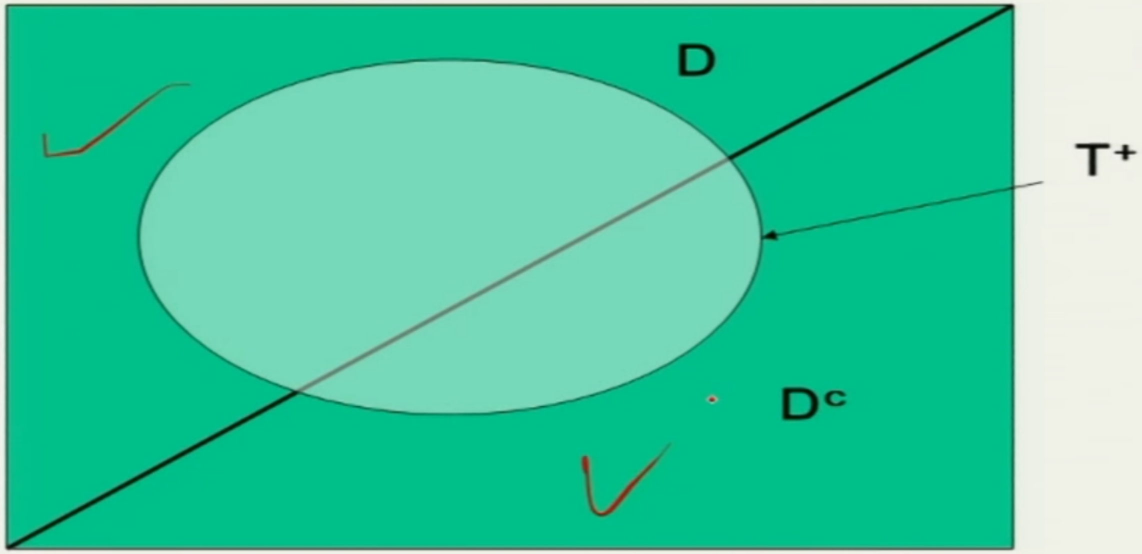
$B = T^+$ = "positive screening result"

