THE TECHNIQUE OF OBSTRUCTION THEORY DEVELOPED IN THE LAST CHAPTER focuses attention on the computation of homotopy groups. In this chapter we obtain some results about the homotopy groups of spheres. The method we follow is due to Serre[1] and uses the technical tool known as a spectral sequence. This algebraic concept is introduced for the study of the homology and cohomology properties of arbitrary fibrations, but it has other important applications in algebraic topology, and the number of these is constantly increasing. Some indication of the power of spectral sequences will be apparent from the results obtained by its use here.

Section 9.1 contains the definition of a spectral sequence, and in Sec. 9.2 the homology spectral sequence of a fibration is established. This is used in Sec. 9.3 to prove generalizations of the Gysin and Wang exact homology sequences. There is also a proof of the homotopy excision theorem, which is used in connection with the Hopf invariant to study in more detail the suspension map for homotopy groups of spheres.

[1] See two basic papers, J.-P. Serre, Homologie singulière des espaces fibrés, *Annals of Mathematics,* vol. 54, pp. 425–505, 1951, and Groupes d'homotopie et classes de groupes abéliens, *Annals of Mathematics,* vol. 58, pp. 258–294, 1953.

Cohomology spectral sequences are considered in Sec. 9.4, and the cohomology spectral sequence of a fibration is established. The multiplicative property of cohomology spectral sequences is applied in Sec. 9.5 to obtain stronger results than were obtained with the homology spectral sequence. Serre classes of abelian groups are introduced in Sec. 9.6, and some technical results, based on spectral sequences, are derived for isomorphisms of groups modulo a Serre class.

In Sec. 9.7 the machinery based on Serre classes is used to prove that all the homotopy groups of spheres are finitely generated and are finite except for stated exceptions. There are also some results concerning p-primary components of homotopy groups of spheres. Further information about homotopy groups of spheres appears in the exercises at the end of the chapter.

1 SPECTRAL SEQUENCES

Corresponding to a subcomplex of a chain complex, there is an associated exact sequence of homology modules. Hence, corresponding to an increasing sequence of subcomplexes of a chain complex, there is an associated sequence of exact sequences of homology modules. This sequence of exact sequences constitutes a new algebraic object, known as a spectral sequence, which provides information about the homology of the chain complex in terms of the homology of the quotient complexes of the sequence of subcomplexes. A spectral sequence consists of a sequence of chain complexes each of which is the homology module of the preceding one. There is an associated limit module, and the spectral sequence itself is viewed as a sequence of approximations to this limit module. In this section we shall define the algebraic concepts involved. In the next section we shall apply these concepts to study the homology of a fibration.

Let us consider modules over a fixed principal ideal domain R. A *bigraded module* E (*over* R) is an indexed collection of R modules $E_{s,t}$ for every pair of integers s and t. A *differential* $d: E \to E$ of *bidegree* $(-r, r-1)$ is a collection of homomorphisms $d: E_{s,t} \to E_{s-r,t+r-1}$, for all s and t, such that $d^2 = 0$. The *homology module* $H(E)$ is the bigraded module defined by

$$H_{s,t}(E) = [\ker (d: E_{s,t} \to E_{s-r,t+r-1})]/d(E_{s+r,t-r+1})$$

Note that if E_q is defined to equal $\bigoplus_{s+t=q} E_{s,t}$, the differential d defines a homomorphism $\partial: E_q \to E_{q-1}$ such that $\{E_q,\partial\}$ is a chain complex. Furthermore, the qth homology module of this chain complex equals $\bigoplus_{s+t=q} H_{s,t}(E)$.

An E^k *spectral sequence* E is a sequence $\{E^r,d^r\}$ for $r \geq k$ such that

(*a*) E^r is a bigraded module and d^r is a differential of bidegree $(-r, r-1)$ on E^r.
(*b*) For $r \geq k$ there is given an isomorphism $H(E^r) \approx E^{r+1}$.

Note that the spectral sequence begins with E^k, and the only role of the integer k is to specify where the spectral sequence starts. In our applications it will usually turn out that $k = 1$ or 2. Clearly, any E^k spectral sequence defines an $E^{k'}$ spectral sequence for every $k' \geq k$.

A *homomorphism* $\varphi: E \to E'$ from one E^k spectral sequence to another is a collection of homomorphisms $\varphi^r: E^r_{s,t} \to E'^r_{s,t}$ for $r \geq k$ and all s and t commuting with the differentials and such that $\varphi^r_*: H(E^r) \to H(E'^r)$ corresponds to $\varphi^{r+1}: E^{r+1} \to E'^{r+1}$ under the isomorphisms of the spectral sequence. The composite of homomorphisms is a homomorphism, and so there is a category of E^k spectral sequences (for fixed k) and homomorphisms.

To define the limit term of a spectral sequence, for $r \geq k$ we regard E^{r+1} as identified with $H(E^r)$ by the isomorphisms of the spectral sequence. Let Z^k be the bigraded module $Z^k_{s,t} = \ker (d^k: E^k_{s,t} \to E^k_{s-k,t+k-1})$ and let B^k be the bigraded module $B^k_{s,t} = d^k(E^k_{s+k,t-k+1})$. Then $B^k \subset Z^k$ and $E^{k+1} = Z^k/B^k$. Let $Z(E^{k+1})$ be the bigraded module $Z(E^{k+1})_{s,t} = \ker (d^{k+1}: E^{k+1}_{s,t} \to E^{k+1}_{s-k-1,t+k})$ and let $B(E^{k+1})$ be the bigraded module $B(E^{k+1})_{s,t} = d^{k+1}(E^{k+1}_{s+k+1,t-k})$. By the Noether isomorphism theorem, there exist bigraded submodules Z^{k+1} and B^{k+1} of Z^k containing B^k such that $Z(E^{k+1})_{s,t} = Z^{k+1}_{s,t}/B^k_{s,t}$ and $B(E^{k+1})_{s,t} = B^{k+1}_{s,t}/B^k_{s,t}$ for all s and t. It follows that $B^{k+1} \subset Z^{k+1}$, and we have

$$B^k \subset B^{k+1} \subset Z^{k+1} \subset Z^k$$

Continuing by induction, we obtain submodules for $r \geq k$

$$B^k \subset B^{k+1} \subset \cdots \subset B^r \subset \cdots \subset Z^r \subset \cdots \subset Z^{k+1} \subset Z^k$$

such that $E^{r+1} = Z^r/B^r$. We define bigraded modules $Z^\infty = \bigcap_r Z^r$, $B^\infty = \bigcup_r B^r$, and $E^\infty = Z^\infty/B^\infty$. The bigraded module E^∞ is called the *limit* of the spectral sequence E, and the terms E^r of the spectral sequence are successive approximations to E^∞.

The spectral sequence E is said to *converge* if for every s and t there exists an integer $r(s,t) \geq k$ such that for $r \geq r(s,t)$, $d^r: E^r_{s,t} \to E^r_{s-r,t+r-1}$ is trivial. Then $E^{r+1}_{s,t}$ is isomorphic to a quotient of $E^r_{s,t}$ and $E^\infty_{s,t}$ is isomorphic to the direct limit of the sequence

$$E^{r(s,t)}_{s,t} \to E^{r(s,t)+1}_{s,t} \to \cdots$$

It is frequently the case that the spectral sequence converges and for given s and t there exists $r(s,t)$ such that all homomorphisms in the sequence displayed above are isomorphisms (so $E^r_{s,t} \approx E^\infty_{s,t}$ for $r \geq r(s,t)$). For example, if E has the property that for some r there exist integers N and N' such that $E^r_{s,t} = 0$ for $s < N$ or $t < N'$, the same is true for $E^{r'}_{s,t}$ for $r' \geq r$. Then, given s and t, if $r' \geq r$ is chosen so that $r' > \sup (s - N, t - N' + 1)$, we have

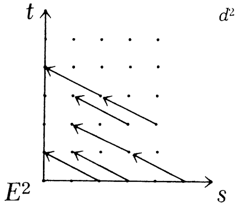
$$E^{r'}_{s+r',t-r'+1} \xrightarrow{d^{r'}} E^{r'}_{s,t} \xrightarrow{d^{r'}} E^{r'}_{s-r',t+r'-1}$$

The first module equals 0 because $t - r' + 1 < N'$ and the last module equals 0 because $s - r' < N$. Therefore, if r' is large enough,

$$E^{r'}_{s,t} \approx E^{r'+1}_{s,t} \approx \cdots \approx E^\infty_{s,t},$$

and E is convergent in this stronger sense.

A particular example of such spectral sequences is a *first-quadrant* spectral sequence, which is defined to be a spectral sequence E having the property that for some r, $E^r_{s,t} = 0$ if $s < 0$ or $t < 0$. Such a spectral sequence is convergent in the strong sense, and for any q there are only a finite number of nontrivial modules $E^\infty_{s,t}$ with $s + t = q$. A first-quadrant spectral sequence is conveniently represented by attaching $E^r_{s,t}$ to the lattice point (s,t) in the first quadrant of the plane and representing the differential d^r by oblique arrows:



Then E^{r+1} is the quotient of the kernel of the arrow which originates at (s,t) by the image of the arrow which terminates at (s,t).

A homomorphism $\varphi: E \to E'$ between E^k spectral sequences induces a homomorphism $\varphi^\infty: E^\infty \to E'^\infty$ between their limit terms. Therefore there is a covariant functor from the category of E^k spectral sequences to the category of bigraded modules which assigns to every spectral sequence its limit. The following useful result is an easy consequence of the fact that a chain transformation which is an isomorphism induces an isomorphism of the corresponding homology modules.

1 THEOREM *Let $\varphi: E \to E'$ be a homomorphism of E^k spectral sequences which is an isomorphism for some $r \geq k$. Then φ is an isomorphism for all $r' \geq r$. Furthermore, if E and E' converge, φ^∞ is an isomorphism of their limits.* ∎

An (*increasing*) *filtration* F on an R module A is a sequence of submodules F_sA for all integers s such that $F_sA \subset F_{s+1}A$. If A is a graded module (that is, $A = \{A_t\}$), the filtration F is required to be compatible with the gradation (that is, F_sA is graded by $\{F_sA_t\}$). Given a filtration F on A, the *associated graded module* $G(A)$ is defined by $G(A)_s = F_sA/F_{s-1}A$. If A is a graded module, the associated module $G(A)$ is bigraded by the modules $G(A)_{s,t} = F_sA_{s+t}/F_{s-1}A_{s+t}$. In this case, s is called the *filtered degree*, t the *complementary degree*, and $s + t$ the *total degree* of an element of $G(A)_{s,t}$. The sequence

$$\cdots \subset F_{s-1}A \subset F_sA \subset F_{s+1}A \subset \cdots$$

is an infinite normal series for A, and the associated module consists of the quotients of this normal series.

A filtration F on A is said to be *convergent* if $\bigcap_s F_sA = 0$ and $\bigcup_s F_sA = A$. For convergent filtrations the associated module $G(A)$ is more closely tied to A than in the case of an arbitrary filtration. However, even if the filtration is finite in the sense that $F_sA = 0$ for some s and $F_{s'}A = A$ for some s', it is not true that $G(A)$ determines A. In the latter case $G(A)$ determines A up to a finite number of module extensions.

A filtration F on a chain complex C is a filtration compatible with the gradation of C and with the differential of C (that is, F_sC is a chain subcomplex of C consisting of $\{F_sC_t\}$). The filtration F on C induces a filtration F on $H_*(C)$ defined by

$$F_sH_*(C) = \operatorname{im}\,[H_*(F_sC) \to H_*(C)]$$

Because the homology functor commutes with direct limits, if F is a convergent filtration of C, it follows that $\bigcup_s F_sH_*(C) = H_*(C)$; however, it is not generally true that $\bigcap_s F_sH_*(C) = 0$. Thus, to ensure that F be a convergent filtration on $H_*(C)$ we need a stronger assumption about the original filtration on C.

A filtration F on a graded module A is said to be *bounded below* if for any t there is $s(t)$ such that $F_{s(t)}A_t = 0$. It is clear that if F is a filtration bounded below on a chain complex C, then the induced filtration on $H_*(C)$ is also bounded below. Thus, if F is convergent and bounded below on C, the same is true of the induced filtration on $H_*(C)$.

The following theorem associates a spectral sequence to a filtration on a chain complex.

2 THEOREM *Let F be a convergent filtration bounded below on a chain complex C. There is a convergent E^1 spectral sequence with*

$$E^1_{s,t} \approx H_{s+t}(F_sC/F_{s-1}C)$$

[and d^1 corresponding to the connecting homomorphism of the triple $(F_sC, F_{s-1}C, F_{s-2}C)$] and E^∞ isomorphic to the bigraded module $GH_*(C)$ (associated to the filtration $F_sH_*(C) = \operatorname{im}\,[H_*(F_sC) \to H_*(C)]$).

PROOF For arbitrary r we define

$$Z^r_s = \{c \in F_sC \mid \partial c \in F_{s-r}C\}$$
$$Z^\infty_s = \{c \in F_sC \mid \partial c = 0\}$$

These are graded modules with $Z^r_{s,t} = \{c \in F_sC_{s+t} \mid \partial c \in F_{s-r}C\}$ and $Z^\infty_{s,t} = \{c \in F_sC_{s+t} \mid \partial c = 0\}$. We then have a sequence of graded modules

$$\cdots \subset \partial Z^{-1}_{s-1} \subset \partial Z^0_s \subset \partial Z^1_{s+1} \subset \cdots \subset \partial C \cap F_sC \subset Z^\infty_s \subset \cdots$$
$$\cdots \subset Z^1_s \subset Z^0_s = F_sC$$

We define

$$E^r_s = Z^r_s/(Z^{r-1}_{s-1} + \partial Z^{r-1}_{s+r-1})$$
$$E^\infty_s = Z^\infty_s/(Z^\infty_{s-1} + \partial C \cap F_sC)$$

The map ∂ sends Z^r_s to Z^r_{s-r} and $Z^{r-1}_{s-1} + \partial Z^{r-1}_{s+r-1}$ to ∂Z^{r-1}_{s-1}. Therefore it induces a homomorphism

$$d^r\colon E^r_s \to E^r_{s-r}$$

Then E^r is a bigraded module and d^r is a differential of bidegree $(-r, r-1)$ on it. For $r < 0$, $d^r = 0$ and $E_s^r = F_sC/F_{s-1}C$. Therefore

$$E_{s,t}^0 = F_sC_{s+t}/F_{s-1}C_{s+t} = G(C)_{s,t}$$

and $d^0\colon F_sC_{s+t}/F_{s-1}C_{s+t} \to F_sC_{s+t-1}/F_{s-1}C_{s+t-1}$ is just the boundary operator of the quotient complex $F_sC/F_{s-1}C$. Furthermore,

$$E_{s,t}^1 = Z_{s,t}^1/(Z_{s-1,t+1}^0 + \partial Z_{s,t+1}^0),$$

where $Z_{s,t}^1 = \{c \in F_sC_{s+t} \mid \partial c \in F_{s-1}C_{s+t-1}\}$. Therefore $Z_{s,t}^1/Z_{s-1,t+1}^0$ is the module of $(s+t)$-cycles of $F_sC/F_{s-1}C$ and $(Z_{s-1,t+1}^0 + \partial Z_{s,t+1}^0)/Z_{s-1,t+1}^0$ is the module of $(s+t)$-boundaries of $F_sC/F_{s-1}C$. By the Noether isomorphism theorem, $E_{s,t}^1 \approx H_{s+t}(F_sC/F_{s-1}C)$. The fact that under this isomorphism d^1 corresponds to the boundary operator of the triple $(F_sC, F_{s-1}C, F_{s-2}C)$ is proved by direct verification, using the definitions.

We prove that $E = \{E^r\}_{r \geq 1}$ is a spectral sequence by computing the homology of E^r with respect to d^r. We have

$$\begin{aligned}\{c \in Z_s^r \mid \partial c \in Z_{s-r-1}^{r-1} + \partial Z_{s-1}^{r-1}\}\\ &= \{c \in Z_s^r \mid \partial c \in F_{s-r-1}C\} + \{c \in Z_s^r \mid \partial c \in \partial Z_{s-1}^{r-1}\}\\ &= Z_s^{r+1} + (Z_{s-1}^{r-1} + Z_s^\infty) = Z_s^{r+1} + Z_{s-1}^{r-1}\end{aligned}$$

Therefore $\ker(d^r\colon E_s^r \to E_{s-r}^r) = (Z_s^{r+1} + Z_{s-1}^{r-1})/(Z_{s-1}^{r-1} + \partial Z_{s+r-1}^{r-1})$. By definition, $\operatorname{im}(d^r\colon E_{s+r}^r \to E_s^r) = (\partial Z_{s+r}^r + Z_{s-1}^{r-1})/(Z_{s-1}^{r-1} + \partial Z_{s+r-1}^{r-1})$. Hence, by the Noether isomorphism theorem, in E_s^r we have

$$\begin{aligned}\ker d^r/\operatorname{im} d^r &\approx (Z_s^{r+1} + Z_{s-1}^{r-1})/(\partial Z_{s+r}^r + Z_{s-1}^{r-1}) \approx Z_s^{r+1}/[Z_s^{r+1} \cap (\partial Z_{s+r}^r + Z_{s-1}^{r-1})]\\ &= Z_s^{r+1}/(\partial Z_{s+r}^r + Z_{s-1}^r) = E_s^{r+1}\end{aligned}$$

Therefore there is an isomorphism $H_*(E^r) \approx E^{r+1}$, and E is a spectral sequence.

We now compute the limit of this spectral sequence. By definition and the Noether isomorphism theorem,

$$E_s^r = Z_s^r/(Z_{s-1}^{r-1} + \partial Z_{s+r-1}^{r-1}) \approx (Z_s^r + F_{s-1}C)/(F_{s-1}C + \partial Z_{s+r-1}^{r-1})$$

In the last expression the numerators decrease as r increases and the denominators increase as r increases. By definition, the limit equals

$$\begin{aligned}\textstyle\bigcap_r (Z_s^r + F_{s-1}C)/\bigcup_r (F_{s-1}C + \partial Z_{s+r-1}^{r-1}) =\\ &(\textstyle\bigcap_r Z_s^r + F_{s-1}C)/(F_{s-1}C + \bigcup_r \partial Z_{s+r-1}^{r-1})\end{aligned}$$

Since $\bigcup_s F_sC = C$, then $\bigcup_r \partial Z_{s+r-1}^{r-1} = \partial C \cap F_sC$. For given t, $\bigcap_r Z_{s,t}^r = Z_{s,t}^\infty$, because $F_sC_t = 0$ for s small enough. Therefore the limit term equals

$$(Z_s^\infty + F_{s-1}C)/(F_{s-1}C + \partial C \cap F_sC) = Z_s^\infty/(Z_{s-1}^\infty + \partial C \cap F_sC) = E_s^\infty$$

To show that the spectral sequence converges, note that because the filtration is bounded below, for fixed $s+t$, $E_{s,t}^r = 0$ for s small enough. Therefore, for fixed s and t there exists r such that for $r' \geq r$, $E_{s,t}^{r'+1}$ is a quotient of $E_{s,t}^{r'}$, and the spectral sequence converges.

To complete the proof we interpret the limit E^∞ as $GH_*(C)$. By definition, $GH_*(C)_{s,t} = F_sH_{s+t}(C)/F_{s-1}H_{s+t}(C)$, where

$$F_sH_{s+t}(C) = \text{im}\,[H_{s+t}(F_sC) \to H_{s+t}(C)]$$

Therefore the graded module $F_sH_*(C) = Z_s^\infty/\partial C \cap F_sC$, and

$$\begin{aligned} F_sH_*(C)/F_{s-1}H_*(C) &= (Z_s^\infty/\partial C \cap F_sC)/(Z_{s-1}^\infty/\partial C \cap F_{s-1}C) \\ &\approx Z_s^\infty/(Z_{s-1}^\infty + \partial C \cap F_sC) \\ &= E_s^\infty \quad \blacksquare \end{aligned}$$

In theorem 2 note that even in the most favorable circumstances E^∞ does not determine $H_*(C)$ completely, but only up to module extensions. Note that we have, in fact, defined an E^0 spectral sequence. The theorem was stated in terms of the corresponding E^1 spectral sequence because the E^1 term contains more information than the E^0 term.

It should be observed that the spectral sequence of theorem 2 is functorial on the category of chain complexes with a convergent filtration bounded below. Combining this with theorem 1, we obtain the following result.

3 THEOREM *Let $\tau\colon C \to C'$ be a chain map preserving filtration between chain complexes having convergent filtrations bounded below. If for some $r \geq 1$ the induced map $\tau^r\colon E^r \to E'^r$ is an isomorphism, then τ induces an isomorphism*

$$\tau_*\colon H_*(C) \approx H_*(C')$$

PROOF By theorem 1, τ^∞ is an isomorphism. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & F_{s-1}H_n(C) & \to & F_sH_n(C) & \to & E_{s,n-s}^\infty & \to & 0 \\ & & {\scriptstyle\tau_*}\downarrow & & {\scriptstyle\tau_*}\downarrow & & \downarrow{\scriptstyle\tau^\infty} & & \\ 0 & \to & F_{s-1}H_n(C') & \to & F_sH_n(C') & \to & E_{s,n-s}'^\infty & \to & 0 \end{array}$$

For fixed n, $F_{s-1}H_n(C)$ and $F_{s-1}H_n(C')$ are both 0 for s small enough (because the filtrations are bounded below). It follows by induction on s, using the five lemma and the fact that τ^∞ is an isomorphism, that $\tau_*\colon F_sH_n(C) \approx F_sH_n(C')$ for all s. Because the filtrations are convergent, $H_n(C) = \bigcup_s F_sH_n(C)$ and $H_n(C') = \bigcup_s F_sH_n(C')$, and so $\tau_*\colon H_n(C) \approx H_n(C')$. $\blacksquare$

We present some examples of spectral sequences.

4 Let C' and C'' be free nonnegative chain complexes with boundary operators ∂' and ∂'', respectively, and let $C = C' \otimes C''$ be their tensor product, with boundary operator ∂. There is a convergent filtration bounded below on C defined by $F_sC = \bigoplus_{q \leq s} C'_q \otimes C''$. For the corresponding spectral sequence,

$$E_{s,t}^1 \approx C'_s \otimes H_t(C'')$$

and for $r \geq 2$ $E^r_{s,t} \approx H_s(C' \otimes H_t(C''))$. A similar result is obtained by filtering the tensor product by the gradation of the second factor.

5 An (*increasing*) *filtration* of a topological pair (X,A) is a sequence of subspaces X_s containing A such that $X_s \subset X_{s+1}$. Such a filtration on (X,A) induces a filtration F on the chain complex $\Delta(X,A)$ by $F_s(\Delta(X,A)) = \Delta(X_s,A)$. If $X_s = A$ for some s, the induced filtration is bounded below. If $X = \cup X_s$ and every compact subset of X is contained in some X_s, then $\cup_s F_s(\Delta(X,A)) = \Delta(X,A)$. Therefore, if the filtration $\{X_s\}$ has both the above properties, the induced filtration on $\Delta(X,A)$ is convergent and bounded below so there is a convergent E^1 spectral sequence with $E^1_{s,t} \approx H_{s+t}(X_s,X_{s-1})$ and in which d^1 corresponds to the boundary operation of the triple (X_s,X_{s-1},X_{s-2}). The limit term of the spectral sequence is the bigraded module associated to the corresponding filtration of $H(X,A)$.

In particular, if (X,A) is a relative CW complex, $X_s = (X,A)^s$ is the s-dimensional skeleton for $s \geq 0$, and $X_s = A$ for $s < 0$, then $E^1_{s,t} \neq 0$ only if $t = 0$, and $E^1_{s,0} \approx H_s(X_s,X_{s-1})$. Therefore, for $r \geq 2$, $E^r_{s,0}$ is the homology of the chain complex $C = \{C_s,\partial\}$, where $C_s = H_s(X_s,X_{s-1})$, and $\partial\colon C_s \to C_{s-1}$ is the boundary operator of the triple (X_s,X_{s-1},X_{s-2}).

Our next example is an alternate description of the spectral sequence of example 5 whose construction does not involve the chain modules. It can also be applied to obtain a spectral sequence corresponding to any sequence of functors having the exactness properties of the homology functors.

6 Let $\{X_s\}$ be an increasing filtration of a pair (X,A). For each s there is an exact homology sequence of (X_s,X_{s-1}) (with some coefficient module)

$$\cdots \to H_q(X_{s-1}) \xrightarrow{i_*} H_q(X_s) \xrightarrow{j_*} H_q(X_s,X_{s-1}) \xrightarrow{\partial} H_{q-1}(X_{s-1}) \to \cdots$$

Therefore we have a sequence of exact sequences. These combine to form a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r, "\partial"] & H_q(X_{s-1}) \arrow[r, "j_*"] \arrow[d, "i_*"'] & H_q(X_{s-1},X_{s-2}) \arrow[r, "\partial"] & H_{q-1}(X_{s-2}) \arrow[r, "j_*"] \arrow[d, "i_*"] & \cdots \\
\cdots \arrow[r, "\partial"] & H_q(X_s) \arrow[r, "j_*"] \arrow[d, "i_*"'] & H_q(X_s,X_{s-1}) \arrow[r, "\partial"] & H_{q-1}(X_{s-1}) \arrow[r, "j_*"] \arrow[d, "i_*"] & \cdots \\
\cdots \arrow[r, "\partial"] & H_q(X_{s+1}) \arrow[r, "j_*"] \arrow[d] & H_q(X_{s+1},X_s) \arrow[r, "\partial"] & H_{q-1}(X_s) \arrow[r, "j_*"] \arrow[d] & \cdots \\
 & \vdots & & \vdots &
\end{tikzcd}$$

in which any sequence consisting of a vertical map i_* followed by two horizontal maps j_* and ∂ and then a vertical map i_* followed again by j_* and ∂ and iteration of this (one possible such sequence is indicated by heavy arrows in the diagram) is exact. From this diagram there is obtained an E^1 spectral sequence in which $E^1_{s,t} = H_{s+t}(X_s,X_{s-1})$ and for $r \geq 2$, $E^r_{s,t}$ is defined to be the quotient $Z^r_{s,t}/B^r_{s,t}$, where

$$Z^r_{s,t} = \partial^{-1}(i_*{}^{r-1}H_{s+t-1}(X_{s-r}))$$
$$B^r_{s,t} = j_*(\ker [i_*{}^{r-1}\colon H_{s+t}(X_s) \to H_{s+t}(X_{s+r-1})])$$

This spectral sequence converges if for fixed q, $H_q(X) \approx \lim_\rightarrow \{H_q(X_s)\}$, and for s small enough $H_q(X_s,A) = 0$. In this case the limit is the bigraded module associated to the filtration of $H(X,A)$ defined by $F_sH(X,A) = \operatorname{im}[H(X_s,A) \to H(X,A)]$.

