1) a 2017 – (1, 2, 3, 4)

1) $\boxed{a}$ i) Evaluate

(i) $\lim_{z \to 1+i} (z^2 - 5z + 10)$

$= (1+i)^2 - 5(1+i) + 10$

$= 1 + 2i + i^2 - 5 - 5i + 10$

$= 1 + 2i - 1 - 5 - 5i + 10$

$= -3i + 5 \quad = 5 + 3i$

ii) $\lim_{z \to 0} \dfrac{z}{\bar{z}}$

$= \lim_{z \to 0} = \lim_{x+iy \to 0} \dfrac{x+iy}{x-iy}$

$= \dfrac{0}{x-iy} = 0$

Scanned with CamScanner

1) b) Show that, $f(z) = (x^3 - 3xy^2) + i(y^3 - 3x^2y)$

$z \neq 0$; 0, $z = 0$ is continous and CR

equations are satisfies but not differentiable

at $z = 0$

Soln: To prove it continous, we have to

check whether its limit exists or not.

$$f(z) = (x^3 - 3xy^2) + i(y^3 - 3x^2y)$$

Along the x axis,

$$\lim_{x \to 0} \frac{(x^3 - 3 \times x \times 0^2) + i(0^3 - 3 \times x^2 \times 0)}{1}$$

$$= \lim_{x \to 0} x^3 = 0$$

Along the y axis,

$$\lim_{y \to 0} (0^3 - 3 \times 0 \times y^2) + i(y^3 - 3 \times 0^2 \times y)$$

$$= \lim_{y \to 0} y^3 = 0$$

Along the $y = mx$ line,

$$\lim_{x \to 0} (x^3 - 3 \times x \times m^2 x^2) + i(m^3 x^3 - 3 \times x^2 \times mx)$$

$$= \lim_{x \to 0} (x^3 - 3m^2 x^3) + i(m^3 x^3 - 3x^3 m)$$

$$= 0$$

The limits are equal in y axis, x axis and $y = mx$ straight line, limit exists.

$f(z) = (x^3 - 3xy^2) + i(y^3 - 3x^2 y)$

$(z = 0)$ put,

$f(0) = 0$, So, the functional value is also equal.

The function is continous.

C-R- equation satisfy:

∂ $u = (x^3 - 3xy^2)$, $v = y^3 - 3x^2 y$

$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 3y^2$

$\Rightarrow \frac{\partial u}{\partial y} = 0 - 6x$

$\frac{\partial v}{\partial x} = -6x$

$\frac{\partial v}{\partial y} = 3y^2 - 3x^2$

$\cancel{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$

Now, $\frac{\partial u}{\partial x} = 3x^2 - 3y^2$

at $(0,0)$, $\frac{\partial u}{\partial x} = 0$

$\frac{\partial v}{\partial y} = 3y^2 - 3x^2$

at $(0,0)$, $\frac{\partial v}{\partial y} = 0$

So, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -6x$

$\therefore$ at $(0,0)$, $\frac{\partial u}{\partial y} = 0$

$\frac{\partial v}{\partial x} = -6x$

at $(0,0)$, $\frac{\partial v}{\partial x} = 0$

So, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

at $z=0$ / $z = x0 + i0$ point, the C-R equations are satisfied.

Differentiability:

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \to 0} \frac{(x-iy)^3 - 0}{x+iy}$$

$\left[(x-iy)^3 = (x^3 - 3xy^2) + i(y^3 - 3x^2y)\right]$

$$= \lim_{\cancel{(x,y)}\, z \to 0} \frac{(x-iy)^3}{x+iy}$$

$$= \lim_{z \to 0} \frac{f(z)^3}{z}$$

$$= \lim_{z \to 0} \frac{(\bar{z})^3}{z} = \frac{(\bar{z})^3}{0}$$ which is infinity

~~The~~ So, at $z = 0$, the function is not differentiable

[Showed]

c) Determine whether the function $3x^2y+2x^2-y^3-2y^2$ is harmonic or not.

