NUMERICAL ANALYSIS

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The Calculus of Finite Differences

§ 1. Introduction. Whenever a technical problem leads to a differential equation which cannot be integrated in closed form, approximate methods of solution must be employed. These methods are based on series expansions or they may be purely numerical methods leading to the evaluation of the unknown integral at specified points of its interval of definition by simple arithmetical means. Initial value and boundary value problems involving either ordinary or partial differential equations, may be solved by such methods. These numerical solutions do not usually allow the determination of general physical laws but do often indicate the dependence of the desired variables on the various parameters of the problem.

Numerical methods for the solution of differential equations have become particularly popular in recent years because modern technical problems lead to complicated equations seldom solvable in finite terms and also because calculating machines and electronic computers have become widely available. The numerical approach also has the advantage of allowing the actual work to be carried out by operators without a knowledge of higher mathematics or of physics, with a resulting economy of effort on the part of highly trained personnel.

The calculus of finite differences is the study of changes that take place in the value of the function or the dependent variable, say $y=f(x)$, with respect to the finite changes in the independent variable x.

Let $y=f(x)=x^2$ be the function of the independent variable x. If there is a finite increase in x, say $\triangle x$, then $f(x)=x^2$ will be increased to $f(x+\triangle x)=(x+\triangle x)^2$, bearing the increment

$$f(x+\triangle x)-f(x)=(x+\triangle x)^2-x^2.$$

Such finite changes are studied in the calculus of finite differences.

value of x in the function $f(x)$. If this increment be denoted by h, then the operation of E on $f(x)$ means that put $x+h$ in the function $f(x)$ wherever there is x *i.e.*

$$E f(x)=f(x+h). \quad \text{(Meerut B.Sc. 1980, 81, 83)}$$

Here we should note that $E f(x)$ does not imply the multiplication of E and $f(x)$ but it implies that E is operated on $f(x)$. The operator E is known as the *shift operator*. By $E^2 f(x)$ we mean that the operator E is applied twice on the function $f(x)$ *i.e.*

$$\begin{aligned} E^2 f(x) &= E\{E f(x)\} \\ &= E\{f(x+h)\}, \text{ by def. of } E \\ &= f(x+h+h), \text{ by def. of } E \\ &= f(x+2h). \end{aligned}$$

Similarly $E^n f(x)$ means that the operator E is applied n times on the function $f(x)$ *i.e.*

$$\begin{aligned} E^n f(x) &= EE\ldots\ldots E \ (n \text{ times}) f(x) \\ &= E^{n-1}\{E f(x)\} = E^{n-1}\{f(x+h)\} \\ &= E^{n-2}\{E f(x+h)\} = E^{n-2} f(x+2h) \\ &= E^{n-3} f(x+3h) = \ldots\ldots = E f(x+\overline{n-1}\,h) \\ &= f(x+nh). \end{aligned}$$

Remark. The operator E^{-1} is the inverse operator of the operator E and is defined as

$$E^{-1} f(x)=f\{x+(-1)h\}=f(x-h).$$

The operator Δ. Let $y=f(x)$ be a function of x. Let the consecutive values of x be a, $a+h$, $a+2h$, ..., $a+nh$ differing by h. Then the corresponding values of y are

$$f(a), f(a+h), f(a+2h), \ldots\ldots, f(a+nh).$$

The independent variable x is known as *argument* and the dependent variable y is known as *entry*. Thus we are given a set of values of argument and entry.

The difference $f(a+h)-f(a)$ is called the first forward difference of the function $f(x)$ at the point $x=a$ and we denote it by $\Delta f(a)$ *i.e.*

$$\Delta f(a)=f(a+h)-f(a).$$

Again the difference $f(a+2h)-f(a+h)$ is called the first forward difference of the function $f(x)$ at the point $x=a+h$ and is denoted by $\Delta f(a+h)$ *i.e.*

$$\Delta f(a+h)=f(a+2h)-f(a+h).$$

Continuing in a similar manner, we ultimately have

$$\Delta f(a+\overline{n-1}\,h)=f(a+nh)-f(a+\overline{n-1}\,h).$$

The operator Δ is called the **forward or descending difference operator.** The differences $\Delta f(a)$, $\Delta f(a+h)$, etc. are called first forward differences.

Thus *the first forward difference of $f(x)$ is defined as*

$$\Delta f(x)=f(x+h)-f(x).$$

The difference $\Delta f(a+h)-\Delta f(a)$ is known as the second forward difference of $f(x)$ at the point $x=a$ and is denoted by $\Delta^2 f(a)$ *i.e.*

$$\begin{aligned} \Delta^2 f(a) &= \Delta f(a+h)-\Delta f(a) \\ &= \{f(a+2h)-f(a+h)\}-\{f(a+h)-f(a)\} \\ &= f(a+2h)-2f(a+h)+f(a). \end{aligned}$$

The difference $\Delta f(a+2h)-\Delta f(a+h)$ is called the second forward difference of $f(x)$ at the point $x=a+h$ and is denoted by $\Delta^2 f(a+h)$.

Thus
$$\begin{aligned} \Delta^2 f(a+h) &= \Delta f(a+2h)-\Delta f(a+h) \\ &= \{f(a+3h)-f(a+2h)\}-\{f(a+2h)-f(a+h)\} \\ &= f(a+3h)-2f(a+2h)+f(a+h). \end{aligned}$$

In general, the second forward difference of $f(x)$ is given by

$$\begin{aligned} \Delta^2 f(x) &= \Delta[\Delta f(x)] = \Delta[f(x+h)-f(x)] \\ &= \Delta f(x+h)-\Delta f(x) \\ &= \{f(x+2h)-f(x+h)\}-\{f(x+h)-f(x)\} \\ &= f(x+2h)-2f(x+h)+f(x). \end{aligned}$$

The differences of the second forward differences are called third forward differences and are denoted by

$$\Delta^3 f(a),\ \Delta^3 f(a+h) \text{ etc.}$$

Thus
$$\begin{aligned} \Delta^3 f(a) &= \Delta^2 f(a+h)-\Delta^2 f(a) \\ &= \{f(a+3h)-2f(a+2h)+f(a+h)\} \\ &\quad -\{f(a+2h)-2f(a+h)+f(a)\} \\ &= f(a+3h)-3f(a+2h)+3f(a+h)-f(a). \end{aligned}$$

In general, the nth forward difference of $f(x)$ is given by

$$\Delta^n f(x)=\Delta^{n-1} f(x+h)-\Delta^{n-1} f(x).$$

Note 1. If the function $f(x)$ is a constant, say $f(x)=c$, then $\Delta f(x)=f(x+h)-f(x)=c-c=0$. Thus the first forward difference of a constant function is zero.

Note 2. It should be noted that by $\Delta^2 f(x)$, we mean that the operator Δ is to be applied twice on the function $f(x)$.

The operator ∇. The difference $f(a+h)-f(a)$ is called the first backward difference of $f(x)$ at $x=a+h$ and is denoted by $\nabla f(a+h)$ *i.e.*

$$\nabla f(a+h)=f(a+h)-f(a).$$

Similarly
$$\begin{aligned} \nabla f(a+2h) &= f(a+2h)-f(a+h) \\ &\ldots\ \ldots\ \ldots \\ \nabla f(a+nh) &= f(a+nh)-f(a+\overline{n-1}\,h). \end{aligned}$$

The operator ∇ is called the **backward or ascending difference operator.** The differences of first backward differences are called second backward differences and are denoted by

$$\nabla^2 f(a+2h),\ \nabla^2 f(a+3h) \text{ etc.}$$

Thus $\nabla^2 f(a+2h)=\nabla f(a+2h)-\nabla f(a+h)$
$$=\{f(a+2h)-f(a+h)\}-\{f(a+h)-f(a)\}$$
$$=f(a+2h)-2f(a+h)+f(a).$$

In general, the first backward difference of $f(x)$ is defined as

$$\nabla f(x)=f(x)-f(x-h).$$

The second backward difference of $f(x)$ is defined as

$$\nabla^2 f(x)=\nabla\{\nabla f(x)\}=\nabla\{f(x)-f(x-h)\}$$
$$=\nabla f(x)-\nabla f(x-h)$$
$$=\{f(x)-f(x-h)\}-\{f(x-h)-f(x-2h)\}$$
$$=f(x)-2f(x-h)+f(x-2h).$$

Similarly the nth backward difference of $f(x)$ is defined as

$$\nabla^n f(x)=\nabla\{\nabla^{n-1} f(x)\}$$
$$=\nabla^{n-1} f(x)-\nabla^{n-1} f(x-h).$$

The identity operator 1. *The operator* 1, *defined by* $1\,f(x)=f(x)$, *is called the* **identity operator.**

§ 3. Algebraic properties of operators E and Δ.

The following algebraic properties of the operators E and Δ follow immediately from the definitions of these operators.

(i) Operators E and Δ are distributive.

Let $u(x)$ be any function which is the sum of the functions $f(x), g(x), p(x), \ldots$, so that

$$u(x)=f(x)+g(x)+p(x)+\cdots$$

Then $Eu(x)=Ef(x)+Eg(x)+Ep(x)+\ldots$

Similarly $\Delta u(x)=\Delta f(x)+\Delta g(x)+\Delta p(x)+\ldots$

(ii) E and Δ are commutative with regard to a constant, *i.e*

$$E\{cf(x)\}=cEf(x)$$

and $\Delta\{cf(x)\}=c\Delta f(x)$.

(iii) E and Δ obey the law of indices, *i.e.*

$$E^m E^n f(x)=E^{m+n} f(x)$$

and $\Delta^m \Delta^n f(x)=\Delta^{m+n} f(x)$.

(iv) E and Δ are not commutative w.r.t. variables, *i.e.* if

$$u(x)=f(x)\,g(x),$$

then $Eu(x)\neq f(x)\,Eg(x)$

and $\Delta u(x)\neq f(x)\,\Delta g(x)$.

(v) $E\{af(x)+bg(x)\}=aEf(x)+bEg(x)$.

(vi) $E^{-n} f(x)=f(x-nh)$.

(vii) $E^2 f(x)\neq\{Ef(x)\}^2$.

(viii) Operators E and Δ cannot stand without the operand.

§ 4. Relations between the operators.

(a) $E\equiv 1+\Delta$ or $\Delta\equiv E-1$. **(Meerut 1981)**

We have $\Delta f(x)=f(x+h)-f(x)$
$$=Ef(x)-f(x)$$
$$=(E-1)f(x).$$

Thus $\Delta f(x)=(E-1)f(x)$, for any function $f(x)$.

$$\therefore\quad \Delta\equiv E-1$$

or $E\equiv 1+\Delta$.

(b) $\nabla\equiv 1-E^{-1}$ or $E^{-1}\equiv 1-\nabla$.

We have $\nabla f(x)=f(x)-f(x-h)$
$$=f(x)-E^{-1}f(x)$$
$$=(1-E^{-1})f(x).$$

Thus $\nabla f(x)=(1-E^{-1})f(x)$, for any function $f(x)$.

$\therefore\quad \nabla\equiv 1-E^{-1}$ or $E^{-1}\equiv 1-\nabla$. **(Rohilkhand B.Sc. 91)**

(c) $E\nabla\equiv\nabla E\equiv\Delta$. **(Meerut 1981)**

We have $(E\nabla)f(x)=E\{\nabla f(x)\}$
$$=E\{f(x)-f(x-h)\}$$
$$=Ef(x)-Ef(x-h)$$
$$=f(x+h)-f(x)$$
$$=\Delta f(x). \qquad \ldots(1)$$

Also $(\nabla E)f(x)=\nabla\{Ef(x)\}=\nabla f(x+h)$
$$=f(x+h)-f(x)$$
$$=\Delta f(x) \qquad \ldots(2)$$

From (1) and (2), we have

$$E\nabla\equiv\Delta \text{ and } \nabla E\equiv\Delta.$$

Thus $E\nabla\equiv\nabla E\equiv\Delta$. **(Rohilkhand B.Sc. 91)**

(d) $\Delta-\nabla\equiv\Delta\nabla$. **(Meerut B.Sc. 1992; Agra 88)**

We have $\Delta f(x)=f(x+h)-f(x)$

and $\nabla f(x)=f(x)-f(x-h)$,

where h is the interval of differencing.

Now $(\Delta\nabla)f(x)=\Delta\{\nabla f(x)\}=\Delta\{f(x)-f(x-h)\}$
$$=\Delta f(x)-\Delta f(x-h)$$
$$=\{f(x+h)-f(x)\}-\{f(x)-f(x-h)\}$$
$$=\Delta f(x)-\nabla f(x)$$
$$=(\Delta-\nabla)f(x).$$

Thus $(\Delta\nabla)f(x)=(\Delta-\nabla)f(x)$, for any function $f(x)$.

$$\therefore \quad \Delta\nabla\equiv\Delta-\nabla.$$

(e) $(1+\Delta)(1-\nabla)\equiv 1$. **(Meerut 1978; Rohilkhand 88, 90)**

We have
$$\begin{aligned}(1+\Delta)(1-\nabla)f(x)&=(1+\Delta)\{(1-\nabla)f(x)\}\\&=(1+\Delta)\{f(x)-\nabla f(x)\}\\&=(1+\Delta)[f(x)-\{f(x)-f(x-h)\}]\\&=(1+\Delta)f(x-h)\\&=Ef(x-h) \qquad [\because E\equiv1+\Delta]\\&=f(x)=1\,.\,f(x).\end{aligned}$$

Thus $(1+\Delta)(1-\nabla)f(x)=1.f(x)$, for any function $f(x)$.

$$\therefore \quad (1+\Delta)(1-\nabla)\equiv1.$$

(f) $E\equiv e^{hD}\equiv1+\Delta$, *where D is the differential operator of differential calculus.*

We have
$$\begin{aligned}Ef(x)&=f(x+h)\\&=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\dots\dots\end{aligned}$$

[by Taylor's theorem of differential calculus]

$$\begin{aligned}&=1.f(x)+h\,D\,f(x)+\frac{h^2}{2!}D^2f(x)+\cdots\\&=\left\{1+h\,D+\frac{h^2D^2}{2!}+\dots\right\}f(x)\\&=e^{hD}f(x).\end{aligned}$$

Thus $Ef(x)=e^{hD}f(x)$, for any function $f(x)$.

$$\therefore \quad E\equiv e^{hD}.$$

Again we know that $E\equiv1+\Delta$. Therefore

$$E\equiv e^{hD}\equiv1+\Delta.$$

Remark. We have $e^{hD}\equiv1+\Delta \Rightarrow hD\equiv\log(1+\Delta)$

$$\equiv\left(\Delta-\frac{\Delta^2}{2}+\frac{\Delta^3}{3}-\frac{\Delta^4}{4}+\dots\right)$$

or $\quad D=\frac{1}{h}\left(\Delta-\frac{\Delta^2}{2}+\frac{\Delta^3}{3}-\frac{\Delta^4}{4}+\dots\right).$

1·5. The difference Table. Suppose $y=f(x)$ is a function of x, x being given at an equal interval. Let the values of x be a, $a+h$, $a+2h$ and so on and the corresponding values of y be $f(a).f(a+h)$, $f(a+2h)$ and so on. Then the forward and backward differences can be calculated from the forward difference table and backward difference table respectively.

Table 1·1

Forward Difference Table

Argument x	Entry $y=f(x)$	First differences $\Delta f(x)$	Second differences $\Delta^2 f(x)$
a	$f(a)$		
		$f(a+h)-f(a)=\Delta f(a)$	
$a+h$	$f(a+h)$		$\Delta f(a+h)-\Delta f(a)=\Delta^2 f(a)$
		$f(a+2h)-f(a+h)=\Delta f(a+h)$	
$a+2h$	$f(a+2h)$		$\Delta f(a+2h)-\Delta f(a+h)=\Delta^2 f(a+h)$
		$f(a+3h)-f(a+2h)=\Delta f(a+2h)$	
$a+3h$	$f(a+3h)$		$\Delta f(a+3h)-\Delta f(a+2h)=\Delta^2 f(a+2h)$
		$f(a+4h)-f(a+3h)=\Delta f(a+3h)$	
$a+4h$	$f(a+4h)$		

Similarly we can calculate third and higher order differences from the table. Here we have taken forward differences therefore this table is known as forward difference table.

Table 1·2
Backward Difference Table

Argument x	Entry $y=f(x)$	First differences $\nabla f(x)$	Second differences $\nabla^2 f(x)$
a	$f(a)$		
		$f(a+h)-f(a)$ $=\nabla f(a+h)$	
$a+h$	$f(a+h)$		$\nabla f(a+2h)-\nabla f(a+h)$ $=\nabla^2 f(a+2h)$
		$f(a+2h)-f(a+h)$ $=\nabla f(a+2h)$	
$a+2h$	$f(a+2h)$		$\nabla f(a+3h)-\nabla f(a+2h)$ $=\nabla^2 f(a+3h)$
		$f(a+3h)-f(a+2h)$ $=\nabla f(a+3h)$	
$a+3h$	$f(a+3h)$		$\nabla f(a+4h)-\nabla f(a+3h)$ $=\nabla^2 f(a+4h)$
		$f(a+4h)-f(a+3h)$ $=\nabla f(a+4h)$	
$a+4h$	$f(a+4h)$		

Similarly we can calculate the differences of higher order from this table.

§ 6. Fundamental theorem of the difference calculus.

The nth difference of a polynomial of degree n is constant and higher order differences are zero i.e. if f(x) is a polynomial of degree n in x then the nth difference of f(x) is constant and (n+1)th difference is zero. **(Meerut M.Sc. 1976, 90, B.Sc. 77, 93, Stat. 90)**

Proof. Let $f(x)=a_0+a_1x+a_2x^2+\ldots\ldots+a_nx^n$, where n is a positive integer and $a_0, a_1, \ldots\ldots, a_n$ are constants.

We have $\Delta f(x)=f(x+h)-f(x)$ [by def. of Δ]

$$=[a_0+a_1(x+h)+a_2(x+h)^2+\ldots+a_n(x+h)^n]$$
$$-[a_0+a_1x+a_2x^2+\ldots+a_nx^n]$$
$$=a_1h+a_2[(x+h)^2-x^2]+a_3[(x+h)^3-x^3]+\ldots$$
$$+a_n[(x+h)^n-x^n]$$

$$=a_1h+a_2[{}^2C_1xh+h^2]+a_3[{}^3C_1x^2h+{}^3C_2xh^2+h^3]+\ldots$$
$$+a_n[{}^nC_1x^{n-1}h+{}^nC_2x^{n-2}h^2+\ldots+{}^nC_nh^n]$$
$$=b_0+b_1x+b_2x^2+\ldots+b_{n-2}x^{n-2}+na_nhx^{n-1}, \quad \ldots(1)$$

where $b_0, b_1, b_2, \ldots, b_{n-2}$ are constant coefficients. From (1) we see that $\Delta f(x)$ is a polynomial of degree $n-1$ in x. Thus the first difference of a polynomial $f(x)$ of degree n is again a polynomial of degree $n-1$ in which the coefficient of x^{n-1}

$$=n.h. \text{ the coefficient of } x^n \text{ in } f(x).$$

Now let $\Delta f(x)=\phi(x)$, where $\phi(x)$ is a polynomial of degree $n-1$.

Then $\quad \Delta^2 f(x)=\Delta[\Delta f(x)]=\Delta\phi(x).$

But by (1), $\Delta\phi(x)$ is a polynomial of degree $n-2$ in which the coefficient of $x^{n-2}=(n-1).h.$ the coefficient of x^{n-1} in $\phi(x)$

$$=(n-1).h.n.h.a_n=n(n-1)\,h^2\,a_n.$$

Thus $\Delta^2 f(x)$ is a polynomial of degree $n-2$ in which the coefficient of $x^{n-2}=n(n-1)\,h^2a_n$.

Continuing the above process, we see that the nth difference of $f(x)$ is a polynomial of degree zero

i.e. $\quad \Delta^n f(x)=n\,(n-1)\,(n-2)\ldots1\,.\,h^na_nx^{n-n}$

$$=n\,!\,h^na_nx^0=n\,!\,h^na_n.$$

Thus the nth difference of $f(x)$ is constant. So all higher order differences are zero, *i.e.* $(n+1)$th and higher differences of a polynomial of degree n are zero.

Note. The converse of the above result is also true *i.e.* if the nth differences of a tabulated function are constant when values of the independent variable are taken at equal intervals, the function is a polynomial of degree n.

§ 7. To express any value of the function in terms of the leading term and the leading differences of the difference table.

Show that $\quad f(a+nh)=f(a)+{}^nC_1\Delta f(a)+{}^nC_2\Delta^2 f(a)+\ldots$
$$+{}^nC_n\Delta^n f(a).$$

We have $\quad f(a+nh)=E^nf(a)$

$$=(1+\Delta)^nf(a)$$
$$=\{1+{}^nC_1\Delta+{}^nC_2\Delta^2+{}^nC_3\Delta^3+\ldots+{}^nC_n\Delta^n\}f(a)$$
$$=f(a)+{}^nC_1\Delta f(a)+{}^nC_2\Delta^2f(a)+\ldots+{}^nC_n\Delta^nf(a).$$

§ 8 One or more missing terms. (Equal intervals).

Sometimes we may be given a set of equidistant terms with some terms (one or two or more) missing. The problem of estimating such terms can be easily tackled by the use of the operators E and Δ.

Let us suppose that we are given $(n+1)$ equidistant arguments, $(x=0, 1, 2, \ldots, n$, say), but the entry y_r corresponding to any one of them, say $(r+1)$th argument, is not given and we want to estimate it. Since we are given n entries, the data can be represented by a polynomial of $(n-1)$th degree. Hence we may take y_x as a polynomial of $(n-1)$th degree.

$$\therefore \quad \Delta^{n-1} y_x = \text{constant}$$

and $$\Delta^n y_x = 0,\ x = 0, 1, 2, \dots, n. \qquad \dots(1)$$

In particular

$$\Delta^n y_0 = 0 \quad i.e. \quad (E-1)^n y_0 = 0$$

$$\Rightarrow \quad [E^n - {}^nC_1 E^{n-1} + {}^nC_2 E^{n-2} - \dots + (-1)^n]\, y_0 = 0$$

$$\Rightarrow \quad E^n y_0 - {}^nC_1 E^{n-1} y_0 + {}^nC_2 E^{n-2} y_0 - \dots + (-1)^n y_0 = 0$$

$$\Rightarrow \quad y_n - {}^nC_1 y_{n-1} + {}^nC_2\, y_{n-2} - \dots + (-1)^n y_0 = 0.$$

From this equation, the missing entry can be easily calculated.

If in a set of $(n+1)$ equidistant arguments, two entries y_r and y_s are missing, then the data can be represented by a polynomial of degree $n-2$. So proceeding as above we have

$$\Delta^{n-1} y_0 = 0 \text{ and } \Delta^{n-1} y_1 = 0$$

$$\Rightarrow \quad (E-1)^{n-1} y_0 = 0 \text{ and } (E-1)^{n-1} y_1 = 0. \qquad \dots(2)$$

Expanding and simplifying as above the two missing terms can be estimated by solving the equations (2). Similarly in a set of $n+1$ equidistant arguments the three missing terms can be estimated by solving the equations

$$\Delta^{n-2} y_0 = 0,\ \Delta^{n-2} y_1 = 0,\ \Delta^{n-2} y_2 = 0.$$

§ 9. **Factorial Notation**. *The product of n consecutive factors each at a constant difference, say h, the first factor being x is called a factorial function or a factorial polynomial of degree n and is denoted by $x^{(n)}$.* Thus

$$x^{(n)} = x(x-h)(x-2h)\dots(x-\overline{n-1}h). \quad \text{(Meerut 1991)}$$

In particular, if $h=1$

$$x^{(n)} = x(x-1)(x-2)\dots(x-n+1).$$

These functions, because of their properties, play an important role in the theory of finite differences.

The factorial function helps in finding the various order differences of a polynomial directly by simple rule of differentiation and similarly given any difference of a function in factorial notation we can find the corresponding function by simple integration.

To show that $\Delta^n x^{(n)} = n!\, h^n$ and $\Delta^{n+1} x^{(n)} = 0$.

By definition of Δ, we have

$$\Delta x^{(n)} = (x+h)^{(n)} - x^{(n)}$$

$$= (x+h)(x+h-h)(x+h-2h)\dots(x+h-\overline{n-1}\,h)$$

$$-[x(x-h)\dots(x-\overline{n-1}\,h)]$$

$$= (x+h)\,x(x-h)\dots(x-\overline{n-2}\,h)$$

$$-x(x-h)\dots(x-\overline{n-2}\,h)(x-\overline{n-1}\,h)$$

$$= x(x-h)\dots(x-\overline{n-2}\,h)[(x+h)-(x-\overline{n-1}\,h)]$$

$$= x^{(n-1)}[x+h-x+nh-h] = nhx^{(n-1)} \qquad \dots(1)$$

or equivalently $\dfrac{\Delta x^{(n)}}{\Delta x} = n\, x^{(n-1)}$ because $\Delta x = x+h-x = h$.

Again $$\Delta^2 x^{(n)} = \Delta\Delta x^{(n)}$$

$$= \Delta\{nhx^{(n-1)}\} \qquad \text{[From (1)]}$$

$$= nh\Delta x^{(n-1)}, \text{ because } nh \text{ is a constant.}$$

Now replacing n by $n-1$ in the relation (1), we have

$$\Delta x^{(n-1)} = (n-1)\, hx^{(n-2)}.$$

$$\therefore \quad \Delta^2 x^{(n)} = nh.(n-1)\, hx^{(n-2)} = n(n-1)\, h^2 x^{(n-2)}.$$

Proceeding in the same manner, we get

$$\Delta^{n-1} x^{(n)} = n(n-1)\dots 2h^{n-1} x.$$

$$\therefore \quad \Delta^n x^{(n)} = n(n-1)\dots 2\, h^{n-1} \Delta x$$

$$= n(n-1)\dots 2\, h^{n-1}(x+h-x)$$

$$= n(n-1)\dots 2.1\, h^{n-1}.h$$

$$= n!\, h^n. \qquad \dots(2)$$

$$\therefore \quad \Delta^{n+1} x^{(n)} = \Delta(n!\, h^n) = n!\, h^n - n!\, h^n = 0.$$

In particular when $h=1$ we get from (1) and (2)

$$\Delta x^{(n)} = nx^{(n-1)} \text{ and } \Delta^n x^{(n)} = n!. \qquad \dots(3)$$

This result leads to the following very important conclusion.

In case of factorial notation, the operator Δ is equivalent to the operator of differentiation if the interval of differencing is unity.

To show that $x^{(-n)} = \dfrac{1}{(x+nh)^{(n)}}$, the interval of differencing being h. (Meerut 1991)

We have

$$x^{(n)} = (x-\overline{n-1}\, h)\, x^{(n-1)}, \qquad \dots(1)$$

when the interval of differencing is h.

In particular, for $n=0$, $x^{(0)} = (x+h)\, x^{(-1)}$.

But by convention we agree to write $x^{(0)} = 1$. $\therefore$ $x^{(-1)} = \dfrac{1}{x+h}$.

Again from (1), for $n=-1$, $x^{(-1)} = (x+2h)\, x^{(-2)}$.

$$\therefore \quad x^{(-2)} = \frac{x^{(-1)}}{x+2h} = \frac{1}{(x+h)(x+2h)}.$$

In general, we shall get

$$x^{(-n)} = \frac{1}{(x+h)(x+2h)\dots(x+nh)} = \frac{1}{(x+nh)^{(n)}} \qquad \dots(2)$$

If the interval of differencing is unity then we have from (2)

$$x^{(-n)} = \frac{1}{(x+1)(x+2)\dots(x+n)}$$

$$= \frac{1}{(x+n)^{(n)}}.$$

Thus $$x^{(-n)}=\frac{1}{(x+n)^{(n)}}\qquad\ldots(3)$$

Now
$$\begin{aligned}
\Delta x^{(-n)}&=(x+h)^{(-n)}-x^{(-n)}\\
&=\frac{1}{(x+2h)(x+3h)\ldots(x+nh)(x+\overline{n+1}h)}-\frac{1}{(x+h)(x+2h)(x+3h)\ldots(x+nh)}\\
&=\frac{1}{(x+2h)(x+3h)\ldots(x+nh)}\left[\frac{1}{(x+\overline{n+1}\,h)}-\frac{1}{x+h}\right]\\
&=\frac{-nh}{(x+h)(x+2h)(x+3h)\ldots(x+nh)(x+\overline{n+1}h)}.
\end{aligned}$$

$$\begin{aligned}
\therefore\quad \Delta x^{(-n)}&=-nhx^{(-\overline{n+1})}\\
&=-nhx^{(-n-1)}\qquad\ldots(4)
\end{aligned}$$

or equivalently $\dfrac{\Delta x^{(-n)}}{\Delta x}=-nx^{(-\overline{n+1})}$ because $\Delta x=x+h-x=h$.

Similarly,
$$\begin{aligned}
\Delta^2x^{(-n)}&=\Delta[-nhx^{(-n-1)}]=(-nh)\Delta[x^{(-n-1)}]\\
&=(-nh)(-n-1)\,hx^{(-n-2)}\\
&=(-1)^2n(n+1)\,h^2x^{(-n-2)},\qquad\ldots(5)
\end{aligned}$$

and so on.

If $h=1$, then from (4) and (5), we get

$$\left.\begin{aligned}
\Delta x^{(-n)}&=-nx^{(-n-1)},\\
\Delta^2x^{(-n)}&=(-n)(-n-1)\,x^{(-n-2)},\text{ etc.}
\end{aligned}\right\}$$

Thus *in the case of negative index for factorial function the operator* $\Delta\equiv D$.

Generalised factorial functions.

The generalised factorial function corresponding to any given function $f(x)$ is defined by

$$[f(x)]^{(n)}=f(x)\,f(x-h)\,f(x-2h)\ldots f(x-\overline{n-1}\,h),\qquad n=1,2,3,\ldots\ldots$$

and $$[f(x)]^{(-n)}=\frac{1}{f(x+h)\,f(x+2h)\ldots f(x+nh)},\quad n=1,2,3,\ldots.$$

Also we agree to write $[f(x)]^{(0)}=1$.

As an illustration let us take $f(x)=ax+b$.

Then
$$\begin{aligned}
f(x-h)&=a(x-h)+b=ax+b-ah,\\
f(x-2h)&=a(x-2h)+b=ax+b-2ah,\\
\ldots&\qquad\ldots\qquad\ldots\qquad\ldots\\
f(x-\overline{n-1}\,h)&=a[x-(n-1)h]+b=ax+b-nah+ah.
\end{aligned}$$

$$\therefore\quad (ax+b)^{(n)}=(ax+b)(ax+b-ah)(ax+b-2ah)\ldots\ldots(ax+h-nah+ah).$$

§ 10. Methods of representing any given polynomial in factorial notation.

First Method (Direct Method)

Express $2x^3-3x^2+3x-10$ *and its differences in factorial notation, the interval of differencing being unity.*

Let
$$\begin{aligned}
2x^3-3x^2+3x-10&\equiv Ax^{(3)}+Bx^{(2)}+Cx^{(1)}+D\\
&=Ax(x-1)(x-2)+Bx(x-1)+Cx+D,\qquad\ldots(1)
\end{aligned}$$

where A, B, C and D are constants to be determined.

Putting $x=0$ in (1), we get $D=-10$.

Again putting $x=1$ in (1), we get

$$2-3+3-10=C+D$$
$$\Rightarrow\quad C=2.$$

Putting $x=2$ in (1), we get

$$16-12+6-10=2B+2C+D$$
$$\Rightarrow\ 0=2B+4-10$$
$$\Rightarrow\ B=3.$$

Equating the coefficients of x^3 on both sides of (1), we get $A=2$.
Putting the values of A, B, C and D in (1), we get

$$f(x)=2x^3-3x^2+3x-10=2x^{(3)}+3x^{(2)}+2x^{(1)}-10.$$

By the rule of simple differentiation, we have

$$\Delta f(x)=6x^{(2)}+6x^{(1)}+2$$
$$\Delta^2f(x)=12x^{(1)}+6$$
$$\Delta^3f(x)=12.$$

Steps in First Method.

(*i*) The given function is expressed term by term in factorial functions with certain unknown coefficients as shown in (1).

(*ii*) To get the values of the unknown coefficients we put $x=0,1,2,\ldots$ successively in the L.H.S. and R.H.S. of (1) and then the resulting equations are solved to find the values of A, B, C, etc.

(*iii*) The values of A, B, C, etc. thus found are substituted in the R.H.S. of (1) to get the given polynomial in factorial notation.

Second Method (Method of detached coefficients or synthetic division).

We have $2x^3-3x^2+3x-10\equiv Ax^{(3)}+Bx^{(2)}+Cx^{(1)}+D$ ......(1)

$$=Ax\,(x-1)\,(x-2)+Bx\,(x-1)+Cx+D$$
$$=x\,[A\,(x-1)\,(x-2)+B\,(x-1)+C]+D.$$

If we divide the given polynomial by x then the remainder will be -10 and the quotient is $2x^2-3x+3$. The value of D in (1) is taken as -10.

Again divide the quotient $2x^2-3x+3$ by $x-1$ as done below :

$$\begin{array}{r|l} & \;\;2x-1 \\ \hline x-1 & 2x^2-3x+3 \\ & 2x^2-2x \\ \cline{2-2} & \quad\;\; -x+3 \\ & \quad\;\; -x+1 \\ \cline{2-2} & \qquad\quad 2 \end{array}$$

$\therefore$ the quotient is $2x-1$ and the remainder is the value of C *i.e.* $C=2$.

Again divide $2x-1$ by $x-2$ as done below :

$$\begin{array}{r|l} & \;\;2 \\ \hline x-2 & 2x-1 \\ & 2x-4 \\ \cline{2-2} & \quad\; 3 \end{array}$$

The quotient 2 is the value of A and the remainder 3 is B. Thus the given polynomial when expressed in factorial notation is $2x^3-3x^2+3x-10=2x^{(3)}+3x^{(2)}+2x-10$.

The above method can be simplified by the procedure of detached coefficients in the following way :

Taking the coefficients of the various powers of x in the given polynomial, we have

$$\begin{array}{c|cccl l} 1 & 2 & -3 & 3 & -10=D & \quad\ldots\text{(a)} \\ & 0 & 2 & -1 & & \\ \hline 2 & 2 & -1 & 2=C & & \quad\ldots\text{(b)} \\ & 0 & 4 & & & \\ \hline 3 & 2 & 3=B & & & \quad\ldots\text{(c)} \\ & 0 & & & & \\ \hline & 2=A & & & & \end{array}$$

Steps in the method of detached coefficients.

(i) First make the given polynomial complete (if it is not so) by supplying the missing terms with zero coefficients. Then write the coefficients of different powers of x in order beginning with the coefficient of highest power of x. The constant term -10 is the value of D.

(ii) Put 1 in the left hand side column of (a) and write zero below the coefficient of highest power of x. In this case we have written 0 below 2 which is the coefficient of x^3. The sum of 2 and 0 is 2 which we write below 0 in the third row. Now we multiply 2 by 1 of the left hand column of (a) and write their product 2 in the second row below -3 of the first row. Adding -3 and 2 we get -1 which we put in the third row below 2 of the second row. Now we multiply -1 by 1 of the first column of (a) and then write their product -1 in the second row below 3 of the first row. Adding 3 and -1 we get 2 which we put below -1. This 2 is the value of C.

(iii) Now we write 2 in the left hand column of (b). Below 2 of the third row we write 0 and adding 2 and 0 we get 2 which we write in the fifth row below 0 of the fourth row. Now we multiply 2 of the fifth row by 2 of the left hand column of (b) and write their product 4 in the fourth row below -1 of the third row. Adding -1 and 4 we get 3 and we write it in the fifth row below 4 of the fourth row. This 3 is the value of B.

(iv) Now we write 3 in the left hand column of (c). Below 2 of the fifth row we write 0 and adding 2 and 0 we get 2 which we write in the seventh row below 0 of the sixth row. This 2 is the value of A.

§ 11. **Differences of zero.** If n and m are positive integers, we have in usual notation of finite difference calculus

$$\Delta^n x^m=(E-1)^n x^m$$
$$=[E^n-{}^nC_1E^{n-1}+{}^nC_2E^{n-2}-\ldots+(-1)^{n-1}\,{}^nC_{n-1}\,E$$
$$+(-1)^n]\,x^m$$
$$=E^n x^m-{}^nC_1\,E^{n-1}\,x^m+{}^nC_2\,E^{n-2}\,x^m-\ldots$$
$$\ldots+{}^nC_{n-1}\,(-1)^{n-1}\,E\,x^m+(-1)^n\,x^m$$
$$=(x+n)^m-{}^nC_1\,(x+\overline{n-1})^m+{}^nC_2\,(x+\overline{n-2})^m-\ldots$$
$$+{}^nC_{n-1}\,(-1)^{n-1}\,(x+1)^m+(-1)^n x^m.$$

Putting $x=0$ and writing $[\Delta^n x^m]_{x=0}$ as $\Delta^n\,0^m$, we have

$$\Delta^n 0^m=n^m-{}^nC_1\,(n-1)^m+{}^nC_2\,(n-2)^m-\ldots$$
$$+{}^nC_{n-1}\,(-1)^{n-1}. \qquad \ldots(1)$$

The quantities $\Delta^n\,0^m$ are known as differences of zero because the leading term is always zero.

We can calculate the values of $\Delta^n 0^m$ for various integral values of n and m. For example,

if $\quad n=1, m=3. \quad \Delta\, 0^3=1^3=1,$

$\quad n=2, m=3, \quad \Delta^2\, 0^3=2^3-2.1^3=6,$

$\quad n=3, m=3, \quad \Delta^3\, 0^3=3^3-3.2^3+3.1^3=6,$ etc.

Recurrence relation between $\Delta^n 0^m$, $\Delta^{n-1} 0^{m-1}$ and $\Delta^n 0^{m-1}$. From (1), we have

$$\begin{aligned}
\Delta^n 0^m &= n^m-{}^nC_1 (n-1)^m+{}^nC_2 (n-2)^m-\ldots+{}^nC_{n-1} (-1)^{n-1}\\
&= n^m-n\,(n-1)^m+\frac{n\,(n-1)}{2\,!}(n-2)^m-\ldots+n\,(-1)^{n-1}\\
&= n\left[n^{m-1}-(n-1)^m+\frac{(n-1)\,(n-2)^m}{2\,!}-\ldots+(-1)^{n-1}\right]\\
&= n\,[n^{m-1}-{}^{n-1}C_1 (n-1)^{m-1}+{}^{n-1}C_2 (n-2)^{m-1}\\
&\qquad -\ldots+(-1)^{n-1}]\\
&= n\,[(1+n-1)^{m-1}-{}^{n-1}C_1 (1+n-2)^{m-1}\\
&\qquad +{}^{n-1}C_2 (1+n-3)^{m-1}-\ldots+(-1)^{n-1}]\\
&= n\,[E^{n-1} (1)^{m-1}-{}^{n-1}C_1 E^{n-2} (1)^{m-1}+{}^{n-1}C_2 E^{n-3} (1)^{m-1}\\
&\qquad -\ldots+(-1)^{n-1} (1)^{m-1}]\\
&\qquad [\because (1)^{m-1}=1]\\
&= n\,[E-1]^{n-1} (1)^{m-1}\\
&= n\,\Delta^{n-1} (1)^{m-1}\\
&= n\,\Delta^{n-1} E(0)^{m-1} \quad [\because E(0)=1]\\
&= n\,\Delta^{n-1} (1+\Delta)\, 0^{m-1}\\
&= n\,\Delta^{n-1} 0^{m-1}+n\,\Delta^n 0^{m-1}.
\end{aligned}$$

This is the required recurrence relation between differences of zero for different values of n and m.

§ 12. Effect of an Error in a Tabular Value.

Let $y_0, y_1, y_2, \ldots, y_n$ be the true values of a function, and suppose the value y_5 to be affected with an error ϵ, so that its erroneous value is $y_5+\epsilon$. Then the successive differences of the y's are as shown in the table that follows.

This table shows that the effect of an error increases with the successive differences, that the coefficients of the ϵ's are the binomial coefficients with alternating signs, and that the algebraic sum of the errors in any difference column is zero. It shows also that the maximum error in the differences is in the same horizontal line as the erroneous tabular value.

Table 1·3

Showing the effect of an error in the tabular values

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
y_0				
	Δy_0			
y_1		$\Delta^2 y_0$		
	Δy_1		$\Delta^3 y_0$	
y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
	Δy_2		$\Delta^3 y_1$	
y_3		$\Delta^2 y_2$		$\Delta^4 y_1+\epsilon$
	Δy_3		$\Delta^3 y_2+\epsilon$	
y_4		$\Delta^2 y_3+\epsilon$		$\Delta^4 y_2-4\epsilon$
	$\Delta y_4+\epsilon$		$\Delta^3 y_3-3\epsilon$	
$y_5+\epsilon$		$\Delta^2 y_4-2\epsilon$		$\Delta^4 y_3+6\epsilon$
	$\Delta y_5-\epsilon$		$\Delta^3 y_4+3\epsilon$	
y_6		$\Delta^2 y_5+\epsilon$		$\Delta^4 y_4-4\epsilon$
	Δy_6		$\Delta^3 y_5-\epsilon$	
y_7		$\Delta^2 y_6$		$\Delta^4 y_5+\epsilon$
	Δy_7		$\Delta^3 y_6$	
y_8		$\Delta^2 y_7$		$\Delta^4 y_6$
	Δy_8		$\Delta^3 y_7$	
y_9		$\Delta^2 y_8$		
	Δy_9			
y_{10}				

Solved Examples

Ex. 1. *Find the value of $E^2 x^2$ when the values of x vary by a constant increment of 2.*

Sol. We have $E^2x^2=EEx^2=E\,(x+2)^2$ [$\because$ interval of differencing is 2]

$$=(x+2+2)^2=(x+4)^2$$
$$=x^2+8x+16.$$

Ex. 2. *Evaluate E^ne^x when interval of differencing is h.*

Sol. We have $E(e^x)=e^{x+h}$

$$E^2(e^x)=EEe^x=Ee^{x+h}=e^{x+2h}$$
$$E^3(e^x)=EE^2e^x=Ee^{x+2h}=e^{x+3h}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$E^ne^x=e^{x+nh}.$$

Ex. 3. *Evaluate the following :*

(i) $\Delta^3\,(1-x)\,(1-2x)\,(1-3x)$, **(Rohilkhand B.Sc. 1991)**

(ii) $\Delta^n\,(e^{ax+b})$,

the interval of differencing being unity.

Sol. (i) Here $f(x)=(1-x)\,(1-2x)\,(1-3x)$

$$=-6x^3+11x^2-6x+1.$$

The polynomial $f(x)$ is of degree 3.

We know that for a polynomial of nth degree, the nth difference is constant being equal to $a_nh^n\,n!$ where a_n is the coefficient of x^n in the polynomial, h is the interval of differencing.

Here $a_n=-6,\ h=1,\ n=3.$

$\therefore\quad \Delta^3f(x)=(-6)\,(1)^3\,3!=-36.$

(ii) Here $f(x)=e^{ax+b}$.

Now $\Delta f(x)=f(x+1)-f(x)$.

$$\therefore\quad \Delta(e^{ax+b})=e^{a(x+1)+b}-e^{ax+b}$$
$$=e^{ax+b}(e^a-1)$$
$$\Delta^2(e^{ax+b})=\Delta(\Delta e^{ax+b})=\Delta\,\{e^{ax+b}(e^a-1)\}$$
$$=(e^a-1)\,(\Delta e^{ax+b})$$
$$=(e^a-1)\,e^{ax+b}(e^a-1)$$
$$=(e^a-1)^2\,e^{ax+b}.$$

Proceeding in this way, we get

$$\Delta^n(e^{ax+b})=(e^a-1)^n\,e^{ax+b}.$$

Ex. 4. *Evaluate*

(i) $\Delta^2(\cos 2x)$; **(Meerut B.Sc. 1992)**

(ii) $\Delta^2(3e^x)$,

the interval of differencing being h. **(Meerut B.Sc. 1978)**

Sol. (i) We have $\Delta^2(\cos 2x)=(E-1)^2\cos 2x$ [$\because\ \Delta\equiv E-1$]

$$=(E^2-2E+1)\cos 2x$$
$$=E^2\cos 2x-2\,E\cos 2x+\cos 2x$$
$$=\cos\{2(x+2h)\}-2\cos 2\,(x+h)+\cos 2x$$
$$=\cos(2x+4h)-2\cos(2x+2h)+\cos 2x$$
$$=\cos(2x+4h)-\cos(2x+2h)-\cos(2x+2h)+\cos 2x$$
$$=2\sin(2x+3h)\sin(-h)+2\sin(2x+h)\sin h$$
$$=-2\sin h\,[\sin(2x+3h)-\sin(2x+h)]$$
$$=-2\sin h\,[2\cos(2x+2h)\sin h]$$
$$=-4\sin^2 h\cos(2x+2h).$$

(ii) We have $\Delta(3e^x)=3(\Delta e^x)=3\,(e^{x+h}-e^x)=3(e^xe^h-e^x)$

$$=3e^x\,(e^h-1).$$

$$\therefore\quad \Delta^2(3e^x)=\Delta(\Delta\,3e^x)=\Delta\,\{3e^x(e^h-1)\}$$
$$=3(e^h-1)(\Delta e^x)=3(e^h-1)(e^{x+h}-e^x)$$
$$=3(e^h-1)\,e^x\,(e^h-1)=3e^x(e^h-1)^2.$$

Ex. 5. *Show that* $e^x=\left(\dfrac{\Delta^2}{E}\right)e^x\cdot\dfrac{Ee^x}{\Delta^2e^x}$; *the interval of differencing being h.* **(Meerut 1978; Bangalore 80)**

Sol. Let $f(x)=e^x$.

We know that $E\,f(x)=f(x+h)$.

$\therefore\quad E\,e^x=e^{x+h}$.

Now $\Delta f(x)=f(x+h)-f(x)$.

$$\therefore\quad \Delta e^x=e^{x+h}-e^x=e^x(e^h-1),$$
$$\Delta^2e^x=\Delta(\Delta e^x)=\Delta\,\{e^x(e^h-1)\}$$
$$=(e^h-1)\,\Delta(e^x)=(e^h-1)^2e^x.$$
$$\therefore\quad \left(\frac{\Delta^2}{E}\right)e^x=(\Delta^2E^{-1})\,e^x=\Delta^2(E^{-1}e^x)=\Delta^2(e^{x-h})$$
$$=\Delta^2\,(e^xe^{-h})=e^{-h}\Delta^2e^x=e^{-h}\,(e^h-1)^2\,e^x.$$
$$\therefore\quad \left(\frac{\Delta^2}{E}\right)e^x\cdot\frac{Ee^x}{\Delta^2e^x}=e^{-h}\,(e^h-1)^2e^x\,\frac{e^{x+h}}{(e^h-1)^2e^x}=e^{-h}e^{x+h}=e^x.$$

Ex. 6. *Evaluate*

$$\left(\frac{\Delta^2}{E}\right)x^3.$$

(Meerut B.Sc. 1974, 92 M.Sc. 88, 91)

Sol. We have $\left(\dfrac{\Delta^2}{E}\right)x^3=\left[\dfrac{(E-1)^2}{E}\right]x^3$ $\{\because\ \Delta\equiv E-1\}$

$$=\left(\frac{E^2-2E+1}{E}\right)x^3$$
$$=\left(E-2+\frac{1}{E}\right)x^3=Ex^3-2x^3+E^{-1}\,x^3$$

$$=(x+1)^3-2x^3+(x-1)^3$$
$$=x^3+3x^2+3x+1-2x^3+x^3-3x^2+3x-1=6x.$$

Ex. 7. *Evaluate*

(i) $\Delta \tan^{-1} x$ (ii) $\Delta\ (x+\cos x)$. **(Meerut B.Sc. 1978)**

Sol. (i) We have $\Delta \tan^{-1} x=\tan^{-1}(x+h)-\tan^{-1} x$

$$=\tan^{-1}\frac{(x+h)-x}{1+(x+h)\,x}$$
$$=\tan^{-1}\left[\frac{h}{1+hx+x^2}\right].$$

(ii) We have $\Delta\ (x+\cos x)=\Delta x+\Delta \cos x$

$$=\{(x+h)-x\}+\{\cos(x+h)-\cos x\}$$
$$=h+2\sin\frac{2x+h}{2}\sin\left(-\frac{h}{2}\right)$$
$$=h-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$$

Ex. 8. *Show that* $\Delta \log f(x)=\log\left\{1+\frac{\Delta f(x)}{f(x)}\right\}$ **(Meerut 1980)**

Sol. We have $\Delta \log f(x)=\log f(x+h)-\log f(x)$

$$=\log\left[\frac{f(x+h)}{f(x)}\right]=\log\left[\frac{E\,f(x)}{f(x)}\right]$$
$$=\log\left[\frac{(1+\Delta)\,f(x)}{f(x)}\right]$$
$$=\log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right]$$
$$=\log\left[1+\frac{\Delta f(x)}{f(x)}\right]$$

Ex. 9. *Evaluate*

(i) $\Delta^{10}\,[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]$ **(Roh. B.Sc. 1991)**

(ii) $\Delta^2\left[\frac{5x+12}{x^2+5x+6}\right]$ **(Meerut 1976)**

(iii) $\Delta^n\left(\frac{1}{x}\right)$

(iv) $\Delta^n\,(ab^{cx})$.

Sol. (i) We know that for a polynomial $f(x)$ of degree n $\Delta^r f(x)=0$ for $r>n$.

