Year - 2016 - 7(a)

Given :

x	-2	-1	0	1	2	3
$p(x)$	0.1	$2k$	0.2	k	0.3	k

As x is a random variable and the function ~~is follows~~ is probability function, so

$$\sum p(x) = 1$$

$$\Rightarrow 0.1 + 2K + 0.2 + K + 0.3 + K = 1$$

$$\Rightarrow K = 0.1$$

So,

x	-2	-1	0	1	2	3
$p(x)$	0.1	0.2	0.2	0.1	0.3	0.1

We know,

mean $= E(x)$ and variance, $\sigma^2 = E(x^2) - \{E(x)\}^2$

Now, $E(x) = \sum x \cdot p(x)$

$$= 0.6$$

$$E(x^2) = \sum x^2 \cdot p(x)$$

$$= 2.8$$

$$\therefore \sigma^2 = 2.8 - (0.6)^2 = 2.44$$

7.(b)

Given $f(x) = \begin{cases} \frac{1}{18}(3+2x) & 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$

Here x is a continuous random variable and

$f(x) \geq 0$.

Now,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{2}^{4} \frac{1}{18}(3+2x)\,dx = \left[\frac{3x}{18} + \frac{x^2}{18}\right]_2^4$$

$$= 1.$$

$\therefore$ Total $f(x) = 1$.

$\therefore$ $f(x)$ is a density function.

The mean of distribution, $\mu_1' = E(x)$ ~~$\sum x\,p(x)$~~

$$= \int_{-\infty}^{\infty} x.\, f(x)\,dx$$

$$= \int_{2}^{4} x.\, \frac{1}{18}(3+2x)\,dx$$

$$= \int_{2}^{4} \left(\frac{x}{6} + \frac{x^2}{9}\right) dx$$

$$= \left[\frac{x^2}{12} + \frac{x^3}{27}\right]_2^4$$

$$= \frac{83}{27}$$

Standard deviation, $\sigma = \sqrt{E(x^2) - \{E(x)\}^2}$

$$E(x^2) = \int_{-\alpha}^{\alpha} x^2 . f(x)\, dx$$

$$= \int_{2}^{4} \left(\frac{x^2}{6} + \frac{x^3}{9}\right) dx$$

$$= \left[\frac{x^3}{6.3} + \frac{x^4}{4.9}\right]_2^4$$

$$= \frac{88}{9}$$

$\therefore \sigma = 0.5726$

mean deviation, $= \int_{-\alpha}^{\alpha} (x - \bar{x})\, f(x)\, dx$

$$= \int_{2}^{4} \left(x - \frac{83}{27}\right) \frac{1}{18} (3 + 2x)\, dx$$

$$= 0$$

(c) Mathematical expectation is the summation or integration of a possible values from a random variable. It's also known as the product of the probability of an event occuring, which is denoted by $P(x)$ and the value corresponding with the actual observed occurance of the event.

The expected value is denoted by $E(x)$.

The formula for mathematical expectation,

$$E(x) = \Sigma(x_1 p_1, x_2 p_2, \ldots x_n p_n)$$

$$= \Sigma x\, p(x)$$

$$E[g(x)] = \Sigma\, g(x) \cdot p(x)$$

The expectation of constant is constant.

$$E(C) = \Sigma\, c \cdot p(x) = c \cdot \Sigma\, p(x) = c.$$

The mean, variance, co-efficient of skewness and co-efficient of Kurtosis can also be obtained from the expectation.

After one toss :

x	1	2	3	4	5	6
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

on the first toss, the probability of getting 4, 5, or 6,

$$P(4,5,6) = P(4) + P(5) + P(6) = 3 \cdot \frac{1}{6} \text{ or } \frac{1}{2}$$

After the second toss the probability of getting 1, 2, 3, or 4,

$$P(1,2,3,4) = \frac{4}{6}$$

As the events are independent, so to get overall probability of tossing a die twice, we have to multiply individual probabilities,

$$P(4,5,6) \times P(1,2,3,4) = \frac{3}{6} \times \frac{4}{6} = \frac{12}{36} = 1/3$$

Year - 2016 - 8(a)

The Problems can be modelled as normal distribution, that is, $N(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

