2018 − (b), 6, 7, 8

5 $\boxed{a}$ Define whether $\lim_{z \to 0} t^{\frac{z}{\bar{z}}}$ exists or not

$\boxed{Sol^n}$: $\lim_{z \to 0} t^{\frac{z}{\bar{z}}}$

If we want to progress along x axis,

$$\lim_{x \to 0} t^{\frac{x+iy}{x-iy}}$$

$$= \lim_{x \to 0} t^{\frac{x}{x}}$$

$$= t^{1} \cancel{=} t$$

$$= t$$

On the other hand, if we want to progress along the y axis,

$$\lim_{y \to 0} t^{\frac{x+iy}{x-iy}}$$

$$\lim_{y \to 0} t^{\frac{iy}{-iy}}$$

$$= t^{-1}$$

$$= \frac{1}{t}$$

So, the limit does not exist.

b) Examine the differentiability and continuity of $f(z) = |z|^2$ at origin

$$f(z) = |z|^2$$

$$\Rightarrow f(z) = \left(\sqrt{x^2+y^2}\right)^2$$

$$\Rightarrow f(z) = x^2+y^2$$

$f(z)$ is continous, because, it is polynomial.

$$f(z) = x^2+y^2$$

$f(0)$ $\quad z_0 = 0$

$$f(z_0) \cancel{= \lim_{z \to z_0}} \frac{\cancel{f(z)} - f(z_0)}{}$$

$$f(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{z \to 0} \frac{x^2+y^2 - [0^2+0^2]}{z - 0}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2+y^2}{x+iy}$$

If we want to progress along the x axis

$$\lim_{x \to 0} \frac{x^2+0}{x+0 \times i}$$

$$= \lim_{x \to 0} (x) = 0$$

$42 = 2$

$z + 42$

$z \pm z_0 -$

$f(z_0)$

$f(z+4z) -$

If we want to progress along y axis

$$\lim_{y \to 0} \frac{x^2+y^2}{x+iy}$$

$$= \lim_{y \to 0} \frac{y^2}{iy}$$

$$= \lim_{y \to 0} \frac{y}{i}$$

$$= 0$$

Again, if we want to progress along $y = mx$

$$\lim_{x \to 0} \frac{x^2+m^2x^2}{x+i(mx)}$$

$$\Rightarrow \lim_{x \to 0} \frac{x^2(1+m^2)}{x(1+mi)}$$

$$= \lim_{x \to 0} \frac{x(m^2+1)}{mi+1} = 0$$

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The function $f(z)$ is continous and differentiable at orgin.

$\boxed{c}$ Prove that the function $f(z) = \frac{xy^2(x+iy)}{x^2+y^4}$, $z \neq 0$; 0, $z = 0$ is not analytic at origin although C.R. equations are satisfied there

$\frac{\delta u}{\delta x}$ $\boxed{\text{Sol}^n}$ according to Cauchy's Reimann formula

We know, $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$ and, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$f(z) = \frac{x^2y^2 + xy^3 i}{x^2+y^4}$$

$$= \frac{x^2y^2}{x^2+y^4} + \left(\frac{xy^3}{x^2+y^4}\right)i$$

Here, $u = \frac{x^2y^2}{x^2+y^4}$, $v = \frac{xy^3}{x^2+y^4}$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2y^2}{x^2+y^4}\right)$$

$$= \frac{(x^2+y^4)\times\frac{\partial}{\partial y}(x^2y^2) - x^2y^2\times\frac{\partial}{\partial y}(x^2+y^4)}{(x^2+y^4)^2}$$

$$= \frac{(x^2+y^4)\times\left[2yx^2\right] - x^2y^2\times 4y^3}{(x^2+y^4)^2}$$

$$= \frac{2x^4y + 2y^5x^2 - 4x^2y^5}{(x^2+y^4)^2}$$

$$= \frac{2x^4y - 2y^5x^2}{(x^2+y^4)^2}$$

$$= \frac{2yx^2(x^2-y^4)}{(x^2+y^4)^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2y^2}{x^2+y^4}\right)$$