$\therefore\quad \Delta^{10}\,[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]=\Delta^{10}\,(abcd\,x^{10})$

$$=abcd\,\Delta^{10}x^{10}$$
$$=abcd\,(10\,!).$$

(ii) We have $\Delta^2\left[\frac{5x+12}{x^2+5x+6}\right]=\Delta^2\left[\frac{2(x+3)+3(x+2)}{(x+2)(x+3)}\right]$

$$=\Delta^2\left[\frac{2}{x+2}+\frac{3}{x+3}\right]$$

Now $\Delta\left[\frac{2}{x+2}+\frac{3}{x+3}\right]=\left[\frac{2}{x+3}+\frac{3}{x+4}\right]-\left[\frac{2}{x+2}+\frac{3}{x+3}\right]$

$$=\left[\frac{2}{x+3}-\frac{2}{x+2}\right]+\left[\frac{3}{x+4}-\frac{3}{x+3}\right]$$
$$=-\frac{2}{(x+2)(x+3)}-\frac{3}{(x+3)(x+4)}.$$

$\therefore\quad \Delta^2\left[\frac{2}{x+2}+\frac{3}{x+3}\right]=\Delta\left[\Delta\left\{\frac{2}{x+2}+\frac{3}{x+3}\right\}\right]$

$$=\Delta\left[-\frac{2}{(x+2)(x+3)}-\frac{3}{(x+3)(x+4)}\right]$$
$$=\left[-\frac{2}{(x+3)(x+4)}-\frac{3}{(x+4)(x+5)}\right]$$
$$-\left[-\frac{2}{(x+2)(x+3)}-\frac{3}{(x+3)(x+4)}\right]$$
$$=-2\left[\frac{1}{(x+3)(x+4)}-\frac{1}{(x+2)(x+3)}\right]$$
$$-3\left[\frac{1}{(x+4)(x+5)}-\frac{1}{(x+3)(x+4)}\right]$$
$$=-2\left[\frac{(x+2)-(x+4)}{(x+2)(x+3)(x+4)}\right]-3\left[\frac{(x+3)-(x+5)}{(x+3)(x+4)(x+5)}\right]$$
$$=-2\left[\frac{-2}{(x+2)(x+3)(x+4)}\right]-3\left[\frac{-2}{(x+3)(x+4)(x+5)}\right]$$
$$=\frac{4}{(x+2)(x+3)(x+4)}+\frac{6}{(x+3)(x+4)(x+5)}.$$

(iii) We have $\Delta^n\left(\frac{1}{x}\right)=\Delta^{n-1}\Delta\left(\frac{1}{x}\right)$.

Now $\Delta\left(\frac{1}{x}\right)=\frac{1}{x+1}-\frac{1}{x}=\frac{x-(x+1)}{x\,(x+1)}=\frac{(-1)}{x\,(x+1)}$

$$\Delta^2\left(\frac{1}{x}\right)=\Delta\Delta\left(\frac{1}{x}\right)=\Delta\left\{\frac{(-1)}{x\,(x+1)}\right\}=(-1)\,\Delta\left\{\frac{1}{x\,(x+1)}\right\}$$
$$=(-1)\left\{\frac{1}{(x+1)(x+2)}-\frac{1}{x\,(x+1)}\right\}$$
$$=(-1)\,\frac{x-(x+2)}{x\,(x+1)(x+2)}=\frac{(-1)(-2)}{x\,(x+1)(x+2)}$$
$$\Delta^3\left(\frac{1}{x}\right)=\frac{(-1)(-2)(-3)}{x\,(x+1)(x+2)(x+3)}.$$

Similarly $\Delta^n\left(\frac{1}{x}\right)=\frac{(-1)(-2)(-3)\ldots(-n)}{x(x+1)(x+2)\ldots(x+n)}$

$$=\frac{(-1)^n n!}{x(x+1)\ldots(x+n)}$$

(iv) We have $\Delta(ab^{cx})=a\Delta b^{cx}=a\{b^{c(x+1)}-b^{cx}\}$

$$=a\{b^{cx}b^c-b^{cx}\}=a(b^c-1)b^{cx}.$$

Now $\Delta^2(ab^{cx})=\Delta\Delta(ab^{cx})=\Delta\{a(b^c-1)b^{cx}\}$

$$=a(b^c-1)\Delta b^{cx}=a(b^c-1)^2 b^{cx}.$$

Proceeding in the same way, we have

$$\Delta^n(ab^{cx})=a(b^c-1)^n b^{cx}.$$

Ex. 10. *Show that* $\sum_{k=0}^{n-1}\Delta^2 f_k=\Delta f_n-\Delta f_0$. **(Meerut M.Sc. 1974, 78)**

Sol. L.H.S. $=\sum_{k=0}^{n-1}\Delta^2 f_k=\sum_{k=0}^{n-1}(E-1)^2 f_k$

$$=\sum_{k=0}^{n-1}(E^2-2E+1)f_k=\sum_{k=0}^{n-1}(f_{k+2}-2f_{k+1}+f_k)$$

$$\begin{aligned}
&=f_2-2f_1+f_0\\
&\quad+f_3-2f_2+f_1\\
&\quad+f_4-2f_3+f_2\\
&\quad+f_5-2f_4+f_3\\
&\quad\ldots\ \ldots\ \ldots\\
&\quad+f_{n-1}-2f_{n-2}+f_{n-3}\\
&\quad+f_n-2f_{n-1}+f_{n-2}\\
&\quad+f_{n+1}-2f_n+f_{n-1}
\end{aligned}$$

$=f_{n+1}-f_n+f_0-f_1$, adding and cancelling the diagonal terms

$$=(f_{n+1}-f_n)-(f_1-f_0)$$

$$=\Delta f_n-\Delta f_0=\text{R.H.S.}$$

Ex. 11. *Show that* $\Delta^r y_k=\nabla^r y_{k+r}$. **(Rohilkhand 1985)**

Sol. We have $\nabla^r y_{k+r}=(1-E^{-1})^r y_{k+r}$ $[\because\ \nabla=1-E^{-1}]$

$$=\left(\frac{E-1}{E}\right)^r y_{k+r}$$

$$=(E-1)^r E^{-r} y_{k+r}$$

$$=(E-1)^r (E^{-r} y_{k+r})$$

$$=(E-1)^r y_{k+r-r}$$

$$=\Delta^r y_k.\quad [\because\ \Delta\equiv E-1]$$

Ex. 12. *Construct a forward difference table for the following values :*

x :	0	5	10	15	20	25
$f(x)$:	7	11	14	18	24	32

Sol.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	7					
		4				
5	11		-1			
		3		2		
10	14		1		-1	
		4		1		0
15	18		2		-1	
		6		0		
20	24		2			
		8				
25	32					

Ex. 13. *Prove that*

(a) $f(4)=f(3)+\Delta f(2)+\Delta^2 f(1)+\Delta^3 f(1)$.

(b) $f(4)=f(0)+4\Delta f(0)+6\Delta^2 f(-1)+10\Delta^3 f(-1)$

as far as third differences.

Sol. (a). We have

$$f(4)-f(3)=\Delta f(3)$$

$$=\Delta[f(2)+\Delta f(2)]\quad \{\because\ \Delta f(2)=f(3)-f(2)\}$$

$$=\Delta f(2)+\Delta^2 f(2)$$

$$=\Delta f(2)+\Delta^2[f(1)+\Delta f(1)]$$

$$\{\because\ \Delta f(1)=f(2)-f(1)\}$$

$$=\Delta f(2)+\Delta^2 f(1)+\Delta^3 f(1).$$

$$\therefore\ f(4)=f(3)+\Delta f(2)+\Delta^2 f(1)+\Delta^3 f(1).$$

(b) We have $f(4)=E^5 f(-1)=(1+\Delta)^5 f(-1)$

$=\{1+{}^5C_1\Delta+{}^5C_2\Delta^2+{}^5C_3\Delta^3\}f(-1),$
taking only upto 3rd differences

$=f(-1)+5\Delta f(-1)+10\Delta^2 f(-1)+10\Delta^3 f(-1)$

$=[f(-1)+\Delta f(-1)]+4[\Delta f(-1)+\Delta^2 f(-1)]+6\Delta^2 f(-1)+10\Delta^3 f(-1)$

$=[f(-1)+\Delta f(-1)]+4\Delta[f(-1)+\Delta f(-1)]+6\Delta^2 f(-1)+10\Delta^3 f(-1)$

$=f(0)+4\Delta f(0)+6\Delta^2 f(-1)+10\Delta^3 f(-1).$

$[\because f(-1)+\Delta f(-1)=f(-1)+f(0)-f(-1)=f(0)]$

Ex. 14. *Obtain the missing terms in the following table :*

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	?	64	?	216	343	512.

(Meerut B.Sc. 1976, M.Sc. 91)

Sol. Here we are given six values, so a polynomial of degree 5 may be fitted which will have its 6th difference as zero,

i.e. $\Delta^6 f(x)=0$ for all values of x

i.e. $(E-1)^6 f(x)=0 \quad \forall x \qquad [\because \Delta\equiv E-1]$

i.e. $(E^6-6E^5+15E^4-20E^3+15E^2-6E+1)f(x)=0 \ \forall x$

i.e. $E^6 f(x)-6E^5 f(x)+15E^4 f(x)-20E^3 f(x)+15E^2 f(x)-6Ef(x)+f(x)=0 \quad \forall x$

i.e. $f(x+6)-6f(x+5)+15f(x+4)-20f(x+3)+15f(x+2)-6f(x+1)+f(x)=0 \quad \forall x, \qquad ...(1)$

here interval of differencing is unity.

Putting $x=1$ and 2 in (1), we get

$f(7)-6f(6)+15f(5)-20f(4)+15f(3)-6f(2)+f(1)=0 \qquad ...(2)$

and $f(8)-6f(7)+15f(6)-20f(5)+15f(4)-6f(3)+f(2)=0. \qquad ...(3)$

Putting the values of $f(8), f(7), f(6), f(4), f(2), f(1)$ in (1) and (2), we get

$343-6\times 216+15f(5)-20\times 64+15f(3)-6\times 8+1=0$

and $512-6\times 343+15\times 216-20f(5)+15\times 64-6f(3)+8=0$

i.e., $15f(5)+15f(3)=2280$

and $20f(5)+6f(3)=2662$

i.e., $f(5)+f(3)=152 \qquad ...(4)$

and $10f(5)+3f(3)=1331 \qquad ...(5)$

Solving (4) and (5), we get

$f(3)=27, f(5)=125.$

Ex. 15. *Obtain the missing terms in the following table :*

x	2·0	2·1	2·2	2·3	2·4	2·5	2·6
$f(x)$	0·135	?	0·111	0·100	?	0·082	0·074

(Agra 1984)

Sol. Since we are given five values so $\Delta^5 f(x)=0 \quad \forall x$

i.e. $(E-1)^5 f(x)=0$ for every value of x

i.e. $(E^5-5E^4+10E^3-10E^2+5E-1)f(x)=0 \quad \forall x$

i.e. $E^5 f(x)-5E^4 f(x)+10E^3 f(x)-10E^2 f(x)+5Ef(x)-f(x)=0$ for every value of x

i.e. $f(x+5\times\cdot 1)-5f(x+4\times\cdot 1)+10f(x+3\times\cdot 1)-10f(x+2\times\cdot 1)+5f(x+\cdot 1)-f(x)=0$ for every value of x,

here interval of differencing is 0·1.

Putting $x=2·0$ and $2·1$, we get

$f(2·5)-5f(2·4)+10f(2·3)-10f(2·2)+5f(2·1)-f(2·0)=0$

and $f(2·6)-5f(2·5)+10f(2·4)-10f(2·3)+5f(2·2)-f(2·1)=0.$

Substituting the values of $f(2·0), f(2·2), f(2·3), f(2·5), f(2·6)$ and then solving the equations simultaneously, we shall have

$f(2·1)=·123$ and $f(2·4)=·090.$

Ex. 16. *Estimate the missing term in the following table :*

x	0	1	2	3	4
$y=f(x)$	1	3	9	?	81

Explain why value differs from 3^3 *or* 27. **(Meerut M.Sc. 1987, 91, 94)**

Sol. Since we are given 4 values, therefore

$\Delta^4 f(x)=0 \quad \forall x$

i.e. $(E-1)^4 f(x)=0 \quad \forall x$

i.e. $(E^4-4E^3+6E^2-4E+1)f(x)=0 \quad \forall x,$

i.e. $E^4 f(x)-4E^3 f(x)+6E^2 f(x)-4Ef(x)+f(x)=0 \quad \forall x,$

i.e. $f(x+4)-4f(x+3)+6f(x+2)-4f(x+1)+f(x)=0 \quad \forall x,$

here interval of differencing is 1.

Putting $x=0$, we get

$f(4)-4f(3)-6f(2)-4f(1)+f(0)=0. \qquad ...(1)$

Substituting the values of $f(0), f(1), f(2), f(4)$ in (1), we get

$81-4f(3)+6\times 9-4\times 3+1=0$

i.e. $4f(3)=124$ *i.e.* $f(3)=31.$

Ex. 17. *Find the first term of the series whose second and subsequent terms are* 8, 3, 0, −1, 0.

(Meerut 1976 ; Rohilkhand 88)

Sol If the interval of differencing is unity, then

$f(1)=E^{-1}f(2)$
$=(1+\Delta)^{-1}f(2)$
$=(1-\Delta+\Delta^2-\Delta^3+...)f(2).$

Since we are given five observations, therefore the 4th differences will be constant and fifth differences will be zero.

Hence $f(1)=f(2)-\Delta f(2)+\Delta^2 f(2)$

[higher order differences are zero]

$=8 \ (-5)+2=15.$ [See difference table on the next page.]

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
2	8		
		-5	
3	3		2
		-3	
4	0		2
		-1	
5	-1		2
		1	
6	0		

Ex. 18. *Given* $u_0=3,\ u_1=12,\ u_2=81,\ u_3=200,\ u_4=100,\ u_5=8$; *find* $\Delta^5 u_0$.

Sol. We have

$$\begin{aligned}\Delta^5 u_0&=(E-1)^5 u_0\\&=(E^5-5E^4+10E^3-10E^2+5E-1)\,u_0\\&=u_5-5u_4+10u_3-10u_2+5u_1-u_0\\&=8-500+2000-810+60-3\\&=755.\end{aligned}$$

Ex. 19. *Given* $u_0=1,\ u_1=11,\ u_2=21,\ u_3=28,\ u_4=29$; *find* $\Delta^4 u_0$.

Sol. We have

$$\begin{aligned}\Delta^4 u_0&=(E-1)^4 u_0\\&=(E^4-4E^3+6E^2-4E+1)\,u_0\\&=u_4-4u_3+6u_2-4u_1+u_0\\&=29-112+126-44+1\\&=0.\end{aligned}$$

Ex. 20. *Find* $f(6)$ *given that* $f(0)=-3,\ f(1)=6,\ f(2)=8,\ f(3)=12$, *the third differences being constant.*

Sol. We construct the following difference table :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	-3			
		9		
1	6		-7	
		2		9
2	8		2	
		4		
3	12			

We have

$$\begin{aligned}f(6)&=f(0+6)=E^6 f(0)=(1+\Delta)^6 f(0)\\&=(1+6\Delta+15\Delta^2+20\Delta^3)\,f(0)\qquad\text{[higher differences being zero]}\\&=f(0)+6\Delta f(0)+15\,\Delta^2 f(0)+20\,\Delta^3 f(0)\\&=-3+6\times 9+15\times(-7)+20\times 9\\&=-3+54-105+180\\&=126.\end{aligned}$$

Ex. 21. *Given* $u_0+u_8=1{\cdot}9243,\ u_1+u_7=1{\cdot}9590,\ u_2+u_6=1{\cdot}9823,\ u_3+u_5=1{\cdot}9956$. *Find* u_4.

Solution. Since 8 entries are given, we have

$$\Delta^8 u_0=0$$

i.e. $(E-1)^8 u_0=0$

i.e. $(E^8-{}^8C_1E^7+{}^8C_2E^6-{}^8C_3E^5+{}^8C_4E^4-{}^8C_5E^3+{}^8C_6E^2-{}^8C_7E+1)\,u_0=0$

i.e. $(E^8-8E^7+28E^6-56E^5+70E^4-56E^3+28E^2-8E+1)\,u_0=0$

i.e. $u_8-8u_7+28u_6-56u_5+70u_4-56u_3+28u_2-8u_1+u_0=0$

i.e. $(u_8+u_0)-8\,(u_7+u_1)+28\,(u_6+u_2)-56\,(u_5+u_3)+70u_4=0$

Putting the given values, we get

$1{\cdot}9243-8\,(1{\cdot}9590)+28\,(1{\cdot}9823)-56\,(1{\cdot}9956)+70u_4=0$

or $-69{\cdot}9969+70u_4=0$

or $u_4=0{\cdot}9999557.$

Ex. 22. *If p, q, r and s be the successive entries corresponding to equidistant arguments in a table, show that when third differences are taken into account, the entry corresponding to the argument half way between the arguments of q and r is $A+\frac{1}{16}B$, where A is the arithmetic mean of q and r and B is the arithmetic mean of $3q-2p-s$ and $3r-2s-p$.*

(Agra B.Sc. 1974; Nagpur B.Sc. 1973; Rohilkhand M.Sc. 1990)

Sol. Let the equidistant arguments be $a,\ a+h,\ a+2h$ and $a+3h$, h being the interval of differencing.

We construct the following difference table :

x	u_x	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$
a	p			
		$q-p$		
$a+h$	q		$r-2q+p$	
		$r-q$		$s-3r+3q-p$
$a+2h$	r		$s-2r+q$	
		$s-r$		
$a+3h$	s			

The argument half way between the arguments of q and r is $\frac{1}{2}(a+h+a+2h)$ *i.e.*, $a+\frac{3}{2}h$.

Hence the required entry is given by

$$u_{a+(3/2)h}=E^{3/2}u_a=(1+\Delta)^{3/2}u_a$$

$$=\left[1+\frac{3}{2}\Delta+\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{2!}\Delta^2+\frac{3}{2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\frac{1}{3!}\Delta^3\right]u_a,$$

higher differences being neglected.

$$\therefore\quad u_{a+(3/2)h}=u_a+\frac{3}{2}\Delta u_a+\frac{3}{8}\Delta^2u_a-\frac{1}{16}\Delta^3u_a$$

$$=p+\frac{3}{2}(q-p)+\frac{3}{8}(r-2q+p)-\frac{1}{16}(s-3r+3q-p)$$

$$=p\left(1-\frac{3}{2}+\frac{3}{8}+\frac{1}{16}\right)+q\left(\frac{3}{2}-\frac{3}{4}-\frac{3}{16}\right)+r\left(\frac{3}{8}+\frac{3}{16}\right)-\frac{1}{16}s$$

$$=-\frac{1}{16}p+\frac{9}{16}q+\frac{9}{16}r-\frac{1}{16}s$$

$$=-\frac{1}{16}p+(q+r)\left(\frac{1}{16}+\frac{1}{2}\right)-\frac{1}{16}s$$

$$=\frac{1}{2}(q+r)+\frac{1}{16}(q+r-p-s)\qquad\ldots(1)$$

Now $A=$ Arithmetic mean of q and $r=\frac{1}{2}(q+r)$

$B=$ Arithmetic mean of $3q-2p-s$ and $3r-2s-p$

$$=\tfrac{1}{2}[3q-2p-s+3r-2s-p]=\tfrac{3}{2}(q+r-s-p).$$

$$\therefore\quad A+\frac{1}{24}B=\frac{q+r}{2}+\frac{1}{16}(q+r-s-p).$$

Substituting this value in (1), we get

$$u_{a+(3/2)h}=A+\frac{1}{24}B.$$

Ex. 23. *Taking fifth order differences of u_n to be constant and given $u_0, u_1, \ldots, u_5$, prove that*

$$u_{2\frac{1}{2}}=\tfrac{1}{2}c+\frac{25(c-b)+3(a-c)}{256}$$

where $a=u_0+u_5,\ b=u_1+u_4,\ c=u_2+u_3.$

(Meerut B. Sc. 1976 ; Rohilkhand M. Sc. 88)

Sol. We have $u_{2\frac{1}{2}}=u_{5/2}=E^{5/2}u_0=(1+\Delta)^{5/2}u_0$

$$=\left[1+\frac{5}{2}\Delta+\frac{1}{2!}\cdot\frac{5}{2}\cdot\frac{3}{2}\Delta^2+\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{3!}\Delta^3\right.$$

$$+\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{4!}\Delta^4$$

$$\left.+\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{5!}\Delta^5\right]u_0$$

$$=u_0+\frac{5}{2}\Delta u_0+\frac{15}{8}\Delta^2u_0+\frac{5}{16}\Delta^3u_0-\frac{5}{128}\Delta^4u_0+\frac{3}{256}\Delta^5u_0$$

$$=u_0+\frac{5}{2}(E-1)u_0+\frac{15}{8}(E-1)^2u_0+\frac{5}{16}(E-1)^3u_0$$

$$-\frac{5}{128}(E-1)^4u_0+\frac{3}{256}(E-1)^5u_0$$

$$=u_0+\frac{5}{2}(u_1-u_0)+\frac{15}{8}(u_2-2u_1+u_0)+\frac{5}{16}(u_3-3u_2+3u_1-u_0)$$

$$-\frac{5}{128}(u_4-4u_3+6u_2-4u_1+u_0)$$

$$+\frac{3}{256}(u_5-5u_4+10u_3-10u_2+5u_1-u_0)$$

$$=\frac{3}{256}(u_0+u_5)-\frac{25}{256}(u_1+u_4)+\frac{75}{128}(u_2+u_3)$$

[on simplification]

$$=\frac{3a}{256}-\frac{25b}{256}+\frac{75c}{128}$$

$$=\frac{3a}{256}-\frac{25b}{256}+\left(\frac{1}{2}+\frac{11}{128}\right)c$$

$$=\frac{c}{2}+\frac{3a-25b+22c}{256}=\frac{c}{2}+\frac{3a-25b+25c-3c}{256}$$

$$=\tfrac{1}{2}c+\frac{3(a-c)+25(c-b)}{256}.$$

Ex. 24. *Represent the function*

$$f(x)=x^4-12x^3+42x^2-30x+9$$

and its successive differences in factorial notation.

(Agra B. Sc. 1983)

Sol. Let $x^4-12x^3+42x^2-30x+9$

$$\equiv Ax^{(4)}+Bx^{(3)}+Cx^{(2)}+Dx^{(1)}+E.$$

Using the method of synthetic division (method of detached coefficients), as shown in the table on the next page we get

$$x^4-12x^3+42x^2-30x+9\equiv x^{(4)}-6x^{(3)}+13x^{(2)}+x^{(1)}+9.$$

Since in the factorial notation, the operator Δ is equivalent to differentiation, we get

$$\Delta f(x)=4x^{(3)}-18x^{(2)}+26x+1$$

$$\Delta^2f(x)=12x^{(2)}-36x^{(1)}+26$$

$$\Delta^3f(x)=24x-36$$

$$\Delta^4f(x)=24.$$

1	1	−12	42	−30	9=E
	0	1	−11	31	
2	1	−11	31	1=D	
	0	2	−18		
3	1	−9	13=C		
	0	3			
4	1	−6=B			
	0				
	1=A				

Note 1. While using the method of synthetic division, we should first of all see whether the given expression is complete or not. If any power of x is missing, we should first make the expression complete by taking its coefficient zero and then employ the method of synthetic division.

2. The reader is advised to use the direct method, viz, by taking

$$f(x)=x^4-12x^3+42x^2-30x+9$$
$$\equiv x^{(4)}+Ax^{(3)}+Bx^{(2)}+Cx^{(1)}+9$$

and calculating the coefficients A, B and C by comparing the coefficients of like powers of x on both sides. It may be pointed out that the coefficients of the highest power of x and the constant terms remain unchanged while transforming to factorial notation. This point reduces calculation work to some extent.

Ex. 25. *Find the function whose first difference is*

$9x^2+11x+5.$ **(Meerut B.Sc. 1983)**

Sol. Let $f(x)$ be the required function.

Then $\Delta f(x)=9x^2+11x+5.$

Let $9x^2+11x+5\equiv 9x^{(2)}+Ax^{(1)}+B$

$$=9x(x-1)+Ax+B.$$

Putting $x=0$, we get $B=5$.

Putting $x=1$, we get $A=20$.

$$\therefore \quad \Delta f(x)=9x^{(2)}+20x^{(1)}+5.$$

Integrating it, we get

$$f(x)=9.\frac{x^{(3)}}{3}+20.\frac{x^{(2)}}{2}+5.\frac{x^{(1)}}{1}+C,$$

where C is a constant.

$$\therefore \quad f(x)=3x^{(3)}+10x^{(2)}+5x^{(1)}+C$$
$$=3x^3+x^2+x+C.$$

Ex. 26. *Find the lowest degree polynomial which takes the following values :*

x	0	1	2	3	4	5
$f(x)$	0	3	8	15	24	35

Sol. We know that

$$f(a+nh)=f(a)+{}^nC_1\Delta f(a)+{}^nC_2\Delta^2 f(a)+\ldots$$
$$+{}^nC_n\Delta^n f(a) \qquad \ldots(1)$$

Putting $a=0$, $h=1$, $n=x$, we get

$$f(x)=f(0)+{}^xC_1\,\Delta f(0)+{}^xC_2\,\Delta^2 f(0)+{}^xC_3\,\Delta^3 f(0)+\ldots$$
$$=f(0)+x^{(1)}\,\Delta f(0)+\frac{x^{(2)}}{2!}\,\Delta^2 f(0)+\ldots \qquad \ldots(2)$$

Now we prepare the difference table for the given data to find $\Delta f(0)$, $\Delta^2 f(0)$, $\Delta^3 f(0)$ etc.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	0			
		3		
1	3		2	
		5		0
2	8		2	
		7		0
3	15		2	
		9		0
4	24		2	
		11		
5	35			

Putting the values of $f(0)$, $\Delta f(0)$, $\Delta^2 f(0)$, $\Delta^3 f(0)$ in (2), we get

$$f(x)=0+3x^{(1)}+2.\frac{x^{(2)}}{2!}+0$$
$$=3x+x(x-1)=x^2+2x.$$

Ex. 27. *Write down the polynomial of lowest degree which satisfies the following set of numbers*

0, 7, 26, 63, 124, 215, 342, 511.

Sol. We have

$$f(x)=f(0)+x^{(1)}\frac{\Delta f(0)}{1\,!}+\frac{x^{(2)}}{2\,!}\Delta^2 f(0)+\frac{x^{(3)}}{3\,!}\Delta^3 f(0)+\ldots$$

By forming the difference table we calculate $\Delta f(0)$, $\Delta^2 f(0)$, $\Delta^3 f(0)$ etc.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	0				
		7			
1	7		12		
		19		6	
2	26		18		0
		37		6	
3	63		24		0
		61		6	
4	124		30		0
		91		6	
5	215		36		0
		127		6	
6	342		42		
		169			
7	511				

Putting the values of $\Delta f(0)$, $\Delta^2 f(0)$,...from the table, we get

$$f(x)=0+7x^{(1)}+\frac{12}{2\,!}\,x^{(2)}+\frac{6}{3\,!}\,x^{(3)}$$

(higher order differences being equal to zero)

$$=7x+6x(x-1)+x(x-1)(x-2)$$
$$=7x+6x^2-6x+x^3-3x^2+2x$$
$$=x^3+3x^2+3x.$$

Ex. 28. *Find the relation between α, β and γ in order that $\alpha+\beta x+\gamma x^2$ may be expressible in one term in the factorial notation.* **(Meerut 1976, 80)**

Solution. Let $f(x)=\alpha+\beta x+\gamma x^2=(a+bx)^{(2)}$

where a and b are some unknown constants.

Now $(a+bx)^{(2)}=(a+bx)\,\{a+b(x-1)\}$
$$=(a+bx)\,(a-b+bx)$$
$$=(a+bx)^2-ab-b^2x$$
$$=(a^2-ab)+(2ab-b^2)x+b^2x^2.$$

$$\therefore\quad \alpha+\beta x+\gamma x^2=(a^2-ab)+(2ab-b^2)x+b^2x^2.$$

Comparing the coefficients of various powers of x, we get

$$\alpha=a^2-ab,\ \beta=2ab-b^2,\ \gamma=b^2.$$

Eliminating a and b from these equations, we get

$$\gamma^2+4\alpha\gamma=\beta^2,$$

which is the required relation between α, β and γ.

Ex. 29. *Find the function whose first difference is e^x.* **(Meerut 1988)**

Sol. We know that $\Delta e^x=e^{x+h}-e^x=e^x\,(e^h-1)$,

where h is the interval of differencing.

$$\therefore\quad e^x=\frac{1}{e^h-1}\,\Delta e^x=\Delta\left(\frac{e^x}{e^h-1}\right).$$

Hence the required function $f(x)$ is $\dfrac{e^x}{e^h-1}$.

Ex. 30. *Find a function u_x for which $\Delta u_x=x\,(x-1)$.*

Solution. We have $\Delta u_x=x\,(x-1)=x^{(2)}$.

$$\therefore\quad u_x=\frac{x^{(3)}}{3}+C,$$

where C is an arbitrary constant

$$=\tfrac{1}{3}x(x-1)\,(x-2)+C.$$

Ex. 31. *Prove $\Delta^2 x^{(m)}=m(m-1)\,x^{(m-2)}$ where m is a positive integer and interval of differencing is unity.*

Sol. We have $x^{(m)}=x\,(x-1)\,(x-2)\ldots(x-m-1)$.

$$\therefore\quad \Delta x^{(m)}=\{(x+1)\,x\,(x-1)\ldots(x+1-\overline{m-1})\}$$
$$-\{x\,(x-1)\ldots(x-\overline{m-1})\}$$
$$=x\,(x-1)\ldots(x-\overline{m-2})\,\{x+1-(x-\overline{m-1})\}$$
$$=mx^{(m-1)}.$$

$$\therefore\quad \Delta^2 x^{(m)}=\Delta\Delta x^{(m)}=\Delta\,\{mx^{(m-1)}\}$$
$$=m\Delta x^{(m-1)}=m(m-1)x^{(m-2)}.$$

Ex. 32. *Prove that $\Delta^2 x^{(-m)}=m(m+1)x^{(-m-2)}$.* **(Meerut 1991)**

Sol. We have

$$x^{(-m)}=\frac{1}{(x+1)\,(x+2)\ldots(x+m)}.$$

$$\therefore\quad \Delta x^{(-m)}=\frac{1}{(x+2)\,(x+3)\ldots(x+m+1)}-\frac{1}{(x+1)\,(x+2)\ldots(x+m)}$$

$$=\frac{1}{(x+2)\ \ (x+m)}\left[\frac{1}{(x+m+1)}-\frac{1}{(x+1)}\right]$$

$$=\frac{-m}{(x+1)(x+2)\ldots(x+m)(x+\overline{m+1})}$$

$$=-m\,x^{(-[m+1])}=-m\,x^{(-m-1)}.$$

$$\therefore\quad \Delta^2x^{(-m)}=\Delta\Delta x^{(-m)}=\Delta\{-m x^{(-m-1)}\}$$

$$=-m\Delta x^{(-m-1)}=-m(-m-1)x^{(-m-2)}$$

$$=m(m+1)\,x^{(-m-2)}.$$

Ex. 33. *Show that* $(x\Delta)^{(n)}u_x=(x+n-1)^{(n)}\,\Delta^n u_x$. **(Patna 1980)**

Sol. We shall prove this result by mathematical induction. For $n=1$, we have

$$(x\Delta)^{(1)}u_x=(x\Delta)u_x=x\,\Delta u_x=x^{(1)}\Delta u_x=(x+1-1)^{(1)}\,\Delta^1 u_x.$$

Thus the result is true for $n=1$.

For $n=2$, the L.H.S. $=(x\Delta)^{(2)}u_x$

$$=(x\Delta)\,(x\Delta-1)\,u_x$$

$$=(x\Delta)\,(x\Delta u_x)-x\Delta u_x$$

$$=x\,\Delta\{(x\Delta u_x)\}-x\Delta u_x$$

$$=x\{(x+1)\,\Delta^2u_x+(\Delta u_x).1\}-x\Delta u_x$$

$$[\because\ \Delta\{f(x)g(x)\}=f(x+h)\,\Delta g(x)+g(x)\Delta f(x)]$$

$$=x(x+1)\,\Delta^2u_x$$

$$=(x+1)^{(2)}\,\Delta^2u_x.$$

Thus the result is true for $n=2$.

Let the result be true for n

i.e., $\quad (x\Delta)^{(n)}u_x=(x+n-1)^{(n)}\,\Delta^n u_x.$ ...(1)

We shall show that it is then also true for the index $n+1$.

Now the operators $x\Delta-r$ and $x\Delta-s$ are commutative, where r and s are any constants. Therefore we have

$$(x\Delta)^{(n+1)}u_x=(x\Delta)\,(x\Delta-1)\ldots(x\Delta-\overline{n-1})\,(x\Delta-n)u_x$$

$$=(x\Delta-n)\,(x\Delta)^{(n)}u_x$$

$$=(x\Delta-n)\,[(x+n-1)^{(n)}\,\Delta^n u_x]\quad \text{[by (1)]}$$

$$=x\Delta[(x+n-1)^{(n)}\,\Delta^n u_x]-n(x+n-1)^{(n)}\,\Delta^n u_x$$

$$=x\,[(x+n)^{(n)}\,\Delta^{n+1}u_x+n(x+n-1)^{(n-1)}\,\Delta^n u_x]-n(x+n-1)^{(n)}\,\Delta^n u_x$$

$$=(x+n)^{(n+1)}\,\Delta^{n+1}u_x+n(x+n-1)^{(n)}\,\Delta^n u_x-n(x+n-1)^{(n)}\,\Delta^n u_x$$

$$=(x+n)^{(n+1)}\,\Delta^{n+1}u_x.$$

This shows that the result is true for $n+1$ if it were true for n. Hence by mathematical induction the result is true for every positive integer n.

Ex. 34. *Prove that*

$$\Delta^n0^{n+1}=\frac{n\,(n+1)}{2}\,\Delta^n0^n.$$

Sol. Using the relation $\Delta^n0^m=n\,[\Delta^{n-1}0^{m-1}+\Delta^n0^{m-1}]$, we get

$$\Delta^n0^{n+1}=n\,[\Delta^{n-1}0^n+\Delta^n0^n]$$

$$\Delta^{n-1}0^n=(n-1)\,[\Delta^{n-2}0^{n-1}+\Delta^{n-1}0^{n-1}]$$

$$\Delta^{n-2}0^{n-1}=(n-2)\,[\Delta^{n-3}0^{n-2}+\Delta^{n-2}0^{n-2}]$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$\Delta^20^3=2\,[\Delta0^2+\Delta^20^2]$$

$$\Delta0^2=1\,[\Delta^00^1+\Delta^10^1].$$

By back substitution of these values, we get

$$\Delta^n0^{n+1}=n\Delta^n0^n+n\,(n-1)\,\Delta^{n-1}0^{n-1}+n\,(n-1)(n-2)\,\Delta^{n-2}0^{n-2}+\ldots+n(n-1)(n-2)\ldots2.\,1\Delta^10^1$$

$$=nn!+n(n-1)\,(n-1)\,!+n(n-1)\,(n-2)\,(n-2)\,!+\ldots+n\,(n-1)\,(n-2)\ldots2.1.1\,!$$

$$=n\,!\,\{n+(n-1)+(n-2)+\ldots+2+1\}$$

$$=n\,!.\,\frac{n\,(n+1)}{2}=\frac{n\,(n+1)}{n}\,.\Delta^n0^n.\qquad [\because\ \ \Delta^n0^n=n\,!]$$

Ex 35. *Sum the series (using differences of zero)*

$$n^2+{}^nC_1\,(n-1)^2+{}^nC_2\,(n-2)^2+\ldots,$$

n being a positive integer.

Sol. The given series

$$=n^2+{}^nC_1\,(n-1)^2+{}^nC_2\,(n-2)^2+\ldots$$

$$=[(x+n)^2+{}^nC_1\,(x+n-1)^2+{}^nC_2\,(x+n-2)^2+\ldots]_{x=0}$$

$$=[E^nx^2+{}^nC_1\,E^{n-1}\,x^2+{}^nC_2\,E^{n-2}x^2+\ldots]_{x=0}$$

$$=[\{E^n+{}^nC_1\,E^{n-1}+{}^nC_2\,E^{n-2}+\ldots\}\,x^2]_{x=0}$$

$$=[(E+1)^nx^2]_{x=0}=[(2+\Delta)^n\,x^2]_{x=0}$$

$$=[\{2^n+{}^nC_1\,2^{n-1}\,\Delta+{}^nC_2\,2^{n-2}\,\Delta^2+\ldots\}\,x^2]_{x=0}$$

$$=[2^n\,x^2+{}^nC_1\,2^{n-1}\,\Delta x^2+{}^nC_2\,2^{n-2}\,\Delta^2x^2+\ldots]_{x=0}$$

$$=0+{}^nC_1\,2^{n-1}\Delta0^2+{}^nC_2\,2^{n-2}\Delta^20^2+{}^nC_3\,2^{n-3}\Delta^30^2+\ldots$$

$$=n.\,2^{n-1}.\,1+\frac{n\,(n-1)}{1.2}.\,2^{n-2}\,2\,!+0+0+\ldots$$

[$\because$ the differences of 0^m of orders higher than m are all zero]

$$=n.\,2^{n-1}+n\,(n-1).\,2^{n-2}$$

$$=n\,2^{n-2}\,[2+n-1]=n\,(n+1).\,2^{n-2}.$$

Ex. 36. *Find and correct by means of differences the error in the following table :*

20736, 28561, 38416, 50625, 65540, 83521, 104976, 130321, 160000.

Sol. We construct the following difference table for the given data.

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
20736					
	7825				
28561		2030			
	9855		324		
38416		2354		28	
	12209		352		-20
50625		2706		8	
	14915		360		40
65540		3066		48	
	17981		408		-40
83521		3474		8	
	21455		416		20
104976		3890		28	
	25345		444		
130321		4334			
	29679				
160000					

From this table we see that the pattern of departure in third differences is quite irregular and the irregularity starts around the horizontal line corresponding to the value $y=65540$.

Since the algebraic sum of the fifth differences is 0, the fifth differences found in this example are accumulated errors.

Referring to table 1·3 page 17, we get

$$-5\epsilon=-20 \Rightarrow \epsilon=4.$$

$\therefore$ the true value of $y_5=65540-4=65536$.

Ex. 37. *Evaluate*

(i) $(2\Delta^2+\Delta-1)(x^2+2x+1)$

(ii) $(\Delta+1)(2\Delta-1)(x^2+2x+1)$

(iii) $(E-2)(E-1)(2^{x/h}+x)$

(iv) $(E^2-3E+2)(2^{x/h}+x)$.

Sol. (i) We have $(2\Delta^2+\Delta-1)(x^2+2x+1)$

$$=2\Delta^2(x^2+2x+1)+\Delta(x^2+2x+1)-1(x^2+2x+1)$$
$$=2\Delta^2x^2+\Delta(x^2+2x)-x^2-2x-1 \qquad [\because \Delta^2(2x+1)=0,\ \Delta 1=0]$$
$$=2\Delta[(x+h)^2-x^2]+[(x+h)^2+2(x+h)-x^2-2x]-x^2-2x-1$$
$$=2\Delta(2hx+h^2)+2hx+h^2+2h-x^2-2x-1 \qquad [\because \Delta h^2=0]$$
$$=4h\Delta x+2hx+h^2+2h-x^2-2x-1$$
$$=4h(x+h-x)+2hx+h^2+2h-x^2-2x-1$$
$$=5h^2+2hx+2h-x^2-2x-1.$$

(ii) We have $(\Delta+1)(2\Delta-1)(x^2+2x+1)$

$$=(\Delta+1)[2\Delta(x^2+2x+1)-1(x^2+2x+1)]$$
$$=(\Delta+1)[2\{(x+h)^2+2(x+h)+1-x^2-2x-1\}-x^2-2x-1]$$
$$=(\Delta+1)[2(2hx+h^2+2h)-x^2-2x-1]$$
$$=(\Delta+1)(4hx+2h^2+4h-x^2-2x-1)$$
$$=\Delta(4hx-x^2-2x)+1(4hx+2h^2+4h-x^2-2x-1) \qquad [\because \Delta(2h^2+4h-1)=0]$$
$$=4h\Delta x-\Delta x^2-2\Delta x+4hx+2h^2+4h-x^2-2x-1$$
$$=4h(x+h-x)-[(x+h)^2-x^2]-2(x+h-x)+4hx+2h^2+4h-x^2-2x-1$$
$$=4h^2-2hx-h^2-2h+4hx+2h^2+4h-x^2-2x-1$$
$$=5h^2+2hx+2h-x^2-2x-1.$$

(iii) We have $(E-2)(E-1)(2^{x/h}+x)$

$$=(E-2)[E(2^{x/h}+x)-(2^{x/h}+x)]$$
$$=(E-2)[2^{(x+h)/h}+x+h-2^{x/h}-x]$$
$$=(E-2)[2^{x/h}(2-1)+h]=(E-2)(2^{x/h}+h)$$
$$=E(2^{x/h}+h)-2(2^{x/h}+h)$$
$$=2^{(x+h)/h}+h-2^{(x+h)/h}-2h=-h.$$

(iv) We have $(E^2-3E+2)(2^{x/h}+x)$

$$=E^2(2^{x/h}+x)-3E(2^{x/h}+x)+2(2^{x/h}+x)$$
$$=[2^{(x+2h)/h}+x+2h]-3[2^{(x+h)/h}+x+h]+2.2^{x/h}+2x$$
$$=4.2^{x/h}+x+2h-3(2.2^{x/h}+x+h)+2.2^{x/h}+2x$$
$$=-h.$$

Ex. 38. *Show that* $B(m+1,n)=(-1)^m\Delta^m\left(\frac{1}{n}\right)$ *where m is a positive integer.*

Sol. We know that $\int_0^\infty e^{-nx}\,dx=\frac{1}{n}$.

$$\therefore \quad \Delta^m\int_0^\infty e^{-nx}\,dx=\Delta^m\left(\frac{1}{n}\right)$$

or $\int_0^\infty \Delta^m e^{-nx}\,dx=\Delta^m\left(\frac{1}{n}\right)$, where for $\Delta^m e^{-nx}$, n is to be regarded as variable and x is to be regarded as constant.

Now $\Delta^m e^{-nx}=\Delta^{m-1}[e^{-(n+1)x}-e^{-nx}]$

$$=\Delta^{m-1}e^{-nx}(e^{-x}-1)=(e^{-x}-1)\Delta^{m-1}e^{-nx}$$
$$=(e^{-x}-1)^2\Delta^{m-2}e^{-nx}=\ldots$$
$$=(e^{-x}-1)^m e^{-nx}.$$

$$\therefore \quad \int_0^\infty (e^{-x}-1)^m e^{-nx}\,dx=\Delta^m\left(\frac{1}{n}\right)$$

Put $e^{-x}=z$ so that $-e^{-x}dx=dz$

or $dx = -(1/z)\, dz.$

Then $\int_1^0 z^n (z-1)^m (-1/z)\, dz = \Delta^m \left(\frac{1}{n}\right)$

or $(-1)^m \int_0^1 z^{n-1} (1-z)^m\, dz = \Delta^m \left(\frac{1}{n}\right)$

or $\int_0^1 z^{n-1} (1-z)^{(m+1)-1}\, dz = (-1)^m \Delta^m \left(\frac{1}{n}\right)$

or $B(m+1,\ n) = (-1)^m \Delta^m \left(\frac{1}{n}\right).$

Ex. 39. *Use the method of separation of symbols to prove that*

$$u_1x + u_2x^2 + u_3x^3 + \ldots = \frac{x}{1-x} u_1 + \frac{x^2}{(1-x)^2} \Delta u_1 + \frac{x^3}{(1-x)^3} \Delta^2 u_1 + \ldots$$

(Agra B.Sc. 1971, 80; Banaras 73; Rohilkhand B.Sc. 91)

Sol. L.H.S. $= u_1x + u_2x^2 + u_3x^3 + \ldots$

$= xu_1 + x^2Eu_1 + x^3E^2u_1 + \ldots$ $\quad [\because\ E^n u_x = u_{x+n}]$

$= x\,[1 + xE + x^2E^2 + \ldots]\, u_1$

$= x \left(\frac{1}{1-xE}\right) u_1$

[Summing as an infinite G.P. with common ratio xE]

$= x \left[\frac{1}{1-x\,(1+\Delta)}\right] u_1$ $\quad [\because\ E \equiv 1+\Delta]$

$= x \left[\frac{1}{1-x-x\Delta}\right] u_1$

$= \frac{x}{1-x} \left[1 - \frac{x\Delta}{1-x}\right]^{-1} u_1$

$= \frac{x}{1-x} \left[1 + \frac{x\Delta}{1-x} + \frac{x^2\Delta^2}{(1-x)^2} + \ldots\right] u_1$

$= \frac{x}{1-x} u_1 + \frac{x^2}{(1-x)^2} \Delta u_1 + \frac{x^3}{(1-x)^3} \Delta^2 u_1 + \ldots$

$=$ R.H.S.

Ex. 40. *Show that*

$$u_0 - u_1 + u_2 - \ldots = \tfrac{1}{2}u_0 - \tfrac{1}{4}\Delta u_0 + \tfrac{1}{8}\Delta^2 u_0 - \ldots$$

Sol. L.H.S. $= u_0 - u_1 + u_2 - u_3 + \ldots$

$= u_0 - Eu_0 + E^2u_0 - E^3u_0 + \ldots$ $\quad [\because\ E^n u_0 = u_n]$

$= (1 - E + E^2 - E^3 + \ldots)\, u_0$

$= \left[\frac{1}{1-(-E)}\right] u_0 = \left(\frac{1}{1+E}\right) u_0$

$= \left(\frac{1}{1+1+\Delta}\right) u_0$ $\quad [\because\ E \equiv 1+\Delta]$

$= \left(\frac{1}{2+\Delta}\right) u_0$

$= \frac{1}{2} \left(1 + \frac{\Delta}{2}\right)^{-1} u_0$

$= \frac{1}{2} \left[1 - \frac{\Delta}{2} + \frac{\Delta^2}{4} - \frac{\Delta^3}{8} + \ldots\right] u_0$

$= \frac{1}{2} u_0 - \frac{1}{4} \Delta u_0 + \frac{1}{8} \Delta^2 u_0 - \frac{1}{16} \Delta^3 u_0 + \ldots$

$=$ R.H.S.

Ex. 41 *Use the method of separation of symbols to prove the following identities :*

(i) $\Delta^n u_x = u_{x+n} - {}^nC_1\, u_{x+n-1} + {}^nC_2\, u_{x+n-2} + \ldots + (-1)^n u_x.$

(ii) $u_x - \frac{1}{8} \Delta^2 u_{x-1} + \frac{1.3}{8.16} \Delta^4 u_{x-2} - \frac{1.3.5}{8.16.24} \Delta^6 u_{x-3} + \ldots$

$= u_{x+(1/2)} - \frac{1}{2} \Delta u_{x+(1/2)} + \frac{1}{4} \Delta^2 u_{x+(1/2)} - \frac{1}{8} \Delta^3 u_{x+(1/2)} + \ldots$

(Meerut M.Sc. 1976, 84, B.Sc. 92)

(iii) $u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \ldots + \Delta^{n-1} u_{x-n} + \Delta^n u_{x-n}.$

(Meerut B.Sc. Stat. 1978, 80, M.Sc. 89; Delhi Hons. 71, 73)

Sol. (i) L.H.S.

$= \Delta^n u_x = (E-1)^n u_x$

$= [E^n - {}^nC_1\, E^{n-1} + {}^nC_2\, E^{n-2} + \ldots + (-1)^n]\, u_x$

$= E^n u_x - {}^nC_1\, E^{n-1} u_x + {}^nC_2\, E^{n-2} u_x + \ldots + (-1)^n u_x$

$= u_{x+n} - {}^nC_1\, u_{x+n-1} + {}^nC_2 u_{x+n-2} + \ldots + (-1)^n u_x$

$=$ R.H.S. $\quad [\because\ E^n u_x = u_{x+n}]$

(ii) L.H.S.

$= u_x - \frac{1}{8} \Delta^2 u_{x-1} + \frac{1.3}{8.16} \Delta^4 u_{x-2} - \frac{1.3.5}{8.16.24} \Delta^6 u_{x-3} + \ldots$

$= u_x - \frac{1}{8} \Delta^2 E^{-1} u_x + \frac{1\cdot 3}{8^2\cdot 1\cdot 2} \Delta^4 E^{-2} u_x - \frac{1\cdot 3\cdot 5}{8^3\cdot 1\cdot 2\cdot 3} \Delta^6 E^{-3} u_x + \ldots$

$= u_x - \frac{1}{2} \left(\frac{\Delta^2}{4E}\right) u_x + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{1.2} \left(\frac{\Delta^2}{4E}\right)^2 u_x$

$+ \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{1.2.3} \left(\frac{\Delta^2}{4E}\right)^3 u_x + \ldots$

$= \left[1 + (-\tfrac{1}{2}) \frac{\Delta^2}{4E} + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2\,!} \left(\frac{\Delta^2}{4E}\right)^2 + \ldots\right] u_x$

$= \left(1 + \frac{\Delta^2}{4E}\right)^{-1/2} u_x = \left(\frac{4E+\Delta^2}{4E}\right)^{-1/2} u_x.$

$= \left\{\frac{4\,(1+\Delta)+\Delta^2}{4E}\right\}^{-1/2} u_x = \left\{\frac{(2+\Delta)^2}{4E}\right\}^{-1/2} u_x$

$$=\left\{\frac{4E}{(2+\Delta)^2}\right\}^{1/2} u_x=2E^{1/2}\,\frac{1}{(2+\Delta)}\,u_x$$

$$=2E^{1/2}\,\frac{1}{2\left(1+\frac{\Delta}{2}\right)}\,u_x=E^{1/2}\left(1+\frac{\Delta}{2}\right)^{-1}u_x$$

$$=E^{1/2}\left\{1-\frac{\Delta}{2}+\frac{\Delta^2}{2^2}-\dots\right\}u_x$$

$$=u_{x+(1/2)}-\tfrac{1}{2}\Delta u_{x+(1/2)}+\tfrac{1}{4}\Delta^2 u_{x+(1/2)}-\dots$$

$=$ R.H.S.