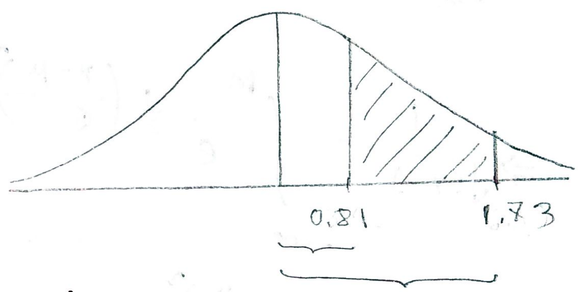
Here, given, $\mu = 1$ and $\sigma = 3$

Let,

$$z = \frac{x-\mu}{\sigma}$$

$$= \frac{x-1}{3}$$

(i) $P(3.43 \le x \le 6.19)$

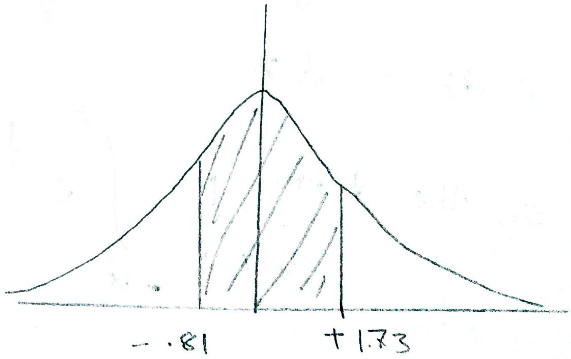


$z_{3.43} = 0.81$

$z_{6.19} = 1.73$

$$P(3.43 \le x \le 6.19) = P(0.81 \le z \le 1.73)$$
$$= P(0 \le z \le 1.73) - P(0 \le z \le 0.81)$$
$$= 0.4582 - 0.2910$$
$$= 0.1672$$

(ii) $P(-1.43 \le x \le 6.19)$



$z_{-1.43} = -0.81$

$z_{6.19} = 1.73$

$$P(-1.43 \le x \le 6.19) = P(-0.81 \le z \le 1.73)$$
$$= P(0 \le z \le .81) + P(0 \le z \le 1.73)$$
$$= 0.7492$$

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8(b)

We have to show that mean and median coincide.

Suppose μ is the mean and M is the median of normal distribution.

As M is the median, that means it's dividing the distribution equally into two parts.

$$\int_{-\infty}^{M} f(x)\,dx = \frac{1}{2}$$ where $f(x)$ is the probability density function, and x is a normally distributed variable.

$$\Rightarrow \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\mu}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$ [if μ lies below M]

Let, $z = \frac{x-\mu}{\sigma}$

x	$-\infty$	μ
z	$-\infty$	0

Note: for a standard normal variable, $\int_{-\infty}^{\mu} (.) = \frac{1}{2}$

$\sigma\,dz = dx$

So, the term, $$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{0} e^{-\frac{z^2}{2}} \sigma\,dz$$

$$= \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} \int_{0}^{\infty} e^{-z^2/2}\, \cancel{\sigma}\,dz$$ [as normal variable]

$$= \frac{1}{2}$$

So,

$$\frac{1}{2} + \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \int_{\mu}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

$$\therefore \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\mu}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 0$$

which means $M = \mu$, these limits are equal.

So for a normal ~~variable~~, ~~mea~~ distribution, mean and median are the same.

8.(c)

[For properties check - 2018.3(a)]

i) The relation between Binomial and Poisson distribution:

The Binomial distribution tends towards the poisson distribution as : $n \to \infty$, $p \to 0$ and $\lambda = np$ stays constant.

We know, Binomial:

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

the mean of Binomial distribution, np.

Suppose, $\lambda = np$; where λ is the mean of Poisson distribution.

$$\Rightarrow p = \frac{\lambda}{n}$$

$$P(X=x) = \binom{n}{x}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

As n gets larger ($n \to \infty$), the Binomial formula tends toward the Poisson formula, that is,

$$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

So this means the poisson distribution can be used to provide a reasonable approximation to the binomial distribution if n is larger and p is small.

For example:

In a random sample of 1000 people, what is the probability ~~of~~ that exactly 2 have albinism. Albinism affects one in 20,000 people.

Binomial: $n = 1000$, $p = \frac{1}{20000}$

$$P(X=2) = \binom{1000}{2}\left(\frac{1}{20000}\right)^2\left(1-\frac{1}{20000}\right)^{1000-2} = 0.001187965$$

Poisson:

$$\lambda = np = 1000 \cdot \frac{1}{20000} = 0.05$$

$$P(X=2) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{(0.05)^2 e^{-0.05}}{2!} = 0.001189037$$

As n is large and p is small, the both probabilities are quite equal.

