Math 2207 Prepared

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1\. a) The n nth roots of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ are given by

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right],$$

where, $k = 0, 1, \ldots\ldots, n-1$.

$$z = (2\sqrt{3}-2i) \Rightarrow z = 4\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

$r = 4$, $\theta = \frac{11\pi}{6}$.

now, $z^{\frac{1}{2}} = (2\sqrt{3}-2i)^{\frac{1}{2}}$

$$= \left\{4\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\right\}^{1/2}$$

$$= 2\left\{\cos\frac{\frac{11\pi}{6}+2k\pi}{2} + i\sin\frac{\frac{11\pi}{6}+2k\pi}{2}\right\}$$

if, $k=0$, $w_1 = 2$

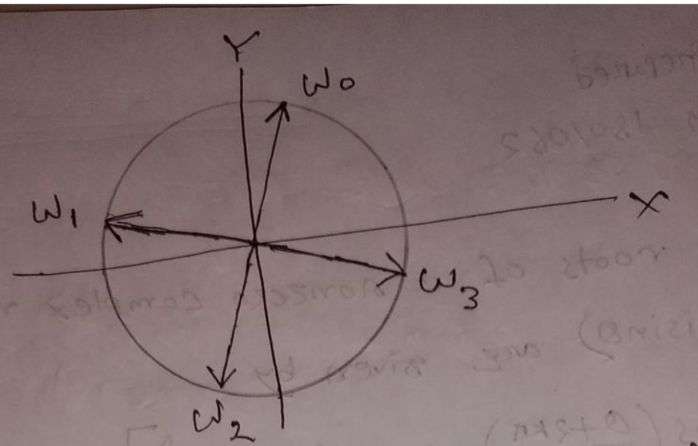
$$\therefore z^{\frac{1}{4}} = \sqrt[4]{4}\left[\cos\left(\frac{\frac{11\pi}{6}+2k\pi}{4}\right) + i\sin\left(\frac{\frac{11\pi}{6}+2k\pi}{4}\right)\right],$$

if, $k=0$, $w_0 = \sqrt{2}\left(\cos\frac{11\pi}{24} + i\sin\frac{11\pi}{24}\right)$

$k=1$, $w_1 = \sqrt{2}\left(\cos\frac{23\pi}{24} + i\sin\frac{23\pi}{24}\right)$

$k=2$, $w_2 = \sqrt{2}\left(\cos\frac{35\pi}{24} + i\sin\frac{35\pi}{24}\right)$

$k=3$, $w_3 = \sqrt{2}\left(\cos\frac{47}{24}\pi + i\sin\frac{47}{24}\pi\right)$



1.(b) Show that $\lim\limits_{z \to z_0} \dfrac{z}{\bar{z}}$ does not exist.

(Same as, Book: A Fist Course in Complex Analysis
Page: 113. Example 1)

EXAMPLE 1 A Limit That Does Not Exist

Show that $\lim\limits_{z\to 0} \dfrac{z}{\bar{z}}$ does not exist.

Solution We show that this limit does not exist by finding two different ways of letting z approach 0 that yield different values for $\lim\limits_{z\to 0} \dfrac{z}{\bar{z}}$. First, we let z approach 0 along the real axis. That is, we consider complex numbers of the form $z = x + 0i$ where the real number x is approaching 0. For these points we have:

$$\lim_{z\to 0} \frac{z}{\bar{z}} = \lim_{x\to 0} \frac{x+0i}{x-0i} = \lim_{x\to 0} 1 = 1. \tag{2}$$

On the other hand, if we let z approach 0 along the imaginary axis, then $z = 0 + iy$ where the real number y is approaching 0. For this approach we have:

$$\lim_{z\to 0} \frac{z}{\bar{z}} = \lim_{y\to 0} \frac{0+iy}{0-iy} = \lim_{y\to 0} (-1) = -1. \tag{3}$$

Since the values in (2) and (3) are not the same, we conclude that $\lim\limits_{z\to 0} \dfrac{z}{\bar{z}}$ does not exist.

1.(c) (Question খুব বড় তাই তুললাম না!)

Part: 1 $u(x,y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

we'll say u harmonic if $\nabla^2 u = 0$.

$$\nabla^2 u$$

$$= \frac{\partial^2}{\partial x^2} u + \frac{\partial^2 u}{\partial y^2}$$

$$= 6x + 6 - 6x - 6 = 0$$

$\therefore \nabla^2 u = 0$, so, $u(x,y)$ is harmonic.