$\boxed{\text{Sol}^n}$: $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$

$\frac{\partial u}{\partial x} = 6xy + 4x$; $\frac{\partial^2 u}{\partial x^2} = 6y + 4$ —— (i)

$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y$ $\quad \frac{\partial^2 u}{\partial y^2} = -6y - 4$ —— (ii)

Adding the equation (i) and (ii) we get,

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial y^2} = 6y+4-6y-4 = 0$$

As we know from Cauchy Reiman's equation,

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, so, $\frac{\partial v}{\partial y} = 6xy + 4x$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ $\Rightarrow$ $\frac{\partial v}{\partial x} = -(3x^2 - 3y^2 - 4y)$

$= 3y^2 + 4y - 3x^2$ —— (III)

$\int \frac{\partial v}{\partial y} = \int 6yx + 4x \Rightarrow \int \partial v = \int (6yx + 4x)\, \partial y$

$\Rightarrow v = 4xy + 6x\frac{y^2}{2} \times x + \overset{h(x)}{\cancel{c}} = 4xy + 3xy^2 + h(x)$

$\Rightarrow \frac{\partial v}{\partial x} = 4y + 3y^2 + h'(x)$ —— (iv)

Comparing the equation (III) and (iv),

$h'(x) = -3x^2$ $\Rightarrow \int h'(x)\, dx = -\int 3x^2\, dx$

$= \boxed{-x^3 + C}$

$\therefore \boxed{v = 4xy + 3xy^2 + C - x^3}$

[Showed]

$u = 3x^2y + 2x^2 - y^3 - 2y^2$

$v = 4xy + 3xy^2 - x^3 + C$

$\frac{\partial u}{\partial x} = 6xy + 4x$

$\frac{\partial v}{\partial y} = 4x + 6xy$

So, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

Again, $\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y$

$\frac{\partial v}{\partial x} = 4y + \cancel{6xy}\ 3y^2 - 3x^2$

So, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ The function satisfies C.R. equations.

2017

2b Evaluate $\oint_C \frac{e^{3z}}{z-\pi i}\,dz$ where

C is the curve $|z-2|+|z+2|=6$

Here,

$$|z-2|+|z+2|=6$$

$$\Rightarrow |x+iy-2|+|x+iy+2|=6$$

$$\Rightarrow \sqrt{(x-2)^2+y^2}+\sqrt{(x+2)^2+y^2}=6$$

$$\Rightarrow \sqrt{(x-2)^2+y^2}=6-\sqrt{(x+2)^2+y^2}$$

$$\Rightarrow (x-2)^2+\cancel{y^2}=36-12\sqrt{(x+2)^2+y^2}+(x+2)^2+\cancel{y^2}$$

$$\Rightarrow x^2-4x+4-x^2-4x-4=36-12\sqrt{(x+2)^2+y^2}$$

$$\Rightarrow -8x=36-12\sqrt{(x+2)^2+y^2}$$

$$\Rightarrow -2x=9-3\sqrt{(x+2)^2+y^2}$$

$$\Rightarrow 3\sqrt{(x+2)^2+y^2}=9+2x$$

$$\Rightarrow 9(x+2)^2+9y^2=81+4x^2+36x$$

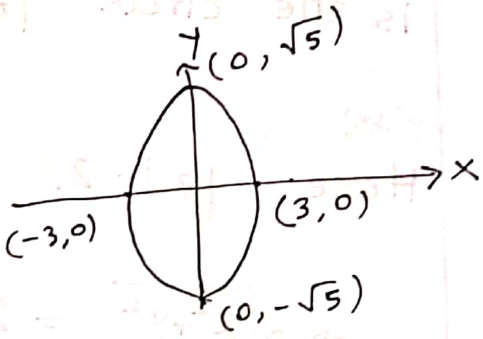
$$\Rightarrow 9x^2+\cancel{36x}+36+9y^2=81+4x^2+\cancel{36x}$$

Scanned with CamScanner

$$\Rightarrow \quad 5x^2 + 9y^2 = 45$$

$$\Rightarrow \quad \frac{x^2}{9} + \frac{y^2}{5} = 1$$

$$\Rightarrow \quad \frac{x^2}{3^2} + \frac{y^2}{(\sqrt{5})^2} = 1$$



Here, $f(z) = e^{3z}$

$z_0 = \pi i$, Here, $x = 0$ and, $y = \pi i$

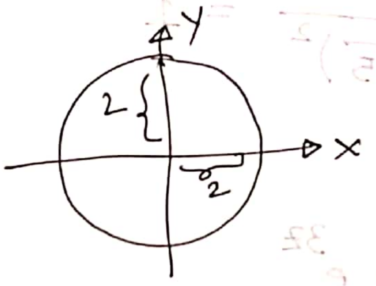
The point ~~z0~~, does not in the closed ~~p~~circle C

So, it would be 0.