The benifits of using Poisson instead Binomial:

1. The factorials and expotentials can be problematic to calcutlate.

2. In Binomial 2 parameters are need to be known, but it poisson we only have to know the mean.

(ii) Relation between Poisson and Normal distribution

If x is a continuous ~~den~~ random variable and $f(x)$ is probability density function and the probability distribution is symmetric about mean, then this distribution is called Normal distribution. It's presented like,

$$N(x;\mu\,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; -\alpha \leq x \leq \alpha$$

Here μ is mean and σ is standard deviation.

Poisson distribution is a discrete distribution ~~aa~~ with the random variable $x \geq 0$.

When the mean of poisson distribution $(\lambda = np)$ is large, it becomes similar to normal distribution. When the mean and variance are same, the normal distribution can be used to approximate poisson distribution.

Year - 2017 - 5 (a)

As the given data is grouped data, so, we know,

$$\mu_r' = \frac{\Sigma f_i (x_i - A)^r}{N}$$

As the observations have to be from origin, so $A = 0$

$$\therefore \mu_r' = \frac{\Sigma f_i \, x_i^r}{N}$$

class interval	f_i	x_i	x_i^2	x_i^3	x_i^4	$f_i x_i$	$f_i x_i^2$	$f_i x_i^3$	$f_i x_i^4$
5 - 7	2	6	36	216	1296	12	72	432	2592
8 - 10	5	9	81	729	6561	45	405	3645	32805
11 - 13	10	12	144	1728	20736	120	1440	17280	207360
14 - 16	3	15	225	3375	50625	45	675	10125	151875
	$\Sigma f_i = 20$	~~Σ = 42~~	~~486~~	~~[illegible]~~	~~[illegible]~~	$\Sigma = 222$	2592	31482	394632

Here $N = 20$

$$\therefore \mu_1' = \frac{\Sigma f_i x_i}{20} = \frac{222}{20} = 11.1$$

$\mu_2' = 129.6$

$\mu_3' = 1574.1$

$\mu_4' = 19731.6$

So these are the four moments.

We know,

$$\text{central moment, } \mu_r = \frac{\sum f_i (x_i - \bar{x})^r}{N}$$

Here, $\bar{x} = \mu_1' = 11.1$

Now, $$\mu_1 = \frac{\sum f_i (x_i - \bar{x})^1}{N}$$

$$= 0$$

$$\mu_2 = \frac{\sum f_i (x_i - \bar{x})^2}{N}$$

$$= 6.39$$

$$\mu_3 = \frac{\sum f_i (x_i - \bar{x})^3}{N}$$

$$= -126.36/20 = -6.318$$

$$\cancel{\mu_4 = \frac{\sum f_i (x_i - \bar{x})^4}{N}}$$

$$\cancel{= 107.5437}$$

$$\beta_1 = \frac{\mu_3}{\sqrt{\mu_2^3}}$$

$$= -0.39$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} \cancel{= 2.634}$$

So the mean is 11.1, variance ~~6.39~~ 6.39 and co-efficient of skewness ~~-6.318~~ -0.39

[এখানে Hence বলা নেই, তাই $\mu_2, \mu_1, \mu_3, \mu_4$ normal equation দিয়ে বের করছি, μ_r' দিয়ে না]

~~Bar diagraph~~ :

Bar diagram :

[Bar diagram এ method এ আমার ভুল থেকে পারে, I'm not sure]

5. b) Here the highest number of cars, $n = 4$

Since p is not given, we have to estimate p.

we know,

$$\bar{x} = \hat{\mu} = n\hat{p}$$

$$\Rightarrow \frac{\sum f_i x_i}{\sum f_i} = n\hat{p}$$

$$\Rightarrow \frac{111}{62} = 4\hat{p}$$

$$\therefore \hat{p} = 0.448$$

So. $\hat{q} = 1 - \hat{p} = 0.552$

So, the Binomial distribution model would be,

$$P(x) = {}^{4}C_{x} \, (0.448)^{x} \cdot (0.552)^{4-x}$$

No. of car/min	f_i	$p(x_i)$	O_i^2	E_i	O_i^2/E_i
0	10	0.093	100	≈ 6	16.67
1	15	0.301	225	≈ 19	11.84
2	20	0.367	400	≈ 23	17.39
3	12	0.199	144	≈ 12	12
4	5	0.0403	25	≈ 3	8.33
	N=62				66.23

$\therefore \chi^2_{cal} = 66.23 - 62 = 4.23$

Here, $\gamma = m - k - 1 = 5 - 1 - 1 = 3$

$\alpha = 0.005$

$\chi^2_{tab} = \chi^2_{(3, 0.005)} = 0.0717$

Since $\chi^2_{cal} > \chi^2_{tab}$

So the fit isn't good at all at this level of significance.