Find $P(D \mid T^+)$

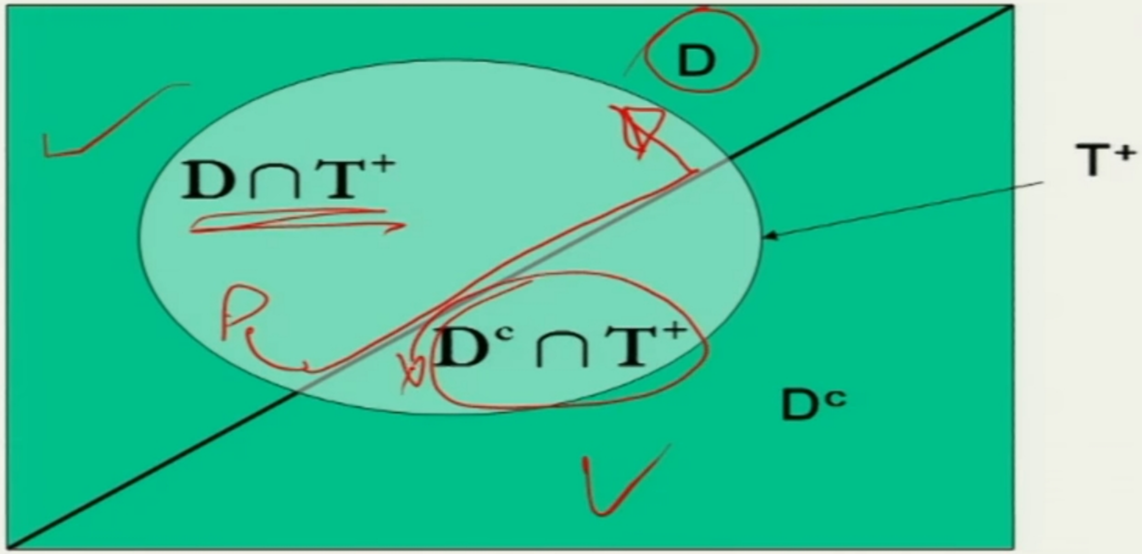
Venn diagram of Bayes' Theorem



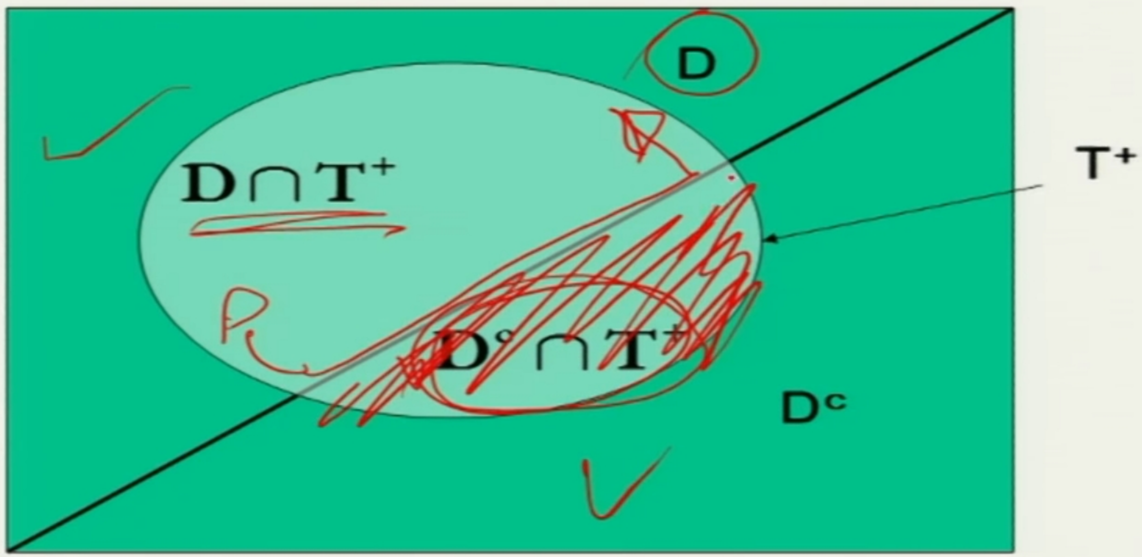
Venn diagram of Bayes' Theorem



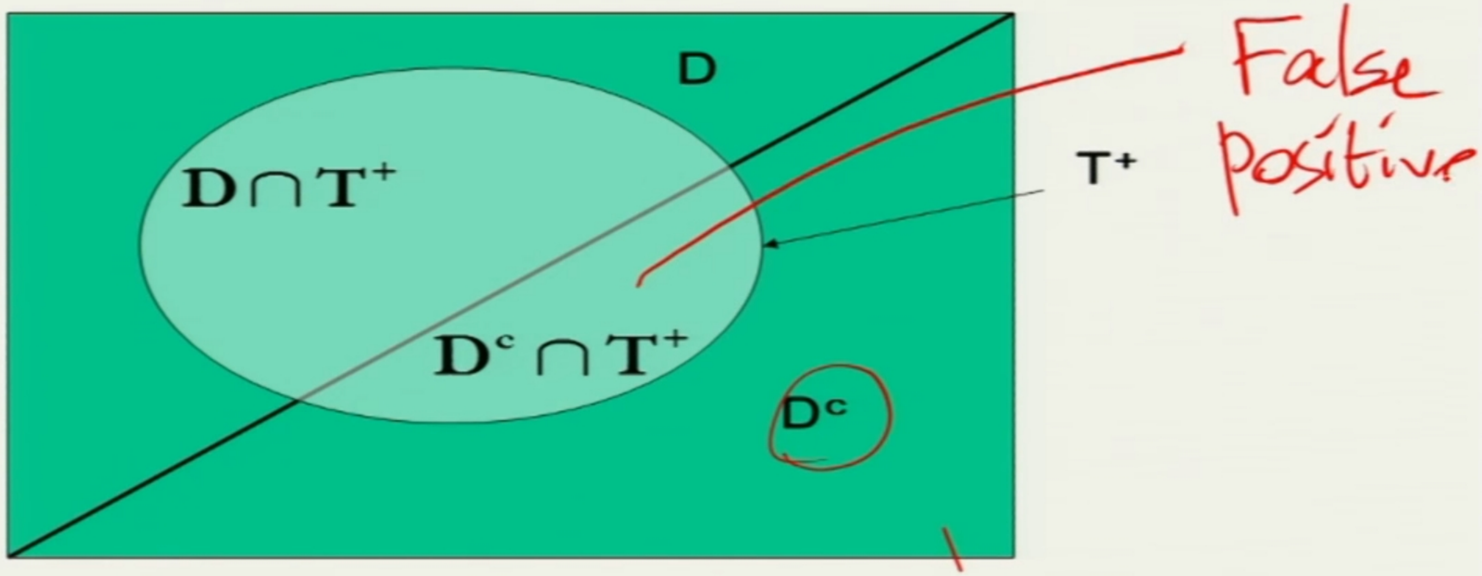
Venn diagram of Bayes' Theorem



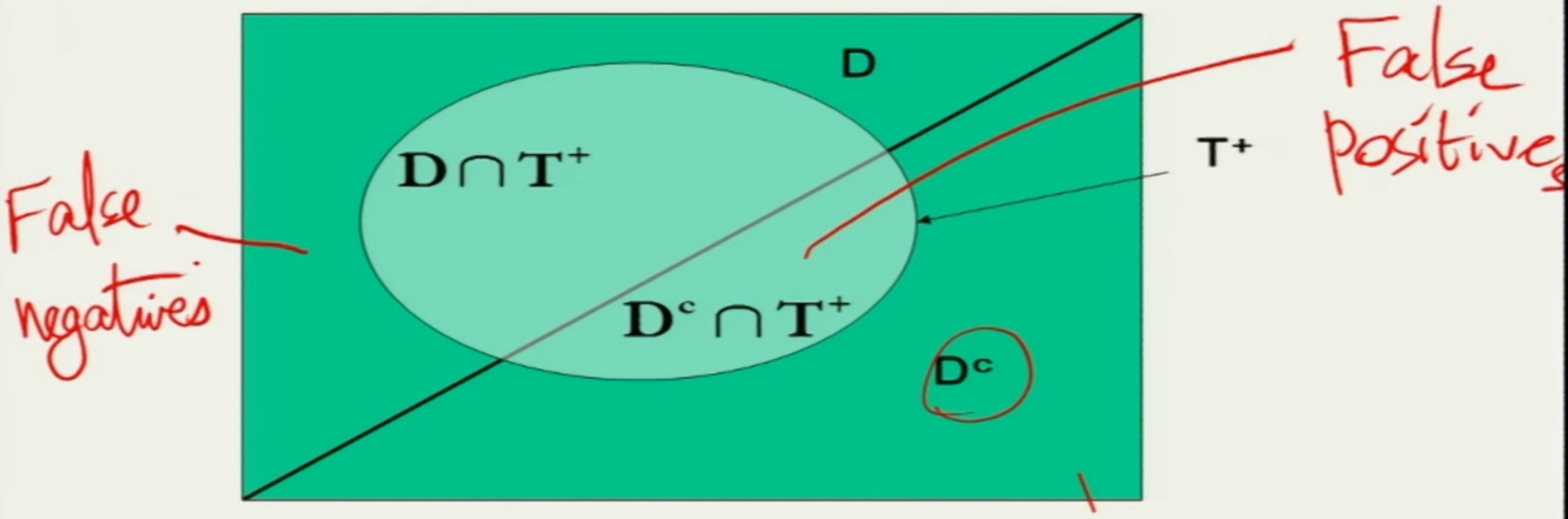
Venn diagram of Bayes' Theorem