It is not hard to verify that the E^1 spectral sequence of example 6 is the same as the E^1 spectral sequence of example 5. The same process can be applied to obtain a spectral sequence from any diagram having the exactness properties of the diagram in example 6. These properties have been formalized in the concept of *exact couple*,[1] but we omit the precise definition.

2 THE SPECTRAL SEQUENCE OF A FIBRATION

One of the most fruitful applications of spectral sequences is to the homology of fibrations. With a suitable orientability assumption on the fibration, there is a spectral sequence converging to the bigraded module associated to a filtration on the homology of the total space and whose E^2 term is isomorphic to the homology of the base space with coefficients in the homology of the fiber. This section is devoted to a construction of the spectral sequence. It depends on a study of the homology of the total space of a fibration whose base space is a relative CW complex utilizing the filtration of the total space consisting of the inverse images of the skeleta of the base space. By using CW approximations, an E^2 spectral sequence is defined for a fibration over any path-connected base space. Some applications of this spectral sequence will be given in the next section.

Let $p\colon E \to B$ be a fibration. For a subspace $A \subset B$ let $E_A = p^{-1}(A) \subset E$. Then p maps (E,E_A) to (B,A). Assume that (B,A) is a relative CW complex and let $E_s = p^{-1}((B,A)^s)$ be the part of E lying over the s-dimensional skeleton of B for $s \geq 0$ and $E_s = E_A$ if $s < 0$. Then $E_s \subset E_{s+1}$, so $\{E_s\}$ is an increasing filtration of (E,E_A). Furthermore, $E_{-1} = E_A$, $\bigcup_s E_s = E$, and every compact subset of E is contained in E_s for some s. By the method of example 9.1.5, we have the following result.

1 THEOREM *Let $p\colon E \to B$ be a fibration over a relative CW complex (B,A). For singular homology with any coefficient module G there is a convergent E^1 spectral sequence with $E^1_{s,t} \approx H_{s+t}(E_s,E_{s-1};\, G)$, d^1 the boundary*

[1] See W. S. Massey, Exact couples in algebraic topology, parts I and II, *Annals of Mathematics,* vol. 56, pp. 364–396. 1952, and parts III, IV, and V, *Annals of Mathematics,* vol. 57, pp. 248–286, 1953.

operator of the triple (E_s,E_{s-1},E_{s-2}), *and* E^∞ *the bigraded module associated to the filtration of* $H_*(E,E_A;\ G)$ *defined by*

$$F_sH_*(E,E_A;\ G) = \operatorname{im}\,[H_*(E_s,E_A;\ G) \to H_*(E,E_A;\ G)] \quad \blacksquare$$

To apply this result we need to compute the module $H_n(E_s,E_{s-1};\ G)$. We let $\{e_\lambda\}$ be the collection of s-cells of $B - A$.

2 LEMMA *The inclusion maps* $i_\lambda\colon (p^{-1}(e_\lambda),p^{-1}(\dot e_\lambda)) \subset (E_s,E_{s-1})$ *induce a direct-sum representation*

$$\{i_{\lambda *}\}\colon \bigoplus_\lambda H_n(p^{-1}(e_\lambda),\ p^{-1}(\dot e_\lambda)) \approx H_n(E_s,E_{s-1})$$

and a direct-product representation

$$\{i_\lambda^*\}\colon H^n(E_s,E_{s-1}) \approx \times\, H^n(p^{-1}(e_\lambda),p^{-1}(\dot e_\lambda))$$

PROOF For each λ let e_λ' be a simplex of dimension s contained in $e_\lambda - \dot e_\lambda$. Then the inclusion maps $((B,A)^s,\ (B,A)^{s-1}) \subset ((B,A)^s,\ (B,A)^s - \bigcup_\lambda (e_\lambda' - \dot e_\lambda'))$ and $(e_\lambda,\dot e_\lambda) \subset (e_\lambda,\ e_\lambda - (e_\lambda' - \dot e_\lambda'))$ are homotopy equivalences. Therefore the corresponding inclusion maps $(E_s,E_{s-1}) \subset (E_s,\ E_s - \bigcup_\lambda p^{-1}(e_\lambda' - \dot e_\lambda'))$ and $(p^{-1}e_\lambda,p^{-1}\dot e_\lambda) \subset (p^{-1}e_\lambda,\ p^{-1}(e_\lambda - (e_\lambda' - \dot e_\lambda')))$ are homotopy equivalences. There is a commutative diagram induced by inclusion maps

$$\begin{tikzcd}
\bigoplus_\lambda H_n(p^{-1}e_\lambda,p^{-1}\dot e_\lambda) \arrow[r, "\{i_{\lambda *}\}"] \arrow[d] & H_n(E_s,E_{s-1}) \arrow[d] \\
\bigoplus_\lambda H_n(p^{-1}e_\lambda,p^{-1}(e_\lambda - (e_\lambda' - \dot e_\lambda'))) \arrow[r] & H_n(E_s,E_s - \bigcup_\lambda p^{-1}(e_\lambda' - \dot e_\lambda')) \\
\bigoplus_\lambda H_n(p^{-1}e_\lambda',p^{-1}\dot e_\lambda') \arrow[r] \arrow[u] & H_n(\bigcup_\lambda p^{-1}e_\lambda',\ \bigcup_\lambda p^{-1}\dot e_\lambda') \arrow[u]
\end{tikzcd}$$

in which all the vertical maps are isomorphisms, the top two because they are induced by homotopy equivalences and the bottom two because they are induced by suitable excision maps. Since e_λ' is disjoint from e_μ' if $\lambda \neq \mu$, the bottom map is an isomorphism because it is induced by a chain isomorphism. This proves the first part of the lemma. A similar argument proves the result for cohomology. $\blacksquare$

Before proceeding further with the computation of $H_n(E_s,E_{s-1})$ and the boundary operator of the triple (E_s,E_{s-1},E_{s-2}), we introduce a subcomplex of the singular chain complex of a relative CW complex which is chain equivalent to the singular complex itself.

3 THEOREM *Let* $\{X_s\}$ *be an increasing sequence of subspaces of a space* X *and let* $\bar\Delta(X)$ *be the subcomplex of* $\Delta(X)$ *generated by singular simplexes* $\sigma\colon \Delta^q \to X$ *such that* $\sigma(\Delta^q)^k \subset X_k$ *for all* k. *If* (X,X_{s-1}) *is* $(s-1)$*-connected for all* s, *then the inclusion map* $\bar\Delta(X) \subset \Delta(X)$ *is a chain equivalence.*

PROOF We shall use lemma 7.4.7. We associate to every singular simplex $\sigma: \Delta^q \to X$ a map $P(\sigma): \Delta^q \times I \to X$ such that the hypotheses of lemma 7.4.7 are satisfied [with $C = \bar{\Delta}(X)$]. This is done by induction on q. If $q = 0$ and $\sigma(\Delta^0) \subset X_0$, define $P(\sigma)$ to be the composite $\Delta^0 \times I \to \Delta^0 \xrightarrow{\sigma} X$. If $\sigma(\Delta^0) \not\subset X_0$, there is a path $\omega: I \to X$ from $\sigma(\Delta^0)$ to some point of X_0 [because (X,X_0) is 0-connected]. Then $P(\sigma): \Delta^0 \times I \to X$ is defined by $P(\sigma)(v_0,t) = \omega(t)$.

Let $q > 0$ and assume inductively that $P(\sigma)$ has been defined to satisfy lemma 7.4.7 for all singular simplexes σ of dimension $< q$. Let $\sigma: \Delta^q \to X$ be a singular q-simplex of X. If $\sigma(\Delta^q)^k \subset X_k$ for all k, define $P(\sigma)$ to be the composite $\Delta^q \times I \to \Delta^q \xrightarrow{\sigma} X$. If $\sigma(\Delta^q)^k \not\subset X_k$ for some k, then a and c of lemma 7.4.7 define $P(\sigma)$ on $\Delta^q \times 0 \cup \dot{\Delta}^q \times I$ and b of lemma 7.4.7 ensures that on $\dot{\Delta}^q \times 1$ the resulting map sends $(\dot{\Delta}^q)^k \times 1$ into X_k for all k.

It is clearly possible to find a homeomorphism $\Delta^q \times I \approx E^q \times I$ which takes $(\Delta^q \times 0 \cup \dot{\Delta}^q \times I,\ \dot{\Delta}^q \times 1)$ onto $(E^q,S^{q-1}) \times 0$ and $\Delta^q \times 1$ onto $S^{q-1} \times I \cup E^q \times 1$. Because (X,X_q) is q-connected, it follows that the given map from $(\Delta^q \times 0 \cup \dot{\Delta}^q \times I,\ \dot{\Delta}^q \times 1)$ to (X,X_q) extends to a map $P(\sigma): \Delta^q \times I \to X$ such that $P(\sigma)(\Delta^q \times 1) \subset X_q$. Then $P(\sigma) \mid \Delta^q \times 1: \Delta^q \times 1 \to X$ is a map such that $(\Delta^q)^k$ is sent into X_k for all k, and so $P(\sigma)$ can be defined for all σ to satisfy the hypotheses of lemma 7.4.7. ∎

Note that theorem 3 applies to the filtration defined by the skeleta of a relative CW complex (X,A). Hence, if $\bar{\Delta}(X) \subset \Delta(X)$ is the subcomplex of cellular singular simplexes, then $\bar{\Delta}(X) \subset \Delta(X)$ is a chain equivalence. Furthermore, if (X',A) is a subcomplex of (X,A), then $\bar{\Delta}(X') = \Delta(X') \cap \bar{\Delta}(X)$. Using theorem 4.6.10 and the five lemma, we see that the inclusion map $\bar{\Delta}(X,X') \subset \Delta(X,X')$ is a chain equivalence. In particular,

$$\bar{\Delta}((X,A)^s, (X,A)^{s-1}) \subset \Delta((X,A)^s, (X,A)^{s-1})$$

is a chain equivalence for any s.

4 COROLLARY *Given a relative CW complex (X,A), let $C(X,A) = \{C_s,\partial\}$ be the chain complex, with*

$$C_s = H_s(\bar{\Delta}((X,A)^s, (X,A)^{s-1}))$$

and with $\partial: C_s \to C_{s-1}$ the boundary operator of the triple $((X,A)^s, (X,A)^{s-1}, (X,A)^{s-2})$. Then $H_*(C(X,A)) \approx H_*(X,A)$.

PROOF Let F be the filtration on $\bar{\Delta}(X,A)$ defined by $F_s\bar{\Delta}(X,A) = \bar{\Delta}((X,A)^s, A)$. Then the corresponding spectral sequence has the property that $E^1_{s,t} \approx H_{s+t}((X,A)^s, (X,A)^{s-1})$ and d^1 corresponds to the boundary operator of the triple $((X,A)^s, (X,A)^{s-1}, (X,A)^{s-2})$. By application of lemma 2 to the trivial fibration $X \to X$, it follows that there is an isomorphism

$$\bigoplus_\lambda H_q(e_\lambda,\dot{e}_\lambda) \approx H_q((X,A)^s, (X,A)^{s-1})$$

where $\{e_\lambda\}$ is the collection of s-cells of (X,A). Then $H_q((X,A)^s, (X,A)^{s-1}) = 0$ if $q \neq s$, and so $E^1_{s,t} = 0$ if $t \neq 0$ and $E^1_{s,0} \approx C_s$. This implies that $E^2_{s,t} = 0$

if $t \neq 0$ and $E^2_{s,0} \approx H_s(C(X,A))$. Therefore, by induction on r, we see that $E^r_{s,t} = 0$ for $t \neq 0$ and $E^r_{s,0} = E^2_{s,0}$ for $r \geq 2$. Hence $E^\infty_{s,t} = 0$ for $t \neq 0$ and $E^\infty_{s,0} \approx H_s(C(X,A))$. Since E^∞ is the bigraded module associated to a filtration on $H(X,A)$, we have $H_s(C(X,A)) \approx H_s(X,A)$. ■

Recall that, by theorem 2.8.12, there is a contravariant functor from the fundamental groupoid of B to the homotopy category which assigns to $b \in B$ the fiber F_b over b and to a path class $[\omega]$ in B the homotopy class $h[\omega] \in [F_{\omega(0)};F_{\omega(1)}]$. Therefore, for fixed R there is a contravariant functor from the fundamental groupoid of B to the category of graded R modules which assigns to $b \in B$ the module $H_*(F_b;R)$ and to a path class $[\omega]$ the homomorphism $h[\omega]_*: H_*(F_{\omega(0)};R) \to H_*(F_{\omega(1)};R)$. The fibration is said to be *orientable over* R if for any closed path ω in B, $h[\omega]_* = 1$. This is a generalization of the concept of orientability of a sphere bundle. (In fact, a sphere bundle ξ is orientable as a sphere bundle if and only if $\dot{p}_\xi: \dot{E}_\xi \to B$ is orientable as a fibration.)

5 THEOREM (*a*) *A fibration over a simply connected base space is orientable over any R.*

(*b*) *A fibration induced from a fibration orientable over R is itself orientable over R.*

PROOF The first statement is immediate from the definition. For the second, let $p': E' \to B'$ be induced from $p: E \to B$ by a map $f: B' \to B$ and let $g': E' \to E$ be the associated map. For any path class $[\omega']$ in B' let $g'_0: F'_{\omega'(0)} \to F_{g\omega'(0)}$ and $g'_1: F'_{\omega'(1)} \to F_{g\omega'(1)}$ be the homeomorphisms defined by g'. It is then easy to verify that $[g'_1] \circ h[\omega'] = h[g \circ \omega'] \circ [g'_0]$, and this implies (*b*). ■

A map $f: p^{-1}(b_0) \to p^{-1}(b_1)$ between fibers of a fibration $p: E \to B$ is said to be *admissible* if there is some path class $[\omega]$ from b_0 to b_1 such that $[f] = h[\omega]$. The following facts are immediate from the definition.

6 *An admissible map is a homotopy equivalence.* ■

7 *The composite of admissible maps is an admissible map.* ■

8 *A homotopy inverse of an admissible map is an admissible map.* ■

9 *If B is path connected, there is an admissible map between any two fibers over B.* ■

10 *If $p: E \to B$ is orientable over R, any two admissible maps from $p^{-1}(b_0)$ to $p^{-1}(b_1)$ induce the same homomorphism from $H_*(p^{-1}(b_0);R)$ to $H_*(p^{-1}(b_1);R)$.* ■

Let $b_0 \in B$ be a base point and let $F = p^{-1}(b_0)$. Given a map $\alpha: X \to B$, an *admissible lifting of* α is a map $\bar{\alpha}: X \times F \to E$ such that $p\bar{\alpha}(x,z) = \alpha(x)$ for $x \in X$ and $z \in F$ and such that for any $x \in X$ the map $f_x: F \to p^{-1}(\alpha(x))$ defined by $f_x(z) = \bar{\alpha}(x,z)$ for $z \in Z$ is an admissible map. The following result is a useful criterion for the admissibility of a lifting.

11 LEMMA *Let p: $E \to B$ be a fibration and let X be a path-connected space. Given maps α: $X \to B$ and $\bar{\alpha}$: $X \times F \to E$ such that $p\bar{\alpha}(x,z) = \alpha(x)$, then $\bar{\alpha}$ is an admissible lifting of α if and only if there is some $x_0 \in X$ such that f_{x_0}: $F \to p^{-1}(\alpha(x_0))$ is admissible.*

PROOF The necessity of the condition is obvious. To prove the sufficiency, let $x_1 \in X$ and let ω: $I \to X$ be a path from x_0 to x_1. Since f_{x_0} is admissible, there is a path ω' in B from b_0 to $\alpha(x_0)$ such that $[f_{x_0}] = h[\omega']$. It is then easy to verify that $[f_{x_1}] = h[\omega' * (\alpha \circ \omega)]$, and so f_{x_1} is admissible. ∎

We want to prove the existence of admissible liftings in certain cases. The following is an alternate version of corollary 7.2.7 valid for a nonpolyhedral pair (X,A).

12 LEMMA *Let p: $E \to B$ be a fibration and let X be a space and A a strong deformation retract of X. Given maps f: $A \to E$ and g: $X \to B$ such that $p \circ f = g \mid A$, there is a map $\bar{g}$: $X \to E$ such that $p \circ \bar{g} = g$ and $\bar{g} \mid A$ is fiber homotopic to f.*

PROOF Let D: $X \times I \to X$ be a homotopy relative to A from some retraction r: $X \to A$ to 1_X (D exists, because A is a strong deformation retract of X). Then $g \circ D$: $X \times I \to B$ and $f \circ r$: $X \to E$ are maps such that $gD(x,0) = gr(x) = pfr(x)$. By the homotopy lifting property of p, there exists a map F: $X \times I \to E$ such that $p \circ F = g \circ D$ and $F(x,0) = fr(x)$. Let $\bar{g}$: $X \to E$ be defined by $\bar{g}(x) = F(x,1)$. Then

$$p\bar{g}(x) = pF(x,1) = gD(x,1) = g(x)$$

and $F \mid A \times I$ is a fiber homotopy from f to $\bar{g} \mid A$. Therefore $\bar{g}$ has the requisite properties. ∎

Let p: $E \to B$ be a fibration over a path-connected base space and let (B,A) be a relative CW complex. Let $b_0 \in B$ be a base point and $F = p^{-1}(b_0)$. For any s, v_0 is a strong deformation retract of Δ^s, and so $v_0 \times F$ is a strong deformation retract of $\Delta^s \times F$. It follows from lemma 12 that, given a singular simplex

$$\sigma: (\Delta^s, \dot{\Delta}^s) \to ((B,A)^s, (B,A)^{s-1})$$

in $\bar{\Delta}((B,A)^s)$, there exist maps $\bar{\sigma}$: $(\Delta^s,\dot{\Delta}^s) \times F \to (E_s,E_{s-1})$ such that $p\bar{\sigma}(x,z) = \sigma(x)$ for $x \in \Delta^s$ and $z \in F$ and such that $\bar{\sigma} \mid v_0 \times F$: $F \to p^{-1}(\sigma(v_0))$ is admissible. By lemma 11, $\bar{\sigma}$ is an admissible lifting of σ.

If $\bar{\sigma}_0$, $\bar{\sigma}_1$: $(\Delta^s,\dot{\Delta}^s) \times F \to (E_s,E_{s-1})$ are two admissible liftings of σ, there is an admissible map f: $F \to F$ such that $\bar{\sigma}_0 \mid v_0 \times F \underset{p}{\simeq} (\bar{\sigma}_1 \mid v_0 \times F) \circ f$. Let g: $\Delta^s \times F \times 0 \cup v_0 \times F \times I \cup \Delta^s \times F \times 1 \to E_s$ be the map defined by $g(x,z,0) = \bar{\sigma}_0(x,z)$, $g(x,z,1) = \bar{\sigma}_1(x,f(z))$, and $g \mid v_0 \times F \times I$: $\bar{\sigma}_0 \mid v_0 \times F \underset{p}{\simeq} (\bar{\sigma}_1 \mid v_0 \times F) \circ f$. Since $p \circ g$ can be extended to a map $\Delta^s \times F \times I \to B$ [by sending (x,z,t) to $\sigma(x)$] and $\Delta^s \times F \times 0 \cup v_0 \times F \times I \cup \Delta^s \times F \times 1$ is a strong deformation retract of $\Delta^s \times F \times I$, it follows from lemma 12 that

$g \mid \Delta^s \times F \times 0 \underset{p}{\simeq} g \mid \Delta^s \times F \times 1$. Therefore $\bar{\sigma}_0$ is homotopic to the composite

$$(\Delta^s,\dot{\Delta}^s) \times F \xrightarrow{1 \times f} (\Delta^s,\dot{\Delta}^s) \times F \xrightarrow{\bar{\sigma}_1} (E_s,E_{s-1})$$

In case $p: E \to B$ is orientable over R, then $f_*: H_*(F) \to H_*(F)$ is the identity map, and so the composite

$$H_*((\Delta^s,\dot{\Delta}^s) \times F) \xrightarrow{(1 \times f)_*} H_*((\Delta^s,\dot{\Delta}^s) \times F) \xrightarrow{\bar{\sigma}_{1*}} H_*(E_s,E_{s-1})$$

equals $\bar{\sigma}_{1*}$. Therefore $\bar{\sigma}_{0*} = \bar{\sigma}_{1*}$, and we have proved the following result.

13 THEOREM *Let $p: E \to B$ be an orientable fibration over a path-connected relative CW complex (B,A), with $F = p^{-1}(b_0)$. For $\sigma: (\Delta^s,\dot{\Delta}^s) \to ((B,A)^s, (B,A)^{s-1})$ there is a well-defined homomorphism*

$$\bar{\sigma}_*: H_*((\Delta^s,\dot{\Delta}^s) \times F) \to H_*(E_s,E_{s-1})$$

defined to be the homomorphism induced by any admissible lifting $\bar{\sigma}: (\Delta^s,\dot{\Delta}^s) \times F \to (E_s,E_{s-1})$ of σ. ∎

The identity map $\xi_s: \Delta^s \subset \Delta^s$ is a cycle modulo $\dot{\Delta}^s$, and its homology class $\{\xi_s\}$ generates $H_s(\Delta^s,\dot{\Delta}^s; R)$. Given $w \in H_n(F;G)$, then $\{\xi_s\} \times w \in H_{n+s}((\Delta^s,\dot{\Delta}^s) \times F; G)$ and $\bar{\sigma}_*(\{\xi_s\} \times w) \in H_{n+s}(E_s,E_{s-1}; G)$. It is clear that for fixed σ the map $w \to \bar{\sigma}_*(\{\xi_s\} \times w)$ is a homomorphism from $H_n(F;G)$ to $H_{n+s}(E_s,E_{s-1}; G)$. Because the cellular σ's form a basis of the free module $\bar{\Delta}_s(B) = \bar{\Delta}_s((B,A)^s)$, there is a homomorphism

$$\psi: \bar{\Delta}_s((B,A)^s) \otimes H_n(F;G) \to H_{n+s}(E_s,E_{s-1}; G)$$

defined by $\psi(\sigma \otimes w) = \bar{\sigma}_*(\{\xi_s\} \times w)$. If $\sigma(\Delta^s) \subset (B,A)^{s-1}$, then $\bar{\sigma}(\Delta^s \times F) \subset E_{s-1}$ for any admissible lifting of σ. Therefore $\bar{\sigma}_*(\{\xi_s\} \times w) = 0$, and so ψ defines a homomorphism

$$\psi: \bar{\Delta}_s((B,A)^s, (B,A)^{s-1}) \otimes H_n(F;G) \to H_{n+s}(E_s,E_{s-1}; G)$$

Next we show that ψ induces a homomorphism from the module $H_s((B,A)^s, (B,A)^{s-1}; H_n(F;G))$.

14 LEMMA *The composite*

$$\begin{array}{c} \bar{\Delta}_{s+1}((B,A)^s, (B,A)^{s-1}) \otimes H_n(F;G) \\ \big\downarrow {\scriptstyle \partial \otimes 1} \\ \bar{\Delta}_s((B,A)^s, (B,A)^{s-1}) \otimes H_n(F;G) \\ \big\downarrow {\scriptstyle \psi} \\ H_{n+s}(E_s,E_{s-1};G) \end{array}$$

is trivial.

PROOF Let $\sigma: (\Delta^{s+1},(\Delta^{s+1})^{s-1}) \to ((B,A)^s,(B,A)^{s-1})$ be a cellular $(s+1)$-simplex of $(B,A)^s$ and let

$$\bar{\sigma}: (\Delta^{s+1},(\Delta^{s+1})^{s-1}) \times F \to (E_s,E_{s-1})$$

be an admissible lifting of σ. For $0 \le i \le s+1$ let $e^i_{s+1}\colon \Delta^s \to \Delta^{s+1}$ omit the ith vertex. Then $\sigma^{(i)} = \sigma \circ e^i_{s+1}$, and the composite

$$\Delta^s \times F \xrightarrow{e^i_{s+1} \times 1} \dot\Delta^{s+1} \times F \xrightarrow{\sigma'} E_s$$

where $\sigma' = \bar\sigma \mid \dot\Delta^{s+1} \times F$, is an admissible lifting of $\sigma^{(i)}$. Therefore

$$\psi(\sigma^{(i)} \otimes w) = \sigma'_*(e^i_{s+1} \times 1)_*(\{\xi_s\} \times w) = \sigma'_*(\{e^i_{s+1}\} \times w)$$

where $\{e^i_{s+1}\} \in H_s(\dot\Delta^{s+1},(\Delta^{s+1})^{s-1})$. It follows that

$$\psi(\partial\sigma \times w) = \sigma'_*(\{\Sigma\,(-1)^i e^i_{s+1}\} \times w)$$

However, in $\Delta(\Delta^{s+1})$ we have the relation $\partial\xi_{s+1} = \Sigma\,(-1)^i e^i_{s+1}$. Hence, if $j\colon (\dot\Delta^{s+1},(\Delta^{s+1})^{s-1}) \subset (\Delta^{s+1},(\Delta^{s+1})^{s-1})$, then $j_*\{\Sigma\,(-1)^i e^i_{s+1}\} = 0$. Because σ' equals the composite

$$\dot\Delta^{s+1} \times F \overset{j \times 1}{\subset} \Delta^{s+1} \times F \xrightarrow{\bar\sigma} E_s$$

it follows that

$$\psi(\partial\sigma \otimes w) = \bar\sigma_*(j \times 1)_*(\{\Sigma\,(-1)^i e^i_{s+1}\} \times w) = \bar\sigma_*(0) = 0 \quad \blacksquare$$

Every element of $\bar\Delta_s((B,A)^s, (B,A)^{s-1}) \otimes H_n(F;G)$ is an s-dimensional cycle of the chain complex $\bar\Delta((B,A)^s, (B,A)^{s-1}) \otimes H_n(F;G)$, and the s-dimensional boundaries are the elements in the image of

$$\partial \otimes 1\colon \bar\Delta_{s+1}((B,A)^s, (B,A)^{s-1}) \otimes H_n(F;G) \to \bar\Delta_s((B,A)^s, (B,A)^{s-1}) \otimes H_n(F;G)$$

It follows from lemma 14 that ψ induces a homomorphism

$$\psi_*\colon H_s((B,A)^s, (B,A)^{s-1}; H_n(F;G)) \to H_{n+s}(E_s,E_{s-1}; G)$$

The computation of the E^1 term of the spectral sequence is completed by the following result.