Part 3:

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \quad \text{——— (i)}$$

and $$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6yx + 6y \quad \text{——— (ii)}$$

(i) $\Rightarrow$ $$V = 3x^2y - y^3 + 6xy + f(x)$$

$$\Rightarrow \frac{\partial V}{\partial x} = 6xy + 6y + f'(x)$$

$$\Rightarrow \cancel{f}\ 6xy + 6y = 6xy + 6y + f'(x) \quad [\text{from (ii)}]$$

$$\Rightarrow f'(x) = 0$$

$$\therefore f(x) = c$$

$$\therefore V = 3x^2y - y^3 + 6xy + c$$

$$\therefore f(z) = u + iv$$

$$= (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(3x^2y - y^3 + 6xy + c)$$

(Ans)

2.(a) Singular Points: A point z at which a complex function $w=f(z)$ fails to be analytic is called a singular point of f.

[* Details (Page-81) Schaums Complex Variable]

3.11 Singular Points

A point at which $f(z)$ fails to be analytic is called a *singular point* or *singularity* of $f(z)$. Various types of singularities exist.

1. **Isolated Singularities.** The point $z = z_0$ is called an *isolated singularity* or *isolated singular point* of $f(z)$ if we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no singular point other than z_0 (i.e., there exists a deleted δ neighborhood of z_0 containing no singularity). If no such δ can be found, we call z_0 a *non-isolated singularity*.

   If z_0 is not a singular point and we can find $\delta > 0$ such that $|z - z_0| = \delta$ encloses no singular point, then we call z_0 an *ordinary point* of $f(z)$.
2. **Poles.** If z_0 is an isolated singularity and we can find a positive integer n such that $\lim_{z\to z_0} (z - z_0)^n f(z) = A \neq 0$, then $z = z_0$ is called a *pole of order n*. If $n = 1$, z_0 is called a *simple pole*.

   EXAMPLE 3.1

   (a) $f(z) = 1/(z-2)^3$ has a pole of order 3 at $z = 2$.

   (b) $f(z) = (3z-2)/(z-1)^2(z+1)(z-4)$ has a pole of order 2 at $z = 1$, and simple poles at $z = -1$ and $z = 4$.

   If $g(z) = (z - z_0)^n f(z)$, where $f(z_0) \neq 0$ and n is a positive integer, then $z = z_0$ is called a *zero of order n* of $g(z)$. If $n = 1$, z_0 is called a *simple zero*. In such a case, z_0 is a pole of order n of the function $1/g(z)$.

3. **Branch Points** of multiple-valued functions, already considered in Chapter 2, are non-isolated singular points since a multiple-valued function is not continuous and, therefore, not analytic in a deleted neighborhood of a branch point.

EXAMPLE 3.2

(a) $f(z) = (z - 3)^{1/2}$ has a branch point at $z = 3$.

(b) $f(z) = \ln(z^2 + z - 2)$ has branch points where $z^2 + z - 2 = 0$, i.e., at $z = 1$ and $z = -2$.

4. **Removable Singularities.** An isolated singular point z_0 is called a *removable singularity* of $f(z)$ if $\lim_{z \to z_0} f(z)$ exists. By defining $f(z_0) = \lim_{z \to z_0} f(z)$, it can then be shown that $f(z)$ is not only continuous at z_0 but is also analytic at z_0.

EXAMPLE 3.3 The singular point $z = 0$ is a removable singularity of $f(z) = \sin z/z$ since $\lim_{z \to 0} (\sin z/z) = 1$.

5. **Essential Singularities.** An isolated singularity that is not a pole or removable singularity is called an *essential singularity*.

EXAMPLE 3.4 $f(z) = e^{1/(z-2)}$ has an essential singularity at $z = 2$.

If a function has an isolated singularity, then the singularity is either removable, a pole, or an essential singularity. For this reason, a pole is sometimes called a *non-essential singularity*. Equivalently, $z = z_0$ is an essential singularity if we cannot find any positive integer n such that $\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$.

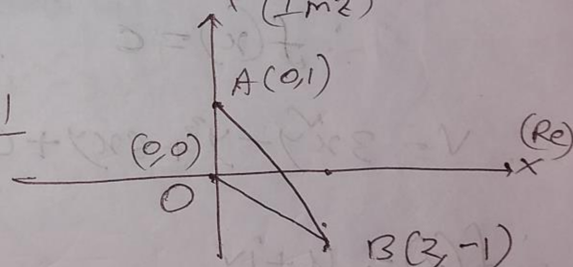
6. **Singularities at Infinity.** The type of singularity of $f(z)$ at $z = \infty$ [the point at infinity; see pages 7 and 47] is the same as that of $f(1/w)$ at $w = 0$.

EXAMPLE 3.5 The function $f(z) = z^3$ has a pole of order 3 at $z = \infty$, since $f(1/w) = 1/w^3$ has a pole of order 3 at $w = 0$.

2.(b) Evaluate $\int_{i}^{2-i} f(3xy+iy^2)\,dz$ along the straight line joining $z=i$ and $z=2-i$.