(iii) We have $u_x-\Delta^n u_{x-n}=u_x-\Delta^n E^{-n}u_x$

$$=(1-\Delta^n E^{-n})\,u_x$$

$$=\left[1-\left(\frac{\Delta}{E}\right)^n\right]u_x$$

$$=\frac{1}{E^n}(E^n-\Delta^n)\,u_x=\frac{1}{E^n}\left\{\frac{E^n-\Delta^n}{E-\Delta}\right\}u_x \qquad [\because\ E\equiv 1+\Delta]$$

$$=E^{-n}\,[E^{n-1}+\Delta E^{n-2}+\Delta^2E^{n-3}+\dots+\Delta^{n-1}]\,u_x$$

$$=[E^{-1}+\Delta E^{-2}+\Delta^2E^{-3}+\dots+\Delta^{n-1}E^{-n}]\,u_x$$

$$=u_{x-1}+\Delta u_{x-2}+\Delta^2u_{x-3}+\dots+\Delta^{n-1}u_{x-n}.$$

$$u_x=u_{x-1}+\Delta u_{x-2}+\Delta^2u_{x-3}+\dots+\Delta^{n-1}u_{x-n}+\Delta^n u_{x-n}.$$

Ex. 42. *Use the method of separation of symbols to prove the following identities :*

(i) $u_0+u_1+u_2+\dots+u_n$

$$={}^{n+1}C_1u_0+{}^{n+1}C_2\Delta u_0+{}^{n+1}C_3\Delta^2u_0+\dots\dots+\Delta^n u_0,$$

(Karnatak 1972)

(ii) $u_{x+n}=u_n+{}^xC_1\Delta u_{n-1}+{}^{x+1}C_2\Delta^2u_{n-2}+{}^{x+2}C_3\Delta^3u_{n-3}+\dots$

(iii) $\Delta^n u_{x-n}=u_x-{}^nC_1u_{x-1}+{}^nC_2u_{x-2}-{}^nC_3u_{x-3}+\dots$

Sol. (i) L.H.S. $=u_0+u_1+u_2+\dots+u_n$

$$=u_0+Eu_0+E^2u_0+\dots+E^nu_0$$

$$=(1+E+E^2+\dots+E^n)u_0$$

$$=\frac{E^{n+1}-1}{E-1}\,u_0$$

$$=\frac{(1+\Delta)^{n+1}-1}{\Delta}\,u_0 \qquad [\because\ \Delta\equiv E-1]$$

$$=\frac{1}{\Delta}\Big[\Big(1+{}^{n+1}C_1\Delta+{}^{n+1}C_2\Delta^2+{}^{n+1}C_3\Delta^3+\dots$$

$$\dots+\Delta^{n+1}\Big)-1\Big]u_0$$

$$=\frac{1}{\Delta}\Big[{}^{n+1}C_1\Delta+{}^{n+1}C_2\Delta^2+{}^{n+1}C_3\Delta^3+\dots$$

$$\dots+\Delta^{n+1}\Big]u_0$$

$$={}^{n+1}C_1u_0+{}^{n+1}C_2\Delta u_0+{}^{n+1}C_3\Delta^2u_0+\dots+\Delta^nu_0$$

$=$ R.H.S.

(ii) R.H.S. $=u_n+{}^xC_1\Delta u_{n-1}+{}^{x+1}C_2\Delta^2u_{n-2}+{}^{x+2}C_3\Delta^3u_{n-3}+\dots$

$$=u_n+{}^xC_1\Delta E^{-1}u_n+{}^{x+1}C_2\Delta^2E^{-2}u_n$$

$$+{}^{x+2}C_3\Delta^3E^{-3}u_n+\dots$$

$$=[1+{}^xC_1\Delta E^{-1}+{}^{x+1}C_2\Delta^2E^{-2}+{}^{x+2}C_3\Delta^3E^{-3}+\dots]u_n$$

$$=[1-\Delta E^{-1}]^{-x}u_n=\left(1-\frac{\Delta}{E}\right)^{-x}u_n=\left(\frac{E-\Delta}{E}\right)^{-x}u_n$$

$$=\left(\frac{1}{E}\right)^{-x}u_n=E^xu_n=u_{x+n}=\text{L.H.S.}$$

(iii) R.H.S. $=u_x-{}^nC_1u_{x-1}+{}^nC_2u_{x-2}-{}^nC_3u_{x-3}+\dots$

$$=u_x-{}^nC_1E^{-1}u_x+{}^nC_2E^{-2}u_x-{}^nC_3E^{-3}u_x+\dots$$

$$=(1-{}^nC_1E^{-1}+{}^nC_2E^{-2}-{}^nC_3E^{-3}+\dots)u_x$$

$$=(1-E^{-1})^n\,u_x=\left(1-\frac{1}{E}\right)^nu_x=\left(\frac{E-1}{E}\right)^nu_x$$

$$=\left(\frac{\Delta}{E}\right)^nu_x=\Delta^nE^{-n}u_x=\Delta^nu_{x-n}$$

$=$ L.H.S.

Ex. 43. *Show that*

$u_{2n}-{}^nC_1\,2u_{2n-1}+{}^nC_2\,2^2u_{2n-2}-\dots+(-2)^n\,u_n=(-1)^n\ (c-2an)$

where $u_n=an^2+bn+c.$

Sol. L.H.S $=u_{2n}-{}^nC_1\,2u_{2n-1}+{}^nC_2\,2^2u_{2n-2}+\dots+(-2)^n\,u_n$

$$=E^nu_n-{}^nC_1.2E^{n-1}u_n+{}^nC_2\,2^2E^{n-2}\,u_n-\dots$$

$$\dots+(-2)^n\,u_n$$

$$=[E^n-{}^nC_1.2E^{n-1}+{}^nC_2.2^2E^{n-2}-\dots+(-2)^n]\,u_n$$

$$=(E-2)^n\,u_n=(\Delta-1)^n\,u_n=(-1)^n\,(1-\Delta)^n\,u_n$$

$$=(-1)^n\left[1-n\Delta+\frac{n\,(n-1)}{1.2}\,\Delta^2\right]u_n$$

[neglecting higher order differences as u_n is a polynomial of second degree]

$$=(-1)^n\left[u_n-n\,\Delta u_n+\frac{n\,(n-1)}{2}\,\Delta^2u_n\right]$$

$$=(-1)^n\Big[(an^2+bn+c)-n\Delta\,(an^2+bn+c)$$

$$+\frac{n^2-n}{2}\,\Delta^2\,(an^2+bn+c)\Big]$$

$$=(-1)^n\Big[(an^2+bn+c)-n\,\{a\Delta n^2+b\Delta n\}$$

$$+\frac{n^2-n}{2}.a\Delta^2n^2\Big]$$

$$=(-1)^n\Big[(an^2+bn+c)-n\{a(n+1)^2-an^2\}$$
$$-bn(n+1-n)+\frac{n^2-n}{2}a\Delta\{(n+1)^2-n^2\}\Big]$$
$$=(-1)^n\Big[(an^2+bn+c)-n(2an+a+b)$$
$$+\frac{n^2-n}{2}.a\,\Delta\,(2n+1)\Big]$$
$$=(-1)^n\Big[(an^2+bn+c)-n(2an+a+b)$$
$$+\frac{n^2-n}{2}.a\{2(n+1)-2n\}\Big]$$
$$=(-1)^n[(an^2+bn+c)-n(2an+a+b)+a(n^2-n)]$$
$$=(-1)^n\{c-2an\}$$
=R.H.S.

Ex. 44. *Show that*

$$\Delta x^m-\tfrac{1}{2}\Delta^2x^m+\frac{1.3}{2.4}\Delta^3x^m-\frac{1.3.5}{2.4.6}\Delta^4x^m+\ldots\text{ to } m \text{ terms}$$
$$=(x+\tfrac{1}{2})^m-(x-\tfrac{1}{2})^m.$$

(Gujrat B.Sc. 1978 ; Meerut M Sc. 86)

Sol. We know that $\Delta^n x^m=0$ for $n>m$. Thus the sum of m terms of the series will be same as the sum of infinite terms of the series.

Now L.H.S.

$$=\Delta x^m-\tfrac{1}{2}\Delta^2x^m+\frac{1.3}{2.4}\Delta^3x^m-\frac{1.3.5}{2.4.6}\Delta^4x^m+\ldots\text{to } m \text{ terms}$$
$$=\Delta\left[1-\frac{\Delta}{2}+\frac{1.3}{2.4}\Delta^2-\frac{1.3.5}{2.4.6}\Delta^3+\ldots m\text{ terms}\right]x^m$$
$$=\Delta\left[1-\tfrac{1}{2}\Delta+\frac{1.3}{2.4}\Delta^2-\frac{1.3.5}{2.4.6}\Delta^3+\ldots\text{to }\infty\right]x^m$$
$$=\Delta[1+\Delta]^{-1/2}x^m=\Delta E^{-1/2}x^m$$
$$=\Delta(x-\tfrac{1}{2})^m,\quad\text{the interval of differencing being unity}$$
$$=(x+\tfrac{1}{2})^m-(x-\tfrac{1}{2})^m$$
=R.H.S.

Ex. 45. *Sum to n terms the series*

$$1.2\Delta x^n-2.3\Delta^2x^n+3.4\Delta^3x^n-4.5\Delta^4x^n+\ldots$$

(Meerut 1978, 80, 94)

Sol. Let $u_n=x^n$. Then we know that nth difference of u_n will be constant and higher order differences will be zero. Thus the given expression will contain tefms upto $\Delta^n x^n$ as the

higher order terms will be zero. Hence the sum of the above series to n terms is the same as upto ∞.

We have

$$1.2\Delta x^n-2.3\Delta^2x^n+3.4\Delta^3x^n-4.5\Delta^4x^n+\ldots$$
$$=2\Delta[1-3\Delta+3.2\Delta^2-2.5\Delta^3+\ldots]x^n$$
$$=2\Delta\left[1-3\Delta+\frac{3.4}{2}\Delta^2-\frac{3.4.5}{2.3}\Delta^3+\ldots\right]x^n$$
$$=2\Delta[1+\Delta]^{-3}x^n=2(E-1)E^{-3}x^n$$
$$=2(E^{-2}-E^{-3})x^n$$
$$=2[E^{-2}x^n-E^{-3}x^n]$$
$$=2[(x-2)^n-(x-3)^n].$$

Ex. 46. *Evaluate*

$$\frac{\Delta^2}{E}\sin(x+h)+\frac{\Delta^2\sin(x+h)}{E\sin(x+h)},$$

h being the interval of differencing. **(Nagpur B.Sc. 82)**

Sol. We have

$$\frac{\Delta^2}{E}\sin(x+h)+\frac{\Delta^2\sin(x+h)}{E\sin(x+h)}$$
$$=\frac{(E-1)^2}{E}\sin(x+h)+\frac{(E-1)^2\sin(x+h)}{\sin(x+2h)}$$
$$=(E-2+E^{-1})\sin(x+h)+\frac{(E^2-2E+1)\sin(x+h)}{\sin(x+2h)}$$
$$=[\sin(x+2h)-2\sin(x+h)+\sin x]$$
$$+\left[\frac{\sin(x+3h)-2\sin(x+2h)+\sin(x+h)}{\sin(x+2h)}\right]$$
$$=2\sin(x+h)[\cos h-1]+\frac{2\sin(x+2h)[\cos h-1]}{\sin(x+2h)}$$
$$=2(\cos h-1)\{\sin(x+h)+1\}.$$

Ex. 47. *Prove by the method of separation of symbols that*

$$u_0+\frac{u_1x}{1!}+\frac{u_2x^2}{2!}+\frac{u_3x^3}{3!}+\ldots=e^x\left[u_0+x\Delta u_0+\frac{x^2}{2!}\Delta^2u_0+\ldots\right].$$

(Baroda 1971; Kanpur 74; Delhi Hons. 75; Meerut 92)

Sol. $\text{L.H.S.}=u_0+\frac{u_1x}{1!}+\frac{u_2x^2}{2!}+\frac{u_3x^3}{3!}+\ldots$

$$=u_0+\frac{x}{1!}Eu_0+\frac{x^2}{2!}E^2u_0+\frac{x^3}{3!}E^3u_0+\ldots$$
$$=\left[1+\frac{xE}{1!}+\frac{x^2E^2}{2!}+\frac{x^3E^3}{3!}+\ldots\right]u_0$$
$$=e^{xE}u_0=e^{x(1+\Delta)}u_0=e^x.e^{x\Delta}u_0$$
$$=e^x\left[1+x\Delta+\frac{x^2\Delta^2}{2!}+\frac{x^3\Delta^3}{3!}+\ldots\right]u_0$$

$$= e^{\circ}\left[u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \dots\right]$$
= R.H.S.

Ex. 48. *Use the method of separation of symbols to prove the following identities :*

(i) $u_0 + {}^xC_1\Delta u_1 + {}^xC_2\Delta^2 u_2 + \dots$
$= u_x + {}^xC_1\Delta^2 u_{x-1} + {}^xC_2\Delta^4 u_{x-2} + \dots$

(ii) $u_x - u_{x+1} + u_{x+2} - u_{x+3} + \dots$
$$= \frac{1}{2}\left[u_{x-(1/2)} - \frac{1}{8}\Delta^2 u_{x-(3/2)} + \frac{1.3}{2!}\left(\frac{1}{8}\right)^2\Delta^4 u_{x-(5/2)} - \frac{1.3.5}{3!}\left(\frac{1}{8}\right)^3\Delta^6 u_{x-(7/2)} + \dots\right]$$
(Lucknow 1980, Agra 81)

(iii) $u_0 + {}^nC_1 u_1 x + {}^nC_2 u_2 x^2 + {}^nC_3 u_3 x^3 + \dots$
$= (1+x)^n u_0 + {}^nC_1 (1+x)^{n-1} x\Delta u_0 + {}^nC_2 (1+x)^{n-2} x^2\Delta^2 u_0 + \dots$

Sol. (i) R.H.S. $= u_x + {}^xC_1\Delta^2 u_{x-1} + {}^xC_2\Delta^4 u_{x-2} + \dots$
$= u_x + {}^xC_1\Delta^2 E^{-1} u_x + {}^xC_2\Delta^4 E^{-2} u_x + \dots$
$= [1 + {}^xC_1\Delta^2 E^{-1} + {}^xC_2\Delta^4 E^{-2} + \dots]\, u_x$
$= (1+\Delta^2 E^{-1})^x u_x = \left(\frac{E+\Delta^2}{E}\right)^x u_x = \left[\frac{E+(E-1)^2}{E}\right]^x u_x$
$= \left(\frac{E^2-E+1}{E}\right)^x u_x = [1+E(E-1)]^x E^{-x} u_x$
$= (1+\Delta E)^x u_0 = (1 + {}^xC_1\Delta E + {}^xC_2\Delta^2 E^2 + \dots)\, u_0$
$= u_0 + {}^xC_1\Delta E u_0 + {}^xC_2\Delta^2 E^2 u_0 + \dots$
$= u_0 + {}^xC_1\Delta u_1 + {}^xC_2\Delta^2 u_2 + \dots =$ L.H.S.

(ii) R.H.S. $= \frac{1}{2}\left[u_{x-(1/2)} - \frac{1}{8}\Delta^2 u_{x-(3/2)} + \frac{1.3}{2!}\left(\frac{1}{8}\right)^2\Delta^4 u_{x-(5/2)} - \frac{1.3.5}{3!}\left(\frac{1}{8}\right)^3\Delta^6 u_{x-(7/2)} + \dots\right]$
$$= \frac{1}{2}\left[E^{-1/2} u_x - \frac{1}{2}\cdot\frac{1}{4}\Delta^2 E^{-3/2} u_x + \frac{(\frac{1}{2})(\frac{3}{2})}{2!}\left(\tfrac{1}{4}\right)^2\Delta^4 E^{-5/2} u_x - \frac{(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})}{3!}\left(\tfrac{1}{4}\right)^3\Delta^6 E^{-7/2} u_x + \dots\right]$$
$$= \tfrac{1}{2} E^{-1/2}\left[1 + (-\tfrac{1}{2})(\tfrac{1}{4}\Delta^2 E^{-1}) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(\tfrac{1}{4}\Delta^2 E^{-1})^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(\tfrac{1}{4}\Delta^2 E^{-1})^3 + \dots\right] u_x$$
$= \frac{1}{2} E^{-1/2}[1 + \frac{1}{4}\Delta^2 E^{-1}]^{-1/2} u_x$
$= \frac{1}{2} E^{-1/2}\left[\frac{4E+\Delta^2}{4E}\right]^{-1/2} u_x = \frac{1}{2} E^{-1/2}\, 2E^{1/2}\, [4(1+\Delta)+\Delta^2]^{-1/2} u_x$

$= [(2+\Delta)^2]^{-1/2} u_x = (2+\Delta)^{-1} u_x = (1+E)^{-1} u_x$
$= [1 - E + E^2 - E^3 + E^4 - E^5 + \dots]\, u_x$
$= u_x - u_{x+1} + u_{x+2} - u_{x+3} + u_{x+4} - u_{x+5} + \dots =$ L.H.S.

(iii) R.H.S. $= (1+x)^n u_0 + {}^nC_1 (1+x)^{n-1} x\Delta u_0 + {}^nC_2 (1+x)^{n-2} x^2\Delta^2 u_0 + \dots$
$= \{(1+x)+x\Delta\}^n u_0 = \{1 + x(1+\Delta)\}^n u_0$
$= (1+xE)^n u_0 = [1 + {}^nC_1 xE + {}^nC_2 x^2E^2 + {}^nC_3 x^3E^3 + \dots]\, u_0$
$= u_0 + {}^nC_1 x u_1 + {}^nC_2 x^2 u_2 + {}^nC_3 x^3 u_3 + \dots =$ L.H.S.

Ex. 49. *Prove that*
$$\sum_{x=0}^{\infty} u_{2x} = \frac{1}{2}\sum_{x=0}^{\infty} u_x + \frac{1}{4}\left(1 - \frac{\Delta}{2} + \frac{\Delta^2}{4} - \dots\right) u_0.$$

Sol. R.H.S. $= \frac{1}{2}\sum_{x=0}^{\infty} u_x + \frac{1}{4}\left(1 - \frac{\Delta}{2} + \frac{\Delta^2}{4} - \dots\right) u_0$
$= \frac{1}{2}(u_0+u_1+u_2+u_3+\dots) + \frac{1}{4}\left(1+\frac{\Delta}{2}\right)^{-1} u_0$
$= \frac{1}{2}(u_0 + E u_0 + E^2 u_0 + E^3 u_0 + \dots) + \frac{1}{4}\left(1+\frac{\Delta}{2}\right)^{-1} u_0$
$= \frac{1}{2}(1+E+E^2+E^3+\dots)\, u_0 + \frac{1}{4}\left(1+\frac{\Delta}{2}\right)^{-1} u_0$
$= \frac{1}{2}(1-E)^{-1} u_0 + \frac{1}{2}(2+\Delta)^{-1} u_0$
$= \frac{1}{2}(1-E)^{-1} u_0 + \frac{1}{2}(1+E)^{-1} u_0$
$= \frac{1}{2}[(1-E)^{-1} + (1+E)^{-1}]\, u_0$
$= \frac{1}{2}.2\,[1+E^2+E^4+E^6+\dots]\, u_0$
$= u_0+u_2+u_4+u_6+\dots$
$= \sum_{x=0}^{\infty} u_{2x} =$ L.H.S.

Ex. 50. *If* $\Delta^3 u_x = 0$, *prove that*
$u_{x+(1/2)} = \frac{1}{2}(u_x+u_{x+1}) - \frac{1}{16}(\Delta^2 u_{x+1} + \Delta^2 u_x)$.
(Kanpur 1975, Agra 84)

Sol. We have
$u_{x+(1/2)} = E^{1/2} u_x = (1+\Delta)^{1/2} u_x$
$= (1+\frac{1}{2}\Delta - \frac{1}{8}\Delta^2)\, u_x$, expanding upto Δ^2 only because $\Delta^3 u_x = 0$
$= u_x + \frac{1}{2}\Delta u_x - \frac{1}{8}\Delta^2 u_x$ ...(1)

Again $\Delta^3 u_x = \Delta(\Delta^2 u_x) = \Delta^2 u_{x+1} - \Delta^2 u_x$.
∴ $\Delta^3 u_x = 0 \Rightarrow \Delta^2 u_{x+1} - \Delta^2 u_x = 0$
$\Rightarrow \Delta^2 u_x = \Delta^2 u_{x+1}$.

Also $\Delta u_x = u_{x+1} - u_x$.

Putting these values in (1), we get

$$u_{x+(1/2)}=u_x+\tfrac{1}{2}(u_{x+1}-u_x)-\tfrac{1}{8}(\tfrac{1}{2}\Delta^2 u_x+\tfrac{1}{2}\Delta^2 u_x)$$

$$=u_x+\tfrac{1}{2}(u_{x+1}-u_x)-\tfrac{1}{8}\left(\frac{\Delta^2 u_x}{2}+\frac{\Delta^2 u_{x+1}}{2}\right)$$

$$=\tfrac{1}{2}(u_x+u_{x+1})-\frac{1}{16}(\Delta^2 u_x+\Delta^2 u_{x+1}).$$

Exercises 1

1. (a) Define operators E and Δ and show that $E\equiv 1+\Delta$. **(Meerut B.Sc. 1969)**

   (b) Discuss if operators E and Δ obey the distributive, commutative, associative and indices laws of algebra. **(Kanpur B.Sc. 81; Meerut B.Sc. 79, Stat. 90)**

2. Find the value of $\Delta^2\left[\dfrac{a^{2x}+a^{4x}}{(a^2-1)^2}\right]$.

3. Evaluate :

   (i) $\Delta\left(\dfrac{2^x}{(x+1)\,!}\right)$ **(Agra B.Sc. 1976)**

   (ii) $\Delta \cot 2^x$, (iii) $\Delta \sinh (a+bx)$,

   (iv) $\Delta \cosh (a+bx)$, (v) $\Delta \tan ax$,

   (vi) $\Delta^2 x^3$, (vii) $\Delta^4 (ae^x)$.

4. Show that

   (i) $\Delta^n \sin (a+bx)=\left(2 \sin \dfrac{b}{2}\right)^n \sin\left[a+bx+n\,\dfrac{(b+\pi)}{2}\right]$.

   (ii) $\Delta^n \cos (a+bx)=\left(2 \sin \dfrac{b}{2}\right)^n \cos\left[a+bx+n\,\dfrac{(b+\pi)}{2}\right]$.

   (iii) $\Delta \cot (a+bx)=\dfrac{-\sin b}{\sin (a+bx) \sin (a+b+bx)}$. **(Bangalore B.Sc. 1972)**

5. If $f(x)=e^{ax}$, show that $f(0)$ and its leading differences form a geometrical progression.

6. Prove that, if $f(x)$ and $g(x)$ are any functions of x,

   (i) $\Delta[f(x)\pm g(x)]=\Delta f(x)\pm \Delta g(x)$,

   (ii) $\Delta[a f(x)]=a\Delta f(x)$, a being a constant,

   (iii) $\Delta[f(x)\, g(x)]=f(x+1)\, \Delta g(x)+g(x)\Delta f(x)$,

   (iv) $\Delta\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{g(x)\Delta f(x)-f(x)\Delta g(x)}{g(x)\, g(x+1)}$. **(Meerut M.Sc. 76)**

7. A third degree polynomial passes through the points (0, −1), (1, 1), (2, 1) and (3, 2). Find the polynomial.

8. Prove that if m is a positive integer, then

   (i) $\dfrac{(x+1)^{(m)}}{m\,!}=\dfrac{x^{(m)}}{m\,!}+\dfrac{x^{(m-1)}}{(m-1)\,!}$.

   (ii) $x^{(m+1)}+mx^{(m)}=x.x^{(m)}$.

9. Find the tenth term of the series

   3, 14, 39, 84, 155, 258, etc.

10. What is the difference between $\left(\dfrac{\Delta u_x}{Eu_x}\right)^2$ and $\left(\dfrac{\Delta^2 u_x}{E^2\, u_x}\right)$?

    If $u_x=x^3$ and interval of differencing is unity, find out the expressions for both. **(Bangalore B Sc. 74; Gujrat B.Sc. 71)**

11. Evaluate the missing term in the following :

x :	100	101	102	103	104
$\log x$:	2·000	2·0043	—	2·0128	2·0170

12. Evaluate the missing term in the following :

x :	1	2	3	4	5	6	7
y ;	2	4	8	—	32	64	128.

13. Find the missing term in the following table :

x :	16	18	20	22	24	26
y :	39	85	—	151	264	388

24. Find the successive differences of

    $$x^4-12x^3+42x^2-30x+9,$$

    when the interval of differencing is unity.

15. Prove that $\Delta^3 u_0=u_3-3u_2+3u_1-u_0$. **(Meerut B.Sc. 1992P)**

16. Show that if $u_x=2^x$ then $\Delta u_x=u_x$.

17. Find a function for which $\Delta u_x=2u_x$.

18. If $u_x=x\,(x-1)\,(x-2)$, prove that

    $$\Delta u_x=3x\,(x-1).$$

19. Find a function u_x for which

    (i) $\Delta u_x=x^{(2)}=x\,(x-1)$, (ii) $\Delta u_x={}^xC_n$.

20. Express x^2-3x+1 in factorials. Hence or otherwise find its third difference.

21. Prove that

    $$\Delta[f(x-1)\,\Delta g(x-1)]=\nabla\,[f(x)\,\Delta g(x)]=\Delta[f(x-1)\,\nabla g(x)].$$

22. Prove that $e^{-hD}=1-\nabla$. **(Rohilkhand 1986)**

23. Show that

    $$Dy=\frac{1}{h}\left[\Delta y-\frac{\Delta^2 y}{2}+\frac{\Delta^3 y}{3}-\frac{\Delta^4 y}{4}+\ldots\right]$$

    and $$D^2 y=\frac{1}{h^2}\left[\nabla^2 y+\nabla^3 y+\frac{11}{12}\nabla^4 y+\ldots\right]$$

    where the symbols have their usual meanings.

24. Denoting $\dbinom{x}{n}=\dfrac{x\,(x-1)\ldots(x-n+1)}{n\,!}$, prove that for any polynomial $\phi(x)$ of degree k,

2

Interpolation with Equal Intervals

§ 1. **Introduction.** Suppose we are given the values of a function $f(x)$, *i.e.*, the entries, for a set of values of the independent variable x, *i.e.*, the arguments. Interpolation is defined as the technique of estimating the value of $f(x)$ for any intermediate value of the argument.

Theile defines interpolation as "*the art of reading between the lines of a table*"

Let us suppose we are given the census figures for the population of India for four years 1931, 1941, 1951 and 1961 and we want to estimate the figure for any intermediate year, *e.g.*, for 1955 or 1958, etc. This can be done by applying the technique of interpolation. Interpolation may be defined as the "*technique of obtaining the most likely estimate of a certain quantity under certain assumptions*".

If the form of the function $f(x)$ is known, we can find $f(x)$ for any value of x by simple substitution. But in case, the form of the function is unknown or if it is quite complicated, then the problem of determining the nature of the function or replacing the function by a comparatively simpler one is also the problem of parabolic or polynomial interpolation.

If we have to estimate the value of $f(x)$ for any value of the argument outside the given range of the arguments, the technique is known as *extrapolation*. For example, the estimation of the population figure for the year 1965 in the above example falls in the domain of extrapolation.

Assumptions. The following are the fundamental assumptions in any method of interpolation :

(i) There are no sudden jumps or falls in values of the entries for the period under consideration. In other words, the given data does not refer to abnormal periods such as periods of famines, wars, epidemics, etc., which may result in sudden changes in the values of $f(x)$. Mathematically it means that the data can be represented by a smooth or continuous curve which implies that the given data can be represented by a polynomial of certain degree, which is determined by the following theorem.

One and only one polynomial curve of degree less than or equal to n passes through a given set of $(n+1)$ distinct points.

All the formulae of interpolation are based upon the fundamental assumption that the data is expressible or can be expressed as a polynomial function, with fair degree of accuracy.

(ii) In the absence of any evidence to the contrary, the rise or fall in the data has been uniform. Thus in the above example of population we assume that the rate of growth of population over the period 1931 to 1961 has been uniform.

§ 2. **The various methods of interpolation are as follows :**

(a) Graphic method.
(b) Method of curve fitting.
(c) Use of the calculus of finite differences formulae.

(a) Graphic Method. We can easily plot a graph between the values of x and the corresponding values of y for a given function $y=f(x)$. From the graph so obtained, we can find out the value of y for given value of x.

For example consider the following data :

Year (x) :	1891	1901	1911	1921	1931
Population (y) : (in thousands)	46	66	81	93	101

Suppose we have to interpolate population for the year 1925.

Steps in Graphic Method.

(i) Take a suitable scale for the values of x and y and plot the various points on the graph paper, for given values of x and y.

(ii) Draw a free hand curve passing through the plotted points.

(iii) Find the point on the curve corresponding to $x=1925$ and find the corresponding value of y.

Drawbacks in the method. It is a very approximate method of estimating the value of y. In most of the cases it is not reliable.

(b) Method of curve fitting. This method can be used only in those cases in which the form of the function is known. Then by the method of least squares we can fit the curve of known form to the given set of observations and with the help of the fitted curve we can calculate the unknown value.

Drawbacks in the method.

(i) The method is not exact.

(ii) The method becomes complicated when the number of observations is sufficiently large.

(iii) The form of the function for the given set of observations is assumed to be known.

(iv) When some additional observations are included in the data then the calculation for finding the unknown constants are to be done afresh.

The only merit of this method lies in the fact that it gives closer approximation than the graphical method.

Let us consider the above example, in which we have to find the population for the year 1925. Let us assume that the function $y=f(x)$ is a first degree polynomial of the form

$$y=a+bx.$$

Now our problem is to find the values of a and b from the given data to get the fitted curve. The calculations can be done in tabular form.

Table 2·1

x	$x-1911$	$z=\frac{x-1911}{10}$	y	yz	z^2
1891	−20	−2	46	−92	4
1901	−10	−1	66	−66	1
1911	0	0	81	0	0
1921	10	1	93	93	1
1931	20	2	101	202	4
Total		0	387	137	10

Then by the method of least squares, we have to minimise

$$S=\Sigma\,(y-a-bz)^2.$$

This gives $\frac{\partial S}{\partial a}=0 \Rightarrow \Sigma\,(y-a-bz)=0$ ...(1)

and $\frac{\partial S}{\partial b}=0 \Rightarrow \Sigma\,(y-a-bz)\,z=0$ ...(2)

i.e. $\Sigma y=na+b\Sigma z$ (3)

and $\Sigma yz=a\Sigma z+b\Sigma z^2$. ...(4)

Putting the values of Σy, Σz, Σyz, Σz^2 from the table 2·1 and putting $n=5$ in equations (3) and (4), we get

$387=5a+0$ or $a=77{\cdot}4$

and $137=0+10b$ or $b=13{\cdot}7$.

$\therefore$ the required fitted polynomial is

$$y=77{\cdot}4+13{\cdot}7\,z. \qquad (5)$$

Now we have to find the population for

$$x=1925 \text{ i.e. for } z=\frac{1925-1911}{10}=1{\cdot}4.$$

Putting $z=1{\cdot}4$ in (5), we get

$$y=77{\cdot}4+(13{\cdot}7)\,(1{\cdot}4)=96{\cdot}58.$$

Hence the population for the year 1925 is estimated to be 96·58 thousand.

(c) Use of the calculus of finite differences formulae. The study of the use of finite difference calculus for the purpose of interpolation can be divided into three cases which are as follows.

(i) The technique of interpolation with equal intervals.

(ii) The technique of interpolation with unequal intervals.

(iii) The technique of central differences.

The use of these methods, though they are approximate, have distinct advantages over the methods of graphs and curve fitting.

Merits. (i) These methods do not assume the form of the function to be known.

(ii) These methods are less approximate than the method of graphs.

(iii) The calculations remain simple even if some additional observations are included in the given data.

Drawbacks in the methods. There is no definite rule to verify whether the assumptions for the application of finite difference calculus are valid for the given set of observations.

§ 3. Interpolation with equal intervals.

(i) Newton-Gregory Formula for Forward Interpolation. (Meerut 1979, 91)

Let $y=f(x)$ be a function which takes the values $f(a), f(a+h), f(a+2h), \ldots, f(a+nh)$ for the $(n+1)$ equidistant values $a, a+h, a+2h, \ldots, a+nh$ of the independent variable x (argument) and let $P_n(x)$ be a polynomial in x of degree n.

Let

$$P_n(x)=A_0+A_1(x-a)+A_2(x-a)(x-a-h)+A_3(x-a)(x-a-h)(x-a-2h)+\ldots+A_n(x-a)(x-a-h)\ldots(x-a-\overline{n-1}\,h) \quad \ldots(1)$$

We choose the coefficients $A_0, A_1, \ldots, A_n$ such that

$$P_n(a)=f(a),\; P_n(a+h)=f(a+h),\; \ldots,\; P_n(a+nh)=f(a+nh).$$

Putting $x=a, a+h, \ldots, a+nh$ in (1) and then also putting the values of $P_n(a), P_n(a+h), \ldots, P_n(a+nh)$, we get

$$f(a)=A_0 \Rightarrow A_0=f(a),$$

$$f(a+h)=A_0+A_1h \Rightarrow A_1=\frac{f(a+h)-f(a)}{h}=\frac{\Delta f(a)}{h}$$

$$f(a+2h)=A_0+2h\,A_1+2h.\,hA_2$$

$$\Rightarrow \quad A_2=\frac{f(a+2h)-2\,[f(a+h)-f(a)]-f(a)}{2h^2}$$

$$=\frac{f(a+2h)-2f(a+h)+f(a)}{2h^2}$$

$$=\frac{1}{2h^2}\,\Delta^2 f(a).$$

Similarly $A_3=\dfrac{1}{3\,!\,h^3}\,\Delta^3 f(a)$,

... ...

$$A_n=\frac{1}{n\,!\,h^n}\,\Delta^n f(a).$$

Putting the values of $A_0, A_1, \ldots, A_n$ found above in (1), we get

$$P_n(x)=f(a)+\frac{\Delta f(a)}{h}(x-a)+\frac{\Delta^2 f(a)}{2\,!\,h^2}(x-a)(x-a-h)$$
$$+\frac{\Delta^3 f(a)}{3\,!\,h^3}(x-a)(x-a-h)(x-a-2h)+\ldots$$
$$+\frac{\Delta^n f(a)}{n\,!\,h^n}(x-a)(x-a-h)(x-a-2h)\ldots(x-a-\overline{n-1}h).$$

This is known as Newton-Gregory formula for forward interpolation.

Putting $\dfrac{x-a}{h}=u$ or $x=a+hu$, the formula takes the form

$$P_n(x)=P_n(a+hu)=f(a)+u\,\Delta f(a)+\frac{u(u-1)}{2\,!}\Delta^2 f(a)$$
$$+\frac{u(u-1)(u-2)}{3\,!}\Delta^3 f(a)+\ldots$$
$$+\frac{u(u-1)(u-2)\ldots(u-n+1)}{n\,!}\Delta^n f(a) \qquad \ldots(2)$$

The result (2) can be written as

$$P_n(x)=P_n(a+hu)=f(a)+u^{(1)}\Delta f(a)+\frac{1}{2\,!}u^{(2)}\Delta^2 f(a)+\ldots$$
$$+\frac{1}{n\,!}u^{(n)}\Delta^n f(a)$$

where $u^{(n)}=u(u-1)(u-2)\ldots(u-n+1)$.

This is the form in which Newton-Gregory formula for forward interpolation is often written.

Note. The reason for the name "forward" interpolation formula lies in the fact that this formula contains values of the tabulated function from $f(a)$ onward to the right and none to the left of this value. This formula is used mainly for interpolating the values of y near the beginning of a set of tabulated values and for extrapolating values of y a short distance backward (*i.e.*, to the left) from $f(a)$.

(ii) Newton-Gregory Formula for Backward Interpolation. **(Meerut 1976)**

To derive this formula, we write

$$P_n(x)=A_0+A_1(x-a-nh)+A_2(x-a-nh)(x-a-nh+h)$$
$$+A_3(x-a-nh)(x-a-nh+h)(x-a-nh+2h)+\ldots$$
$$+A_n(x-a-nh)(x-a-nh+h)\ldots(x-a-h), \quad \ldots(1)$$

where $A_0, A_1, \ldots, A_n$ are constants which are to be determined so as to make

$$P_n(a+nh)=f(a+nh), \ldots, P_n(a)=f(a).$$

Putting in (1), $x=a+nh,\ a+nh-h, \ldots$ and also putting $P_n(a+nh)=f(a+nh), \ldots$, we get

$$f(a+nh)=A_0 \quad \text{or} \quad A_0=f(a+nh),$$

$$f(a+nh-h)=A_0+A_1(-h) \quad \text{or} \quad A_1=\frac{f(a+nh)-f(a+nh-h)}{h}$$

$$=\frac{1}{h}\,\nabla f(a+nh),$$

$$f(a+nh-2h)=A_0+A_1(-2h)+A_2(-2h)(-h)$$

or $$A_2=\frac{-A_0+2A_1h+f(a+\overline{n-2}h)}{2h^2}$$

$$=\frac{-f(a+nh)+2\,[f(a+nh)-f(a+nh-h)]+f(a+\overline{n-2}h)}{2h^2}$$

$$=\frac{f(a+nh)-2f(a+\overline{n-1}h)+f(a+\overline{n-2}h)}{2h^2}$$

$$=\frac{1}{2\,!\,h^2}\,\nabla^2 f(a+nh).$$

Similarly, $A_3=\dfrac{1}{3\,!\,h^3}\,\nabla^3 f(a+nh), \ldots,$

$$A_n=\frac{1}{n\,!\,h^n}\,\nabla^n f(a+nh).$$

Substituting the values of $A_0, A_1, \ldots, A_n$ in (1), we get

$$P_n(x)=f(a+nh)+\frac{\nabla f(a+nh)}{h}(x-\overline{a+nh})$$

$$+\frac{\nabla^2 f(a+nh)}{2!\,h^2}(x-\overline{a+nh})(x-\overline{a+nh-h})+\ldots$$
$$+\frac{\nabla^n f(a+nh)}{n!\,h^n}(x-\overline{a+nh})(x-\overline{a+nh-h})\ldots(x-a-h).$$

This is Newton-Gregory formula for backward interpolation.

Putting $u=\dfrac{x-(a+nh)}{h}$ or $x=a+nh+uh$, **we get**

$$P_n(x)=P_n(a+nh+hu)=f(a+nh)+u\,\nabla f(a+nh)$$
$$+\frac{u(u+1)}{2!}\nabla^2 f(a+nh)+\frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh)$$
$$+\ldots+\frac{u(u+1)\ldots(u+n-1)}{n!}\nabla^n f(a+nh). \qquad \ldots(2)$$

This is the form in which Newton-Gregory formula for backward interpolation is often written.

Note. It is called the formula for "backward" interpolation because it contains values of the tabulated function from $f(a+nh)$ backward to the left and none to the right of $f(a+nh)$. This formula is used mainly for interpolating values of y near the end of a set of tabulated values, and also for extrapolating values of y a short distance ahead (to the right) of $f(a+nh)$.

Solved Examples

Ex. 1. *From the following table of yearly premiums for policies maturing at quinquennial ages, estimate the premiums for policies maturing at the age of 46 years.*

Age x :	45	50	55	60	65
Premium f (x) :	2·871	2·404	2·083	1·862	1·712

Sol. The difference table of the given data is as follows :

Age x	Premium $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
45	2·8710				
		−0·467			
50	2·4040		0·146		
		−0·321		−0·046	
55	2·0830		0·100		0·017
		−0·221		−0·029	
60	1·8620		0·071		
		−0·150			
65	1·7120				

We want $f(46)=f(a+uh)$, (say), where $a=45$ and $h=5$.

$\therefore\ a+uh=46 \Rightarrow 45+u\times 5=46$

$\Rightarrow u=\frac{1}{5}=0.2.$

Substituting $u=\frac{1}{5}$ in Newton's formula for forward interpolation, we get

$$f(46)=f(45)+\frac{1}{5}\Delta f(45)+\frac{1}{5}\left(-\frac{4}{5}\right)\frac{1}{2!}\Delta^2 f(45)$$
$$+\frac{1}{5}\left(-\frac{4}{5}\right)\left(-\frac{9}{5}\right)\frac{1}{3!}\Delta^3 f(45)+\frac{1}{5}\left(-\frac{4}{5}\right)\left(-\frac{9}{5}\right)\left(-\frac{14}{5}\right)\frac{1}{4!}\Delta^4 f(45)$$
$$=2.871+\frac{1}{5}(-0.467)-\frac{2}{25}(0.146)+\frac{6}{125}(-0.046)-\frac{21}{625}(0.017)$$
$$=2.871-0.0934-.01168-.002208-.0005712$$
$$=2.763 \text{ (approx).}$$

Ex. 2. *From the following table, estimate the number of students who obtained marks between 40 and 45.*

Marks	*No. of students*
30−40	31
40−50	42
50−60	51
60−70	35
70−80	31

(Meerut M.Sc. 1989; I.A.S. 77)

Sol. First we prepare the cumulative frequency table, as given below :

Marks above 30 but less than x	No. of students $f(x)$
40	31
50	73
60	124
70	159
80	190

Now we prepare the difference table for this data

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
40	31				
		42			
50	73		9		
		51		25	
60	124		−16		37
		35		12	
70	159		−4		
		31			
80	190				

We shall find :

$f(45)$ = number of students with marks less than 45. Taking $a+uh=45$, we get

$$40+u\times 10=45 \Rightarrow u=\frac{1}{2}.$$

Using Newton's formula for forward interpolation, we get

$$f(45)=f(40)+\frac{1}{2}\,\Delta f(40)+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{\Delta^2 f(40)}{2!}$$
$$+\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{\Delta^3 f(40)}{3!}+\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{\Delta^4 f(40)}{4!}$$
$$=31+\frac{1}{2}\times 42+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2!}\times 9+\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{3!}(-25)$$
$$+\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{1}{4!}\times 37$$

$=47{\cdot}868$ (on simplification).

Hence the number of students with marks less than 45 is $47{\cdot}868$ *i.e.*, 48.

But the number of students with marks less than 40 is 31.

Hence the required number of students getting marks between 40 and 45 is $=48-31=17$.

Ex. 3. *The following table gives the population of a town during the last six censuses. Estimate using any suitable interpolation formula, the increase in the population during the period from 1946 to 1948 :*

Year :	1911	1921	1931	1941	1951	1961
Population : *(in thousands)*	12	15	20	27	39	52

Sol. The difference table for the given data is as follows ;

Year x	Population y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1911	12					
		3				
1921	15		2			
		5		0		
1931	20		2		3	
		7		3		−10
1941	27		5		−7	
		12		−4		
1951	39		1			
		13				
1961	52					

We have to find $f(1946)$ and $f(1948)$.

To find $f(1946)$, $u=\dfrac{1946-1911}{10}=3{\cdot}5.$

Applying Newton's formula for forward interpolation, we get

$$f(1946)=12+3{\cdot}5\times 3+\frac{(3{\cdot}5)(2{\cdot}5)}{2!}\times 2+0+\frac{(3{\cdot}5)(2{\cdot}5)(1{\cdot}5)(\cdot 5)}{4!}\times 3$$
$$+\frac{(3{\cdot}5)(2{\cdot}5)(1{\cdot}5)(\cdot 5)(-\cdot 5)}{5!}(-10)$$
$$=12+10{\cdot}5+8{\cdot}75+0{\cdot}8203+{\cdot}2734$$
$$=32{\cdot}3437.$$

To find $f(1948)$, $u=\dfrac{1948-1911}{10}=3{\cdot}7.$

$$\therefore\ f(1948)=12+3{\cdot}7\times 3+\frac{(3{\cdot}7)(2{\cdot}7)}{2!}\times 2+0$$
$$+\frac{(3{\cdot}7)(2{\cdot}7)(1{\cdot}7)(\cdot 7)}{4!}\times 3+\frac{(3{\cdot}7)(2{\cdot}7)(1{\cdot}7)(\cdot 7)(-\cdot 3)}{5!}\times(-10)$$
$$=12+11{\cdot}1+9{\cdot}99+1{\cdot}4860+0{\cdot}2972$$
$$=34{\cdot}8732.$$

Therefore increase in the population during the period from 1946 to 1948 $=f(1948)-f(1946)$

$=34{\cdot}8732-32{\cdot}3437$

$=2{\cdot}5295$ thousand

$=2{\cdot}53$ thousand approximately.

Ex. 4. *Given*

$\sin 45^\circ=0{\cdot}7071,\quad \sin 50^\circ=0{\cdot}7660$

$\sin 55^\circ=0{\cdot}8192,\quad \sin 60^\circ=0{\cdot}8660.$

Find $\sin 52^\circ$ *by using any method of interpolation.*

Sol. Here, we have

x :	45°	50°	55°	60°
$f(x)$:	0·7071	0·7660	0·8192	0·8660

The difference table for the given data is as follows :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
45°	0·7071			
		0·0589		
50°	0·7660		−0·0057	
		0·0532		−0·0007
55°	0·8192		−0·0064	
		0·0468		
60°	0·8660			

We want $f(52)=f(a+uh)$, say.

$\therefore\ 52^\circ=a+uh \Rightarrow 52^\circ=45^\circ+u\times 5^\circ \Rightarrow u=\frac{7}{5}=1\cdot 4.$

By Newton's forward interpolation formula, we get

$$f(a+uh)=f(a)+u\,\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(a).$$

$$\therefore\ f(52^\circ)=f(45^\circ)+1\cdot 4\,\Delta f(45^\circ)+\frac{(1\cdot 4)(0\cdot 4)}{2!}\Delta^2 f(45^\circ)+\frac{(1\cdot 4)(0\cdot 4)(-0\cdot 6)}{3!}\Delta^3 f(45^\circ)$$

$$=0\cdot 7071+1\cdot 4\times 0\cdot 0589+\frac{(1\cdot 4)(0\cdot 4)}{2!}(-0\cdot 0057)+\frac{(1\cdot 4)(0\cdot 4)(-0\cdot 6)}{3!}\times(-0\cdot 0007)$$

$$=0\cdot 7071+0\cdot 08246-0\cdot 001596+0\cdot 0000392$$
$$=0\cdot 7880032.$$

Thus $\sin 52^\circ=0\cdot 7880032=0\cdot 7880$ approx.

Ex. 5. *Using Newton's formula for interpolation, estimate the population for the year* 1905 :

Year	*Population*
1891	98, 752
1901	132, 285
1911	168, 076
1921	195, 690
1931	246, 050

Sol. The difference table is as given below :

Year x	Population $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1891	98752				
		33533			
1901	132285		2258		
		35791		-10435	
1911	168076		-8177		41358
		27614		30923	
1921	195690		22746		
		50360			
1931	246050				

Here $a=1891,\ u=\frac{x-a}{h}=\frac{1905-1891}{10}=1\cdot 4.$

By Newton's interpolation formula, we have

$$f(1905)=98725+(1\cdot 4)\,(33533)+\frac{(1\cdot 4)\,(0\cdot 4)}{2!}\,(2258)+\frac{(1\cdot 4)(0\cdot 4)(-0\cdot 6)}{3!}\,(-10435)+\frac{(1\cdot 4)\,(0\cdot 4)\,(-0\cdot 6)\,(-1\cdot 6)}{4!}\,(41358)$$

$$=98752+46946\cdot 2+632\cdot 24+584\cdot 36+926\cdot 42$$
$=147841$ approximately.

Ex. 6. *The area A of a circle of diameter d is given for the following values :*

d :	80	85	90	95	100
A :	5026	5674	6362	7088	7854.

Find approximate values for the areas of circles of diameters 82 *and* 91 *respectively.* **(Meerut M.Sc. 1976, 86)**

Sol. First we shall find the value of A corresponding to $d=82$. Let $A=f(d)$.

The difference table is given below :

d	$f(d)$	$\Delta f(d)$	$\Delta^2 f(d)$	$\Delta^3 f(d)$	$\Delta^4 f(d)$
80	5026				
		648			
85	5674		40		
		688		-2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7854				

Now $f(82)=$area of circle of diameter $82=f(a+uh)$.

$\therefore\ 82=a+uh \Rightarrow 82=80+u\times 5 \Rightarrow u=\frac{2}{5}.$

By Newton's forward interpolation formula, we get

$$f(82)=f(80)+\frac{2}{5}\,\Delta f(80)+\frac{2}{5}\left(\frac{2}{5}-1\right)\frac{\Delta^2 f(80)}{2!}+\frac{2}{5}\left(\frac{2}{5}-1\right)\left(\frac{2}{5}-2\right)\frac{\Delta^3 f(80)}{3!}+\frac{2}{5}\left(\frac{2}{5}-1\right)\left(\frac{2}{5}-2\right)\left(\frac{2}{5}-3\right)\frac{\Delta^4 f(80)}{4!}$$

$$=5026+\frac{2}{5}\times 648+\frac{2}{5}\left(-\frac{3}{5}\right)\times\frac{40}{2}+\frac{2}{5}\left(-\frac{3}{5}\right)\left(-\frac{8}{5}\right)\frac{(-2)}{6}$$

$$+\frac{2}{5}\left(-\frac{3}{5}\right)\left(-\frac{8}{5}\right)\left(-\frac{13}{5}\right)\cdot\frac{4}{24}$$

$=5026+259{\cdot}2-4{\cdot}8-0{\cdot}128-0{\cdot}1664=5280{\cdot}1056$

$=5280$ approx.

Similarly we can find $f(91)$ *i.e.* the area of circle of diameter 91.

We find that $f(91)=6504$.

Ex. 7. *The population of a country in the decennial census were as under. Estimate the population for the year 1925.*

Year x	:	1891	1901	1911	1921	1931
Population y *(in thousands)*	:	46	66	81	93	101

Sol. Let us introduce a new variable u given by

$$u=\frac{x-1891}{10}.$$

$\therefore$ u takes the values 0, 1, 2, 3, 4.

The difference table is as follows :

u	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	46				
		20			
1	66		−5		
		15		2	
2	81		−3		−3
		12		−1	
3	93		−4		
		8			
4	101				

Since five values are given, we must assume that fourth differences are constant. We want the entry for $x=1925$, *i.e.*, for

$$u=\frac{1925-1891}{10}=3{\cdot}4.$$

We have

$$y_{3\cdot4}=E^{3\cdot4}\,y_0=(1+\Delta)^{3\cdot4}\,y_0$$

$$=y_0+3{\cdot}4\,\Delta\,y_0+\frac{3{\cdot}4\times2{\cdot}4}{1\times2}\,\Delta^2 y_0+\frac{3{\cdot}4\times2{\cdot}4\times1{\cdot}4}{1\times2\times3}\,\Delta^3 y_0$$

$$+\frac{3{\cdot}4\times2{\cdot}4\times1{\cdot}4\times{\cdot}4}{1\times2\times3\times4}\Delta^4 y_0$$

$$=46+3{\cdot}4\times20+\frac{3{\cdot}4\times2{\cdot}4}{2}(-5)+\frac{3{\cdot}4\times2{\cdot}4\times1{\cdot}4}{6}\times2$$

$$+\frac{3{\cdot}4\times2{\cdot}4\times1{\cdot}4\times{\cdot}4}{24}(-3)$$

$=46+68-20{\cdot}4+3{\cdot}808-0{\cdot}5712$

$=96{\cdot}8368$ thousand.

