~~Ne~~ Numerical Methods
Question 2017

5.(a) Partial 'Differential' Equation:

A partial differential equation is a differential equation involving more than one independent variable. These variables determine the behaviour of the dependent variable as described by their partial derivatives contained in the equation. Such as:

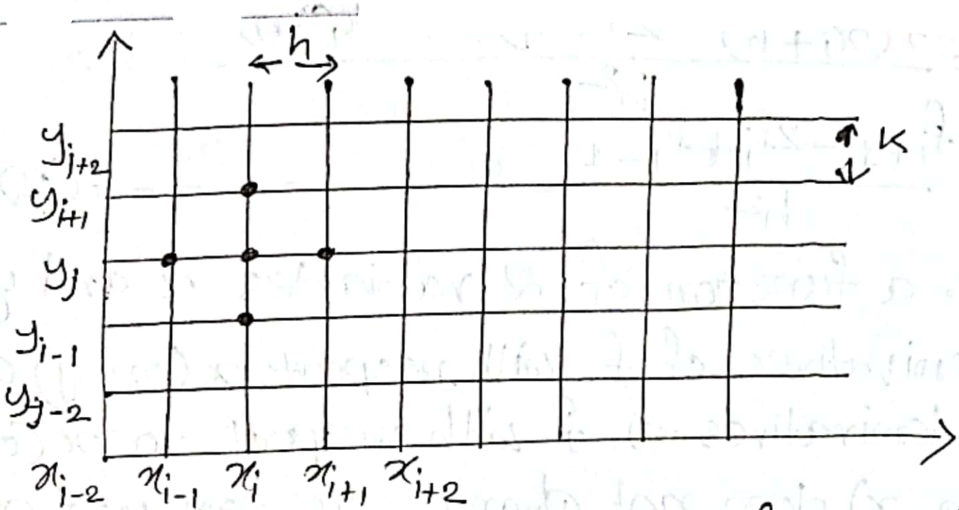
1. Study of displacement of a vibrating string.
2. Heat flow problems
3. Fluid flow analysis
4. Electrical potential distribution
5. Analysis of torsion in a bar subject to twisting
6. Study of diffusion of matter and so on.

5.b) In numerical analysis, the Crank-Nicolson method is a finite difference method used for numerically solving the heat equation and similar partial differential equations. It is a second order method in time. It is implicit in time, can be written as an Implicit Runge-kutta method and it is numerically stable. The Crank-Nicholson formula is,

$$-rf_{i-1,j+1}+(2+2r)f_{i,j+1}-rf_{i,j+1}=rf_{i-1,j}+(2-2r)f_{ij}+rf_{i+1,j}$$

The Crank-Nicholson formula is called an implicit method because the values to be computed are not just a function of values at the previous time step, but also involve the values at the same time step which are not readily available. This requires us to solve a set of simultaneous equations at each time step.

2017
5.(c)



$x_{i+1} = x_i + h$
$y_{j+1} = y_j + h$

fig: Two dimensional finitie difference grid

f_{ij} = A function of the two space variables x and y (The pivotal values at the points of intersection)

In the finite difference method, we replace derivatives that occur in the PDE by their finite difference equivalents. We then write the difference equation corresponding to each "grid point" (where derivative is required) using function values at the surrounding grid points. Solving these equations simultaneously gives values for the function at each grid point.

If the function $f(x)$ has a continuous 4th derivative, then its 1st and 2nd derivatives are given by the following central difference approxim...

$$f'(x_i) = \frac{f(x_i+h) - f(x_i-h)}{2h}$$

$$\Rightarrow f_i' = \frac{f_{i+1} - f_{i-1}}{2h} \quad \text{-------------- (1)}$$

$$f''(x_i) = \frac{f(x_i+h) - 2f(x_i) + f(x_i-h)}{h^2}$$

$$\Rightarrow f_i'' = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} \quad \text{---------- (2)}$$

When f is a function of 2 variables x and y, the partial derivatives of f with respect x (or y) are the ordinary derivatives of f with respect to x (or y) when y (or x) does not change. We can use eqn (1) & (2) in the x-direction to determine derivatives with respect to y. Thus we have,

$$\frac{\partial f(x_i, y_j)}{\partial x} = f_x(x_i, y_j) = \frac{f(x_{i+1}, y_j) - f(x_{i-1}, y_j)}{2h}$$

$$\frac{\partial f(x_i, y_j)}{\partial y} = f_y(x_i, \cancel{y}\, y_j) = \frac{f(x_i, y_{j+1}) - f(x_i, y_{j-1})}{\cancel{2h}\ 2k}$$