Ans: OA বরাবর, $z = iy$

AB, এর eq^n $\quad \dfrac{x-0}{0-2} = \dfrac{y-1}{1+1}$



$\therefore x+y-1=0.$

$\Rightarrow y = 1-x$

$dy = -dx$

$$I = \int_0^2 [3x(1-x) + i(1-x^2)][dx + i(-dx)]$$

$$= \int_0^2 (3x - 3x^2 + i - 2xi + ix^2)(1-i)\,dx$$

since $\int_C f(z)\,dz = \int (u+iv)(dx+idy)$

$$= \int_0^2 \left(3x - 3xi - 3x^2 + 3ix^2 + i + 1 - 2xi + 2x + ix^2 - x^2\right) dx$$

$$= \int_0^2 \left(5x - 5xi - 4x^2 + 4ix^2 + i + 1\right) dx$$

$$= \left[\frac{5x^2}{2} - \frac{5x^2 i}{2} - \frac{4x^3}{3} + \frac{4ix^3}{3} + \frac{ix}{2} + \frac{x}{2}\right]_0^2$$

$$= 10 - 10i - \frac{32}{3} + \frac{32}{3}i + i + 1$$

$$= \frac{1}{3} + i\,\frac{5}{3}$$

$$= \frac{1}{3}(1 + 5i) \quad \text{(Ans.)}$$

2.(c) Define Cauchy's integral theorem. Evaluate $\oint_C \frac{dz}{z-a}$ where C is any simple closed curve and $z = a$ is (i) inside C. (ii) Outside C.

Ans: Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then for any point z_0 within C,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz.$$

(i) when ~~a~~ inside C

when inside, Cauchy's integral formula

$$\oint \frac{dz}{z-a} = 2\pi i f(a)$$

$$= 2\pi i \quad \text{since } f(z) = 1$$

$$\therefore f(a) = 1.$$

(ii) when outside C.

$$\oint \frac{dz}{z-a} = 0 \quad \text{[Cauchy's theorem]}$$

3. (a)

let, $\phi = 4x^2 y + z^3$

$$\nabla\phi = 8xy\hat{i} + 3z^2\hat{k} + 4x^2\hat{j}$$

at point $(1, -1, 2)$, $\nabla\phi = -8\hat{i} + 12\hat{k} + 4\hat{j}$

let, $r = ax^2 - (a+2)x - byz$

$$\nabla r = \{2ax - (a+2)\}\hat{i} - bz\hat{j} - by\hat{k}$$

at point $(1, -1, 2)$

$$\nabla r = (a-2)\hat{i} - 2b\hat{j} + b\hat{k}$$

since, φ and r are orthogonal

$\nabla\varphi \cdot \nabla r = 0$

$\Rightarrow$ $2a + 5b - 4 = 0$ ——— (i)

given, $ax^2 - byz = (a+2)x$

in $(1,-1,2)$.

$a + 2b = a + 2$

$\therefore b = 1$

from (i) $a = -\frac{1}{2}$

(Ans)

3.(b) given,

$$\bar{F} = (2xy^3z - y + 2)\hat{i} + (3x^2y^2z - x + z)\hat{j} + (x^2y^2 + y - 2)\hat{k}$$

$\bar{F}$ is rotational if curl $\bar{F} \neq 0$.

$$\bar{\nabla} \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy^3z - y + 2) & (3x^2y^2z - x + z) & (x^2y^2 + y - 2) \end{vmatrix}$$

$$= \hat{i}\left[3x^2y^2+1-3x^2y^2-1\right]+\hat{j}\left[2xy^3-2xy^3\right]$$

$$+\hat{k}\left[6xy^2z-1-6xy^2z+1\right]$$

$$=0$$

$\therefore F$ is irrotational

$$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial z}=F$$

$$\therefore \frac{\partial\phi}{\partial x} = 2xy^3z-y+2 \quad \text{——— (i)}$$

$$\therefore \phi = x^2y^3z-xy+2x+h(y,z)$$

$$\frac{\partial\phi}{\partial y} = 3x^2y^2z-x+z \quad \text{——— (ii)}$$

$$\therefore \phi = x^2y^3z-xy+yz+k(x,z)$$

$$\frac{\partial\phi}{\partial z} = x^2y^3+y-2 \quad \text{——— (iii)}$$

$$\therefore \phi = x^2y^3z+yz-2z+L(x,y)$$

$$\therefore \frac{\partial\phi}{\partial x} = 2xy^3z+L'(x,y)$$

$$L(x,y) = -xy+2x \quad [\text{From (i)}]$$

similarly, $h(y,z) = yz - 2z$

$g(x,z) = 2x - 2z$

$\therefore \varphi = x^2y^3z + yz - 2z - xy + 2x$.

at $(1,-2,3)$

$\varphi = -32$.