Year - 2017 . - 6 (a)

(i) Here, $p(-1) = -3K$

$p(0) = 2K$

$p(1) = 4K$

$p(2) = 0.5$

~~Suppose the function is probability distribution fun~~

For being probability distribution function,

$$\sum p(x) = 1$$

$$\Rightarrow -3K + 2K + 4K + 0.5 = 1$$

$$\therefore K = \frac{1}{6}$$

Now, if $K = \frac{1}{6}$, $p(-1) = -\frac{1}{2} < 0$

So this function is not probability function.

(ii) $p(-2) = 2K$, $p(0) = 2K$, $p(2) = K$, $p(4) = 0.5$

Now, $\sum p(x) = 1$

$$\Rightarrow 2K + 2K + K + 0.5 = 1$$

$$\Rightarrow K = 0.1$$

Now, $p(-2) = 0.2$, $p(0) = 0.2$, $p(2) = 0.1$, $p(4) = 0.5$

~~So~~. As $p(x) \geq 0$; So it's probability function.

Healthcare

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x	-2	0	2	4
$P(x)$	0.2	0.2	0.1	0.5

Now, $P(-2 \leq x \leq 2) = p(x=-2) + p(x=0)$

$$= 0.2 + 0.2$$

$$= 0.4$$

As the function is probability mass function, so we can find expectation.

$$E[g(x)] = \sum g(x) \cdot p(x)$$

$$\Rightarrow E(2x+3) = \sum (2x+3) \cdot p(x)$$

$$= \sum 2x \cdot p(x) + 3 \cdot \sum p(x)$$

$$= \{(-2 \cdot (0.2)) + (2 \cdot (0.1)) + (4 \cdot (0.5))\} \times 2 + 3\{0.2 + 0.2 + 0.1 + 0.5\}$$

$$= 6.6$$

(III) $p(x) = \begin{cases} kx & -1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$

Here, $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\Rightarrow \int_{-\infty}^{-1} f(x)\, dx + \int_{-1}^{2} f(x)\, dx + \int_{2}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow \left[\frac{kx^2}{2}\right]_{-1}^{2} = 1$$

$$\Rightarrow \frac{K}{2}\left(\frac{2^2}{\cancel{3}} - (-1)^2\right) = 1$$

$$\Rightarrow K = \frac{2}{3}$$

Which gives, $p(x) = \begin{cases} \frac{2x}{3} & -1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$

confusion

$$P(-1) = \frac{-2}{3} < 0$$

So it's not a probability distribution

6. (b)

According to normal distribution, we know,

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; -\alpha \le x \le \alpha$$

Here given,

$\mu = \bar{x} = 60$ and $\sigma^2 = 4$

$\therefore \sigma = 2$

So, $f(x;60,2) = \dfrac{1}{\sqrt{2\pi}\,.2} e^{-\frac{1}{2}\left(\frac{x-60}{2}\right)^2}$

Here x is a continuous random variable. For being continuous probability distribution, it has to justify two conditions:

1. $f(x) \ge 0$: In the function $f(x;60,2)$; e^{-x} can never be negative. ~~Also~~ the standard deviation is never negative, so $f(x)$ can never be negative.

2. $$\int_{-\infty}^{\infty} f(x)\, dx = \cancel{\tfrac{1}{2}}\ 1$$

Proof :

At first assume, $$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

Suppose, $$z = \frac{x-\mu}{\sqrt{2}\,\sigma} = \frac{x-60}{2\sqrt{2}}$$

$$\Rightarrow \frac{dz}{dx} = \frac{1}{2\sqrt{2}}$$

$$\therefore\ dx = 2\sqrt{2}\, dz$$

So, $$\int_{-\infty}^{\infty} f(x)\, dx = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-60}{2\sqrt{2}}\right)^2} dx$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2}\, 2\sqrt{2}\, dz$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2}\, dz$$

$$= \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi}$$

$$= 1$$

So $f(x)$ is a probability density function.