Hence the population for 1925 is estimated to be 96·84 thousand.

Ex. 8. *If l_x represents the number of persons living at age x in a life table, find as accurately as data will permit, l_x for values of $x=35$, 42 and 47. Given*

$$l_{20}=512,\ l_{30}=439,\ l_{40}=346 \text{ and } l_{50}=243.$$

(Agra 1986)

Sol. The difference table for the given data is as follows :

x	l_x	$\Delta\, l_x$	$\Delta^2\, l_x$	$\Delta^3\, l_x$
20	512			
		−73		
30	439		−20	
		−93		10
40	346		−10	
		−103		
50	243			

To find l_{35}.

We have $l_{35}=l_{20+10\times(3/2)}=E^{3/2}\,l_{20}$

$$=(1+\Delta)^{3/2}\,l_{20} \qquad [\because\ E\equiv1+\Delta]$$

$$=\left[1+\frac{3}{2}\Delta+\frac{\frac{3}{2}\cdot\frac{1}{2}}{2\,!}\Delta^2+\frac{\frac{3}{2}\cdot\frac{1}{2}\left(-\frac{1}{2}\right)}{3\,!}\Delta^3\right]l_{20}$$

{neglecting higher order differences}

$$=l_{20}+\frac{3}{2}\Delta l_{20}+\frac{\frac{3}{2}\cdot\frac{1}{2}}{2\,!}\Delta^2 l_{20}+\frac{\frac{3}{2}\cdot\frac{1}{2}\left(-\frac{1}{2}\right)}{3\,!}\Delta^3 l_{20}$$

$$=512+\frac{3}{2}(-73)+\frac{3}{8}(-20)-\frac{1}{16}\times10$$

$=512-109{\cdot}5-7{\cdot}5-0{\cdot}625$

$=394{\cdot}375$

$=394$ nearly.

Similarly, we can find $l_{42}=326$ and $l_{47}=274$.

Ex. 9. *Find the number of men getting wages between Rs. 10 and 15 from the following table :*

Wages in Rs.	*Frequency*
0—10	9
10—20	30
20—30	35
30—40	42

(Meerut M.Sc. 1988, 89)

Sol. First we prepare the cumulative frequency table, as given below :

Wages less than rupees x	No. of men $f(x)$
10	9
20	39
30	74
40	116

The difference table is as given below :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
10	9			
		30		
20	39		5	
		35		2
30	74		7	
		42		
40	116			

Now $f(15)$ = no. of men getting wages less than Rs. 15
$=f(a+uh)$, (say).

$\therefore \quad 15=a+uh \;\Rightarrow\; 15=10+u\times 10 \;\Rightarrow\; u=\frac{1}{2}$.

$\therefore$ By Newton's forward interpolation formula,

$$f(15)=f(10)+\frac{1}{2}\Delta f(10)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\Delta^2 f(10)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\Delta^3 f(10)$$

$$=9+\frac{1}{2}\times 30+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2!}\times 5+\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{3!}\times 2$$

$=9+15-0\cdot625+0\cdot125=23\cdot5$

$=24$ nearly.

Therefore no. of persons getting wages between Rs. 10 and Rs. $15=24-9=15$.

Ex. 10. *Use Newton formula for interpolation to find the net premium at age 25 from the table given below :*

Age	*Annual net premium*
20	0·01427
24	0·01581
28	0·01772
32	0·01996

Sol. The difference table for the given data is as follows :

Age x	Premium $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
20	0·01427			
		0·00154		
24	0·01581		0·00037	
		0·00191		−0·00004
28	0·01772		0·00033	
		0·00224		
32	0·01996			

We have to find $f(25)=f(a+uh)$, (say).

$\therefore \quad 25=a+uh \;\Rightarrow\; 25=20+u\times 4 \;\Rightarrow u=\frac{5}{4}$.

$\therefore \qquad u=1\cdot25.$

By Newton's formula, we get

$$f(a+uh)=f(a)+u\,\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

[Leaving higher order differences]

$$\therefore f(25)=f(20)+(1\cdot25)\times\Delta f(20)+\frac{(1\cdot25)(1\cdot25-1)}{2!}\Delta^2 f(20)+\frac{(1\cdot25)(1\cdot25-1)(1\cdot25-2)}{3!}\Delta^3 f(20)$$

$$=0{\cdot}01427+(1{\cdot}25)\,(0{\cdot}00154)+\frac{(1{\cdot}25)\,({\cdot}25)}{2}(0{\cdot}00037)$$

$$+\frac{(1{\cdot}25)\,({\cdot}25)\,(-0{\cdot}75)}{6}\,(-0{\cdot}00004)$$

$$=0{\cdot}01427+0{\cdot}001925+0{\cdot}0000578+0{\cdot}0000016$$

$$=0{\cdot}0162544=0{\cdot}01625 \text{ approx.}$$

Ex. 11. *The following are the numbers of deaths in four successive ten year age groups. Find the number of deaths at* 45 – 50 *and* 50 – 55.

Age group :	25 – 35	35 – 45	45 – 55	55 – 65
Deaths :	13229	18139	24225	31496

(**Agra B.Sc. 1977**)

Sol. First we shall form the cumulative frequency table.

Age above 25 and below x	No. of deaths $f(x)$
35	13229
45	31368
55	55593
65	87089

The difference table is as under :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
35	13229			
		18139		
45	31368		6086	
		24225		1185
55	55593		7271	
		31496		
65	87089			

First we shall find $f(50)$ *i.e.* the no. of deaths above the age 25 and below 50.

Here $\quad u=\dfrac{x-a}{h}=\dfrac{50-35}{10}=1{\cdot}5.$

By Newton's formula, we get

$$f(50)=f(35)+(1{\cdot}5)\,\Delta f(35)+\frac{(1{\cdot}5)\,(1{\cdot}5-1)}{2\,!}\,\Delta^2 f(35)$$

$$+\frac{(1{\cdot}5)\,(1{\cdot}5-1)\,(1{\cdot}5-2)}{3\,!}\,\Delta^3 f(35)$$

$$=13229+(1{\cdot}5)\,(18139)+\frac{(1{\cdot}5)\,({\cdot}5)}{2}\times 6086$$

$$+\frac{(1{\cdot}5)\,({\cdot}5)\,(-{\cdot}5)}{6}\times(1185)$$

$$=13229+27208{\cdot}5+2282{\cdot}25-74{\cdot}0625$$

$$=42646.$$

Hence the required number of deaths between 45 and 50

$$=42646-31368=11278.$$

Therefore the number of deaths between 50 and 55

$$=24225-11278=12947.$$

Ex. 12. *Given*

x :	1	2	3	4	5	6	7	8
$f(x)$:	1	8	27	64	125	216	343	512

Find $f(7{\cdot}5)$.

Sol. The value to be interpolated lies at the end of the given observations *i.e.* near $x=8$. So in this case Newton's backward formula will be more suitable.

Here $\quad u=\dfrac{x-(a+nh)}{h}=\dfrac{7{\cdot}5-8}{1}=-0{\cdot}5.$

To calculate backward differences $\nabla f(a+nh)$, $\nabla^2 f(a+nh)$, ..., we prepare the following difference table.

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		6
4	64		24	
		61		6
5	125		30	
		91		6
6	216		36	
		127		6
7	343		42	
		169		
8	512			

Since $\nabla^3 f(x)$ is constant, so we can leave higher order differences.

By Newton's backward interpolation formula,

$$f(a+nh+uh)=f(a+nh)+u\,\nabla f(a+nh)+\frac{u(u+1)}{2!}\nabla^2 f(a+nh)+\frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh).$$

$$\therefore\quad f(7\cdot5)=f(8)+(-0\cdot5)\,\nabla f(8)+\frac{(-0\cdot5)\,(-0\cdot5+1)}{2}\nabla^2 f(8)+\frac{(-0\cdot5)\,(-0\cdot5+1)\,(-0\cdot5+2)}{6}\nabla^3 f(8)$$

$$=512+(-0\cdot5)\times169+\frac{(-0\cdot5)\,(0\cdot5)}{2}\times42+\frac{(-0\cdot5)\,(0\cdot5)\,(1\cdot5)}{6}\times6$$

$$=512-84\cdot5-5\cdot25-\cdot375$$

$$=421\cdot875.$$

Ex. 13. *A second degree polynomial passes through* (0, 1), (1, 3), (2, 7) *and* (3, 13). *Find the polynomial.*

(Agra B.Sc. 1976; Meerut 79, 89, 92)

Sol. The difference table is as follows :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		
3	13			

We know that $f(a+uh)=f(a)+u\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)$,

expanding upto second order differences only.

Put $a=0$, $h=1$, $u=x$, so that

$$f(x)=f(0)+x\Delta f(0)+\frac{x(x-1)}{2!}\Delta^2 f(0)$$

$$=1+x\times2+\frac{x(x-1)}{2!}\times2$$

$$=1+2x+x^2-x\quad=x^2+x+1.$$

$x=x_0+ph$
$=0+p.1$

Ex. 14. *The following table is given* **(Mysore 78)**

x :	0	1	2	3	4
$f(x)$:	3	6	11	18	27

What is the form of the function $f(x)$?

Sol. The difference table for the given data is as follows :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	3			
		3		
1	6		2	
		5		0
2	11		2	
		7		0
3	18		2	
		9		
4	27			

As in the previous example, we have

$$f(x)=f(0)+x\Delta f(0)+\frac{x(x-1)}{2!}\Delta^2 f(0)$$

$$=3+x\times3+\frac{x(x-1)}{2!}\times2$$

$$=x^2+2x+3.$$

$x=x_0+ph$
$=0+p.1$
$\Rightarrow p=x$

Ex. 15. *Show that Newton-Gregory Interpolation formula can be put in the form*

$$u_x=u_0+x\,\Delta u_0-ax\Delta^2 u_0+abx\Delta^3 u_0-abcx\Delta^4 u_0+\ldots$$

where $a=1-\frac{1}{2}(x+1)$, $b=1-\frac{1}{3}(x+1)$, $c=1-\frac{1}{4}(x+1)$ *etc.*

(Meerut M.Sc. Maths 1980)

Sol. Here, $a=1-\frac{1}{2}(x+1)$.

$$\therefore\quad -a=\frac{1}{2}(x+1)-1=\frac{1}{2}(x-1).$$

Also $b=1-\frac{1}{3}(x+1)$. $\therefore\ -b=\frac{1}{3}(x+1)-1=\frac{1}{3}(x-2)$.

Similarly $-c=\frac{1}{4}(x-3)$, $-d=\frac{1}{5}(x-4)$, etc.

Newton-Gregory interpolation formula gives

$$u_x=u_0+\frac{x}{1!}\Delta u_0+\frac{x(x-1)}{2!}\Delta^2 u_0+\frac{x(x-1)(x-2)}{3!}\Delta^3 u_0$$
$$+\frac{x(x-1)(x-2)(x-3)}{4!}\Delta^4 u_0+\ldots$$
$$=u_0+x\Delta u_0+x\left(\frac{x-1}{2}\right)\Delta^2 u_0+x\left(\frac{x-1}{2}\right)\left(\frac{x-2}{3}\right)\Delta^3 u_0$$
$$+x\left(\frac{x-1}{2}\right)\left(\frac{x-2}{3}\right)\left(\frac{x-3}{4}\right)\Delta^4 u_0+\ldots$$
$$=u_0+x\Delta u_0+x(-a)\Delta^2 u_0+x(-a)(-b)\Delta^3 u_0$$
$$+x(-a)(-b)(-c)\Delta^4 u_0+\ldots$$
$$=u_0+x\Delta u_0-ax\Delta^2 u_0+abx\Delta^3 u_0-abcx\Delta^4 u_0+\ldots$$

Ex. 16. *Given*

$$\sum_1^{10} u_x=500426,\quad \sum_4^{10} u_x=329240$$

$$\sum_7^{10} u_x=175212 \text{ and } u_{10}=40365.$$

Find u_1. **(Kanpur B.Sc. 1974)**

Sol. Here we are given the cumulative function, say $F(x)$. We are given

$$F(1)=\sum_1^{10} u_x=500426,\quad F(4)=\sum_4^{10} u_x=329240,$$

$$F(7)=\sum_7^{10} u_x=175212,\ F(10)=u_{10}=40365.$$

The difference table for $F(x)$ is as follows :

x	$F(x)$	$\Delta F(x)$	$\Delta^2 F(x)$	$\Delta^3 F(x)$
1	500426			
		−171186		
4	329240		17158	
		−154028		2023
7	175212		19181	
		−134847		
10	40365			

Now we shall find $F(2)=\sum_2^{10} u_x$.

Taking $a+xh=2$, we get $1+3.x=2 \Rightarrow x=\frac{1}{3}$.

By Newton's formula, we get

$$F(2)=F(1)+x\Delta F(1)+{}^xC_2\,\Delta^2 F(1)+{}^xC_3\,\Delta^3 F(1)+\ldots$$
$$=500426+\frac{1}{3}(-171186)+\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\frac{1}{2!}\times 17158$$
$$+\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\cdot\frac{1}{3!}\times 2023$$
$$=500426-57062-1906\cdot4444+124\cdot8765$$
$$=441582\cdot432.$$

Hence

$$u_1=\sum_1^{10} u_x-\sum_2^{10} u_x=F(1)-F(2)$$
$$=500426-441582\cdot432$$
$$=58843\cdot568.$$

Exercises 2

1. What do you mean by Interpolation ? What are the underlying assumptions for the validity of the various methods used for interpolation ? **(Delhi Hons. 1972)**
2. Derive an interpolation formula for equal intervals. **(Delhi Hons. 1972; Meerut B.Sc. 1980, 92M)**
3. By constructing a difference table, find the 7th term as well as the general term of the sequence
   0, 0, 2, 6, 12, 20,... .
4. The length of the day was 12 hours, on March 19th, 14 hours on April 18th, and 15 hours 40 minutes on May 18th. Compute (a) the length of the day on May 3rd, (b) the mean length of the day during the period, March 19th to May 18th. **(Meerut M.Sc. 1987, 91)**

[**Hint.** (a) Here we have $y_0=12$, $y_{30}=14$, $y_{60}=\frac{47}{3}$.

We require y_{45}. Now $y_{45}=E^{3/2}y_0=y_0+\frac{3}{2}\Delta y_0+\frac{\frac{3}{2}\times\frac{1}{2}}{2}\Delta^2 y_0$

$$=14\frac{7}{8}\text{ hours.}$$

3

Interpolation with unequal Intervals of the Argument

§ 1. Introduction. The interpolation formulae derived in the preceding chapter are applicable only when the values of the function are given at equidistant intervals of the independent variable or argument. It is sometimes inconvenient, or even impossible, to obtain values of a function at equidistant values of its argument, and in such cases it is desirable to have interpolation formulae which are applicable when the functional values are given at unequal intervals of the argument. Two such formulae are Newton's formula for unequal intervals of the argument and Lagrange's formula. The former employs differences, but the latter does not. The differences used in the Newton formula are called *divided differences* which are differences obtained in the usual manner and then divided by certain differences of the values of the argument. Hence the name.

§ 2. Divided Differences. Let $f(x_0), f(x_1), \ldots, f(x_n)$ be the entries corresponding to the arguments $x_0, x_1, \ldots, x_n$ where the intervals x_1-x_0, x_2-x_1, $\ldots$, x_n-x_{n-1} may not be equal, *i.e.*, not necessarily equally spaced. Then the *first divided difference* of $f(x)$ for the arguments x_0, x_1 is defined as

$\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$ or $\dfrac{f(x_0)-f(x_1)}{x_0-x_1}$ and is denoted by $f(x_0, x_1)$ or by $\underset{x_1}{\Delta} f(x_0)$.

Similarly the other first divided differences of $f(x)$ for the arguments x_1, x_2 ; x_2, x_3 ; $\ldots$, x_{n-1}, x_n are

$$f(x_1, x_2)=\frac{f(x_2)-f(x_1)}{x_2-x_1}=\underset{x_2}{\Delta} f(x_1)$$

$$f(x_2, x_3)=\frac{f(x_3)-f(x_2)}{x_3-x_2}=\underset{x_3}{\Delta} f(x_2)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$f(x_{n-1}, x_n)=\frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}=\underset{x_n}{\Delta} f(x_{n-1}).$$

The second divided difference of $f(x)$ for the three arguments x_0, x_1 and x_2 is defined as

The nth divided difference is given by

$$f(x_0, x_1, x_2, \ldots, x_n) = \frac{f(x_1, x_2, \ldots, x_n) - f(x_0, x_1, \ldots, x_{n-1})}{x_n - x_0}$$

$$= \frac{f(x_0, x_1, \ldots, x_{n-1}) - f(x_1, x_2, \ldots, x_n)}{x_0 - x_n}$$

$$= \underset{x_1, x_2, \ldots, x_n}{\Delta^n} f(x_0).$$

Note. If two of the arguments coincide, the divided difference can be given a meaning assigned by taking the limit. Thus

$$f(x_0, x_0) = \lim_{\epsilon \to 0} f(x_0, x_0 + \epsilon) = \lim_{\epsilon \to 0} \frac{f(x_0 + \epsilon) - f(x_0)}{\epsilon} = f'(x_0).$$

Similarly, $\underbrace{f(x_0, x_0, \ldots, x_0)}_{(r+1) \text{ arguments}} = \frac{1}{r!} f^{(r)}(x_0).$

§ 3. Properties of divided differences.

Theorem 1. *The divided differences are symmetrical in all their arguments, that is, the value of any difference is independent of the order of the arguments.* **(Meerut M.Sc. 1987, B.Sc. 93)**

Proof. We have

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f(x_1, x_0)$$

$$= \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} = \Sigma \frac{f(x_0)}{x_0 - x_1}, \text{ showing that } f(x_0, x_1)$$

is symmetrical in x_0, x_1.

$f(x_1)$ / $(x_1-x_0)(x_1-x_0)$

$f(x_1)$ / $(x_1-x_0)(x_1-x_2)$ / $x_1 - x_0$ / (x_1-x_2)

$$\text{Again } f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$= \frac{1}{(x_2 - x_0)} \left[\left\{ \frac{f(x_1)}{x_1 - x_2} + \frac{f(x_2)}{x_2 - x_1} \right\} - \left\{ \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} \right\} \right]$$

$$= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

$$\text{or } f(x_0, x_1, x_2) = \Sigma \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)}, \text{ showing that } f(x_0, x_1, x_2)$$

is symmetrical in x_0, x_1, x_2.

Let us assume similar symmetrical expressions for the $(n-1)$th divided differences *i.e.* let us assume that

$$f(x_0, x_1, \ldots, x_{n-1}) = \frac{f(x_0)}{(x_0 - x_1) \ldots (x_0 - x_{n-1})}$$

$$+ \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \ldots (x_1 - x_{n-1})} + \ldots$$

$$+ \frac{f(x_{n-1})}{(x_{n-1} - x_0)(x_{n-1} - x_1) \ldots (x_{n-1} - x_{n-2})}$$

$$= \Sigma \frac{f(x_0)}{(x_0 - x_1) \ldots (x_0 - x_{n-1})},$$

and similar expressions for the other $(n-1)$th divided differences.

$$\text{Then } f(x_0, x_1, \ldots, x_n) = \frac{f(x_0, \ldots, x_{n-1}) - f(x_1, \ldots, x_n)}{x_0 - x_n}$$

$$= \frac{1}{(x_0 - x_n)} \left[\left\{ \frac{f(x_0)}{(x_0 - x_1) \ldots (x_0 - x_{n-1})} + \frac{f(x_1)}{(x_1 - x_0) \ldots (x_1 - x_{n-1})} + \ldots + \frac{f(x_{n-1})}{(x_{n-1} - x_0) \ldots (x_{n-1} - x_{n-2})} \right\} - \left\{ \frac{f(x_1)}{(x_1 - x_2) \ldots (x_1 - x_n)} + \frac{f(x_2)}{(x_2 - x_1) \ldots (x_2 - x_n)} + \ldots + \frac{f(x_n)}{(x_n - x_1) \ldots (x_n - x_{n-1})} \right\} \right]$$

$$= \frac{f(x_0)}{(x_0 - x_1) \ldots (x_0 - x_n)} + \frac{f(x_1)}{(x_1 - x_0) \ldots (x_1 - x_n)} + \ldots + \frac{f(x_n)}{(x_n - x_0) \ldots (x_n - x_{n-1})}$$

$$= \Sigma \frac{f(x_0)}{(x_0 - x_1) \ldots (x_0 - x_n)},$$

showing that the nth divided difference $f(x_0, x_1, \ldots, x_n)$ is also symmetrical in $x_0, x_1, \ldots, x_n$ and thus completing the proof of the theorem by mathematical induction.

Thus we see that a divided difference is a symmetrical function of all the arguments involved and it follows that for any function $f(x)$ the value of a divided difference remains unaltered when any of the arguments involved are interchanged, *i.e.*, the value of a divided difference depends only on the values of the arguments involved and not on the order in which they are taken. Thus

$$f(x_0, x_1) = f(x_1, x_0),$$

$$f(x_0, x_1, x_2) = f(x_1, x_0, x_2) = f(x_2, x_1, x_0) = \ldots, \text{ and so on.}$$

Theorem 2. *The nth divided differences of a polynomial of the nth degree are constant.* **(Meerut B.Sc. Stat. 80, 86, 94)**

Proof. First consider the function $f(x) = x^n$. The first divided differences of this function are given by

$$f(x_r, x_{r+1}) = \frac{f(x_{r+1}) - f(x_r)}{x_{r+1} - x_r} = \frac{x_{r+1}^n - x_r^n}{x_{r+1} - x_r}$$

$$= x_{r+1}^{n-1} + x_r x_{r+1}^{n-2} + \ldots + x_r^{n-2} x_{r+1} + x_r^{n-1},$$

which is a homogeneous expression of degree $(n-1)$ in x_r, x_{r+1}. The second divided differences are given by

$$f(x_r, x_{r+1}, x_{r+2}) = \frac{f(x_r, x_{r+1}) - f(x_{r+1}, x_{r+2})}{x_r - x_{r+2}}$$

$$=\frac{f(x_{r+1}, x_{r+2})-f(x_r, x_{r+1})}{x_{r+2}-x_r}$$

$$=\frac{1}{(x_{r+2}-x_r)}\Big[\Big(x_{r+2}^{n-1}+x_{r+1}\,x_{r+2}^{n-2}+\ldots+x_{r+1}^{n-2}\,x_{r+2}+x_{r+1}^{n-1}\Big)$$
$$-\Big(x_{r+1}^{n-1}+x_r\,x_{r+1}^{n-2}+\ldots+x_r^{n-2}x_{r+1}+x_r^{n-1}\Big)\Big]$$

$$=\frac{x_{r+2}^{n-1}-x_r^{n-1}}{x_{r+2}-x_r}+x_{r+1}\frac{x_{r+2}^{n-2}-x_r^{n-2}}{x_{r+2}-x_r}+\ldots+x_{r+1}^{n-2}\frac{(x_{r+2}-x_r)}{x_{r+2}-x_r}$$

$$=\Big(x_{r+2}^{n-2}+\ldots+x_r^{n-2}\Big)+x_{r+1}\Big(x_{r+1}^{n-3}+\ldots+x_r^{n-3}\Big)+\ldots+x_{r+1}^{n-2},$$

which is a homogeneous expression of degree $n-2$ in
x_r, x_{r+1} and x_{r+2}.

By induction it can be shown that $f(x_r, x_{r+1},\ldots, x_{r+m})$ is a homogeneous expression of degree $n-m$. In particular, the nth divided difference of $f(x)=x^n$ is an expression of degree zero, *i e.*, is a constant, and is therefore independent of the values of $x_r, x_{r+1}, x_{r+2},\ldots, x_{r+n}$.

Since the nth divided differences of x^n are constant, therefore the divided differences of x^n of order greater than n will all be zero.

If $f(x)=ax^n$, where a is a constant, then the nth divided difference of $f(x)$

$=a$. (the nth divided difference of x^n),

which is a constant.

Hence if $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$ be a polynomial of degree n, then the nth divided differences of all the terms except a_0x^n will be zero and so the nth divided difference of the whole polynomial will be constant.

Theorem 3. *The nth divided difference can be expressed as the quotient of two determinants each of order* $n+1$. **(Roh. B.Sc. 90)**

Proof. Let us consider the third divided difference.

We have $$f(x_0, x_1, x_2, x_3)=\Sigma\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$
$$=\frac{\Sigma\,[f(x_0).\,(\text{difference product of } x_1, x_2, x_3)]}{\text{difference-product of } x_0, x_1, x_2, x_3}.$$

By the theorem of determinants due to Vander-Monde,

the difference product of $x_1, x_2, x_3=\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}$

Therefore,

$$f(x_0, x_1, x_2, x_3)=\Sigma\left[f(x_0)\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}\right]\div\begin{vmatrix} x_0^3 & x_1^3 & x_2^3 & x_3^3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0 & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

or $$f(x_0, x_1, x_2, x_3)=\begin{vmatrix} f(x_0) & f(x_1) & f(x_2) & f(x_3) \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0 & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}\div\begin{vmatrix} x_0^3 & x_1^3 & x_2^3 & x_3^3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0 & x_1 & x_2 & x_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

Thus the third divided difference has been expressed as the quotient of two determinants each of order 4.

Proceeding the same way, we can express the higher order differences as the quotient of two determinants each of order one more than the order of the difference.

Theorem 4. *The divided differences can be expressed as the product of multiple integrals i.e.*

$$f(x_1, x_2,\ldots, x_n)=\int_0^1 dt_1\int_0^{t_1} dt_2\int_0^{t_2} dt_3\ldots\int_0^{t_{n-2}} f^{n-1}(u_n)\,dt_{n-1}$$

where $u_n=(1-t_1)\,x_1+(t_1-t_2)\,x_2+\ldots+(t_{n-2}-t_{n-1})\,x_{n-1}+t_{n-1}\,x_n$. $t_1, t_2, t_3,\ldots, t_n$ *are independent variables and* f^{n-1} *means the* $(n-1)^{th}$ *derivative of* f. **(Rohilkhand 1988)**

Proof. First, we shall prove the result for $n=2$, and $n=3$.

For $n=2$, the R.H.S$=\int_0^1 f'(u_2)\,dt_1=\int_0^1 f'\{(1-t_1)\,x_1+t_1\,x_2\}\,dt_1$

$$=\left[\frac{f\{(1-t_1)\,x_1+t_1\,x_2\}}{x_2-x_1}\right]_0^1$$

$$=\frac{f(x_2)-f(x_1)}{x_2-x_1}=f(x_1,\ x_2)=\text{the L.H.S.}$$

Again for $n=3$, the R.H.S.

$$=\int_0^1 dt_1\int_0^{t_1} f''(u_3)\ dt_2$$

$$=\int_0^1 dt_1\int_0^{t_1} f''\{(1-t_1)x_1+(t_1-t_2)x_2+t_2x_3\}\ dt_2$$

$$=\int_0^1 dt_1\left[f'\{(1-t_1)x_1+(t_1-t_2)x_2+t_2x_3\}\right]_0^{t_1}\Big/(x_3-x_2)$$

$$=\frac{1}{x_3-x_2}\left[\int_0^1 f'\{(1-t_1)x_1+t_1x_3\}\ dt_1-\int_0^1 f'\{(1-t_1)x_1+t_1x_2\}\ dt_1\right]$$

$$=\frac{1}{(x_3-x_2)\ (x_3-x_1)}\ [f(x_3)-f(x_1)]-\frac{1}{(x_3-x_2)\ (x_2-x_1)}\ [f(x_2)-f(x_1)]$$

$$=\frac{f(x_1)}{(x_1-x_2)\ (x_1-x_3)}+\frac{f(x_2)}{(x_2-x_1)\ (x_2-x_3)}+\frac{f(x_3)}{(x_3-x_1)\ (x_3-x_2)}$$

$=f(x_1,\ x_2,\ x_3)=$ the L.H.S.

Thus the result is true for $n=2$ and $n=3$.

Now let the result be true for n arguments. We shall show that it is also true for $(n+1)$ arguments.

We have $\int_0^{t_{n-1}} f^n\ (u_{n+1})dt_n$

$$=\int_0^{t_{n-1}} f^n\{(1-t_1)x_1+(t_1-t_2)x_2+\dots+(t_{n-1}-t_n)x_n+t_n\ x_{n+1}\}\ dt_n$$

$$=\frac{1}{x_{n+1}-x_n}\left[f^{n-1}\{(1-t_1)x_1+(t_1-t_2)x_2+\dots+(t_{n-1}-t_n)x_n+t_nx_{n+1}\}\right]_0^{t_{n-1}}$$

$$=\frac{-1}{x_n-x_{n+1}}\Big[f^{n-1}\{(1-t_1)x_1+\dots+(t_{n-2}-t_{n-1})x_{n-1}+t_{n-1}x_n\}$$
$$-f^{n-1}\{(1-t_1)x_1+\dots+(t_{n-2}-t_{n-1})x_{n-1}+t_{n-1}x_{n+1}\}\Big].$$

$$\therefore\ \int_0^1 dt_1\int_0^{t_1} dt_2\dots\int_0^{t_{n-1}} f^n\ (u_{n+1})\ dt_n$$
$$=\frac{f(x_1,\ x_2,\ \dots\ x_n)-f(x_1,\ x_2,\ \dots,\ x_{n-1},\ x_{n+1})}{x_n-x_{n+1}}$$

$$=\frac{f(x_n,\ x_2,\dots\ x_1)-f(x_n,\ x_2,\ \dots,\ x_{n-1},\ x_{n+1})}{x_1-x_{n+1}}$$

{interchanging x_1 and x_n}

$$=\frac{f(x_1,\ x_2,\dots,\ x_n)-f(x_2,\ x_3,\dots,\ x_n,\ x_{n+1})}{x_1-x_{n+1}}$$

$$=f(x_1,\ x_2,\ \dots,\ x_{n+1}).$$

Thus the result is true for $(n+1)$ arguments. We have already shown that the result is true for 2 and 3 arguments. Hence by mathematical induction the result is true for n arguments where n is any +ive integer.

§ 4. Newton's Formula for Unequal Intervals.

(Agra B.Sc. 1975; Meerut B.Sc. 92, Stat. 90, M.Sc. 87, 91)

Let $f(x_0),\ f(x_1),\ \dots,f(x_n)$ be the values of $f(x)$ corresponding to the arguments $x_0,\ x_1,\dots,\ x_n$, not necessarily equally spaced. From the definition of divided differences,

$$f(x,\ x_0)=\frac{f(x)-f(x_0)}{x-x_0}$$

or $f(x)=f(x_0)+(x-x_0)\ f(x,\ x_0)$. ...(1)

Also $f(x,\ x_0,\ x_1)=\dfrac{f(x,\ x_0)-f(x_0,\ x_1)}{x-x_1}$.

or $f(x,\ x_0)=f(x_0,\ x_1)+(x-x_1)\ f(x,\ x_0,\ x_1)$. ...(2)

Similarly $f(x,\ x_0,\ x_1)=f(x_0,\ x_1,\ x_2)+(x-x_2)\ f(x,\ x_0,\ x_1,\ x_2)$ ...(3)

... ...

... ...

$$f(x,\ x_0,\ x_1,\ \dots,\ x_{n-1})=f(x_0,\ x_1,\dots,\ x_n)+(x-x_n)\ f(x,\ x_0,\ x_1,\dots,\ x_n)\ \dots(4)$$

Multiplying the equation (2) by $(x-x_0)$, (3) by $(x-x_0)\ (x-x_1)$ and so on and finally the equation (4) by $(x-x_0)\ (x-x_1)\dots(x-x_{n-1})$ and adding to the equation (1), we have

$$f(x)=f(x_0)+(x-x_0)\ f(x_0,\ x_1)+(x-x_0)(x-x_1)\ f(x_0,\ x_1,\ x_2)+\dots$$
$$+(x-x_0)(x-x_1)\dots(x-x_{n-1})\ f(x_0,\ x_1,\dots,x_n)+R_n,$$

where the remainder R_n is given by

$$R_n=(x-x_0)(x-x_1)\dots(x-x_n)\ f(x,\ x_0,\ x_1,\dots,\ x_n).$$

Assuming that $f(x)$ is a polynomial of degree n, $f\ (x,\ x_0,\ x_1,\dots,\ x_n)$ vanishes so that

$$f(x)=f(x_0)+(x-x_0)\ f(x_0,\ x_1)+(x-x_0)(x-x_1)\ f(x_0,\ x_1,\ x_2)+\dots$$
$$+(x-x_0)(x-x_1)\dots(x-x_{n-1})\ f(x_0,\ x_1,\ \dots,\ x_n).\quad \dots(5)$$

This formula is called Newton's divided difference interpolation formula.

§ 5. Relation between divided differences and ordinary differences.

Let the arguments $x_0, x_1, x_2, \ldots, x_n$ be equally spaced, *i.e.*, $h=x_1-x_0=x_2-x_1=\ldots=x_n-x_{n-1}$ and let $x=x_0+uh$. Then

$$f(x_0, x_1)=\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{1}{h}\,\Delta f(x_0),$$

$$f(x_0, x_1, x_2)=\frac{f(x_0, x_1)-f(x_1, x_2)}{x_0-x_2}$$

$$=\frac{\frac{1}{h}\Delta f(x_0)-\frac{1}{h}\Delta f(x_1)}{-2h}=\frac{1}{2!\,h^2}\,\Delta^2 f(x_0).$$

Similarly

$$f(x_0, x_1, \ldots, x_n)=\frac{1}{n!\,h^n}\,\Delta^n f(x_0).$$

Substituting these values of the divided differences in Newton's formula *i.e.* the equation (5) of § 4, we get

$$f(x_0+uh)=f(x_0)+\frac{uh}{1!\,h}\,\Delta f(x_0)+\frac{uh(uh-h)}{2!\,h^2}\,\Delta^2 f(x_0)+\ldots$$
$$+\frac{uh\,(uh-h)\,(uh-2h)\ldots(uh-\overline{n-1}\,h)}{n!\,h^n}\,\Delta^n f(x_0)$$

$$=f(x_0)+u\Delta f(x_0)+\frac{u(u-1)}{2!}\,\Delta^2 f(x_0)+\ldots$$
$$+\frac{u\,(u-1)(u-2)\ldots(u-n+1)}{n!}\,\Delta^n f(x_0)$$

$$=f(x_0)+{}^uC_1\,\Delta f(x_0)+{}^uC_2\,\Delta^2 f(x_0)+\ldots+{}^uC_n\,\Delta^n f(x_0),$$

which is Newton's formula for advancing differences.

§ 6. Sheppard's Rule. In Newton's divided difference formula the function $f(x)$ is expressed in terms of leading term and leading differences of a divided difference table. But now we shall give a rule known as Sheppard's rule which can be used to write a divided difference formula with any value of $f(x)$ as the initial term.

Newton's divided difference formula for interpolation is $f(x)=P(x)$

$$=f(x_0)+(x-x_0)\underset{x_1}{\Delta} f(x_0)+(x-x_0)(x-x_1)\underset{x_1x_2}{\Delta^2} f(x_0)+\ldots$$
$$+(x-x_0)(x-x_1)\ldots(x-x_{n-1})\underset{x_1\ldots x_n}{\Delta^n} f(x_0) \quad \ldots(1)$$

In (1), put

$$x-x_0=X_0,\; x-x_1=X_1,\; \ldots,\; x-x_{n-1}=X_{n-1}.$$

Then, we have

$$f(x)=f(x_0)+X_0\underset{x_1}{\Delta} f(x_0)+X_0X_1\underset{x_1x_2}{\Delta^2} f(x_0)+\ldots+X_0\ldots X_{n-1}\underset{x_1\ldots x_n}{\Delta^n} f(x_0) \quad \ldots(2)$$

In the R.H.S. of (2) we observe that the coefficients of various terms are written in capital letters $X_0, X_1, \ldots$etc. and the suffixes of the operator and the operand are given in small letters x_0, x_1 etc. In the first term of R.H.S. of (2), the coefficient of $f(x_0)$ is one and the suffix of the operand is x_0. In the second term, the coefficient is X_0 and there are two suffixes x_0 and x_1 in the operator and the operand. In the third term the coefficient consists of two letters X_0, X_1 and there are three suffixes x_0, x_1 and x_2 in the operator and operand and so on. We conclude from this that small letters in each term of R.H.S. of (2) are one more in number than the number of capital letters. Now we give the small and capital letters term by term.

Small letters :	x_0	x_0x_1	$x_0x_1x_2\ldots\ldots x_0x_1\ldots x_n$
Capital letters :	1	X_0	$X_0X_1\ldots\ldots X_0X_1\;\; X_{n-1}$.

This characteristic of Newton's divided difference formula has given rise to a rule which is called **Sheppard's rule** or **Zigzag rule**, and is as follows :

(i) Start with any initial term.

(ii) In order to get the second term take any first order difference, either moving upward or downward in the divided difference table, which contains the suffix of the initial term and multiply this first order difference by the term obtained by subtracting from x the value of the suffix of the initial term.

(iii) In the same way find the third term, keeping in mind the suffixes of second term. In this way complete the formula.

The rule will be clear from the following example.

Ex. *The following observations are given for the function* $y=f(x)$:

x :	x_0	x_1	x_2	x_3	x_4	x_5
$f(x)$:	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	$f(x_5)$.

Then find a divided difference formula with $f(x_2)$ *as the initial term.*

Sol. For the given data we construct the following difference table.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
x_0	$f(x_0)$					
		$\underset{x_1}{\Delta} f(x_0)$				
x_1	$f(x_1)$		$\underset{x_1x_2}{\Delta^2} f(x_0)$			
		$\underset{x_2}{\Delta} f(x_1)$		$\underset{x_1x_2x_3}{\Delta^3} f(x_0)$		
x_2	$f(x_2)$		$\underset{x_2x_3}{\Delta^2} f(x_1)$		$\underset{x_1x_2x_3x_4}{\Delta^4} f(x_0)$	
		$\underset{x_3}{\Delta} f(x_2)$		$\underset{x_2x_3x_4}{\Delta^3} f(x_1)$		$\underset{x_1x_2x_3x_4x_5}{\Delta^5} f(x_0)$
x_3	$f(x_3)$		$\underset{x_3x_4}{\Delta^2} f(x_2)$		$\underset{x_2x_3x_4x_5}{\Delta^4} f(x_1)$	
		$\underset{x_4}{\Delta} f(x_3)$		$\underset{x_3x_4x_5}{\Delta^3} f(x_2)$		
x_4	$f(x_4)$		$\underset{x_4x_5}{\Delta^2} f(x_3)$			
		$\underset{x_5}{\Delta} f(x_4)$				
x_5	$f(x_5)$					

Beginning with $f(x_2)$ as the initial term and moving one step upward and one step downward alternately in the difference table, applying Sheppard's Rule, we get

$$f(x)=f(x_2)+X_2 \underset{x_2}{\Delta} f(x_1)+X_2X_1 \underset{x_2x_3}{\Delta^2} f(x_1)+X_3X_2X_1 \underset{x_1x_2x_3}{\Delta^3} f(x_0)$$
$$+X_2X_2X_1X_0 \underset{x_1x_2x_3x_4}{\Delta^4} f(x_0)+X_4X_3X_2X_1X_0 \underset{x_1x_2x_3x_4x_5}{\Delta^5} f(x_0)+\ldots$$
$$=f(x_2)+(x-x_2) \underset{x_2}{\Delta} f(x_1)+(x-x_2)(x-x_1) \underset{x_2x_3}{\Delta^2} f(x_1)$$
$$+(x-x_3)(x-x_2)(x-x_1) \underset{x_1x_2x_3}{\Delta^3} f(x_0)+\ldots$$

This is the required interpolation formula with $f(x_2)$ as the initial term.

If we take our first step in downward direction of the difference table, the formula will be

$$f(x)=f(x_2)+X_2 \underset{x_3}{\Delta} f(x_2)+X_2X_3 \underset{x_2x_3}{\Delta^2} f(x_1)+X_1X_2X_3 \underset{x_2x_3x_4}{\Delta^3} f(x_1)$$
$$+X_1X_2X_3X_4 \underset{x_1x_2x_3x_4}{\Delta^4} f(x_0)+\ldots$$
$$=f(x_2)+(x-x_2) \underset{x_3}{\Delta} f(x_2)+(x-x_2)(x-x_3) \underset{x_2x_3}{\Delta^2} f(x_1)$$
$$+(x-x_1)(x-x_2)(x-x_3) \underset{x_2x_3x_4}{\Delta^3} f(x_1)+\ldots$$

§ 7. Lagrange's Interpolation formula for unequal intervals.

(Meerut B.Sc. 1991, 91P, 92, 92M, M.Sc. 86, 92P 93, 93P Roh. B.Sc. 91)

Let $f(x)$ denote a polynomial of the nth degree which takes the values $y_0, y_1, y_2, \ldots, y_n$ when x has the values $x_0, x_1, x_2, \ldots, x_n$ respectively. Then the $(n+1)$th differences of this polynomial are zero. Hence

$$f(x, x_0, x_1, x_2, \ldots, x_n)=0. \qquad \ldots(1)$$

But we have

$$f(x, x_0, x_1, x_2, \ldots, x_n)$$
$$=\frac{y}{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_n)}+\frac{y_0}{(x_0-x)(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}$$
$$+\frac{y_1}{(x_1-x)(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}+\ldots$$
$$+\frac{y_n}{(x_n-x)(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}.$$

$\therefore$ using (1), we get

$$\frac{y}{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_n)}+\frac{y_0}{(x_0-x)(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}$$
$$+\frac{y_1}{(x_1-x)(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}+\ldots$$
$$+\frac{y_n}{(x_n-x)(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}=0.$$

Transposing all the terms except the first to the right hand side, we get

$$\frac{y}{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_n)}$$
$$=\frac{y_0}{(x-x_0)(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}$$
$$+\frac{y_1}{(x-x_1)(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}+\ldots$$

$$+\frac{y_n}{(x-x_n)(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}.$$

Solving for y and then removing the common factors $(x-x_0)$, $(x-x_1)$, $(x-x_2)$,..., $(x-x_n)$ in the respective terms, we get

$$y=\frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}y_0$$
$$+\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}y_2+\ldots$$
$$+\frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})}y_n. \quad \ldots(2)$$

This is Lagrange's formula and is seen to give $y=y_0$, y_1,..., y_n when $x=x_0$, x_1, ..., x_n respectively. The values of the independent variable may or may not be equidistant.

Since Lagrange's formula is merely a relation between two variables, either of which may be taken as the independent variable, it is evident that by considering y as the independent variable we can write a formula giving x as a function of y. Hence on interchanging x and y in (2), we get

$$x=\frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_n)}x_0$$
$$+\frac{(y-y_0)(y-y_2)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots(y_1-y_n)}x_1$$
$$+\frac{(y-y_0)(y-y_1)\ldots(y-y_n)}{(y_2-y_0)(y_2-y_1)\ldots(y_2-y_n)}x_2+\ldots$$
$$+\frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots(y_n-y_{n-1})}x_n. \quad \ldots(3)$$

The main uses of Lagrange's formula are : (1) to find any value of a function when the given values of the independent variable are not equidistant, and (2) to find the value of the independent variable corresponding to a given value of the function.

The reader will notice that Lagrange's formula is tedious to apply and involves a great deal of computation. It must also be used with care and caution, for if the values of the independent variable are not taken close together the results are liable to be very inaccurate.

Alternative proof of Lagrange's interpolation formula without using the divided differences.

Let the given function be $y=f(x)$. Let corresponding to the values x_0, x_1, x_2, ..., x_{n-1}, x_n of the argument x, the values of the function $f(x)$ be $f(x_0)$, $f(x_1)$, $f(x_2)$, ..., $f(x_{n-1})$, $f(x_n)$ respectively,

where the intervals x_1-x_0, x_2-x_1,..., x_n-x_{n-1} are not necessarily equally spaced. As we have taken $n+1$ values of $f(x)$, therefore $f(x)$ can be assumed as a polynomial of degree n.

Let $f(x)=A_0(x-x_1)(x-x_2)\ldots(x-x_{n-1})(x-x_n)$
$$+A_1(x-x_0)(x-x_2)\ldots(x-x_n)+\ldots$$
$$\ldots+A_n(x-x_0)(x-x_1)\ldots(x-x_{n-1}), \quad \ldots(1)$$
where A's are constants.

To find A_0, A_1, A_2, ..., A_n, we put $x=x_0$, x_1, x_2, ..., x_n respectively in (1). Thus putting $x=x_0$ in (1), we get

$$f(x_0)=A_0(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)+0+0+\ldots$$

or
$$A_0=\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}.$$

Similarly by putting $x=x_1$, we get

$$A_1=\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} \text{ and so on.}$$

Thus
$$A_n=\frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}.$$

Substituting these values of A_0, A_1, A_2, ..., A_n in (1), we get

$$f(x)=\frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}f(x_0)$$
$$+\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}f(x_1)+\ldots$$
$$\ldots+\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}f(x_n).$$

This is Lagrange's interpolation formula and can be used for both equal and unequal intervals.

§ 8. Hermite's Interpolation Formula. (Meerut 89, 92P, 93)

So far we have considered the interpolation formulae which make use only of a certain number of function values. We now derive an interpolation formula in which both the function and its first derivative are to be assigned at each point of interpolation.

The problem of interpolation is then : Given the set of data points (x_i, y_i, y_i'), $i=0, 1, \ldots, n$, it is required to find a polynomial of the least degree, say $\phi_{2n+1}(x)$, such that

$$\left.\begin{aligned} &\phi_{2n+1}(x_i)=y_i, \\ \text{and} \quad &\phi'_{2n+1}(x_i)=y_i', \\ &i=0, 1, \ldots, n \end{aligned}\right\} \quad \ldots(1)$$

Here, we have $(2n+2)$ conditions and we note that a polynomial of degree $(2n+1)$ has $(2n+2)$ coefficients to be determined.

We write

$$\phi_{2n+1}(x)=\sum_{i=0}^{n} u_i(x)\, y_i+\sum_{i=0}^{n} v_i(x)\, y_i', \qquad \ldots(2)$$

where $u_i(x)$ and $v_i(x)$ are polynomials in x of degree $(2n+1)$. Using the conditions (1), we get

$$\left.\begin{aligned} u_i(x_j)&=1 \text{ if } i=j, & v_i(x_j)&=0\ ;\\ &=0 \text{ if } i\neq j\ ; & & \\ u_i'(x_j)&=0 & v_i'(x_j)&=1 \text{ if } i=j\\ & & &=0 \text{ if } i\neq j \end{aligned}\right\} \qquad \ldots(3)$$

We therefore choose

$$\left.\begin{aligned} u_i(x)&=a_i(x)\,[l_i(x)]^2\\ \text{and}\quad v_i(x)&=b_i(x)\,[l_i(x)]^2, \end{aligned}\right\} \qquad \ldots(4)$$

where $l_i(x)$ is defined as

$$l_i(x)=\frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}.$$

Since $l_i(x)$ is a polynomial of degree n, it follows that $a_i(x)$ and $b_i(x)$ are both linear functions.

By using the conditions (3), we get

$$\left.\begin{aligned} u_i(x_i)&=1 \quad ; & b_i(x_i)&=0\ ;\\ a_i'(x_i)&=-2\,l_i'(x_i)\ ; & b_i'(x_i)&=1 \end{aligned}\right\} \qquad \ldots(5)$$

from which it follows that

$$a_i(x)=1-2\,l_i'(x_i)(x-x_i)$$

and $$b_i(x)=x-x_i.$$

Hence (2) becomes

$$\phi_{2n+1}(x)=\sum_{i=0}^{n}\left[\left\{1-2l_i'(x_i)(x-x_i)\right\}\left\{l_i(x)\right\}^2 y_i\right]$$
$$+\sum_{i=0}^{n}\left[(x-x_i)\left\{l_i(x)\right\}^2 y_i'\right], \qquad \ldots(6)$$

which is Hermite's interpolation formula.

Solved Examples

Ex. 1. *Construct a divided difference table for the following :*

x	1	2	4	7	12
$f(x)$	22	30	82	106	216

Sol. The divided difference table for the given data is as follows :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	22				
		$\frac{30-22}{2-1}=8$			
2	30		$\frac{26-8}{4-1}=6$		
		$\frac{82-30}{4-2}=26$		$\frac{(-3\cdot6)-6}{7-1}=-1\cdot6$	
4	82		$\frac{8-26}{7-2}=-3\cdot6$		$\frac{0\cdot535-(-1\cdot6)}{12-1}=0\cdot194$
		$\frac{106-82}{7-4}=8$		$\frac{1\cdot75-(-3\cdot6)}{12-2}=0\cdot535$	
7	106		$\frac{22-8}{12-4}=1\cdot75$		
		$\frac{216-106}{12-7}=22$			
12	216				

Ex. 2. *Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x)=x^3-2x$.* **(Kanpur B.Sc. 1976)**

Sol. The divided difference table is :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	4			
		$\frac{56-4}{4-2}=26$		
4	56		$\frac{131-26}{9-2}=15$	
		$\frac{711-56}{9-4}=131$		$\frac{23-15}{10-2}=1$
9	711		$\frac{269-131}{10-4}=23$	
		$\frac{980-711}{10-9}=269$		
10	980			

Hence the third divided difference is 1.