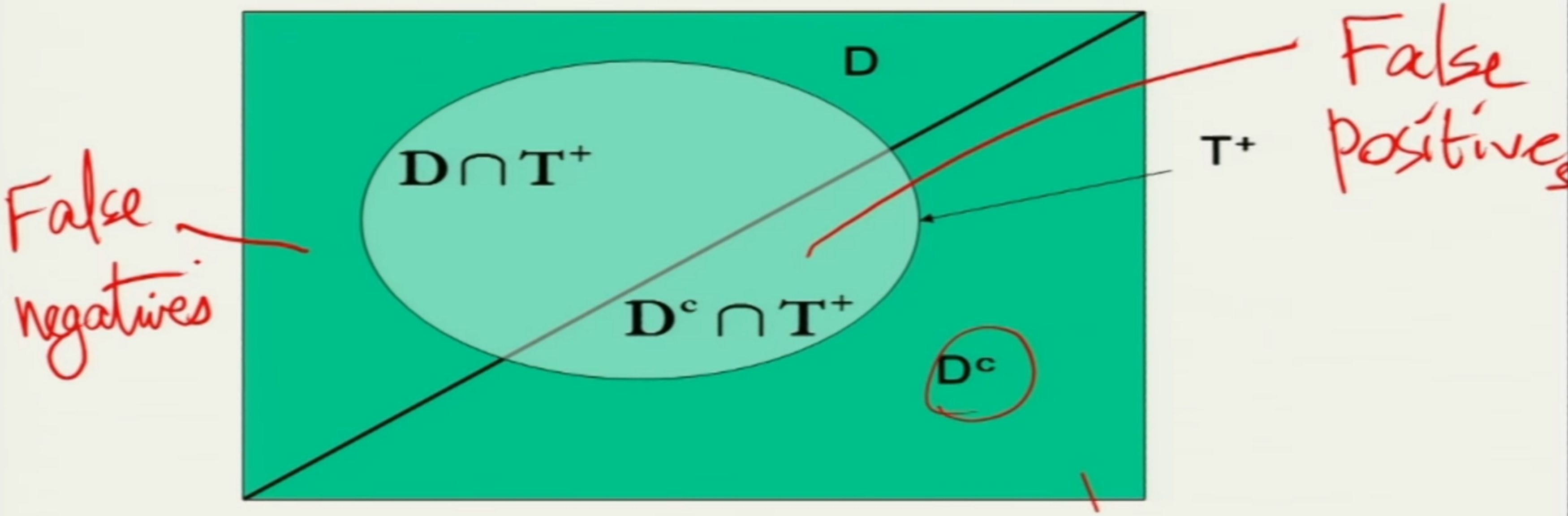
Venn diagram of Bayes' Theorem



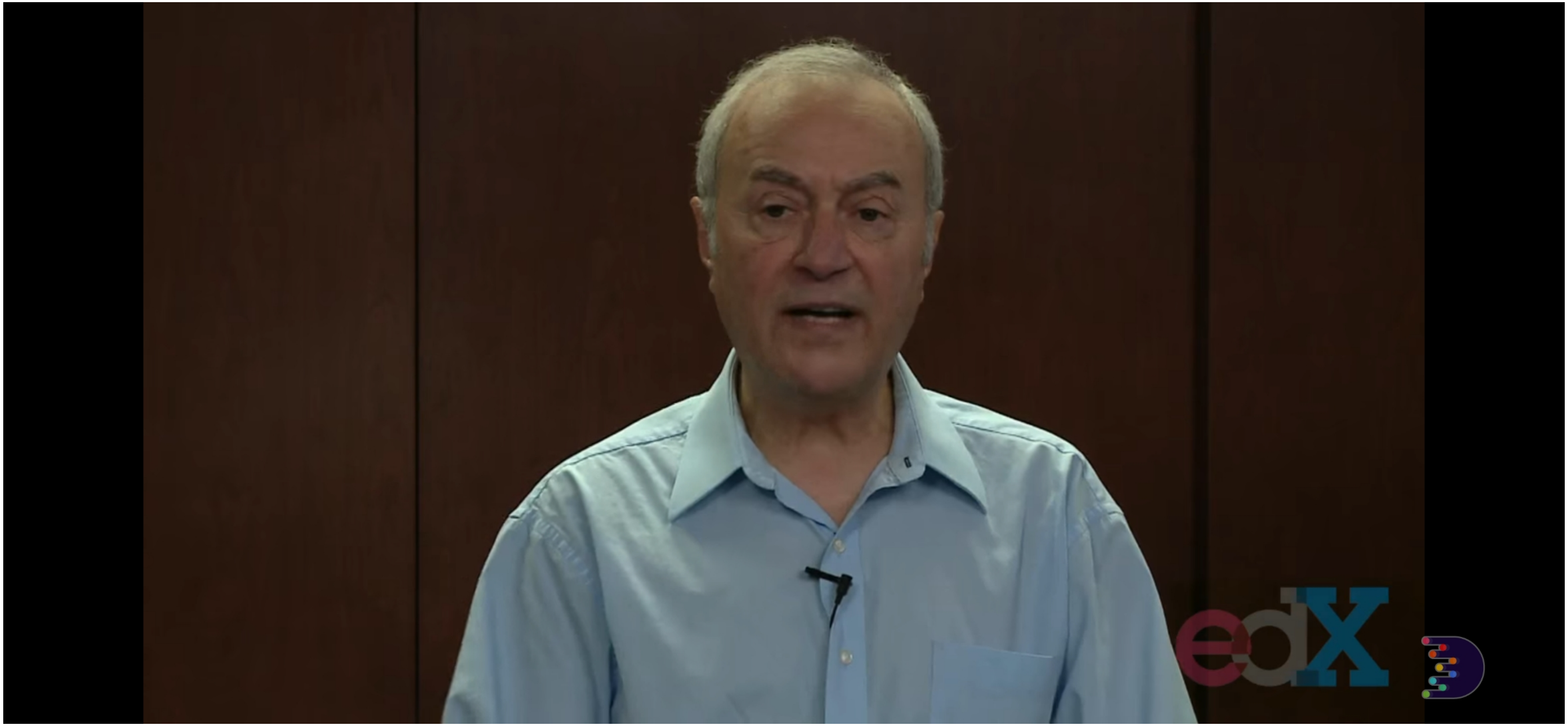
Venn diagram of Bayes' Theorem



Venn diagram of Bayes' Theorem



$$P(D|T^+) = \frac{P(D \cap T^+)}{P(T^+)} = \frac{P(D \cap T^+)}{P(D \cap T^+) + P(D^c \cap T^+)}$$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Prior to testing