$$\frac{\partial^2 f(x_i, \cancel{y}\, y_j)}{\partial x^2} = f_{xx}(x_i, y_j) = \frac{f(x_{i+1}, y_j) - 2f(x_i, y_j) + f(x_{i-1}, y_j)}{h^2}$$

$$\frac{\partial^2 f(x_i, y_j)}{\partial y^2} = f_{yy}(x_i, y_j) = \frac{f(x_i, y_{j+1}) - 2f(x_i, y_j) + f(x_i, y_{j+1})}{\cancel{k^2}\ k^2}$$

$$\frac{\partial^2 f(x_i, y_j)}{\partial x \partial y} = \frac{f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_{j-1}) - f(x_{i-1}, y_{j+1}) + f(x_{i-1}, y_{j-1})}{4hk}$$

It is convenient to use double subscripts i, j on f to indicate x and y values. Then, the above equations become, ~~becor~~

$$f_{x,ij} = \frac{f_{i+1,j} - f_{i-1,j}}{2h}$$

$$f_{y,ij} = \frac{f_{i,j+1} - f_{i,j-1}}{2k}$$

$$f_{xx,ij} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{k^2}$$

$$f_{y,ij} = \frac{f_{i,j+1} - 2f_{ij} + f_{i,j-1}}{k^2}$$

$$f_{xy,ij} = \frac{f_{i+1,j+1} - f_{i+1,j-1} - f_{i-1,j+1} + f_{i-1,j-1}}{4hk}$$

We can write a second order equation involving two independent variables in general form as

$$\cancel{a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial y^2}}$$

$$a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial x \partial y} + c\frac{\partial^2 f}{\partial y^2} = f\left(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

When we put $a = 1$, $b = 0$, $c = 1$ and $f(x, y, f, f_x, f_y) = 0$ the above equation becomes,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \nabla^2 f = 0 \quad \text{------} \quad (1)$$

The operator $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is called the Laplacian operator and eqn (1) is called the Laplace's equation.

Replacing the second order derivatives by their finite difference equivalents we obtain,

$$\nabla^2 f_{ij} = \frac{f_{i+1,j} - 2f_{ij} + f_{i-1,j}}{h^2} + \frac{f_{i,j+1} - 2f_{ij} + f_{i,j-1}}{k^2}$$

$$\left[\because \frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{f_{i+1,j} - 2f_{ij} + f_{i-1,j}}{h^2}\right.$$

$$\left.\because \frac{d^2 f}{\partial y^2} = f_{yy} = \frac{f_{i,j+1} - 2f_{ij} + f_{i,j-1}}{k^2}\right]$$

If we assume for simplicity $h=k$ then,

$$\nabla^2 f_{ij} = \frac{1}{h^2}\left(f_{i+1,j} + f_{i-1,j} - 4f_{ij} + f_{i,j+1} + f_{i,j-1}\right) = 0$$

This equation contains four neighbouring points around the ~~er~~ central points as shown in the figure below. This equation is known as the five points difference formula for Laplace's equation.