Ex. 3. *By means of Newton's divided difference formula, find the values of $f(8)$ and $f(15)$ from the following table :*

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

(Meerut B.Sc. 1992, M.Sc. 85, 92)

Sol. The divided difference table is :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		$\frac{100-48}{5-4}=52$			
5	100		$\frac{97-52}{7-4}=15$		
		$\frac{294-100}{7-5}=97$		$\frac{21-15}{10-4}=1$	
7	294		$\frac{202-97}{10-5}=21$		0
		$\frac{900-294}{10-7}=202$		$\frac{27-21}{11-5}=1$	
10	900		$\frac{310-202}{11-7}=27$		0
		$\frac{1210-900}{11-10}=310$		$\frac{33-27}{13-7}=1$	
11	1210		$\frac{409-310}{13-10}=33$		
		$\frac{2028-1210}{13-11}=409$			
13	2028				

Using Newton's interpolation formula for unequal intervals, we get

$$f(8)=48+(8-4)\times 52+(8-4)(8-5)\times 15+(8-4)(8-5)(8-7)\times 1$$
$$=448,$$

and

$$f(15)=48+(15-4)\times 52+(15-4)(15-5)\times 15+(15-4)(15-5)(15-7)\times 1$$
$$=3150.$$

Ex. 4. *If $f(x)=\frac{1}{x^2}$, find the divided differences $f(a, b)$, $f(a, b, c)$ and $f(a, b, c, d)$.*

Sol. We have $f(a, b)=\frac{f(b)-f(a)}{b-a}=\frac{\frac{1}{b^2}-\frac{1}{a^2}}{b-a}$

$$=-\frac{(b^2-a^2)}{(b-a)\,b^2a^2}=-\frac{b+a}{a^2b^2}. \quad \text{...(1)}$$

Again $f(a, b, c)=\frac{f(b, c)-f(a, b)}{c-a}$

$$=\frac{1}{c-a}\left[-\frac{b+c}{b^2c^2}+\frac{a+b}{a^2b^2}\right], \text{ using (1)}$$

$$=-\frac{1}{(c-a)}\left[\frac{b+c}{b^2c^2}-\frac{a+b}{a^2b^2}\right]$$

$$=-\frac{1}{(c-a)}\left[\frac{a^2(b+c)-c^2(a+b)}{a^2b^2c^2}\right]$$

$$=-\left[\frac{b(a^2-c^2)+ac(a-c)}{(c-a)\,a^2b^2c^2}\right]$$

$$=-\frac{(a-c)\{b(a+c)+ac\}}{(c-a)\,a^2b^2c^2}$$

$$=\frac{ab+bc+ca}{a^2b^2c^2}. \quad \text{...(2)}$$

Now $f(a, b, c, d)$

$$=\frac{f(b, c, d)-f(a, b, c)}{d-a}$$

$$=\frac{1}{(d-a)}\left[\frac{bc+cd+db}{b^2c^2d^2}-\frac{ab+bc+ca}{a^2b^2c^2}\right]$$

$$=\frac{1}{(d-a)}\left[\frac{a^2(bc+cd+db)-d^2(ab+bc+ca)}{a^2b^2c^2d^2}\right]$$

$$=\frac{1}{(d-a)}\left[\frac{bc(a^2-d^2)+acd(a-d)+abd(a-d)}{a^2b^2c^2d^2}\right]$$

$$=\frac{(a-d)}{(d-a)}\left[\frac{bc(a+d)+acd+abd}{a^2b^2c^2d^2}\right]$$

$$=-\frac{abc+bcd+acd+abd}{a^2b^2c^2d^2}.$$

Ex. 5. *Show that* $\underset{bcd}{\Delta^3}\left(\frac{1}{a}\right)=-\frac{1}{abcd}$.

(Meerut M.Sc. 1988, 89, 91, 92P)

Sol. The divided difference table is as follows :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
a	$\frac{1}{a}$			
		$\dfrac{\frac{1}{b}-\frac{1}{a}}{b-a}=-\frac{1}{ab}$		
b	$\frac{1}{b}$		$(-1)\dfrac{\frac{1}{bc}-\frac{1}{ab}}{c-a}$ $=(-1)^2\frac{1}{abc}$	
		$\dfrac{\frac{1}{c}-\frac{1}{b}}{c-b}=-\frac{1}{bc}$		$(-1)^2\dfrac{\frac{1}{bcd}-\frac{1}{abc}}{d-a}$ $=(-1)^3\frac{1}{abcd}$
c	$\frac{1}{c}$		$(-1)\dfrac{\frac{1}{dc}-\frac{1}{bc}}{d-b}$ $=(-1)^2\frac{1}{bcd}$	
		$\dfrac{\frac{1}{d}-\frac{1}{c}}{d-c}=-\frac{1}{dc}$		
d	$\frac{1}{d}$			

Hence, $\underset{bcd}{\Delta^3}\ \frac{1}{a}=(-1)^3\frac{1}{abcd}=-\frac{1}{abcd}$

Ex. 6. *Show that* $\underset{yz}{\Delta^2}\ x^3=x+y+z.$ **(Meerut B.Sc. 92M, M.Sc. 80)**

Sol. We construct the following divided difference table :

Argument	Entry	First divided diff.	Second divided diff.
x	x^3		
		$\dfrac{y^3-x^3}{y-x}=y^2+x^2+xy$	
y	y^3		$\dfrac{(z^2+y^2+zy)-(y^2+x^2+yx)}{z-x}$ $=\dfrac{(z^2-x^2)+y(z-x)}{z-x}$ $=x+y+z$
		$\dfrac{z^3-y^3}{z-y}=z^2+y^2+zy$	
z	z^3		

From the table, we observe that

$$\underset{yz}{\Delta^2}\ x^3=x+y+z.$$

Ex. 7. *Find the polynomial of the lowest possible degree which assumes the values* 3, 12, 15, -21 *when x has the values* 3, 2, 1, -1 *respectively.* **(Meerut M.Sc. 1987, 92P, 93)**

Sol. For the given data the divided defference table is as given delow :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	-21			
		$\dfrac{15+21}{1+1}=18$		
1	15		$\dfrac{-3-18}{2+1}=-7$	
		$\dfrac{12-15}{2-1}=-3$		$\dfrac{-3+7}{3+1}=1$
2	12		$\dfrac{-9+3}{3-1}=-3$	
		$\dfrac{3-12}{3-2}=-9$		
3	3			

By Newton's divided difference formula, we get

$$f(x)=f(x_0)+(x-x_0)\,\Delta f(x_0)+(x-x_0)(x-x_1)\,\Delta^2 f(x_0)$$
$$+(x-x_0)(x-x_1)(x-x_2)\,\Delta^3 f(x_0)$$

or $f(x)=-21+\{x-(-1)\}\,(18)+\{x-(-1)\}\,(x-1)\,(-7)$
$+\{x-(-1)\}\,(x-1)\,(x-2)\,(1)$
$=-21+(x+1)\,18+(x+1)\,(x-1)\,(-7)$
$+(x+1)\,(x-1)\,(x-2)$
$=x^3-9x^2+17x+6.$

Ex. 8. *Using the following table, find $f(x)$ as a polynomial in powers of $(x-6)$.*

x	-1	0	2	3	7	10
$f(x)$	-11	1	1	1	141	561

Sol. The divided difference table is as given below :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	-11			
		$\frac{1+11}{0+1}=12$		
0	1		$\frac{0-12}{2+1}=-4$	
		$\frac{1-1}{2-0}=0$		$\frac{0+4}{3+1}=1$
2	1		0	
		$\frac{1-1}{3-2}=0$		$\frac{7-0}{7-0}=1$
3	1		$\frac{35-0}{7-2}=7$	
		$\frac{141-1}{7-3}=35$		$\frac{15-7}{10-2}=1$
7	141		$\frac{140-35}{10-3}=15$	
		$\frac{561-141}{10-7}=140$		$\frac{a-15}{6-3}=1$
10	561		$\frac{b-140}{6-7}=a$	
...	...	....	...	...
		$\frac{c-561}{6-10}=b$		$\frac{a'-a}{6-7}=1$
6	c		$\frac{b'-b}{6-10}=a'$	
		b'		$\frac{a''-a'}{6-10}=1$
6	c		a''	
		b''		
6	c			

Since the third differences are constant, $f(x)$ is a polynomial of degree three and hence the argument 6 should appear 3 times successively in the difference table. The extended table has been shown below the dotted line.

From the column of third differences,

$$\frac{a-15}{6-3}=1 \quad \text{or} \quad a=15+3=18.$$

Also $\quad \frac{a'-a}{6-7}=1 \;\Rightarrow\; a'=a-1=17,$

and $\quad \frac{a''-a'}{6-10}=1 \;\Rightarrow\; a''=a'-4=13.$

From the column of second differences, we have

$$\frac{b-140}{6-7}=a \;\Rightarrow\; b=140-a=122,$$

and $\quad \frac{b'-b}{6-10}=a' \;\Rightarrow\; b'=b-4a'=54.$

Again from the column of first differences, we have

$$\frac{c-561}{6-10}=b \;\Rightarrow\; c=561-4b=73.$$

Newton's formula with arguments x_0, x_1, x_2, x_3 is

$$f(x)=f(x_0)+(x-x_0)\,f(x_0,x_1)+(x-x_0)\,(x-x_1)\,f(x_0,x_1,x_2)$$
$$+(x-x_0)\,(x-x_1)\,(x-x_2)f(x_0,x_1,x_2,x_3).$$

Putting $x_0=x_1=x_2=x_3=6$, we get

$$f(x)=f(6)+(x-6)\,b'+(x-6)^2\,a''+(x-6)^3\cdot 1$$
$$=73+54\,(x-6)+13\,(x-6)^2+(x-6)^3.$$

Ex. 9 *The observed values of a function are respectively 168, 120, 72 and 63 at the four positions 3, 7, 9 and 10 of the independent variable. What is the best estimate you can give for the value of the function at the position 6 of the independent variable ?* **(Delhi Hons. 75 ; Gujrat B. Sc. 73)**

Sol. The divided difference table is as given below :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
3	168			
		$\frac{120-168}{7-3}=-12$		
7	120		$\frac{-24+12}{9-3}=-2$	
		$\frac{72-120}{9-7}=-24$		$\frac{5+2}{10-3}=1$
9	72		$\frac{-9+24}{10-7}=5$	
		$\frac{63-72}{10-9}=-9$		
10	63			

Hence

$$f(x)=f(x_0)+(x-x_0)\,\Delta f(x_0)+(x-x_0)(x-x_1)\,\Delta^2 f(x_0)$$
$$+(x-x_0)(x-x_1)(x-x_2)\,\Delta^3 f(x_0)$$
$$=168+(x-3)(-12)+(x-3)(x-7)(-2)$$
$$+(x-3)(x-7)(x-9) \quad (1)$$
$$=x^3-21x^2+119x-27.$$

When $x=6$, the estimate of the function is given by

$$f(6)=6^3-21.6^2+119.6-27=147.$$

Ex. 10. *Given the values :*

x	:	4	5	7	10	11	13
u_x	:	48	100	294	900	1210	2028

Form the table of divided differences and extend it to include the values $x=2$ and $x=15$.

Sol. The divided difference table is as follows :

x	u_x	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$	$\Delta^4 u_x$	$\Delta^5 u_x$
2	**4**					
		22				
4	48		**10**			
		52		**1**		
5	100		15		**0**	
		97		1		**0**
7	294		21		0	
		202		1		0
10	900		27		0	
		310		1		**0**
11	1210		33		**0**	
		409		**1**		
13	2028		**38**			
		561				
15	**3150**					

$$\therefore \quad u_x=48+52(x-4)+15(x-4)(x-5)$$
$$+1.(x-4)(x-5)(x-7)$$
$$=48+52(x-4)+15(x^2-9x+20)$$
$$+(x^3-16x^2+83x-140)$$
$$=x^3-x^2=x^2(x-1).$$

$$\therefore \quad u_2=2^3-2^2=4 \text{ and } u_{15}=15^3-15^2=3150.$$

The original divided difference table is for the argument values 4 to 13, leaving 2 and 15 and the figures in antique.

The complete table, with arguments 2 and 15 and the figures in antique, gives the original divided difference table extended to include the values $x=2$ and $x=15$.

Ex. 11. *The following are the mean temperatures (Fahrenheit) on three days, 30 days apart, round the periods of summer and winter. Estimate the approximate dates and values of the maximum and minimum temperature.*

	Summer		Winter	
Day	Date	Temperature	Date	Temperature
0	15th June	58·8	16th Dec.	40·7
30	15th July	63·4	15th Jan.	38·1
60	14th Aug.	62·5	14th Feb.	39·3

(Delhi B.Sc. Hons. 1975; Rohilkhand M.Sc. 90)

Sol. It is obvious that the dates and values of maximum and minimum temperatures are to be obtained from the summer and winter records respectively.

Let us choose 1 unit$=30$ days, then we are given the entries (temperatures) corresponding to equidistant arguments (dates), the interval of differencing being unity.

The divided difference table for summer is as below :

x	temperature	$\Delta f(x)$	$\Delta^2 f(x)$
0	58·8		
		4·6	
1	63·4		$-\frac{5\cdot 5}{2}=-2\cdot 75$
		$-0\cdot 9$	
2	62·5		

$$\therefore\ f(x)=f(x_0)+(x-x_0)\ \Delta f(x_0)+(x-x_0)\ (x-x_1)\ \Delta^2 f(x_0)$$
$$=58\cdot 8+(x-0)\ (4\cdot 6)+(x-0)\ (x-1)\ (-2\cdot 75).$$
$$=-2\cdot 75x^2+7\cdot 35x+58\cdot 8 \qquad \ldots(1)$$

For maxima or minima, we must have $f'(x)=0$.

We have $f'(x)=0 \Rightarrow -5\cdot 5x+7\cdot 35=0 \Rightarrow x=1\cdot 336$.

Now $f''(x)=-5\cdot 5$, which is negative.

Hence $f(x)$ is maximum at $x=1\cdot 336$.

Since 1 unit$=30$ days,

$\therefore$ 1·336 units$=30\times 1\cdot 336=40\cdot 08$ days.

Thus the maximum temperature was on 15th June$+40$ days *i.e.*, on 25th July and the value of maximum temperature is

$$\max f(x)=[f(x)]_{x=1\cdot 336}$$
$$=-2\cdot 75\times(1\cdot 336)^2+7\cdot 35\times 1\cdot 336+58\cdot 8$$
$$=63\cdot 71\ F \text{ approximately.}$$

The divided difference table for winter temperature is as follows :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
0	40·7		
		$-2\cdot 6$	
1	38·1		1·9
		1·2	
2	39·3		

$$\therefore\ f(x)=f(x_0)+(x-x_0)\ \Delta f(x_0)+(x-x_0)\ (x-x_1)\ \Delta^2 f(x_0)$$
$$=40\cdot 7+(x-0)\ (-2\cdot 6)+(x-0)\ (x-1)\ (1\cdot 9)$$
$$=40\cdot 7-2\cdot 6x+(1\cdot 9x^2-1\cdot 9x)$$
$$=1\cdot 9x^2-4\cdot 5x+40\cdot 7.$$

We have

$$f'(x)=3\cdot 8x-4\cdot 5=0 \Rightarrow x=1\cdot 184.$$

Also

$$f''(x)=3\cdot 8, \text{ which is positive.}$$

Hence $f(x)$ is minimum at $x=1\cdot 184$.

Since 1 unit$=30$ days,

$\therefore$ 1·184 units$=30\times 1\cdot 184$ days $=35\cdot 52$ days.

Thus the date of minimum temperature is 16th December $+35\cdot 5$ days *i.e.* on the mid-night of 20th January.

Minimum temperature$=[f(x)]_{x=1\cdot 184}$
$$=(1\cdot 9)\ (1\cdot 184)^2-4\cdot 5\times 1\cdot 184+40\cdot 7$$
$$=38\cdot 036\ F \text{ approximately.}$$

Ex. 12. *Find a polynomial satisfied by* $(-4, 1245)$, $(-1, 33)$, $(0, 5)$, $(2, 9)$ *and* $(5, 1335)$.

Sol. The scheme of divided differences is as below :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-4	1245				
		-404			
-1	33		94		
		-28		-14	
0	5		10		3
		2		13	
2	9		38		
		442			
5	1335				

By Newton's divided difference formula

$$f(x)=1245-404\ (x+4)+94\ (x+4)\ (x+1)$$
$$-14\ (x+4)\ (x+1)\ x+3\ (x+4)\ (x+1)\ x\ (x-2)$$
$$=3x^4-5x^3+6x^2-14x+5.$$

Ex. 13. *Given* $\log_{10} 654=2\cdot 8156$, $\log_{10} 658=2\cdot 8182$, $\log_{10} 659=2\cdot 8189$, $\log_{10} 661=2\cdot 8202$; *find* $\log_{10} 656$. **(Agra 1984)**

Sol. For the values x_0, x_1, x_2, x_3, Lagrange's formula is

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3).$$

Here $x_0=654$, $x_1=658$, $x_2=659$, $x_3=661$, $x=656$.

Substituting these values in Lagrange's formula, we get

$$\log_{10} 656=\frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)}\times 2\cdot 8156+\frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)}\times 2\cdot 8182$$
$$+\frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)}\times 2\cdot 8189+\frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)}\times 2\cdot 8202.$$

$$\therefore \log_{10} 656=\frac{(-2)(-3)(-5)}{(-4)(-5)(-7)}\times 2\cdot 8156+\frac{(2)(-3)(-5)}{(4)(-1)(-3)}\times 2\cdot 8182+\frac{(2)(-2)(-5)}{(5)(1)(-2)}\times 2\cdot 8189+\frac{(2)(-2)(-3)}{(7)(3)(2)}\times 2\cdot 8202$$
$$=0\cdot 6033+7\cdot 0455-5\cdot 6378+0\cdot 8058$$
$$=2\cdot 8168.$$

Hence the estimated value of $\log_{10} 656=2\cdot 8168$.

Ex. 14. *The values of y and x are given as below:*

x:	5	6	9	11
y:	12	13	14	16

Find the value of y when $x=10$. **(Meerut B.Sc. 93, M.Sc. 83, 88)**

Sol. Applying Lagrange's formula for $x_0=5$, $x_1=6$, $x_2=9$, $x_3=11$ and $x=10$, we get

$$y_{10}=\frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}\times 12+\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}\times 13$$
$$+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}\times 14+\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}\times 16$$
$$=\frac{(4)(1)(-1)}{(-1)(-4)(-6)}\times 12+\frac{(5)(1)(-1)}{(1)(-3)(-5)}\times 13+\frac{(5)(4)(-1)}{(4)(3)(-2)}\times 14+\frac{(5)(4)(1)}{(6)(5)(2)}\times 16$$
$$=2-4\cdot 33+11\cdot 66+5\cdot 33=14\cdot 66.$$

Ex. 15. *For the following table find the form of the function* $f(x)$.

x:	0	1	2	5
$f(x)$:	2	3	12	147

(Meerut 1976, 89)

Sol. Here $x_0=0$, $x_1=1$, $x_2=2$, $x_3=5$.

By Lagrange's formula, we have

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3).$$

Substituting the values of x_0, x_1, x_2, x_3 in this, we get

$$f(x)=\frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}\times 2+\frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}\times 3$$
$$+\frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}\times 12+\frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}\times 147$$

or
$$\frac{f(x)}{x(x-1)(x-2)(x-5)}=-\frac{1}{5x}+\frac{3}{4(x-1)}-\frac{2}{x-2}+\frac{49}{20(x-5)}$$
$$=\frac{20x^3+20x^2-20x+40}{20x(x-1)(x-2)(x-5)}$$
$$\Rightarrow f(x)=x^3+x^2-x+2.$$

Ex. 16. *The following table is given:*

x:	0	1	2	3	4
$f(x)$:	3	6	11	18	27

What is the form of the function $f(x)$?

Sol. Proceeding as in the above example, we get

$$\frac{f(x)}{x(x-1)(x-2)(x-3)(x-4)}=\frac{3}{(-1)(-2)(-3)(-4)}\cdot\frac{1}{x}+\frac{6}{1(-1)(-2)(-3)}\cdot\frac{1}{(x-1)}$$
$$+\frac{11}{2\cdot 1(-1)(-2)}\cdot\frac{1}{(x-2)}+\frac{18}{3.2.1(-1)}\cdot\frac{1}{(x-3)}+\frac{27}{4.3.2.1}\cdot\frac{1}{x-4}$$
$$=\frac{1}{8x}-\frac{1}{(x-1)}+\frac{11}{4(x-2)}-\frac{3}{(x-3)}+\frac{9}{8(x-4)}$$

$$=\frac{x^2+2x+3}{x(x-1)(x-2)(x-3)(x-4)}.$$

$\therefore\ f(x)=x^2+2x+3.$

Ex. 17. *The mode of a certain frequency curve $y=f(x)$ is very near $x=9$ and the values of the frequency density $f(x)$ for $x=8{\cdot}9$, $9{\cdot}0$ and $9{\cdot}3$ are respectively equal to $0{\cdot}30$, $0{\cdot}35$ and $0{\cdot}25$. Calculate the approximate value of the mode.* **(Agra 1987; Roh. M.Sc. 90)**

Sol. We have

x :	8·9	9·0	9·3
$f(x)$:	0·30	0·35	0·25

First we shall find the form of the frequency density $f(x)$.

By Lagrange's formula, we get

$$f(x)=\frac{(x-9)(x-9{\cdot}3)}{(8{\cdot}9-9)(8{\cdot}9-9{\cdot}3)}\times 0{\cdot}30+\frac{(x-8{\cdot}9)(x-9{\cdot}3)}{(9-8{\cdot}9)(9-9{\cdot}3)}\times 0{\cdot}35+\frac{(x-8{\cdot}9)(x-9)}{(9{\cdot}3-8{\cdot}9)(9{\cdot}3-9)}\times 0{\cdot}25$$

$$=-\frac{25}{12}x^2+\frac{453{\cdot}5}{12}x-\frac{2052{\cdot}3}{12}.$$

This is the form of the frequency density.

Now $f(x)$ will be maximum for that value of x for which $f'(x)=0$ and $f''(x)$ is negative.

We have

$$f'(x)=0 \Rightarrow -\frac{50}{12}x+\frac{453{\cdot}5}{12}=0$$

$$\Rightarrow -\frac{5}{12}(10x-90{\cdot}7)=0$$

$$\Rightarrow x=9{\cdot}07.$$

Also, $f''(x)=-\dfrac{50}{12}=-\dfrac{25}{6}$ which is negative.

Hence the mode of the frequency curve $y=f(x)$ is $9{\cdot}07$.

Ex. 18. *Four equidistant values u_{-1}, u_0, u_1 and u_2 being given, a value is interpolated by Lagrange's formula. Show that it may be written in the form*

$$u_x=yu_0+xu_1+y\frac{(y^2-1)}{3!}\Delta^2 u_{-1}+\frac{x(x^2-1)}{3!}\Delta^2 u_0,$$

where $x+y=1$. **(Baroda B.Sc. 1972; Nagpur B.Sc. 72; Meerut 92P; Agra 87; Bangalore 88)**

Sol. We have

$$\Delta^2u_{-1}=(E-1)^2u_{-1}=(E^2-2E+1)\,u_{-1}$$
$$=u_1-2u_0+u_{-1}.$$

Similarly, $\Delta^2u_0=(E^2-2E+1)\,u_0$
$$=u_2-2u_1+u_0.$$

Now the R.H.S.

$$=yu_0+xu_1+\frac{y(y^2-1)}{3!}\Delta^2u_{-1}+\frac{x(x^2-1)}{3!}\Delta^2u_0$$

$$=(1-x)\,u_0+xu_1+\frac{(1-x)\,x\,(x-2)}{3!}(u_1-2u_0+u_{-1})+\frac{x(x+1)(x-1)}{3!}(u_2-2u_1+u_0)$$

$$=-\frac{x(x-1)(x-2)}{6}u_{-1}+\left[-(x-1)+\frac{x(x-1)(x-2)}{3}+\frac{(x-1)\,x\,(x+1)}{6}\right]u_0+\left[x-\frac{x(x-1)(x-2)}{6}-\frac{(x-1)\,x\,(x+1)}{3}\right]u_1+\frac{(x-1)\,x\,(x+1)}{6}u_2$$

$$=-\frac{x(x-1)(x-2)}{6}u_{-1}+\frac{(x-2)(x-1)(x+1)}{2}u_0-\frac{(x-2)\,x\,(x+1)}{2}u_1+\frac{(x-1)\,x\,(x+1)}{6}u_2. \quad ...(1)$$

For u_{-1}, u_0, u_1, u_2 as known values, by Lagrange's formula, we have

$$u_x=\frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)}u_{-1}+\frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)}u_0+\frac{(x+1)(x-0)(x-2)}{(1+1)(1-0)(1-2)}u_1+\frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)}u_2$$

$$=-\frac{x(x-1)(x-2)}{6}u_{-1}+\frac{(x+1)(x-1)(x-2)}{2}u_0-\frac{(x+1)\,x\,(x-2)}{2}u_1+\frac{(x+1)\,x\,(x-1)}{6}u_2. \quad ...(2)$$

From (1) and (2), we have

$$u_x=y\,u_0+x\,u_1+\frac{y(y^2-1)}{3!}\Delta^2u_{-1}+\frac{x(x^2-1)}{3!}\Delta^2u_0.$$

Ex. 19. *By means of Lagrange's formula, prove that $y_1=y_3-{\cdot}3\,(y_5-y_{-3})+{\cdot}2\,(y_{-3}-y_{-5})$, approximately.* **(Meerut 1978, 92)**

Sol. For given y_{-5}, y_{-3}, y_3, y_5 we have to obtain y_1.

By Lagrange's formula, we have

$$y_1=\frac{(1+3)\,(1-3)\,(1-5)}{(-5+3)(-5-3)(-5-5)}\,y_{-5}+\frac{(1+5)\,(1-3)\,(1-5)}{(-3+5)\,(-3-3)\,(-3-5)}\,y_{-3}$$
$$+\frac{(1+5)\,(1+3)\,(1-5)}{(3+5)\,(3+3)\,(3-5)}\,y_3+\frac{(1+5)\,(1+3)\,(1-3)}{(5+5)\,(5+3)\,(5-3)}\,y_5$$
$$=\frac{(4)\,(-2)\,(-4)}{(-2)\,(-8)\,(-10)}\,y_{-5}+\frac{(6)\,(-2)\,(-4)}{(2)\,(-6)\,(-8)}\,y_{-3}+\frac{(6)\,(4)\,(-4)}{(8)\,(6)\,(-2)}\,y_3+\frac{(6)\,(4)\,(-2)}{(10)\,(8)\,(2)}\,y_5$$
$$=-\frac{1}{5}\,y_{-5}+\frac{1}{2}\,y_{-3}+y_3-\frac{3}{10}\,y_5$$
$$=-\cdot 2y_{-5}+\cdot 5y_{-3}+y_3-\cdot 3y_5$$
$$=y_3-\cdot 3\,(y_5-y_{-3})+\cdot 2\,(y_{-3}-y_{-5}).$$

Ex. 20. *The following values of the function $f(x)$ for values of x are given*

$$f(1)=4,\ f(2)=5,\ f(7)=5,\ f(8)=4.$$

Find the value of $f(6)$ and also the value of x for which $f(x)$ is maximum or minimum. **(Meerut M.Sc. 1990)**

Sol. Using Lagrange's formula for the arguments 1, 2, 7 and 8, we get

$$f(x)=\frac{(x-2)\,(x-7)\,(x-8)}{(1-2)\,(1-7)\,(1-8)}\,f(1)+\frac{(x-1)\,(x-7)\,(x-8)}{(2-1)\,(2-7)\,(2-8)}\,f(2)$$
$$+\frac{(x-1)\,(x-2)\,(x-8)}{(7-1)\,(7-2)\,(7-8)}\,f(7)+\frac{(x-1)\,(x-2)\,(x-7)}{(8-1)\,(8-2)\,(8-7)}\,f(8)$$
$$=-\frac{4}{42}\,(x-2)\,(x-7)\,(x-8)+\frac{5}{30}\,(x-1)\,(x-7)\,(x-8)$$
$$-\frac{5}{30}\,(x-1)\,(x-2)\,(x-8)+\frac{4}{42}\,(x-1)\,(x-2)\,(x-7)$$
$$=\frac{4}{42}\,(x-2)\,(x-7)\left\{(x-1)-(x-8)\right\}$$
$$+\frac{5}{30}\,(x-1)\,(x-8)\left\{(x-7)-(x-2)\right\}$$
$$=\frac{2}{3}\,(x-2)\,(x-7)-\frac{5}{6}\,(x-1)\,(x-8)$$
$$=\frac{1}{6}\left[4\,(x^2-9x+14)-5\,(x^2-9x+8)\right]$$
$$=\frac{1}{6}\left[-x^2+9x+16\right].$$
$$\therefore\quad f(6)=\frac{1}{6}\left[-6^2+9\cdot 6+16\right]=\frac{1}{6}\left[-36+54+16\right]$$

$$=\frac{1}{6}\times 34=5\cdot 66.$$

Again for maximum or minimum of $f(x)$, we have $f'(x)=0$ *i.e.*, $-2x+9=0$ or $x=4\cdot 5$.

Since $f''(x)=\frac{1}{6}\times(-2)$ which is <0, therefore $f(x)$ is maximum at the point $x=4\cdot 5$.

Ex. 21. *The values of $f(x)$ are given at a, b and c. Show that the maximum or minimum is attained by*

$$x=\frac{f(a)\,(b^2-c^2)+f(b)\,(c^2-a^2)+f(c)\,(a^2-b^2)}{2\,\{f(a)\,(b-c)+f(b)\,(c-a)+f(c)\,(a-b)\}}.$$

(Nagpur B.Sc. 73, 76 ; Sri Venketeshwara 78)

Sol. By Lagrange's formula, for the arguments a, b, c, the function $f(x)$ is given by

$$f(x)=\frac{(x-b)\,(x-c)}{(a-b)\,(a-c)}\,f(a)+\frac{(x-a)\,(x-c)}{(b-a)\,(b-c)}\,f(b)+\frac{(x-a)\,(x-b)}{(c-a)\,(c-b)}\,f(c)$$
$$=\frac{x^2-(b+c)\,x+bc}{(a-b)\,(a-c)}\,f(a)+\frac{x^2-(a+c)\,x+ac}{(b-a)\,(b-c)}\,f(b)+\frac{x^2-(a+b)\,x+ab}{(c-a)\,(c-b)}\,f(c).$$

For maxima or minima, we have $f'(x)=0$, which gives

$$\frac{2x-(b+c)}{(a-b)\,(a-c)}\,f(a)+\frac{2x-(a+c)}{(b-a)\,(b-c)}\,f(b)+\frac{2x-(a+b)}{(c-a)\,(c-b)}\,f(c)=0$$
$$\Rightarrow 2x\,\{(b-c)\,f(a)+(c-a)\,f(b)+(a-b)\,f(c)\}$$
$$-\{(b^2-c^2)\,f(a)+(c^2-a^2)\,f(b)+(a^2-b^2)\,f(c)\}=0$$
$$\Rightarrow x=\frac{(b^2-c^2)\,f(a)+(c^2-a^2)\,f(b)+(a^2-b^2)\,f(c)}{2\,\{(b-c)\,f(a)+(c-a)\,f(b)+(a-b)\,f(c)\}}.$$

Ex. 22. *If all terms except y_5 of the sequence $y_1, y_2, \ldots, y_9$ be given, show that the value of y_5 is*

$$\frac{56\,(y_4+y_6)-28\,(y_3+y_7)+8\,(y_2+y_8)-(y_1+y_9)}{70}.$$

(Meerut 1978, 80)

Sol. Lagrange's formula is

$$f(x)=\frac{(x-x_1)\,(x-x_2)\ldots(x-x_n)}{(x_0-x_1)\,(x_0-x_2)\ldots(x_0-x_n)}\,f(x_0)+\frac{(x-x_0)\,(x-x_2)\ldots(x-x_n)}{(x_1-x_0)\,(x_1-x_2)\ldots(x_1-x_n)}\,f(x_1)$$
$$+\ldots+\frac{(x-x_0)\,(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)\,(x_n-x_1)\ldots(x_n-x_{n-1})}\,f(x_n)$$

or
$$\frac{f(x)}{(x-x_0)(x-x_1)\ldots(x-x_n)}=\frac{f(x_0)}{(x-x_0)(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}$$
$$+\frac{f(x_1)}{(x-x_1)(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}$$
$$+\ldots+\frac{f(x_n)}{(x-x_n)(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}$$

Putting $x_0=1,\ x_1=2,\ x_2=3,\ x_3=4,\ x_4=6,\ x_5=7,\ x_6=8,\ x_7=9$, $x=5,\ f(1)=y_1,\ f(2)=y_2,\ f(3)=y_3,\ f(4)=y_4,\ f(5)=y_5,\ f(6)=y_6,\ f(7)=y_7, f(8)=y_8, f(9)=y_9$, we get

$$\frac{y_5}{(5-1)(5-2)(5-3)(5-4)(5-6)(5-7)(5-8)(5-9)}$$
$$=\frac{y_1}{(5-1)(1-2)(1-3)(1-4)(1-6)(1-7)(1-8)(1-9)}$$
$$+\frac{y_2}{(5-2)(2-1)(2-3)(2-4)(2-6)(2-7)(2-8)(2-9)}$$
$$+\frac{y_3}{(5-3)(3-1)(3-2)(3-4)(3-6)(3-7)(3-8)(3-9)}$$
$$+\frac{y_4}{(5-4)(4-1)(4-2)(4-3)(4-6)(4-7)(4-8)(4-9)}$$
$$+\frac{y_6}{(5-6)(6-1)(6-2)(6-3)(6-4)(6-7)(6-8)(6-9)}$$
$$+\frac{y_7}{(5-7)(7-1)(7-2)(7-3)(7-4)(7-6)(7-8)(7-9)}$$
$$+\frac{y_8}{(5-8)(8-1)(8-2)(8-3)(8-4)(8-6)(8-7)(8-9)}$$
$$+\frac{y_9}{(5-9)(9-1)(9-2)(9-3)(9-4)(9-6)(9-7)(9-8)}$$

or
$$\frac{y_5}{1.2.3.4.1.2.3.4}=\frac{-y_1}{1.2.3.4.5.6.7.8}+\frac{y_2}{1.2.3.4.5.6.7}$$
$$-\frac{y_3}{2.1.2.3.4.5.6}+\frac{y_4}{6.1.2.3.4.5}+\frac{y_6}{6.1.2.3.4.5}$$
$$-\frac{y_7}{2.1.2.3.4.5.6}+\frac{y_8}{1.2.3.4.5.6.7}-\frac{y_9}{1.2.3.4.5.6.7.8}$$

or
$$\frac{y_5}{24}=-\frac{y_1}{1680}+\frac{y_2}{210}-\frac{y_3}{60}+\frac{y_4}{30}+\frac{y_6}{30}-\frac{y_7}{60}+\frac{y_8}{210}-\frac{y_9}{1680}$$
$$=\frac{1}{30}(y_4+y_6)-\frac{1}{60}(y_3+y_7)+\frac{1}{210}(y_2+y_8)$$
$$-\frac{1}{1680}(y_1+y_9)$$

or
$$y_5=\frac{4}{5}(y_4+y_6)-\frac{2}{5}(y_3+y_7)+\frac{8}{70}(y_2+y_8)-\frac{1}{70}(y_1+y_9)$$
$$=\frac{56(y_4+y_6)-28(y_3+y_7)+8(y_2+y_8)-(y_1+y_9)}{70}.$$

Ex. 23. *Prove that the Lagrange's formula can be put in the form*
$$P_n(x)=\sum_{r=0}^{n}\frac{\phi(x)\,f(x_r)}{(x-x_r)\,\phi'(x_r)}$$
where
$$\phi(x)=\prod_{r=0}^{n}(x-x_r).$$
(Agra 1982)

Sol. We can write Lagrange's formula as
$$P_n(x)=\sum_{r=0}^{n}\frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(x_r-x_0)(x_r-x_1)\ldots(x_r-x_n)}f(x_r)$$
$$=\sum_{r=0}^{n}\left[\frac{\phi(x)\,f(x_r)}{(x-x_r)}\right]\left[\frac{1}{(x_r-x_0)(x_r-x_1)\ldots(x_r-x_n)}\right].$$

Now $\phi(x)=\prod_{r=0}^{n}(x-x_r)$.

$$\therefore\quad \phi'(x_r)=[\phi'(x)]_{x=x_r}$$
$$=(x_r-x_0)(x_r-x_1)\ldots(x_r-x_{r-1})(x_r-x_{r+1})\ldots(x_r-x_n).$$
$$\therefore\quad P_n(x)=\sum_{r=0}^{n}\frac{\phi(x)\,f(x_r)}{(x-x_r)\,\phi'(x_r)}.$$

Ex. 24. *Values of y_x are given for all integral values of x from 0 to $n-1$. Show that y_x is capable of being expressed in the form :*
$$\frac{x\,!}{(x-n)\,!\,(n-1)\,!}\left[\frac{y_{n-1}}{x-n+1}-{}^{n-1}C_1\frac{y_{n-2}}{x-n+2}+{}^{n-1}C_2\frac{y_{n-3}}{x-n+3}\right.$$
$$\left.+\ldots\pm{}^{n-1}C_{n-1}\frac{y_0}{x}\right].$$
(Banglore 1988, Patna 86)

Sol. The values of the argument are $0, 1, 2, \ldots, n-1$ and the corresponding entries are $y_0, y_1, y_2, \ldots, y_{n-1}$. By Lagrange's formula, we get
$$y_x=\frac{(x-0)(x-1)(x-2)\ldots(x-\overline{n-2})}{(\overline{n-1}-0)(\overline{n-1}-1)(\overline{n-1}-2)\ldots(\overline{n-1}-\overline{n-2})}\,y_{n-1}$$
$$+\frac{(x-0)(x-1)\ldots(x-\overline{n-3})(x-\overline{n-1})}{(\overline{n-2}-0)(\overline{n-2}-1)\ldots(\overline{n-2}-\overline{n-3})(\overline{n-2}-\overline{n-1})}\,y_{n-2}+\ldots$$

$$+\frac{(x-0)(x-1)\ldots(x-\overline{n-4})(x-\overline{n-2})(x-\overline{n-1})}{(\overline{n-3}-0)(n-3-1)(\overline{n-3}-\overline{n-4})(\overline{n-3}-\overline{n-2})(\overline{n-3}-\overline{n-1})}\,y_{n-1}$$

$$+\ldots+\frac{(x-1)(x-2)\ldots(x-\overline{n-1})}{(0-1)(0-2)\ldots(0-\overline{n-1})}\,y_0$$

or
$$y_x=\frac{x(x-1)(x-2)\ldots(x-n+2)}{(n-1)!}\,y_{n-1}$$

$$+\frac{x(x-1)\ldots(x-n+3)(x-n+1)}{(n-2)!\,(-1)}\,y_{n-2}$$

$$+\frac{x(x-1)\quad(x-n+4)(x-n+2)(x-n+1)}{(n-3)!\,(-1)(-2)}\,y_{n-3}$$

$$+\ldots+\frac{(x-1)(x-2)\quad(x-n+1)}{(-1)(-2)\ldots\{-(n-1)\}}\,y_0$$

or
$$y_x=\frac{x(x-1)\ldots(x-n+2)(x-n+1)}{(n-1)!}\,\frac{y_{n-1}}{x-n+1}$$

$$+\frac{x(x-1)\ldots(x-n+3)(x-n+2)(x-n+1)}{(n-2)!\,(-1)}\,\frac{y_{n-2}}{x-n+2}$$

$$+\frac{x(x-1)\ldots(x-n+4)(x-n+3)(x-n+2)(x-n+1)}{(n-3)(-1)(-2)}$$

$$\times\frac{y_{n-3}}{x-n+3}+\ldots+\frac{x(x-1)\ldots(x-n+1)}{(-1)(-2)\ldots\{-(n-1)\}}\,\frac{y_0}{x}$$

$$=\frac{x!}{(x-n)!}\,\frac{1}{(n-1)!}\,\frac{y_{n-1}}{x-n+1}+(-1)\frac{x!}{(x-n)!\,(n-2)!}\cdot\frac{y_{n-2}}{x-n+2}$$

$$+(-1)^2\frac{x!}{(x-n)!}\,\frac{1}{(n-3)!}\cdot\frac{1}{2!}\,\frac{y_{n-3}}{x-n+3}+\ldots$$

$$+(-1)^{n-1}\frac{x!}{(x-n)!}\,\frac{1}{(n-1)!}\,\frac{y_0}{x}$$

$$=\frac{x!}{(x-n)!\,(n-1)!}\left[\frac{y_{n-1}}{x-n+1}+(-1)(n-1)\frac{y_{n-2}}{x-n+2}\right.$$

$$\left.+(-1)^2\frac{(n-1)(n-2)}{2!}\,\frac{y_{n-3}}{x-n+3}+\ldots+(-1)^{n-1}\frac{y_0}{x}\right]$$

$$=\frac{x!}{(x-n)!\,(n-1)!}\left[\frac{y_{n-1}}{x-n+1}+(-1)^{n-1}C_1\frac{y_{n-2}}{x-n+2}\right.$$

$$\left.+(-1)^2\,{}^{n-1}C_2\,\frac{y_{n-3}}{x-n+3}+\ldots+(-1)^{n-1}\,{}^{n-1}C_{n-1}\,\frac{y_0}{x}\right].$$

Ex. 25. *Find the polynomial of fifth degree from the following data* $u_0=-18$, $u_1=0$, $u_3=0$, $u_5=-248$, $u_6=0$, $u_9=13104$.

(Meerut M.Sc. 1984, 86)

Sol. From the given data we see that the value of the function is zero at $x=1, 3, 6$. Let the function $f(x)$ be a polynomial in x. Then $(x-1)$, $(x-3)$ and $(x-6)$ will be factors of $f(x)$.

Let $\qquad f(x)=(x-1)(x-3)(x-6)\,P(x). \qquad \ldots(1)$

Since we are given 6 values, so we can fit a polynomial of fifth degree. As such $P(x)$ will be a polynomial of second degree in x.

From (1), $P(x)=\dfrac{f(x)}{(x-1)(x-3)(x-6)}$.

$$\therefore\quad P(0)=\frac{f(0)}{(0-1)(0-3)(0-6)}=\frac{-18}{-18}=1.$$

$$P(5)=\frac{f(5)}{(5-1)(5-3)(5-6)}=\frac{-248}{-8}=31,$$

$$\text{and } P(9)=\frac{f(9)}{(9-1)(9-3)(9-6)}=\frac{13104}{144}=91.$$

Now we shall find $P(x)$. The divided difference table for $P(x)$ is as follows :

x	P	$\Delta P(x)$	$\Delta^2 P(x)$
0	1		
		6	
5	31		1
		15	
9	91		

By Newton's divided difference formula, we get

$$P(x)=P(0)+(x-0)\,\Delta P(0)+(x-0)(x-5)\,\Delta^2 P(0)$$
$$=1+x.6+x(x-5).1=x^2+x+1.$$

Thus $\qquad f(x)=(x-1)(x-3)(x-6)(x^2+x+1).$

Ex. 26. *If* $y_0, y_1, y_2, \ldots, y_6$ *are the consecutive terms of a series, then using Lagrange's formula prove that*

$$y_3=0{\cdot}05\,(y_0+y_6)-0{\cdot}3\,(y_1+y_5)+0{\cdot}75\,(y_2+y_4).$$

(Meerut 1976)

Sol. Lagrange's interpolation formula is

$$f(x)=\frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}\,f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}\,f(x_1)$$

$$+\ldots+\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}\,f(x_n)$$

or
$$\frac{f(x)}{(x-x_0)(x-x_1)\ldots(x-x_n)}$$

$$= \frac{1}{(x-x_0)(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} f(x_0)$$
$$+ \frac{1}{(x-x_1)(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} f(x_1)+\ldots$$
$$+ \frac{1}{(x-x_n)(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} f(x_n).$$

Putting $x=3$, $f(x)=y_3$; $x_0=0, f(x_0)=y_0$; $x_1=1$, $f(x_1)=y_1$; $x_2=2, f(x_2)=y_2$; $x_3=4, f(x_3)=y_4$; $x_4=5, f(x_4)=y_5$; and $x_5=6$, $f(x_5)=y_6$, we get

$$\frac{y_3}{(3-0)(3-1)(3-2)(3-4)(3-5)(3-6)}$$
$$=\frac{y_0}{(3-0)(-1)(-2)(-4)(-5)(-6)}$$
$$+\frac{y_1}{(3-1)(1)(-1)(-3)(-4)(-5)}$$
$$+\frac{y_2}{(3-2)(2)(1)(-2)(-3)(-4)}+\frac{y_4}{(3-4)(4)(3)(2)(-1)(-2)}$$
$$+\frac{y_5}{(3-5)(5)(4)(3)(1)(-1)}+\frac{y_6}{(3-6)(6)(5)(4)(2)(1)}$$

or $$-\frac{y_3}{3.2.1.1.2.3}=-\frac{y_0}{1.2.3.4\,5.6}+\frac{y_1}{2\,3.4.5}-\frac{y_2}{1.2.1.2.3.4}$$
$$-\frac{y_4}{2.1.2.3.4}+\frac{y_5}{1.2.3.4.5}-\frac{y_6}{1.2.3.4.5.6}$$
$$=-\frac{y_0+y_6}{1.2.3.4.5.6}+\frac{y_1+y_5}{1.2.3.4.5}-\frac{y_2+y_4}{2.1.2.3.4}$$

or $$y_3=\frac{6\,(y_0+y_6)}{4.5.6}-\frac{6\,(y_1+y_5)}{4.5}+\frac{6\,(y_2+y_4)}{2.4}$$
$$=0.05\,(y_0+y_6)-0.3\,(y_1+y_5)+0.75\,(y_2+y_4).$$

Ex. 27. *If the data are u_0, u_3, u_4, u_7, u_{11} and the interpolation formula is*

$$u_x=u_4+c_1 \underset{3}{\Delta} u_4+c_2 \underset{4,\,7}{\Delta^2} u_3+c_3 \underset{3,\,4,\,7}{\Delta^3} u_0+c_4 \underset{3,\,4,\,7,\,11}{\Delta^4} u_0,$$

find the values of c_1, c_2, c_3 and c_4. **(Agra 1986)**

Sol. Newton's divided difference formula is

$$u_x=u\,(x_0)+(x-x_0)\,u\,(x_0,x_1)+(x-x_0)\,(x-x_1)\,u\,(x_0,x_1,x_2)$$
$$+(x-x_0)\,(x-x_1)\,(x-x_2)\,u\,(x_0,x_1,x_2,x_3)$$
$$+(x-x_0)\,(x-x_1)\,(x-x_2)\,(x-x_3)\,u\,(x_0,x_1,x_2,x_3,x_4).$$

$$\therefore\quad u_x=u_4+(x-4)\underset{3}{\Delta} u_4+(x-4)\,(x-3)\underset{4,\,7}{\Delta^2} u_3$$
$$+(x-7)\,(x-4)\,(x-3)\underset{3,\,4,\,7}{\Delta^3} u_0$$
$$+(x-0)\,(x-7)\,(x-4)\,(x-3)\underset{3,\,4,\,7,\,11}{\Delta^4} u_0$$

Comparing it with the given result, we get
$c_1=x-4$, $c_2=(x-4)\,(x-3)$, $c_3=(x-7)\,(x-4)\,(x-3)$,
$c_4=x\,(x-7)\,(x-4)\,(x-3)$.

Exercises 3

1. Given that $f(0)=8$, $f(1)=68$ and $f(5)=123$, construct a divided difference table. Using the table determine the value of $f(2)$. **(Meerut B.Sc. 1992)**

2. If $\log 2=0{\cdot}30103$, $\log 3=0{\cdot}47712$, $\log 5=0{\cdot}69897$, $\log 7=0{\cdot}84510$, find the value of log 4·7 to four places of decimals.

3. Show that nth order divided difference $f(x_0, x_1, \ldots, x_n)$ for the function $u_x=\dfrac{1}{x}$ is equal to $\dfrac{(-1)^n}{x_0\,x_1\,x_2\ldots x_n}$. **(Meerut M.Sc. 1986)**

4. Given the following data, find $f(x)$ as a polynomial in powers of $(x-5)$.

x :	0	2	3	4	7	9
$f(x)$:	4	26	58	112	466	922

5. From the following table, find $f(x)$ in powers of $(x-3)$.

x :	5	11	27	34	42
$f(x)$:	23	899	17315	35606	68510

6. Deduce Lagrange's formula for interpolation. The observed values of a function are respectively 168, 120, 72 and 63 at the four positions 3, 7, 9 and 10 of the independent variable. What is the best estimate you can give for the value of the function at the position 6 of the independent variable ?

7. The following table gives the normal weights of babies during the first 12 months of life :

Age in months :	0	2	5	8	10	12
Weight in lbs :	$7\frac{1}{2}$	$10\frac{1}{4}$	15	16	18	21

Estimate the weight of the baby at the age of 7 months.

8. Determine by Lagrange's formula the percentage number of criminals under 35 years :

Age	% numbers of criminals
Under 25 years	52
Under 30 years	67·3
Under 40 years	84·1
Under 50 years	94·4

9. Find the form of the function $f(x)$ for the following table

x :	0	1	4	5
$f(x)$:	8	11	68	123

10. Using Lagrange's formula find the form of the function given by

x :	3	2	1	-1
$f(x)$:	3	12	15	-21

(Meerut 1993)

4

Central Difference Interpolation Formulae

§ 1. Introduction. Earlier we have obtained Newton's and Lagrange's interpolation formulae. Lagrange's formula is not easy to apply. There is a great deal of computational work in its application and it does not give accurate results if the values of the independent variable are quite apart. Newton's formulae are fundamental and are applicable to almost all cases of interpolation, but in general they do not converge as rapidly as central difference formulae.

In particular the central difference formulae are used for interpolating values of the function near the middle of a tabulated set. Before we derive various central difference formulae, we introduce some operators besides Δ, E and ∇.

The operator δ, called **the central difference operator**, is defined by the operator equation

$$\delta = E^{1/2} - E^{-1/2}.$$

The first central difference $\delta f(x)$ of $f(x)$ is given by

$$\delta f(x) = f(x+\tfrac{1}{2}h) - f(x-\tfrac{1}{2}h)$$
$$= E^{1/2} f(x) - E^{-1/2} f(x), \quad \text{where } E \equiv 1+\Delta$$
$$= (E^{1/2} - E^{-1/2}) f(x).$$

Thus $\delta = E^{1/2} - E^{-1/2} = E^{-1/2}(E-1) = E^{-1/2}\,\Delta.$ ...(1)

The relation connecting E and ∇ is $\nabla \equiv 1 - E^{-1}$.