$$= \frac{(x^2+y^4)\times 2xy^2 - x^2y^2\times 2x}{(x^2+y^4)^2}$$

$$= \frac{2x^3y^2 + 2xy^6 - 2x^3y^2}{(x^2+y^4)^2}$$

$$= \frac{2xy^6}{(x^2+y^4)^2}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\frac{xy^3}{x^2+y^4}\right)$$

$$= \frac{(x^2+y^4)\times 3y^2x - xy^3\times(4y^3)}{(x^2+y^4)^2}$$

$$= \frac{3x^3y^2 + 3y^6x - 4y^6x}{(x^2+y^4)^2}$$

$* \frac{y^4}{x^2}$

$$= \frac{3x^3y^2 - y^6x}{(x^2+y^4)^2}$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{xy^3}{x^2+y^4}\right)$$

$$= \frac{(x^2+y^4)\times y^3 - xy^3 \times 2x}{(x^2+y^4)^2}$$

$$= \frac{x^2y^3 + y^7 - 2x^2y^3}{(x^2+y^4)^2}$$

$$= \frac{y^7 - x^2y^3}{(x^2+y^4)^2}$$

$$= \frac{y^3(y^4-x^2)}{(x^2+y^4)^2} \quad \cancel{=0}$$

Here, at $(x,y)\equiv(0,0)$ or $z=0$ point

$$\frac{\partial v}{\partial x}=0, \quad \frac{\partial v}{\partial y}=0, \quad \frac{\partial u}{\partial x}=0, \quad \frac{\partial u}{\partial y}=0$$

satisfying C.R. equations,

2018) 5c

In order to be an analytic function, the function needs to satisfy CR equation, differiable, continu

$$f(z) = \begin{cases} \dfrac{xy^2(x+iy)}{x^2+y^4} ; & z \neq 0 \\ 0 ; & z = 0 \end{cases}$$

Differentiability:

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \to 0} \frac{\frac{xy^2(x+iy)}{x^2+y^4} - 0}{z - 0}$$

$$= \lim_{z \to 0} \frac{xy^2(x+iy)}{(x+iy)(x^2+y^4)}$$

$$= \cancel{\lim_{x \to 0}} \lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy^2}{(x^2+y^4)}$$

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Along the y axis,

$$\lim_{y \to 0} \frac{0 \times y^2}{0^2 + y^4} = \lim_{y \to 0} \frac{0}{y^4} = 0$$

Along the x axis,

$$\lim_{x \to 0} \frac{x \times 0^2}{x^2 + 0^4}$$

$= 0$

Along the $y = \sqrt{x}$

$$= \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4}$$

$y^2 = x$

$y^4 = x$

$$= \lim_{x \to 0} \frac{x \times x}{x^2 + x^2}$$

$$= \lim_{x \to 0} \frac{x^2}{2x^2}$$

$$= \frac{1}{2}$$

So, the limits are not equal, It

is not differentiable. ~~at origin~~.

So, the function is not analytic.

6) i) Singular Point :

A point at which a given function a complex variable has no derivative of which every neighborhood contains points at which the function has derivatives.

iv) Removable Singular Point : A removable singular point of a function which it is possible to assign a complex number in such a way, that becomes analytic.

ii) Pole : The pole of a function is an isolated singular point "a" of a single-valued character of an analytic function $f(z)$ of the complex variable z for which $|f(z)|$ increases without bound when z approaches a : $\lim_{z \to a} f(z) = \infty$

6) b

i) $f(z) = \dfrac{e^{-z}}{(z-2)^4}$

$= \dfrac{1}{e^z (z-2)^4}$

$\lim_{z \to z_0} (z-2)^4 f(z)$

Here, $z_0 = 2$

$\Rightarrow \lim_{z \to 2} (z-2)^4 \times \dfrac{1}{e^z (z-2)^4}$

$\Rightarrow \dfrac{1}{e^z} \neq 0$; [which is non zero value]

isolated singular point.

pole of order 4

i) $f(z) = \sin\left(\frac{1}{z-1}\right)$

We know,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\Rightarrow \sin\left(\frac{1}{z-1}\right) = \frac{1}{(z-1)} - \frac{1}{(z-1)^3 \times 3!} + \frac{1}{(z-1)^5 \times 5!} - \frac{1}{(z-1)^7 \times 7!} + \cdots \infty$$