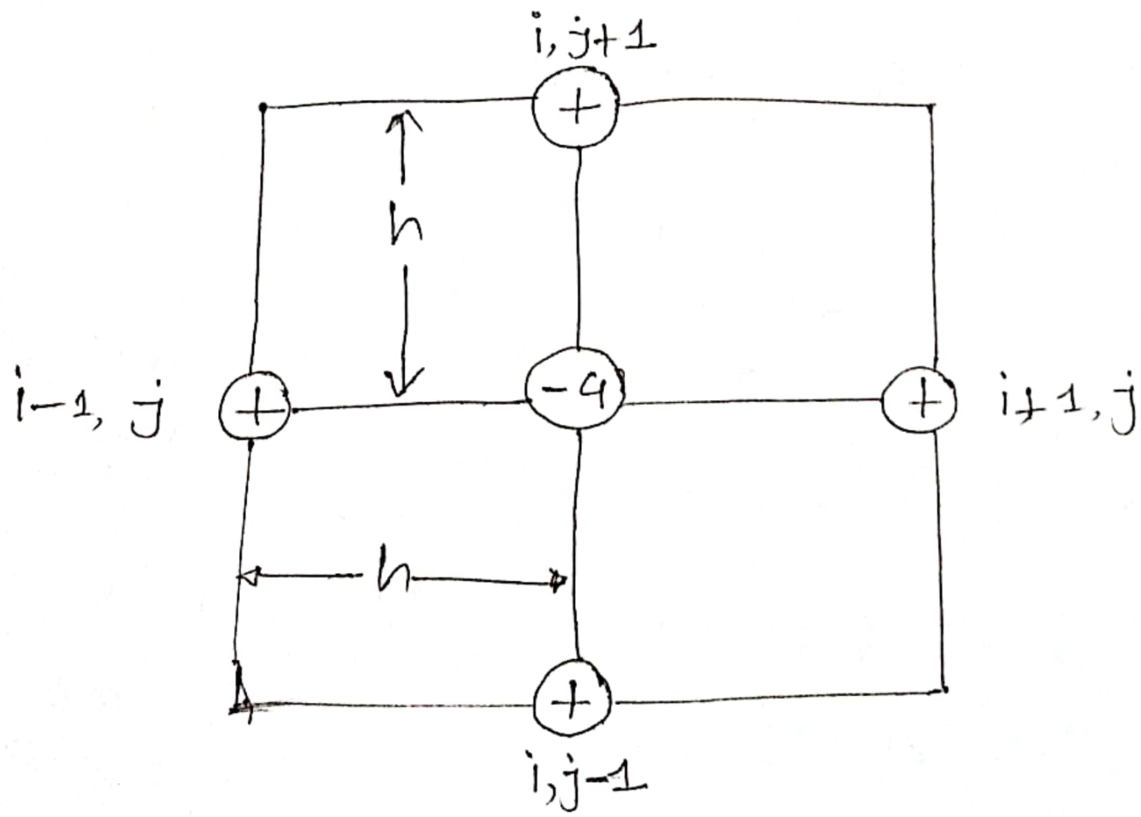
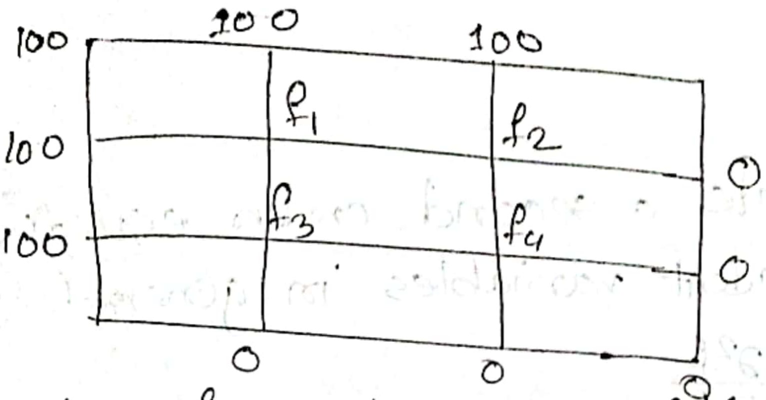


fig : Grid for laplace's equation

5.d) A problem with the values known on each boundary is said to have Dirichlet Boundary Conditions. The problem is illustrated below:



The system of equations is as follows:

At point 1: $f_2+f_3-4f_1+100+100=0$

" " 2: $f_1+f_4-4f_2+100+0=0$

" " 3: $f_1+f_4-4f_3+100+0=0$

" " 4: $f_2+f_3-4f_4+0+0=0$

That is,
$$-4f_1+f_2+f_3+0=-200$$
$$f_1-4f_2+0+f_4=-100$$
$$f_1+0-4f_3+f_4=-100$$
$$0+f_2+f_3-4f_4=0$$

Solutions of this system are, $f_1=75$, $f_2=50$

$f_3=50$, $f_4=25$

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2017

6. a) The process of computing the value of a definite integral from a set of numerical values of the integrand is called Numerical Integration.

There are some functions which do not have an anti derivative which can be expressed in terms of familiar functions such as polynomials, exponentials and trigonometric functions. One such example is e^{-x^2}; of course this is an important function since it is the probability density function for the normal distribution. In this types of functions numerical integration comes into account, Numerical integration is the approximation of the definite ~~integral~~ integration.

To calculate numerical integration there are 2 methods. They are:

1) Trapezoidal Rule
2) Simpson's Rule

Example:

We will consider the integral $\int_0^1 (1-x^2)\,dx$. Of course

We can evaluate this integral directly with the fundamental Theorem of Calculus:

$$\int_0^1 (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

We will now build the trapezoidal approximation so that we might see explicitly how ~~of~~ much of an error we are making.

First, we will consider, $h = \frac{1-0}{1} = 1$; $n = 1$

In this case, there are only two points $x_0 = 0$ and $x_0 = 1$, then we have,

$y_0 = 1 - x_0^2 = 1$ and

$y_1 = 1 - x_1^2 = 0$

Then the trapezoidal rule produces,

$$\int_0^1 (1-x^2)\,dx = \frac{h}{2}(y_0 + y_1) = \frac{1}{2}(1+0) = \frac{1}{2}$$

$\therefore$ This means that the Trapezoidal Rule with $n=1$ produces an approximation of $\frac{1}{2}$ to the integral which we know is $\frac{2}{3}$.