15 THEOREM *(a) For all $s \ge 0$ there is an isomorphism*

$$\psi_*\colon H_s((B,A)^s, (B,A)^{s-1}; H_n(F;G)) \approx H_{n+s}(E_s,E_{s-1}; G)$$

(b) For $s \ge 1$ there is a commutative square

$$\begin{array}{ccc} H_s((B,A)^s, (B,A)^{s-1}; H_n(F;G)) & \xrightarrow{\psi_*} & H_{n+s}(E_s,E_{s-1}; G) \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} \\ H_{s-1}((B,A)^{s-1}, (B,A)^{s-2}; H_n(F;G)) & \xrightarrow{\psi_*} & H_{n+s-1}(E_{s-1},E_{s-2}; G) \end{array}$$

PROOF (a) Because of direct-sum properties of both modules (the right-hand one by lemma 2) and an obvious naturality property of ψ_*, it suffices to prove that for an s-cell e of $B - A$ the map

$$\psi_*\colon H_s(e,\dot e; H_n(F;G)) \to H_{n+s}(p^{-1}(e),p^{-1}(\dot e); G)$$

is an isomorphism. Let $f\colon (E^s,S^{s-1}) \to (e,\dot e)$ be a characteristic map for e and let $p'\colon E' \to E^s$ be the induced fibration over E^s with corresponding map $f'\colon (E', p'^{-1}(S^{s-1})) \to (p^{-1}(e), p^{-1}(\dot e))$. Then there is a commutative square

$$\begin{array}{ccc} H_s(E^s,S^{s-1};\, H_n(F;G)) & \xrightarrow{\psi_*} & H_{n+s}(E',p'^{-1}(S^{s-1});\, G) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f'_*} \\ H_s(e,\dot{e};\, H_n(F;G)) & \xrightarrow{\psi_*} & H_{n+s}(p^{-1}(e),p^{-1}(\dot{e});\, G) \end{array}$$

in which the vertical maps are isomorphisms (by excision and homotopy properties). Therefore it suffices to prove the result for a trivial fibration over E^s, and for such a fibration ψ_* is an isomorphism by the Künneth theorem.

(*b*) Given $\sigma\colon (\Delta^s,\dot{\Delta}^s) \to ((B,A)^s, (B,A)^{s-1})$, let $\{\sigma \otimes w\}$ be the element of $H_s((B,A)^s, (B,A)^{s-1};\, H_n(F;G))$ determined by the cycle $\sigma \otimes w$. Then in $H_{s-1}((B,A)^{s-1}, (B,A)^{s-2};\, H_n(F;G))$, we have $\partial\{\sigma \otimes w\} = \{\sum (-1)^i\sigma^{(i)} \otimes w\}$. Let $\bar{\sigma}\colon (\Delta^s,\dot{\Delta}^s) \times F \to (E_s,E_{s-1})$ be an admissible lifting of σ. For $0 \le i \le s$ the composite

$$(\Delta^{s-1},\dot{\Delta}^{s-1}) \times F \xrightarrow{e_s^i \times 1} (\dot{\Delta}^s,(\Delta^s)^{s-2}) \times F \xrightarrow{\sigma'} (E_{s-1},E_{s-2})$$

where $\sigma' = \bar{\sigma} \mid (\dot{\Delta}^s,(\Delta^s)^{s-2})$, is an admissible lifting of $\sigma^{(i)}$. Therefore

$$\begin{aligned} \psi_*\,\partial\{\sigma \otimes w\} &= \textstyle\sum (-1)^i\sigma'_*(e_s^i \times 1)_*(\{\xi_{s-1}\} \times w) = \sigma'_*(\{\sum (-1)^i e_s{}^i\} \times w) \\ &= \sigma'_*(\partial\{\xi_s\} \times w) = \sigma'_*\,\partial(\{\xi_s\} \times w) = \partial\bar{\sigma}_*(\{\xi_s\} \times w) \\ &= \partial\psi_*\{\sigma \otimes w\} \quad \blacksquare \end{aligned}$$

Because $H_s((B,A)^s, (B,A)^{s-1})$ is a free module, it follows from the universal-coefficient theorem that

$$\begin{aligned} H_s((B,A)^s, (B,A)^{s-1};\, H_n(F;G)) &\approx H_s((B,A)^s, (B,A)^{s-1}) \otimes H_n(F;G) \\ &= C_s(B,A) \otimes H_n(F;G) \end{aligned}$$

Under this isomorphism, it is easy to see that the boundary operator of the triple $((B,A)^s, (B,A)^{s-1}, (B,A)^{s-2})$ corresponds to the map

$$\partial \otimes 1\colon C_s(B,A) \otimes H_n(F;G) \to C_{s-1}(B,A) \otimes H_n(F;G)$$

Therefore theorem 15 can be interpreted as asserting that ψ induces an isomorphism of the bigraded chain complex $C_*(B,A) \otimes H_*(F;G)$, with the E^1 term of the spectral sequence of theorem 1. This, together with corollary 4, gives the following result about the E^2 term.

16 THEOREM *Let $p\colon E \to B$ be an orientable fibration over a path-connected relative CW complex (B,A) and let $F = p^{-1}(b_0)$. There is a convergent E^2 spectral sequence with $E^2_{s,t} \approx H_s(B,A;\, H_t(F;G))$ and E^∞ the bigraded module associated to the filtration of $H_*(E,E_A;\, G)$ defined by*

$$F_sH_*(E,E_A;\, G) = \operatorname{im}\,[H_*(E_s,E_A;\, G) \to H_*(E,E_A;\, G)] \quad \blacksquare$$

Note that the spectral sequence of theorem 16 is a first-quadrant spectral sequence and is functorial on the category of orientable fibrations $p\colon E \to B$ over a path-connected relative CW complex (B,A) and fiber-preserving maps $f'\colon E' \to E$ such that the base space pair is mapped by a cellular map $f\colon (B',A') \to (B,A)$.

To extend the spectral sequence to fibrations with more general base spaces, let $p\colon E \to B$ be an orientable fibration over a path-connected base space B and let $A \subset B$. Let $f\colon (B',A) \to (B,A)$ be a relative CW approximation to (B,A) (which exists by theorem 7.8.1). Let $p'\colon E' \to B'$ be the induced fibration and $f'\colon E' \to E$ the fiber-preserving map induced by f. It follows from the exactness of the homotopy sequence of a fibration and the five lemma that f' is a weak homotopy equivalence. Therefore f' induces an isomorphism of the homology sequence of (E',E'_A) with the homology sequence of (E,E_A). Because B' is path connected and $p'\colon E' \to B'$ is orientable, there is a convergent E^2 spectral sequence with

$$E^2_{s,t} \approx H_s(B',A;\ H_t(F;G)) \approx H_s(B,A;\ H_t(F;G))$$

and E^∞ associated to a filtration of $H_*(E',E'_A;\ G) \approx H_*(E,E_A;\ G)$. If $g\colon (B'',A) \to (B,A)$ is another relative CW approximation to (B,A), there is a cellular map $h\colon (B'',A) \to (B',A)$ such that $f \circ h \simeq g$ rel A. The map h induces an isomorphism of the E^2 spectral sequences of $p''\colon E'' \to B''$ and $p'\colon E' \to B'$ (but not an isomorphism of the E^1 terms). It follows that the filtration of $H_*(E,E_A;\ G)$ induced by the isomorphisms $H_*(E',E'_A;\ G) \approx H_*(E,E_A;\ G)$ and $H_*(E'',E''_A;\ G) \approx H_*(E,E_A;\ G)$ correspond, and we have the following main result.

17 THEOREM *Let $p\colon E \to B$ be an orientable fibration with B path connected and fiber F over $b_0 \in B$. Given $A \subset B$, there is a convergent E^2 spectral sequence, with $E^2_{s,t} \approx H_s(B,A;\ H_t(F;G))$ and E^∞ the bigraded module associated to some filtration of $H_*(E,E_A;\ G)$. This spectral sequence is a first-quadrant spectral sequence functorial on the category of orientable fibrations and fiber-preserving maps.* ∎

3 APPLICATIONS OF THE HOMOLOGY SPECTRAL SEQUENCE

In this section we shall consider applications of the spectral sequence of a fibration and show that it leads to generalized Gysin and Wang homology sequences in case the fiber or base is a homology sphere. We shall also use the spectral sequence in the proof of the homotopy excision theorem. The section concludes with a definition of the Hopf invariant homomorphism and an exact sequence connecting it and the suspension homomorphism of homotopy groups of spheres.

1 THEOREM *Let $p\colon E \to B$ be a fibration which is orientable over a field with path-connected base and with fiber F. Assume that the Euler characteristics $\chi(F)$ and $\chi(B)$ are defined (over the field). Then $\chi(E)$ is defined, and $\chi(E) = \chi(B)\chi(F)$.*

PROOF We use the spectral sequence of theorem 9.2.17. For a finitely generated bigraded module E^r we define the Euler characteristic $\chi(E^r) = \sum_{s,t}(-1)^{s+t} \dim E^r_{s,t}$. Because we are considering a field as coefficients, it follows from the Künneth formula that

$$E^2_{s,t} \approx H_s(B;H_t(F)) \approx H_s(B) \otimes H_t(F)$$

Therefore $\chi(E^2) = \chi(B)\chi(F)$. Because $E^{r+1} \approx H(E^r)$, it follows (as in theorem 4.3.14) that

$$\chi(E^2) = \chi(E^3) = \cdots = \chi(E^r)$$

Because $E^2_{s,t} = 0$ if s and t are large enough, the same is true of $E^r_{s,t}$ for any r. Therefore $E^\infty = E^r$ for large enough r, and so $\chi(E^\infty) = \chi(B)\chi(F)$. By a standard property of dimension,

$$\dim [H_n(E)] = \sum_{s+t=n} \dim E^\infty_{s,t}$$

and so $\chi(E) = \chi(E^\infty) = \chi(B)\chi(F)$. ∎

We now compute the homomorphism induced by $i\colon F \subset E$ in terms of the spectral sequence. For $r \geq 2$, $E^{r+1}_{0,t}$ is a quotient of $E^r_{0,t}$ (because $E^r_{-r,t+r-1} = 0$ in a first-quadrant spectral sequence). Therefore there is an epimorphism $E^2_{0,t} \to E^\infty_{0,t}$. Because B is path connected, there is an isomorphism $H_t(F;G) \approx H_0(B;\ H_t(F;G))$. By using the spectral sequence of the fibration $F \to b_0$ and the functorial property of the spectral sequence, it follows that $i_*\colon H_t(F;G) \to H_t(E;G)$ is the composite

$$H_t(F;G) \approx H_0(B;\ H_t(F;G)) \approx E^2_{0,t} \to E^\infty_{0,t} = F_0H_t(E;G) \subset H_t(E;G)$$

This leads to the following *generalized Wang homology sequence.*

2 THEOREM *Let $p\colon E \to B$ be a fibration, with fiber F and simply connected base B which is a homology n-sphere (over R) for some $n \geq 2$ [that is, $H_q(B) = 0$ if $q \neq 0$ or n and $H_0(B) \approx R \approx H_n(B)$]. Then there is an exact sequence*

$$\cdots \to H_t(F;G) \xrightarrow{i_*} H_t(E;G) \to H_{t-n}(F;G) \to H_{t-1}(F;G) \xrightarrow{i_*} \cdots$$

PROOF Because $H_*(B)$ has no torsion, $E^2_{s,t} \approx H_s(B) \otimes H_t(F;G)$ in the spectral sequence of p. Therefore $E^2_{s,t} = 0$ unless $s = 0$ or n, and the only nonzero differential is $d^n\colon E^2_{n,t} \to E^2_{0,t+n-1}$. Hence there are exact sequences

$$0 \to E^\infty_{n,t} \to E^2_{n,t} \xrightarrow{d^n} E^2_{0,t+n-1} \to E^\infty_{0,t+n-1} \to 0$$

and

$$0 \to E^\infty_{0,t} \to H_t(E;G) \to E^\infty_{n,t-n} \to 0$$

These fit together into an exact sequence

$$\cdots \to H_t(E;G) \to E^2_{n,t-n} \xrightarrow{d^n} E^2_{0,t-1} \to H_{t-1}(E;G) \to \cdots$$

The result follows on observing that

$$E^2_{n,t-n} \approx H_n(B) \otimes H_{t-n}(F;G) \approx H_{t-n}(F;G)$$
$$E^2_{0,t-1} \approx H_0(B) \otimes H_{t-1}(F;G) \approx H_{t-1}(F;G)$$

and that on replacing $E^2_{0,t-1}$ by $H_{t-1}(F;G)$ in the exact sequence, the resulting map $H_{t-1}(F;G) \to H_{t-1}(E;G)$ is i_*. ∎

Let $p\colon E \to B$ be an orientable fibration with path-connected base and let $B' \subset B$ and $E' = p^{-1}(B')$. We now show how the homomorphism induced by $p\colon (E,E') \to (B,B')$ is determined from the spectral sequence. For $r \geq 2$, $E^{r+1}_{s,0}$ is a submodule of $E^r_{s,0}$ (because $E^r_{s+r,-r+1} = 0$). Therefore there is a monomorphism $E^\infty_{s,0} \to E^2_{s,0}$. The augmentation homomorphism $H_0(F;G) \to G$ induces a homomorphism $H_s(B,B';\ H_0(F;G)) \to H_s(B,B';\ G)$. By using the spectral sequence of the fibration $B \subset B$ and the functorial property of the spectral sequence, it follows that $p_*\colon H_s(E,E';\ G) \to H_s(B,B';\ G)$ is the composite

$$H_s(E,E';\ G) = F_sH_s(E,E';\ G) \to E^\infty_{s,0} \to E^2_{s,0} \approx H_s(B,B';\ H_0(F;G)) \to H_s(B,B';\ G)$$

This leads to the following *generalized Gysin homology sequence.*

3 THEOREM *Let $p\colon E \to B$ be an orientable fibration with path-connected base space and with fiber F a homology n-sphere (over R), where $n \geq 1$. If $B' \subset B$ and $E' = p^{-1}(B')$, there is an exact sequence*

$$\cdots \to H_s(E,E';\ G) \xrightarrow{p_*} H_s(B,B';\ G) \to H_{s-n-1}(B,B';\ G) \to H_{s-1}(E,E';\ G) \xrightarrow{p_*} \cdots$$

PROOF Because, in the spectral sequence of p,

$$E^2_{s,t} \approx H_s(B,B';\ H_t(F;G)) = 0 \qquad t \neq 0 \text{ or } n$$

the only nonzero differential is $d^{n+1}\colon E^2_{s,0} \to E^2_{s-n-1,n}$. Hence there are exact sequences

$$0 \to E^\infty_{s,0} \to E^2_{s,0} \xrightarrow{d^{n+1}} E^2_{s-n-1,n} \to E^\infty_{s-n-1,n} \to 0$$

and

$$0 \to E^\infty_{s-n,n} \to H_s(E,E';\ G) \to E^\infty_{s,0} \to 0$$

These fit together into an exact sequence

$$\cdots \to H_s(E,E';\ G) \to E^2_{s,0} \xrightarrow{d^{n+1}} E^2_{s-n-1,n} \to H_{s-1}(E,E';\ G) \to \cdots$$

The result follows on observing that

$$E^2_{s,0} \approx H_s(B,B';\ H_0(F;G)) \approx H_s(B,B';\ G)$$
$$E^2_{s-n-1,n} \approx H_{s-n-1}(B,B';\ H_n(F;G)) \approx H_{s-n-1}(B,B';\ G)$$

and that on replacing $E^2_{s,0}$ by $H_s(B,B';\ G)$ in the exact sequence, the resulting map $H_s(E,E';\ G) \to H_s(B,B';\ G)$ is p_*. ∎

4 LEMMA *Let $p\colon E \to B$ be an orientable fibration with path-connected base space and with path-connected fiber F. Assume that $H_q(B,B') = 0$ for $q < n$ and $H_q(F) = 0$ for $0 < q < m$ (all coefficients R). Then the homomor-*

phism $p_*: H_q(E,E') \to H_q(B,B')$ *is an isomorphism for* $q \le n + m - 1$ *and an epimorphism for* $q = n + m$.

PROOF For the spectral sequence we have

$$E^2_{s,t} \approx H_s(B,B'; H_t(F)) \approx H_s(B,B') \otimes H_t(F) \oplus H_{s-1}(B,B') * H_t(F)$$

By the hypotheses, $E^2_{s,t} = 0$ if $s < n$ or $0 < t < m$. Therefore, if $q \le n + m - 1$, then $E^2_{s,q-s} = 0$, except possibly for the term $E^2_{q,0}$. It follows that $E^r_{s,q-s} = 0$, except for the term $E^r_{q,0}$, and $E^r_{q,0} \approx E^2_{q,0}$. Therefore $E^\infty_{q,0} \approx E^2_{q,0}$ and $E^\infty_{s,q-s} = 0$ if $s \ne q$. Hence

$$H_q(E,E') \approx H_q(B,B'; H_0(F)) \approx H_q(B,B')$$

and the isomorphism is induced by p_*.

If $q = n + m$, then $E^2_{s,n+m-s} = 0$ except for the terms $E^2_{n+m,0}$ and $E^2_{n,m}$. Since $E^2_{n+m-r,r-1} = 0$ for $r \ge 2$, it follows that

$$E^\infty_{n+m,0} \approx E^2_{n+m,0} \approx H_{n+m}(B,B'; H_0(F)) \approx H_{n+m}(B,B')$$

Therefore $p_*(H_{n+m}(E,E')) = H_{n+m}(B,B')$. ∎

We use this to prove the following *homotopy excision theorem.*[1]

5 **THEOREM** *Let A, B, and* $A \cap B$ *be path-connected subspaces of a simple space X such that*

(*a*) *Either* $X = \operatorname{int} A \cup \operatorname{int} B$, *or* $X = A \cup B$ *where A and B are closed subsets of X such that* $A \cap B$ *is a strong deformation retract of some neighborhood in A (or in B).*

(*b*) $A \cap B$, *A, B, and X have isomorphic fundamental groups.*

(*c*) $(A, A \cap B)$ *is n-connected and* $(B, A \cap B)$ *is m-connected, where* $n, m \ge 1$.

Then the homomorphism

$$j_\#: \pi_q(A, A \cap B) \to \pi_q(X,B)$$

induced by the excision map $j: (A, A \cap B) \subset (X,B)$ *is an isomorphism for* $q \le n + m - 1$ *and an epimorphism for* $q = n + m$.

PROOF First we reduce consideration to the case $X = \operatorname{int} A \cup \operatorname{int} B$. If A and B are closed in X and $A \cap B$ is a strong deformation retract of some neighborhood U in B, let $A' = A \cup U$ and observe that A is a strong deformation retract of A'. Furthermore, $A' \cap B = U$, and the inclusion map $(A, A \cap B) \subset (A', A' \cap B)$ is a homotopy equivalence, so that $(A', A' \cap B)$ is n-connected. By the exactness of the homotopy sequence of the triple $(B, A' \cap B, A \cap B)$ and the fact that $(A' \cap B, A \cap B)$ is k-connected for all k, we see that $(B, A' \cap B)$ is m-connected. Note that

$$X = A \cup (B - A) \subset \operatorname{int} A' \cup \operatorname{int} B,$$

[1]A more general form of this theorem can be found in A. L. Blakers and W. S. Massey, The homotopy groups of a triad, II, *Annals of Mathematics*, vol. 55, pp. 192–201, 1952.

and so A' and B satisfy conditions (a), (b), and (c). Since there is a commutative triangle

$$\begin{tikzcd} \pi_q(A, A \cap B) \arrow[rr, "\approx"] \arrow[dr, "j_\#"'] & & \pi_q(A', A' \cap B) \arrow[dl, "j'_\#"] \\ & \pi_q(X,B) & \end{tikzcd}$$

we are reduced to proving that $j'_\#$ has the desired properties.

Similarly, if $A \cap B$ is a strong deformation retract of some neighborhood V in A, let $B' = V \cup B$ and observe that B is a strong deformation retract of B'. Then $A \cap B' = V$, and it follows, as in the case above, that $(A, A \cap B')$ is n-connected and $(B', A \cap B')$ is m-connected. Since $X = (A - B) \cup B$ is contained in int $A \cup$ int B', we see that A and B' satisfy conditions (a), (b), and (c). From the commutativity of the square

$$\begin{tikzcd} \pi_q(A, A \cap B) \arrow[r, "\approx"] \arrow[d, "j_\#"'] & \pi_q(A, A \cap B') \arrow[d, "j''_\#"] \\ \pi_q(X,B) \arrow[r, "\approx"] & \pi_q(X,B') \end{tikzcd}$$

we are reduced to proving that $j''_\#$ has the desired properties.

In either case we have shown that it suffices to prove the theorem under the hypothesis that $X =$ int $A \cup$ int B, and we make this assumption now. By corollary 8.3.8, there is a fibration $p: E \to X$ such that E is simply connected and $p_\#: \pi_q(E) \approx \pi_q(X)$ for $q > 1$. Let E_A and E_B be the parts of E over A and B, respectively, and note that $E_A \cap E_B$ is the part of E over $A \cap B$. From theorem 7.2.8 it follows that $(E_A, E_A \cap E_B)$ is n-connected and $(E_B, E_A \cap E_B)$ is m-connected. Using (b) and the exactness of the homotopy sequence of a fibration, it is easy to see that $E_A \cap E_B$, E_A, and E_B are all simply connected. Since it is obvious that $E \subset p^{-1}(\text{int } A) \cup p^{-1}(\text{int } B) \subset \text{int } E_A \cup \text{int } E_B$, we have reduced the theorem to the case where all the spaces in question are simply connected by virtue of the commutativity of the square

$$\begin{tikzcd} \pi_q(E_A, E_A \cap E_B) \arrow[r, "\approx"] \arrow[d, "\bar{j}_\#"'] & \pi_q(A, A \cap B) \arrow[d, "j_\#"] \\ \pi_q(E,E_B) \arrow[r, "\approx"] & \pi_q(X,B) \end{tikzcd}$$

Thus, assume $X =$ int $A \cup$ int B and that $A \cap B$, A, B, and X are all simply connected. We replace the inclusion map $A \subset X$ by the homotopically equivalent mapping path fibration $p: P \to X$ as in theorem 2.8.9. Then P is the space of paths $\omega: (I,0) \to (X,A)$ in the compact-open topology, and $p(\omega) = \omega(1)$. The fiber F of p over a point $a_0 \in A \cap B$ is the space of paths in X which start in A and end at a_0. If $p': PX \to X$ is the path fibration of all paths in X which end at a_0 and $p'(\omega) = \omega(0)$, then $F = p'^{-1}(A)$. Since PX is contractible, there are isomorphisms

$$\pi_q(X,A) \xleftarrow[\approx]{p'_\#} \pi_q(PX,F) \xrightarrow[\approx]{\partial} \pi_{q-1}(F)$$

Because $X = \text{int } A \cup \text{int } B$, the excision map j': $(B, A \cap B) \subset (X,A)$ induces isomorphisms in homology. It follows from the relative Hurewicz isomorphism theorem and the m-connectedness of $(B, A \cap B)$ that (X,A) is also m-connected. Therefore F is $(m - 1)$-connected, and so $H_q(F) = 0$ for $0 < q < m$.

Let $E' = p^{-1}(B)$ and observe that since X is simply connected, the fibration p: $P \to X$ is orientable. Since $j_* : H_q(A, A \cap B) \approx H_q(X,B)$, it follows that $H_q(X,B) = 0$ for $q < n + 1$. By lemma 4, the homomorphism

$$p_* : H_q(P,E') \to H_q(X,B)$$

is an isomorphism for $q \leq n + m$ and an epimorphism for $q = n + m + 1$. The map j: $(A, A \cap B) \subset (X,B)$ has a lifting $\bar{j}$: $(A, A \cap B) \to (P,E')$, where $\bar{j}(a)$ is the constant path at a for all $a \in A$. There is a commutative triangle

$$\begin{tikzcd} H_q(A, A \cap B) \arrow[rr, "\bar{j}_*"] \arrow[dr, "j_*"', "\approx"] & & H_q(P,E') \arrow[dl, "p_*"] \\ & H_q(X,B) & \end{tikzcd}$$

Therefore $\bar{j}_*$ is an isomorphism for $q \leq n + m$. Since $\bar{j} \mid A$: $A \to P$ is a homotopy equivalence, it follows from the five lemma that the homomorphism

$$(\bar{j} \mid A \cap B)_* : H_q(A \cap B) \to H_q(E')$$

is an isomorphism for $q \leq n + m - 1$.

Because $\pi_1(E') \approx \pi_1(F) \approx \pi_2(X,A)$, and the latter group is a quotient group of $\pi_2(X)$ since $\pi_1(A) \approx \pi_1(X)$, we see that E' has an abelian fundamental group. Since $A \cap B$ is simply connected, it follows from the absolute Hurewicz isomorphism theorem that E' is also simply connected. By the Whitehead theorem, the homomorphism

$$(\bar{j} \mid A \cap B)_\#: \pi_q(A \cap B) \to \pi_q(E')$$

is an isomorphism for $q \leq n + m - 2$ and an epimorphism for $q = n + m - 1$. Since $\bar{j} \mid A$: $A \to E$ is a homotopy equivalence, it follows from the five lemma that the homomorphism

$$\bar{j}_\#: \pi_q(A, A \cap B) \to \pi_q(P,E')$$

is an isomorphism for $q \leq n + m - 1$ and an epimorphism for $q = n + m$. The result follows from this and the commutativity of the triangle

$$\begin{tikzcd} \pi_q(A, A \cap B) \arrow[rr, "\bar{j}_\#"] \arrow[dr, "j_\#"'] & & \pi_q(P,E') \arrow[dl, "p_\#", "\approx"'] \\ & \pi_q(X,B) & \end{tikzcd} \quad \blacksquare$$

It should be noted that the main argument above involved the case where A and B satisfy (c), satisfy (b) in the stronger form that all the spaces in question are simply connected, and satisfy the condition that $\{A,B\}$ is an excisive couple of subsets of X, which is a weak form of (a). It should also be observed that if A and B satisfy condition (a) of theorem 5, then if $(A, A \cap B)$ is n-connected [or $(B, A \cap B)$ is m-connected], it is easy to show that (X,B) is also n-connected [or (X,A) is m-connected]. Furthermore, if A and B satisfy a and c and $A \cap B$ is simply connected, then it follows that A and B are each simply connected and also that X is simply connected. Hence condition b is also satisfied, and theorem 5 is valid in this case.