(2a) See 2018's $\underline{\underline{6(a)}}$

c) Evaluate, $\oint_C \frac{e^{iz}}{z^3} dz$ where C is the circle $|z| = 2$

Here, $|z| = 2$



$\Rightarrow \sqrt{x^2+y^2} = 2$

$\Rightarrow x^2+y^2 = 2^2$

By comparing,

Here, $n+1 = 3$

$\therefore \boxed{n = 2}$

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_c \frac{f(z)}{(z-z_0)^{n+1}} dz$$

So, $\boxed{z_0 = 0}$

$$\frac{f^2(0) \times 2\pi i}{2!}$$

$$= \frac{-e^{i(z_0)} \times 2\pi i}{2}$$

$$= -e^{i\times 0} \times \pi i$$

$f(z) = e^{iz}$

$\Rightarrow f^1(z) = i e^{iz}$

$\Rightarrow f^2(z) = i \times i \times e^{iz}$

$= -e^{iz}$

$\Rightarrow f''(0) = -e^{i\times 0}$

$= -e^0$

$= 1$

2017

3a Define dependence of vectors.

Determine whether the vectors are linearly ~~it~~ independent or not.

$A = 2\underline{i} + 2\underline{j} - 3\underline{k}$

$B = \underline{i} - 4\underline{k} + \cancel{0k}\ 0j$

$C = 4\underline{i} + 3\underline{j} - \underline{k}$

Sol^n:

Linearly Dependent: Let $V(F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ of vectors of V is said to be linearly dependent if there exists scalar $\{a_1, a_2 \dots a_n \in F\}$ not all of them 0 (some of them may be zero) such that, $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \dots + a_n\alpha_n = 0$

Linearly Independent: Let, $V(F)$ be a vector

2017

3a Define dependence of vectors.

Determine whether the vectors are linearly ~~il~~ independent or not.

$A = 2\underline{i} + 2\underline{j} - 3\underline{k}$

$B = \underline{i} - 4\underline{k} + \cancel{0k}\ 0j$

$C = 4\underline{i} + 3\underline{j} - \underline{k}$

Sol^n:

Linearly Dependent: Let $V(F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ of vectors of V is said to be linearly dependent if there exists scalar $\{a_1, a_2 \ldots a_n \in F)$ not all of them 0 (some of them may be zero) such that, $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \ldots + a_n\alpha_n = 0$

Linearly Independent: Let, $V(F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ of vectors V is said to be linearly independent if every relation of the form $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \dots + a_n\alpha_n = 0$

$a_i \in F,\ 1 \leq i \leq n \Rightarrow a_i = 0$ for each $1 \leq i \leq n$

Taking the vectors in the coloumn matrix :

$$\begin{bmatrix} 2 & 1 & 4 \\ 1 & -4 & 3 \\ -3 & 0 & -1 \end{bmatrix} \quad R_2' = R_2 - \frac{1}{2}R_1 \quad \begin{bmatrix} 2 & 1 & 4 \\ 0 & -\frac{9}{2} & 1 \\ -3 & 0 & -1 \end{bmatrix}$$

$-9 - \frac{1}{2}$

$$R_3' = R_3 + \frac{3}{2}R_1 \Rightarrow \begin{bmatrix} 2 & 1 & 4 \\ 0 & -\frac{9}{2} & 1 \\ 0 & & \end{bmatrix}$$

$-3 + 2 \times \frac{3}{2}$

$$\begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & 3 \\ -3 & -4 & -1 \end{bmatrix} \quad R_2' = R_2 - \frac{1}{2}R_1 \Rightarrow \begin{bmatrix} 2 & 1 & 4 \\ 0 & -\frac{1}{2} & 1 \\ -3 & -4 & -1 \end{bmatrix}$$

$$R_3' = R_3 + \frac{3}{2}R_1 \Rightarrow \begin{bmatrix} 2 & 1 & 4 \\ 0 & -\frac{1}{2} & 1 \\ 0 & -\frac{5}{2} & -5 \end{bmatrix}$$