Now, in Normal distribution, mean, mode and median coincide,

So, $\bar{x} = \hat{x} = \tilde{x} = 60$.

And co-efficient of skewness, $\beta_1 = 0$; as it's symmetric.

Year - 2018 - 3 (a)

The binomial distribution is a probability distribution that summarizes the likelihood that a value will take one of two independent values under a given set of parameters or assumptions.

There are mainly two assumptions needed regarding Binomial distribution:

1. Benoulli Trial: Bernoulli Trial is a random experiment which has to satify the below conditions:

a) The trial must have only two outputs.

For example: Tossing a coin, for a coin the output would be either Head or tail.

If a family gets a newborn baby, their are two possibilities, either the child is boy or a girl.

b) Every trial has to be independent: For example, suppose, when we toss a coin for the first time we got H. When we will toss the same coin again we don't have any idea about output, it can be either head (H) or tail (T), which means this output doesn't depend on the previous output.

c) Probability of such is each trial is fixed.

For example: After tossing 'a coin, the probability of getting head & tail is ½ and ½. When we toss same coin the probabilities will remain same, their probability is fixed.

When a trial satisfies all these conditions it would be a bernoulli trial.

2) Total number of trials are fixed and finite.

For example: When we say, we are tossing a coin 10 times, this means there will be 10 trial. This trial number has to be fixed. And also it can't be infinite.

Poisson distribution:

If in burnoulli distribution n is very large ($n \rightarrow \infty$) and probability for a single trial is very small ($p \rightarrow 0$), then the binomial distribution is converted into poisson distribution.

4 properties of Poisson distribution:

1. Poisson distribution is a discrete distribution.
2. Poisson distribution is given by,

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

So it's visible that it has only one parameter, λ.

③ Mean of poisson distribution, $\lambda = np$

④ The mean and variance of poisson distribution are equal. $\lambda = np = \sigma^2$

Normal Distribution:

Normal distribution, also known as Gaussian distribution, is a probability distribution that is symmetric about the mean, showing that data near the mean are more frequent in occurance than the data far away from the mean.

4 properties of Normal distribution:

1. A normal distribution comes with a perfect symmetrical shape. This means that the distribution curve can be divided in the middle to produce two equal halves.

2. The mean ($\bar{x}$), mode ($\hat{x}$) and median ($\tilde{x}$) are equal.

3. The skewness of normal distribution is zero.

4. The normal distribution has a kurtosis of 3, that's why it's mesokurtic.

3.(b)

Here the problem can be a model of poisson process.

we know, poisson process $P_x(t_1, t_2) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$

λt = 8 virus per min. $= \frac{8}{60}$ /sec

$\lambda t = \frac{8}{60} \times 30 = 4$

(i) Probability of virus attack below traffic load,

$$P_K = \sum_{K=0}^{3} \left[\frac{e^{-\lambda t} (\lambda t)^K}{K!} \right]$$

$$= \sum_{K=0}^{3} \left[\frac{e^{-4} (4)^K}{K!} \right] = 0.156$$

confusion

(ii) probability of virus attack exactly at the level of critical load, $P_4 = \frac{e^{-4} (4)^4}{4!}$

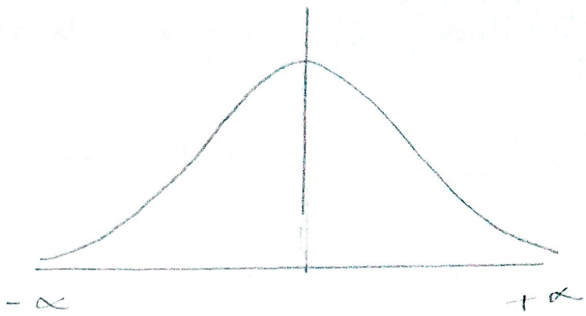
confusion

$= 0.195$

3(c)

Given, $\mu = 13$ and standard deviation, $\sigma = 4$

Let, $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-13}{4}$



Score	f_i	L.B.	z_i	$P(0 \le z \le z_i)$	$\Delta P(z_i)$	E	O_i^2	O_i^2/E_i
$-\alpha - (-1)$	0	$-\alpha$	$-\alpha$	0.5	0.0004	0		
0 - 5	3	-0.5	-3.38	0.4996	0.0297	1	9	9
6 - 10	8	5.5	-1.88	0.4699	0.1965	6	64	10.67
10 - 15	13	10	-0.75	0.2734	0.5091	15	169	11.27
16 - 20	6	15.5	0.63	0.2357	0.2342	7	36	5.14
21 - α	0	20.5	1.88	0.4699				
	N = 30							$\Sigma = 36.08$