4.(a)

Surface integral $= \iint_S A\,n\,ds$

$= \iint_{S_1} A\,n\,ds_1 + \iint_{S_2} A\,n\,ds_2 + \iint_{S_3} A\,n\,ds_3$

On S_1 $(z=0)$

$n = -k$, $F = 6xz\hat{i} + 2x\hat{j} - y\hat{k}$. $\bar{F}\cdot\bar{n} = 0$.

$\therefore \iint_{S_1} \bar{F}\cdot\bar{n}\,ds_1 = 0$.

On S_2 $(z=8)$

$n = k$, $F = 6xz\hat{i} + 2x\hat{j} - y\hat{k}$, $\bar{F}\cdot\bar{n} = -y$

$\iint_{S_2} \bar{F}\cdot\bar{n}\,ds_2 = \iint -y \cdot \frac{dx\,dy}{n\cdot k}$

$= \int_{x=0}^{2} \int_{y=0}^{\sqrt{4-x^2}} -y\,dy\,dx$

$$= -\int_{x=0}^{2} \left[\frac{y^2}{2}\right]_0^{\sqrt{4-x^2}} dx$$

$$= -\frac{1}{2}\int_{x=0}^{2} (4-x^2)\,dx$$

$$= -\frac{1}{2}\left[4x - \frac{x^3}{3}\right]_0^2$$

$$= -\frac{1}{2}\left(8 - \frac{8}{3}\right)$$

$$= -\frac{8}{3}.$$

On S_3 $x^2+y^2=4$

$$\nabla(x^2+y^2) = 2x\hat{i}+2y\hat{j},\ n = \frac{2x\hat{i}+2y\hat{i}}{\sqrt{4x^2+4y^2}} = \frac{x\hat{i}+y\hat{i}}{2}$$

$$F.n = (6xz\hat{i}+2x\hat{j}-y\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{i}}{2}\right)$$

$$= 3x^2z+xy$$

$x = 2\cos\theta, \quad y = 2\sin\theta, \quad ds_3 = 2\,d\theta\,dz$

$$\iint_{S_3} F\cdot n \, ds_3 = \int_{\theta=0}^{2\pi}\int_{z=0}^{8} 3(2\cos\theta)^2 z + (2\cos\theta)(2\sin\theta) \; 2dz\, d\theta$$

$$= 2\int_{\theta=0}^{2\pi} \left[6\cos^2\theta \, z^2 + 4\cos\theta\sin\theta \, z \right]_0^8 d\theta$$

$$= 16\int_{\theta=0}^{2\pi} \left[48\cos^2\theta + 4\cos\theta . \sin\theta \right] d\theta$$

$$= 64\int_{\theta=0}^{2\pi} \left[16\cos^2\theta + \cos\theta\sin\theta \right] d\theta$$

$$= 64\int_{\theta=0}^{2\pi} (1-\cos 2\theta)\, d\theta + 32\int_{\theta=0}^{2\pi} \sin 2\theta \, d\theta$$

$$= 64 \times 8 \left[\theta\right]_0^{2\pi} - 64.4\left[\sin 2\theta\right]_0^{2\pi} + 16\left[-\cos 2\theta\right]_0^{2\pi}$$

$$= 64 \cdot 8 \cdot 2\pi + 16(-2)$$

$$= 32(32\pi - 1)$$

$$\therefore \iint_S F\cdot \hat{n} \cdot ds = 32(32\pi - 1) - \frac{8}{3} \; \text{(Ans)}$$

4.(b) The divergence $\nabla \cdot F$ of ~~the~~ a vector field $\bar{F}$ is defined as the net amount of the flux of the vector field diverging or converging per unit volume.

Given, $\phi = x^3 y^2 z$,

$\nabla\phi = 3x^2y^2z\hat{i} + 2x^3yz\hat{j} + x^3y^2\hat{k}$

at $(1, 4, -1)$, $\nabla\phi = -48\hat{i} - 32\hat{j} + 16\hat{k}$

Directional directives along z axis.

$\nabla\phi \cdot \hat{k} = 16$

$\frac{d\phi}{ds} = \nabla\phi \frac{dr}{ds}$ is the projection of $\nabla\phi$ in the direction $\frac{dr}{ds}$. This projection will be maximum when $\nabla\phi$ and $\frac{dr}{ds}$ have the same direction. Then the maximum value of $\frac{d\phi}{ds}$ takes place in the direction of $\nabla\phi$.

1st Part:

4(c) Green's theorem

If R is a closed region of the xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then.

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

where C is traversed in the positive (counter clockwise) direction.

2nd Part: In Green's theorem,

Put, $M=-y$, $N=x$ then,

$$\oint_C x dy - y dx = \iint_R \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\, dx\, dy$$

$$= 2 \iint_R dx\, dy$$

$= 2A$, where A is the required area.

$$\therefore A = \frac{1}{2} \oint_C x dy - y dx$$

(showed)

Thank You