Exercise Sheet 5

Exercise 16

For a radioactive compound X is the delay time for the next decay with $X \sim E_{1/\tau}$. Therefore, τ is the average lifetime of a particle and the expectation of X. Note, that the exponential is also parametrized by $\lambda = 1/\tau$, with expectation $1/\lambda$ in some literature. Compute the maximum-likelihood estimator for τ. Hint: the density of X is given by

$$f_X(x) = \frac{1}{\tau} \cdot e^{-x/\tau}\,, \mathrm{for} x \geq 0$$

Compare your results to the Method-of-Moments estimator for τ.

Exercise 17

Given is a fish pond with m fishes and m unknown. The population size is to be estimated. Therefore, at first c fishes are angled from the pond, marked with a spot of color and thrown back into the pond, a week later r fishes are angled and the number t of the r fishes determined which have a spot of color.

(a) Simulate the example: a virtual fish pond with $m = 500$ fishes and the values $c, r \in \{10, 20, 40, 80\}$. Execute each simulation 100 times and compare the variations of the results in consideration of the values c, r respectively.

(b) Show that the estimator $\hat{m} = \frac{c \cdot r}{t}$ is a maximum-likelihood estimator. (hint: hypergeometric distribution for the number t). The likelihood function depends on m and one tries to show that the quotient $L(m)/L(m-1)$ exactly then is > 1, if $m < \frac{c \cdot r}{t}$. This means that L increases up to this value, afterwards L decreases and therefore this m is the value maximizing the likelihood, i.e., the maximum likelihood estimator.

Exercise 18

Let $X \sim Par_{k,x_{min}}$ be Pareto-distributed with parameters $k > 0$ and $x_{min} > 0$. This distribution is often used in economics. Its expectation is given by:

$$E[X] = x_{min} \cdot \frac{k}{k-1}$$

Assume, we are given data $30, 100, 120, 250, 57$ sampled from a Pareto and let $x_{min} = 30$, the minimum value of the data. Find an estimator for the parameter k using the Method-of-Moments and calculate its value. Plot data and distribution density in a chart.

Exercise 19

Plot the distribution function and the density of a (student-) t-distribution with varying degrees of freedom $n \in \{1, 2, 3, 5, 10\}$ on top of the density of a standard normal distribution and discuss the similarities/differences you can find. In a simulation, generate 3 random numbers from the standard normal distribution $\mathcal{N}(0, 1)$, estimate mean and variance by $\hat{\mu} =$ `numpy.mean(data)` and $\hat{\sigma}^2 =$ `numpy.var(data, ddof=1)`. Finally compute

$$S = \frac{\hat{\mu}}{\sqrt{\hat{\sigma}^2}/\sqrt{n}}$$

Repeat this experiment $m = 1000$ times and add the empirical distribution function of $S_1, \ldots, S_m$ to the plot of the distribution function. To which distribution does the resulting data compare best?