6 COROLLARY *Let (X,A) be an n-connected relative CW complex, where $n \geq 2$, such that A is m-connected, where $m \geq 1$. Then the collapsing map k: $(X,A) \to (X/A,x_0)$ induces a homomorphism*

$$k_\#: \pi_q(X,A) \to \pi_q(X/A)$$

which is an isomorphism for $q \leq m + n$ and an epimorphism for $q = m + n + 1$.

PROOF Let CA be the unreduced cone over A and regard it as a space whose intersection with X is A. Since A is m-connected and CA is contractible, it follows that (CA,A) is $(m + 1)$-connected. We shall apply theorem 5, with A and B replaced by X and CA, respectively. Since $X \cap CA = A$ is a strong deformation retract of some neighborhood in CA, a of theorem 5 is satisfied. Since A is simply connected and c is also satisfied, it follows, as in the remarks above, that b is satisfied too. Hence the hypotheses of theorem 5 are satisfied, and it follows that j: $(X,A) \subset (X \cup CA, CA)$ induces a homomorphism

$$j_\#: \pi_q(X,A) \to \pi_q(X \cup CA, CA)$$

which is an isomorphism for $q \leq n + m$ and an epimorphism for $q = n + m + 1$. It follows from lemma 7.1.5 that the collapsing map k': $(X \cup CA, CA) \to (X \cup CA, CA)/CA$ is a homotopy equivalence. The result follows from the commutativity of the triangle

$$\begin{tikzcd} \pi_q(X,A) \arrow[r, "j_\#"] \arrow[dr, "k_\#"'] & \pi_q(X \cup CA, CA) \arrow[d, "k'_\#", "\approx"'] \\ & \pi_q(X/A) \end{tikzcd} \qquad \blacksquare$$

7 COROLLARY *Let f: $(X',A') \to (X,A)$ be a relative homeomorphism between relative CW complexes both of which are n-connected, with $n \geq 2$, and such that A' and A are m-connected, with $m \geq 1$. Then f induces an isomorphism*

$$f_\#: \pi_q(X',A') \approx \pi_q(X,A) \qquad q \leq n + m$$

PROOF Let k': $(X',A') \to (X'/A',x_0')$ and k: $(X,A) \to (X/A,x_0)$ be the collapsing maps. Then f induces a homeomorphism f': $X'/A' \to X/A$ such that $f' \circ k' = k \circ f$. Since f' induces isomorphisms of the homotopy groups in all dimensions, the result follows from corollary 6. ■

We use this last result to study the suspension map

$$S: \pi_q(S^n) \to \pi_{q+1}(S^{n+1})$$

in more detail. Since $S^{n+1} = S(S^n)$, there is a characteristic map $\mu'\colon S^n \to \Omega S^{n+1}$ for the path fibration $PS^{n+1} \to S^{n+1}$. From the commutativity of the triangle

$$\begin{tikzcd} \pi_q(S^n) \arrow[rr, "\mu'_\#"] \arrow[dr, "S"'] & & \pi_q(\Omega S^{n+1}) \\ & \pi_{q+1}(S^{n+1}) \arrow[ur, "\approx"', "\partial"] & \end{tikzcd}$$

it suffices to study the map $\mu'_\#$.

Let X^{2n} be the space obtained from $S^n \times S^n$ by identifying (z,z_0) with (z_0,z) for all $z \in S^n$ (where z_0 is a base point of S^n). We regard S^n as imbedded in X^{2n} as the set of points corresponding to $S^n \times z_0$ in $S^n \times S^n$. Then X^{2n} is a CW complex consisting of S^n and a single $2n$-cell attached by a map $\alpha_n\colon S^{2n-1} \to S^n$.

8 LEMMA *There is a map $g\colon X^{2n} \to \Omega S^{n+1}$, where $n \geq 2$, which is a $(3n-1)$-equivalence such that $g \mid S^n = \mu'$.*

PROOF Let $\mu\colon S^n \times \Omega S^{n+1} \to \Omega S^{n+1}$ be the map defined by $\mu(z,\omega) = \omega * \mu'(z)$. By corollary 8.5.8, μ is homotopic to a clutching function for the fibration $PS^{n+1} \to S^{n+1}$. Let $f\colon S^n \times S^n \to \Omega S^{n+1}$ be defined by $f(z,z') = \mu'(z') * \mu'(z)$. There is a commutative diagram

$$\begin{tikzcd} H_{n+1}(C_-S^n,S^n) \otimes H_n(\Omega S^{n+1}) \arrow[rr, "\approx"'] \arrow[d, "\partial \otimes 1"', "\approx"] & & H_{2n+1}((C_-S^n,S^n) \times \Omega S^{n+1}) \arrow[dr, "\approx"', "\mu_* \partial"] & \\ H_n(S^n) \otimes H_n(\Omega S^{n+1}) \arrow[r, "\approx"'] & H_{2n}(S^n \times \Omega S^{n+1}) \arrow[rrr, "\mu_*"] & & & H_{2n}(\Omega S^{n+1}) \\ H_n(S^n) \otimes H_n(S^n) \arrow[u, "1 \otimes \mu'_*", "\approx"'] \arrow[rr, "\approx"'] & & H_{2n}(S^n \times S^n) \arrow[ul, "(1 \times \mu')_*"] \arrow[urr, "f_*"'] & & \end{tikzcd}$$

Therefore $f_*\colon H_{2n}(S^n \times S^n) \approx H_{2n}(\Omega S^{n+1})$. Since $f \mid S^n \vee S^n$ is homotopic to the map sending (z,z_0) to $\mu'(z)$ and (z_0,z) to $\mu'(z)$, f is homotopic to a map f' such that $f'(z,z_0) = \mu'(z) = f'(z_0,z)$. Then f' defines a map $g\colon X^{2n} \to \Omega S^{n+1}$ such that $g \circ k = f'$, where $k\colon S^n \times S^n \to X^{2n}$ is the quotient map. Then $g \mid S^n = \mu'$, and since $H_n(S^n) \approx H_n(X^{2n})$, $g_*\colon H_n(X^{2n}) \approx H_n(\Omega S^{n+1})$. Since $k_*\colon H_{2n}(S^n \times S^n) \approx H_{2n}(X^{2n})$, it follows that $g_*\colon H_{2n}(X^{2n}) \approx H_{2n}(\Omega S^{n+1})$. The only nontrivial homology groups of X^{2n} are in degrees, 0, n, and $2n$, and in degrees $< 3n$ the only nontrivial homology groups of ΩS^{n+1} are in degrees 0, n, and $2n$. Therefore $g_*\colon H_q(X^{2n}) \approx H_q(\Omega S^{n+1})$ for $q < 3n$. Since $n \geq 2$, X^{2n} and ΩS^{n+1} are both simply connected. By the Whitehead theorem, the homomorphism

$$g_\#\colon \pi_q(X^{2n}) \to \pi_q(\Omega S^{n+1})$$

is an isomorphism for $q < 3n - 1$ and an epimorphism for $q = 3n - 1$. ∎

Let $\bar{\alpha}_n\colon (E^{2n},S^{2n-1}) \to (X^{2n},S^n)$ be the characteristic map for the $2n$-cell of X^{2n} corresponding to the attaching map $\alpha_n\colon S^{2n-1} \to S^n$. Then $\bar{\alpha}_n$ is a relative homeomorphism between $(2n-1)$-connected pairs such that S^{2n-1} and S^n are both $(n-1)$-connected. It follows from corollary 7 that $\bar{\alpha}_{n\#}\colon \pi_q(E^{2n},S^{2n-1}) \to \pi_q(X^{2n},S^n)$ is an isomorphism for $q \le 3n-2$.

The *Hopf invariant*[1] is the homomorphism

$$H\colon \pi_{q+1}(S^{n+1}) \to \pi_{q-1}(S^{2n-1}) \qquad q \le 3n-2$$

defined so that the following diagram is commutative [where $j\colon X^{2n} \subset (X^{2n},S^n)$]:

$$\begin{tikzcd}
\pi_{q+1}(S^{n+1}) \arrow[r, "\bar{\partial}", "\approx"'] \arrow[d, "H"'] & \pi_q(\Omega S^{n+1}) \arrow[r, "g_\#^{-1}", "\approx"'] & \pi_q(X^{2n}) \arrow[d, "j_\#"] \\
\pi_{q-1}(S^{2n-1}) & \pi_q(E^{2n},S^{2n-1}) \arrow[l, "\partial"', "\approx"] & \pi_q(X^{2n},S^n) \arrow[l, "\bar{\alpha}_{n\#}^{-1}"', "\approx"]
\end{tikzcd}$$

The Hopf invariant plays an important role in the study of the suspension homomorphism by virtue of the following exactness property.[2]

9 THEOREM *For $n \ge 2$ there is an exact sequence*

$$\pi_{3n-2}(S^n) \xrightarrow{S} \cdots \to \pi_q(S^n) \xrightarrow{S} \pi_{q+1}(S^{n+1}) \xrightarrow{H} \pi_{q-1}(S^{2n-1}) \xrightarrow{\alpha_{n\#}} \pi_{q-1}(S^n) \to \cdots$$

PROOF The result follows from the exactness of the homotopy sequence of (X^{2n},S^n) and the commutativity of the following diagram:

$$\begin{tikzcd}
 & \pi_q(S^n) \arrow[dl, "S"'] \arrow[d, "\mu_\#'"] \arrow[dr, "i_\#"] & \\
\pi_{q+1}(S^{n+1}) \arrow[r, "\bar{\partial}", "\approx"'] \arrow[d, "H"'] & \pi_q(\Omega S^{n+1}) \arrow[r, "g_\#^{-1}", "\approx"'] & \pi_q(X^{2n}) \arrow[d, "j_\#"] \\
\pi_{q-1}(S^{2n-1}) \arrow[dr, "\alpha_{n\#}"'] & \pi_q(E^{2n},S^{2n-1}) \arrow[l, "\partial"', "\approx"] & \pi_q(X^{2n},S^n) \arrow[l, "\bar{\alpha}_{n\#}^{-1}"', "\approx"] \arrow[dl, "\partial"] \\
 & \pi_{q-1}(S^n) &
\end{tikzcd}$$
∎

If G is an infinite cyclic group, we define a function $|\cdot|$ from G to the set of nonnegative integers by the condition $|g| = m$ if and only if there is a generator $g' \in G$ such that $g = mg'$. Since $\pi_{2n-1}(S^{2n-1}) \approx \mathbf{Z}$, we can define $|H[\alpha]|$ for $[\alpha] \in \pi_{2n+1}(S^{n+1})$. The following is an interpretation of $|H[\alpha]|$

10 THEOREM *Let $\alpha\colon S^{2n+1} \to S^{n+1}$ be a base-point-preserving map and let $E_\alpha \to S^{2n+1}$ be the principal fibration induced by α. Then $|H[\alpha]| = m$ if and only if the integral homology group $H_{2n}(E_\alpha)$ is isomorphic to $\mathbf{Z}_m$ (where $\mathbf{Z}_0 = \mathbf{Z}$).*

[1] See H. Hopf, Über die Abbildungen von Sphären auf Sphären niedrigerer Dimension, *Fundamenta Mathematica*, vol. 25, pp. 427–440, 1935, and G. W. Whitehead, A generalization of the Hopf invariant, *Annals of Mathematics*, vol. 51, pp. 192–237, 1950.

[2] See G. W. Whitehead, On the Freundenthal theorems, *Annals of Mathematics*, vol. 57, pp. 209–228, 1953.

PROOF From the definition of H and the naturality of the Hurewicz homomorphism φ, it is easily seen that $|H[\alpha]| = |\varphi\bar{\partial}[\alpha]|$, where

$$\varphi\bar{\partial}[\alpha] \in H_{2n}(\Omega S^{n+1}) \approx \mathbf{Z}$$

Since α induces a map $\bar{\alpha}$: $(E_\alpha,\Omega S^{n+1}) \to (PS^{n+1},\Omega S^{n+1})$, there is a commutative diagram

$$\begin{tikzcd}
\pi_{2n+1}(S^{2n+1}) \arrow[d, "\alpha_\#"'] \arrow[dr, "\bar{\partial}"] & & \\
& \pi_{2n}(\Omega S^{n+1}) \arrow[r, "\varphi"] & H_{2n}(\Omega S^{n+1}) \\
\pi_{2n+1}(S^{n+1}) \arrow[ur, "\bar{\partial}"'] & &
\end{tikzcd}$$

Therefore $|\varphi\bar{\partial}[\alpha]| = |\varphi\bar{\partial}\alpha_\#[1_{S^{2n+1}}]| = |\varphi\bar{\partial}[1_{S^{2n+1}}]|$. There is also a commutative diagram

$$\begin{tikzcd}
\pi_{2n+1}(S^{2n+1}) \arrow[d, "\varphi"', "\approx"] & \pi_{2n+1}(E_\alpha,\Omega S^{n+1}) \arrow[l, "\approx"'] \arrow[r, "\partial"] \arrow[d, "\varphi"'] & \pi_{2n}(\Omega S^{n+1}) \arrow[d, "\varphi"] \\
H_{2n+1}(S^{2n+1}) & H_{2n+1}(E_\alpha,\Omega S^{n+1}) \arrow[l, "\approx"'] \arrow[r, "\partial"] & H_{2n}(\Omega S^{n+1})
\end{tikzcd}$$

from which it follows that $|\varphi\bar{\partial}[1_{S^{2n+1}}]| = |\partial(z)|$, where z is a generator of $H_{2n+1}(E_\alpha,\Omega S^{n+1})$. By lemma 4, $H_{2n}(E_\alpha,\Omega S^{n+1}) \approx H_{2n}(S^{2n+1}) = 0$, and so

$$H_{2n}(E_\alpha) \approx H_{2n}(\Omega S^{n+1})/\partial H_{2n+1}(E_\alpha,\Omega S^{n+1})$$

and this gives the result. ■

4 MULTIPLICATIVE PROPERTIES OF SPECTRAL SEQUENCES

This section is devoted to pairings from two spectral sequences to a third. This will be applied, by means of the cross product, to pair the homology spectral sequences of two fibrations to the spectral sequence of their product. We shall also consider cohomology spectral sequences. There is a cohomology spectral sequence for a fibration and a cross-product pairing of the cohomology spectral sequences of two fibrations to the cohomology spectral sequence of their product. The diagonal map then endows the cohomology spectral sequence with a multiplicative structure, which will be applied in the next section.

Let p: $(E,E_A) \to (B,A)$ and p': $(E',E'_{A'}) \to (B',A')$ be fibrations over relative CW complexes and let p'': $E \times E' \to B \times B'$ be the product fibration (that is, $p'' = p \times p'$). There is a filtration of the pair $(E \times E',\ E_A \times E' \cup E \times E'_{A'})$ defined by $(E \times E')_k = E_A \times E' \cup E \times E'_{A'} \cup \bigcup_{i+j=k} E_i \times E'_j$, where $\{E_i\}$ and $\{E'_j\}$ are the filtrations of (E,E_A) and $(E',E'_{A'})$ corresponding to the skeleta of (B,A) and (B',A'), respectively. Then $E \times E' = \bigcup_k (E \times E')_k$, and every compact subset of $E \times E'$ is contained in $(E \times E')_k$ for some k. By the method of example 9.1.5, there is a convergent E^1 spectral sequence with

$E^1_{s,t} \approx H_{s+t}((E \times E')_s, (E \times E')_{s-1}; G)$ and E^∞ the bigraded module associated to the filtration of $H_* = H_*((E,E_A) \times (E',E'_{A'}); G)$ defined by

$$F_sH_* = \operatorname{im}\,[H_*((E \times E')_s, E \times E'_{A'} \cup E_A \times E'; G) \to H_*]$$

We relate this spectral sequence to the cross product of the spectral sequences of p and of p'. If E, E', and E'' are E^k spectral sequences, a *pairing* from E and E' to E'' is a sequence of homomorphisms

$$h^r\colon E^r_{s,t} \otimes E'^r_{s',t'} \to E''^r_{s+s',t+t'}$$

for all $r \geq k$ such that for $x \in E^r_{s,t}$

$$d^rh^r(x \otimes y) = h^r(d^rx \otimes y) + (-1)^{s+t}h^r(x \otimes d'^ry)$$

and such that h^{r+1} is the composite

$$E^{r+1} \otimes E'^{r+1} \approx H(E^r) \otimes H(E'^r) \to H(E^r \otimes E'^r) \xrightarrow{h^r_*} H(E''^r) \approx E''^{r+1}$$

For the sequence of submodules used to define E^∞

$$B^k \subset B^{k+1} \subset \cdots \subset B^r \subset \cdots \subset Z^r \subset \cdots \subset Z^{k+1} \subset Z^k$$

it is clear that h^k pairs Z^k and Z'^k to Z''^k in such a way that $Z^r \otimes Z'^r$ is mapped to Z''^r and $B^r \otimes Z'^r + Z^r \otimes B'^r$ is mapped to B''^r for all $r \geq k$. It follows that h^k maps $Z^\infty \otimes Z'^\infty = (\bigcap Z^r) \otimes (\bigcap Z'^r)$ to $\bigcap Z''^r = Z''^\infty$ and maps $B^\infty \otimes Z'^\infty + Z^\infty \otimes B'^\infty = \bigcup_r (B^r \otimes \bigcap_j Z'^j) + \bigcup_r (\bigcap_j Z^j \otimes B'^r)$ to $\bigcup B''^r = B''^\infty$. There is induced a pairing

$$h^\infty\colon E^\infty \otimes E'^\infty \to E''^\infty$$

which is compatible with the pairings $\{h^r\}$.

1 THEOREM *Let $p\colon E \to B$ and $p'\colon E' \to B'$ be orientable fibrations over path-connected relative CW complexes (B,A) and (B',A') with fibers $F = p^{-1}(b_0)$ and $F' = p'^{-1}(b_0')$, respectively. There is a pairing $\{h^r\}$ from the E^1 spectral sequences of p and p' to the E^1 spectral sequence of $p \times p'$ such that h^∞ is induced by the cross-product pairing*

$$H_*(E,E_A; G) \otimes H_*(E',E'_{A'}; G') \to H_*((E,E_A) \times (E',E'_{A'}); G \otimes G')$$

PROOF An Eilenberg-Zilber chain map

$$\Delta(E,E_A) \otimes \Delta(E',E'_{A'}) \to \Delta((E,E_A) \times (E',E'_{A'}))$$

induces a map from $F_s\Delta(E,E_A) \otimes F_{s'}\Delta(E',E'_{A'})$ to $F_{s+s'}\Delta((E,E_A) \times (E',E'_{A'}))$ for all s and s'. Therefore it induces in a natural way a pairing of the corresponding spectral sequences. Since an Eilenberg-Zilber chain map induces the homology cross product, the result follows. ∎

To interpret this result on the E^2 level, let $C_*(B,A)$ and $C_*(B',A')$ be the chain complexes of the relative CW complexes (B,A) and (B',A'), respectively, defined as in corollary 9.2.4. If $\sigma \in \bar\Delta_s((B,A)^s, (B,A)^{s-1})$, then

$$\{\sigma\} \in H_s(\bar\Delta((B,A)^s, (B,A)^{s-1})) = C_s(B,A)$$

and these elements $\{\sigma\} \in C_s(B,A)$ generate $C_s(B,A)$. We define a homomorphism

$$\psi'': C_s(B,A) \otimes C_{s'}(B',A') \otimes H_n(F \times F'; G'') \to H_{s+s'+n}((E \times E')_{s+s'}, (E \times E')_{s+s'-1}; G'')$$

by $$\psi''(\{\sigma\} \otimes \{\sigma'\} \otimes w) = (\bar\sigma \times \bar\sigma')_* T_*(\{\xi_s\} \times \{\xi_{s'}\} \times w)$$

where $\bar\sigma: (\Delta^s,\dot\Delta^s) \times F \to (E_s,E_{s-1})$ and $\bar\sigma': (\Delta^{s'},\dot\Delta^{s'}) \times F' \to (E'_{s'},E'_{s'-1})$ are admissible liftings of σ and σ', respectively, and

$$T: (\Delta^s,\dot\Delta^s) \times (\Delta^{s'},\dot\Delta^{s'}) \times F \times F' \to (\Delta^s,\dot\Delta^s) \times F \times (\Delta^{s'},\dot\Delta^{s'}) \times F'$$

is the map which interchanges the second and third coordinates. The fact that ψ'' is well-defined follows by an argument similar to that of lemma 9.2.14.

2 LEMMA *The map ψ'' induces an isomorphism*

$$\psi'': [C_*(B,A) \otimes C_*(B',A')]_s \otimes H_n(F \times F'; G'') \approx E''^1_{s,n}$$

such that $\psi'' \circ (\partial \otimes 1) = d^1 \circ \psi''$ and such that there is a commutative square

$$\begin{array}{ccc} C_s(B,A) \otimes H_t(F;G) \otimes C_{s'}(B,A) \otimes H_{t'}(F';G') & \xrightarrow{\psi \otimes \psi'} & E^1_{s,t} \otimes E'^1_{s',t'} \\ \varphi\downarrow & & \downarrow h^1 \\ C_s(B,A) \otimes C_{s'}(B',A') \otimes H_{t+t'}(F \times F'; G'') & \xrightarrow{\psi''} & E''^1_{s+s',t+t'} \end{array}$$

where $\varphi(c \otimes w \otimes c' \otimes w) = (-1)^{ts'} c \otimes c' \otimes (w \times w')$, with G and G' paired to G''.

PROOF The first part follows by an argument similar to that of theorem 9.2.15. For the second part we have

$$\begin{aligned} \psi''\varphi(\{\sigma\} \otimes w \otimes \{\sigma'\} \otimes w') &= (-1)^{ts'}\psi''(\{\sigma\} \otimes \{\sigma'\} \otimes (w \times w')) \\ &= (-1)^{ts'}(\bar\sigma \times \bar\sigma')_* T_*(\{\xi_s\} \times \{\xi_{s'}\} \times (w \times w')) \\ &= (\bar\sigma \times \bar\sigma')_*((\{\xi_s\} \times w) \times (\{\xi_{s'}\} \times w')) \\ &= h^1(\psi(\{\sigma\} \otimes w) \otimes \psi'(\{\sigma'\} \otimes w')) \quad \blacksquare \end{aligned}$$

It follows from theorem 1 and lemma 2 that

$$E''^2_{s,t} \approx H_s((B,A) \times (B',A'); H_t(F \times F'; G''))$$

and that the pairing h^2 from $E^2_{s,t}$, $E'^2_{s',t'}$ to $E''^2_{s+s',t+t'}$ corresponds to $(-1)^{ts'}$ times the pairing given by cross product

$$H_s(B,A; H_t(F;G)) \otimes H_{s'}(B',A'; H_{t'}(F';G')) \to H_{s+s'}((B,A) \times (B',A'); H_{t+t'}(F \times F'; G''))$$

where the coefficients are themselves paired by the cross product. That is, the left-hand side is isomorphic to

$$H_s((B,A)^s,(B,A)^{s-1}) \otimes H_t(F;G) \otimes H_{s'}((B',A')^{s'}, (B',A')^{s'-1}) \otimes H_{t'}(F';G')$$

the right-hand side is isomorphic to

$$H_{s+s'}(((B,A) \times (B',A'))^s, ((B,A) \times (B',A'))^{s-1}) \otimes H_{t+t'}(F \times F'; G'')$$

and the map sends $x \otimes y \otimes x' \otimes y'$ to $(-1)^{ts'}(x \times x') \otimes (y \times y')$.

3 THEOREM *Let p: $E \to B$ and p': $E' \to B'$ be orientable fibrations with path-connected base spaces and with fibers F and F', respectively. Let $A \subset B$ and $A' \subset B'$ and assume that $\{B \times A', A \times B'\}$ is an excisive couple in $B \times B'$ and $\{E_A \times E', E \times E'_{A'}\}$ is an excisive couple in $E \times E'$. Given a pairing $G \otimes G' \to G''$, there is a pairing of the E^2 spectral sequences of p and p' to the E^2 spectral sequence of $p \times p'$, which on E^2 corresponds to $(-1)^{ts'}$ times the cross-product pairing*

$$H_s(B,A; H_t(F;G)) \otimes H_{s'}(B',A'; H_{t'}(F';G')) \\ \to H_{s+s'}((B,A) \times (B',A'); H_{t+t'}(F \times F'; G''))$$

and on E^∞ is compatible with the cross-product pairing

$$H_n(E,E_A; G) \otimes H_{n'}(E',E'_{A'}; G') \to H_{n+n'}((E,E_A) \times (E',E'_{A'}); G'')$$

PROOF Let $f:(X,A) \to (B,A)$ and f': $(X',A') \to (B',A')$ be relative CW approximations to (B,A) and (B',A'), respectively. Let E_X and $E'_{X'}$ be the induced fibrations over X and X', respectively, with corresponding maps $\bar f$: $(E_X,E_A) \to (E,E_A)$ and $\bar f'$: $(E'_{X'},E'_{A'}) \to (E',E'_{A'})$. The excisiveness hypotheses ensure that the Künneth formula can be applied to deduce isomorphisms

$$(f \times f')_*: H_*((X,A) \times (X',A')) \approx H_*((B,A) \times (B',A'))$$
$$(\bar f \times \bar f')_*: H_*((E_X,E_A) \times (E'_{X'},E'_{A'})) \approx H_*((E,E_A) \times (E',E'_{A'}))$$

The result now follows from application of theorem 1 and lemma 2 to the fibrations $E_X \to X$ and $E'_{X'} \to X'$ and from the remarks made above about the pairing induced on the E^2 terms (the resulting E^2 spectral sequence being independent of the choices of X and X'). ∎

The pairing of theorem 3 has properties analogous to those of the cross-product pairing. In particular, it is functorial on fiber-preserving maps and commutes up to sign with the homomorphism induced by interchanging the factors of $p \times p'$.