Here, $\gamma = m - k - 1 = 4 - 0 - 1 = 3$ Let, $\alpha = 0.05$

So $\chi^2_{(\gamma,\alpha)} = \chi^2_{(3, 0.05)}$

$$\chi^2_{cal} = \sum \frac{O_i^2}{E_i} - N = 36.08 - 30 = 6.08$$

$$\chi^2_{tab} = \chi^2_{3, 0.05} = 0.352$$

As $\chi^2_{cal} > \chi^2_{tab}$, so the fit is not good at all at this level of significance.

Year - 2018 - 4(a).

This problem can be modeled as normal distribution, that is,

$$N(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} ; -\infty \le x \le \infty$$

Let, $z = \frac{x-\mu}{\sigma} = \frac{x-40}{7}$

Here, $\mu = 40$

$\sigma = \sqrt{49} = 7$

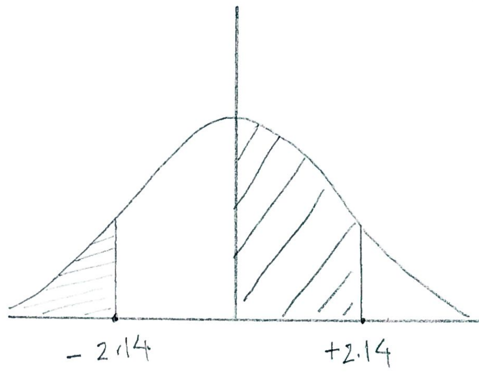
(i) $P(\text{not eligible}) = P(x < 25)$

$z_{25} = -2.14$

$P(x < 25) = P(z < -2.14)$

$= P(0 \le z \le 2.14)$

$= 0.4838$



(ii) $P(25 < x < 60)$

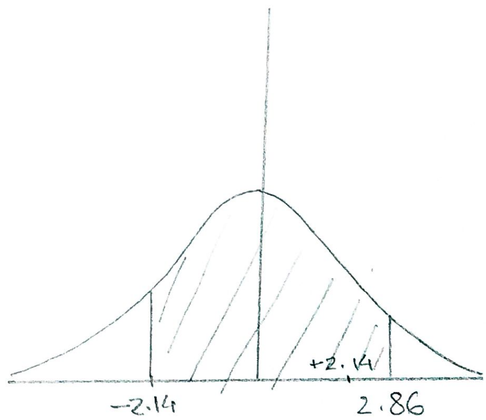
$z_{25} = -2.14$, $z_{60} = 2.86$

$P(-2.14 \le z < 2.86) =$

$P(0 \le z \le 2.14) + P(0 \le z \le 2.86)$

$= 0.4838 + 0.4979$

$= 0.9817$



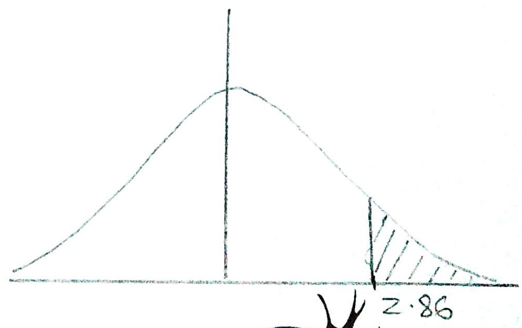
(iii) $P(x > 60)$

$z_{60} = 2.86$

$P(z > 2.86) = 0.5 - P(0 \le z \le 2.86)$

$= 0.5 - 0.4979$

$= 0.0021$



Year - 2018 Ques. 4(b)

Scaling is a linear transformation that enlarges or shrinks or reverses a physical quantity (object) by a scaling factor. A scaling can be represented by a scaling matrix.

To scale an object by a vector $\bar{v} = (v_x, v_y, v_z)$, whose each point $p = (p_x, p_y, p_z)$ would need to be multiplied with this scaling matrix:

$$S_v = \begin{bmatrix} v_x & 0 & 0 \\ 0 & v_y & 0 \\ 0 & 0 & v_z \end{bmatrix}$$

So the expected result would be:

$$S_v \cdot p = \begin{bmatrix} v_x & 0 & 0 \\ 0 & v_y & 0 \\ 0 & 0 & v_z \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} v_x p_x \\ v_y p_y \\ v_z p_z \end{bmatrix}$$

There are two types of scaling.