	Has Disease D	Disease Free D^C		
Test Positive T^+	$P(T^+ \mid D)$ sensitivity	$P(T^+ \mid D^C)$		$P(T^+)$
Test Negative T^-	$P(T^- \mid D)$	$P(T^- \mid D^C)$ specificity		$P(T^-)$
	$P(D)$ prevalence	$P(D^C) = 1 - P(D)$		

Prior to testing

≈ 1 Ideal:

	Has Disease D	Disease Free D^C		
Test Positive T^+	$P(T^+ \mid D)$ sensitivity	$P(T^+ \mid D^C)$		$P(T^+)$
Test Negative T^-	$P(T^- \mid D)$	$P(T^- \mid D^C)$ specificity		$P(T^-)$
	$P(D)$ prevalence	$P(D^C)$ $=1- P(D)$		

Post testing

	Has Disease D	Disease Free D^C		
Test Positive T^+	$P(D \mid T^+)$ PPV	$P(D^C \mid T^+)$		$P(T^+)$
Test Negative T^-	$P(D \mid T^-)$	$P(D^C \mid T^-)$ NPV		$P(T^-) = 1 - P(T^+)$
	$P(D)$ prevalence	$P(D^C) = 1 - P(D)$		

PPV=positive predictive value NPV=negative predictive value

From Bayes' theorem:

Positive predictive value:

$$P(D \mid T^+) = \frac{P(D)\,P(T^+ \mid D)}{P(D)P(T^+ \mid D) + P(D^c)P(T^+ \mid D^c)}$$

=

Example: X-ray screening for tuberculosis

	Tuberculosis
X-ray	Yes
Positive	22
Negative	8
Total	30

$$\text{Sensitivity} = \frac{22}{30} = .7333$$

Example: X-ray screening for tuberculosis

X-ray	Tuberculosis: Yes	Tuberculosis: No
Positive	22	51
Negative	8	1739
Total	30	1790

$$\text{Sensitivity} = \frac{22}{30} = .7333$$

$$\text{Specificity} = \frac{1739}{1790} = .9715$$

Example: X-ray screening for tuberculosis

X-ray	Tuberculosis: Yes	Tuberculosis: No
Positive	22	51
Negative	8	1739
Total	30	1790

$$\text{Sensitivity} = \frac{22}{30} = .7333$$

$$\text{Specificity} = \frac{1739}{1790} = .9715$$

Example: X-ray screening for tuberculosis

X-ray	Tuberculosis: Yes	Tuberculosis: No	Total
Positive	22	51	73
Negative	8	1739	1747
Total	30	1790	1820