We next consider cohomology spectral sequences. Let $C^* = \{C^q,\delta\}$ be a cochain complex. A (*decreasing*) *filtration* F on C^* is a sequence of subcomplexes F^sC^* such that $F^sC^* \supset F^{s+1}C^*$ for all s. The filtration is *convergent* if $\cup F^sC^* = C^*$ and $\cap F^sC^* = 0$. It is *bounded above* if for each n there is $s(n)$ such that $F^{s(n)}C^n = 0$. Given a convergent filtration bounded above on a cochain complex C^*, there is an analogue of theorem 9.1.2 which asserts the existence of a convergent E_1 spectral sequence $\{E_r,d_r\}$, where E_r is bigraded by $E_r^{s,t}$ and d_r is a differential on E_r of bidegree $(r, 1 - r)$. Furthermore, we have $E_1^{s,t} \approx H^{s+t}(F^sC^*/F^{s+1}C^*)$ and d_1 corresponds to the coboundary operator of the triple $(F^sC^*,F^{s+1}C^*,F^{s+2}C^*)$. The limit term E_∞ is the bigraded module associated to the filtration on $H^*(C^*)$ defined by

$$F^sH^*(C^*) = \ker [H^*(C^*) \to H^*(F^{s-1}C^*)]$$

(that is, $E_\infty^{s,t} \approx \ker [H^{s+t}(C^*) \to H^{s+t}(F^{s-1}C^*)]/\ker [H^{s+t}(C^*) \to H^{s+t}(F^sC^*)]$).

4 EXAMPLE Let $\{X_s\}$ be an increasing filtration of a pair (X,A) and let $\bar{\Delta}(X,A)$ be the subcomplex of $\Delta(X,A)$ generated by singular simplexes $\sigma\colon \Delta^q \to X$ such that $\sigma((\Delta^q)^k) \subset X_k$ for all k. Let $\bar{C}^* = \operatorname{Hom}(\bar{\Delta}(X,A), G)$. A decreasing filtration on $\bar{C}^*$ is defined by

$$F^s\bar{C}^* = \{c \in \bar{C}^* \mid c \mid \bar{\Delta}(X_{s-1},A) = 0\}$$

where $\bar{\Delta}(X_{s-1},A) = \bar{\Delta}(X,A) \cap \Delta(X_{s-1},A)$. Since $\bar{\Delta}_s(X,A) = \bar{\Delta}_s(X_s,A)$, it follows that $F^{s+1}\bar{C}^s = 0$, and so the filtration is bounded above. In case the original filtration on (X,A) is bounded below (that is, $X_s = A$ for some s), then $\cup F^s\bar{C}^* = \{c \in \bar{C}^* \mid c \mid \bar{\Delta}(A,A) = 0\} = \bar{C}^*$. Hence, in the latter case there is an associated convergent E_1 spectral sequence. In case the inclusion maps $\bar{\Delta}(X,A) \subset \Delta(X,A)$ and $\bar{\Delta}(X_s,A) \subset \Delta(X_s,A)$ are chain equivalences, this spectral sequence has the property that $E_1^{s,t} \approx H^{s+t}(X_s, X_{s-1};\, G)$ and E_∞ is the bigraded module associated to the filtration on $H^*(X,A;\, G)$ defined by

$$F^sH^*(X,A;\, G) = \ker [H^*(X,A;\, G) \to H^*(X_{s-1},A;\, G)]$$

In particular, if (X,A) is a relative CW complex, $X_s = (X,A)^s$ if $s \geq 0$, and $X_s = A$ if $s < 0$, it follows from theorem 9.2.3 that the hypotheses are satisfied and that

$$E_1^{s,t} \approx H^{s+t}((X,A)^s, (X,A)^{s-1};\, G) = 0 \qquad t \neq 0$$

Therefore the spectral sequence collapses and $H^s(X,A;\, G)$ is isomorphic to $E_2^{s,0} \approx H^*(C^*)$, where $C^* = \{C^q,\delta\}$ is the cochain complex

$$C^q = H^q((X,A)^q, (X,A)^{q-1};\, G)$$

and δ is the coboundary operator of the triple $((X,A)^q, (X,A)^{q-1}, (X,A)^{q-2})$. By the universal-coefficient theorem for cohomology, $C^* = \operatorname{Hom}(C_*(X,A), G)$. Hence we have proved that $H^*(X,A;\, G) \approx H^*(C_*(X,A);\, G)$.

5 THEOREM *Let $p\colon E \to B$ be a fibration over a relative CW complex (B,A). There is a convergent E_1 cohomology spectral sequence, with $E_1^{s,t} \approx H^{s+t}(E_s,E_{s-1};\, G)$ and E_∞ the bigraded module associated to the filtration of $H^*(E,E_A;\, G)$ defined by*

$$F^sH^*(E,E_A;\, G) = \ker [H^*(E,E_A;\, G) \to H^*(E_{s-1},E_A;\, G)]$$

PROOF Since $(B, (B,A)^s)$ is s-connected for all s, it follows easily from theorem 7.2.8 that (E,E_s) is s-connected for all s. By theorem 9.2.3, the chain complex $\bar{\Delta}(E,E_A)$ is chain equivalent to $\Delta(E,E_A)$ and $\bar{\Delta}(E_s,E_A)$ is chain equivalent to $\Delta(E_s,E_A)$. The result now follows by the method of example 4. ∎

To compute $E_1^{s,t}$ we assume that B is path connected and that $p\colon E \to B$ is an orientable fibration. Let $F = p^{-1}(b_0)$ and let $\sigma\colon (\Delta^s,\dot{\Delta}^s) \to ((B,A)^s, (B,A)^{s-1})$

be a singular simplex in $\bar{\Delta}(B,A)$. If $\bar{\sigma}$: $(\Delta^s,\dot{\Delta}^s) \times F \to (E_s,E_{s-1})$ is an admissible lifting of σ, the homomorphism

$$\bar{\sigma}^*: H^n(E_s,E_{s-1};\, G) \to H^n((\Delta^s,\dot{\Delta}^s) \times F;\, G)$$

depends only on σ and not the particular choice of the lifting $\bar{\sigma}$ (because the fibration is orientable). Let $\{\xi_s\}^* \in H^s(\Delta^s,\dot{\Delta}^s)$ be the generator characterized by the condition $\langle \{\xi_s\}^*,\{\xi_s\}\rangle = 1$. It follows from theorem 5.6.1 that the map $v \to \{\xi_s\}^* \times v$ is an isomorphism

$$H^q(F;G) \approx H^{s+q}((\Delta^s,\dot{\Delta}^s) \times F;\, G)$$

As in theorem 9.2.13 and lemma 9.2.14, it can be shown that there is a well-defined homomorphism

$$\psi^*: H^n(E_s,E_{s-1};\, G) \to H^s((B,A)^s,\, (B,A)^{s-1};\, H^{n-s}(F;G))$$

characterized by the equation

$$\{\xi_s\}^* \times \langle \psi^*(u),\{\sigma\}\rangle = \bar{\sigma}^*(u)$$

where σ: $(\Delta^s,\dot{\Delta}^s) \to ((B,A)^s,\, (B,A)^{s-1})$ and $\bar{\sigma}$: $(\Delta^s,\dot{\Delta}^s) \times F \to (E_s,E_{s-1})$ is an admissible lifting of σ, and $\langle \psi^*(u),\{\sigma\}\rangle \in H^{n-s}(F;G)$. Analogous to theorem 9.2.15 is the result that ψ^* is an isomorphism (this uses the second part of lemma 9.2.2 instead of the first part) and that it commutes with the differentials d_1 and the coboundary operator of the triple $((B,A)^s,\, (B,A)^{s-1},\, (B,A)^{s-2})$. Using the technique of relative CW approximation, we have the following analogue of theorem 9.2.17.

6 THEOREM *Let p: $E \to B$ be an orientable fibration over a path-connected base and let $F = p^{-1}(b_0)$. Given $A \subset B$, there is a convergent E_2 cohomology spectral sequence, with $E_2^{s,t} \approx H^s(B,A;\, H^t(F;G))$ and E_∞ the bigraded module associated to some filtration of $H^*(E,E_A;\, G)$. This spectral sequence is a first-quadrant spectral sequence functorial on the category of orientable fibrations and fiber-preserving maps.* ■

For the multiplicative properties of cohomology spectral sequences we shall use the following result about pairings of cohomology spectral sequences.

7 THEOREM *Let p: $E \to B$ and p': $E' \to B'$ be orientable fibrations over path-connected relative CW complexes (B,A) and (B',A'), with fibers F and F', respectively. There is a pairing $\{h_r\}$ from the E_1 cohomology spectral sequences of p and p' to the E_1 cohomology spectral sequence of $p \times p'$ such that h_2 is induced by $(-1)^{ts'}$ times the cross-product pairing*

$$H^s(B,A;\, H^t(F;G)) \otimes H^{s'}(B',A';\, H^{t'}(F';G')) \to H^{s+s'}((B,A) \times (B',A');\, H^{t+t'}(F \times F';\, G''))$$

and h_∞ is induced by the cross-product pairing

$$H^*(E,E_A;\, G) \otimes H^*(E',E'_{A'};\, G') \to H^*((E,E_A) \times (E',E'_{A'});\, G'')$$

where G and G' are paired to G''.

PROOF There are chain equivalences

$$\bar{\Delta}(E,E_A) \otimes \bar{\Delta}(E',E'_{A'}) \subset \Delta(E,E_A) \otimes \Delta(E',E'_{A'}) \to \Delta((E,E_A) \times (E',E'_{A'}))$$

and therefore an isomorphism

$$H^*((E,E_A) \times (E',E'_{A'});\, G'') \approx H^*(\bar{\Delta}(E,E_A) \otimes \bar{\Delta}(E',E'_{A'});\, G'')$$

We define a filtration on $C^* = \operatorname{Hom}(\bar{\Delta}(E,E_A) \otimes \bar{\Delta}(E',E'_{A'}), G'')$ by

$$F^sC^* = \{c \in C^* \mid c \mid \bar{\Delta}(E_i,E_A) \otimes \bar{\Delta}(E'_j, E'_{A'}) = 0,\ i + j = s\}$$

Then the cross product

$$\operatorname{Hom}(\bar{\Delta}(E,E_A), G) \otimes \operatorname{Hom}(\bar{\Delta}(E',E'_{A'}), G') \to C^*$$

maps $F^s \otimes F^{s'}$ to $F^{s+s'}C^*$. It follows easily that there is an induced pairing of the corresponding cohomology spectral sequences and that h_∞ has the stated property.

To prove the statement about the pairing h_2, let C_* and C'_* be the chain complexes of (B,A) and (B',A'), respectively, and let $C''_* = C_* \otimes C'_*$. We define a homomorphism

$$\psi''^*\colon E_1''^{s,t} \to \operatorname{Hom}(C''_s, H^t(F \times F';\, G''))$$

by the condition

$$\{\xi_i\}^* \times \{\xi_j\}^* \times \langle \psi''^*(u), \{\sigma\} \times \{\sigma'\}\rangle = (\bar{\sigma} \times \bar{\sigma}')^*(u)$$

where $\sigma\colon (\Delta^i,\dot{\Delta}^i) \to ((B,A)^i, (B,A)^{i-1})$, $\sigma'\colon (\Delta^j,\dot{\Delta}^j) \to ((B',A')^j, (B',A')^{j-1})$, with $i + j = s$, $\bar{\sigma}\colon (\Delta^i,\dot{\Delta}^i) \times F \to (E_i,E_{i-1})$, and $\bar{\sigma}'\colon (\Delta^j,\dot{\Delta}^j) \times F' \to (E'_j,E'_{j-1})$ are admissible liftings, and where $u \in H^{s+t}((E \times E')_s, (E \times E')_{s-1};\, G \otimes G')$. Then ψ'' is an isomorphism taking d_1 into the coboundary operator of the cochain complex $\operatorname{Hom}(C''_*, H^*(F \times F';\, G''))$.

Furthermore, if $v \in E_1^{s,t}$ and $v' \in E_1'^{s',t'}$, then $v \times v' \in E_1^{s+s',t+t'}$, and from the definitions we have

$$\begin{aligned}
&\{\xi_s\}^* \times \{\xi_{s'}\}^* \times \langle \psi''^*(v \times v'), \{\sigma\} \times \{\sigma'\}\rangle \\
&\quad = (\bar{\sigma} \times \bar{\sigma}')^*(v \times v') = \bar{\sigma}^*(v) \times \bar{\sigma}'^*(v') \\
&\quad = (\{\xi_s\}^* \times \langle \psi^*(v),\{\sigma\}\rangle) \times (\{\xi_{s'}\}^* \times \langle \psi'^*(v'),\{\sigma'\}\rangle \\
&\quad = (-1)^{ts'}\{\xi_s\}^* \times \{\xi_{s'}\}^* \times \langle \psi^*(v) \times \psi'^*(v'), \{\sigma\} \times \{\sigma'\}\rangle
\end{aligned}$$

Therefore $\psi''^*(v \times v') = (-1)^{ts'}\psi^*(v) \times \psi'^*(v')$, and this implies the result about the pairing h_2. ∎

This gives the following important multiplicative property for the cohomology spectral sequence of a fibration.

8 THEOREM *Let $p\colon E \to B$ be an orientable fibration over a path-connected base, with fiber F. Let $\{A_1,A_2\}$ be an excisive couple of path-connected subspaces of B such that $\{E_{A_1},E_{A_2}\}$ is an excisive couple in E. Then there is a functorial pairing of the E_2 cohomology spectral sequences of (E,E_{A_1}) and (E,E_{A_2}) to the E_2 cohomology spectral sequence of $(E, E_{A_1} \cup E_{A_2})$, which on E_2 is isomorphic to $(-1)^{ts'}$ times the cup-product pairing (G and G' paired to G'')*

$$H^s(B,A_1;\ H^t(F;G)) \otimes H^{s'}(B,A_2;\ H^{t'}(F;G')) \to H^{s+s'}(B,\ A_1 \cup A_2;\ H^{t+t'}(F;G''))$$

and on E_∞ *is induced by the cup-product pairing*

$$H^*(E,E_{A_1};\ G) \otimes H^*(E,E_{A_2};\ G') \to H^*(E,E_{A_1} \cup E_{A_2};\ G'')$$

PROOF We begin by showing that there exists a CW complex X, with subcomplexes X_1 and X_2, and a weak homotopy equivalence $f\colon X \to B$ such that $f \mid X_1\colon X_1 \to A_1$ and $f \mid X_2\colon X_2 \to A_2$ are also weak homotopy equivalences. In fact, let $g\colon Y \to B$, $g_1\colon Y_1 \to A_1$, and $g_2\colon Y_2 \to A_2$ be CW approximations. Then there exist maps $g_1'\colon Y_1 \to Y$ and $g_2'\colon Y_2 \to Y$ (which can be taken to be cellular) such that $g \circ g_1'\colon Y_1 \to B$ and $g \circ g_2'\colon Y_2 \to B$ are homotopic, respectively, to the composites $Y_1 \xrightarrow{g_1} A_1 \overset{i}{\subset} B$ and $Y_2 \xrightarrow{g_2} A_2 \overset{i'}{\subset} B$. Let X be the CW complex obtained from the disjoint union $Y_1 \times I \cup Y \cup Y_2 \times I$ by identifying $(y_1,0)$ with $g_1'(y_1) \in Y$ for all $y_1 \in Y_1$ and $(y_2,0)$ with $g_2'(y_2) \in Y$ for all $y_2 \in Y_2$. Let $k\colon Y_1 \times I \cup Y \cup Y_2 \times I \to X$ be the collapsing map and define a map $f\colon X \to B$ such that $(f \circ k) \mid Y = g$, $(f \circ k) \mid Y_1 \times I$ is a homotopy from $g \circ g_1'$ to $i \circ g_1$, and $(f \circ k) \mid Y_2 \times I$ is a homotopy from $g \circ g_2'$ to $i' \circ g_2$. Let $X_1 = k(Y_1 \times 1)$ and $X_2 = k(Y_2 \times 1)$ and observe that X_1 and X_2 are subcomplexes of X such that $f \mid X_1\colon X_1 \to A_1$ and $f \mid X_2\colon X_2 \to A_2$ are weak homotopy equivalences. Furthermore, $k(Y)$ is a strong deformation retract of X, and since $f \mid k(Y)\colon k(Y) \to B$ is a weak homotopy equivalence, so is $f\colon X \to B$. Therefore the map $f\colon X \to B$ has the desired properties.

The excisiveness assumption about $\{A_1,A_2\}$ implies that f induces an isomorphism

$$f^*\colon H^*(B,\ A_1 \cup A_2) \approx H^*(X,\ X_1 \cup X_2)$$

Let $p'\colon E_X \to X$ be the induced fibration over X and let $\bar{f}\colon E_X \to E$ be the corresponding map. Then $\bar{f}$ induces isomorphisms

$$H^*(E,E_{A_1}) \approx H^*(E_X,E_{X_1}) \qquad \text{and} \qquad H^*(E,E_{A_2}) \approx H^*(E_X,E_{X_2})$$

The excisiveness assumption about $\{E_{A_1},E_{A_2}\}$ ensures that $\bar{f}$ also induces an isomorphism

$$H^*(E,\ E_{A_1} \cup E_{A_2}) \approx H^*(E_X,\ E_{X_1} \cup E_{X_2})$$

By theorem 7, there is a pairing of the E_2 cohomology spectral sequences of (E_X,E_{X_1}) and (E_X,E_{X_2}) to the E_2 cohomology spectral sequence of $(E_X,E_{X_1}) \times (E_X,E_{X_2})$, which corresponds to cross product on the E_2 and E_∞ terms. There is a commutative square (whose horizontal maps are diagonal maps)

$$\begin{array}{ccc} E_X & \to & E_X \times E_X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p' \times p'} \\ X & \longrightarrow & X \times X \end{array}$$

Let $d\colon X \to X \times X$ be a cellular approximation to the diagonal map having the property that $d(X_1) \subset X_1 \times X_1$ and $d(X_2) \subset X_2 \times X_2$ (such maps exist). It follows that there is a lifting $\bar{d}\colon E_X \to E_X \times E_X$ of $d \circ p'\colon E_X \to X \times X$

which is homotopic to the diagonal map $E_X \to E_X \times E_X$. Then $\bar{d}$ maps the filtration of $(E_X, E_{X_1} \cup E_{X_2})$ into the filtration of $(E_X,E_{X_1}) \times (E_X,E_{X_2})$ and so induces a homomorphism from the E_2 cohomology spectral sequence of $(E_X,E_{X_1}) \times (E_X,E_{X_2})$ into the E_2 cohomology spectral sequence of $(E_X, E_{X_1} \cup E_{X_2})$. Since $\bar{d}$ takes cross products in $E_X \times E_X$ into cup products in E_X, the composite of this homomorphism with the pairing above is a pairing from the spectral sequences of (E_X,E_{X_1}) and (E_X,E_{X_2}) to the spectral sequence of $(E_X, E_{X_1} \cup E_{X_2})$, which is induced by $\pm$ cup product on the E_2 and E_∞ terms.

By means of the isomorphisms induced by f and $\bar{f}$, this gives a pairing from the E_2 cohomology spectral sequences of (E,E_{A_1}) and (E,E_{A_2}) to the E_2 cohomology spectral sequence of $(E, E_{A_1} \cup E_{A_2})$, which is induced by $\pm$ cup product on the E_2 and E_∞ terms. The resultant pairing is independent of the choice of X. ∎

9 COROLLARY *Let $p: E \to B$ be an orientable fibration with path-connected base B, with fiber F. For any $A \subset B$ there is a convergent E_2 cohomology spectral sequence of bigraded algebras with $E_2^{s,t} \approx H^s(B,A;\, H^t(F;R))$ and E_∞ the bigraded algebra associated to some filtration of $H^*(E,E_A;\, R)$. This spectral sequence is functorial on the category of such fibrations and fiber-preserving maps.* ∎

5 APPLICATIONS OF THE COHOMOLOGY SPECTRAL SEQUENCE

Because the cohomology spectral sequence of a fibration has a multiplicative structure, it is a more powerful tool than the homology spectral sequence. We shall use it in deriving the generalized Wang and Gysin cohomology sequences and then apply the cohomology spectral sequence to obtain another description of the Hopf invariant in a particular dimension. The section closes with some results about the homology and cohomology of spaces of type $(\pi,1)$.

Let $p: E \to B$ be an orientable fibration over a path-connected base and with fiber F. First we shall determine $i^*: H^*(E;G) \to H^*(F;G)$, where $i: F \subset E$, in terms of the cohomology spectral sequence of E. Because this is a first-quadrant spectral sequence, there is a monomorphism $E_\infty^{0,t} \to E_2^{0,t}$. Since B is path connected, there is an isomorphism $H^0(B;\, H^t(F;G)) \approx H^t(F;G)$. Using the fact that the cohomology spectral sequence is functorial, it follows that i^* maps the spectral sequence of $E \to B$ to the spectral sequence of $F \to b_0$. Therefore $i^*: H^*(E;G) \to H^*(F;G)$ is the composite

$$H^t(E;G) = F^0H^t(E;G) \to E_\infty^{0,t} \to E_2^{0,t} \approx H^0(B;\, H^t(F;G)) \approx H^t(F;G)$$

This leads to the following *generalized Wang cohomology sequence.*

1 THEOREM *Let $p: E \to B$ be a fibration, with fiber F and simply connected base B, which is a cohomology n-sphere (over R) for some $n \geq 2$. There is an exact sequence*

$$\cdots \to H^t(E;G) \xrightarrow{i^*} H^t(F;G) \xrightarrow{\theta} H^{t-n+1}(F;G) \to H^{t+1}(E;G) \xrightarrow{i^*} \cdots$$

in which $\theta(u \smile v) = \theta(u) \smile v + (-1)^{(n+1)\deg u}\, u \smile \theta(v)$, *the coefficients being suitably paired.*

PROOF Since B has no torsion, for the cohomology spectral sequence of $E \to B$ we have

$$E_2^{s,t} \approx H^s(B) \otimes H^t(F;G) = 0 \qquad s \neq 0, n$$

As in the proof of theorem 9.3.2, this leads to an exact sequence

$$\cdots \to H^t(E;G) \to E_2^{0,t} \xrightarrow{d_n} E_2^{n,t-n+1} \to H^{t+1}(E;G) \to \cdots$$

Let $1 \in H^0(B)$ be the unit class and let $w \in H^n(B)$ be a generator of $H^n(B)$. The map $u \to 1 \otimes u$ is an isomorphism of $H^t(F;G)$, with $E_2^{0,t}$, and the map $v \to w \otimes v$ is an isomorphism of $H^{t-n+1}(F;G)$, with $E_2^{n,t-n+1}$. Define θ: $H^t(F;G) \to H^{t-n+1}(F;G)$ by the condition

$$d_n(1 \otimes u) = w \otimes \theta(u)$$

Then the desired exact sequence is obtained from the exact sequence above on replacing $E_2^{0,t}$ by $H^t(F;G)$ and $E_2^{n,t-n+1}$ by $H^{t-n+1}(F;G)$ and interpreting the resulting homomorphisms. To verify that θ has the stated multiplicative property, we use the fact that d_n is a derivation. Then we have

$$\begin{aligned} w \otimes \theta(u \smile v) &= d_n(1 \otimes (u \smile v)) = d_n(1 \otimes u \smile 1 \otimes v) \\ &= d_n(1 \otimes u) \smile 1 \otimes v + (-1)^{\deg u}\, 1 \otimes u \smile d_n(1 \otimes v) \\ &= w \otimes [\theta(u) \smile v + (-1)^{(n+1)\deg u} u \smile \theta(v)] \quad \blacksquare \end{aligned}$$

Let p: $E \to B$ be an orientable fibration with path-connected base and let $B' \subset B$ and $E' = p^{-1}(B')$. We show how the homomorphism

$$p^*: H^*(B,B';\, G) \to H^*(E,E';\, G)$$

can be interpreted in terms of the cohomology spectral sequence of (E,E'). Because the spectral sequence is a first-quadrant spectral sequence, there is an epimorphism $E_2^{s,0} \to E_\infty^{s,0}$. The augmentation $G \to H^0(F;G)$ induces a homomorphism $H^s(B,B';\, G) \to H^s(B,B';\, H^0(F;G))$. Using the spectral sequence of the fibration $B \subset B$ and the functorial property of the cohomology spectral sequence, it follows that $p^*: H^*(B,B';\, G) \to H^*(E,E';\, G)$ is the composite

$$\begin{aligned} &H^s(B,B';\, G) \to H^s(B,B';\, H^0(F;G)) \\ &\qquad \approx E_2^{s,0} \to E_\infty^{s,0} \approx F^sH^s(E,E';\, G) \subset H^s(E,E';\, G) \end{aligned}$$

This leads to the following *generalized Gysin cohomology sequence.*

2 THEOREM *Let* p: $E \to B$ *be an orientable fibration with path-connected base space and with fiber* F *a cohomology* n*-sphere (over* R*), with* $n \geq 1$. *If* $B' \subset B$ *and* $E' = p^{-1}(B')$, *there is an exact sequence*

$$\cdots \xrightarrow{p^*} H^s(E,E';\, G) \to H^{s-n}(B,B';\, G) \xrightarrow{\Psi} H^{s+1}(B,B';\, G) \xrightarrow{p^*} H^{s+1}(E,E';\, G) \to \cdots$$

in which $\Psi(u) = u \smile \Omega$ *for some* $\Omega \in H^{n+1}(B;R)$. *If* n *is even,* $2\Omega = 0$.