So, it i-

essential singular point.

c) Using Cauchy's integral formula evaluate, $\oint_c \frac{z\,dz}{(9-z^2)(z+i)}$, where C is the circle $|z|=2$

$$\oint \frac{\frac{z}{(9-z^2)}}{(z+i)}\,dz$$

Here, $f(z) = \frac{z}{9-z^2}$

$z_0 = -i$

$= 2\pi i \times f(-i)$

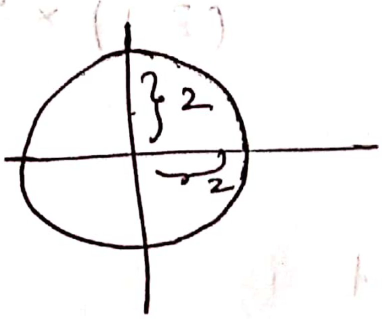
$= 2\pi i \times \left(\frac{x+iy}{9-x^2+2xyi+y^2}\right)$

$= 2\pi i \times \left(\frac{-i}{9-0+0+1}\right)$

Here, $|z|=2$

$\Rightarrow \sqrt{x^2+y^2} = 2$

$\Rightarrow x^2+y^2 = 2^2$



We know,

$x+iy$

$\boxed{y=-1}$

$f(z_0) \times 2\pi i = \oint_c \frac{f(z)}{z-z_0}$

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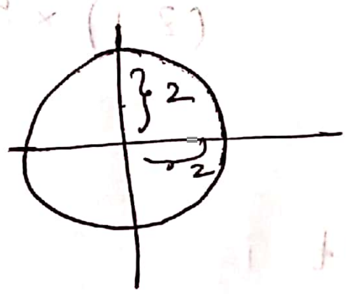
c) Using Cauchy's integral formula evaluate, $\oint_c \frac{z\,dz}{(9-z^2)(z+i)}$, where C is the circle $|z|=2$