Now Let's see what happens when $n=2$.

$\therefore h = \frac{1-0}{2} = 1/2$

$\therefore x_0 = 0$ $\quad\therefore y_0 = 1 - x_0^2 = 1$

$\therefore x_1 = 1/2$ $\quad\therefore y_1 = 1 - (1/2)^2 = 3/4$

$\therefore x_2 = 1$ $\quad\therefore y_2 = 1 - 1^2 = 0$

$$\therefore \int_0^1 (1-x^2)\,dx = \frac{h}{2}[y_0 + 2y_1 + y_2] = \frac{5}{8} = 0{\cdot}625$$

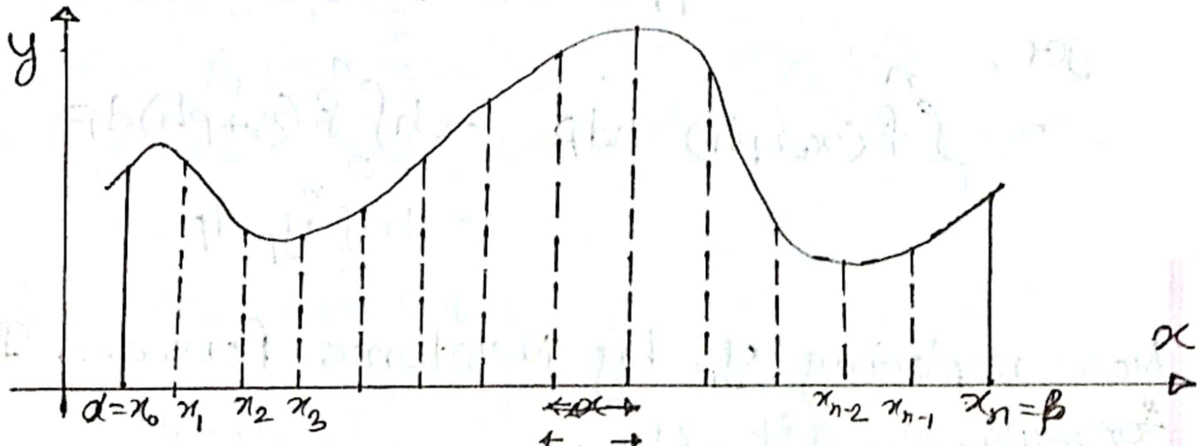
By this example it is clear that numerical integration is just approximation of definite integration.

6.b) We derive the general quadrature formula for equidistant ordinates.
We consider the definite integral $I = \int_{\alpha}^{\beta} y\,dx$

We assume that $y = f(x)$ is continuous on $[\alpha, \beta]$ and we divide $[\alpha, \beta]$ into n subintervals of same length.

$$\Delta x = \frac{\beta - \alpha}{n} = h$$

So there will $n+1$ points.



$x_0 = \alpha$, $x_1 = x_0 + \Delta x$, ... $x_n = x_0 + n\Delta x = \beta$
We can compute the value of $y = f(x)$ at these points,
$y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$, ... $y_n = f(x_n)$.
Then,

$$I = \int_{\alpha}^{\beta} y\,dx = \int_{x_0}^{x_0+nh} y\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx \quad \text{-------- (1)}$$

Putting $x = x_0 + ph$ .... (2)

$$1 = 0 + h\frac{dp}{dx}$$

$$\therefore dx = h\,dp$$

Lower limit x_0 , upper limit x_0+nh

From eqn (2),

$$x_0 = x_0 + ph$$
$$\Rightarrow 0 = ph$$
$$\cancel{\therefore h = 0} \quad \therefore p = 0$$

Again from eqn (2), $x_0 + nh = x_0 + ph$

$$\Rightarrow nh = ph$$
$$\therefore p = n$$

Putting x, dx, upper limit and lower limit in (1) we get,

$$I = \int_0^n f(x_0+ph)\, h\,dp \quad = h\int_0^n f(x_0+ph)\,dp$$
$$= h\int_0^n y_p\, dp$$

Now replacing y_p by Newton's Forward Interpolation formula, we get, y_p.