PROOF For the cohomology spectral sequence of (E,E') we have

$$E_2^{s,t} \approx H^s(B,B';\, H^t(F;G)) = 0 \qquad t \neq 0, n$$

As in the proof of theorem 9.3.3, this leads to an exact sequence

$$\cdots \to H^s(E,E';\, G) \to E_2^{s-n,n} \xrightarrow{d_{n+1}} E_2^{s+1,0} \to H^{s+1}(E,E';\, G) \to \cdots$$

Let $1 \in H^0(F;R)$ be the unit class and let $w \in H^n(F;R)$ be a generator of $H^n(F;R)$. Corresponding to these generators are isomorphisms $G \approx H^0(F;G)$ and $G \approx H^n(F;G)$. Thus we have isomorphisms

$$H^s(B,B';\, G) \approx H^s(B,B';\, H^0(F;G)) \approx E_2^{s,0}$$

whose composite will be denoted by α: $H^s(B,B';\, G) \approx E_2^{s,0}$, and

$$H^s(B,B';\, G) \approx H^s(B,B';\, H^n(F;G)) \approx E_2^{s,n}$$

whose composite will be denoted by β: $H^s(B,B';\, G) \approx E_2^{s,n}$. Define the homomorphism Ψ: $H^{s-n}(B,B';\, G) \to H^{s+1}(B,B';\, G)$ by the equation

$$\alpha\Psi(u) = (-1)^{\deg u}\, d_{n+1}\beta(u)$$

The desired exact sequence is obtained from the exact sequence above on replacing $E_2^{s-n,n}$ by $H^{s-n}(B,B';\, G)$, $E_2^{s+1,0}$ by $H^{s+1}(B,B';\, G)$ and interpreting the resulting homomorphisms.

In the spectral sequence of E with coefficients R there are similar isomorphisms α: $H^s(B,R) \approx E_2^{s,0}$ and β: $H^s(B;R) \approx E_2^{s,n}$. Let 1 also denote the unit class of $H^0(B;R)$ and define $\Omega \in H^{n+1}(B;R)$ by the equation

$$\alpha(\Omega) = d_{n+1}\beta(1)$$

To verify that $\Psi(u) = u \smile \Omega$, we use the cup-product pairing from the spectral sequence of (E,E') with coefficients G, and the spectral sequence of E with coefficients R, to the spectral sequence of (E,E') with coefficients G. Then

$$\begin{aligned} \alpha\Psi(u) &= (-1)^{\deg u} d_{n+1}\beta(u) = (-1)^{\deg u} d_{n+1}(\alpha(u) \smile \beta(1)) \\ &= \alpha(u) \smile d_{n+1}\beta(1) = \alpha(u) \smile \alpha(\Omega) = \alpha(u \smile \Omega) \end{aligned}$$

Therefore $\Psi(u) = u \smile \Omega$. Since $w \smile w = 0$, $\beta(1) \smile \beta(1) = 0$ in the spectral sequence of E. Therefore, if n is even,

$$0 = d_{n+1}(\beta(1) \smile \beta(1)) = \alpha(\Omega) \smile \beta(1) + \beta(1) \smile \alpha(\Omega) = \beta(2\Omega)$$

showing that $2\Omega = 0$. ∎

We use the cohomology spectral sequence to give another interpretation of the integer $|H[\alpha]|$, where $[\alpha] \in \pi_{2n+1}(S^{n+1})$ and H: $\pi_{2n+1}(S^{n+1}) \to \pi_{2n-1}(S^{2n-1})$ is the Hopf invariant defined in Sec. 9.3.

3 THEOREM *Let α: $S^{2n+1} \to S^{n+1}$ be a base-point-preserving map and let Y_α be the CW complex obtained by attaching a $(2n + 2)$-cell to S^{n+1} by the map α. Then $H^{n+1}(Y_\alpha)$ and $H^{2n+2}(Y_\alpha)$ are both infinite cyclic, and if u and v are generators, respectively, then $u \smile u = \pm|H[\alpha]|v$.*

PROOF If Z is the mapping cylinder of α, then $Y_\alpha = Z/S^{2n+1}$, and so $H^*(Y_\alpha) \approx H^*(Z,S^{2n+1})$. Let $u \in H^{n+1}(Z,S^{2n+1})$ and $v \in H^{2n+2}(Z,S^{2n+1})$ be respective generators. It suffices to prove that $u \smile u = \pm|H[\alpha]|v$.

Let $r: Z \to S^{n+1}$ be the retraction and let $E \to Z$ be the principal fibration induced by r. Since r is a homotopy equivalence, the induced map $E \to PS^{n+1}$ induces isomorphisms of homology. Therefore $\tilde{H}_*(E) = 0$ and $\tilde{H}^*(E) = 0$. The restriction of E to S^{2n+1} is the principal fibration $E_\alpha \to S^{2n+1}$ induced by α. By theorem 9.3.10, $|H[\alpha]| = m$ if and only if $H_{2n}(E_\alpha) \approx \mathbf{Z}_m$. From the following portion of the Wang homology sequence of E_α

$$0 \to H_{2n+1}(E_\alpha) \to H_0(\Omega S^{n+1}) \to H_{2n}(\Omega S^{n+1}) \to H_{2n}(E_\alpha) \to 0$$

it follows that if $m \neq 0$, $H_{2n+1}(E_\alpha) = 0$, and if $m = 0$, then $H_{2n+1}(E_\alpha) \approx \mathbf{Z}$. By the universal-coefficient formula for cohomology, $H^{2n+1}(E_\alpha) \approx \mathbf{Z}_m$ no matter whether $m = 0$ or not (recall that we have adopted the convention that $\mathbf{Z}_0 = \mathbf{Z}$).

Since $\tilde{H}^*(E) = 0$, there is an isomorphism

$$\delta: H^{2n+1}(E_\alpha) \approx H^{2n+2}(E,E_\alpha)$$

and so $H^{2n+2}(E,E_\alpha) \approx \mathbf{Z}_m$, where $m = |H[\alpha]|$. We compute the order of $H^{2n+2}(E,E_\alpha)$ by using the cohomology spectral sequence.

For $s + t = 2n + 2$ the only nonzero term $E_2^{s,t}$ is the term

$$E_2^{2n+2,0} \approx H^{2n+2}(Z,S^{2n+1}) \otimes H^0(\Omega S^{n+1}),$$

and for $s + t = 2n + 1$ the only nonzero term $E_2^{s,t}$ is the term

$$E_2^{n+1,n} \approx H^{n+1}(Z,S^{2n+1}) \otimes H^n(\Omega S^{n+1}).$$

It follows that

$$H^{2n+2}(E,E_\alpha) \approx E_\infty^{2n+2,0} \approx E_2^{2n+2,0}/d_{n+1}(E_2^{n+1,n})$$

Let $u' \in H^{n+1}(Z)$ be the generator defined by $u' = u \mid Z$. Then, since $\tilde{H}^*(E) = 0$, there is a generator $w \in H^n(\Omega S^{n+1})$ such that in the spectral sequence of E we have $d_{n+1}(1 \otimes w) = u' \otimes 1$. Using the pairing of the cohomology spectral sequences of (E,E_α) and E_α to that of (E,E_α), we see that

$$\begin{aligned} d_{n+1}(u \otimes w) &= d_{n+1}(u \otimes 1 \smile 1 \otimes w) = \pm u \otimes 1 \smile d_{n+1}(1 \otimes w) \\ &= \pm u \otimes 1 \smile u' \otimes 1 = \pm(u \smile u') \otimes 1 \\ &= \pm(u \smile u) \otimes 1 \end{aligned}$$

Therefore $H^{2n+2}(E,E_\alpha)$ is infinite cyclic if and only if $u \smile u = 0$, and $H^{2n+2}(E,E_\alpha)$ has order m if and only if $u \smile u = \pm\, mv$. Comparing this with the earlier calculation of $H^{2n+2}(E,E_\alpha)$ gives the result. ∎

4 COROLLARY *For any integer $m \geq 1$ the Hopf invariant*

$$H: \pi_{4m+1}(S^{2m+1}) \to \pi_{4m-1}(S^{4m-1})$$

is the trivial homomorphism.

PROOF For any α: $S^{4m+1} \to S^{2m+1}$, if Y_α is the CW complex obtained by attaching a $(4m + 2)$-cell to S^{2m+1} by the map α and if $u \in H^{2m+1}(Y_\alpha)$ is arbitrary, then $u \smile u = -u \smile u$, and so $u \smile u = 0$. By theorem 3, $|H[\alpha]| = 0$, and so $H[\alpha] = 0$ for all $[\alpha] \in \pi_{4m+1}(S^{2m+1})$. ■

5 COROLLARY *For any $m \geq 1$, if α_{2m}: $S^{4m-1} \to S^{2m}$ is the map used in forming the CW complex X^{4m}, then $|H[\alpha_{2m}]| = 2$.*

PROOF Recall the definition of $X^{4m} = Y_{\alpha_{2m}}$ in Sec. 9.3. There is a collapsing map k: $S^{2m} \times S^{2m} \to X^{4m}$ with the property that if $u' \in H^{2m}(S^{2m})$ is a generator, there are generators $u \in H^{2m}(X^{4m})$ and $v \in H^{4m}(X^{4m})$ such that $k^* u = u' \times 1 + 1 \times u'$ and $k^* v = u' \times u'$. Then

$$k^*(u \smile u) = (u' \times 1 + 1 \times u') \smile (u' \times 1 + 1 \times u') = 2u' \times u'$$

Since k^*: $H^{4m}(X^{4m}) \approx H^{4m}(S^{2m} \times S^{2m})$, it follows that $u \smile u = 2v$, and the result follows from theorem 3. ■

If π is a group, we define $H_*(\pi)$ [and $H^*(\pi)$] to be the integral homology [and cohomology] groups of a space of type $(\pi,1)$. Since any two spaces of type $(\pi,1)$ are easily seen to have the same weak homotopy type, these groups are independent (up to canonical isomorphism) of the space of type $(\pi,1)$ chosen. Furthermore, any homomorphism $\pi \to \pi'$ induces homomorphisms $H_*(\pi) \to H_*(\pi')$ and $H^*(\pi') \to H^*(\pi)$. We use the cohomology spectral sequence to obtain information about these groups.

6 THEOREM *For $n > 1$ there are isomorphisms*

$$H^q(\mathbf{Z}_n) \approx \begin{cases} 0 & q \text{ odd} \\ \mathbf{Z} & q = 0 \\ \mathbf{Z}_n & q \text{ even}, q > 0 \end{cases}$$

PROOF Let X be a CW complex of type $(\mathbf{Z},2)$ and let $PX \to X$ be the path fibration. Then the fiber ΩX of this fibration is a space of type $(\mathbf{Z},1)$. Therefore ΩX is a cohomology 1-sphere, and since PX is contractible, it follows from theorem 2 that $H^*(X)$ is a polynomial algebra on a generator Ω of degree 2, characterized by the equation $\Omega \otimes 1 = d_2(1 \otimes w)$ [where w is a generator of $H^1(\Omega X)$ and d_2 is the differential operator in E_2 of the cohomology spectral sequence of the fibration $PX \to X$]. Let f: $X \to X$ be a map such that $f^* \iota = n\iota$ for some 2-characteristic element $\iota \in H^2(X)$ (such a map exists, by theorem 8.1.10). It follows that $f^*(u) = nu$ for any $u \in H^2(X)$ and $f_\#$: $\pi_2(X) \to \pi_2(X)$ is the homomorphism $f_\#[\alpha] = n[\alpha]$. Let p: $E \to X$ be the principal fibration induced by f. Then p has fiber ΩX, and from the functorial property of the cohomology spectral sequence, we have $d_2(1 \otimes w) = f^*\Omega \otimes 1 = n\Omega \otimes 1$ in the spectral sequence of p: $E \to X$. Therefore, in the Gysin sequence of p the homomorphism

$$\Psi: H^{s-1}(X) \to H^{s+1}(X)$$

equals the cup product by $n\Omega$, and so

$$\ker \Psi = 0 \qquad \operatorname{coker} \Psi \approx \mathbf{Z}_n, \text{ for odd } s$$

Therefore $H^q(E) = 0$ unless q is even, and $H^0(E) \approx \mathbf{Z}$ and $H^q(E) \approx \mathbf{Z}_n$ if q is even and $q > 0$.

It merely remains to verify that E is a space of type $(\mathbf{Z}_n,1)$. This follows from the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\cdots \to & \pi_q(X) & \xrightarrow{\bar\partial} & \pi_{q-1}(\Omega X) & \to & \pi_{q-1}(E) & \xrightarrow{p_\#} & \pi_{q-1}(X) & \to \cdots \\
& \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle =} & & \downarrow & & \downarrow & \\
\cdots \to & \pi_q(X) & \xrightarrow{\bar\partial} & \pi_{q-1}(\Omega X) & \to & \pi_{q-1}(PX) & \to & \pi_{q-1}(X) & \to \cdots
\end{array} \quad \blacksquare$$

7 COROLLARY *For $n > 1$ there are isomorphisms*

$$H_q(\mathbf{Z}_n) \approx \begin{cases} 0 & q \text{ even}, q \neq 0 \\ \mathbf{Z} & q = 0 \\ \mathbf{Z}_n & q \text{ odd}, q > 0 \end{cases}$$

PROOF This will follow from theorem 6 and the universal-coefficient formula 5.5.12 once we have verified that $H_*(\mathbf{Z}_n)$ is of finite type. We use the particular space E of type $(\mathbf{Z}_n,1)$ constructed in the proof of theorem 6. Since the fiber ΩX of the fibration $p\colon E \to X$ is a homology 1-sphere, there is, by theorem 9.3.3, an exact Gysin homology sequence

$$\cdots \to H_s(E) \xrightarrow{p_*} H_s(X) \to H_{s-2}(X) \to H_{s-1}(E) \to \cdots$$

Since $H_*(X)$ is of finite type [in fact, $H_q(X) = 0$ if q is odd and $H_q(X) \approx \mathbf{Z}$ if q is even and $q > 0$, as can be seen by using the Gysin sequence of the fibration $PX \to X$], it follows that $H_*(E)$ is of finite type. ■

Because S^1 is a space of type $(\mathbf{Z},1)$, we now know $H_*(\pi)$ if π is a cyclic group. The groups $H_*(\pi)$ for π a finite direct sum of cyclic groups can be computed by induction, using the Künneth formula and the following result.

8 LEMMA *If Y and Y' are spaces of type $(\pi,1)$ and $(\pi',1)$, respectively, then $Y \times Y'$ is a space of type $(\pi \times \pi', 1)$.*

PROOF It is easily verified from the definitions of the homotopy groups that $\pi_q(Y \times Y') \approx \pi_q(Y) \times \pi_q(Y')$. ■

In this way we can determine $H_*(\pi)$ if π is a finitely generated abelian group. The following result gives information about $H_*(\pi)$ for an arbitrary abelian group π.

9 THEOREM *Let $\{\pi_\alpha\}$ be the family of finitely generated subgroups directed by inclusion of an abelian group π. Then*

$$H_*(\pi) \approx \lim_{\to} \{H_*(\pi_\alpha)\}$$

PROOF For each element $\lambda \in \pi$ let S_λ^1 be a 1-sphere. Let $X^1 = \bigvee S_\lambda^1$ and define a homomorphism $\beta\colon \pi_1(X^1) \to \pi$ by the condition $\beta[\omega_\lambda] = \lambda$, where

$[\omega_\lambda] \in \pi_1(X')$ is determined by the inclusion map $S_\lambda^1 \subset X^1$ ($\pi_1(X')$ is the free group generated by the collection $\{[\omega_\lambda]\}_\lambda$). For every base-point-preserving map ω: $S^1 \to X^1$ such that $\beta[\omega] = 0$, attach a 2-cell to X^1 and let X^2 be the space obtained by adjoining all these 2-cells to X^1. Continue inductively, defining X^m for $m \geq 3$ to be the space obtained from X^{m-1} by attaching m-cells for every map $S^{m-1} \to X^{m-1}$. Let X be the CW complex whose m-skeleton is X^m for all $m \geq 1$ and whose 0-skeleton is the base point of X^1. Then X is a space of type $(\pi,1)$.

For any finite subset a of π let X_a be the largest subcomplex of X such that $X_a{}^1 = \bigvee_{\lambda \in a} S_\lambda{}^1$. Then it is clear from the construction of X that X_a is a space of type $(\pi_a,1)$, where π_a is the subgroup of π generated by the set a. Since every compact subset of X is contained in X_a for some finite subset a of π, it follows that

$$H_*(\pi) \approx H_*(X) \approx \lim_{\rightarrow} \{H_*(X_a)\} \approx \lim_{\rightarrow} \{H_*(\pi_a)\}$$

Since π_a is a finitely generated subgroup of π and every finitely generated subgroup of π is of this form, the right-hand side above is isomorphic to $\lim_{\rightarrow} \{H_*(\pi_\alpha)\}$. ■

These results on $H_*(\pi)$ will be used in the next section in the proof of the generalized Hurewicz isomorphism theorem.

6 SERRE CLASSES OF ABELIAN GROUPS

The spectral sequence of a fibration is well suited for inductive arguments based on the lowest (or highest) dimension in which a particular phenomenon occurs. Such arguments can be simplified further by systematically neglecting certain abelian groups in order to carry along just that portion of a given group which is relevant to the phenomenon in question. For example, in studying the p-primary components of finitely generated abelian groups, it is convenient to neglect finite summands whose order is not divisible by p. The process of neglecting certain groups will be formalized in this section by means of a study of groups "modulo a Serre class of abelian groups." The machinery will be applied, by means of the spectral sequence of a fibration, to the study of the homotopy groups of a space. In particular, the section closes with interesting generalizations of the Hurewicz and Whitehead theorems.

A *Serre class* of abelian groups is a nonempty class $\mathcal{C}$ of abelian groups having the property that for any exact three-term sequence of abelian groups $A \to B \to C$, if $A, C \in \mathcal{C}$, then $B \in \mathcal{C}$.

1 THEOREM *A class $\mathcal{C}$ of abelian groups is a Serre class if and only if it has the following properties:*

(*a*) *$\mathcal{C}$ contains a trivial group.*
(*b*) *If $A \in \mathcal{C}$ and $A \approx A'$, then $A' \in \mathcal{C}$.*
(*c*) *If $A \subset B$ and $B \in \mathcal{C}$, then $A \in \mathcal{C}$.*

(*d*) *If* $A \subset B$ *and* $B \in \mathcal{C}$, *then* $B/A \in \mathcal{C}$.
(*e*) *If* $0 \to A \to B \to C \to 0$ *is a short exact sequence, with* $A, C \in \mathcal{C}$, *then* $B \in \mathcal{C}$.

PROOF If $\mathcal{C}$ is a Serre class, it is nonempty, and if $A \in \mathcal{C}$, then (*a*) follows from the exactness of $A \to 0 \to A$. Properties (*b*), (*c*), and (*d*) follow from (*a*) and the exactness of the sequences $0 \to A' \to A$, $0 \to A \to B$, and $B \to B/A \to 0$, respectively, while (*e*) follows from the defining property of a Serre class.

Conversely, if $\mathcal{C}$ satisfies properties (*a*) to (*e*), then $\mathcal{C}$ is nonempty, by (*a*). If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is an exact sequence, there is a short exact sequence

$$0 \to \text{im } \alpha \to B \to \text{coker } \alpha \to 0$$

and isomorphisms $A/\ker \alpha \approx \text{im } \alpha$ and $\text{coker } \alpha \approx \text{im } \beta \subset C$. If $A \in \mathcal{C}$, it follows from properties (*d*) and (*b*) that $\text{im } \alpha \in \mathcal{C}$. If $C \in \mathcal{C}$, it follows from properties (*c*) and (*b*) that $\text{coker } \alpha \in \mathcal{C}$. If $A, C \in \mathcal{C}$, it follows on using (*e*) that $B \in \mathcal{C}$. Hence, $\mathcal{C}$ is a Serre class. ∎

Note that it follows from *a* and *b* of theorem 1 that a Serre class does not form a set. We list some examples of Serre classes.

2 The class of all abelian groups

3 The class of trivial groups

4 The class of finitely generated abelian groups

5 The class of finite abelian groups

6 The class of torsion abelian groups

7 The class of p-groups for a given prime p

8 The class of torsion groups having no element with order a positive power of a given prime p

Given a class $\mathcal{C}$, we are interested in computing modulo groups in $\mathcal{C}$. Thus a homomorphism $\varphi: A_1 \to A_2$ is defined to be a $\mathcal{C}$-*monomorphism* (or $\mathcal{C}$-*epimorphism*) if $\ker \varphi \in \mathcal{C}$ (or $\text{coker } \varphi \in \mathcal{C}$) and is a $\mathcal{C}$-*isomorphism* if both conditions are satisfied. It is easily verified that the composite of $\mathcal{C}$-isomorphisms is a $\mathcal{C}$-isomorphism. Two abelian groups A_1 and A_2 are said to be $\mathcal{C}$-*isomorphic*, denoted by $A_1 \underset{\mathcal{C}}{\approx} A_2$, if there exists an abelian group A and $\mathcal{C}$-isomorphisms $A \to A_1$ and $A \to A_2$. Note the similarity between the definition of $\mathcal{C}$-isomorphic abelian groups and the definition of spaces of the same weak homotopy type.

9 **LEMMA** *The relation of being* $\mathcal{C}$-*isomorphic is an equivalence relation.*

PROOF The relation is clearly reflexive and symmetric. To show that it is transitive, assume $A_1 \underset{\mathcal{C}}{\approx} A_2$ and $A_2 \underset{\mathcal{C}}{\approx} A_3$. There exist abelian groups B and B' and $\mathcal{C}$-isomorphisms $\varphi_1: B \to A_1$, $\varphi_2: B \to A_2$, $\varphi_2': B' \to A_2$, and $\varphi_3': B' \to A_3$.

Let $C = \{(b,b') \in B \oplus B' \mid \varphi_2(b) = \varphi_2'(b')\}$ and let $p\colon C \to B$ and $p'\colon C \to B'$ be the projections (C is the fibered product of φ_2 and φ_2' in the category of abelian groups). Because there is an exact sequence

$$\ker \varphi_2' \to C \xrightarrow{p} B \to \operatorname{coker} \varphi_2'$$

it follows that p is a $\mathcal{C}$-isomorphism. Similarly, p' is a $\mathcal{C}$-isomorphism. Therefore the composites $C \xrightarrow{p} B \xrightarrow{\varphi_1} A_1$ and $C \xrightarrow{p'} B' \xrightarrow{\varphi_3'} A_3$ are $\mathcal{C}$-isomorphisms, showing that $A_1 \underset{\mathcal{C}}{\approx} A_3$. ■

A topological space X is said to be $\mathcal{C}$-*acyclic* if its integral homology groups $H_q(X) \in \mathcal{C}$ for $q > 0$. In order to ensure that the product of two $\mathcal{C}$-acyclic spaces be $\mathcal{C}$-acyclic, we need $\mathcal{C}$ to have the additional property that A, $B \in \mathcal{C}$ imply $A \otimes B$, $A * B \in \mathcal{C}$. A Serre class with this additional property is called a *ring of abelian groups*.

A pair (X,X') with X' nonempty is said to be $\mathcal{C}$-*acyclic* if the integral groups $H_q(X,X') \in \mathcal{C}$ for all q. In order to ensure that the product of a $\mathcal{C}$-acyclic pair and an arbitrary space is a $\mathcal{C}$-acyclic pair, we need $\mathcal{C}$ to have the property that $A \in \mathcal{C}$ implies $A \otimes B$, $A * B \in \mathcal{C}$ for arbitrary B. A Serre class $\mathcal{C}$ with this additional property is called an *ideal of abelian groups*. Obviously, an ideal of abelian groups is a ring of abelian groups. Examples 2, 3, 6, 7, and 8 are ideals of abelian groups, while examples 4 and 5 are rings of abelian groups which are not ideals of abelian groups.

In the sequel some of the results will be valid for a ring of abelian groups and somewhat stronger results will be valid for an ideal of abelian groups. The results will usually be stated in pairs, one for a ring of abelian groups, and the other for an ideal of abelian groups. The proofs of the two pairs will usually differ only in minor details. The following generalization of lemma 9.3.4 is the main result obtained from a spectral-sequence argument.

10 THEOREM *Let $p\colon E \to B$ be an orientable fibration with path-connected fiber F and path-connected base B and let B' be a nonempty subspace of B. Define $E' = p^{-1}(B')$ and let $\mathcal{C}$ be a Serre class. We assume that $H_i(B,B'; R) \in \mathcal{C}$ for $0 \le i < n$ and $H_j(F;G) \in \mathcal{C}$ for $0 < j < m$. We define an integer $r \ge 0$ as follows:*

(*a*) *If $\mathcal{C}$ is a ring of abelian groups and $H_1(B,B'; R) = 0$, let $r = \inf(n, m + 1)$.*
(*b*) *If $\mathcal{C}$ is an ideal of abelian groups, let $r = n + m - 1$.*

Then the homomorphism $p_*\colon H_q(E,E'; G) \to H_q(B,B\ ; G)$ is a $\mathcal{C}$-monomorphism for $q \le r$ and a $\mathcal{C}$-epimorphism for $q \le r + 1$.

PROOF We use the spectral sequence of (E,E') and show first that $E^2_{s,t} \in \mathcal{C}$ if $s + t \le r$ and $t \ge 1$. We know that

$$E^2_{s,t} \approx H_s(B,B'; R) \otimes H_t(F;G) \oplus H_{s-1}(B,B'; R) * H_t(F;G)$$

In case (a), because $H_0(B,B'; R) = 0 = H_1(B,B'; R)$, it follows that $E^2_{s,t} \in \mathcal{C}$ if

$s = 0$ or 1. If $s > 1$, then $t < m$ (because $s + t \leq m + 1$). Therefore $H_t(F;G) \in \mathcal{C}$, and because it is also true that $s < n$, $H_s(B,B'; R)$ and $H_{s-1}(B,B'; R)$ are both in $\mathcal{C}$. Because $\mathcal{C}$ is a ring of abelian groups, $E^2_{s,t} \in \mathcal{C}$.

In case (*b*),

$$s + t \leq r = n + m - 1$$

implies $s \leq n - 1$ or $t \leq m - 1$. Because $\mathcal{C}$ is an ideal of abelian groups, it again follows that $E^2_{s,t} \in \mathcal{C}$.