$\therefore\ \delta = E^{1/2} - E^{-1/2} \Rightarrow \delta = E^{1/2}(1-E^{-1}) = E^{1/2}\,\nabla.$ ...(2)

From (1) and (2), we get

$$\delta = E^{-1/2}\,\Delta = E^{1/2}\,\nabla. \quad \text{(Meerut 88, 91)} \ldots(3)$$

The operator μ, called **the mean or average operator**, is defined by the operator equation

$$\mu = \tfrac{1}{2}(E^{1/2} + E^{-1/2}).$$

$\therefore\ \mu f(x) = \tfrac{1}{2}(E^{1/2} + E^{-1/2}) f(x).$

Also $\mu^2 f(x) = \tfrac{1}{4}(E^{1/2} + E^{-1/2})^2 f(x)$

$= \frac{1}{4}[(E^{1/2} - E^{-1/2})^2 + 4] f(x) = \frac{1}{4}(\delta^2+4) f(x) = \left(1+\frac{\delta^2}{4}\right) f(x).$

$$\therefore \qquad \mu^2 \equiv 1 + \frac{\delta^2}{4} \qquad \text{(Meerut 92)} \qquad \ldots(4)$$

Let y_x be a function of x. Then

$$\delta y_x = (E^{1/2} - E^{-1/2})\, y_x = y_{x+(1/2)} - y_{x-(1/2)}$$

and $$\mu y_x = \tfrac{1}{2}(E^{1/2} + E^{-1/2})\, y_x = \tfrac{1}{2}(y_{x+(1/2)} + y_{x-(1/2)}).$$

$$\therefore\ 2\mu y_x + \delta y_x = 2 y_{x+(1/2)} \Rightarrow (2\mu + \delta)\, y_x = 2E^{1/2} y_x$$

$$\Rightarrow 2\mu + \delta \equiv 2E^{1/2} \Rightarrow \mu + \tfrac{1}{2}\delta \equiv E^{1/2}. \qquad \ldots(5)$$

(Rohilkhand 1986; Meerut 87, 90, 92P)

We have $E \equiv e^{hD} = e^U$ where $U = hD$.

$$\therefore\ \delta = E^{1/2} - E^{-1/2} = e^{U/2} - e^{-U/2} = 2 \sinh \frac{U}{2}$$

and $$\mu = \tfrac{1}{2}(E^{1/2} + E^{-1/2}) = \frac{e^{U/2} + e^{-U/2}}{2} = \cosh \frac{U}{2} \qquad \ldots(6)$$

The operator σ, called **the central sum operator** is defined by

$$\sigma f(x) = \sigma f(x-h) + f(x - \tfrac{1}{2}h)$$

or $$(\sigma - \sigma E^{-1}) f(x) = E^{-1/2} f(x)$$

or $$\sigma - \sigma E^{-1} \equiv E^{-1/2} \quad \text{or} \quad \sigma(1 - E^{-1}) \equiv E^{-1/2};$$

or $$\sigma \equiv \frac{E^{-1/2}}{1 - E^{-1}} \equiv \frac{E^{1/2}}{E - 1} \qquad \ldots(7)$$

(Meerut M.Sc. 1986, 90)

The operator σ is inverse to the operator δ.

§ 2. Gauss's Central Difference Formulae.

(a) Gauss's forward formula. The general Newton formula is

$$y = f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$
$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4)$$
$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4) f(x_0, x_1, x_2, x_3, x_4, x_5)$$
$$+ (x - x_0)(x - x_1) \ldots (x - x_5) f(x_0, x_1, x_2, x_3, x_4, x_5, x_6) + \ldots \qquad \ldots(1)$$

Putting $x_0 = x_0$, $x_1 = x_0 + h$, $x_2 = x_0 - h$, $x_3 = x_0 + 2h$, $x_4 = x_0 - 2h$, $x_5 = x_0 + 3h$, $x_6 = x_0 - 3h$ etc., we get

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_0 + h)$$
$$+ (x - x_0)(x - x_0 - h) f(x_0, x_0 + h, x_0 - h)$$
$$+ (x - x_0)(x - x_0 - h)(x - x_0 + h) f(x_0, x_0 + h, x_0 - h, x_0 + 2h)$$
$$+ (x - x_0)(x - x_0 - h)(x - x_0 + h)(x - x_0 - 2h)$$
$$\times f(x_0, x_0 + h, x_0 - h, x_0 + 2h, x_0 - 2h)$$
$$+ (x - x_0)(x - x_0 - h)(x - x_0 + h)(x - x_0 - 2h)(x - x_0 + 2h)$$
$$\times f(x_0, x_0 + h, x_0 - h, x_0 + 2h, x_0 - 2h, x_0 + 3h) + \ldots$$

Now put $u = \dfrac{x - x_0}{h}$ *i.e.* $x - x_0 = hu$. Then, we have

$y_n(x) = y_0 + P\Delta y_0 + \dfrac{P(P-1)}{2!}\Delta^2 y_0 + \dfrac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \ldots$

$$f(x) = f(x_0) + hu\, f(x_0, x_0 + h) + hu\,(hu - h)\, f(x_0 - h, x_0, x_0 + h)$$
$$+ hu\,(hu - h)(hu + h)\, f(x_0 - h, x_0, x_0 + h, x_0 + 2h)$$
$$+ hu\,(hu - h)(hu + h)(hu - 2h)\, f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h)$$
$$+ hu\,(hu - h)(hu + h)(hu - 2h)(hu + 2h)$$
$$\times f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h, x_0 + 3h) + \ldots \qquad \ldots(2)$$

But, we have

$$f(x_0, x_0 + h) = \frac{\Delta y_0}{h}, \quad f(x_0 - h, x_0, x_0 + h) = \frac{\Delta^2 y_{-1}}{2h^2},$$

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$$f(x_0 - h, x_0, x_0 + h, x_0 + 2h) = \frac{\Delta^3 y_{-1}}{3!\, h^3},$$

$$f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h) = \frac{\Delta^4 y_{-2}}{4!\, h^4},$$

$$f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h, x_0 + 3h) = \frac{\Delta^5 y_{-2}}{5!\, h^5}, \text{ etc.}$$

Substituting these values in (2), we get

$$y = y_0 + hu \frac{\Delta y_0}{h} + h^2 u(u-1) \frac{\Delta^2 y_{-1}}{2h^2} + h^3 u(u-1)(u+1) \frac{\Delta^3 y_{-1}}{3!\, h^3}$$
$$+ h^4 u(u-1)(u+1)(u-2) \frac{\Delta^4 y_{-2}}{4!\, h^4}$$
$$+ h^5 u(u-1)(u+1)(u-2)(u+2) \frac{\Delta^5 y_{-2}}{5!\, h^5} + \ldots$$

or $$y = y_0 + u\,\Delta y_0 + u(u-1)\frac{\Delta^2 y_{-1}}{2!} + u(u^2 - 1)\frac{\Delta^3 y_{-1}}{3!}$$
$$+ u(u^2 - 1)(u - 2)\frac{\Delta^4 y_{-2}}{4!} + u(u^2 - 1)(u^2 - 2^2)\frac{\Delta^5 y_{-2}}{5!} + \ldots \qquad \ldots(3)$$

This result is known as **Gauss's forward formula** for equal **intervals.**

(b) Gauss's backward formula. Substituting $x_0 = x_0$, $x_1 = x_0 - h$, $x_2 = x_0 + h$, $x_3 = x_0 - 2h$, $x_4 = x_0 + 2h$, $x_5 = x_0 - 3h$, $x_6 = x_0 + 3h$ etc. in the general Newton formula (1), we get

$$y = y_0 + (x - x_0) f(x_0, x_0 - h) + (x - x_0)(x - x_0 + h) f(x_0, x_0 - h, x_0 + h)$$
$$+ (x - x_0)(x - x_0 + h)(x - x_0 - h) f(x_0, x_0 - h, x_0 + h, x_0 - 2h)$$
$$+ (x - x_0)(x - x_0 + h)(x - x_0 - h)(x - x_0 + 2h) f(x_0, x_0 - h,$$
$$x_0 + h, x_0 - 2h, x_0 + 2h)$$
$$+ (x - x_0)(x - x_0 + h)(x - x_0 - h)(x - x_0 + 2h)(x - x_0 - 2h)$$
$$\times f(x_0, x_0 - h, x_0 + h, x_0 - 2h, x_0 + 2h, x_0 - 3h) + \ldots.$$

Now put $u = \dfrac{x - x_0}{h}$ *i.e.* $x - x_0 = uh$. Then we get

$$y = y_0 + hu\, f(x_0 - h, x_0) + hu\,(hu+h)\, f(x_0 - h, x_0, x_0 + h)$$
$$+ hu\,(hu+h)(hu-h)\, f(x_0 - 2h, x_0 - h, x_0, x_0 + h)$$
$$+ hu\,(hu+h)(hu-h)(hu+2h)\, f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h)$$
$$+ hu\,(hu+h)(hu-h)(hu+2h)(hu-2h)$$
$$\times f(x_0 - 3h, x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h) + \ldots \qquad \ldots(4)$$

But, we know that

$$f(x_0 - h, x_0) = \frac{\Delta y_{-1}}{h},\ f(x_0 - h, x_0, x_0 + h) = \frac{\Delta^2 y_{-1}}{2h^2},$$

$$f(x_0 - 2h, x_0 - h, x_0, x_0 + h) = \frac{\Delta^3 y_{-2}}{3!\, h^3},$$

$$f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h) = \frac{\Delta^4 y_{-2}}{4!\, h^4},$$

$$f(x_0 - 3h, x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h) = \frac{\Delta^5 y_{-3}}{5!\, h^5}, \text{ etc.}$$

Substituting these in (4), we get

$$y = y_0 + u\, \Delta y_{-1} + u(u+1) \frac{\Delta^2 y_{-1}}{2!} + u(u^2-1) \frac{\Delta^3 y_{-2}}{3!}$$
$$+ u(u^2-1)(u+2) \frac{\Delta^4 y_{-2}}{4!} + u(u^2-1)(u^2-2^2) \frac{\Delta^5 y_{-3}}{5!} + \ldots \qquad \ldots(5)$$

This result is known as **Gauss's backward formula.**

(c) Third formula due to Gauss. It is a formula which starts with y_1 and runs parallel to the backward formula (5).

We have $u = \dfrac{x - x_0}{h}$ $\therefore$ $\dfrac{x - x_k}{h} = \dfrac{x - (x_0 + kh)}{h} = u - k.$

To derive this third formula we increase the subscripts of x and y in (5) by one unit and change the u's by putting $k = 1$ in the formula $u - k = \dfrac{x - x_k}{h}$. We get $x - x_1 = hu - h$ *i.e.* we have to replace u by $u - 1$. These changes reduce (5) to

$$y = y_1 + (u-1)\, \Delta y_0 + u(u-1) \frac{\Delta^2 y_0}{2!} + u(u-1)(u-2) \frac{\Delta^3 y_{-1}}{3!}$$
$$+ u(u^2-1)(u-2) \frac{\Delta^4 y_{-1}}{4!} + u(u^2-1)(u-2)(u-3) \frac{\Delta^5 y_{-2}}{5!} + \ldots \qquad \ldots(6)$$

This result is known as **third formula due to Gauss.**

In the following difference table the paths of the three Gauss formulae have been shown. The designations A, B, C are corresponding to the formulae (3), (5) and (6) respectively.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$	
x_{-6}	y_{-6}							
		Δy_{-6}						
x_{-5}	y_{-5}		$\Delta^2 y_{-6}$					
		Δy_{-5}		$\Delta^3 y_{-6}$				
x_{-4}	y_{-4}		$\Delta^2 y_{-5}$		$\Delta^4 y_{-6}$			
		Δy_{-4}		$\Delta^3 y_{-5}$		$\Delta^5 y_{-6}$		
x_{-3}	y_{-3}		$\Delta^2 y_{-4}$		$\Delta^4 y_{-5}$		$\Delta^6 y_{-6}$	
		Δy_{-3}		$\Delta^3 y_{-4}$		$\Delta^5 y_{-5}$		
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4}$		$\Delta^6 y_{-5}$	
		Δy_{-2}		$\Delta^3 y_{-3}$		$\Delta^5 y_{-4}$		
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		$\Delta^6 y_{-4}$	
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$		A
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$	
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$		B
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$	
		Δy_1		$\Delta^3 y_0$		$\Delta^5 y_{-1}$		C
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		$\Delta^6 y_{-1}$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$		
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$	
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$		
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$			
		Δy_4		$\Delta^3 y_3$				
x_5	y_5		$\Delta^2 y_4$					
		Δy_5						
x_6	y_6							

§ 3. Stirling's Interpolation Formula. (Meerut M.Sc. 1990)

Taking the mean of Gauss's forward formula and Gauss's backward formula, we get

$$y = y_0 + u \frac{(\Delta y_{-1} + \Delta y_0)}{2} + \frac{u^2}{2} \Delta^2 y_{-1} + \frac{u(u^2-1)}{3!} \frac{(\Delta^3 y_{-2} + \Delta^3 y_{-1})}{2}$$
$$+ \frac{u^2(u^2-1)}{4!} \Delta^4 y_{-2} + \frac{u(u^2-1)(u^2-2^2)}{5!} \frac{(\Delta^5 y_{-3} + \Delta^5 y_{-2})}{2} + \ldots \qquad \ldots(7)$$

This is Stirling's formula. The more general form of this formula is

$$y=y_0+u\frac{(\Delta y_{-1}+\Delta y_0)}{2}+\frac{u^2}{2!}\Delta^2 y_{-1}+\frac{u(u^2-1^2)}{3!}\cdot\frac{(\Delta^3 y_{-2}+\Delta^3 y_{-1})}{2}+\frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2}+\dots$$

$$+\frac{u(u^2-1^2)(u^2-2^2)\dots[u^2-(k-1)^2]}{(2k-1)!}\frac{\Delta^{2k-1}y_{-k+1}+\Delta^{2k-1}y_{-k}}{2}+\frac{u^2(u^2-1^2)(u^2-2^2)\dots[u^2-(k-1)^2]}{(2k)!}\Delta^{2k}y_{-k}.$$

The quantities that occur in Stirling's formula are shown in the difference table given below:

$$\begin{array}{ccccccccc}
 & \Delta y_{-1} & & \Delta^3 y_{-2} & & \Delta^5 y_{-3} & & \Delta^7 y_{-4} & \\
y_0 & & \Delta^2 y_{-1} & & \Delta^4 y_{-2} & & \Delta^6 y_{-3} & & \Delta^8 y_{-4} \\
 & \Delta y_0 & & \Delta^3 y_{-1} & & \Delta^5 y_{-2} & & \Delta^7 y_{-3} &
\end{array}$$

§ 4. Bessel's Interpolation Formula. Taking the mean of Gauss's forward formula and the third formula due to Gauss, we get

$$y=\frac{y_0+y_1}{2}+\left(u-\frac{1}{2}\right)\Delta y_0+\frac{u(u-1)}{2}\frac{(\Delta^2 y_{-1}+\Delta^2 y_0)}{2}$$

$$+\frac{u(u-\frac{1}{2})(u-1)}{3!}\Delta^3 y_{-1}+\frac{u(u^2-1)(u-2)}{4!}\frac{(\Delta^4 y_{-2}+\Delta^4 y_{-1})}{2}$$

$$+\frac{u(u-\frac{1}{2})(u^2-1)(u-2)}{5!}\Delta^5 y_{-2}+\dots \quad \dots(8)$$

This is Bessel's formula. It can be written in a slightly different form by transforming the first two terms to $y_0+u\Delta y_0$ because $\Delta y_0=y_1-y_0$. Then (8) becomes

$$y=y_0+u\Delta y_0+\frac{u(u-1)}{2}\frac{(\Delta^2 y_{-1}+\Delta^2 y_0)}{2}+\frac{u(u-\frac{1}{2})(u-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{u(u^2-1)(u-2)}{4!}\frac{(\Delta^4 y_{-2}+\Delta^4 y_{-1})}{2}+\dots \quad \dots(9)$$

The more general form of Bessel's formula is

$$y=y_0+u\Delta y_0+\frac{u(u-1)}{2}\frac{(\Delta^2 y_{-1}+\Delta^2 y_0)}{2}+\frac{(u-\frac{1}{2})u(u-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{(u+1)u(u-1)(u-2)}{4!}\frac{(\Delta^4 y_{-1}+\Delta^4 y_{-2})}{2}$$

$$+\frac{u(u-\frac{1}{2})(u^2-1)(u-2)}{5!}\Delta^5 y_{-2}$$

$$+\frac{u(u^2-1)(u^2-4)(u-3)}{6!}\frac{(\Delta^6 y_{-3}+\Delta^6 y_{-2})}{2}+\dots$$

$$+\frac{u(u^2-1)(u^2-4)\dots(u-n)(u+n-1)}{(2n)!}\frac{(\Delta^{2n}y_{-n}+\Delta^{2n}y_{-n+1})}{2}$$

$$+\frac{u(u-\frac{1}{2})(u^2-1)(u^2-4)\dots(u-n)(u+n-1)}{(2n+1)!}\Delta^{2n+1}y_{-n}. \quad \dots(10)$$

Putting $u=\frac{1}{2}$ in (10), we get

$$y=\frac{y_0+y_1}{2}-\frac{1}{8}\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}+\frac{3}{128}\frac{\Delta^4 y_{-2}+\Delta^4 y_{-1}}{2}$$

$$-\frac{5}{1024}\frac{\Delta^6 y_{-3}+\Delta^6 y_{-2}}{2}+\dots$$

$$+(-1)^n\frac{[1.3.5\dots(2n-1)]^2}{2^{2n}(2n)!}\frac{\Delta^{2n}y_{-n}+\Delta^{2n}y_{-n+1}}{2} \quad \dots(11)$$

This is the special case of Bessel's formula. It is called the formula for interpolating to halves. It is used for computing values of the function mid-way between any two given values.

Putting $u-\frac{1}{2}=v$ or $u=v+\frac{1}{2}$ in (10), we get a more symmetrical and convenient form of Bessel's formula:

$$y=\frac{y_0+y_1}{2}+v\Delta y_0+\frac{(v^2-\frac{1}{4})}{2}\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}+\frac{v(v^2-\frac{1}{4})}{3!}\Delta^3 y_{-1}$$

$$+\frac{(v^2-\frac{1}{4})(v^2-9/4)}{4!}\frac{\Delta^4 y_{-2}+\Delta^4 y_{-1}}{2}+\frac{v(v^2-\frac{1}{4})(v^2-9/4)}{5!}\Delta^5 y_{-2}$$

$$+\dots+\frac{(v^2-\frac{1}{4})(v^2-9/4)\dots\{v^2-(2n-1)^2/4\}}{(2n)!}\times\frac{\Delta^{2n}y_{-n}+\Delta^{2n}y_{-n+1}}{2}$$

$$+\frac{v(v^2-\frac{1}{4})(v^2-9/4)\dots\{v^2-(2n-1)^2/4\}}{(2n+1)!}\Delta^{2n+1}y_{-n}.$$

(Meerut B.Sc. Stat. 90)

The following table shows the quantities which occur in this formula.

$$\begin{array}{ccccccccc}
y_0 & & \Delta^2 y_{-1} & & \Delta^4 y_{-2} & & \Delta^6 y_{-3} & & \Delta^8 y_{-4} \\
 & \Delta y_0 & & \Delta^3 y_{-1} & & \Delta^5 y_{-2} & & \Delta^7 y_{-3} & \\
y_1 & & \Delta^2 y_0 & & \Delta^4 y_{-1} & & \Delta^6 y_{-2} & & \Delta^8 y_{-3}
\end{array}$$

§ 5. Laplace-Everett Formula. The Gauss forward formula is

$$y_u=y_0+u\Delta y_0+u(u-1)\frac{\Delta^2 y_{-1}}{2!}+u(u^2-1)\frac{\Delta^3 y_{-1}}{3!}$$

$$+u(u^2-1)(u-2)\frac{\Delta^4 y_{-2}}{4!}+u(u^2-1)(u^2-2^2)\frac{\Delta^5 y_{-2}}{5!}+\ldots \qquad \ldots(12)$$

We have

$$\Delta^{2k+1}y_{-k}=\Delta^{2k}(\Delta y_{-k})=\Delta^{2k}(y_{-k+1}-y_{-k})=\Delta^{2k}y_{1-k}-\Delta^{2k}y_{-k} \qquad \ldots(13)$$

Putting the values of odd differences in (12) in terms of even differences by (13), we get the following expansion in factorial notation :

$$y_u=y_0+u(y_1-y_0)+\frac{u^{(2)}}{2!}\Delta^2y_{-1}+\frac{(u+1)^{(3)}}{3!}(\Delta^2y_0-\Delta^2y_{-1})$$
$$+\frac{(u+1)^{(4)}}{4!}\Delta^4y_{-2}+\ldots+\frac{(u+k-1)^{(2k)}}{(2k)!}\Delta^{2k}y_{-k}$$
$$+\frac{(u+k)^{(2k+1)}}{(2k+1)!}(\Delta^{2k}y_{1-k}-\Delta^{2k}y_{-k})+\ldots$$
$$=(1-u)\,y_0+uy_1+\left\{\frac{u^{(2)}}{2!}-\frac{(u+1)^{(3)}}{3!}\right\}\Delta^2y_{-1}+\frac{(u+1)^{(3)}}{3!}\Delta^2y_0$$
$$+\left\{\frac{(n+1)^{(4)}}{4!}-\frac{(u+2)^{(5)}}{5!}\right\}\Delta^4y_{-2}+\frac{(u+2)^{(5)}}{5!}\Delta^4y_{-1}+\ldots$$
$$+\left\{\frac{(u+k-1)^{(2k)}}{(2k)!}-\frac{(u+k)^{(2k+1)}}{(2k+1)!}\right\}\Delta^{2k}y_{-k}$$
$$+\frac{(u+k)^{(2k+1)}}{(2k+1)!}\Delta^{2k}y_{1-k}+\ldots \qquad \ldots(14)$$

Now

$$\frac{(u+k-1)^{(2k)}}{(2k)!}-\frac{(u+k)^{(2k+1)}}{(2k+1)!}=\frac{(u+k-1)^{(2k)}}{(2k+1)!}\{(2k+1)-(u+k)\}$$
$$=\frac{(k+1-u)\,(u+k-1)^{(2k)}}{(2k+1)!}=-\frac{(u-k-1)\,(u+k-1)^{(2k)}}{(2k+1)!}$$
$$=-\frac{(u+k-1)^{(2k+1)}}{(2k+1)!}=-\frac{(k-v)^{(2k+1)}}{(2k+1)!} \qquad [\text{Put } v=1-u]$$
$$=\frac{(v+k)^{(2k+1)}}{(2k+1)!}=\frac{v\,(v^2-1)\,(v^2-4)\ldots(v^2-k^2)}{(2k+1)!}.$$

Introducing this variable v, the equation (14) takes the form

$$y_u=vy_0+\frac{v\,(v^2-1)}{3!}\Delta^2y_{-1}+\frac{v\,(v^2-1)\,(v^2-4)}{5!}\Delta^4y_{-2}+\ldots$$
$$+\frac{v\,(v^2-1)\,(v^2-4)\ldots(v^2-k^2)}{(2k+1)!}\Delta^{2k}y_{-k}+\ldots$$
$$+uy_1+\frac{u\,(u^2-1)}{3!}\Delta^2y_0+\frac{u(u^2-1)\,(u^2-4)}{5!}\Delta^4y_{-1}+\ldots$$

$$+\frac{u\,(u^2-1)\,(u^2-4)\ldots(u^2-k^2)}{(2k+1)!}\Delta^{2k}y_{1-k}+\ldots \qquad \ldots(15)$$

This is the most common form of Laplace-Everett's formula. This is convenient especially while using tables in which only differences of even order are tabulated.

Solved Examples

Ex. 1. *Prove that*

(i) $\Delta\equiv\frac{1}{2}\delta^2+\delta\sqrt{\left(1+\frac{\delta^2}{4}\right)}$, **(Rohilkhand 1988; Meerut 92)**

(ii) $\delta^3\,y_{1/2}=y_2-3y_1+3y_0-y_{-1}$,

(iii) $\Delta\nabla\equiv\nabla\Delta\equiv\Delta-\nabla\equiv\delta^2$, **(Rohilkhand 1988; Meerut 92P)**

(iv) $\delta^n y_n\equiv\Delta^n y_{n-(n/2)}$.

Sol. (i) We have R.H.S. $=\frac{1}{2}\delta^2+\delta\sqrt{\left(1+\frac{\delta^2}{4}\right)}$

$$=\frac{(E^{1/2}-E^{-1/2})^2}{2}+(E^{1/2}-E^{-1/2})\sqrt{\left\{1+\frac{(E^{1/2}-E^{-1/2})^2}{4}\right\}}$$
$$[\because\ \delta\equiv E^{1/2}-E^{-1/2}]$$
$$=\frac{(E^{1/2}-E^{-1/2})^2}{2}+(E^{1/2}-E^{-1/2})\sqrt{\left\{\frac{4+(E^{1/2}-E^{-1/2})^2}{4}\right\}}$$
$$=\frac{(E+E^{-1}-2)}{2}+\frac{(E^{1/2}-E^{-1/2})\,(E^{1/2}+E^{-1/2})}{2}$$
$$=\frac{E+E^{-1}}{2}-1+\frac{(E-E^{-1})}{2}$$
$$=E-1=\Delta=\text{L.H.S.}$$

(ii) We have L.H.S. $=\delta^3y_{1/2}=(\Delta E^{-1/2})^3y_{1/2}$ $[\because\ \delta\equiv\Delta E^{-1/2}]$

$$=\Delta^3E^{-3/2}E^{1/2}y_0=\Delta^3E^{-1}y_0=(E-1)^3E^{-1}y_0 \qquad [\because\ \Delta\equiv E-1]$$
$$=(E^3-3E^2+3E-1)\,E^{-1}y_0=(E^2-3E+3-E^{-1})\,y_0$$
$$=y_2-3y_1+3y_0-y_{-1}=\text{R.H.S.}$$

(iii) We have $\Delta\nabla=(\delta E^{1/2})\,(\delta E^{-1/2})=\delta^2$,

$$\nabla\Delta=(\delta E^{-1/2})\,(\delta E^{1/2})=\delta^2,$$

and $\Delta-\nabla=\delta E^{1/2}-\delta E^{-1/2}=\delta\,(E^{1/2}-E^{-1/2})=\delta.\delta=\delta^2$.

Thus $\Delta\nabla\equiv\nabla\Delta\equiv\Delta-\nabla\equiv\delta^2$.

(iv) We have $\delta^ny_n=(\Delta E^{-1/2})^ny_n=(\Delta^nE^{-n/2})\,y_n$

$$=\Delta^n\,(E^{-n/2}y_n)=\Delta^n\,y_{n-(n/2)}.$$

Ex. 2. *Use Gauss's forward formula to find the value of y when $x=3{\cdot}75$, given the following table :*

x :	2·5	3·0	3·5	4·0	4·5	5·0
y_x :	24·145	22·043	20·225	18·644	17·262	16·047

Sol. Let us take 3·5 as the origin and ·5 as the unit.

We know $u=\dfrac{x-x_0}{h}$. Here $x=3{\cdot}75$, $x_0=3{\cdot}5$, $h={\cdot}5$.

$$\therefore \quad u=\frac{3{\cdot}75-3{\cdot}5}{{\cdot}5}={\cdot}5.$$

We require the value of y for $u={\cdot}5$.

The difference table is given below :

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$
2·5	−2	24·145					
			−2·102				
3·0	−1	22·043		·284			
			−1·818		−·047		
3·5	0	20·225		·237		·009	
			−1·581		−·038		−·003
4·0	1	18·644		·199		·006	
			−1·382		−·032		
4·5	2	17·262		·167			
			−1·215				
5·0	3	16·047					

Gauss's forward formula is

$$y_u=y_0+u\,\Delta y_0+\frac{u\,(u-1)}{2}\Delta^2 y_{-1}+\frac{(u+1)\,u\,(u-1)}{6}\Delta^3 y_{-1}$$
$$+\frac{(u+1)\,u\,(u-1)\,(u-2)}{24}\Delta^4 y_{-2}$$
$$+\frac{(u+2)\,(u+1)\,u\,(u-1)\,(u-2)}{120}\Delta^5 y_{-2}+\cdots$$

$$\therefore \quad y_{\cdot 5}=20{\cdot}225+{\cdot}5\;(-1{\cdot}581)+\frac{({\cdot}5)\,({\cdot}5-1)}{2}\times{\cdot}237$$
$$+\frac{({\cdot}5+1)\,({\cdot}5)\,({\cdot}5-1)}{6}\,(-{\cdot}038)+\frac{({\cdot}5+1)\,({\cdot}5)\,({\cdot}5-1)\,({\cdot}5-2)}{24}\times({\cdot}009)$$
$$+\frac{({\cdot}5+2)\,({\cdot}5+1)\,({\cdot}5)\,({\cdot}5-1)\,({\cdot}5-2)}{120}\times(-{\cdot}003)$$
$$=20{\cdot}225-0{\cdot}7905-0{\cdot}029625+{\cdot}002375+{\cdot}00210938-{\cdot}0000352$$
$$=19{\cdot}40 \text{ approx.}$$

Hence the value of y for $x=3{\cdot}75$ is 19·40 approximately.

Ex. 3. *Use Gauss's forward formula to find y_{30}, given that* $y_{21}=18{\cdot}4708$; $y_{25}=17{\cdot}8144$; $y_{29}=17{\cdot}1070$; $y_{33}=16{\cdot}3432$; $y_{37}=15{\cdot}5154$. **(Meerut 1980)**

Sol. Take $x_0=29$, $h=4$.

We require the value of y for $x=30$ *i.e.* for

$$u=\frac{x-x_0}{h}=\frac{30-29}{4}=0{\cdot}25.$$

The difference table is given below :

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
21	−2	18·4708				
			−0·6564			
25	−1	17·8144		−0·0510		
			−0·7074		−0·0054	
29	0	17·1070		−0·0564		−0·0022
			−0·7638		−0·0076	
33	1	16·3432		−0·0640		
			−0·8278			
37	2	15·5154				

Gauss's forward formula gives

$$y_{0\cdot 25}=17{\cdot}1070+(0{\cdot}25)\;(-0{\cdot}7638)+(0{\cdot}25)\;({\cdot}25-1)\;\frac{(-0{\cdot}0564)}{2}$$
$$+\frac{({\cdot}25+1)\,({\cdot}25)\,({\cdot}25-1)}{6}\,(-0{\cdot}0076)$$
$$+\frac{({\cdot}25+1)\,({\cdot}25)\,({\cdot}25-1)\,({\cdot}25-2)}{24}\,(-0{\cdot}0022)$$
$$=17{\cdot}1070-{\cdot}19095+{\cdot}0052875+{\cdot}0002968-{\cdot}0000375$$
$$=16{\cdot}9216 \text{ approx.}$$

Thus the value of y_{30} is 16·9216 approximately.

Ex. 4. *Interpolate by means of Gauss's backward formula the population for the year 1936, given the following table :*

Year :	1901	1911	1921	1931	1941	1951
Population :	12	15	20	27	39	52
(in thousands)						

(Meerut M. Sc. 1987)

Sol. Take 1931 as the origin and 10 years as the unit, then the population is to be estimated for $u=\dfrac{1936-1931}{10}={\cdot}5$.

The difference table is given below :

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$
1901	−3	12					
			3				
1911	−2	15		2			
			5		0		
1921	−1	20		2		3	
			7		3		−10
1931	0	27		5		−7	
			12		−4		
1941	1	39		1			
			13				
1951	2	52					

Now Gauss's backward formula is

$$y_u = y_0 + u\Delta y_{-1} + \frac{(u+1)u}{2}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{6}\Delta^3 y_{-2} + \frac{(u+2)(u+1)u(u-1)}{24}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{120}\Delta^5 y_{-3} + \cdots$$

$$\therefore\ y_{.5} = 27 + (\cdot 5)\times 7 + \frac{(\cdot 5+1)(\cdot 5)}{2}\times 5 + \frac{(\cdot 5+1)(\cdot 5)(\cdot 5-1)}{6}\times 3 + \frac{(\cdot 5+2)(\cdot 5+1)(\cdot 5)(\cdot 5-1)}{24}\times(-7) + \frac{(\cdot 5+2)(\cdot 5+1)(\cdot 5)(\cdot 5-1)(\cdot 5-2)}{120}\times(-10)$$

$$= 27 + 3\cdot 5 + 1\cdot 875 - 0\cdot 1875 + 0\cdot 2734 - 0\cdot 1172 = 32\cdot 3437 \text{ thousand.}$$

Thus the estimated population for 1936 is 32·3437 thousand.

Ex. 5. *Use Stirling's formula to find y_{35}, given*

$y_{20}=512,\ y_{30}=439,\ y_{40}=346,\ y_{50}=243.$

Sol. Take $x_0=30$, $h=10$. Then we require the value of y for $u=\dfrac{35-30}{12}=\cdot 5$. The difference table is given below :

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
20	-1	512			
			-73		
30	0	439		-20	
			-93		10
40	1	346		-10	
			-103		
50	2	243			

Stirling's formula is

$$y_u = y_0 + u\,\frac{\Delta y_0+\Delta y_{-1}}{2} + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\,\frac{(\Delta^3 y_{-2}+\Delta^3 y_{-1})}{2} + \cdots$$

$$\therefore\ y_{.5} = 439 + (\cdot 5)\,\frac{(-93-73)}{2} + \frac{(\cdot 5)^2}{2}(-20)$$

$$= 439 - 41\cdot 5 - 2\cdot 5 = 395.$$

Hence the estimated value of y_{35} is 395.

Ex. 6. *Use Stirling's formula to find y_{28}, given*

$y_{20}=49225,\ y_{25}=48316,\ y_{30}=47236,\ y_{35}=45926,\ y_{40}=44306.$

(Agra 1987 ; Kanpur B. Sc. 73)

Sol. Take $x=30$ as the origin and $h=5$ as the unit, the value of y required will be for $u=\dfrac{28-30}{5}=-0\cdot 4$.

The difference table is given below :

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
20	-2	49225				
			-909			
25	-1	48316		-171		
			-1080		-59	
30	0	47236		-230		-21
			-1310		-80	
35	1	45926		-310		
			-1620			
40	2	44306				

Putting the values in Stirling's formula, we get

$$y_{-0\cdot 4} = 47236 + (-0\cdot 4)\,\frac{(-1310-1080)}{2} + \frac{(-0\cdot 4)^2}{2}(-230) + \frac{(-0\cdot 4)(\cdot 16-1)}{6}\cdot\frac{(-59-80)}{2} + \frac{(0\cdot 16)(0\cdot 16-1)}{24}(-21)$$

$$= 47236 + 478 - 18\cdot 4 - 3\cdot 8920 + \cdot 1176 = 47692 \text{ approx.}$$

Thus the estimated value of y_{28} is 47692 approximately.

Ex. 7. *Apply Bessel's formula to obtain y_{25}, given*

$y_{20}=2854,\ y_{24}=3162,\ y_{28}=3544,\ y_{32}=3992.$

(Delhi Hons. 74, 75)

Sol. Take $x=24$ as origin and 4 as unit. We are to find the value of y for $u=\dfrac{25-24}{4}=0\cdot 25$.

The difference table is given below :

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
20	-1	2854			
			308		
24	0	3162		74	
			382		-8
28	1	3544		66	
			448		
32	2	3992			

Bessel's formula is

$$y_u = \tfrac{1}{2}(y_0+y_1) + (u-\tfrac{1}{2})\,\Delta y_0 + \frac{u(u-1)}{2!}\,\frac{(\Delta^2 y_{-1}+\Delta^2 y_0)}{2} + \frac{(u-\frac{1}{2})\,u(u-1)}{6}\,\Delta^3 y_{-1} + \cdots$$

$$\therefore \quad y_{0\cdot25}=\tfrac{1}{2}(3162+3544)+(\tfrac{1}{4}-\tfrac{1}{2})\times 382+\tfrac{1}{4}(\tfrac{1}{4}-1)\frac{(74+66)}{4}$$

$$+(\tfrac{1}{4}-\tfrac{1}{2})\,\tfrac{1}{4}.(\tfrac{1}{4}-1)\frac{(-8)}{6}$$

$$=3353-95\cdot5-6\cdot5625-0\cdot0625=3250\cdot875.$$

Thus the estimated value of y_{26} is 3250 875.

Ex. 8. *Apply Bessel's formula to find y_{25}, given*

$y_{20}=24,\ y_{24}=32,\ y_{28}=35,\ y_{32}=40.$

(Agra 1987 ; Meerut M.Sc. 88)

Sol. Take $x=24$ as the origin and 4 as the unit.

The value of y required will be for $u=\dfrac{25-24}{4}=0\cdot25.$

The difference table is given below :

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
20	−1	24			
			8		
24	0	32		−5	
			3		7
28	1	35		2	
			5		
32	2	40			

Bessel's formula is

$$y_u=\tfrac{1}{2}(y_0+y_1)+(u-\tfrac{1}{2})\,\Delta y_0+\frac{u(u-1)}{2!}\,\frac{(\Delta^2 y_{-1}+\Delta^2 y_0)}{2}$$

$$+\frac{(u-\tfrac{1}{2})\,u(u-1)}{6}\,\Delta^3 y_{-1}+\cdots$$

$$\therefore \quad y_{0\cdot25}=\tfrac{1}{2}(35+32)+(\tfrac{1}{4}-\tfrac{1}{2})\times 3+\tfrac{1}{4}(\tfrac{1}{4}-1)\frac{(-5+2)}{4}$$

$$+(\tfrac{1}{4}-\tfrac{1}{2}).\tfrac{1}{4}.(\tfrac{1}{4}-1).\tfrac{7}{6}$$

$$=33\cdot5-\cdot75+0\cdot140625+0\cdot0546875=32\cdot945313.$$

Thus the estimated value of y_{25} is 32·945313.

Ex. 9. *Given the table :*

x	$\log x$
310	2·4913617
320	2·5051500
330	2·5185139
340	2·5314789
350	2·5440680
360	2·5563025

find the value of log 337·5 by Laplace-Everett formula. **(Meerut 1978)**

Sol. Take $x=330$ as the origin and 10 as the unit.

The value of log x is required for $u=\dfrac{337\cdot5-330}{10}=\cdot75.$

$\therefore \quad v=1-u=1-\cdot75=\cdot25.$

The difference table is given below :

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$
310	−2	2·4913617					
			·0137883				
320	−1	2·5051500		−·0004244			
			·0133639		·0000255		
330	0	2·5185139		−·0003989		$-2\cdot5.10^{6}$	
			·0129650		·0000230		8.10^{-7}
340	1	2·5314789		−·0003759		-17.10^{-6}	
			·0125891		·0000213		
350	2	2·5440680		−·0003546			
			·0122345				
360	3	2·5563025					

Laplace-Everett's formula where $v=1-u=\frac{1}{4}$ is

$$y_u=vy_0+\frac{v(v^2-1)}{3!}\,\Delta^2 y_{-1}+\frac{(v^2-1)(v^2-4)}{5!}\,\Delta^4 y_{-2}+\cdots$$

$$+uy_1+\frac{u(u^2-1)}{3!}\,\Delta^2 y_0+\frac{u(u^2-1)(u^2-4)}{5!}\,\Delta^4 y_{-1}+\cdots$$

$$\therefore \quad y_{\cdot75}=\tfrac{1}{4}(2\cdot5185139)+\left(-\frac{5}{128}\right)(-\cdot0003989)$$

$$+\frac{63}{8192}(-\cdot0000025)+\tfrac{3}{4}(2\cdot5314789)$$

$$+\left(-\frac{\cdot7}{128}\right)(-\cdot0003759)+\frac{77}{8192}(-\cdot0000017)$$

$$=\cdot6296284+\cdot0000155+1\cdot8986092+\cdot0000206$$

$$=2\cdot5282738 \text{ approx.}$$

$\therefore \quad \log 337\cdot5=2\cdot5282738.$

Ex. 10. *Apply Everett's formula to find y_{25}, given*

$y_{20}=2854,\ y_{24}=3162,\ y_{28}=3544,\ y_{32}=3992.$

(Kanpur B.Sc. 1975; Meerut M.Sc. 89)

Sol. Take $x=24$ as the origin and $h=4$ as the unit.

We are to find the value of y for $u=\dfrac{25-24}{4}=\dfrac{1}{4}.$

Now $\quad v=1-u=1-\frac{1}{4}=(3/4).$

Using the difference table of Ex. 7, and putting the values in Everett's formula, we get

$$y_{\cdot25}=\frac{3}{4}(3162)+\frac{3}{4}\left(\frac{9}{16}-1\right)\times\frac{74}{6}+\frac{1}{4}(3544)+\frac{1}{4}\left(\frac{1}{16}-1\right)\times\frac{66}{6}$$

$=2371{\cdot}5-4{\cdot}046875+886-2{\cdot}578\ 25=3250{\cdot}875.$

Thus the estimated value of y_{25} is 3250·875.

Ex. 11. *Given*

θ	0°	5°	10°	15°	20°	25°	30°
$\tan\theta$	0·0000	0·0875	0·1763	0·2679	0·3640	0·4663	0·5774

Using Stirling formula show that $\tan 16°=0{\cdot}28676$.

(Rohilkhand 1988)

Sol. Take $\theta=15°$ as the origin and $h=5°$ as the unit.

We are to find the value of tan θ for $u=\dfrac{16°-15°}{5°}={\cdot}2.$

The difference table is given below :

	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$	$\Delta^6 y_u$
0°	−3	0·0000						
			·0875					
5°	−2	0·0875		·0013				
			·0888		·0015			
10°	−1	0·1763		·0028		·0002		
			·0916		·0017		−·0002	
15°	0	0·2679		·0045		·0000		·0011
			·0961		·0017		·0009	
20°	1	0·3640		·0062		·0009		
			·1023		·0026			
25°	2	0·4663		·0088				
			·1111					
30°	3	0·5774						

Stirling's formula is

$$y_u=y_0+u.\tfrac{1}{2}\,[\Delta y_0+\Delta\, y_{-1}]+\frac{u^2}{2}\,\Delta^2 y_{-1}$$

$$+\frac{u\,(u^2-1)}{3\,!}\,\tfrac{1}{2}\,[\Delta^3 y_{-1}+\Delta^3 y_{-2}]+\frac{u^2\,(u^2-1)}{4\,!}\,\Delta^4 y_{-2}$$

$$+\frac{u\,(u^2-1)(u^2-4)}{5\,!}\,\tfrac{1}{2}\,[\Delta^5 y_{-2}+\Delta^5 y_{-3}]$$

$$+\frac{u^2\,(u^2-1)\,(u^2-4)}{6\,!}\,\Delta^6 y_{-3}.$$

$$\therefore\quad y_{.2}=0{\cdot}2679+\frac{\cdot 2}{2}\,[{\cdot}0961+{\cdot}0916]+\frac{(\cdot 2)^2}{2}\,({\cdot}0045)$$

$$+\frac{(\cdot 2)\,(\cdot 04-1)}{6}\,\frac{[{\cdot}0017+{\cdot}0017]}{2}+\frac{(\cdot 04)\,(\cdot 04-1)}{24}\,({\cdot}0000)$$

$$+\frac{(\cdot 2)\,(\cdot 04-1)\,(\cdot 04-4)}{120}\cdot\frac{(-{\cdot}0002+{\cdot}0009)}{2}$$

$$+\frac{(\cdot 04)\,(\cdot 04-1)\,(\cdot 04-4)}{720}\,({\cdot}0011)$$

$=0{\cdot}2679+{\cdot}01877+{\cdot}00009+$negligible quantities

$=0{\cdot}28676.$

Thus the estimated value of tan 16°=0·28676.

Ex. 12. *If third differences are constant, prove that*

$$y_{x+(1/2)}=\tfrac{1}{2}\,(y_x+y_{x+1})-\frac{1}{16}\,(\Delta^2 y_{x-1}+\Delta^2 y_x).$$

Sol. Bessel's formula is

$$y_x=\frac{y_0+y_1}{2}+(x-\tfrac{1}{2})\,\Delta\,y_0+\frac{x\,(x-1)}{2\,!}\,\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}$$

$$+\frac{(x-\frac{1}{2})\,x\,(x-1)}{3\,!}\,\Delta^3 y_{-1}. \quad ...(1)$$

Here we have taken terms upto third differences since third differences are given to be constant.

Putting $x=\frac{1}{2}$ in (1), we get

$$y_{1/2}=\frac{y_0+y_1}{2}+\tfrac{1}{2}\,(\tfrac{1}{2}-1)\,\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{4} \quad ...(2)$$

Changing the origin to x, the equation (2) becomes

$$y_{x+(1/2)}=\frac{y_x+y_{x+1}}{2}-\frac{1}{16}\,(\Delta^2 y_{x-1}+\Delta^2 y_x).$$

Ex. 13. *Given* $y_0, y_1, y_2, y_3, y_4, y_5$ *(fifth differences constant), prove that*

$$y_{2\frac{1}{2}}=\tfrac{1}{2}c+\frac{25\,(c-b)+3\,(a-c)}{256},$$

where $a=y_0+y_5,\ b=y_1+y_4,\ c=y_2+y_3.$

(Meerut B. Sc. 1975, Agra 86)

Sol. Putting $x=\frac{1}{2}$ in Bessel's formula, we get

$$y_{1/2}=\tfrac{1}{2}\,(y_0+y_1)-\frac{1}{8}\,\frac{(\Delta^2 y_{-1}+\Delta^2 y_0)}{2}+\frac{3}{128}\,\frac{(\Delta^4 y_{-2}+\Delta^4 y_{-1})}{2}. \quad ...(1)$$

Here we have taken terms upto fifth differences only because fifth differences are given to be constant. Shifting the origin to 2 in (1), we get

$$y_{2\frac{1}{2}}=\tfrac{1}{2}\,(y_2+y_3)-\frac{1}{16}\,(\Delta^2 y_1+\Delta^2 y_2)+\frac{3}{256}\,(\Delta^4 y_0+\Delta^4 y_1) \quad ..(2)$$

Now $\Delta^2 y_0=(E-1)^2\,y_0=(E^2-2E+1)\,y_0=y_2-2y_1+y_0,$

$\Delta^2 y_1=(E-1)^2\,y_1=y_3-2y_2+y_1,$

$\Delta^2 y_2=(E-1)^2\,y_2=y_4-2y_3+y_2,$

$\Delta^4 y_0=(E-1)^4\,y_0=y_4-4y_3+6y_2-4y_1+y_0,$

$\Delta^4 y_1=(E-1)^4\,y_1=(E^4-4E^3+6E^2-4E+1)\,y_1$

$=y_5-4y_4+6y_3-4y_2+y_1.$

Putting these values in (2), we get

$$y_{2\frac{1}{2}}=\tfrac{1}{2}(y_2+y_3)-\frac{1}{16}\{(y_3-2y_2+y_1)+(y_4-2y_3+y_2)\}$$
$$+\frac{3}{256}\{(y_4-4y_3+6y_2-4y_1+y_0)+(y_5-4y_4+6y_3-4y_2+y_1)\}$$
$$=\tfrac{1}{2}(y_2+y_3)-\frac{1}{16}(y_4-y_3-y_2+y_1)$$
$$+\frac{3}{256}(y_5-3y_4+2y_3+2y_2-3y_1+y_0)$$
$$=\tfrac{1}{2}c-\frac{1}{16}(b-c)+\frac{3}{256}(a-3b+2c)$$
$$=\tfrac{1}{2}c+\frac{1}{256}[16(c-b)+3\{(a-c)+3(c-b)\}]$$
$$=\tfrac{1}{2}c+\frac{1}{256}[3(a-c)+25(c-b)].$$

§ 6. Uses of various interpolation formulae.

Lagrange's formula for interpolation is applicable for any type of observations. But when the observations are given at equal spaced values of the argument, the formulae using various order differences are found to be more convenient and easy to use than Lagrange's formula.

Newton's forward formula is appropriate to use when the unknown value lies near the beginning of the table and Newton's backward formula is mainly used when the unknown value to be interpolated lies near the end of the table.

The central difference formulae are more appropriate when the value of the argument for unknown entry lies somewhere in the middle of the difference table. The basic reason is that the central difference formulae are likely to converge more rapidly.

Among these Gauss's forward or backward formulae are suitable when the interpolated value lies near to the right or left of the central value in the table. Stirling's formula is appropriate in the situation when the value of u in the expression $x=a+uh$ is very near to zero. Bessel's formula will give a more accurate result when interpolating near the middle of an interval.

Exercises 4

1. Prove that

(i) $\delta=\Delta E^{-1/2}=\nabla E^{1/2}$ (Meerut M.Sc. 1988, 91)

(ii) $\mu^2=1+\frac{1}{4}\delta^2$ (Meerut M. Sc. 1978, 80, 87)

(iii) $E^{1/2}=\mu+\frac{1}{2}\delta$ (Meerut M.Sc. 1987, 90)

(iv) $\sqrt{(1+\delta^2\mu^2)}=1+\frac{1}{2}\delta^2$.

(v) $\delta^2 y_0=y_1-2y_0+y_{-1}$.

2. Establish the following relations :

(i) $\mu\delta=\frac{1}{2}(\nabla+\Delta)=\frac{1}{2}\Delta E^{-1}+\frac{1}{2}\Delta$

(ii) $\Delta+\nabla=\Delta/\nabla-\nabla/\Delta$.

3. If D, E, δ and μ be the operators with usual meanings and if $hD=U$, where h is the interval of differencing, prove that

(i) $E=e^U$. (ii) $\delta=2\sinh(U/2)$

(iii) $\mu=\cosh(U/2)$ (iv) $e^{-U}=1-\nabla$

(v) $(E+1)\delta=2\mu(E-1)$. (Meerut M. Sc. 1986)

4. Prove that if third differences are assumed to be constant

$$y_x=xy_1+\frac{x(x^2-1)}{3!}\Delta^2 y_0+uy_0+\frac{u(u^2-1)}{3!}\Delta^2 y_{-1},$$

where $u=1-x$. Apply this formula to find the values of y_{11} and y_{16} given that

$y_0=3010$, $y_5=2710$, $y_{10}=2285$, $y_{15}=1860$, $y_{20}=1560$, $y_{25}=1510$ and $y_{30}=1835$. (Meerut 1976)

[Hint. This formula can be proved with the help of Everett's formula.]

5. Given $y_2=10$, $y_1=8$, $y_0=5$, $y_{-1}=10$, find $y_{1/2}$ by Gauss's forward formula.

6. Given that :

$\sqrt{12500}=111{\cdot}803399$; $\sqrt{12510}=111{\cdot}848111$;

$\sqrt{12520}=111{\cdot}892806$; $\sqrt{12530}=111{\cdot}937483$

show by Gauss's backward formula that

$\sqrt{12516}=111{\cdot}874930$.