$$\oint \frac{\frac{z}{(9-z^2)}}{(z+i)}\,dz$$

Here, $|z|=2$

$\Rightarrow \sqrt{x^2+y^2}=2$

$\Rightarrow x^2+y^2=2^2$



Here, $f(z)=\frac{z}{9-z^2}$

$9-(i)^2$
$9-i^2$
$=9+1$

$z_0=-i$

We know, $x+iy$ $\boxed{y=-1}$

$f(z_0)\times 2\pi i = \oint_c \frac{f(z)}{z-z_0}$

$= 2\pi i \times f(-i)$

$\frac{i}{9-i^2}$

$\frac{-i}{10}$

$(x+iy)^2 = x^2+2xiy-y^2$

$= 2\pi i \times \left(\frac{x+iy}{9-x^2+2xyi+y^2}\right)$

$= 2\pi i \times \left(\frac{-i}{9-0+0+1}\right)$

$-i^2$
$=-(-1)$

$= \frac{2\pi i \times -i}{10} = \frac{2\pi}{10} = \frac{\pi}{5}$

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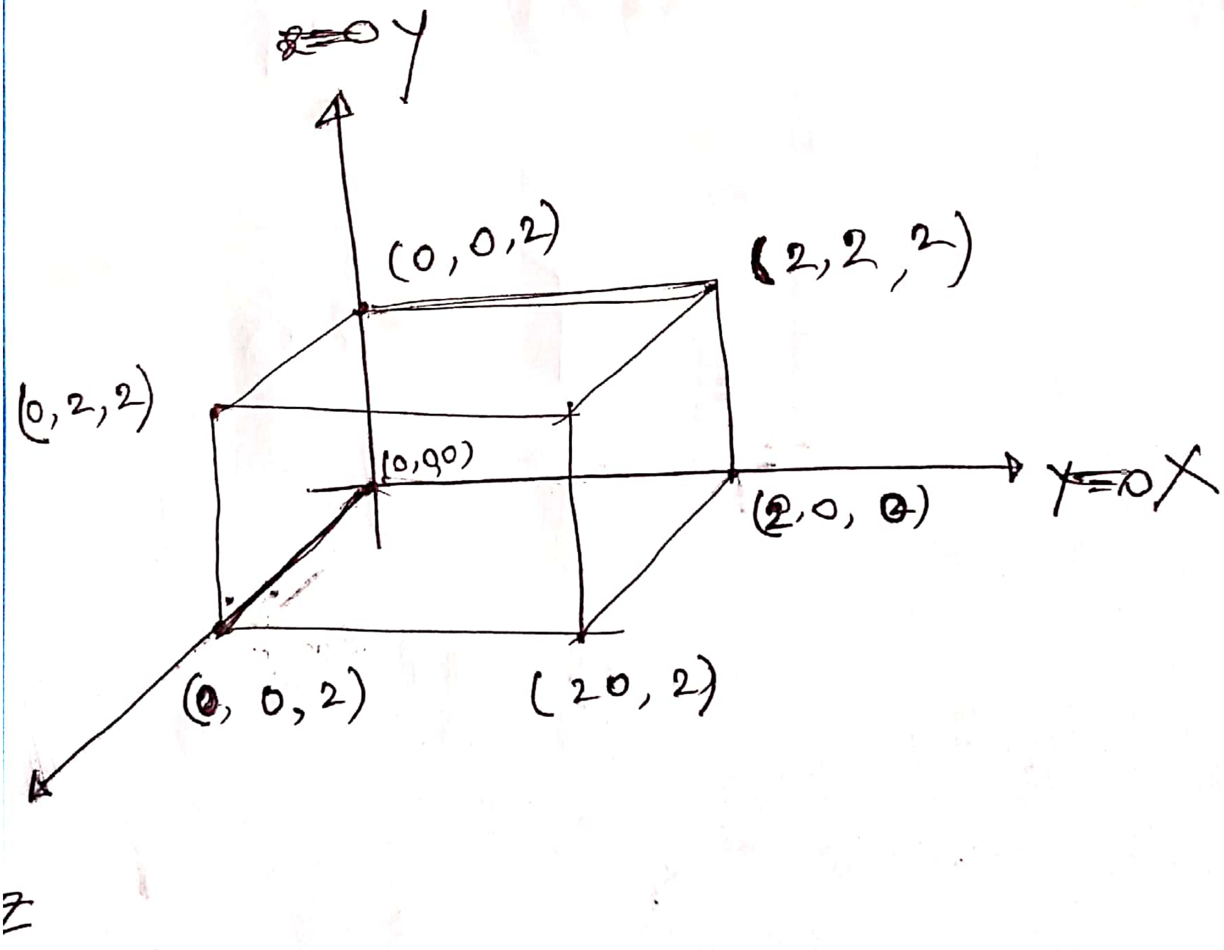
7) [a] Using appropriate theorem evaluate

$\oiint_S \vec{F} \cdot n\, ds$ where s is the surface bounded

by the planes : $x=0,\ y=0,\ z=0,\ x=2,\ y=2,$

$z=2$ and $\vec{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$



The appropriate theorem is divergence theorem.

$$\iiint_V \nabla \cdot \vec{F}\, dv = \iint_S F\, n\, dS$$

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot \left(x^2\hat{i} + y^2\hat{j} + z^2\hat{k} \right)$$

$$= \frac{\partial (x^2)}{\partial x} + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$

$$\iiint_V (2x + 2y + 2z)\, dV$$

$$= \iiint_V (2x + 2y + 2z)\, dx\, dy\, dz$$

$$\cancel{= \int_0^2\int_0^2\int_0^2 (2x}$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2}\int_{z=0}^{z=2} (2x + 2y + 2z)\, dx\, dy\, dz$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2} \left[\left(2xz + 2yz + z^2 \right) \right]_0^{z=2} dy\, dx$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=2} \left[2x(2-0) + 2y(2-0) + (2-0)^2 \right] dy\, dx$$

$$= \int_{x=0}^{x=2} \int_{y=0}^{y=2} (4x + 4y + 4)\, dy\, \cancel{dz}\, dx$$

$$= \int_{x=0}^{x=2} \left[4xy + 2y^2 + 4y \right]_{y=0}^{y=2} d\cancel{z}$$

$$= \int_{x=0}^{x=2} \left[4x(2-0) + 2\times(2-0)^2 + 4(2-0) \right] dx$$

$$= \int_{x=0}^{x=2} \left[8x + 8 + 8 \right] dx$$

$$= \int_{x=0}^{x=2} (8x + 16)\, dx$$

$$= \left[8 \times \frac{x^2}{2} \right]_0^2 + 16 \left[x \right]_0^2$$

$$= 4 \times \left[2^2 - 0^2 \right] + 16 \times \left[2 - 0 \right]$$

$$= 4 \times 4 + 32 = 48$$

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7) b)