-5

$-9 + \frac{3}{2}$

$= \frac{-8+3}{2}$

$-1 + \frac{3}{2} \times 2$

$= -5$

Now, $R_3'' = R_3' - R_2' \times 5 = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -\frac{1}{2} & 1 \\ 0 & 0 & -10 \end{bmatrix}$

Rank = 3 , linearly independent

Scanned with CamScanner

2017

3) b] Find equations for the tangent plane and normal line to the surface $z-x^2-y^2=0$ at the point $(2,-1,5)$

$$\nabla (z-x^2-y^2)$$

$$= \hat{i} \frac{\partial}{\partial x} (z-x^2-y^2) + \hat{j} \frac{\partial}{\partial y} (z-x^2-y^2) + \hat{k} \frac{\partial}{\partial z} (z-x^2-y^2)$$

$$= \hat{i}\,[0-2x-0] + \hat{j}\,[0-0-2y] + \hat{k}\times(1)$$

$$= -2x\hat{i} - 2y\hat{j} + \hat{k}$$

$$= -4\hat{i} + 2\hat{j} + \hat{k}$$

c) ~~Let~~, Find the most general differentiable function $f(r)$ so that $\underline{r} f(r)$ is solenoidal

$\boxed{Sol^n}$: We know that, $\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\nabla \underline{r} \cdot f(r) = 0$$

$$\nabla (f(r) \cdot \underline{r}) = 0$$

$$\Rightarrow \nabla \cdot (f(r) \cdot \underline{r}) = (\nabla f(r)) \cdot r - f(r)(\nabla \cdot r)$$

$$= r f'(r) + 3f(r)$$

If $f(r)\underline{r}$ is solenoidal, then $\nabla \cdot (f(r)\underline{r}) = 0$

so that, $u = f(r)$ will satisfy

$$r \frac{du}{dr} + 3u = 0$$

$$\Rightarrow \frac{du}{u} = \frac{-3dr}{r}$$

$$\Rightarrow \ln|u| = -3\ln|r| + \ln|c|$$

$$\therefore u = Cr^{-3} \qquad \therefore f(r) = Cr^{-3}$$

2017

4) a $\int_C \vec{F}\, d\vec{r}$, where c is the curve in the xy plane, $y = x^3$ from the plan (1,1) to (2,8)

$\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$

$\vec{F} \cdot d\vec{r}$

$= \{(5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}\} \cdot \{(dx\hat{i} + dy\hat{j} + dz\hat{k})\}$

$= (5xy - 6x^2)\,dx + (2y - 4x)\,dy$

Here,

$y = x^3$

$\Rightarrow \frac{dy}{dx} = 3x^2$

$\Rightarrow \boxed{dy = 3x^2\,dx}$

$= (5 \times x \times x^3 - 6x^2)\,dx + (2x^3 - 4x)\,dy$

$= (5x^4 - 6x^2)\,dx + (2x^3 - 4x) \times 3x^2\,dx$

$= x^2(5x^2 - 6)\,dx + 3x^3(2x^2 - 4)\,dx$

$= \int_1^2 x^2(5x^2 - 6)\,dx + 3\int_1^2 x^3(2x^2 - 4)\,dx$

$= \int_1^2 5x^4\,dx - \int_1^2 6x^2\,dx + 6\int_1^2 x^5\,dx - 12\int_1^2 x^3\,dx$

$$\begin{array}{r} 97 \\ 62 \\ \hline 35 \end{array}$$

$$= 5\left[\frac{x^5}{5}\right]_1^2 + 6 \times \left[\frac{x^6}{6}\right]_1^2 - 6\left[\frac{x^3}{3}\right]_1^2$$

$$- 12\left[\frac{x^4}{4}\right]_1^2$$

$$= \left[x^5\right]_1^2 + \left[x^6\right]_1^2 - 2\left[x^3\right]_1^2 - 3\left[x^4\right]_1^2$$