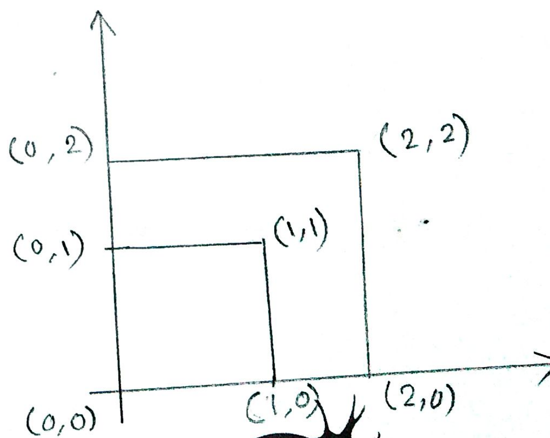
1. Uniform scaling: This scaling enlarges or shrinks objects by a scaling factor that is the same in all directions.

For example:

If we want to do two times uniform scaling, the final matrix would be,

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$



2. Non-uniform matrix : In this scaling at least one of the scaling factors is different from the others. For example :

$$\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

if

$K_1 = 2$, $K_2 = 3$

then the final matrix would be,

$$= \begin{pmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 \end{pmatrix}$$

Non-uniform scaling method can ~~make~~ transform a square into a rectangle.

Scaling can also reverse an object.

For example : In $\begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$, the value of K is smaller than 0,

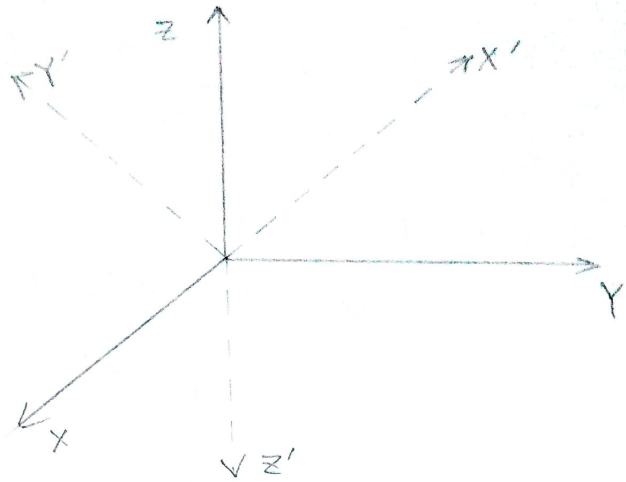
then,

$$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Now the whole square would be reversed.

Year - 2018 - 4(c)

Here $\phi(x,y,z)$ and $\phi'(x',y',z')$ are two coordinates that has the same origin but the axes are rotated with each other.



Here $\phi(x,y,z)$ is a scalar invarient with respect to a rotation of axes.

We know for scalar the physical quantity stays the same after rotation. So, for scalar

$$\phi(x,y,z) = \phi(x',y',z')$$

Now, we have show that the gradient of ϕ is a vector invarient under this transformation. Meaning;

$$\nabla \phi(x,y,z) = \nabla \phi'(x',y',z')$$

$$\Rightarrow \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} = \hat{i}'\frac{\partial \phi'}{\partial x'} + \hat{j}'\frac{\partial \phi'}{\partial y'} + \hat{k}'\frac{\partial \phi'}{\partial z'}$$

We know,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

So,

$$x' = l_{11}x + l_{12}y + l_{13}z \longrightarrow (\mathrm{I})$$

$$y' = l_{21}x + l_{22}y + l_{23}z \longrightarrow (\mathrm{II})$$

$$z' = l_{31}x + l_{32}y + l_{33}z \longrightarrow (\mathrm{III})$$

Now, $$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial x'}\cdot\frac{\partial x'}{\partial x} + \frac{\partial\phi}{\partial y'}\cdot\frac{\partial y'}{\partial x} + \frac{\partial\phi}{\partial z'}\cdot\frac{\partial z'}{\partial x}$$