To complete the proof, note that the spectral sequence gives a normal series

$$0 \subset D_0 \subset D_1 \subset \cdots \subset D_q = H_q(E,E'; G)$$

where $D_0, D_1/D_0, \ldots, D_q/D_{q-1}$ are the limit terms of the spectral sequence (that is, $D_i/D_{i-1} \approx E^\infty_{i,q-i}$). Therefore

$$H_q(E,E'; G)/D_{q-1} \approx E^\infty_{q,0} \subset E^2_{q,0} = H_q(B,B'; G)$$

and the kernel of the homomorphism $p_*: H_q(E,E'; G) \to H_q(B,B'; G)$ equals the kernel of the map $D_q \to D_q/D_{q-1}$. To show that p_* is a $\mathcal{C}$-monomorphism, therefore, we must show that $D_{q-1} \in \mathcal{C}$. By a simple induction (on D_k for $0 \leq k \leq q - 1$), it suffices to show that $E^\infty_{s,t} \in \mathcal{C}$ for $s + t \leq r$ and $t \geq 1$. This follows from the corresponding property of $E^2_{s,t}$ already established.

To prove that p_* is a $\mathcal{C}$-epimorphism, we must show that $E^2_{q,0}/E^\infty_{q,0} \in \mathcal{C}$ if $q \leq r + 1$. However, there is a sequence

$$E^2_{q,0} \supset E^3_{q,0} \supset \cdots \supset E^{q+1}_{q,0} = E^\infty_{q,0}$$

and again by a simple induction, it suffices to show that $E^k_{q,0}/E^{k+1}_{q,0} \in \mathcal{C}$ for $q \leq r + 1$ and $k \geq 2$. By definition,

$$E^{k+1}_{q,0} \approx \ker (d^k: E^k_{q,0} \to E^k_{q-k,k-1})$$

Therefore $E^k_{q,0}/E^{k+1}_{q,0}$ is isomorphic to a submodule of $E^k_{q-k,k-1}$, and it suffices to show that $E^k_{q-k,k-1} \in \mathcal{C}$ for $q \leq r + 1$ and $k \geq 2$. This follows from the fact (already established) that $E^2_{q-k,k-1} \in \mathcal{C}$ for $q \leq r + 1$ and $k \geq 2$. ∎

By specializing to the case where B' is a point, we get the following interesting applications of this last result.

11 COROLLARY *Let $p: E \to B$ be a fibration with path-connected fiber F and simply connected base B. Assume that E is $\mathcal{C}$-acyclic and $H_i(B) \in \mathcal{C}$ for $0 < i < n$. Then $H_i(F) \in \mathcal{C}$ for $0 < i < n - 1$, and*

(*a*) *If $\mathcal{C}$ is a ring of abelian groups, $H_{n-1}(F) \underset{\mathcal{C}}{\approx} H_n(B)$.*

(*b*) *If $\mathcal{C}$ is an ideal of abelian groups, $H_i(F) \underset{\mathcal{C}}{\approx} H_{i+1}(B)$ for $i < 2n - 2$.*

PROOF Let $B' = \{b_0\}$ and $E' = p^{-1}(b_0) = F$ and use induction on n. Inductively we can assume $H_i(F) \in \mathcal{C}$ for $0 < i < n - 1$. We apply theorem 10, with $m = n - 1$. In case (*a*) $r = n$ and in case (*b*) $r = 2n - 2$, and $H_i(E,F) \underset{\mathcal{C}}{\approx} H_i(B,b_0)$ for $i \leq r$. Because E is $\mathcal{C}$-acyclic, $H_i(E,F) \underset{\mathcal{C}}{\approx} H_{i-1}(F)$ for

$i \geq 2$, and because b_0 is a point, $H_i(B) \approx H_i(B,b_0)$ for $i > 0$. The result follows by combining these $\mathcal{C}$-isomorphisms. ■

12 COROLLARY *Let p: $E \to B$ be a fibration with path-connected fiber F and simply connected base B. If $\mathcal{C}$ is a ring of abelian groups and two of the three spaces E, B, and F are $\mathcal{C}$-acyclic, so is the third.*

PROOF If B and F are $\mathcal{C}$-acyclic, let $B' = \{b_0\}$ and $E' = p^{-1}(b_0) = F$ and apply theorem 10a, with $n = m = \infty$. We find that (E,F) is $\mathcal{C}$-acyclic, and, since F is $\mathcal{C}$-acyclic, E is $\mathcal{C}$-acyclic. If E and B are $\mathcal{C}$-acyclic, it follows from corollary 11a that F is $\mathcal{C}$-acyclic. If E and F are $\mathcal{C}$-acyclic, let $n \geq 2$ be the smallest integer such that $H_n(B) \notin \mathcal{C}$ (if such integers exist). By corollary 11a, $H_{n-1}(F) \notin \mathcal{C}$, which is a contradiction. Therefore B is $\mathcal{C}$-acyclic. ■

The following special case of this last result is worth explicit mention.

13 COROLLARY *Let X be a simply connected space and let $\mathcal{C}$ be a ring of abelian groups. Then X is $\mathcal{C}$-acyclic if and only if its loop space ΩX is $\mathcal{C}$-acyclic.* ■

For our next application of the spectral sequence of a fibration (namely, to prove the generalized Hurewicz isomorphism theorems) we need another property of Serre classes of abelian groups. A Serre class $\mathcal{C}$ of abelian groups is said to be an *acyclic class* if any space of type $(\pi,1)$ with $\pi \in \mathcal{C}$ is $\mathcal{C}$-acyclic. Thus $\mathcal{C}$ is an acyclic class if and only if $\pi \in \mathcal{C}$ implies $H_q(\pi) \in \mathcal{C}$ for $q > 0$. From the remarks and results at the end of Sec. 9.5, it follows that each of examples 2 to 8 is an acyclic class.

The loop space of a space of type (π,n), with $n \geq 2$, is a space of type $(\pi, n-1)$. Hence we have the following result by induction on n from corollary 13.

14 LEMMA *If $\mathcal{C}$ is an acyclic ring of abelian groups, any space of type (π,n), with $n \geq 1$ and $\pi \in \mathcal{C}$, is $\mathcal{C}$-acyclic.* ■

Using this and a Postnikov system for X, it can be shown that if X is a simply connected space whose homotopy groups belong to an acyclic ring $\mathcal{C}$ of abelian groups, then X is $\mathcal{C}$-acyclic. This is also a consequence of the following *generalized absolute Hurewicz isomorphism theorem.*

15 THEOREM *Let $\mathcal{C}$ be an acyclic ring of abelian groups and let X be a simply connected space. The following are equivalent:*

(*a*) $\pi_i(X) \in \mathcal{C}$ *for* $2 \leq i < n$.
(*b*) $H_i(X) \in \mathcal{C}$ *for* $2 \leq i < n$.

Furthermore, either implies that the Hurewicz homomorphism

$$\varphi\colon \pi_i(X) \to H_i(X)$$

is a $\mathcal{C}$-isomorphism for $i \leq n$.

PROOF It clearly suffices to prove that (b) implies φ: $\pi_n(X) \underset{\mathcal{C}}{\approx} H_n(X)$. For

$n = 2$ this is a consequence of the absolute Hurewicz isomorphism theorem. We assume $n \geq 3$ and prove the result by induction on n. It follows that $\pi_i(X) \in \mathcal{C}$ for $i < n$.

By corollary 8.3.8, there exists a sequence of fibrations

$$E_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} E_1 = X$$

such that E_j is j-connected and $p_j\colon E_j \to E_{j-1}$ has a fiber F_j which is a space of type $(\pi_j(X), j-1)$.

It follows from the acyclicity of $\mathcal{C}$ that F_j is $\mathcal{C}$-acyclic for $2 \leq j \leq n-1$ [because $\pi_j(X) \in \mathcal{C}$ for $j < n$]. By induction on j, for $2 \leq j \leq n-1$, we prove that $p_{j*}\colon H_i(E_j) \underset{\mathcal{C}}{\approx} H_i(E_{j-1})$ for $i \leq n$. Assuming it for $j-1$, where $j > 1$, we see that $H_i(E_{j-1},e_0) \in \mathcal{C}$ for $i < n$ and $H_i(F) \in \mathcal{C}$ for $0 < i$. We deduce from theorem 10a that $p_{j*}\colon H_i(E_j,F_j) \underset{\mathcal{C}}{\approx} H_i(E_{j-1})$ for $0 < i \leq n$. Since F_j is $\mathcal{C}$-acyclic, this implies that $p_{j*}\colon H_i(E_j) \underset{\mathcal{C}}{\approx} H_i(E_{j-1})$ for $i \leq n$. This completes the induction.

Therefore the composite $f = p_2 \circ \cdots \circ p_{n-1}\colon E_{n-1} \to X$ has the property that

$$f_*\colon H_i(E_{n-1}) \underset{\mathcal{C}}{\approx} H_i(X) \qquad i \leq n$$

We have a commutative diagram

$$\begin{array}{ccc} \pi_n(E_{n-1}) & \xrightarrow{\varphi} & H_n(E_{n-1}) \\ {\scriptstyle f_\#}\downarrow & & \downarrow{\scriptstyle f_*} \\ \pi_n(X) & \xrightarrow{\varphi} & H_n(X) \end{array}$$

and, by the absolute Hurewicz isomorphism theorem, the top homomorphism is an isomorphism. Since both vertical maps are $\mathcal{C}$-isomorphisms, the result follows. ∎

This theorem clearly implies that a simply connected space is $\mathcal{C}$-acyclic (for an acyclic ring $\mathcal{C}$ of abelian groups) if and only if all its homotopy groups are in $\mathcal{C}$. Taking $\mathcal{C}$ to be the acyclic ring of all finitely generated abelian groups, we obtain the following result.

16 COROLLARY *A simply connected space has finitely generated homology groups in every dimension if and only if it has finitely generated homotopy groups in every dimension.* ∎

In particular, it follows from corollary 16 that any sphere S^n has finitely generated homotopy groups. Corollary 16 is not true if $\pi_1(X)$ is assumed to be finitely generated (instead of 0), as shown by the following example.

17 EXAMPLE Let $X = S^2 \vee S^1$. Then $\pi_1(X) \approx \mathbf{Z}$ is finitely generated and $H_i(X)$ is finitely generated for all i, but $\pi_2(X)$ is a free abelian group on a countable set of generators and so is not finitely generated.

Nevertheless, there is a generalization of corollary 16 valid for spaces which are not assumed to be simply connected. If X is a path-connected space, let j: $X \subset B$ be an imbedding of X in a space B of type $(\pi_1(X),1)$ such that $j_\#$: $\pi_1(X) \approx \pi_1(B)$. Let p: $P_j \to B$ be the mapping path fibration corresponding to j, as defined in Sec. 2.8 (so X and P_j have the same homotopy type). The space X is said to be *strongly simple* if $\pi_1(X)$ is abelian and if p: $P_j \to B$ is orientable over $\mathbf{Z}$.

18 EXAMPLE Assume that X is a space such that for every element $a \in \pi_1(X)$ there is a map $\bar{\omega}$: $S^1 \times X \to X$, with $\bar{\omega} \mid S^1 \times x_0$ representing a and $\bar{\omega} \mid p_0 \times X$ homotopic to 1_X. Then X is strongly simple (because P_j also has the same property as X). In particular, any H-space is strongly simple.

19 LEMMA *Let $\mathcal{C}$ be an acyclic ring of abelian groups and assume that X is a strongly simple space such that $\pi_1(X) \in \mathcal{C}$ and $H_i(X) \in \mathcal{C}$ for $0 < i < n$, where $n \geq 2$. If F is the fiber of the fibration p: $P_j \to B$, then $H_q(F) \to H_q(P_j)$ is a $\mathcal{C}$-isomorphism for $q \leq n$.*

PROOF Since X and P_j have the same homotopy type, $H_i(P_j) \in \mathcal{C}$ for $0 < i < n$. Let $m < n$ and assume inductively that $H_q(F) \underset{\mathcal{C}}{\approx} H_q(P_j)$ for $q \leq m$. Then $H_q(F) \in \mathcal{C}$ for $0 < q \leq m$.

We now prove that $H_{m+1}(F) \underset{\mathcal{C}}{\approx} H_{m+1}(P_j)$. From the spectral sequence of the fibration (the fibration being orientable, since X is strongly simple), there is a composition series

$$0 \subset D_0 \subset D_1 \subset \cdots \subset D_{m+1} = H_{m+1}(P_j)$$

where $D_s/D_{s-1} \approx E^\infty_{s,m+1-s}$ and $D_0 = \text{im}\,[H_{m+1}(F) \to H_{m+1}(P_j)]$. To show that $H_{m+1}(F) \to H_{m+1}(P_j)$ is a $\mathcal{C}$-epimorphism, it suffices to show that $E^\infty_{s,m+1-s} \in \mathcal{C}$ for $s > 0$. This will be so if $E^2_{s,m+1-s} \in \mathcal{C}$ for $s > 0$. However,

$$E^2_{s,m+1-s} \approx H_s(B) \otimes H_{m+1-s}(F) \oplus H_{s-1}(B) * H_{m+1-s}(F)$$

Since $\mathcal{C}$ is an acyclic Serre class, $H_s(B) \in \mathcal{C}$ for $s > 0$ [and, of course, $H_0(B) \approx \mathbf{Z}$]. Since, by the inductive hypothesis, $H_{m+1-s}(F) \in \mathcal{C}$ for $s > 0$, we see that $E^2_{s,m+1-s} \in \mathcal{C}$ for $s \geq 0$ because $\mathcal{C}$ is a ring of abelian groups.

To show that $H_{m+1}(F) \to H_{m+1}(P_j)$ is a $\mathcal{C}$-monomorphism, we have a sequence of homomorphisms

$$H_{m+1}(F) \approx E^2_{0,m+1} \to E^3_{0,m+1} \to \cdots \to E^\infty_{0,m+1} \approx D_0$$

and it suffices to prove ker $(E^r_{0,m+1} \to E^{r+1}_{0,m+1}) \in \mathcal{C}$ for $r \geq 2$. This is equivalent to showing that $d^r(E^r_{r,m+2-r}) \subset E^r_{0,m+1}$ is in $\mathcal{C}$ for $r \geq 2$. This will be true if $E^2_{r,m+2-r} \in \mathcal{C}$ for $r \geq 2$. However,

$$E^2_{r,m+2-r} \approx H_r(B) \otimes H_{m+2-r}(F) \oplus H_{r-1}(B) * H_{m+2-r}(F)$$

and because $m + 2 - r \leq m$, $H_{m+2-r}(F) \in \mathcal{C}$, by the inductive assumption. The result follows because $\mathcal{C}$ is an acyclic ring of abelian groups. ∎

We now have the following strengthened version of theorem 15.

20 THEOREM *Let X be a strongly simple space and let $\mathcal{C}$ be an acyclic ring of abelian groups. If $\pi_1(X) \in \mathcal{C}$, the following are equivalent:*

(*a*) $\pi_i(X) \in \mathcal{C}$ *for* $2 \leq i < n$.
(*b*) $H_i(X) \in \mathcal{C}$ *for* $2 \leq i < n$.

Either implies that $\varphi\colon \pi_i(X) \to H_i(X)$ *is a* $\mathcal{C}$*-isomorphism for* $i \leq n$.

PROOF It suffices to prove that (*b*) implies $\varphi\colon \pi_n(X) \underset{\mathcal{C}}{\approx} H_n(X)$. Let F be the fiber of the fibration $p\colon P_j \to B$. Since X and P_j have the same homotopy type, there is a map $f\colon F \to X$, equivalent to $F \subset P_j$. Since $\pi_i(F) \approx \pi_i(P_j)$ for $i \geq 2$, it follows that $f_\#\colon \pi_i(F) \approx \pi_i(X)$ for $i \geq 2$. By lemma 19, $f_*: H_i(F) \to H_i(X)$ is a $\mathcal{C}$-isomorphism for $i \leq n$. Since F is simply connected, it follows from theorem 15 that $\varphi\colon \pi_n(F) \underset{\mathcal{C}}{\approx} H_n(F)$. Since $\varphi \circ f_\# = f_* \circ \varphi$, this gives the result. ∎

We use this result to establish the following *generalized relative Hurewicz isomorphism theorem.*

21 THEOREM *Let* $\mathcal{C}$ *be an acyclic ideal of abelian groups, let* $A \subset X$, *and assume that A and X are simply connected. The following are equivalent:*

(*a*) $\pi_i(X,A) \in \mathcal{C}$ *for* $2 \leq i < n$.
(*b*) $H_i(X,A) \in \mathcal{C}$ *for* $2 \leq i < n$.

Either property implies

(*c*) $\varphi\colon \pi_n(X,A) \underset{\mathcal{C}}{\approx} H_n(X,A)$.

PROOF It suffices to prove that (*b*) implies (*c*) by induction on n. For $n = 2$ this follows from the relative Hurewicz isomorphism theorem. Therefore we assume $n \geq 3$ and $\pi_i(X,A) \in \mathcal{C}$ for $i < n$. Let $x_0 \in A$, let PX be the space of paths in X with origin x_0, and let $p\colon PX \to X$ be the fibration sending a path to its terminal point. The fiber $p^{-1}(x_0)$ is the loop space ΩX. By theorem 7.2.8, $p_\#\colon \pi_k(PX,p^{-1}(A)) \approx \pi_k(X,A)$ for $k \geq 1$. Because PX is contractible, $\pi_k(PX,p^{-1}(A)) \approx \pi_{k-1}(p^{-1}(A))$ for $k \geq 2$ and $H_k(PX,p^{-1}(A)) \approx H_{k-1}(p^{-1}(A))$ for $k \geq 2$. For $i \geq 2$ there is a commutative diagram

$$\begin{tikzcd}
\pi_{i-1}(p^{-1}(A)) \arrow[d, "\varphi"'] & \pi_i(PX,p^{-1}(A)) \arrow[l, "\partial"', "\approx"] \arrow[r, "p_\#", "\approx"'] \arrow[d, "\varphi"'] & \pi_i(X,A) \arrow[d, "\varphi"] \\
H_{i-1}(p^{-1}(A)) & H_i(PX,p^{-1}(A)) \arrow[l, "\partial"', "\approx"] \arrow[r, "p_*"] & H_i(X,A)
\end{tikzcd}$$

Applying theorem 10*b*, where $H_i(X,A) \in \mathcal{C}$ for $i < n$, and taking $m = 1$ (ΩX is path connected because X was assumed to be simply connected), we see that $p_*: H_i(PX,p^{-1}(A)) \underset{\mathcal{C}}{\approx} H_i(X,A)$ for $i \leq n$. Therefore, for $i = n$ all the horizontal maps in the above diagram are $\mathcal{C}$-isomorphisms, and to complete the proof it suffices to prove $\varphi\colon \pi_{n-1}(p^{-1}(A)) \underset{\mathcal{C}}{\approx} H_{n-1}(p^{-1}(A))$. This will follow from theorem 20 once we have verified that $p^{-1}(A)$ is strongly simple.

Because $p^{-1}(A)$ is a principal fibration with fiber ΩX, there is a continuous map $\Omega X \times p^{-1}(A) \to p^{-1}(A)$. Since $\pi_2(X) \to \pi_2(X,A)$ is an epimorphism, so is $\pi_1(\Omega X) \to \pi_1(p^{-1}(A))$. Therefore the existence of the map

$$\Omega X \times p^{-1}(A) \to p^{-1}(A)$$

implies that $p^{-1}(A)$ is strongly simple, as in example 18. ■

By using the mapping cylinder (as in the proof of theorem 7.5.9), the following *generalized Whitehead theorem* can be deduced from theorem 21.

22 THEOREM *Let $\mathcal{C}$ be an acyclic ideal of abelian groups and let $f\colon X \to Y$ be a map between simply connected spaces. For $n \geq 1$ the following are equivalent:*

(a) $f_\#\colon \pi_i(X) \to \pi_i(Y)$ is a $\mathcal{C}$-isomorphism for $i \leq n$ and a $\mathcal{C}$-epimorphism for $i = n + 1$.
(b) $f_*\colon H_i(X) \to H_i(Y)$ is a $\mathcal{C}$-isomorphism for $i \leq n$ and a $\mathcal{C}$-epimorphism for $i = n + 1$. ■

7 HOMOTOPY GROUPS OF SPHERES

The results of the last section were obtained by using the homology spectral sequence of a fibration. In this section we shall use the multiplicative properties of the cohomology spectral sequence to obtain some specific results about the homotopy groups of spheres. These homotopy groups are finitely generated, and we shall obtain information about their p-primary components. The first main result is that the only homotopy groups of S^n which are infinite are $\pi_n(S^n)$, and if n is even, $\pi_{2n-1}(S^n)$. The next main result concerns the double suspension. It will be shown that for odd n the double suspension

$$S^2\colon \pi_m(S^n) \to \pi_{m+2}(S^{n+2})$$

induces an isomorphism of the p-primary components of these groups for a wider range of values of m and n (depending on p) than the range for which it is an isomorphism between the groups. Combining this with specific computations of p-primary components of $\pi_m(S^3)$, we determine the lowest dimension $m > n$ for which $\pi_m(S^n)$ for n odd has a nontrivial p-primary component.

We begin with the following useful technical result about the cohomology spectral sequence of a fibration.

1 LEMMA *Let X be a simply connected space and assume that there is an element $u \in H^n(X;R)$, with $n \geq 2$, such that $u^{m-1} \neq 0$ for some $m \geq 2$ and $\{1,u,u^2, \ldots ,u^{m-1}\}$ form a basis for $H^*(X;R)$ in degrees $< mn$. Then there is an element $v \in H^{n-1}(\Omega X;R)$ such that $\{1,v\}$ form a basis for $H^*(\Omega X;R)$ in degrees $< mn - 2$.*

PROOF We use the spectral sequence of theorem 9.4.7, with A empty, for the fibration $PX \to X$. Because PX is contractible, $E_\infty^{s,t} = 0$ if $(s,t) \neq (0,0)$, and because X has no torsion in degrees $< mn$, $E_2^{s,t} \approx H^s(X) \otimes H^t(\Omega X)$ for $s < mn$ (all coefficients R). Then we have $E_2^{s,t} = 0$ if $s < mn$ and

$s \neq 0, n, 2n, \ldots, (m-1)n$. Because d_r has bidegree $(r, 1-r)$, it follows that for $s < mn$, $d_r\colon E_r^{s,t} \to E_r^{s+r,t-r+1}$ is zero unless $r = n, 2n, \ldots, (m-1)n$. Therefore $E_n^{s,t} \approx E_2^{s,t}$ for $s < mn$. If $t < n-1$, $E_n^{0,t} \approx E_\infty^{0,t}$, and if $0 < t < n-1$, we see that

$$H^t(\Omega X) \approx E_2^{0,t} \approx E_\infty^{0,t} = 0$$

and so $H^t(\Omega X) = 0$ for $0 < t < n-1$. Furthermore, there is an exact sequence

$$0 \to E_\infty^{0,n-1} \to H^0(X) \otimes H^{n-1}(\Omega X) \xrightarrow{d_n} H^n(X) \otimes H^0(\Omega X) \to E_\infty^{n,0} \to 0$$

Because $E_\infty^{0,n-1} = 0 = E_\infty^{n,0}$, it follows that there is an element $v \in H^{n-1}(\Omega X)$ such that $d_n(1 \otimes v) = u \otimes 1$. Because d_n is a derivation, $d_n(u^k \otimes v) = (-1)^{kn}u^{k+1} \otimes 1$. The assumption about the cohomology of X ensures that for $s < mn$ the map $d_n\colon E_n^{s-n,n-1} \to E_n^{s,0}$ is an isomorphism. Because d_n is a differential, the composite

$$E_n^{s-2n,2n-2} \xrightarrow{d_n} E_n^{s-n,n-1} \xrightarrow{d_n} E_n^{s,0}$$

is trivial. Therefore $d_n\colon E_n^{s-2t,2n-2} \to E_n^{s-n,n-1}$ is trivial for $s < mn$ and $E_{n+1}^{s-n,n-1} = 0 = E_{n+1}^{s,0}$ for $s < mn$. Hence

$$E_r^{s,t} = 0 \qquad s < mn,\ t \leq n-1,\ r \geq n+1$$
$$E_{n+1}^{0,2n-2} = E_2^{0,2n-2}$$

Assume the lemma false and let q be the smallest integer such that $n - 1 < q < mn - 2$ and $H^q(\Omega X) \neq 0$. We shall show that

$$E_\infty^{0,q} = E_2^{0,q} \approx H^q(\Omega X),$$

which is a contradiction. We know that $E_n^{0,q} \approx E_2^{0,q}$. Furthermore, $d_n\colon E_n^{0,q} \to E_n^{n,q-n+1}$ is trivial, because if $q-n+1 \neq n-1$, then $H^{q-n+1}(\Omega X)=0$ and $E_r^{s,q-n+1} = 0$ for all r and s, and if $q - n + 1 = n - 1$, then $E_{n+1}^{0,2n-2} = E_2^{0,2n-2}$. Therefore $E_{n+1}^{0,q} \approx E_n^{0,q}$. From the assumption that q is the smallest degree larger than $n - 1$ for which $H^q(\Omega X) \neq 0$, it follows that $E_r^{s,t} = 0$ if $s < mn$, $t < q$, and $r \geq n + 1$ (in case $t \leq n - 1$ this was noted above). Therefore $d_r\colon E_r^{0,q} \to E_r^{r,q-r+1}$ is trivial for all $r \geq n + 1$ and $E_\infty^{0,q} \approx E_{n+1}^{0,q}$. Hence we have the isomorphisms

$$E_\infty^{0,q} \approx E_{n+1}^{0,q} \approx E_n^{0,q} \approx E_2^{0,q} \quad \blacksquare$$

By using the generalized Gysin sequence of theorem 9.5.2, it is easy to show that if ΩX is a cohomology n-sphere for some odd $n \geq 1$, then $H^*(X)$ is a polynomial algebra on a generator of degree $n + 1$. The following converse is an immediate consequence of lemma 1 for the case $m = \infty$.

2 COROLLARY *Let X be a simply-connected space such that $H^*(X;R)$ is a polynomial algebra on a generator of degree n (n is then necessarily even). Then the loop space ΩX is a cohomology $(n-1)$-sphere.* $\blacksquare$

We shall also need the following consequence of the generalized Wang sequence of theorem 9.5.1.