$F = 3xy\hat{i} - y^2\hat{j}$, evaluate $\int_C F dr$, where C is the curve in the xy plane, $y = 2x^2$ from (0,0, to (1,2)

$F.dr$

$= (3xy\hat{i} - y^2\hat{j})(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$= 3xy\,dx - y^2 dy$

Here,

$y = 2x^2$

$\Rightarrow dy = 2 \times 2x$

$\int_C F.dr$

$= \int 3xy\,dx - y^2 dy$

$= \int 3x \times 2x^2\,dx - 4x^4 \times 4x\,dx$

$= \int_0^1 6x^3\,dx - 16\int_0^1 x^5\,dx$

$= 6 \times \left[\frac{x^4}{4}\right]_0^1 - \frac{16}{6}\left[x^6\right]_0^1$

$= \frac{6}{4} - \frac{16}{6} \qquad = -\frac{7}{6}$

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7c) Evaluate grad div $\left(\frac{\vec{r}}{r}\right)$

$$\left(\frac{\vec{r}}{r}\right) = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}$$

$$\nabla\cdot\left(\frac{\vec{r}}{r}\right) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot$$

$$\left[\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\hat{i} + \left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\hat{j} + \left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\hat{k}\right]$$

$$\nabla\cdot\left(\frac{\vec{r}}{r}\right) = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) +$$

$$\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\nabla\cdot\left(\frac{\vec{r}}{r}\right) = \frac{\sqrt{x^2+y^2+z^2}\times 1 - x\times\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$$

$$+ \frac{\sqrt{x^2+y^2+z^2}\times 1 - \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2+z^2}}\times y}{x^2+y^2+z^2} +$$

$$\frac{\sqrt{x^2+y^2+z^2}\times 1-\left(\frac{2x\times z}{\cancel{2}\sqrt{x^2+y^2+z^2}}\times z\right)}{x^2+y^2+z^2}$$

$$\Rightarrow \nabla\cdot\left(\frac{\vec{r}}{r}\right)=\frac{\frac{x^2+y^2+z^2-x^2}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}+\frac{\frac{x^2+y^2+z^2-y^2}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}+\frac{\frac{x^2+y^2+z^2-z^2}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$$

$$\Rightarrow \nabla\cdot\left(\frac{\vec{r}}{r}\right)=\frac{y^2+z^2+x^2+z^2+x^2+y^2}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \nabla\cdot\left(\frac{\vec{r}}{r}\right)=\frac{2\cancel{(x^2+y^2+z^2)}}{\cancel{(x^2+y^2+z^2)}\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \phi(x,y,z)=\frac{2(x^2+y^2+z^2)^1}{(x^2+y^2+z^2)^{3/2}}$$

Let,

$$\phi(x,y,z)=\frac{2}{\sqrt{x^2+y^2+z^2}}$$

$$\nabla \cdot \phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right)$$

$$\left(\frac{2}{\sqrt{x^2+y^2+z^2}} \right)$$

$$= 2x \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}} \right)\hat{i} + 2x \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{x^2+y^2+z^2}} \right)$$

$$+ 2x \frac{\partial}{\partial z}\left(\frac{1}{\sqrt{x^2+y^2+z^2}} \right)\hat{k}$$

$$= \left(2x - \frac{1}{2} \times (x^2+y^2+z^2)^{-3/2} \times 2x\right) +$$

$$\left(2x - \frac{1}{2} \times (x^2+y^2+z^2)^{-3/2} \times 2y\right) +$$

$$\left(2x - \frac{1}{2} \times (x^2+y^2+z^2)^{-3/2} \times 2z\right)$$

$$= -2(x^2+y^2+z^2)^{-3/2}\left[x+y+z\right]$$

$$= -2x(x^2+y^2+z^2)^{-1/2}$$

8 [a]