$$\text{Sensitivity} = \frac{22}{30} = .7333$$

$$\text{Specificity} = \frac{1739}{1790} = .9715$$

Screening for TB

Population: 1,000,000

T^+
28,565

Screening for TB

Population: 1,000,000

T^+
28,565

T^-
971,435

Screening for TB

Population:	1,000,000

Prevalence = 9.3 per 100,000

Screening for TB

Population: 1,000,000

Prevalence = 9.3 per 100,000

TB: 93

No TB: 999,907

Screening for TB

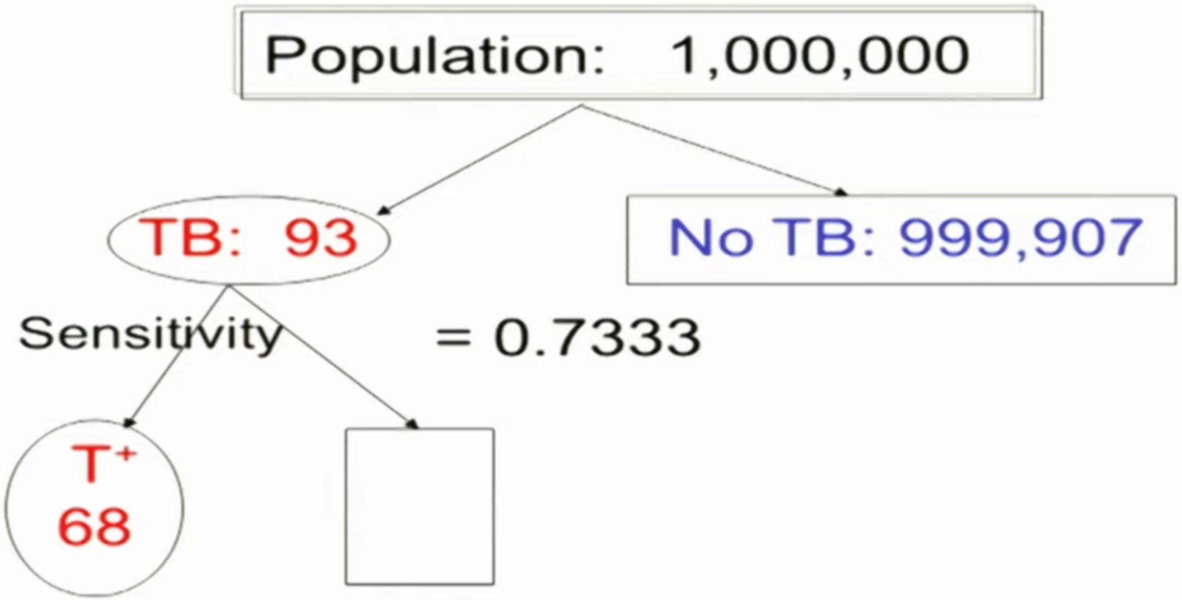
Population: 1,000,000

TB: 93

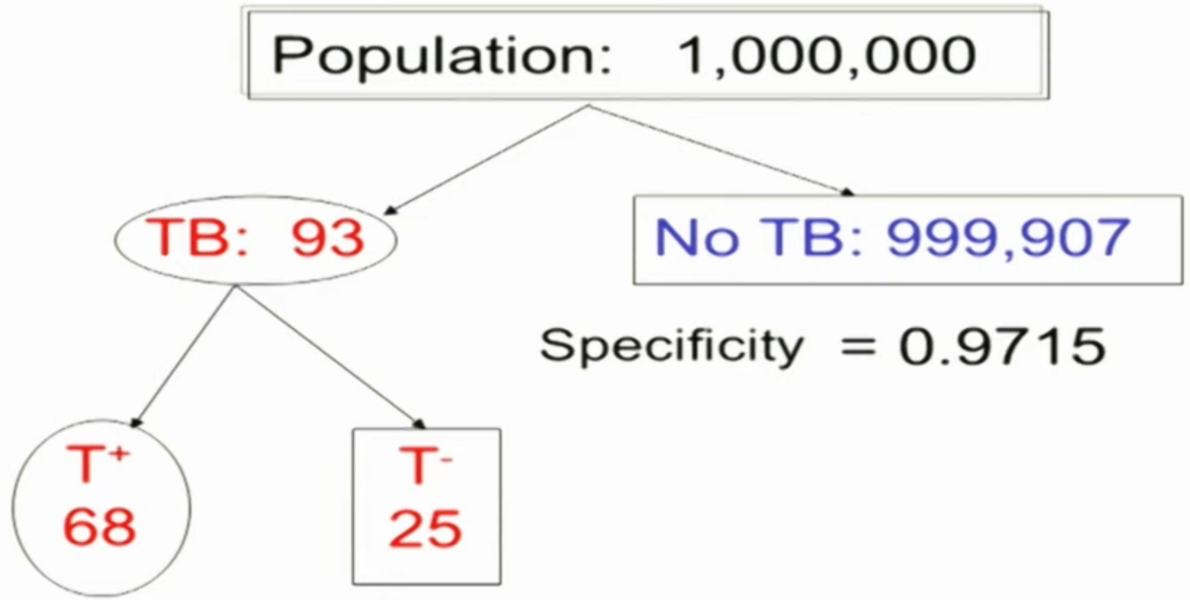
No TB: 999,907

Sensitivity = 0.7333

Screening for TB

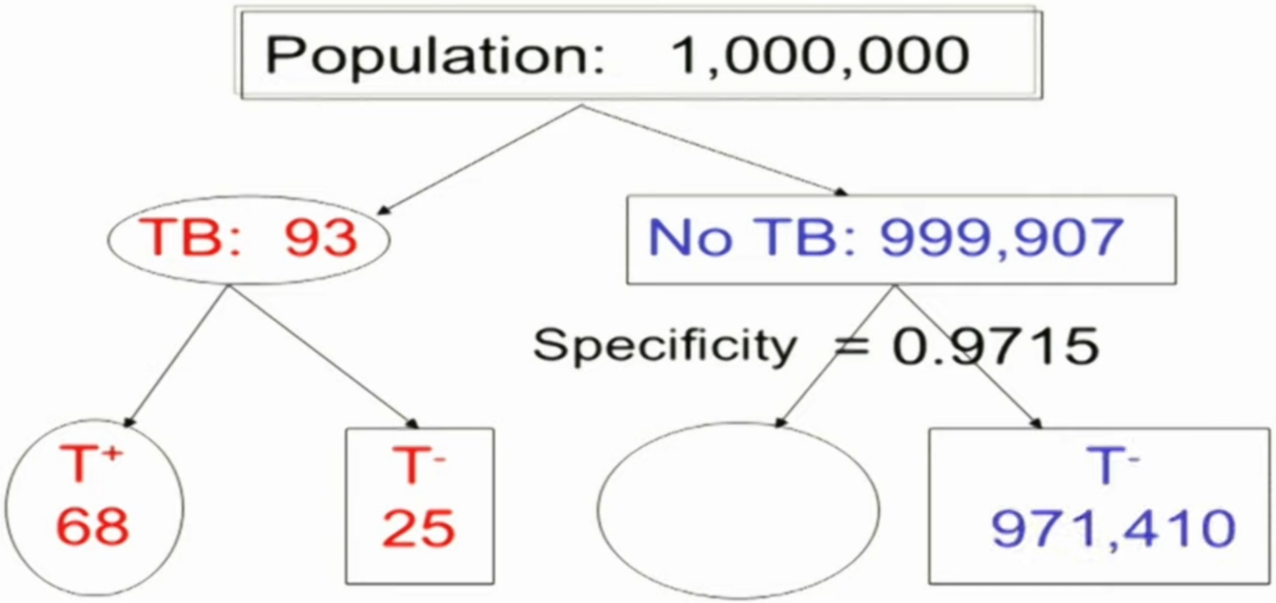