Newton's Forward Interpolation formula:

We assume that $y=f(x)$ be a function compute the values of $y=f(x)$ at $n+1$ values of the independent variables x be equidistant $x_i = x_0 + ih$

$[i = 0, 1, 2, \ldots, n]$

ie. $x_0 = x_0 \quad x_1 = x_0 + h \quad x_2 = x_0 + 2h \ldots\ldots x_n = x_0 + nh$

$y_0 = f(x_0) \quad y_1 = f(x_1) \quad y_2 = f(x_2) \quad\quad y_n = f(x_n)$

And let $y(x)$ be the polynomial in x of nth degree

$y(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)(x-x_1) + A_3(x-x_0)(x-x_1)(x-x_2)$
$+ \ldots\ldots\ldots\ldots + A_n(x-x_0)\ldots\ldots(x-x_n)$ —— (3)

such that,

$y_i = f(x_i)$ [where $i = 0, 1, 2, \ldots\ldots n$]

Putting $x = x_0, x_1, x_2, x_3, \ldots\ldots, n$ in above eqn [eqn (3)]
we get,

$y_0 = A_0$

$y_1 = A_0 + A_1(x_1 - x_0)$

$y_2 = A_0 + A_1(x_2 - x_0) + A_2(x_2 - x_0)(x_2 - x_1)$

$y_3 = A_0 + A_1(x_3 - x_0) + A_2(x_3 - x_0)(x_3 - x_1) + A_3(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)$

$\vdots$

And so, on, from these,

$y_0 = A_0$

$$A_1 = \frac{y_1 - A_0}{x - x_0} = \frac{y_1 - y_0}{h} = \frac{\Delta y}{h}$$

$$A_2 = \frac{y_2 - A_0 - A_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{y_2 - y_0 - \frac{\Delta y}{h}(2h)}{2h \cdot h}$$

$$= \frac{y_2 - y_0 - 2(y_1 - y_0)}{2 \cdot h^2} \quad [\because \Delta y = y_1 - y_0]$$

$$= \frac{y_2 - y_0 - 2y_1 + 2y_0}{2!\, h^2} = \frac{y_2 - 2y_1 + y_0}{2!\, h^2}$$

$$= \frac{1}{2!\, h^2} \Delta^2 y_0 \quad [\because \Delta^2 y_0 = y_2 - 2y_1 + y_0]$$

Similarly, $A_3 = \frac{1}{3!\,h^3} \Delta^3 y_0$

$\vdots$

$$A_n = \frac{1}{n!\,h^n} \Delta^n y_0$$

Putting these values $A_0, A_1, A_2, \dots A_n$ in eqn (3), we get,

$$y(x) = y_0 + \frac{\Delta y_0}{h}(x-x_0) + \frac{1}{2!\,h^2}\Delta^2 y_0 (x-x_0)(x-x_1) + \frac{1}{3!\,h^3}\Delta^3 y_0 (x-x_0)(x-x_1)(x-x_2) + \dots + \frac{1}{n!\,h^n}(x-x_0)\dots(x-x_n) \quad (4)$$

Putting $\frac{x-x_0}{h} = p$ i.e. $x = x_0 + ph$ where p is a real number. Since,

$$\frac{(x-x_0)(x-x_1)}{h^2} = \frac{x-x_0}{h} \cdot \frac{x-x_1}{h} = p\,\frac{x-x_1}{h}$$

$$= p\,\frac{x-x_0-x_1+x_0}{h}$$

$$= p\,\frac{(x-x_0)-(x_1-x_0)}{h}$$

$$= p\left[\left(\frac{x-x_0}{h}\right) - \left(\frac{x_1-x_0}{h}\right)\right]$$

$$= p\left(p - \frac{h}{h}\right) = p(p-1).$$

Similarly, $\frac{(x-x_0)(x-x_1)(x-x_2)}{h^3} = p(p-1)(p-2)$

So above eqn (eqn 4) takes the form,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots + \frac{p(p-1)(p-2)\dots[p-(n-1)]}{n!}\Delta^n y_0$$

Where $y_p = y(x_0+ph)$ is known as Newton forward Interpolation formula.

We have,

$$I = h\int_0^n f(x_0+ph)\,dp = \int_0^n y_p\,dp$$

Now replacing y_p by Newton's forward interpolation we get,

$$I = h\int_0^n\left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots\right]dp$$

Now integrating it term by term we get,

$$I = h\Big[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\Big(\frac{n^3}{3} - \frac{n^2}{2}\Big)\Delta^2 y_0 + \frac{1}{6}\Big(\frac{n^4}{4} - n^3 + n^2\Big)\Delta^3 y_0$$
$$+ \frac{1}{24}\Big(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2\Big)\Delta^4 y_0 + \cdots\cdots\Big]$$

This is known as the General Quadrature formula.