Test whether the vector field $\vec{F}$ is rotational or not. If possible find the scalar potential such that the scalar solution be zero at (0,0,0). Given, $\vec{F} = \frac{\vec{r}}{r^2}$

Sol^n : $\vec{F} = \frac{\vec{r}}{r^2} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\right] -$$

$$\hat{j}\left[\frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right.$$

$$\left. + \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right]\right.$$

$$= \hat{i}\left[\cancel{\frac{(\sqrt{x^2+y^2+z^2})\times 0 - z \times}{x^2+y^2+z^2}} - \frac{\sqrt{x^2+y^2+z^2}}{(x^2+y^2+z^2)}\right.$$

$$= \hat{i}\left\{\left[\frac{\sqrt{x^2+y^2+z^2}\times 0 - z\times \frac{1\times y}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2} - \frac{\sqrt{x^2+y^2+z^2}\times 0 - y\times \frac{\cancel{2}\times z}{\cancel{2}\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}\right]\right\} -$$

$$\hat{j}\left\{\left[\frac{\sqrt{x^2+y^2+z^2}\times 0 - z\times \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2} - \frac{\sqrt{x^2+y^2+z^2}\times 0 - x\times \frac{\cancel{2}z}{\cancel{2}\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}\right]\right\} +$$

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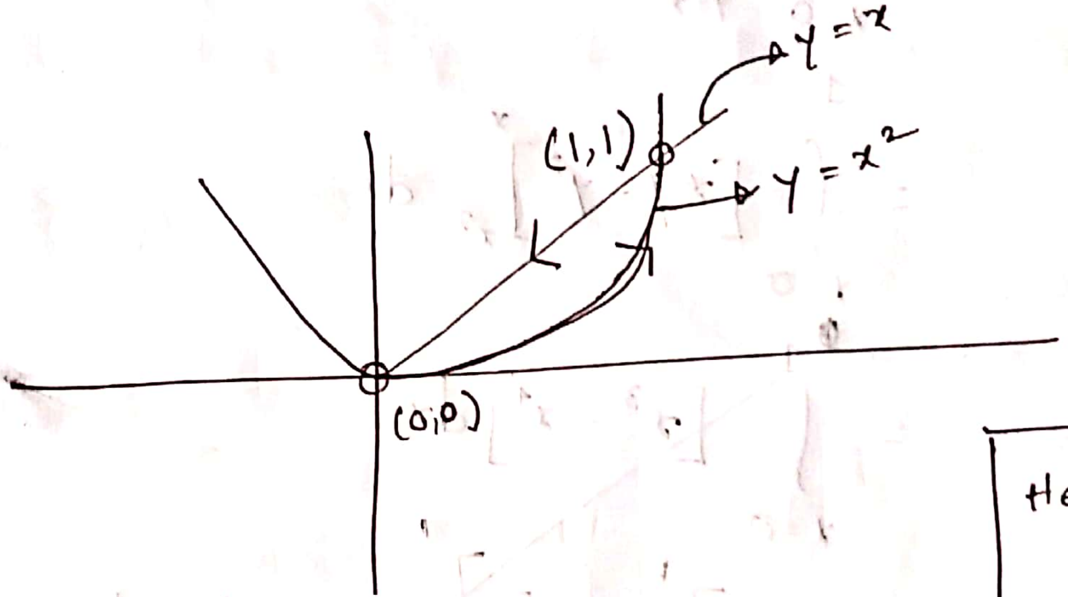
$$\hat{k}\left\{\left[\frac{\sqrt{x^2+y^2+z^2}\times 0 - y\times\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}\right] - \left[\frac{\sqrt{x^2+y^2+z^2}\times 0 - x\times\frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}\right]\right\}$$

$$= 0 - 0 + 0$$

So, it is irrotational.