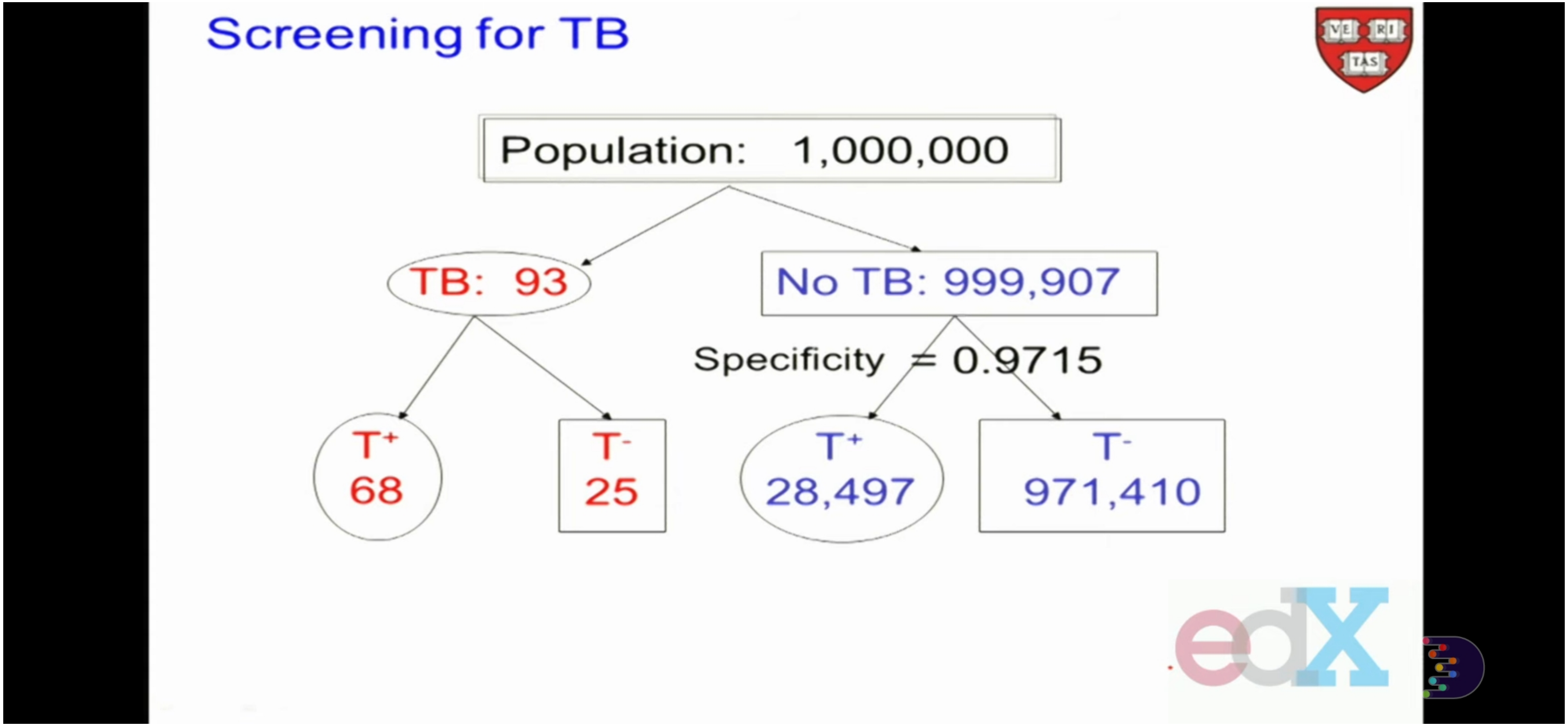
Screening for TB



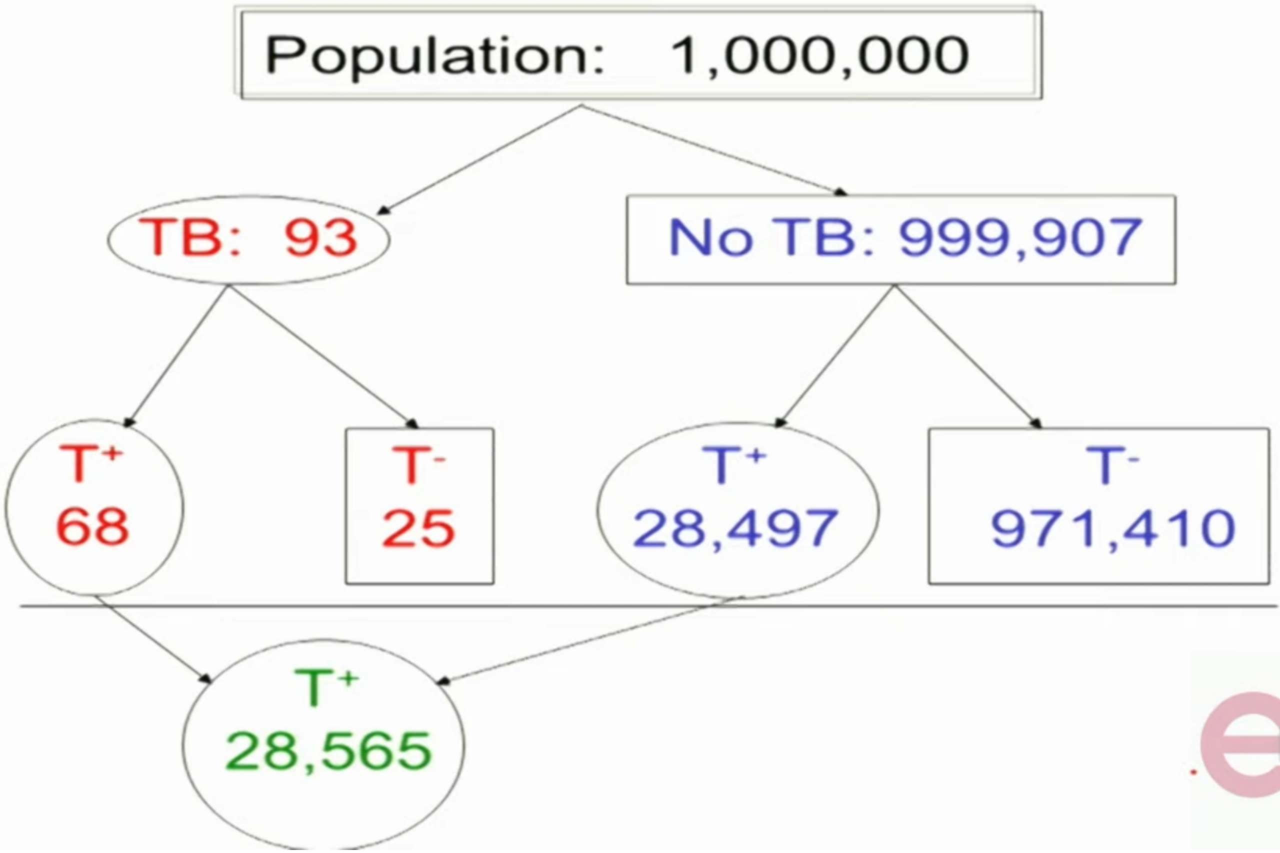
Specificity = 0.9715

Screening for TB

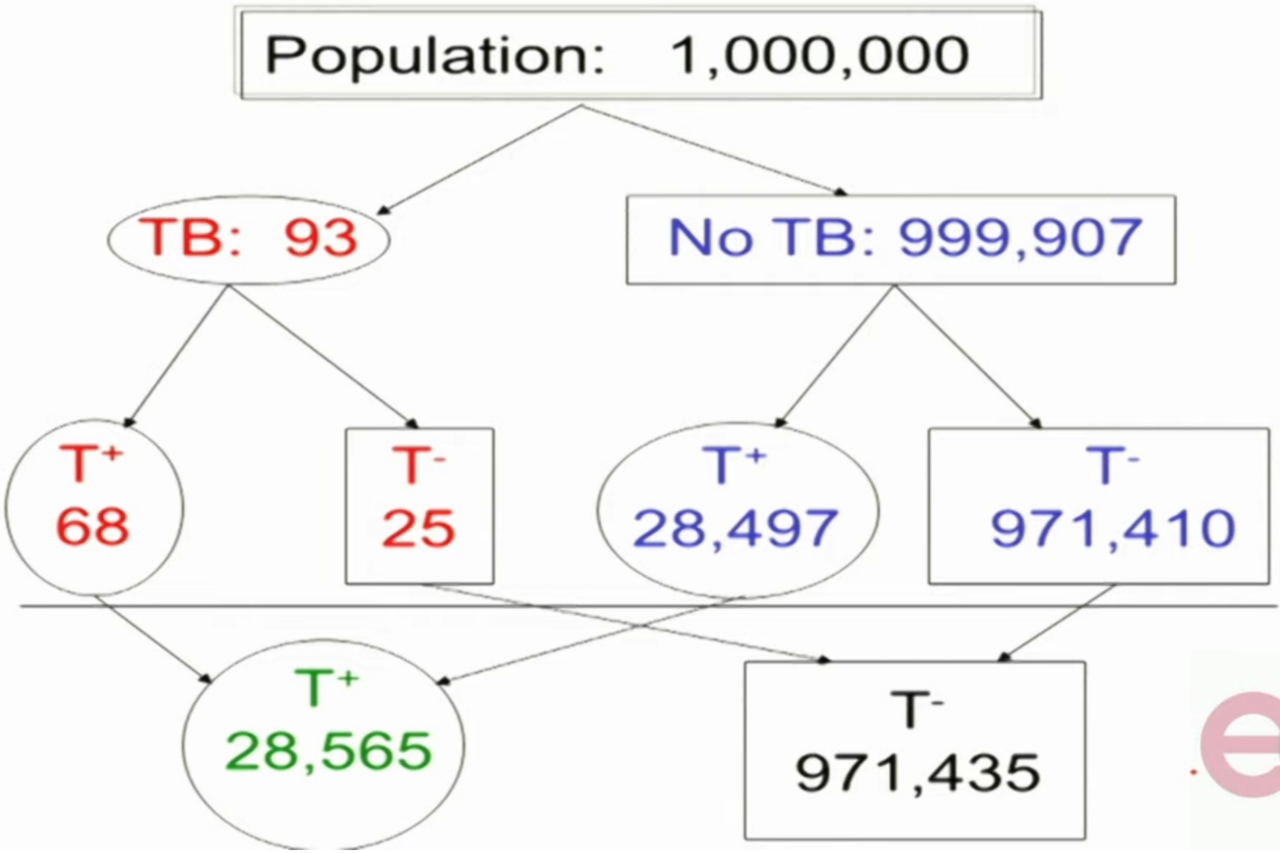



Screening for TB
Population: 1,000,000
TB: 93
No TB: 999,907
Specificity = 0.9715
T+
68
T-
25
T+
28,497
T-
971,410