Putting $n=3$ in the above ~~form~~ formula and neglecting all differences above the third we get,

$$\int_{x_0}^{x_0+3h} y\,dx = h\Big[3y_0 + \frac{3^2}{2}\Delta y_0 + \frac{1}{2}\Big(\frac{27}{3} - \frac{9}{2}\Big)\Delta^2 y_0 + \frac{1}{6}\Big(\frac{81}{4} - 27 + 9\Big)\Delta^3 y_0\Big]$$

$$= h\Big[3y_0 + \frac{9}{2}\Delta y_0 + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\Big]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Similarly,

$$\int_{x_0+3h}^{x_0+6h} y\,dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right]$$

$$\int_{x_0+(n-3)h}^{x_0+nh} y\,dx = \frac{3h}{8}\left[y_{n-3}+3y_{n-2}+3y_{n-1}+y_n\right]$$

Adding all these integrals where n is a multiple of 3, we have,

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\Big[(y_0+y_n)+3[y_1+y_2+y_4+y_5+\cdots+y_{n-2}+y_{n-1})+$$
$$+2(y_3+y_6+y_9+y_{12}+\cdots\cdots+y_{n-3})\Big]$$

This formula is known as Simpson's 3/8 Rule.

2017

6.c) $\int_4^{5.2} \tan x\,dx$.

Taking $h=0.2$ divide the whole range of integration $(4, 5.2)$ into 6 equal parts. The values of $\tan x$ for each point of subdivision are given below: [calculator radian mode]

x	$y=\tan x$
$x_0=4$	$y_0=1.1578$
$x_1=4.2$	$y_1=1.777$
$x_2=4.4$	$y_2=3.0963$
$x_3=4.6$	$y_3=8.8602$
$x_4=4.8$	$y_4=-11.3849$
$x_5=5.0$	$y_5=-3.3805$
$x_6=5.2$	$y_6=-1.8856$

By Simpson's 1/3 rule, we have,

$$\int_4^{5.2} \tan x \, dx = \frac{h}{3}\left[y_0+y_6+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{0.2}{3}\left[1.1578+(-1.8856)+4(1.777+8.8602 + -3.3805)+2(3.0963-11.3849)\right]$$

$$= 0.78145$$

$$\int_4^{5.2} \tan x \, dx = \Big[-\ln|\cos x|\Big]_4^{5.2}$$

$$= -\left[\ln\left|\cos\left(\frac{5.2\times180}{\pi}\right)\right| - \left|\ln \cos\left(\frac{4\times180}{\pi}\right)\right|\right]$$

$$= -\left[\ln|0.4685| - \ln|-0.6536|\right]$$

$$= -(-0.33296) = 0.33296$$

So the standard result is 0.33296 and we got 0.78145 by Simpson's 1/3 rule.

$\therefore$ Error $= |0.78145-0.33296| = 0.4485$ (Ans)

2017

7.a) Differences between homogeneous and non-homogeneous system of equations are given below:

Homogeneous	Non homogeneous
1) A homogeneous system of linear equations is one in which all constant terms are zero.	1) A non homogeneous system of linear equations is one in which all constant terms are not zero.
2) This system always has at least one solution, namely the zero vector.	2) A $n \times n$ nonhomogeneous system of linear equations has a unique non-trivial solution if and only if its determinant is non zero. If this determinant is zero, then the system has either no non-trivial solutions or an infinite numbers of solutions.

Homogeneous	Non Homogeneous
3) When a row operation is applied to a homogeneous system, the new system is still homogeneous.	3) A non homogeneous system has an associated homogeneous system which you get by replacing the constant terms in each equation with zero.
4) When ~~a~~ we represent a homogeneous system as a matrix, we often leave ~~the~~ off the final column of constant terms, since applying row operations would not modify that column. So we use ~~an~~ a regular matrix instead of an augmented matrix.	~~F~~4) We represent a non-homogeneous system as a matrix in augmented matrix form.

2017

7. b) $a_{11} \neq 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0 \quad |A| \neq 0$ etc.

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{23} & a_{33} \end{bmatrix}$$

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 &= b_1 \\ a_{21} x_2 + a_{22} x_2 + a_{23} x_3 &= b_2 \quad \cdots (1) \\ a_{31} x_3 + a_{32} x_2 + a_{33} x_3 &= b_3 \end{aligned}$$

This eqn (1) can be put in the form

$AX = B$ - - - - - - - - - - (2)

Let $A = LU$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Now we shall discuss the procedure of computing the matrices L and U. from the relation $A = LU$. we get,

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \times \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the matrices on the left hand side and then equating the corresponding elements on both sides we have,

$1 \times u_{11} + 0 \times 0 + 0 \times 0 = a_{11}$ $\quad \therefore \boxed{u_{11} = a_{11}}$

$\boxed{u_{12} = a_{12}}$; $\boxed{u_{13} = a_{13}}$

$l_{21} \times u_{11} + 0 \times 1 + 0 \times 0 = a_{21}$; $\therefore \boxed{l_{21} = \frac{a_{21}}{u_{11}} = \frac{a_{21}}{a_{11}}}$