7. The values of e^{-x} for certain equidistant values of x are given in the following table. Find the value of e^{-x} when $x=1{\cdot}7489$ by Bessel's as well as Stirling's formula

x	e^{-x}
1·72	0·1790661479
1·73	0·1772844100
1·74	0·1755204006
1·75	0·1737739435
1·76	0·1720448638
1·77	0·1703329888
1·78	0·1686381473

5

Numerical Differentiation

§ 1. Introduction. The process of finding the derivative or derivatives of a function at some value of the independent variable, when we know the values of the function corresponding to the given values of the independent variable, is called **numerical differentiation.** The problem of differentiation is solved by the principle "Fit up an interpolation polynomial to the given set of values of the function and then differentiate it as many times as desired". In case the values of the argument are equally spaced, we represent the function by Newton-Gregory formula. If it is required to find the derivative of the function at a point near the beginning (end) of a set of tabular values, we use Newton-Gregory forward (backward) formula. To find the derivative at a point near the middle of the table, we should use one of central difference formulae. If the values of the argument are unequally spaced, Newton's divided difference formula should be used to represent the function.

To find the maximum or minimum value of a tabulated function we compute the necessary differences from the given table, and substitute them in some suitable interpolation formula. Then we put the first derivative of the function obtained from this formula equal to zero and solve for u. The required value of x is obtained from the relation $x=x_0+hu$.

Solved Examples

Ex. 1. *Find the first, second and third derivatives of the function tabulated below, at the point* $x=1{\cdot}5$

x :	1·5	2·0	2·5	3·0	3·5	4·0
$y=f(x)$:	3·375	7·000	13·625	24·000	38·875	59·000

(Meerut M.Sc. 1978, Ajmer 90)

Sol. Since the derivatives are required at $x=1{\cdot}5$, which is near the beginning of the table, therefore we shall use Newton's forward formula. The difference table is given below :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1·5	3·375				
		3·625			
2·0	7·000		3·000		
		6·625		·750	
2·5	13·625		3·750		0
		10·375		·750	
3·0	24·000		4·500		0
		14·875		·750	
3·5	38 875		5·250		
		20·125			
4·0	59·000				

Newton's forward formula is

$$f(a+xh)=f(a)+{}^xC_1\Delta f(a)+{}^xC_2\Delta^2 f(a)+{}^xC_3\Delta^3 f(a),$$

upto third differences

$$=f(a)+x\Delta f(a)+\frac{x(x-1)}{2!}\Delta^2 f(a)+\frac{x(x-1)(x-2)}{3!}\Delta^3 f(a).$$

Differentiating w.r.t. x thrice and then putting $x=0$ in the equations obtained, we get

$$hf'(a)=\Delta f(a)-\tfrac{1}{2}\Delta^2 f(a)+\tfrac{1}{3}\Delta^3 f(a),$$
$$h^2 f''(a)=\Delta^2 f(a)-\Delta^3 f(a),$$
$$h^3 f'''(a)=\Delta^3 f(a).$$

Putting $a=1{\cdot}5$, $h={\cdot}5$ and the values of various differences in these equations, we get

$$f'(1{\cdot}5)=\frac{1}{{\cdot}5}[3{\cdot}625-\tfrac{1}{2}(3{\cdot}000)+\tfrac{1}{3}({\cdot}750)]=4{\cdot}750,$$

$$f''(1{\cdot}5)=\frac{1}{{\cdot}25}[3{\cdot}000-({\cdot}750)]=9{\cdot}000,$$

$$f'''(1{\cdot}5)=\frac{1}{{\cdot}125}[{\cdot}750]=6{\cdot}000.$$

Note. The function tabulated in the given table of this problem is

$$y=x^3-2x+3.$$

Hence $\dfrac{dy}{dx}=3x^2-2,\ \dfrac{d^2y}{dx^2}=6x,\ \dfrac{d^3y}{dx^3}=6.$

Putting $x=1{\cdot}5$, we get

$$\left(\frac{dy}{dx}\right)_{1\cdot5}=3\times(1{\cdot}5)^2-2=4{\cdot}750,\ \left(\frac{d^2y}{dx^2}\right)_{x=1\cdot5}=9{\cdot}000,\ \left(\frac{d^3y}{dx^3}\right)_{x=1\cdot5}=6{\cdot}000.$$

Thus we get the same values as obtained by numerical differentiation.

Ex. 2. *Find the first and second derivatives of the function tabulated below, at the point $x=3.0$.*

x :	3.0	3.2	3.4	3.6	3.8	4.0
y :	−14.000	−10.032	−5.296	0.256	6.672	14.000

Sol. Since the derivatives are required at $x=3.0$, which is near the beginning of the table, therefore we shall use Newton's forward formula. The difference table is given below :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3.0	−14.000				
		3.968			
3.2	−10.032		.768		
		4.736		.048	
3.4	−5.296		.816		0
		5.552		.048	
3.6	0.256		.864		0
		6.416		.048	
3.8	6.672		.912		
		7.328			
4.0	14.000				

Newton's forward formula is

$$f(a+xh)=f(a)+x\Delta f(a)+\frac{x(x-1)}{2}\Delta^2 f(a)+\frac{x(x-1)(x-2)}{6}\Delta^3 f(a),$$

taking upto third differences only.

Differentiating w. r. t. x twice and then putting $x=0$ in the equations obtained, we get

$$hf'(a)=\Delta f(a)-\tfrac{1}{2}\Delta^2 f(a)+\tfrac{1}{3}\Delta^3 f(a),$$

$$h^2 f''(a)=\Delta^2 f(a)-\Delta^3 f(a).$$

Putting $a=3$, $h=.2$ and the values of various differences in these equations, we get

$$f'(3)=\frac{1}{.2}[3.968-\tfrac{1}{2}(.768)+\tfrac{1}{3}(.048)]=18$$

$$f''(3)=\frac{1}{.04}[.768-.048]=18.$$

Note. The function tabulated above is

$$y=x^3-9x-14.$$

$$\therefore\quad \frac{dy}{dx}=3x^2-9,\ \frac{d^2y}{dx^2}=6x.$$

$$\therefore\quad \left(\frac{dy}{dx}\right)_{x=3}=3.3^2-9=18,\ \left(\frac{d^2y}{dx^2}\right)_{x=3}=18.$$

Hence we get the same values as obtained by numerical differentiation. **(Meerut M.Sc. 1992P, 93)**

Ex. 3. *Find the first and second derivatives of the function tabulated below at the point $x=1.1$*

x :	1	1.2	1.4	1.6	1.8	2.0
$f(x)$:	0	.1280	.5440	1.2960	2.4320	4.00

(Agra 1987; Meerut B.Sc. Stat. 90)

Sol. Since the derivatives are required at $x=1.1$, which is near the beginning of the table, we shall use Newton's Forward formula. The difference table is as below :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.0	0				
		.1280			
1.2	.1280		.2880		
		.4160		.0480	
1.4	.5440		.3360		0
		.7520		.0480	
1.6	1.2960		.3840		0
		1.1360		.0480	
1.8	2.4320		.4320		
		1.5680			
2.0	4.00				

Newton's Forward formula is

$$f(a+xh)=f(a)+x\Delta f(a)+\frac{x(x-1)}{2}\Delta^2 f(a)+\frac{x(x-1)(x-2)}{6}\Delta^3 f(a),$$

taking upto third differences only.

Differentiating w.r.t. x twice, we get

$$hf'(a+xh)=\Delta f(a)+\frac{(2x-1)}{2}\Delta^2 f(a)+\frac{(3x^2-6x+2)}{6}\Delta^3 f(a),$$

$$h^2 f''(a+xh)=\Delta^2 f(a)+(x-1)\Delta^3 f(a).$$

Putting $a=1.0$, $h=.2$, $x=\frac{1}{2}$ and the values of differences in these equations, we get

$$f'(1.1)=\frac{1}{.2}\left[.1280+0+\frac{1}{6}(3.\tfrac{1}{4}-6.\tfrac{1}{2}+2)(.0480)\right]$$

$$=.630$$

and $\quad f''(1.1)=\frac{1}{.04}[.2880+(\frac{1}{2}-1)(.0480)]=6\ 60.$

Note. The function tabulated above is

$$y=x^3-3x+2.$$

$$\therefore\quad \frac{dy}{dx}=3x^2-3,\ \frac{d^2y}{dx^2}=6x.$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=1\cdot1} = 3\,(1\cdot1)^2 - 3 = \cdot630, \quad \left(\frac{d^2y}{dx^2}\right)_{x=1\cdot1} = 6\cdot60.$$

Hence the actual values of the derivatives are the same as obtained by numerical differentiation.

Ex. 4. *Find the derivative of $f(x)$ at $x=\cdot4$ from the following table :*

x :	$\cdot1$	$\cdot2$	$\cdot3$	$\cdot4$
$f(x)$:	1·10517	1·22140	1·34986	1·49182

Sol. Since the derivative is required at $x=\cdot4$, which is near the end of the table, therefore we shall use Newton's Backward formula. The difference table is given below :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
·1	1·10517			
		·11623		
			·01223	
·2	1·22140			
		·12846		·00127
·3	1·34986		·01350	
		·14196		
·4	1·49182			

Newton's Backward formula is

$$f(a+nh+xh)=f(a+nh)+x\nabla f(a+nh)+\frac{x\,(x+1)}{2\,!}\,\nabla^2 f(a+nh)$$
$$+\frac{x\,(x+1)\,(x+2)}{3\,!}\,\nabla^3 f(a+nh),$$

taking upto third differences only.

Differentiating w.r.t. x, we get

$$h\,f'(a+nh+xh)=\nabla f(a+nh)+\frac{2x+1}{2}\,\nabla^2 f(a+nh)$$
$$+\frac{3x^2+6x+2}{6}\,\nabla^3 f(a+nh).$$

Putting $a+nh=\cdot4$, $h=\cdot1$, $x=0$, we get

$$(\cdot1)\,f'(\cdot4)=\nabla f(\cdot4)+\tfrac{1}{2}\nabla^2 f(\cdot4)+\tfrac{1}{3}\nabla^3 f(\cdot4)$$
$$=\cdot14196+\tfrac{1}{2}(\cdot01350)+\tfrac{1}{3}(\cdot00127)=\cdot14913.$$

$$\therefore \quad f'(\cdot4)=1\cdot4913.$$

Ex. 5. *Find $f'(7\cdot50)$ from the following table :*

x :	7·47	7·48	7·49	7·50	7·51	7·52	7·53
$y=f(x)$:	·193	·195	·198	·201	·203	·206	·208

(Robilkhand 1984, 88)

Sol. Here we want to find the derivative at $x=7\cdot50$ which lies near the middle of the table and so we shall use one of the central difference formulae. Take $x=7\cdot50$ as the origin. Then $x_0=7\cdot50$, $h=\cdot01$.

$$\therefore \quad u=\frac{x-x_0}{h}=\frac{x-7\cdot50}{\cdot01}.$$

The Bessel's formula with the variable u is given by

$$y_u=\tfrac{1}{2}\,[y_0+y_1]+(u-\tfrac{1}{2})\,\Delta y_0+\frac{u\,(u-1)}{2\,!}\,\frac{[\Delta^2 y_{-1}+\Delta^2 y_0]}{2}$$
$$+\frac{(u-\frac{1}{2})\,u\,(u-1)}{3\,!}\,\Delta^3 y_{-1}+\frac{(u+1)\,u\,(u-1)\,(u-2)}{4\,!}\,\frac{[\Delta^4 y_{-1}+\Delta^4 y_{-2}]}{2}$$
$$+\frac{(u-\frac{1}{2})\,(u+1)\,u\,(u-1)\,(u-2)}{5\,!}\,\Delta^5 y_{-2}$$
$$+\frac{(u+2)\,(u+1)\,u\,(u-1)\,(u-2)\,(u-3)}{6\,!}\,\frac{[\Delta^6 y_{-3}+\Delta^6 y_{-2}]}{2}+\ldots \qquad \ldots(1)$$

The difference table is given below :

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$	$\Delta^6 y_u$
7·47	−3	·193						
			·002					
7·48	−2	·195		·001				
			·003		−·001			
7·49	−1	·198		·000		·000		
			·003		−·001		·003	
7·50	0	·201		−·001		·003		−·010
			·002		·002		−·007	
7·51	1	·203		·001		−·004		
			·003		−·002			
7·52	2	·206		−·001				
			·002					
7·53	3	·208						

Differentiating (1) w.r.t. u and putting $u=0$, we get

$$f'(0)=\cdot002-\tfrac{1}{2}\,[-\cdot001+\cdot001]+\frac{1}{12}\,(\cdot002)$$
$$+\frac{1}{24}\,[-\cdot004+\cdot003]-\frac{1}{120}\,(-\cdot007)-\frac{1}{120}\,(-\cdot010)$$
$$=\cdot002+0+\cdot0001666-\cdot0000416+\cdot0000583+\cdot0000833$$
$$=\cdot0022666.$$

We have $u=\dfrac{x-x_0}{h}$. $\quad \therefore \quad \dfrac{du}{dx}=\dfrac{1}{h}$.

$$\therefore \quad \frac{d}{dx}\,\{f(x)\}=\frac{d}{du}\,\{f(x)\}.\,\frac{du}{dx}=\frac{1}{h}\,f'(u).$$

When $x=7\cdot50$, we have $u=0$. Therefore $f'\,(7\cdot50)=\dfrac{1}{\cdot01}f'(0)$.

$$\therefore \quad f'(7.50)=\frac{1}{.01}\,(.0022666)=.22666.$$

Ex. 6. *Find the value of $f'(.04)$ from the following table :*

x :	.01	.02	.03	.04	.05	.06
$y=f(x)$:	.1023	.1047	.1071	.1096	.1122	.1148

Sol. Here we require the derivative at the point $x=.04$ which lies near the middle of the table, so we may use Gauss's forward formula.

With a new variable $u=\dfrac{x-.04}{.01}$, Gauss's forward formula is

$$f(u)=f(0)+u\Delta f(0)+\frac{u\,(u-1)}{2\,!}\,\Delta^2 f(-1)+\frac{u\,(u+1)\,(u-1)}{3\,!}\,\Delta^3 f(-1)$$
$$+\frac{u\,(u+1)\,(u-1)\,(u-2)}{4\,!}\,\Delta^4 f(-2)+\ldots \qquad \ldots(1)$$

The difference table is given below :

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
.01	-3	.1023				
			.0024			
.02	-2	.1047		0		
			.0024		.0001	
.03	-1	.1071		.0001		$-.0001$
			.0025		0	
.04	0	.1096		.0001		$-.0001$
			.0026		$-.0001$	
.05	1	.1122		0		
			.0026			
.06	2	.1148				

Since $\quad u=\dfrac{x-.04}{h}, \qquad \therefore\ \dfrac{du}{dx}=\dfrac{1}{h}.$

Hence $\dfrac{d}{dx}\{f(x)\}=\dfrac{d}{dx}\{f(x)\}.\dfrac{du}{dx}=\dfrac{1}{h}f'(u).$

When $x=.04$, we have $u=0$.

Differentiating (1) w.r.t. u and putting $u=0$, we get

$$f'(0)=\Delta f(0)-\tfrac{1}{2}\Delta^2 f(-1)-\frac{1}{6}\,\Delta^3 f(-1)+\frac{1}{12}\,\Delta^4 f(-2),$$

leaving higher order differences

$$\doteq .0026-\tfrac{1}{2}\,(.0001)-\frac{1}{6}\,(0)+\frac{1}{12}\,(-.0001)=.0025417.$$

$$\therefore\quad hf'(.04)=.0025417 \Rightarrow f'(.04)=\frac{.0025417}{.10}=0.25417.$$

Ex. 7. *For the following pairs of values of x and y find numerically the first derivative at $x=4$.*

x :	1	2	4	8	10
y :	0	1	5	21	27

(Meerut M.Sc. 1987, 90, 91)

Sol. Here the values of x are not equally spaced, therefore we shall use Newton's divided difference formula. The divided difference table is given below :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	0				
		1			
2	1		$\frac{1}{3}$		
		2		0	
4	5		$\frac{1}{3}$		$-\frac{1}{144}$
		4		$-\frac{1}{16}$	
8	21		$-\frac{1}{6}$		
		3			
10	27				

We know that Newton's divided difference formula is

$$f(x)=f(x_0)+(x-x_0)\,f(x_0,\,x_1)+(x-x_0)\,(x-x_1)\,f(x_0,\,x_1,\,x_2)$$
$$+(x-x_0)\,(x-x_1)\,(x-x_2)\,f(x_0,\,x_1,\,x_2,\,x_3)$$
$$+(x-x_0)\,(x-x_1)\,(x-x_2)\,(x-x_3)\,f(x_0,\,x_1,\,x_2,\,x_3,\,x_4),$$

neglecting terms of higher order differences.

Differentiating this w.r.t. x, we get

$$f'(x)=f(x_0,\,x_1)+\{(x-x_1)+(x-x_0)\}\,f(x_0,\,x_1,\,x_2)+\{(x-x_1)(x-x_2)$$
$$+(x-x_0)(x-x_2)+(x-x_0)(x-x_1)\}\,f(x_0,\,x_1,\,x_2,\,x_3)$$
$$+\{(x-x_1)(x-x_2)(x-x_3)+(x-x_0)(x-x_2)(x-x_3)$$
$$+(x-x_0)(x-x_1)(x-x_3)+(x-x_0)(x-x_1)(x-x_2)\}$$
$$\times f(x_0,\,x_1,\,x_2,\,x_3,\,x_4).$$

Putting $x=4$, $x_0=1$, $x_1=2$, $x_2=4$, $x_3=8$, $x_4=10$ and the values of divided differences, we get

$$f'(4)=1+\{(4-2)+(4-1)\}\,\tfrac{1}{3}+\{(4-1)(4-2)\}\times 0$$
$$+\{(4-1)\,(4-2)(4-8)\}\left(-\frac{1}{144}\right)$$

$$=1+\frac{5}{3}+0+\frac{24}{144}=1+1.666+0.166=2.8326.$$

Ex. 8: *Find $f'(5)$ from following table :*

x :	0	2	3	4	7	9
$f(x)$:	4	26	58	112	466	922

Sol. Since the values of the argument are unequally spaced therefore we shall use Newton's divided difference formula. The divided difference table is given below :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	4			
		11		
2	26		7	
		32		1
3	58		11	
		54		1
4	112		16	
		118		1
7	466		22	
		228		
9	922			

Newton's divided difference formula if third differences are constant, is

$$f(x)=f(x_0)+(x-x_0)\,f(x_0,x_1)+(x-x_0)\,(x-x_1)\,f(x_0,x_1,x_2)$$
$$+(x-x_0)\,(x-x_1)\,(x-x_2)\,f(x_0,x_1,x_2,x_3).$$

Differentiating this w.r.t. x, we get

$$f'(x)=f(x_0,x_1)+\{(x-x_1)+(x-x_0)\}\,f(x_0,x_1,x_2)+\{(x-x_1)(x-x_2)$$
$$+(x-x_0)\,(x-x_2)+(x-x_0)\,(x-x_1)\}\,f(x_0,x_1,x_2,x_3).$$

Putting $x=5$, $x_0=0$, $x_1=2$, $x_2=3$, $x_3=4$ and the values of various divided differences, we get

$$f'(5)=11+\{(5-2)+(5-0)\}\times 7+\{(5-2)(5-3)+(5-0)(5-3)$$
$$+(5-0)(5-2)\}\times 1$$
$$=11+56+31=98.$$

(Meerut M.Sc. 1993P)

Ex. 9. *Assuming Bessel's interpolation formula, show that*

$$\frac{d}{dx}(y_x)=\Delta y_{x-(1/2)}-\frac{1}{24}\Delta^3 y_{x-(3/2)}+\ldots$$

(Meerut M.Sc. 1977, 80, 84, 85, 86, 91, 92, 92P; Rohilkhand 86)

$$\frac{d^2}{dx^2}(y_x)=\tfrac{1}{2}[\Delta^2 y_{x-(3/2)}+\Delta^2 y_{x-(1/2)}]+\ldots$$

$$\frac{d^3}{dx^3}(y_x)=\Delta^3 y_{x-(3/2)}+\ldots$$

Sol. We know that Bessel's formula is

$$y_x=\frac{y_0+y_1}{2}+(x-\tfrac{1}{2})\Delta y_0+\frac{x\,(x-1)}{2\,!}\,\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}$$
$$+\frac{(x-\frac{1}{2})\,x\,(x-1)}{3\,!}\Delta^3 y_{-1}+\ldots \qquad \ldots(1)$$

Putting $x+\frac{1}{2}$ in place of x in (1), we have

$$y_{x+(1/2)}=\frac{y_0+y_1}{2}+x\Delta y_0+\frac{(x+\frac{1}{2})\,(x-\frac{1}{2})}{2\,!}\,\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}$$
$$+\frac{x\,(x+\frac{1}{2})\,(x-\frac{1}{2})}{3\,!}\Delta^3 y_{-1}+\ldots \qquad \ldots(2)$$

Differentiating (2) w.r.t. x, we get

$$\frac{d}{dx}(y_{x+1/2})=\Delta y_0+x\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}+\left(\frac{x^2}{2}-\frac{1}{24}\right)\Delta^3 y_{-1}+\ldots$$

Putting $x=0$ in this we get

$$\frac{d}{dx}(y_{1/2})=\Delta y_0-\frac{1}{24}\Delta^3 y_{-1}+\ldots$$

Changing the origin to $x-\frac{1}{2}$, we get

$$\frac{d}{dx}(y_x)=\Delta y_{x-(1/2)}-\frac{1}{24}\Delta^3 y_{x-(3/2)}+\ldots$$

Differentiating (2) twice w.r.t. x, we have

$$\frac{d^2}{dx^2}(y_{x+1/2})=\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}+x\Delta^3 y_{-1}+\ldots$$

Putting $x=0$ and then shifting the origin from $x=0$ to $x-\frac{1}{2}$ as earlier, we get

$$\frac{d^2}{dx^2}(y_x)=\tfrac{1}{2}[\Delta^2 y_{x-(3/2)}+\Delta^2 y_{x-(1/2)}]+\ldots$$

In the same way, we can show that

$$\frac{d^3}{dx^3}(y_x)=\Delta^3 y_{x-(3/2)}+\ldots$$

Ex. 10. *From Stirling's formula, obtain the following result.*

$$\frac{d}{dx}(y_x)=\frac{2}{3}[y_{x+1}-y_{x-1}]-\frac{1}{12}[y_{x+2}-y_{x-2}],$$

upto third differences. (Meerut 1978, 93P; Rohilkhand 84, 88)

Sol. We know that Stirling's formula is

$$y_x=y_0+x\,\frac{\Delta y_0+\Delta y_{-1}}{2}+\frac{x^2}{2\,!}\Delta^2 y_{-1}+\frac{x\,(x^2-1)}{3\,!}\,\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2},$$

taking upto third differences only.

Differentiating w.r.t. x, we get

$$\frac{d}{dx}(y_x)=\frac{\Delta y_0+\Delta y_{-1}}{2}+x\,\Delta^2 y_{-1}+\frac{3x^2-1}{3\,!}\,\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}.$$

Putting $x=0$, we get

$$\frac{d}{dx}(y_0)=\frac{\Delta y_0+\Delta y_{-1}}{2}-\frac{1}{12}(\Delta^3 y_{-1}+\Delta^3 y_{-2}) \qquad \ldots(1)$$

Now we have $\Delta y_0=y_1-y_0$, $\Delta y_{-1}=y_0-y_{-1}$,

$\Delta^3 y_{-1}=(E-1)^3 y_{-1}=(E^3-3E^2+3E-1)\,y_{-1}=y_2-3y_1+3y_0-y_{-1}$,

and $\Delta^3 y_{-2}=y_1-3y_0+3y_{-1}-y_{-2}$.

Putting these values in (1), we get

$$\frac{d}{dx}(y_0)=\tfrac{1}{2}[y_1-y_0+y_0-y_{-1}]-\frac{1}{12}[(y_2-3y_1+3y_0-y_{-1})$$
$$+(y_1-3y_0+3y_{-1}-y_{-2})]$$

$$=\tfrac{1}{2}(y_1-y_{-1})-\frac{1}{12}(y_2-2y_1+2y_{-1}-y_{-2})$$

$$=\frac{2}{3}(y_1-y_{-1})-\frac{1}{12}(y_2-y_{-2}).$$

Now shifting the origin to x, we get

$$\frac{d}{dx}(y_n)=\frac{2}{3}(y_{n+1}-y_{n-1})-\frac{1}{12}(y_{n+2}-y_{n-2}).$$

Ex. 11. *Prove that*

$$y'=\frac{1}{h}\left(\delta y-\frac{1}{24}\delta^3 y+\frac{3}{640}\delta^5 y-\dots\right)$$

and $$y''=\frac{1}{h^2}\left(\delta^2 y-\frac{1}{12}\delta^4 y+\frac{1}{90}\delta^6 y-\dots\right)$$

(Delhi Hons. 1971; Meerut M.Sc. 73, 91, 93)

Sol. We have

$$\delta=E^{1/2}-E^{-1/2}=e^{hD/2}-e^{-hD/2}=2\sinh\frac{hD}{2}.$$

$$\therefore\ \sinh\frac{hD}{2}=\frac{\delta}{2}\ \Rightarrow\ \frac{hD}{2}=\sinh^{-1}\frac{\delta}{2} \qquad \dots(1)$$

By Taylor's series expansion, we know that

$$\sinh^{-1}x=x-\frac{x^3}{6}+\frac{3}{40}x^5-\dots$$

$$\therefore\ \sinh^{-1}\frac{\delta}{2}=\frac{\delta}{2}-\frac{1}{6}\left(\frac{\delta}{2}\right)^3+\frac{3}{40}\left(\frac{\delta}{2}\right)^5-\dots$$

From (1), we have

$$\frac{hD}{2}=\frac{\delta}{2}-\frac{1}{6}\left(\frac{\delta}{2}\right)^3+\frac{3}{40}\left(\frac{\delta}{2}\right)^5-\dots$$

$$\Rightarrow\ D=\frac{1}{h}\left[\delta-\frac{\delta^3}{24}+\frac{3}{640}\delta^5-\dots\right] \qquad \dots(2)$$

Now operating these operators on y, we get

$$Dy=\frac{1}{h}\left[\delta-\frac{\delta^3}{24}+\frac{3}{640}\delta^5-\dots\right]y$$

or $$y'=\frac{1}{h}\left[\delta y-\frac{1}{24}\delta^3 y+\frac{3}{640}\delta^5 y-\dots\right].$$

Squaring (2), we get

$$D^2=\frac{1}{h^2}\left[\delta^2-\frac{1}{12}\delta^4+\frac{1}{90}\delta^6-\dots\right]$$

$$\Rightarrow\ D^2y=\frac{1}{h^2}\left[\delta^2-\frac{1}{12}\delta^4+\frac{1}{90}\delta^6-\dots\right]y, \text{ operating on } y$$

$$\Rightarrow\ y''=\frac{1}{h^2}\left[\delta^2 y-\frac{1}{12}\delta^4 y+\frac{1}{90}\delta^6 y-\dots\right].$$

Ex. 12. *Show that*

$$y'=\frac{\mu}{h}\left(\delta y-\frac{1}{6}\delta^3 y+\frac{1}{30}\delta^5 y-\dots\right)$$

and $$y'_0=\frac{1}{2h}\left[(y_1-y_{-1})-\frac{1}{6}(\delta^2 y_1-\delta^2 y_{-1})+\frac{1}{30}(\delta^4 y_1-\delta^4 y_{-1})\dots\right].$$

Sol. We know that

$$\mu^2=1+\frac{\delta^2}{4}.$$

$$\therefore\ \mu^{-1}=\left(1+\frac{\delta^2}{4}\right)^{-1/2}=1-\frac{\delta^2}{8}+\frac{3}{128}\delta^4-\dots$$

Also from the equation (2) of Ex. 11., we have

$$\frac{hD}{\delta}=1-\frac{\delta^2}{24}+\frac{3}{640}\delta^4-\dots$$

Now $$\frac{hD}{\delta\mu}=\frac{hD}{\delta}\mu^{-1}=\left[1-\frac{\delta^2}{24}+\frac{3}{640}\delta^4-\dots\right]\left[1-\frac{\delta^2}{8}+\frac{3}{128}\delta^4-\dots\right]$$

$$=1-\frac{\delta^2}{6}+\frac{1}{30}\delta^4-\dots$$

or $$D=\frac{\delta\mu}{h}\left[1-\frac{\delta^2}{6}+\frac{1}{30}\delta^4-\dots\right]$$

$$=\frac{\mu}{h}\left[\delta-\frac{\delta^3}{6}+\frac{\delta^5}{30}-\dots\right].$$

Operating these operators on y, we get

$$Dy=\frac{\mu}{h}\left[\delta-\frac{\delta^3}{6}+\frac{\delta^5}{30}-\dots\right]y$$

$$\Rightarrow\ y'=\frac{\mu}{h}\left[\delta y-\frac{\delta^3 y}{6}+\frac{\delta^5 y}{30}-\dots\right] \qquad \dots(1)$$

By definition, we have

$\delta=E^{1/2}-E^{-1/2}$ and $\mu=\tfrac{1}{2}(E^{1/2}+E^{-1/2})$.

$\therefore\ \delta\mu=\tfrac{1}{2}(E-E^{-1})=\mu\delta.$

$\therefore\ (\delta\mu)\,y_0=\tfrac{1}{2}(E-E^{-1})\,y_0=\tfrac{1}{2}(y_1-y_{-1})=\mu\delta y_0 \qquad \dots(2)$

From (1), we have

$$y_0'=\frac{1}{h}\left[\mu\delta y_0-\frac{\mu\delta^3}{6}y_0+\frac{\mu\delta^5}{30}y_0-\dots\right]$$

$$=\frac{1}{2h}\Big[(y_1-y_{-1})-\frac{1}{6}(\delta^3y_1-\delta^3y_{-1})+\frac{1}{30}(\delta^4y_1-\delta^4y_{-1})-\dots\Big].$$

Ex. 13. *Prove that* $y'=\frac{1}{h}(\Delta y-\frac{1}{2}\Delta^2y+\frac{1}{3}\Delta^3 y-\frac{1}{4}\Delta^4y+\dots)$

and $$y''=\frac{1}{h^2}(\nabla^2y+\nabla^3y+\frac{11}{12}\nabla^4y+\dots).$$

Sol. We know that $E=e^{hD}$.

$$\therefore \quad 1+\Delta=e^{hD}$$

or $$hD=\log(1+\Delta)$$

$$\Rightarrow \quad D=\frac{1}{h}\log(1+\Delta)$$

$$\Rightarrow Dy=\frac{1}{h}\log(1+\Delta)\,y$$

$$=\frac{1}{h}\left\{\Delta-\frac{\Delta^2}{2}+\frac{\Delta^3}{3}-\frac{\Delta^4}{4}+\dots\right\}y.$$

$$\therefore \quad y'=\frac{1}{h}\left\{\Delta y-\frac{\Delta^2y}{2}+\frac{\Delta^3y}{3}-\frac{\Delta^4y}{4}+\dots\right\}.$$

Again $e^{-hD}=1-\nabla$. $\therefore$ $D=-\frac{1}{h}\log(1-\nabla)$.

Now $$y''=D^2y=\left\{-\frac{1}{h}\log(1-\nabla)\right\}^2y$$

$$=\frac{1}{h^2}\{\log(1-\nabla)\}^2y$$

$$=\frac{1}{h^2}\{\nabla+\tfrac{1}{2}\nabla^2+\tfrac{1}{3}\nabla^3+\dots\}^2y$$

$$=\frac{1}{h^2}\left\{\nabla^2+\nabla^3+\frac{11}{12}\nabla^4+\dots\right\}y$$

$$=\frac{1}{h^2}\left\{\nabla^2y+\nabla^3y+\frac{11}{12}\nabla^4y+\dots\right\}.$$

Ex. 14. *Prove that*

$$\frac{d}{dx}(y_x)=\frac{1}{h}(y_{x+h}-y_{x-h})-\frac{1}{2h}(y_{x+2h}-y_{x-2h})+\frac{1}{3h}(y_{x+3h}-y_{x-3h})-\dots$$

Sol. We have the R.H.S. $=\frac{1}{h}\{y_{x+h}-\frac{1}{2}y_{x+2h}+\frac{1}{3}y_{x+3h}-\dots\}$

$$-\frac{1}{h}\{y_{x-h}-\tfrac{1}{2}y_{x-2h}+\tfrac{1}{3}y_{x-3h}-\dots\}$$

$$=\frac{1}{h}[\{Ey_x-\tfrac{1}{2}E^2y_x+\tfrac{1}{3}E^3y_x-\dots\}-\{E^{-1}y_x-\tfrac{1}{2}E^{-2}y_x+\tfrac{1}{3}E^{-3}y_x-\dots\}]$$

$$=\frac{1}{h}[\{\log(1+E)\}y_x-\{\log(1+E^{-1})\}y_x]$$

$$=\frac{1}{h}\log\left(\frac{1+E}{1+E^{-1}}\right)y_x$$

$$=\left\{\frac{1}{h}\log E\right\}y_x=\left(\frac{1}{h}\log e^{hD}\right)y_x \qquad [\because \ E=e^{hD}]$$

$$=Dy_x=\frac{d}{dx}(y_x)$$

= the L.H.S.

Ex. 15. *Find first and second derivatives of the function given below at the point* $x=1\cdot2$

x :	1	2	3	4	5
y :	0	1	5	6	8

(Meerut M. Sc. 1988)

Sol. Here the derivatives are required at $x=1\cdot2$, which is near the beginning of the table so we shall use Newton's Forward formula.

The difference table is given below :

x	y	Δy	Δ^2y	Δ^3y	Δ^4y
1	0				
		1			
2	1		3		
		4		-6	
3	5		-3		10
		1		4	
4	6		1		
		2			
5	8				

Newton's Forward formula is

$$f(a+xh)=f(a)+x\Delta f(a)+\frac{x(x-1)}{2!}\Delta^2f(a)+\frac{x(x-1)(x-2)}{3!}\Delta^3f(a)+\frac{x(x-1)(x-2)(x-3)}{4!}\Delta^4f(a),$$

taking upto 4th differences only.

Differentiating w.r.t. x twice, we get

$$hf'(a+xh)=\Delta f(a)+\frac{(2x-1)}{2}\Delta^2f(a)+\frac{(3x^2-6x+2)}{6}\Delta^3f(a)+\frac{(4x^3-18x^2+22x-6)}{24}\Delta^4f(a)$$

and $h^2 f''(a+xh)=\Delta^2 f(a)+(x-1)\,\Delta^3 f(a)+\frac{(12x^2-36x+22)}{24}\Delta^4 f(a)$.

Putting $a=1$, $h=1$, $x=\cdot2$ and the values of various differences in these equations, we get

$$f'(1\cdot2)=1+\tfrac{1}{2}\{(2\times\cdot2)-1\}\times3+\frac{1}{6}\{(3\times\cdot04)-(6\times\cdot2)+2\}(-6)$$

$$+\frac{1}{24}\{(4\times\cdot008)-(18\times\cdot04)+(22\times\cdot2)-6\}\times10$$

$$=1-0\cdot9-0\cdot92-0\cdot953=-1\cdot773$$

$$f''(1\cdot2)=3+(\cdot2-1)\,(-6)+\frac{1}{24}\{(12\times\cdot04)-(36\times\cdot2)+22\}\times10$$

$$=3+4\cdot8+6\cdot366=14\cdot16.$$

Exercises 5

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x=1$ from the following table :

x :	1	2	3	4	5	6
y :	198669	295520	389418	479425	564642	644217

2. Find $f'''(5)$ given

x :	2	4	9	13	16	21	29
$f(x)$:	57	1345	66340	402052	1118209	4287844	21242820

3. Find the first three derivatives of the function tabulated below at the point $x=2\cdot5$:

x :	1·5	1·9	2·5	3·2	4·3	5·9
$f(x)$:	3·375	6·059	13·625	29·368	73·907	196·579

4. From the following data, find $f'(10)$:

x :	3	5	11	27	34
$f(x)$:	-13	23	899	17315	35606

(Meerut M.Sc. 1986; Rohilkhand 86, 87, 90)

5. Find $\frac{dy}{dx}$ at $x=1$ from the following table :

x :	·7	·8	·9	1·0	1·1	1·2	1·3
y :	0·644218	0·717356	0·783327	0·841471	0·891207	0·932039	0·963558

(Meerut M.Sc. 1974)

6. Find the first and second derivatives of $\log_e x$ at $x=500$.

x	$\log x$
500	6·214608
510	6·234411
520	6·253829
530	6·272877
540	6·291569
550	6·309918

7. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of $y=x^{1/3}$ at $x=50$ from the following table :

x	$y=x^{1/3}$
50	3·6840
51	3·7084
52	3·7325
53	3·7563
54	3·7798
55	3·8030
56	3·8259

8. Find the first two derivatives of $f(x)$ at $x=1$ from the following table :

x :	-2	-1	0	1	2	3	4
$f(x)$:	104	17	0	-1	8	69	272

9. Find the value of $f'(8)$ from the table given below :

x :	6	7	9	12
$f(x)$:	1·556	1·690	1·908	2·158

10. Find $f'(6)$ from the following table :

x :	0	1	3	4	5	7	9
$f(x)$:	150	108	0	-54	-100	-144	-84

11. The following table gives the values of an empirical function $f(x)$ for certain values of x. Find (a) $f'(93)$, (b) the value of x for which $f(x)$ is maximum, (c) the maximum value of $f(x)$ in the range of x :

x :	60	75	90	105	120
$f(x)$:	28·2	38·2	43·2	40·9	37·7

12. By using Stirling's formula of degree six, show that

(i) $y'(x_0)=\frac{1}{h}\left(\delta\mu y_0-\frac{1}{6}\delta^3\mu y_0+\frac{1}{30}\delta^5\mu y_0+\dots\right)$;

(ii) $y''(x_0)=\frac{1}{h^2}\left(\delta^2 y_0-\frac{1}{12}\delta^4 y_0+\dots\right)$. **(Meerut M.Sc. 1972)**

13. Show that the expressions given below are approximations to the third derivative of y_x :

(i) $\Delta^3 y_0+\left(x-\frac{3}{2}\right)\Delta^4 y_0$;

(ii) $\Delta^3 y_{-1}+(x-\frac{1}{2}).\frac{1}{2}\,(\Delta^4 y_{-2}+\Delta^4 y_{-1})$.

14. Use Stirling's formula, to find the first derivative of the function $y=2e^x-x-1$ tabulated below at the point $x=0\cdot6$.

x	y
0·4	1·5836494
0·5	1·7974426
0·6	2·0442376
0·7	2·3275054
0·8	2·6510818

Compare with the true value which is 2·644238.

Answers

1. 98008, −1986. **2.** 1626. **3.** 16·750, 15·00, 6·00. **4.** 233. **5.** 0·54030. **6.** ·002000, −0·0000040. **7.** ·02455, −·0003. **8.** 1, 6. **9.** 0·109. **10.** −23. **11.** $f'(93)=-\cdot036271$, $f(91\cdot92189)=43\cdot2641$. **14.** 2·644225.

6
Numerical Integration

§ 1. Introduction. The process of computing the value of a definite integral from a set of numerical values of the integrand is called *Numerical Integration*. When applied to the integration of a function of a single variable, the process is known as quadrature.

The problem of numerical integration, like that of numerical differentiation, is solved by representing the integrand by an interpolation formula and then integrating this formula between the desired limits. Thus, to find the value of the definite integral $\int_a^b y\,dx$, we replace the function y by an interpolation formula, usually one involving differences, and then integrate this formula between the limits a and b. In this way we can derive quadrature formulas for the approximate integration of any function for which numerical values are known.

§ 2. A General Quadrature Formula for Equidistant Ordinates.
(Meerut M.Sc. 1986, 88, 89, 90, 91, 92P)

Let $I=\int_a^b y\,dx$ where $y=f(x)$. Let $f(x)$ be given for certain equidistant values of x say x_0, x_0+h, x_0+2h,... . Let the range (a, b) be divided into n equal parts, each of width h so that $b-a=nh$.

Let $x_0=a$, $x_1=x_0+h=a+h$, $x_2=a+2h$, ..., $x_n=a+nh=b$. We have assumed that the $n+1$ ordinates $y_0, y_1, \ldots, y_n$ are at equal intervals.

$$\therefore \quad I=\int_a^b y\,dx=\int_{x_0}^{x_0+nh} y_x\,dx=\int_0^n y_{x_0+uh}\,h\,du$$

$$\text{where } u=\frac{x-x_0}{h},\ dx=h\,du$$

$$\text{or } I=h\int_0^n\left[y_0+u\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+\frac{u(u-1)(u-2)}{3!}\Delta^3 y_0+\ldots\right]du$$

$$=h\left[ny_0+\frac{n^2}{2}\Delta y_0+\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!}+\left(\frac{n^4}{4}-n^3+n^2\right)\frac{\Delta^3 y_0}{3!}+\ldots\text{upto } (n+1) \text{ terms}\right] \quad \ldots(1)$$

This is the general quadrature formula. We can deduce a number of formulae from this by putting $n=1, 2, \ldots\ldots$.

§ 3. The Trapezoidal Rule. **(Meerut 80, 89, 91; Roh. 91)**

Putting $n=1$ in the formula (1) of § 2 and neglecting second and higher order differences, we get

$$\int_{x_0}^{x_0+h} y\,dx=h\left[y_0+\frac{1}{2}\Delta y_0\right]$$

$$=h\left[y_0+\frac{y_1-y_0}{2}\right]=h\left[\frac{y_0+y_1}{2}\right].$$

Similarly $$\int_{x_0+h}^{x_0+2h} y\,dx=h\left[\frac{y_1+y_2}{2}\right].$$

... ...

$$\int_{x_0+\overline{n-1}h}^{x_0+nh} y\,dx=h\left[\frac{y_{n-1}+y_n}{2}\right].$$

Adding these n integrals, we get

$$\int_{x_0}^{x_0+nh} y\,dx=h\,[\tfrac{1}{2}(y_0+y_n)+(y_1+y_2+\ldots+y_{n-1})]$$

=distance between two consecutive ordinates

$\times$\{mean of the first and the last ordinates+sum of all the intermediate ordinates\}.

This rule is known as the Trapezoidal rule.

Note. Here we have assumed that y is a function of x of first degree, *i.e.*, the equation of the curve is of the form

$$y=a+bx.$$

§ 4. Simpson's One-third Rule.

(Meerut B.Sc. Stat. 1990, M.Sc. 87, 90, 92; Rohilkhand 88)

Putting $n=2$ in the formula (1) of § 2 and neglecting third and higher order differences, we get

$$\int_{x_0}^{x_0+2h} y\,dx=h\left[2y_0+2\Delta y_0+\frac{\left(\frac{8}{3}-2\right)}{2}\Delta^2 y_0\right]$$

$$=h\,[2y_0+2(y_1-y_0)+\tfrac{1}{3}(y_2-2y_1+y_0)]$$

$$=\frac{h}{3}(y_0+4y_1+y_2).$$

Similarly, $$\int_{x_0+2h}^{x_0+4h} y\,dx=\frac{h}{3}(y_2+4y_3+y_4)$$

... ...

$$\int_{x_0+(n-2)h}^{x_0+nh} y\,dx=\frac{h}{3}(y_{n-2}+4y_{n-1}+y_n),$$

when n is even.

Adding all these integrals, we get

$$\int_{x_0}^{x_0+nh} y\,dx=\frac{h}{3}\,[(y_0+y_n)+4(y_1+y_3+\ldots+y_{n-1})$$
$$+2(y_2+y_4+\ldots+y_{n-2})].$$

This formula is known as Simpson's one-third rule.

Note. Here we have neglected all differences above the second, so y must be a polynomial of second degree only, that is

$$y=ax^2+bx+c.$$

§ 5. Simpson's Three-eighth's Rule. **(Meerut M.Sc. 80, 86, 91, 92P)**

Putting $n=3$ in the formula (1) of § 2 and neglecting all differences above the third, we get

$$\int_{x_0}^{x_0+3h} y\,dx=h\left[3y_0+\frac{9}{2}\Delta y_0+\left(\frac{27}{3}-\frac{9}{2}\right)\frac{\Delta^2 y_0}{2\,!}\right.$$
$$\left.+\left(\frac{81}{4}-27+9\right)\frac{\Delta^3 y_0}{3\,!}\right]$$

$$=h\left[3y_0+\frac{9}{2}(y_1-y_0)+\frac{9}{4}(y_2-2y_1+y_0)\right.$$
$$\left.+\frac{3}{8}(y_3-3y_2+3y_1-y_0)\right]$$

$$=\frac{3h}{8}[y_0+3y_1+3y_2+y_3].$$ **(Meerut M.Sc. 94)**

Similarly, $$\int_{x_0+3h}^{x_0+6h} y\,dx=\frac{3h}{8}[y_3+3y_4+3y_5+y_6]$$

... ...

$$\int_{x_0+(n-\ )h}^{x_0+nh} y\,dx=\frac{3h}{8}[y_{n-3}+3y_{n-2}+3y_{n-1}+y_n].$$

Adding all these integrals where n is a multiple of 3, we have

$$\int_{x_0}^{x_0+nh} y\,dx=\frac{3h}{8}[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\ldots+y_{n-1})$$
$$+2(y_3+y_6+\ldots+y_{n-3})].$$

This formula is known as Simpson's three-eighth's rule.

Note. Here we have neglected all differences above the third so y is a polynomial of the third degree, *i.e.*,

$$y=ax^3+bx^2+cx+d.$$

§ 6. Weddle's Rule. Putting $n=6$ in the formula (1) of § 2 and neglecting all differences of seventh and higher order, we get

$$\int_{x_0}^{x_0+6h} y\,dx=h\left[6y_0+18\Delta y_0+27\Delta^2 y_0+24\Delta^3 y_0\right.$$
$$\left.+\frac{123}{10}\Delta^4 y_0+\frac{33}{10}\Delta^5 y_0+\frac{41}{140}\Delta^6 y_0\right].$$

Here the coefficient of $\Delta^6 y_0$ differs from 3/10 by the small fraction 1/140. Hence if we replace this coefficient by 3/10, we commit an error of only $\frac{h}{140}\Delta^6 y_0$. If the value of h is such that the sixth differences are small, the error committed will be negligible. We therefore change the last term to $(3/10)\Delta^6 y_0$ and replace all differences by their values in terms of the given y's. The result becomes

$$\int_{x_0}^{x_0+6h} y\,dx=\frac{3h}{10}[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6].$$

Similarly,

$$\int_{x_0+6h}^{x_0+12h} y\,dx=\frac{3h}{10}[y_6+5y_7+y_8+6y_9+y_{10}+5y_{11}+y_{12}]$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\int_{x_0+(n-6)h}^{x_0+nh} y\,dx=\frac{3h}{10}[y_{n-6}+5y_{n-5}+y_{n-4}+6y_{n-3}+y_{n-2}+5y_{n-1}+y_n],$$

if n is a multiple of 6.

Adding all these integrals, we have if n is a multiple of 6,

$$\int_{x_0}^{x_0+nh} y\,dx=\frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+2y_6+5y_7+y_8+\ldots\right].$$

This formula is known as Weddle's Rule. It is more accurate, in general, than Simpson's Rule, but it requires at least seven consecutive values of the function.

Note. Here we have assumed that the function y is of the form $y=ax^6+bx^5+cx^4+dx^3+ex^2+fx+g$.

§ 7. Cote's Method. (Agra 1987; Meerut 84, 85, 89, 91, 92P, 93)

This is also a method for evaluating the integral $I=\int_a^b f(x)dx$, when $f(x)$ is known at equidistant values of x. In this method we integrate Lagrange's interpolation formula.

Let $y_0, \ldots, y_n$ be the values of the function $y=f(x)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$. Let the values of the arguments be at equal intervals say h, *i.e.*, $x_i=x_0+ih$.

By Lagrange's formula,

$$f(x)\simeq P(x)=\sum_{k=0}^{n} L_k(x)\,y_k$$

where $L_k(x)=\dfrac{(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}$

Thus $I=\int_a^b f(x)\,dx=\int_{x_0}^{x_0+nh} P(x)\,dx$

$$=\int_{x_0}^{x_0+nh}\left\{\sum_{k=0}^{n} L_k(x)\,y_k\right\}dx$$

$$=\int_{x_0}^{x_0+nh}\{L_0(x)\,y_0+\ldots+L_n(x)\,y_n\}\,dx$$

$$=y_0\int_{x_0}^{x_0+nh} L_0(x)\,dx+\ldots+y_n\int_{x_0}^{x_0+nh} L_n(x)\,dx$$

$$=\sum_{k=0}^{n}\left\{y_k\int_{x_0}^{x_0+nh} L_k(x)\,dx\right\}.$$

Putting $x=x_0+uh$, so that $dx=h\,du$, we get

$$I=h\sum_{k=0}^{n} y_k\int_0^n L_k\,du=nh\sum_{k=0}^{n} y_k\cdot\frac{1}{n}\int_0^n L_k\,du \qquad \ldots(1)$$

where $L_k=L_k(x_0+uh)$.

Now $L_k(x_0+uh)$

$$=\frac{(x_0+uh-x_0)\ldots\{x_0+uh-x_0-(k-1)h\}\{x_0+uh-x_0-(k+1)h\}\ldots(x_0+uh-x_0-nh)}{(x_0+kh-x_0)\ldots\{x_0+kh-x_0-(k-1)h\}\{x_0+kh-x_0-(k+1)h\}\ldots\ldots(x_0+kh-x_0-nh)}$$

$$=\frac{u(u-1)\ldots(u-k+1)(u-k-1)\ldots(u-n)}{k(k-1)\ldots(1)(-1)\ldots(k-n)} \qquad \ldots(2)$$

Now putting $\frac{1}{n}\int_0^n L_k\,du=C_k^n$, we get

$$I=nh\sum_{k=0}^{n} y_k\,C_k^n$$

$$=(b-a)\sum_{k=0}^{n} y_k\,C_k^n \qquad \ldots(3)$$

which is the required formula. It is known as **Cote's formula.**

The numbers C_k^n, $0\le k\le n$ are called *Cote's numbers*. We can evaluate I from (3) if we know the values of Cote's numbers.