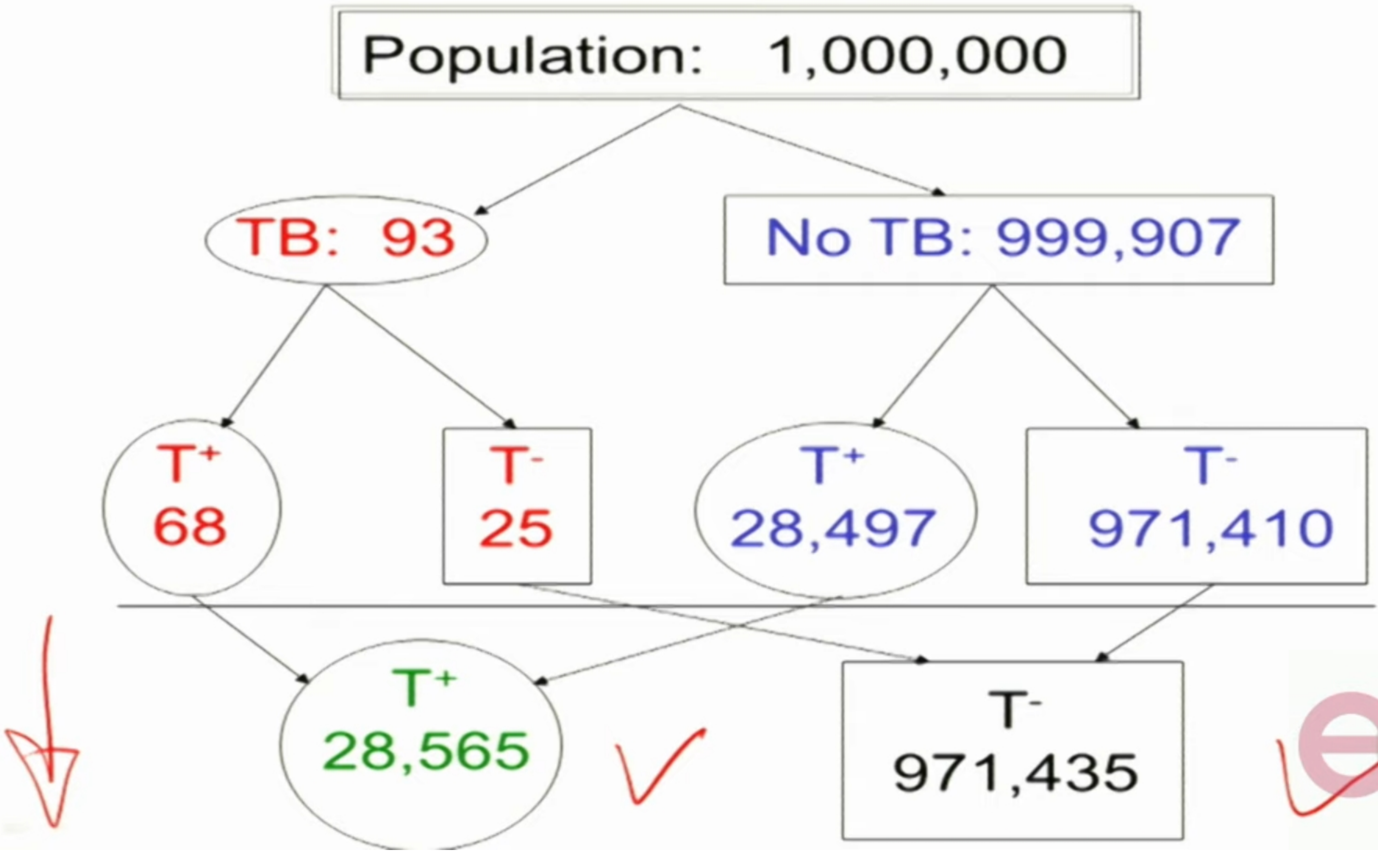
Screening for TB



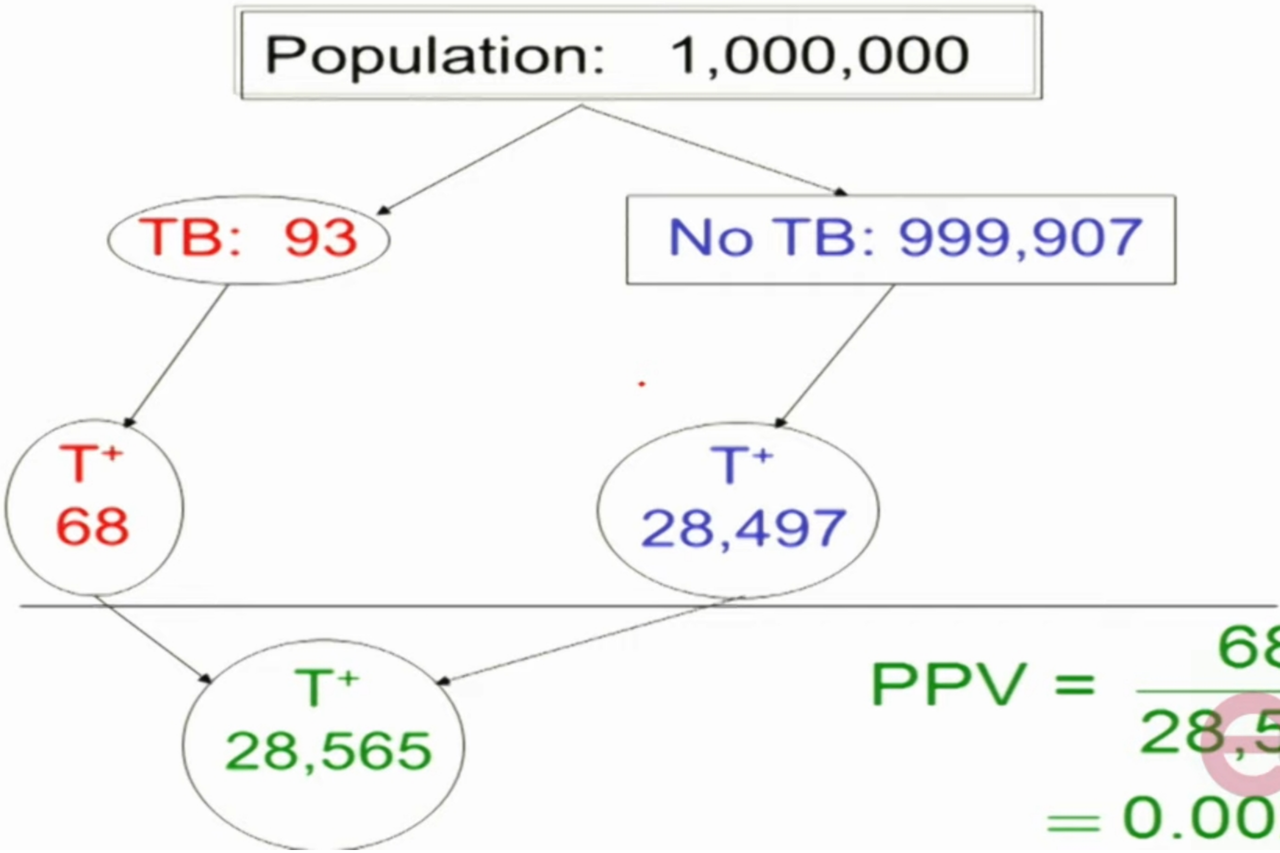
Screening for TB



Screening for TB



Positive Predictive Value



$$PPV = \frac{68}{28,565} = 0.00238$$

Prior to the test we have:

$$P(D) = \frac{93}{1{,}000{,}000} = 0.000093$$

Post(erior) to the test we have:

$$P(D \mid T^{+}) = \frac{68}{28{,}565} = 0.00238$$

Ratio:

$$\frac{0.00238}{0.000093} = 25.6$$