$l_{21} u_{12} + u_{22} = a_{22}$

$\Rightarrow \frac{a_{21}}{a_{11}} \times a_{12} + u_{22} = a_{22}$ $\quad \therefore \boxed{u_{22} = a_{22} - \frac{a_{21}}{a_{11}} \times a_{12}}$

$l_{21} u_{13} + u_{23} = a_{23}$

$\Rightarrow \frac{a_{21}}{a_{11}} \times a_{13} + u_{23} = a_{23}$ $\quad \therefore \boxed{u_{23} = a_{23} - \frac{a_{21}}{a_{11}} \times a_{13}}$

$l_{31} \times u_{11} + 0 + 0 = a_{31}$

$\therefore \boxed{l_{31} = \frac{a_{31}}{u_{11}} = \frac{a_{31}}{a_{11}}}$

$l_{31} \times u_{12} + l_{32} \times u_{22} + 0 = a_{32}$

$\Rightarrow \cancel{l_{31} \times a_{12}} + \frac{a_{31}}{a_{11}} \times a_{12} + l_{32}\left(a_{22} - \frac{a_{21}}{a_{11}} \times a_{12}\right) = a_{32}$

$\therefore \boxed{l_{32} = \frac{a_{32} - (a_{31}/a_{11}) \cdot a_{12}}{a_{22} - \frac{a_{21}}{a_{11}} \times a_{12}}}$

$l_{31} \times u_{13} + l_{32} \times u_{23} + 1 \times u_{33} = a_{33}$

$\boxed{u_{33}} = a_{33} - u_{13} l_{31} - l_{32} u_{23}$

$$= a_{33} - a_{13} \times \frac{a_{31}}{a_{11}} - \frac{a_{32} - (a_{31}/a_{11}) a_{12}}{a_{22} - \frac{a_{21}}{a_{11}} \times a_{12}} \left(a_{23} - \frac{a_{21}}{a_{11}} \times a_{13}\right)$$

2017

7. c) Let us take $h = 0.1$, Here $f(x, y) = x + y$

Now $k_1 = hf(x_0, y_0) = 0.1\ (0 + 1) = 0.1$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1 (0 + 0.05 + 1 + 0.05) = 0.11$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1 (0 + 0.05 + 1 + 0.055) = 0.1105$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 (0 + 0.1 + 1 + 0.1105) = 0.12105$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.1 + 2\times 0.11 + 2\times 0.1105 + 0.12105) = 0.11034$

Thus, $x_1 = x_0 + h = 0 + 0.1 = 0.1$ and $y_1 = y_0 + \Delta y = 1 + 0.11034$
$= 1.11034$

Now for the second interval, we have.

$k_1 = hf(x_1, y_1)$

$= 0.1 [0.1 + \cancel{1} + 1.11034] = 0.121034$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$= 0.1 (0.1 + 0.05 + 1.11034 + \frac{0.121034}{2}) = 0.13208$

$k_3 = hf(x_1 + h/2, y_1 + k_2/2)$

$= 0.1 (0.1 + 0.05 + 1.11034 + \frac{0.13208}{2}) = 0.13263$

$\cancel{k_4 = hf(x_1 + h/2, y_1 + k_3}$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1 (0.1 + 0.1 + 1.11034 + 0.13263)$
$= 0.14429$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.121034 + 2\times 0.13208 + 2\times 0.13263 + 0.14429)$

$= 0.13246$

Hence $x_2 = 0.2$ and $y_2 = y_1 + \Delta y = 1.11034 + 0.13246$
$= 1.2428$

2017

8. a) $2x + 3y + z = 9$

$x + 2y + 3z = 6$

$3x + y + 2z = 8$

Here

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Let $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$\therefore u_{11} = a_{11} = 2 \qquad \therefore u_{12} = a_{12} = 3 \qquad \therefore u_{13} = a_{13} = 1$

$\therefore l_{21} = \dfrac{a_{21}}{u_{11}} = \dfrac{1}{2}$

$\therefore u_{22} = a_{22} - \dfrac{a_{21}}{a_{11}} \times a_{12} = 2 - \dfrac{1}{2} \times 3 = \dfrac{1}{2}$

$\therefore u_{23} = a_{23} - \dfrac{a_{21}}{a_{11}} \times a_{13} = 3 - \dfrac{1}{2} \times 1 = \dfrac{5}{2}$

$\therefore l_{31} = \dfrac{a_{31}}{a_{11}} = \dfrac{3}{2}$

$\therefore l_{32} = \dfrac{a_{32} - (a_{31}/a_{11})\, a_{12}}{a_{22} - (a_{21}/a_{11})\, a_{12}} = -7$

$\therefore u_{33} = a_{33} - u_{13} l_{31} - l_{32} u_{23} = 18$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$$

Now the system can be written as.