8 c) $\oint_C (xy + y^2)\,dx + x^2\,dy$ by using Green's theorem, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$

Ans



Here $y = x$

$y = x^2$ $x =$ $x =$

$\Rightarrow x = x^2$

$\Rightarrow x - x^2 = 0$

$\Rightarrow x(1-x) = 0$

We know from Green's theorem,

$$\int_C P\,dx + Q\,dy = \iint \left(\frac{dQ}{dx} - \frac{dP}{dy} \right) dx\,dy$$

Here, $Q = x^2$ | $P = xy + y^2$

$\Rightarrow \frac{dQ}{dx} = 2x$ | $\frac{dP}{dy} = x + 2y$

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$$\oint_C P\,dx + Q\,dy = \int_{x=0}^{1}\int_{y=x^2}^{y=x} (2x - x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{y=x} (x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{1} \left[xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left[x^3 - x^4\right] dx$$

$$= \left[\frac{x^4}{4}\right]_0^1 - \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

$$= \int_{x=0}^{1} \left\{ \left[xy\right]_{x^2}^{x} - \left[y^2\right]_{x^2}^{x} \right\} dx$$

$$= \int_{x=0}^{1} \left\{ \left[x \times x - x \times x^2\right] - \left[x^2 - x^4\right] \right\} dx$$

$$= \int_{x=0}^{1} \left[x^2 - x^3\right] - \left[x^2 + x^4\right] dx$$

$$= \int_{x=0}^{1} \left[x^4 - x^3 \right] dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}$$

8b) Given the function,

$$\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$$

let, $\phi(x,y,z) = 2x+y+2z-6$

Now, $$\nabla\phi = \left\{ \hat{i}\frac{\partial}{\partial x}(2x+y+2z-6) + \hat{j}\frac{\partial}{\partial y}(2x+y+2z-6) + \hat{k}\frac{\partial}{\partial z}(2x+y+2z-6) \right\}$$

$$\Rightarrow \nabla\phi = 2\hat{i} + \hat{j} + 2\hat{k}$$

$\therefore$ The unit vector normal to the given surface is,

$$\hat{n} = \frac{\text{grad }\phi}{|\text{grad }\phi|} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{4+4+1}} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{3}$$

and $$\vec{A}\cdot\hat{n} = \left[(x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}\right]\cdot\left[\frac{2\hat{i}}{3} + \frac{\hat{j}}{3} + \frac{2\hat{k}}{3}\right]$$

$$= \frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{4}{3}yz$$

$$= \frac{2}{3}y^2 + \frac{4}{3}yz$$

$$= \frac{2}{3}(y^2+2yz)$$

let, R is the projection on yz plane of given plane $2x+y+2z=6$

$2x+y+2z=6$

$\Rightarrow y+2z=6$

$\Rightarrow z=\frac{6-y}{2}$

$z=0$ to $\frac{6-y}{2}$

$0 \le z \le \frac{6-y}{2}$

and $0 \le y \le 0$

$$\therefore \iint_S \vec{A}\cdot\hat{n}\, dS$$

$$= \int_{y=0}^{y=6}\int_{z=0}^{z=\frac{6-y}{2}} \frac{2}{3}(y^2+2yz)\, dy\, dz$$

$$\Rightarrow \int_{y=0}^{y=6} \frac{2}{3}\int_{z=0}^{z=\frac{6-y}{2}} (y^2+2yz)\, dz\, dy$$

$$= \int_{y=0}^{y=6} \frac{2}{3}\times\left[y^2\times\left(\frac{6-y}{2}-0\right)+\frac{2y}{2}\times\left(\frac{6-y}{2}\right)^2\right] dy$$

y^2z

yz^2

$$= \int_{y=0}^{y=6}\left[\frac{\cancel{2}}{3}\times y^2\times\frac{(6-y)}{\cancel{2}} + \frac{2}{3}\times\frac{y}{4}\times(6-y)^2\right] dy$$

$$= \int_{y=0}^{y=6}\left[\frac{y^2(6-y)}{3}+\frac{y}{6}(6-y)^2\right] dy$$

$$= \frac{1}{3}\int_0^{y=6} y^2(6-y)\, dy + \frac{1}{6}\int_0^6 y(6-y)^2\, dy$$

$\Rightarrow$ $36+18 = 54$