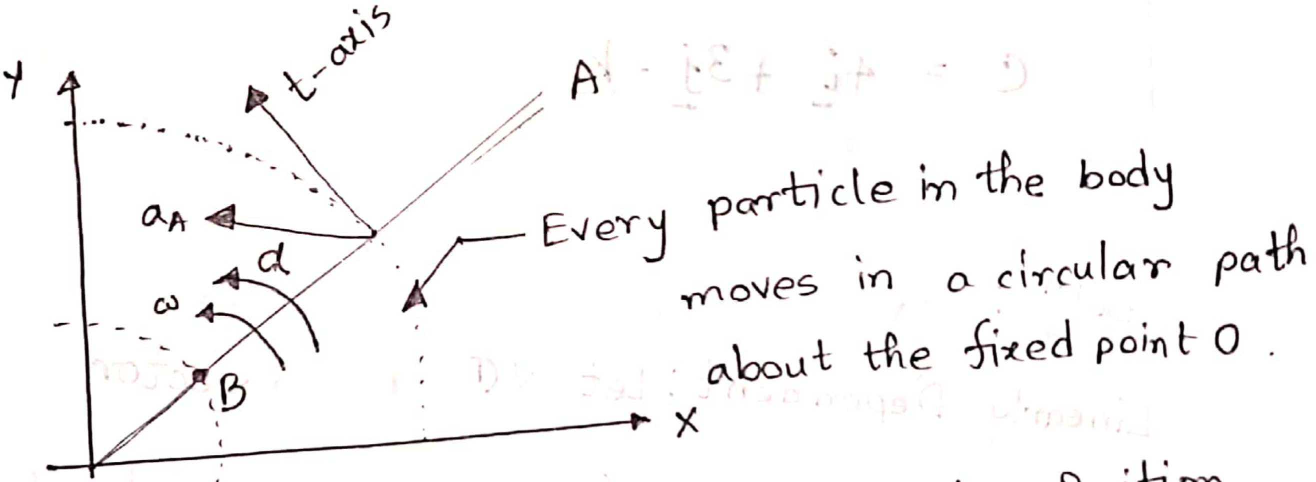
$$= [32-1] + [64-1] - 2[8-1] - 3[16-1]$$

$$= 31 + 63 - \cancel{2\times}\, 14 - 45 \qquad = 35$$

2017

4c) Define pure rotation and rotation plus translation.

Pure rotation occurs when a body rotates about a ~~non~~ fixed non-moving axes.



θ = Angular Position

ω = Angular velocity

α = Angular acceleration.

4) b $\quad A = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$

taken over the region in the first octant bounded by $y^2+z^2=3^2$ and $x=2$

$$\nabla \cdot A = \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2)$$

$$= 4yx - 2y + 8xz$$

For the limits on z axis as the radius is 3,

$$\Rightarrow \iiint_V \nabla \cdot F \, dv = \int_{z=-3}^{z=3} \int_{y=-\sqrt{9-z^2}}^{\sqrt{9-z^2}} \int_{x=0}^{2} (4yx - 2y + 8xz)\,dx\,dy\,dz$$

$$= \int_{z=-3}^{z} \int_{y=-\sqrt{9-z^2}}^{\sqrt{9-z^2}} \int_{x=0}^{2} (4yx - 2y + 8xz)\,dx\,dy\,dz$$

$$= \int_{z=-3}^{z=3} \int_{y=-\sqrt{9-z^2}}^{\sqrt{9-z^2}} \left[4x\frac{x^2y}{2} - 2yx + 4x^2z\right]_{x=0}^{2} dy\,dz$$

4) b

$$\int_{z=0}^{3}\int_{y=0}^{\sqrt{9-z^2}}\int_{x=0}^{2} (4xy - 2y + 8xz)\, dx\, dy\, dz$$

$$= \int_{0}^{3}\int_{y=0}^{y=\sqrt{9-z^2}} [4y + 16z]\, dy\, dz$$

$$= \int_{0}^{3} [2y^2 + 16yz]_0^{\sqrt{9-z^2}}\, dz$$

$$= \int_{0}^{2} \left(2(9-z^2) + 16\sqrt{9-z^2}\right) dz$$

$$= \cancel{18\pi} \left[18z - \frac{2z^3}{3} + 16(9-z^2)^{3/2}\left(\frac{2}{3} \times -\frac{1}{2}\right)\right]_0^3$$

$$= 18\times 3 - 2\times 3^2 + 0 - 16(9)^{2/3} \cancel{\times}\left(-\frac{1}{3}\right)$$

$$= 18\times 2 + 16(9)$$

$$= 180$$

Scanned with CamScanner