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1/2 & 0 & 0 \\ 3/2 & -7 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}}_{B} \qquad [\because LUX = B]$$

$$\overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix}} \overset{Y}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} = \overset{B}{\begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}}$$

[$\because$ UX = Y]
~~[∵ UX = B]~~

This system is equivalent to $y_1 = 9$

$$\frac{1}{2}y_1 + y_2 = 6 \Rightarrow y_2 = 3/2$$

$$3/2\, y_1 - 7y_2 + y_3 = 8 \quad \therefore y_3 = 5$$

Now the solution of the original system is given by.

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

[$\because$ UX = B]

$\therefore z = 5/18$

$\Rightarrow 1/2\, y + \frac{5}{2} z = 3/2 \quad \therefore y = \frac{29}{18}$

$\Rightarrow 2x + 3y + z = 9 \quad \cancel{\therefore 2 =} \therefore x = \frac{35}{18}$

2017

8.b) $27x + 6y - z = 85$ $\therefore x = \frac{1}{27}(85 - 6y + z)$ —— (2)

$x + 2y + 3z = 6$ $\therefore y = \frac{1}{2}(6 - x - 3z)$ —— (3)

$3x + y + 2z = 8$ $\therefore z = \frac{1}{2}(8 - 3x - y)$ —— (4)

Starting with $x = 0$, $y = 0$, $z = 0$, we get,

$x^{(1)} = 85/27 = 3.148$

Putting $x^{(1)} = 3.148$ and $z = 0$ in (3), we get

$$y^{(1)} = 1/2\,(6 - x^{(1)} - 3z) = 1.426$$

Now putting $x^{(1)} = 3.148$ and $y^{(1)} = 1.426$ in (4), we get,

$$z^{(1)} = \frac{1}{2}(8 - 3x^{(1)} - y^{(1)}) = -1.435$$

Second approximation,

$$x^{(2)} = \frac{1}{27}(85 - 6y^{(1)} + z^{(1)}) = \frac{1}{27}(85 - 6\times 1.426 - 1.435)$$
$$= 2.778$$

$$y^{(2)} = \frac{1}{2}(6 - x^{(2)} - 3z^{(1)}) = \frac{1}{2}(6 - 2.778 - 3\times -1.435)$$
$$= 3.7635$$

$$z^{(2)} = \frac{1}{2}(8 - 3x^{(2)} - y^{(2)}) = \frac{1}{2}(8 - 3\times 2.778 - 3.7635)$$
$$= -2.04875$$

Third approximation,

$$x^{(3)} = \frac{1}{27}(85 - 6y^{(2)} + z^{(2)}) = \frac{1}{27}(85 - 6\times 2.778 + (-2.04875))$$
$$= 2.4549$$

$$y^{(3)} = \frac{1}{2}(6 - x^{(3)} - 3z^{(2)}) = \frac{1}{2}(6 - 2.4549 - 3\times - 2.04875)$$
$$\cancel{= 4.8456} = 4.8457$$

$$z^{(3)} = \frac{1}{2}(8 - 3x^{(3)} - y^{(3)}) = \frac{1}{2}(8 - 3\times 2.4549 - \cancel{4.85}\ 4.8457)$$
$$= -2.1052$$

Fourth approximation,

$$x^{(4)} = \frac{1}{27}(85 - 6y^{(3)} + z^{(3)}) = \frac{1}{27}(85 - 6 \times 4.8457 - 2.1052)$$
$$= 1.99$$

$$y^{(4)} = \frac{1}{2}(6 - x^{(4)} - 3z^{(3)}) = \frac{1}{2}(6 - 1.99 - 3 \times -2.1052)$$
$$= 5.16$$

$$z^{(4)} = \frac{1}{2}(8 - 3x^{(4)} - y^{(4)}) = -1.565$$

Since $x^{(4)}, y^{(4)}, z^{(4)}$ are sufficiently close to $x^{(3)}, y^{(3)}, z^{(3)}$ respectively, so the values 1.99, 5.16, −1.565 can be taken as the solution of the given system.

2017

8.(c) $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$[A : I] = \left[\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} R_2' \leftarrow R_2 - R_1 \\ R_3' \leftarrow R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \cancel{R_1' \leftarrow R_2 - 3R_1} \\ R_1' \leftarrow R_1 - 3R_2 \end{array}$$

$$= \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \quad R_1' \leftarrow R_1 - 3R_3$$

$$\therefore A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{Ans.}$$