Now from (1) $\Rightarrow$

$$x' = l_{11}x + l_{12}y + l_{13}z$$

$$\Rightarrow \frac{\partial x'}{\partial x} = l_{11}\frac{\partial x}{\partial x} + 0 + 0$$

$$= l_{11}$$

from (II) $\Rightarrow$

$$\frac{\partial y'}{\partial x} = l_{21}$$

from (III) $\Rightarrow$

$$\frac{\partial z'}{\partial x} = l_{31}$$

So, $$\frac{\partial\phi}{\partial x} = l_{11}\frac{\partial\phi}{\partial x'} + l_{21}\frac{\partial\phi}{\partial y'} + l_{31}\frac{\partial\phi}{\partial z'}$$

as $\phi = \phi'$ we can write,

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi'}{\partial x'}l_{11} + \frac{\partial\phi'}{\partial y'}\cdot l_{21} + \frac{\partial\phi'}{\partial z'}l_{31} \longrightarrow (4)$$

In the same way,

$$\frac{\partial\phi}{\partial y} = l_{12}\frac{\partial\phi'}{\partial x'} + l_{22}\frac{\partial\phi'}{\partial y'} + l_{32}\frac{\partial\phi'}{\partial z'} \longrightarrow (5)$$

and, $$\frac{\partial\phi}{\partial z} = l_{13}\frac{\partial\phi'}{\partial x'} + l_{23}\frac{\partial\phi'}{\partial y'} + l_{33}\frac{\partial\phi'}{\partial z'} \longrightarrow (6)$$

Now multiply (4) with $\hat{i}$, (5) with $\hat{j}$ and (6) with $\hat{k}$ and add them,

$$\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = \frac{\partial\phi'}{\partial x'}(\hat{i}l_{11} + \hat{j}l_{12} + \hat{k}l_{13}) + \frac{\partial\phi'}{\partial y'}(\hat{i}l_{21} + \hat{j}l_{22} + \hat{k}l_{23}) + \frac{\partial\phi'}{\partial z'}(\hat{i}l_{31} + \hat{j}l_{32} + \hat{k}l_{33}) \longrightarrow (7)$$

From 1,

$x = \hat{i}$, $x' = \hat{i}'$, $y = \hat{j}$, $y' = \hat{j}'$, $z = \hat{k}$, $z = \hat{k}'$

So from 7 ⇒

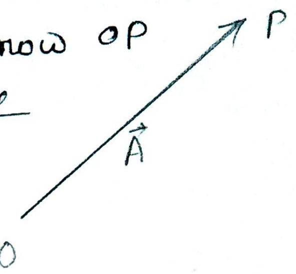
$$\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} = \hat{i}'\frac{\partial \phi'}{\partial x'} + \hat{j}'\frac{\partial \phi'}{\partial y'} + \hat{k}'\frac{\partial \phi'}{\partial z'}$$

(proved)

Year - 2018, Ques - 4(d)

A vector is a quantity having both magnitude and direction, such as displacement, velocity, force. and acceleration.

Graphycally a vector can be represented by an arrow OP defining the direction and the magnitude of the vector is indicated by the length of the arrow. The vector is represented as $\vec{A}$ and the magnitude is denoted by $|\vec{A}|$.



Also the tail end O of the arrow is called the origin of the vector and the head P is called the terminal point.

Point function and Field:
Every physical quantity can be expressed as a continuous function of the position of a point in the region of the space.
Such function is called a point function and the region in which it specifies the physical quantity is known as field.

Vector point function and Vector field:
If the physical quantity is vector then the function is called vector point function and the corresponding domain will be vector field.

For example :

$\cancel{\bar{v}(x, y, z) = 2x\hat{i} + y\hat{j} + \hat{k}}$

~~Here $\bar{v}(x, y, z)$ defines a vector~~

If to each point (x, y, z) of a region R in space there is assigned a vector $\bar{v} = \bar{v}(x, y, z)$, then $\bar{v}$ is a vector point function.

$$\bar{v}(x, y, z) = v_1(x, y, z)\hat{i} + v_2(x, y, z)\hat{j} + v_3(x, y, z)\hat{k}$$

suppose, if we get to know the exact velocity at any point (x, y, z) at a certain ~~point~~ time, then a vector field is defined.

A vector field that varies with time can be represent -ed as,

$\bar{v} \cancel{=} \cancel{\bar{v}(x, y, z) = v_1(x.}$

$$\bar{v} = v_1(x, y, z, t)\hat{i} + v_2(x, y, z, t)\hat{j} + v_3(x, y, z, t)\hat{k}$$

or as

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1(x, y, z, t) \\ v_2(x, y, z, t) \\ v_3(x, y, z, t) \end{bmatrix}$$