3 LEMMA *Let X be a simply connected space which is a cohomology n-sphere for some odd $n > 1$. Then the cohomology ring $H^*(\Omega X)$ of its loop space ΩX has a basis consisting of elements $\{1,u_1,u_2, \ldots\}$ with degree $u_k = k(n-1)$ and $u_p \smile u_q = [(p+q)!/p!q!]u_{p+q}$.*

PROOF We use the Wang exact sequence of the fibration $PX \to X$. Because PX is contractible, the map

$$\theta: H^t(\Omega X) \to H^{t-n+1}(\Omega X)$$

is an isomorphism for $t \neq 0$. Define $u_k \in H^{k(n-1)}(\Omega X)$ for $k \geq 0$ by induction by the equations

$$u_0 = 1$$
$$\theta(u_k) = u_{k-1} \qquad k > 0$$

Then the set $\{1,u_1,u_2, \ldots\}$ is a basis for $H^*(\Omega X)$, and we verify that it has the stated multiplicative property by double induction on p and q. If $i = 0$ or $j = 0$, then $u_i \smile u_j = u_{i+j}$. Let $p > 0$ and $q > 0$ and assume that $u_i \smile u_j = [(i+j)!/i!j!]u_{i+j}$ if $i + j < p + q$, $i \geq 0$, and $j \geq 0$. Because n is odd,

$$\begin{aligned}\theta(u_p \smile u_q) &= \theta(u_p) \smile u_q + u_p \smile \theta(u_q) = u_{p-1} \smile u_q + u_p \smile u_{q-1} \\ &= \left[\frac{(p+q-1)!}{(p-1)!q!} + \frac{(p+q-1)!}{p!(q-1)!}\right] u_{p+q-1} = \frac{(p+q)!}{p!q!} u_{p+q-1}\end{aligned}$$

Because θ is a monomorphism,

$$u_p \smile u_q = \frac{(p+q)!}{p!q!} u_{p+q} \quad \blacksquare$$

It follows from lemma 3 that $(u_1)^p = p!u_p$. Over a field of characteristic 0, the elements $\{1,u_1,u_1^2, \ldots\}$ also form a basis of $H^*(\Omega X)$, and so we obtain the next result.

4 COROLLARY *Let X be a simply connected space which is a rational cohomology n-sphere for some odd $n > 1$. The rational cohomology algebra of the loop space ΩX is a polynomial algebra with one generator of degree $n - 1$.* $\blacksquare$

Let X be a space of type $(\mathbf{Z},3)$ and let $f: S^3 \to X$ be a map such that $f_\#: \pi_3(S^3) \approx \pi_3(X)$. Let $p: E \to S^3$ be the principal fibration induced by f. Then the fiber F of $p: E \to S^3$ is a space of type $(\mathbf{Z},2)$. We shall need the following computation of the homology groups of E.

5 LEMMA *Let $p: E \to S^3$ be a fibration with fiber F a space of type $(\mathbf{Z},2)$ such that $\partial: \pi_3(S^3) \approx \pi_2(F)$. Then the integral homology of E is given by*

$$H_q(E) \approx \begin{cases} 0 & q \text{ odd} \\ \mathbf{Z} & q = 0 \\ \mathbf{Z}_n & q = 2n > 0 \end{cases}$$

PROOF We know that $H^*(F)$ is a polynomial algebra with one generator u of degree 2. Because $\bar{\partial}$: $\pi_3(S^3) \approx \pi_2(F)$, it follows that E is 3-connected, and so $H^2(E) = 0$. By the exact Wang sequence of the fibration, θ: $H^2(F) \approx H^0(F)$. Without loss of generality, we can assume that u has been chosen so that $\theta(u) = 1$. Then $\theta(u^n) = nu^{n-1}$. By the exact Wang sequence again, $H^q(E) = 0$ if q is even and $q < 0$, and $H^{2n+1}(E) \approx \mathbf{Z}_n$ if $n \geq 1$. The result then follows from the universal-coefficient theorem. ■

If $\mathcal{C}$ is the Serre class of groups having no element with order a positive power of a given prime p, then $H_i(E) \in \mathcal{C}$ for $0 < i < 2p$. By theorem 9.6.15, $\pi_i(E) \in \mathcal{C}$ for $i < 2p$ and $\pi_{2p}(E) \underset{\mathcal{C}}{\approx} \mathbf{Z}_p$. Because $\pi_i(E) \approx \pi_i(S^3)$ for $i > 3$, we obtain the following result.

6 **COROLLARY** *The p-primary component of $\pi_i(S^3)$ is zero if $3 < i < 2p$ and is $\mathbf{Z}_p$ if $i = 2p$.* ■

We are now ready to prove the finiteness of the higher homotopy groups of odd-dimensional spheres.

7 **THEOREM** *If n is odd, $\pi_m(S^n)$ is finite for $m \neq n$.*

PROOF We use induction on n. If $n = 1$, we know that $\pi_m(S^1) = 0$ if $m \neq 1$, and the result is valid in this case. For $n = 3$, if E is the space of lemma 5 and $\mathcal{C}$ is the Serre class of finite groups, then E is $\mathcal{C}$-acyclic. By theorem 9.6.15, $\pi_i(E) \in \mathcal{C}$ for all i. Because $\pi_i(E) \approx \pi_i(S^3)$ for $i > 3$, $\pi_i(S^3)$ is finite for $i > 3$.

Assume $n > 3$ and $\pi_m(S^{n-2})$ finite for $m \neq n - 2$. We compute the rational cohomology algebras of ΩS^n and $\Omega^2 S^n$. By corollary 4, that of ΩS^n is a polynomial algebra with one generator of degree $n - 1$. By corollary 2, $\Omega^2 S^n$ is a rational cohomology $(n - 2)$-sphere. By the universal-coefficient theorem, the integral group $H_i(\Omega^2 S^n)$ is a torsion group if $0 < i \neq n - 2$, and $H_{n-2}(\Omega^2 S^n)$ is isomorphic to a direct sum of $\mathbf{Z}$ and a torsion group. Furthermore, $\pi_k(\Omega^2 S^n) \approx \pi_{k+2}(S^n)$ for all k. Therefore $\Omega^2 S^n$ is $(n - 3)$-connected and φ: $\pi_{n-2}(\Omega^2 S^n) \approx H_{n-2}(\Omega^2 S^n)$. If α: $S^{n-2} \to \Omega^2 S^n$ is a generator and $\mathcal{C}$ is the Serre class of torsion groups, it follows that α_*: $H_i(S^{n-2}) \underset{\mathcal{C}}{\approx} H_i(\Omega^2 S^n)$ for all i. Because $\mathcal{C}$ is an acyclic ideal of abelian groups and S^{n+2} and $\Omega^2 S^n$ are both simply connected, we can apply the generalized Whitehead theorem 9.6.22 to deduce that $\alpha_\#$: $\pi_i(S^{n-2}) \underset{\mathcal{C}}{\approx} \pi_i(\Omega^2 S^n)$. By the inductive assumption $\pi_i(S^{n-2})$ is finite for $i > n - 2$. Therefore $\pi_i(\Omega^2 S^n) \underset{\mathcal{C}}{\approx} \pi_{i+2}(S^n)$ is a torsion group for $i > n - 2$. Because $\pi_m(S^n)$ is known to be finitely generated, $\pi_m(S^n)$ is finite for $m \neq n$. ■

We want to establish a result similar to theorem 7 for even-dimensional spheres. This will be done by considering a suitable $(n - 1)$-sphere bundle over S^n. Let W^{2n-1} be the subspace of $R^n \times R^n$ consisting of pairs of unit vectors (z_1,z_2) which are orthogonal and let p: $W^{2n-1} \to S^n$ map (z_1,z_2) to z_1. Then p: $W^{2n-1} \to S^n$ is a fiber bundle with fiber S^{n-1} (it is the unit tangent bundle of S^n), as can be verified by constructing an explicit homeomorphism $p^{-1}(U) \approx U \times S^{n-1}$ for any proper open subset $U \subset S^n$.

8 LEMMA *If n is even, the integral homology groups of W^{2n-1} are all finite except for $H_0(W^{2n-1})$ and $H_{2n-1}(W^{2n-1})$, which are infinite cyclic.*

PROOF Because n is even, there is no map $f\colon S^n \to S^n$ which sends each point of S^n to an orthogonal point of S^n (by corollary 4.7.11). It follows that $p\colon W^{2n-1} \to S^n$ has no section. If $[\alpha] \in \pi_n(S^n)$ is a generator, $[\alpha]$ is not in the image of

$$p_\#\colon \pi_n(W^{2n-1}) \to \pi_n(S^n)$$

(because there is no section). Therefore, $\bar\partial[\alpha] \neq 0$ in $\pi_{n-1}(S^{n-1})$. Because $\pi_{n-1}(S^{n-1})$ is infinite cyclic, $\bar\partial\colon \pi_n(S^n) \to \pi_{n-1}(S^{n-1})$ is a monomorphism and $\pi_{n-1}(W^{2n-1}) \approx \pi_{n-1}(S^{n-1})/\bar\partial(\pi_n(S^n))$ is a finite group. Because W^{2n-1} is $(n-2)$-connected, $\pi_{n-1}(W^{2n-1}) \approx H_{n-1}(W^{2n-1})$. Therefore $H_{n-1}(W^{2n-1})$ is a finite group, and by the exact Wang sequence

$$0 \to H_n(W^{2n-1}) \to H_0(S^{n-1}) \xrightarrow{\theta} H_{n-1}(S^{n-1}) \to H_{n-1}(W^{2n-1}) \to 0$$

we see that $H_n(W^{2n-1}) = 0$. From this exact sequence it also follows that $H_i(W^{2n-1}) = 0$ for $n < i < 2n-1$ and $H_{2n-1}(W^{2n-1})$ is infinite cyclic. ∎

9 THEOREM *If n is even, $\pi_m(S^n)$ is finite for $m \neq n$* and *$m \neq 2n-1$, and $\pi_{2n-1}(S^n)$ is the direct sum of an infinite cyclic group and a finite group.*

PROOF Let $\mathcal{C}$ be the acyclic ideal of abelian groups consisting of the torsion groups. By lemma 8, $H_i(W^{2n-1}) \in \mathcal{C}$ for $0 < i < 2n-1$. By theorem 9.6.15, $\pi_{2n-1}(W^{2n-1}) \underset{\mathcal{C}}{\approx} H_{2n-1}(W^{2n-1})$. Because $\pi_{2n-1}(W^{2n-1})$ is finitely generated (by corollary 9.6.16) and $H_{2n-1}(W^{2n-1})$ is infinite cyclic, $\pi_{2n-1}(W^{2n-1})$ is a direct sum of an infinite cyclic group and a finite group. If $\alpha\colon S^{2n-1} \to W^{2n-1}$ represents a generator of the infinite cyclic summand of $\pi_{2n-1}(W^{2n-1})$, then $\alpha_*\colon H_i(S^{2n-1}) \underset{\mathcal{C}}{\approx} H_i(W^{2n-1})$ for all i. By the generalized Whitehead theorem, $\alpha_\#\colon \pi_i(S^{2n-1}) \underset{\mathcal{C}}{\approx} \pi_i(W^{2n-1})$ for all i. Using this and theorem 7, $\pi_i(W^{2n-1})$ is finite for $i \neq 2n-1$. The theorem now follows from the exact homotopy sequence of the fibration $W^{2n-1} \to S^n$ and the fact that, by theorem 7, $\pi_i(S^{n-1})$ is finite for $i \neq n-1$. ∎

We now consider the double suspension

$$\pi_i(S^n) \xrightarrow{S} \pi_{i+1}(S^{n+1}) \xrightarrow{S} \pi_{i+2}(S^{n+2})$$

where n is odd. This involves a study of the composite

$$S^n \xrightarrow{\rho} \Omega S^{n+1} \xrightarrow{\Omega\rho} \Omega(\Omega S^{n+2})$$

We begin with the following partial computation of the $\mathbf{Z}_p$ homology of $\Omega^2 S^{n+2}$.

10 LEMMA *Let n be odd and p be prime. Then $H_q(\Omega^2 S^{n+2};\mathbf{Z}_p) = 0$ for $n < q < p(n+1) - 2$.*

PROOF By lemma 3, the set of elements $\{1,u_1,u_1^2, \ldots ,u_1^{p-1}\}$ forms a basis for $H^*(\Omega S^{n+2};\mathbf{Z}_p)$ in degrees $< p(n+1)$. By lemma 1, there is an element $v \in H^n(\Omega^2 S^{n+2};\mathbf{Z}_p)$ such that $\{1,v\}$ forms a basis for $H^*(\Omega^2 S^{n+2};\mathbf{Z}_p)$ in degrees $< p(n+1) - 2$. The lemma then follows by the universal-coefficient theorem. ■

This implies the following result about the double suspension.

11 THEOREM *Let n be an odd integer, p a prime, and $\mathcal{C}$ the acyclic ideal of torsion groups with trivial p-primary component. Then*

$$S^2\colon \pi_i(S_n) \to \pi_{i+2}(S^{n+2})$$

is a $\mathcal{C}$-isomorphism for $i < p(n+1) - 3$ and a $\mathcal{C}$-epimorphism for $i = p(n+1) - 3$.

PROOF The composite

$$S^n \xrightarrow{\rho} \Omega S^{n+1} \xrightarrow{\Omega\rho} \Omega^2 S^{n+2}$$

induces an isomorphism of $\pi_n(S^n)$ with $\pi_n(\Omega^2 S^{n+2})$ and, by the Whitehead theorem, an isomorphism of $H_n(S^n)$ with $H_n(\Omega^2 S^{n+2})$. From this and lemma 10 it follows that the above composite induces an isomorphism of $H_q(S^n;\mathbf{Z}_p)$, with $H_q(\Omega^2 S^{n+2};\mathbf{Z}_p)$ for $q \le p(n+1) - 3$. By the universal-coefficient theorem, it induces a $\mathcal{C}$-isomorphism of $H_q(S^n)$, with $H_q(\Omega^2 S^{n+2})$ for $q \le p(n+1) - 3$. From the generalized Whitehead theorem, it induces a $\mathcal{C}$-isomorphism of $\pi_q(S^n)$ with $\pi_q(\Omega^2 S^{n+2})$ for $q < p(n+1) - 3$, and a $\mathcal{C}$-epimorphism of $\pi_{p(n+1)-3}(S^n)$ to $\pi_{p(n+1)-3}(\Omega^2 S^{n+2})$. The theorem follows from the fact that S^2 corresponds to the above induced homomorphism under the isomorphism

$$\pi_q(\Omega^2 S^{n+2}) \approx \pi_{q+2}(S^{n+2}) \quad ■$$

12 COROLLARY *Let $n \ge 3$ be odd and p prime. Then $\pi_i(S^n)$ and $\pi_{i-n+3}(S^3)$ have isomorphic p-primary components if $i < 4p + n - 6$.*

PROOF We use induction on n. If $n = 3$, there is nothing to be proved. If $n \ge 5$, we need only prove that $S^2\colon \pi_{i-2}(S^{n-2}) \to \pi_i(S^n)$ induces an isomorphism of p-primary components. By theorem 11, this will be true if $i - 2 < p(n-1) - 3$. Hence we need only verify that

$$4p + n - 6 \le p(n-1) - 1$$

But this is equivalent to $(p-1)(n-5) \ge 0$. ■

Combining corollary 12 with corollary 6, we have the following result.

13 COROLLARY *Let $n \ge 3$ be odd and p prime. For $0 < m < 2p - 3$, $\pi_{n+m}(S^n)$ has trivial p-primary component and $\pi_{n+2p-3}(S^n)$ has $\mathbf{Z}_p$ as p-primary component.* ■

EXERCISES

A SPECTRAL SEQUENCES AND SUSPENSION

In this group of exercises all spaces will be assumed to be finite pointed CW complexes and all pairs will be finite pointed CW pairs.

1 Prove that $\{X;Y\}$ is finitely generated.

2 For spaces X and Y prove that there is a convergent E^2 spectral sequence $\{E^r\}$ with

$$E^2_{s,t} \approx H_s(Y; \{X;S^0\}_t)$$

and with E^∞ isomorphic to the graded group associated to the increasing filtration on $\{X;Y\}_*$ defined by

$$F_s\{X;Y\}_* = \text{im}\,(\{X;Y^s\}_* \to \{X;Y\}_*)$$

3 For spaces X and Y prove that there is a convergent E_2 spectral sequence $\{E_r\}$ with

$$E_2^{s,t} \approx H^s(X; \{S^0;Y\}_{-t})$$

and with E_∞ isomorphic to the graded group associated to the decreasing filtration of $\{X;Y\}_*$ defined by

$$F^s\{X;Y\}_* = \ker\,(\{X;Y\}_* \to \{X^{s-1};Y\}_*)$$

B THE TRANSGRESSION HOMOMORPHISM

Let $p\colon E \to B$ be a fibration with path-connected base and path-connected fiber $F = p^{-1}(b_0)$. Consider the homomorphisms

$$H^q(F;G) \xrightarrow{\delta} H^{q+1}(E,F;\, G) \xleftarrow{p^*} H^{q+1}(B,b_0;\, G) \xrightarrow{j^*} H^{q+1}(B;G)$$

The *transgression* τ is the homomorphism [from a subgroup of $H^q(F;G)$ to a quotient group of $H^{q+1}(B;G)$]

$$\tau\colon \delta^{-1}(\text{im}\, p^*) \to H^{q+1}(B;\, G)/j^*(\ker p^*)$$

defined by $\tau(u) = j^*p^{*\,-1}\delta(u)$, where $u \in H^q(F;G)$ is such that $\delta(u) \in p^*(H^{q+1}(B,b_0;\, G))$.

1 Prove that τ commutes with the Steenrod squaring operations Sq^i and with induced homomorphisms for induced fibrations.

2 Assume that B is $(n-1)$-connected for $n \geq 2$ and consider the path fibration $p\colon PB \to B$ with fiber ΩB. Prove that $\tau\colon H^{n-1}(\Omega B;G) \approx H^n(B;G)$ and that $\iota \in H^{n-1}(\Omega B;G)$ is $(n-1)$-characteristic for ΩB if and only if $\tau(\iota)$ is n-characteristic for B.

For the remainder of this group of exercises we assume that the fibration is orientable over R and the coefficient module is R.

3 For the spectral sequence of the fibration prove the following:

(*a*) $\delta^{-1}(\text{im}\, p^*) \approx E_{q+1}^{0,q} \subset E_2^{0,q} \approx H^q(F)$.
(*b*) $H^{q+1}(B)/j^*(\ker p^*) \approx E_{q+1}^{q+1,0}$ and is a quotient of $E_2^{q+1,0} \approx H^{q+1}(B;G)$.
(*c*) Under these isomorphisms τ corresponds to $d_{q+1}\colon E_{q+1}^{0,q} \to E_{q+1}^{q+1,0}$.

4 If $\tilde H^*(E) = 0$, prove that $H^i(F) = 0$ for $0 < i < q$ if and only if $H^i(B) = 0$ for $0 < i < q+1$ and, in this case, $\tau\colon H^q(F) \approx H^{q+1}(B)$.

5 Assume that $H^i(B) = 0$ for $0 < i < s$ and $H^j(F) = 0$ for $0 < j < t$. Prove the exactness of the following *Serre cohomology sequence:*

$$\cdots \to H^q(F) \xrightarrow{\tau} H^{q+1}(B) \xrightarrow{p^*} H^{q+1}(E) \xrightarrow{i^*} H^{q+1}(F) \to \cdots \to H^{s+t-1}(F)$$

Consider the homomorphisms

$$H_q(B) \xrightarrow{j_*} H_q(B,b_0) \xleftarrow{p_*} H_q(E,F) \xrightarrow{\partial} H_{q-1}(F)$$

and define the *homology transgression*

$$\tau_* : j_*{}^{-1}(\text{im } p_*) \to H_{q-1}(F)/\partial(\ker p_*)$$

by $\tau_*(z) = \partial p_*{}^{-1} j_*(z)$, where $z \in H_q(B)$ is such that $j_*(z) \in p_*(H_q(E,F))$.

6 If $H_i(B) = 0$ for $0 < i < s$ and $H_j(F) = 0$ for $0 < j < t$, prove the exactness of the following *Serre homology sequence:*

$$H_{s+t-1}(F) \to \cdots \to H_q(F) \xrightarrow{i_*} H_q(E) \xrightarrow{p_*} H_q(B) \xrightarrow{\tau_*} H_{q-1}(F) \to \cdots$$

C SERRE CLASSES OF ABELIAN GROUPS

A *chain complex modulo* $\mathcal{C}$ is a graded group $C = \{C_q\}$ and a sequence of homomorphisms $\{\partial_q: C_q \to C_{q-1}\}$ such that $(\partial_{q-1} \circ \partial_q)(C_q) \in \mathcal{C}$ for all q. The *homology group of* C is the graded group $H(C) = \{H_q(C)\}$, where

$$H_q(C) = \ker \partial_q/(\ker \partial_q \cap \text{im } \partial_{q+1}) \approx (\ker \partial_q \cup \text{im } \partial_{q+1})/\text{im } \partial_{q+1}$$

A three-term sequence of groups and homomorphisms

$$G' \xrightarrow{\alpha} G \xrightarrow{\beta} G''$$

is said to be $\mathcal{C}$*-exact* if $(\text{im } \alpha \cup \ker \beta)/\text{im } \alpha \in \mathcal{C}$ and if $(\text{im } \alpha \cup \ker \beta)/\ker \beta \in \mathcal{C}$. Longer sequences are $\mathcal{C}$*-exact* if every three-term sequence is $\mathcal{C}$-exact.

1 Let C be a chain complex modulo $\mathcal{C}$, let C' be a subcomplex of C (that is, $C'_q \subset C_q$ and $\partial'_q = \partial_q \mid C'_q$ for all q), and define the quotient complex $C/C' = \{C_q/C'_q, \partial''_q\}$, where ∂''_q is induced from ∂_q. Prove that there is a $\mathcal{C}$-exact sequence

$$\cdots \to H_q(C') \to H_q(C) \to H_q(C/C') \to H_{q-1}(C') \to \cdots$$

2 Let $0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$ be a short $\mathcal{C}$-exact sequence of chain complexes modulo $\mathcal{C}$ and chain maps (α and β commute with the boundary homomorphisms of the chain complexes). Prove that there is a $\mathcal{C}$-exact sequence

$$\cdots \to H_q(C') \xrightarrow{\alpha_*} H_q(C) \xrightarrow{\beta_*} H_q(C'') \xrightarrow{\partial_*} H_{q-1}(C') \to \cdots$$

3 Prove the *five lemma modulo* $\mathcal{C}$. That is, given a commutative diagram

$$\begin{array}{ccccccccc}
G_5 & \xrightarrow{\alpha_5} & G_4 & \xrightarrow{\alpha_4} & G_3 & \xrightarrow{\alpha_3} & G_2 & \xrightarrow{\alpha_2} & G_1 \\
\downarrow{\scriptstyle\gamma_5} & & \downarrow{\scriptstyle\gamma_4} & & \downarrow{\scriptstyle\gamma_3} & & \downarrow{\scriptstyle\gamma_2} & & \downarrow{\scriptstyle\gamma_1} \\
H_5 & \xrightarrow{\beta_5} & H_4 & \xrightarrow{\beta_4} & H_3 & \xrightarrow{\beta_3} & H_2 & \xrightarrow{\beta_2} & H_1
\end{array}$$

with $\mathcal{C}$-exact rows such that γ_1, γ_2, γ_4, and γ_5 are $\mathcal{C}$-isomorphisms, prove that γ_3 is also a $\mathcal{C}$-isomorphism.

For the rest of this group of exercises assume that $p: E \to B$ is a fibration with path-connected fiber and simply connected base space and that $\mathcal{C}$ is an ideal of abelian groups.

4 If $H_i(B) \in \mathcal{C}$ for $0 < i$, prove that $H_i(F) \underset{\mathcal{C}}{\approx} H_i(E)$ for all i.

5 *Vietoris-Begle mapping theorem modulo* $\mathcal{C}$. If F is $\mathcal{C}$-acyclic, prove that $H_i(E) \underset{\mathcal{C}}{\approx} H_i(B)$ for all i.

D HOMOTOPY GROUPS OF SPHERES

1 If S^n is an H-space, prove that there is a short exact sequence

$$0 \to \pi_q(S^n) \xrightarrow{S} \pi_{q+1}(S^{n+1}) \xrightarrow{H} \pi_{q-1}(S^{2n-1}) \to 0 \qquad q \leq 3m - 2$$

2 Prove that $\pi_7(S^4) \approx \pi_6(S^3) \oplus \pi_7(S^7)$ and that the order of $\pi_8(S^5)$ is twice the order of $\pi_6(S^3)$.

3 Let X_p^n be a CW complex consisting of an n-sphere with an $(n+1)$-cell attached by a map of degree p. If $n \geq 2$ and p is a prime, prove that $\pi_q(X_p^n)$ is a finite p-group for all q, and if $q \leq 2n - 2$, prove there is an exact sequence

$$0 \to \pi_q(S^n) \otimes \mathbf{Z}_p \to \pi_q(X_p^n) \to \pi_{q-1}(S^n) * \mathbf{Z}_p \to 0$$

4 Prove that $S(X_p^n)$ has the same homotopy type as X_p^{n+1} and $\{X_p^n;X_p^n\} \approx \mathbf{Z}_p$ if $p \neq 2$ and $\{X_2^n;X_2^n\}$ is a group of order 4.

5 Let $p\colon E \to S^3$ be a fibration, with fiber F a space of type $(\mathbf{Z},2)$, such that $\partial\colon \pi_3(S^3) \approx \pi_2(F)$, as in lemma 9.7.5. Let $f\colon X_2^4 \to E$ be a map such that $f_\#\colon \pi_4(X_2^4) \approx \pi_4(E)$ [such a map exists, because $\pi_4(E) \approx \mathbf{Z}_2$]. Prove that $f_\#\colon \pi_5(X_2^4) \approx \pi_5(E)$ and $f_\#$ is a monomorphism of $\pi_6(X_2^4)$ onto the 2-primary component of $\pi_6(E)$. [*Hint:* Show that $f_*\colon H_q(X_2^4) \to H_q(E)$ is an isomorphism of 2-primary components for $q < 8$ and use the generalized Whitehead theorem.]

6 Prove that $\pi_{n+2}(S^n) \approx \mathbf{Z}_2$ for $n \geq 2$.

7 Prove the following:

(*a*) $\pi_5(S^2) \approx \mathbf{Z}_2$.
(*b*) $\pi_6(S^3)$ is a group of order 12.
(*c*) $\pi_7(S^4) \approx \pi_6(S^3) \oplus \mathbf{Z}$.
(*d*) $\pi_{n+3}(S^n)$ is a group of order 24 for $n \geq 5$.

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