Properties of Cote's numbers.

(i) $C_k^n=C_{n-k}^n$ (Meerut 93) (ii) $\sum_{k=0}^{n} C_k^n=1$.

We shall prove the first property.

We have $C_k^n=\frac{1}{n}\int_0^n L_k\,du$

$$=\frac{1}{n}\int_0^n\frac{u(u-1)\ldots\{u-(k-1)\}\{u-(k+1)\}\ldots(u-n)}{k(k-1)\ldots(1)(-1)\ldots(k-n)}\,du$$

$$=\frac{1}{n}\int_0^n\frac{u(u-1)\ldots\{u-(k-1)\}\{u-(k+1)\}\ldots(u-n)}{k!\,(-1)^{n-k}\,(n-k)!}\,du$$

Now C^n_{n-k}

$$= \frac{1}{n}\int_0^n \frac{u(u-1)\ldots\{u-(n-k-1)\}\{u-(n-k+1)\}\ldots(u-n)}{(n-k)!\,(-1)^k\,k!}\,du.$$

Putting $u-n=-t$ or $u=n-t$, $du=-dt$, we get

$$C^n_{n-k} = \frac{1}{n}\int_n^0 \frac{(n-t)(n-t-1)\ldots(-t+k+1)(-t+k-1)\ldots(-t)}{k!\,(n-k)!\,(-1)^k}(-dt)$$

$$= \frac{1}{n}\int_0^n \frac{(-1)^n\, t(t-1)\ldots(t-k+1)(t-k-1)\ldots(t-n)}{k!\,(n-k)!\,(-1)^k}\,dt$$

$$= \frac{1}{n}\int_0^n \frac{(-1)^{n-k}\, t(t-1)\ldots\{t-(k-1)\}\{t-(k+1)\}\ldots(t-n)}{k!\,(n-k)!}\,dt$$

$$= \frac{1}{n}\int_0^n \frac{t(t-1)\ldots\{t-(k-1)\}\{t-(k+1)\}\ldots(t-n)}{k!\,(n-k)!\,(-1)^{n-k}}\,dt$$

{multiplying Nr. and Dr. by $(-1)^{n-k}$}

$$= \frac{1}{n}\int_0^n \frac{u(u-1)\ldots\{u-(k-1)\}\{u-(k+1)\}\ldots(u-n)}{k!\,(n-k)!\,(-1)^{n-k}}\,du$$

$\left\{\because \int_a^b f(x)\,dx = \int_a^b f(t)\,dt,\right.$ by a property of definite integrals$\left.\right\}$

$= C^n_k$.

Some deductions from Cote's formula.

Putting $n=1$ in (3), we get

$$I = \int_{x_0}^{x_1} f(x)\,dx = 1.h\sum_{k=0}^{1} C^1_k\, y_k$$

$$= h\left[C^1_0\, y_0 + C^1_1\, y_1\right].$$

From $C^n_k = \frac{1}{n}\int_0^n L_k\,du$, we get

$$C^1_0 = \frac{1}{1}\int_0^1 L_0\,du = \int_0^1 \frac{(u-1)}{(0-1)}\,du$$

$$= (-1)\left[\frac{u^2}{2}-u\right]_0^1 = \tfrac{1}{2}$$

and $$C^1_1 = \frac{1}{1}\int_0^1 L_1\,du = \int_0^1 \frac{(u-0)}{(1-0)}\,du$$

$$= \left[\frac{u^2}{2}\right]_0^1 = \tfrac{1}{2}.$$

Thus $$I = \frac{h}{2}(y_0+y_1).$$

This is the Trapezoidal rule. **(Meerut 84, 85, 89, 91)**

Putting $n=2$ in (3), we get

$$I = \int_{x_0}^{x_0+2h} f(x)\,dx = 2h\sum_{k=0}^{2} C^2_k\, y_k$$

$$= 2h\left(y_0 C^2_0 + y_1 C^2_1 + y_2 C^2_2\right).$$

Now

$$C^2_0 = \tfrac{1}{2}\int_0^2 L_0\,du = \tfrac{1}{2}\int_0^2 \frac{(u-1)(u-2)}{(0-1)(0-2)}\,du = \frac{1}{6}$$

$$C^2_1 = \tfrac{1}{2}\int_0^2 L_1\,du = \tfrac{1}{2}\int_0^2 \frac{u(u-2)}{(1-0)(1-2)}\,du = \frac{2}{3}$$

$$C^2_2 = C^2_0 = \frac{1}{6} \qquad \left[\because\ C^n_k = C^n_{n-k}\right]$$

$$\therefore\ I = 2h\left[\frac{1}{6}y_0 + \frac{2}{3}y_1 + \frac{1}{6}y_2\right] = \frac{h}{3}(y_0+4y_1+y_2).$$

This is the Simpson's '⅓' rule. **(Agra 1987; Meerut 84, 85, 89, 91, 93)**

Similarly putting $n=3$, we can find the Simpson's '3/8' rule.

Here we give the *Cote's numbers* for some values of n.

	C^n_0	C^n_1	C^n_2	C^n_3	C^n_4	C^n_5	C^n_6
$n=1$	$\frac{1}{2}$	$\frac{1}{2}$					
$n=2$	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$				
$n=3$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$			
$n=4$	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$		
$n=5$	$\frac{19}{288}$	$\frac{75}{288}$	$\frac{50}{288}$	$\frac{50}{288}$	$\frac{75}{288}$	$\frac{19}{288}$	
$n=6$	$\frac{41}{840}$	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	$\frac{27}{840}$	$\frac{216}{840}$	$\frac{41}{840}$

§ 8. The Euler-Maclaurin's Summation Formula.

This formula is derived by the method of expansion of operators.

Let $\Delta F(x) = f(x)$. Then we have an operator Δ^{-1}, known as inverse operator, which is defined by

$$F(x)=\Delta^{-1}f(x).$$

We have $\Delta F(x_0)=f(x_0)$

$\Rightarrow$ $F(x_1)-F(x_0)=f(x_0)$.

Similarly $F(x_2)-F(x_1)=f(x_1)$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$F(x_n)-F(x_{n-1})=f(x_{n-1})$.

On adding all these equations, we get

$$F(x_n)-F(x_0)=\sum_{i=0}^{n-1} f(x_i) \qquad \ldots(1)$$

where $x_0, \ldots, x_n$ are the $(n+1)$ equidistant values of x with difference h.

Also, we have

$$\begin{aligned}
F(x) &= \Delta^{-1}f(x) \\
&= (E-1)^{-1}f(x) & [\because \Delta\equiv E-1] \\
&= (e^{hD}-1)^{-1}f(x) & [\because E\equiv e^{hD}] \\
&= \left\{\left(1+hD+\frac{h^2D^2}{2!}+\frac{h^3D^3}{3!}+\ldots\right)-1\right\}^{-1} f(x) \\
&= \left(hD+\frac{h^2D^2}{2!}+\frac{h^3D^3}{3!}+\ldots\right)^{-1} f(x) \\
&= (hD)^{-1}\left\{1+\left(\frac{hD}{2!}+\frac{h^2D^2}{3!}+\ldots\right)\right\}^{-1} f(x) \\
&= \frac{1}{h}D^{-1}\left\{1-\left(\frac{hD}{2!}+\frac{h^2D^2}{3!}+\ldots\right)\right. \\
&\qquad \left.+\frac{(-1)(-2)}{2!}\left(\frac{hD}{2!}+\frac{h^2D^2}{3!}+\ldots\right)\right\} f(x) \\
&= \frac{1}{h}D^{-1}\left(1-\frac{hD}{2!}+\frac{h^2D^2}{12}-\frac{h^4D^4}{720}+\ldots\right) f(x) \\
&= \left(\frac{1}{h}D^{-1}-\frac{1}{2}+\frac{hD}{12}-\frac{h^3D^3}{720}+\ldots\right) f(x) \\
&= \frac{1}{h}\int f(x)\,dx-\tfrac{1}{2}f(x)+\frac{h}{12}f'(x)-\frac{h^3}{720}f'''(x)+\ldots \qquad \ldots(2)
\end{aligned}$$

$\{D^{-1}$ is integral operator$\}$

Putting $x=x_n$ and $x=x_0$ successively in (2) and then subtracting, we get

$$F(x_n)-F(x_0)=\frac{1}{h}\int_{x_0}^{x_n} f(x)\,dx-\tfrac{1}{2}[f(x_n)-f(x_0)]$$
$$+\frac{h}{12}[f'(x_n)-f'(x_0)]-\frac{h^3}{720}[f'''(x_n)-f'''(x_0)]+\ldots \qquad \ldots(3)$$

From (1) and (3), we get

$$\sum_{i=0}^{n-1} f(x_i)=\frac{1}{h}\int_{x_0}^{x_n} f(x)\,dx-\tfrac{1}{2}[f(x_n)-f(x_0)]$$
$$+\frac{h}{12}[f'(x_n)-f'(x_0)]-\frac{h^3}{720}[f'''(x_n)-f'''(x_0)]+\ldots$$

$$\therefore \quad \frac{1}{h}\int_{x_0}^{x_n} f(x)\,dx=\sum_{i=0}^{n-1} f(x_i)+\tfrac{1}{2}[f(x_n)-f(x_0)]$$
$$-\frac{h}{12}[f'(x_n)-f'(x_0)]+\frac{h^3}{720}[f'''(x_n)-f'''(x_0)]-\ldots$$

But $\sum_{i=0}^{n-1} f(x_i)=\sum_{i=0}^{n} f(x_i)-f(x_n)$ and $x_n=x_0+nh$.

$$\therefore \quad \frac{1}{h}\int_{x_0}^{x_0+nh} f(x)\,dx=\sum_{i=0}^{n} f(x_i)-\tfrac{1}{2}[f(x_n)+f(x_0)]$$
$$-\frac{h}{12}[f'(x_0+nh)-f'(x_0)]+\frac{h^3}{720}[f'''(x_0+nh)-f'''(x_0)]-\ldots$$

This is called Euler-Maclaurin's summation formula.

§ 9. Stirling's Formula for Approximation to Factorials. (Roh. 90)

In Euler-Maclaurin formula, putting $x_0=1$, $h=1$, $n-1$ for n,

$f(x)=\log_e x, f'(x)=\dfrac{1}{x}, f'''(x)=\dfrac{2}{x^3}, \ldots$, we get

$$\int_1^n \log x\,dx=\tfrac{1}{2}\log 1+\log 2+\log 3+\ldots+\log(n-1)+\tfrac{1}{2}\log n$$
$$-\frac{1}{12}\left(\frac{1}{n}-\frac{1}{1}\right)+\frac{1}{720}\left(\frac{2}{n^3}-\frac{2}{1^3}\right)-\ldots$$

or $\quad \Big[x\log x-x\Big]_1^n=\log(n\,!)-\tfrac{1}{2}\log n-\dfrac{1}{12n}+\dfrac{1}{360n^3}-\ldots$

or $\quad \log(n\,!)=(n+\tfrac{1}{2})\log n-n+\dfrac{1}{12n}-\dfrac{1}{360n^3}+\ldots+c \qquad \ldots(1)$

where c is a constant independent of n.

To find c, we make use of Walli's formula in Trigonometry, viz.

$$\frac{\pi}{2}=\lim_{n\to\infty}\left[\frac{2.2}{1.3}\cdot\frac{4.4}{3.5}\cdot\frac{6.6}{5.7}\cdots\frac{2n.2n}{(2n-1)\,(2n+1)}\right]$$
$$=\lim_{n\to\infty}\frac{2^{4n}\,(n\,!)^4}{[(2n)\,!]^2\,(2n+1)}$$

$$\therefore \quad \log\frac{\pi}{2}=\lim_{n\to\infty}[4n\log 2+4\log(n\,!)-2\log[(2n)\,!]-\log(2n+1)].$$

Substituting the values of $\log(n\,!)$ and $\log[(2n)\,!]$ from (1), we get

$$\log\frac{\pi}{2}=\lim_{n\to\infty}\Big[4n\log 2+4\{(n+\tfrac12)\log n-n+c\}$$
$$-2\{(2n+\tfrac12)\log(2n)-2n+c\}-\log(2n+1)\Big]$$
$$=\lim_{n\to\infty}[2c+2\log n-\log(2n)-\log(2n+1)]$$
$$=\lim_{n\to\infty}\left[2c+\log\frac{n^2}{4n^2+2n}\right]$$
$$=2c+\log(\tfrac14)=2c-\log 4.$$
$$\therefore\quad c=\tfrac12\log(2\pi)=\log\sqrt{(2\pi)}.$$

Substituting the value of c in (1), we get

$$\log(n\,!)=(n+\tfrac12)\log n+\log\sqrt{(2\pi)}-n+\frac{1}{12n}-\frac{1}{360n^3}+\ldots$$
$$=\log\left\{\sqrt{(2\pi)}.n^{n+\frac12}\right\}+\log\exp\left\{-n\left(1-\frac{1}{12n^2}+\frac{1}{360\,n^4}-\ldots\right)\right\}$$
$$\therefore\quad n\,!=\sqrt{(2\pi)}.n^{n+\frac12}\exp.\left\{-n\left(1-\frac{1}{12n^2}+\frac{1}{360\,n^4}-\ldots\right)\right\}$$

or $$n\,!\sim\sqrt{(2\pi)}.n^{n+\frac12}\,e^{-n}.$$

This is Stirling's formula for factorial n, when n is large.

§ 10. Gaussian Integration.

For evaluating the integral $I=\int_a^b f(x)\,dx$, we derived some integration formulae which require values of the function at equally spaced points of the interval. Gauss derived a formula which uses the same number of function values but with different spacing and gives better accuracy.

Gauss's formula is expressed in the form

$$\int_{-1}^{1}F(u)\,du=w_1F(u_1)+w_2F(u_2)+\ldots+w_nF(u_n)$$
$$=\sum_{i=1}^{n}w_iF(u_i),\qquad\ldots(1)$$

where w_i and u_i are called the *weights* and *abscissae* respectively. In this formula the abscissae and weights are symmetrical with respect to the middle point of the interval.

In equation (1), there are $2n$ unknowns. Thus $2n$ relations between them are necessary which can be obtained such that the formula is exact for all polynomials of degree not exceeding $(2n-1)$. Hence, we take

$$F(u)=c_0+c_1u+c_2u^2+c_3u^3+\ldots+c_{2n-1}u^{2n-1}.\qquad\ldots(2)$$

Then from the left band side of (1), we obtain

$$\int_{-1}^{1}F(u)\,du=\int_{-1}^{1}\left[c_0+c_1u+c_2u^2+c_3u^3+\ldots+c_{2n-1}u^{2n-1}\,du\right]$$
$$=2c_0+\frac23c_2+\frac25c_4+\ldots\ldots\qquad\ldots(3)$$

Putting $u=u_i$ in (2), we get

$$F(u_i)=c_0+c_1u_i+c_2u_i^2+c_3u_i^3+\ldots+c_{2n-1}u_i^{2n-1}.$$

Substituting these values on the right hand side of (1), we get

$$\int_{-1}^{1}F(u)\,du=w_1\left[c_0+c_1u_1+c_2u_1^2+c_3u_1^3+\ldots+c_{2n-1}u_1^{2n-1}\right]$$
$$+w_2\left[c_0+c_1u_2+c_2u_2^2+c_3u_2^3+\ldots+c_{2n-1}u_2^{2n-1}\right]$$
$$+w_3\left[c_0+c_1u_3+c_2u_3^2+c_3u_3^3+\ldots+c_{2n-1}u_3^{2n-1}\right]$$
$$+\quad\ldots\quad\ldots\quad\ldots$$
$$+w_n\left[c_0+c_1u_n+c_2u_n^2+c_3u_n^3+\ldots+c_{2n-1}u_n^{2n-1}\right]$$

or
$$\int_{-1}^{1}F(u)\,du=c_0\left[w_1+w_2+w_3+\ldots+w_n\right]$$
$$+c_1\left[w_1u_1+w_2u_2+w_3u_3+\ldots+w_nu_n\right]$$
$$+c_2\left[w_1u_1^2+w_2u_2^2+w_3u_3^2+\ldots+w_nu_n^2\right]$$
$$+\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$+c_{2n-1}\left[w_1u_1^{2n-1}+w_2u_2^{2n-1}+w_3u_3^{2n-1}+\ldots+w_nu_n^{2n-1}\right]\qquad\ldots(4)$$

Now, the equations (3) and (4) are identical for all values of c_i and hence comparing the coefficients of c_i, we obtain the $2n$ equations in $2n$ unknowns w_i and u_i $(i=1,2,\ldots,n)$:

$$\left.\begin{array}{l}w_1+w_2+w_3+\ldots+w_n=2\\ w_1u_1+w_2u_2+w_3u_3+\ldots+w_nu_n=0\\ w_1u_1^2+w_2u_2^2+w_3u_3^2+\ldots+w_nu_n^2=\dfrac23\\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\ w_1u_1^{2n-1}+w_2u_2^{2n-1}+w_3u_3^{2n-1}+\ldots+w_nu_n^{2n-1}=0.\end{array}\right\}\quad\ldots(5)$$

We can think to obtain the $2n$ unknowns w_i's and u_i's at least theoretically, by solving the above system of non-linear equations by the usual algebra method, but in practice it would be quite difficult even for small values of n. However, the above

system can be reduced to a system of linear equations, if we know $u_1, u_2, \ldots, u_n$.

It can, however, be shown that the u_i's, are the zeros of a set of polynomials called *Legender polynomials*. Here it is sufficient to mention that the abscissae u_i and the weights w_i are extensively tabulated for different values of n. We list below the abscissae and weights for values of n upto $n=6$.

Table 6·1. Abscissae and Weights For Gaussian Integration.

n	$\pm u_i$	w_i
2	0·5773502692	1·0000000000
3	0·0000000000	0·8888888889
	0·7745966692	0·5555555556
4	0·3399810436	0·6521451549
	0·8611363116	0·3478548451
5	0·0000000000	0·5688888889
	0·5384693101	0·4786286705
	0·9061798459	0·2369268851
6	0·2386191861	0·4679139346
	0·6612093865	0·3607615730
	0·9324695142	0·1713244924

In the general case, the limits of the integral $\int_a^b f(x)\,dx$ have to be changed to those in (1) by means of the transformation

$$x=\tfrac{1}{2}u\,(b-a)+\tfrac{1}{2}\,(a+b).$$

Solved Examples

Ex. 1 *Calculate the value of the integral*

$$\int_4^{5\cdot 2} \log x\,dx \text{ by}$$

(a) *Trapezoidal rule*, **(b)** *Simpson's '$\frac{1}{3}$' rule*
(c) *Simpson's '$\frac{3}{8}$' rule*, **(d)** *Weddle's rule.* **(Meerut 90)**

After finding the true value of the integral, compare the errors in the four cases.

Sol. Taking $h=\cdot 2$ divide the whole range of integration (4, 5·2) into six equal parts. The values of $\log x$ for each point of sub-division are given below :

x	$y=\log_e x$	x	$y=\log_e x$
$x_0=4\cdot 0$	$y_0=1\cdot 3862944$	$x_0+4h=4\cdot 8$	$y_4=1\cdot 5686159$
$x_0+h=4\cdot 2$	$y_1=1\cdot 4350845$	$x_0+5h=5\cdot 0$	$y_5=1\cdot 6094379$
$x_0+2h=4\cdot 4$	$y_2=1\cdot 4816045$	$x_0+6h=5\cdot 2$	$y_6=1\cdot 6486586$
$x_0+3h=4\cdot 6$	$y_3=1\cdot 5260563$		

(a) By Trapezoidal rule, we have

$$\int_4^{5\cdot 2} \log_e x\,dx=\frac{h}{2}\left[y_0+y_6+2\left\{y_1+y_2+y_3+y_4+y_5\right\}\right]$$

$$=\frac{\cdot 2}{2}\left[3\cdot 034953+2\times 7\cdot 6207991\right]$$

$$=\cdot 1\,(18\cdot 276551)=1\cdot 8276551.$$

(b) By Simpson's '$\frac{1}{3}$' rule, we have

$$\int_4^{5\cdot 2} \log_e x\,dx=\frac{h}{3}\left[y_0+y_6+4\,(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$=\frac{\cdot 2}{3}\left[3\cdot 034953+4\,(4\cdot 5705787)+2\,(3\cdot 0502204)\right]$$

$$=\frac{\cdot 2}{3}\left[3\cdot 034953+18\cdot 282315+6\cdot 1004408\right]$$

$$=\frac{\cdot 2}{3}\times 27\cdot 417709=1\cdot 8278472.$$

(c) By Simpson's $\frac{3}{8}$ rule, we have

$$\int_4^{5\cdot 2} \log_e x\,dx=\frac{3h}{8}\left[y_0+y_6+3\,(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$=\frac{3\,(\cdot 2)}{8}\left[3\cdot 034953+3\,(6\cdot 0947428)+2\,(1\cdot 5260563)\right]$$

$$=\frac{\cdot 6}{8}\left[3\cdot 034953+18\cdot 284228+3\cdot 0521126\right]=\frac{\cdot 6}{8}\times 24\cdot 371294$$

$$=1\cdot 827847.$$

(d) By Weddle's rule, we have

$$\int_4^{5\cdot 2} \log_e x\,dx=\frac{3h}{10}\left[y_0+y_6+5\,(y_1+y_5)+y_2+y_4+6y_3\right]$$

$$=\frac{3\,(\cdot 2)}{10}\Big[3\cdot034953+5\,(3\cdot0445224)+3\cdot0502204+6\,(1\cdot5260563)\Big]$$

$$=\frac{\cdot 6}{10}\Big[3\cdot034953+15\cdot222612+3\cdot0502204+9\cdot1563378\Big]$$

$$=\frac{\cdot 6}{10}\times 30\cdot464123=1\cdot8278474.$$

Actual value of $\displaystyle\int_4^{5\cdot2}\log_e x\,dx=\Big[x\,(\log x-1)\Big]_4^{5\cdot2}$

$$=[5\cdot2\,(\log 5\cdot2-1)-4\,(\log 4-1)]$$
$$=[3\cdot3730249-1\cdot5451774]$$
$$=1\cdot8278475.$$

Hence the errors due to different formulae are

(a) $\cdot0001924$ (b) $\cdot0000003$

(c) $\cdot0000005$ (d) $\cdot0000001$.

We observe that Weddle's rule is more accurate.

Ex. 2. *Find* $\displaystyle\int_0^1\frac{dx}{1+x^2}$ *by using Simpson's* $\frac{1}{3}$ *and* $\frac{3}{8}$ *rule. Hence obtain the approximate value of* π *in each case.*

(Meerut 1980)

Sol. Divide the range of integration (0, 1) into 6 equal parts each of width $\frac{1-0}{6}=\frac{1}{6}$, so that $h=\frac{1}{6}$. The values of $f(x)$ at each point of sub-division are given below :

x	$y=\frac{1}{1+x^2}$
$x_0=0$	$1/1=1\cdot0000000$
$x_0+h=1/6$	$36/37=0\cdot9729729$
$x_0+2h=2/6$	$36/40=0\cdot9000000$
$x_0+3h=3/6$	$36/45=0\cdot8000000$
$x_0+4h=4/6$	$36/52=0\cdot6923076$
$x_0+5h=5/6$	$36/61=0\cdot5901639$
$x_0+6h=1$	$1/2=0\cdot5000000$

By Simpson's '1/3' rule, we get

$$\int_0^1\frac{dx}{1+x^2}=\int_{x_0}^{x_0+6h}\frac{dx}{1+x^2}=\frac{h}{3}\Big[y_0+y_6+4\,(y_1+y_3+y_5)+2\,(y_2+y_4)\Big]$$

$$=\frac{1}{18}\Big[1\cdot5000000+4\,(2\cdot3631369)+2\,(1\cdot5923077)\Big]$$

$$=\frac{1}{18}\,(14\cdot137163)=0\cdot7853979. \qquad \ldots(1)$$

By Simpson's '3/8' rule, we get

$$\int_0^1\frac{dx}{1+x^2}=\frac{3h}{8}\Big[y_0+y_6+3\,(y_1+y_2+y_4+y_5)+2y_3\Big]$$

$$=\frac{1}{16}\Big[1\cdot5000000+3\,(3\cdot1554446)+2\,(\cdot8000000)\Big]$$

$$=\frac{1}{16}\,(12\cdot566334)=0\cdot7853958. \qquad \ldots(2)$$

But $\displaystyle\int_0^1\frac{dx}{1+x^2}=\Big[\tan^{-1}x\Big]_0^1=\tan^{-1}1-\tan^{-1}0=\pi/4$ $\qquad\ldots(3)$

Now from (1) and (3), we get

$\pi/4=0\cdot7853979$ or $\pi=3\cdot1415916.$

From (2) and (3), we get

$\pi/4=0\cdot7853958$ or $\pi=3\cdot1415835.$

Ex. 3. *Evaluate the value of the integral*

$$\int_{\cdot2}^{1\cdot4}(\sin x-\log_e x+e^x)\,dx$$

by (a) *Trapezoidal rule,* (b) *Simpson's '⅓' rule,* (c) *Simpson's '⅜' rule,* (d) *Weddle's rule.*

After finding the true value of the integral, compare the errors in the four cases.

Sol. Divide the range of integration ($\cdot2$, $1\cdot4$) into 12 equal parts each of width $=\frac{1\cdot4-\cdot2}{12}=\cdot1$. Hence $h=\cdot1$. The values of the function at each point of sub-division are given below :

x	$\sin x$	$\log_e x$	e^x	$y=\sin x - \log_e x + e^x$
$x_0=\cdot 2$	·19867	−1·60943	1·22140	3·02950
$x_0+h=\cdot 3$	·29552	−1·20397	1·34986	2·84935
$x_0+2h=\cdot 4$	·38942	−·91629	1·49182	2·79753
$x_0+3h=\cdot 5$	·47943	−·69315	1·64872	2·82130
$x_0+4h=\cdot 6$	·56464	−·51083	1·82212	2·89759
$x_0+5h=\cdot 7$	·64422	−·35667	2·01375	3·01464
$x_0+6h=\cdot 8$	·71736	−·22314	2·22554	3·16604
$x_0+7h=\cdot 9$	·78333	−·10536	2·45960	3·34829
$x_0+8h=1\cdot 0$	·84147	·00000	2·71828	3·55975
$x_0+9h=1\cdot 1$	·89121	·09531	3·00417	3·80007
$x_0+10h=1\cdot 2$	·93204	·18232	3·32012	4·06984
$x_0+11h=1\cdot 3$	·96356	·26236	3·66930	4·37050
$x_0+12h=1\cdot 4$	·98545	·33647	4·05520	4·70418

(*a*) By Trapezoidal rule, we get

$$\int_{\cdot 2}^{1\cdot 4} y\,dx=\frac{h}{2}\,[y_0+y_{12}+2\,(y_1+y_2+\ldots+y_{10}+y_{11})]$$

$$=\frac{\cdot 1}{2}\,[7\cdot 73368+2\,(36\cdot 69481)]=\frac{\cdot 1}{2}\,(81\cdot 1233)$$

$$=4\cdot 056165=4\cdot 05617.$$

(*b*) By Simpson's '$\frac{1}{3}$' rule, we get

$$\int_{\cdot 2}^{1\cdot 4} y\,dx=\frac{h}{3}\,[y_0+y_{12}+4\,(y_1+y_3+y_5+y_7+y_9+y_{11})$$
$$+2\,(y_2+y_4+y_6+y_8+y_{10})]$$

$$=\frac{\cdot 1}{3}\,[7\cdot 73368+4\,(20\cdot 20415)+2\,(16\cdot 49075)]$$

$$=\frac{\cdot 1}{3}\,(121\cdot 53178)=4\cdot 0510593=4\cdot 05106.$$

(*c*) By Simpson's '$\frac{3}{8}$' rule, we get

$$\int_{\cdot 2}^{1\cdot 4} y\,dx=\frac{3h}{8}\,[y_0+y_{12}+3\,(y_1+y_2+y_4+y_5+y_7+y_8+y_{10}+y_{11})$$
$$+2\,(y_3+y_6+y_9)]$$

$$=\frac{3\,(\cdot 1)}{8}\,[7\cdot 73368+3\,(26\cdot 90749)+2\,(9\cdot 78741)]$$

$$=\frac{\cdot 3}{8}\,(108\cdot 03097)=4\cdot 0511614$$

$$=4\cdot 05116.$$

(*d*) By Weddle's rule, we get

$$\int_{\cdot 2}^{1\cdot 4} y\,dx=\frac{3\,(\cdot 1)}{10}\,[y_0+y_{12}+5\,(y_1+y_5+y_7+y_{11})+2y_6$$
$$+y_2+y_4+y_8+y_{10}+6\,(y_3+y_9)]$$

$$=\frac{\cdot 3}{10}\,[7\cdot 73368+5\,(13\cdot 58278)+2\,(3\cdot 16604)$$
$$+13\cdot 32471+6\,(6\cdot 62137)]$$

$$=\frac{\cdot 3}{10}\,(135\cdot 03259)$$

$$=4\cdot 0509777=4\cdot 05098.$$

Actual value of $\int_{\cdot 2}^{1\cdot 4} (\sin x-\log_e x+e^x)\,dx$

$$=\Big[-\cos x-x\,(\log_e x-1)+e^x\Big]_{\cdot 2}^{1\cdot 4}$$

$$=[\{-\cos 1\cdot 4-(1\cdot 4)\,(\log_e 1\cdot 4-1)+e^{1\cdot 4}\}$$
$$-\{-\cos \cdot 2-(\cdot 2)\,(\log_e \cdot 2-1)+e^{\cdot 2}\}]$$

$$=4\cdot 05095.$$

Hence the errors are :

due to Trapezoidal rule	−·00522
due to Simpson's '$\frac{1}{3}$' rule	−·00011
due to Simpson's '$\frac{3}{8}$' rule	−·00021
due to Weddle's rule	−·00003.

Thus we observe that Weddle's rule gives more accurate result than other rules.

Ex. 4. *Calculate by Simpson's rule an approximate value of $\int_{-3}^{3} x^4\,dx$ by taking seven equidistant ordinates. Compare it with the exact value and the value obtained by using the Trapezoidal rule.*

Sol. Divide the range of integration (−3, 3) into six equal parts each of width$=\frac{3-(-3)}{6}=1$. Hence $h=1$. The values of the function for each point of sub-division are given below :

x	$y=x^4$
$x_0=-3$	$(-3)^4=81$
$x_0+h=-2$	$(-2)^4=16$
$x_0+2h=-1$	$(-1)^4=1$
$x_0+3h=0$	$0^4=0$
$x_0+4h=1$	$1^4=1$
$x_0+5h=2$	$2^4=16$
$x_0+6h=3$	$3^4=81$

By Simpson's rule, we get

$$\int_{-3}^{3} x^4\,dx=\frac{h}{3}\,[y_0+y_6+4\,(y_1+y_3+y_5)+2\,(y_2+y_4)]$$
$$=\tfrac{1}{3}\,[162+4\times 32+2\times 2]$$
$$=\tfrac{1}{3}\times 294=98.$$

The exact value of

$$\int_{-3}^{3} x^4\,dx=\left[\frac{x^5}{5}\right]_{-3}^{3}=\frac{1}{5}\left[(3)^5-(-3)^5\right]$$
$$=\frac{1}{5}\times 486=97\cdot 2.$$

By Trapezoidal rule, we get

$$\int_{-3}^{3} x^4\,dx=\frac{h}{2}\left[y_0+y_6+2\,(y_1+y_2+y_3+y_4+y_5)\right]$$
$$=\frac{1}{2}\left[162+2\times 34\right]=115.$$

In this case we observe that the Trapezoidal rule does not give an accurate result. In general Simpson's rule gives a better result than the Trapezoidal rule.

Ex. 5. *A curve is drawn to pass through the points given by the following table :*

x :	1	1·5	2	2·5	3	3·5	4
y :	2	2·4	2·7	2·8	3	2·6	2·1

Find the area bounded by the curve, the x-axis and the lines $x=1$, $x=4$.

Sol. In order to find the required area we shall compute the value of the integral

$$I=\int_1^4 y\,dx.$$

Here $n=6$, $h=\cdot 5$.

By Simpson's '$\frac{1}{3}$' rule, we get

$$I=\frac{h}{3}\left[y_0+y_6+4\,(y_1+y_3+y_5)+2\,(y_2+y_4)\right]$$
$$=\frac{\cdot 5}{3}\left[4\cdot 1+4\,(7\cdot 8)+2\,(5\cdot 7)\right]$$
$$=\frac{\cdot 5}{3}\times 46\cdot 7=7\cdot 78 \text{ units of area.}$$

By Weddle's rule, we get

$$I=\frac{3h}{10}\left[y_0+y_6+5\,(y_1+y_5)+y_2+y_4+6y_3\right]$$
$$=\frac{3\,(\cdot 5)}{10}\left[4\cdot 1+25+5\cdot 7+16\cdot 8\right]$$
$$=\cdot 15\times 51\cdot 6=7\cdot 74 \text{ units of area.}$$

Ex. 6. *Show that* $\int_0^1 \frac{dx}{1+x}=\log 2=0\cdot 69315.$

(Meerut B. Sc. 74)

Sol. To use Simpson's '1/3' rule, divide the range of integration (0, 1) into ten equal parts each of width 1/10. Hence $h=1/10=\cdot 1$, $n=10$.

The values of y at each point of sub-division are given below :

x	$y=\dfrac{1}{1+x}$
$x_0=0$	1
$x_0+h=\cdot1$	$1/1\cdot1=\cdot9090909$
$x_0+2h=\cdot2$	$1/1\cdot2=\cdot8333333$
$x_0+3h=\cdot3$	$1/1\cdot3=\cdot7692307$
$x_0+4h=\cdot4$	$1/1\cdot4=\cdot7142857$
$x_0+5h=\cdot5$	$1/1\cdot5=\cdot6666666$
$x_0+6h=\cdot6$	$1/1\cdot6=\cdot6250000$
$x_0+7h=\cdot7$	$1/1\cdot7=\cdot5882352$
$x_0+8h=\cdot8$	$1/1\cdot8=\cdot5555555$
$x_0+9h=\cdot9$	$1/1\cdot9=\cdot5263157$
$x_0+10h=1$	$1/2=\cdot5000000$

By Simpson's '$\frac{1}{3}$' rule, we get

$$\int_0^1 \frac{dx}{1+x}=\frac{h}{3}\Big[y_0+y_{10}+4\,(y_1+y_3+y_5+y_7+y_9)$$
$$+2\,(y_2+y_4+y_6+y_8)\Big]$$
$$=\frac{\cdot1}{3}\,[1\cdot5+4\times3\cdot4595391+2\times2\cdot7281745]$$
$$=\frac{\cdot1}{3}\times20\cdot794505=\cdot6931501$$
$$=\cdot69315.$$

The actual value of $\displaystyle\int_0^1 \frac{dx}{1+x}=\Big[\log(1+x)\Big]_0^1$
$$=\log 2-\log 1=\log 2.$$

Hence $\displaystyle\int_0^1 \frac{dx}{1+x}=\log 2=\cdot69315.$

Ex. 7. *Calculate (upto 4 places of decimal)* $\displaystyle\int_2^{10}\frac{dx}{1+x}$ *by dividing the range into eight equal parts.*

(**Meerut B. Sc. 1975, 76, 77, 80 ; M. Sc. 86**)

Sol. Divide the range of integration (2, 10) into eight equal parts each of width $\dfrac{10-2}{8}=1$. Hence $h=1$. The values of y for each point of sub-division are given below :

x	$y=\dfrac{1}{1+x}$
$x_0=2$	$1/3$
$x_0+h=3$	$1/4$
$x_0+2h=4$	$1/5$
$x_0+3h=5$	$1/6$
$x_0+4h=6$	$1/7$
$x_0+5h=7$	$1/8$
$x_0+6h=8$	$1/9$
$x_0+7h=9$	$1/10$
$x_0+8h=10$	$1/11$

By Simpson's '1/3' rule, we get

$$\int_2^{10}\frac{dx}{1+x}=\frac{h}{3}\Big[y_0+y_8+4\,(y_1+y_3+y_5+y_7)+2\,(y_2+y_4+y_6)\Big]$$
$$=\frac{1}{3}\Big[\frac{1}{3}+\frac{1}{11}+4\Big(\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}\Big)$$
$$+2\Big(\frac{1}{5}+\frac{1}{7}+\frac{1}{9}\Big)\Big]$$
$$=\frac{1}{3}\Big[\cdot4242424+4\,(\cdot6416666)+2\,(\cdot4539682)\Big]$$
$$=\tfrac{1}{3}\times3\cdot8988453=1\cdot2996151=1\cdot2996.$$

Ex. 8. *Evaluate* $\displaystyle\int_0^6\frac{dx}{1+x^2}$ *by using (i) Simpson's '1/3' rule, (ii) Simpson's '3/8' rule, (iii) Weddle's rule.*

(**Meerut 1980, 92; Kanpur B.Sc. 73**)

Sol. Divide the range of integration (0, 6) into six equal parts each width $\dfrac{6-0}{6}=1$. Hence $h=1$. The values of y for each point of sub-division are given below :

x	$y=\frac{1}{1+x^2}$
$x_0=0$	$1/1=1\cdot0000000$
$x_0+h=1$	$1/2=\cdot5000000$
$x_0+2h=2$	$1/5=\cdot2000000$
$x_0+3h=3$	$1/10=\cdot1000000$
$x_0+4h=4$	$1/17=\cdot0588235$
$x_0+5h=5$	$1/26=\cdot0384615$
$x_0+6h=6$	$1/37=\cdot0270270$

By Simpson's '1/3' rule, we get

$$\int_0^6 \frac{dx}{1+x^2}=\frac{h}{3}\left[y_0+y_6+4\,(y_1+y_3+y_5)+2\,(y_2+y_4)\right]$$

$$=\frac{1}{3}\left[1\cdot0270270+2\cdot5538462+\cdot517647\right]$$

$$=1\cdot3661734.$$

By Simpson's '3/8' rule, we get

$$\int_0^6 \frac{dx}{1+x^2}=\frac{3h}{8}\left[y_0+y_6+3\,(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$=\frac{3}{8}\left[1\cdot0270270+2\cdot391855+\cdot200000\right]$$

$$=\frac{3}{8}\times3\cdot618882=1\cdot3570808.$$

By Weddle's rule, we get

$$\int_0^6 \frac{dx}{1+x^2}=\frac{3h}{10}\left[y_0+y_6+5\,(y_1+y_5)+y_2+y_4+6y_3\right]$$

$$=\frac{3}{10}\left[1\cdot0270270+2\cdot6923075+\cdot8588235\right]$$

$$=1\cdot3734474.$$

Ex. 9. *Evaluate* $\int_0^4 e^x\,dx$, *by Simpson's rule, given that* $e=2\cdot72$, $e^2=7\cdot39$, $e^3=20\cdot09$, $e^4=54\cdot60$, *and compare it with the actual value.* **(Meerut M.Sc. 1971, 88)**

Sol. Divide the whole range (0, 4) into 4 equal parts taking $h=1$.

By Simpson's '1/3' rule, we get

$$\int_0^4 e^x\,dx=\frac{h}{3}\left[y_0+y_4+4\,(y_1+y_3)+2y_2\right]$$

$$=\frac{1}{3}\left[1+54\cdot60+4\,(2\cdot72+20\cdot09)+2\times7\cdot39\right]$$

$$=\frac{1}{3}\times161\cdot62=53\cdot873333=53\cdot87.$$

The actual value of $\int_0^4 e^x\,dx=\left[e^x\right]_0^4=e^4-e^0$

$$=54\cdot60-1=53\cdot60.$$

Ex. 10. *Use Simpson's rule to prove that* $\log_e 7$ *is approximately* $1\cdot9587$ *using* $\int_1^7 \frac{dx}{x}$. **(Rohilkhand M.Sc. 1990)**

Sol. Divide the range (1, 7) into six equal parts each of width 1. Hence $h=1$. The values of y for each point of sub-division are given below :

x : $x_0=1$ $x_1=2$ $x_2=3$ $x_3=4$ $x_4=5$ $x_5=6$ $x_6=7$

y : $y_0=1$ $y_1=1/2$ $y_2=1/3$ $y_3=1/4$ $y_4=1/5$ $y_5=1/6$ $y_6=1/7$

By Simpson's '1/3' rule, we have

$$\int_1^7 \frac{dx}{x}=\frac{h}{3}\left[y_0+y_6+4\,(y_1+y_3+y_5)+2\,(y_2+y_4)\right]$$

$$=\frac{1}{3}\left[1\cdot1428571+3\cdot6666667+1\cdot0666667\right]=\frac{1}{3}\times5\cdot8761905$$

$$=1\cdot9587302=1\cdot9587$$

The exact vaule of $\int_1^7 \frac{dx}{x}=\left[\log x\right]_1^7=\log_e 7.$

Thus $\log_e 7=1\cdot9587.$

Ex. 11. *Evaluate* $\int_{0\cdot5}^{0\cdot7} x^{1/2}e^{-x}\,dx$ *approximately by using a suitable formula.* **(Rohilkhand 1988)**

Sol. Divide the range of integration (·5, ·7) into 4 equal parts each of width $\frac{\cdot7-\cdot5}{4}=\cdot05$. Hence $h=\cdot05$. The values of y for e ach point of sub-division are given below :

x	$y=x^{1/2}e^{-x}$
$x_0=\cdot50$	·4288818
$x_0+h=\cdot55$	·4278774
$x_0+2h=\cdot60$	·4251076
$x_0+3h=\cdot65$	·4208867
$x_0+4h=\cdot70$	·4154730

By Simpson's '1/3' rule, we have

$$\int_{\cdot5}^{\cdot7} x^{1/2}e^{-x}\,dx=\frac{h}{3}\left[y_0+y_4+4(y_1+y_3)+2y_2\right]$$

$$=\frac{\cdot05}{3}\left[\cdot8413548+3\cdot3950564+\cdot8502152\right]$$

$$=\cdot0848271.$$

Ex. 12. *Calculate an approximate value of* $\int_0^{\pi/2}\sin x\,dx$ *by (a) the Trapezoidal rule, (b) Simpson's '⅓' rule, using 11 ordinates.*

Sol. First we divide the range of integration into ten equal parts by taking the interval of differencing $h=\pi/20$ and then we compute the values of the function $f(x)=\sin x$ for each point of sub-division. These computed values are as shown in the following table.

x	$y=\sin x$	x	$y=\sin x$
$x_0=0$	·00000	$x_0+6h=6\pi/20$	·80902
$x_0+h=\pi/20$	·15643	$x_0+7h=7\pi/20$	·89101
$x_0+2h=2\pi/20$	·30902	$x_0+8h=8\pi/20$	·95106
$x_0+3h=3\pi/20$	·45399	$x_0+9h=9\pi/20$	·98769
$x_0+4h=4\pi/20$	·58778	$x_0+10h=10\pi/20$	1·00000
$x_0+5h=5\pi/20$	·70711		

(a) **By Trapezoidal rule, we have**

$$\int_0^{\pi/2}\sin x\,dx=\frac{h}{2}\left[y_0+y_{10}+2(y_1+y_2+\ldots+y_9)\right]$$

$$=\frac{(\pi/20)}{2}\left[1\cdot00000+2(5\cdot85311)\right]=\frac{\pi}{40}\left[12\cdot70622\right]$$

$$=\cdot9981.$$

(b) **By Simpson's '⅓' rule, we have**

$$\int_0^{\pi/2}\sin x\,dx=\frac{h}{3}\left[y_0+y_{10}+4(y_1+y_3+\ldots+y_9)+2(y_2+y_4+\ldots+y_8)\right]$$

$$=\frac{\pi}{60}\left[1\cdot00000+4(3\cdot19623)+2(2\cdot65688)\right]$$

$$=\frac{\pi}{60}\left[1\cdot00000+12\cdot78492+5\cdot31376\right]$$

$$=\frac{\pi}{60}\left[19\cdot09868\right]=1\cdot0006.$$

Again the exact value of the integral

$$\int_0^{\pi/2}\sin x\,dx=\Big[-\cos x\Big]_0^{\pi/2}=1\cdot0000.$$

Therefore the error

(i) Due to Trapezoidal rule $=\cdot0019$.

(ii) Due to Simpson's '⅓' rule $=-\cdot0006$.

Ex. 13. *If $f(x)$ is a polynomial in x of the third degree, find an expression for $\int_0^t f(x)\,dx$ in terms of $f(0), f(1), f(2)$ and $f(3)$. Use this result to show that*

$$\int_1^2 f(x)\,dx=\frac{1}{24}\left[-f(0)+13f(1)+13f(2)-f(3)\right].\ \textbf{(Meerut 93P)}$$

Sol. By Lagrange's interpolation formula, we know that

$$f(x)=\frac{(x-1)(x-2)(x-3)}{(-1)(-2)(-3)}f(0)+\frac{(x-0)(x-2)(x-3)}{(1)(-1)(-2)}f(1)+\frac{x(x-1)(x-3)}{(2)(1)(-1)}f(2)+\frac{x(x-1)(x-2)}{(3)(2)(1)}f(3)$$

$$=-\tfrac{1}{6}[x^3-6x^2+11x-6]f(0)+\tfrac{1}{2}[x^3-5x^2+6x]f(1)-\tfrac{1}{2}[x^3-4x^2+3x]f(2)+\tfrac{1}{6}[x^3-3x^2+2x]f(3).$$

$$\therefore\quad \int_0^t f(x)\,dx=\left[-\frac{1}{6}\left(\frac{x^4}{4}-\frac{6x^3}{3}+\frac{11x^2}{2}-6x\right)f(0)+\frac{1}{2}\left(\frac{x^4}{4}-\frac{5x^3}{3}+\frac{6x^2}{2}\right)f(1)-\frac{1}{2}\left(\frac{x^4}{4}-\frac{4x^3}{3}+\frac{3x^2}{2}\right)f(2)+\frac{1}{6}\left(\frac{x^4}{4}-\frac{3x^3}{3}+\frac{2x^2}{2}\right)f(3)\right]_0^t$$

$$=-\frac{1}{6}\left(\frac{t^4}{4}-2t^3+\frac{11t^2}{2}-6t\right)f(0)+\frac{1}{2}\left(\frac{t^4}{4}-\frac{5t^3}{3}+3t^2\right)f(1)-\frac{1}{2}\left(\frac{t^4}{4}-\frac{4t^3}{3}+\frac{3t^2}{2}\right)f(2)+\frac{1}{6}\left(\frac{t^4}{4}-t^3+t^2\right)f(3)\quad \ldots(1$$

Now putting $t=2$ in (1), we get

$$\int_0^2 f(x)\,dx=\tfrac{1}{3}f(0)+\tfrac{4}{3}f(1)+\tfrac{1}{3}f(2). \qquad ...(2)$$

Again putting $t=1$ in (1), we get

$$\int_0^1 f(x)\,dx=\frac{3}{8}f(0)+\frac{19}{24}f(1)-\frac{5}{24}f(2)+\frac{1}{24}f(3). \qquad ...(3)$$

Subtracting (3) from (2), we have

$$\int_1^2 f(x)\,dx=-\frac{1}{24}f(0)+\frac{13}{24}f(1)+\frac{13}{24}f(2)-\frac{1}{24}f(3)$$

$$=\frac{1}{24}[-f(0)+13f(1)+13f(2)-f(3)].$$

Ex. 14. *If $U_x=a+bx+cx^2$, prove that*

$$\int_1^3 U_x\,dx=2U_2+\frac{1}{12}(U_0-2U_2+U_4)$$

and hence find an approximate value for

$$\int_{-1/2}^{1/2} exp\,(-x^2/10)\,dx.$$

(Nagpur B.Sc. 1973 ; Kurukshetra M.A. 76)

Sol. Shifting the origin to -2, we have to prove that

$$\int_{-1}^1 U_x\,dx=2U_0+\frac{1}{12}(U_{-2}-2U_0+U_2) \qquad ...(1)$$

$$\text{L.H.S. of (1)}=\int_{-1}^1 (a+bx+cx^2)\,dx=\left[ax+b\frac{x^2}{2}+c\frac{x^3}{3}\right]_{-1}^1$$

$$=2\left(a+\frac{c}{3}\right).$$

Now $U_x=a+bx+cx^2$.

$\therefore\quad U_0=a,\ U_{-2}=a-2b+4c,\ U_2=a+2b+4c.$

$$\therefore\ \text{R.H.S. of (1)}=2U_0+\frac{1}{12}(U_{-2}-2U_0+U_2)$$

$$=2a+\frac{1}{12}[2(a+4c)-2a]$$

$$=2a+\frac{8}{12}c=2\left(a+\frac{c}{3}\right)$$

$$=\text{L.H.S.}$$

Changing the scale to $\frac{1}{2}$ in (1), we get

$$\int_{-1/2}^{1/2} U_x\,dx=\tfrac{1}{2}\left[2U_0+\frac{1}{12}(U_{-1}-2U_0+U_1)\right]$$

Taking $U_x=\exp(-x^2/10)$, we get

$$\int_{-1/2}^{1/2}\exp\left(\frac{-x^2}{10}\right)dx=1+\frac{1}{24}[2e^{-1/10}-2]$$

$$=1+\frac{1}{12}[e^{-1/10}-1].$$

Ex. 15. *If the third order differences are constant, prove that*

$$\int_0^2 U_x\,dx=\frac{1}{24}\left[U_{-1/2}+23U_{1/2}+23U_{3/2}+U_{5/2}\right].$$

Sol. Since the third order differences are given to be constant, we may take U_x as a polynomial of degree 3.

Let $\qquad U_x=a+bx+cx^2+dx^3 \qquad ...(1)$

Changing the scale to two times, we have to prove that

$$\int_0^4 U_x\,dx=\frac{1}{12}[U_{-1}+23U_1+23U_3+U_5].$$

Now changing the origin to -2, we have to prove that

$$\int_{-2}^2 U_x\,dx=\frac{1}{12}[U_{-3}+23U_{-1}+23U_1+U_3] \qquad ...(2)$$

$$\text{L.H.S. of (2)}=\int_{-2}^2 U_x\,dx=\int_{-2}^2 (a+bx+cx^2+dx^3)\,dx$$

$$=\left[ax+\frac{bx^2}{2}+\frac{cx^3}{3}+\frac{dx^4}{4}\right]_{-2}^2=2\left(2a+\frac{8c}{3}\right)=4a+\frac{16c}{3}.$$

$$\text{R.H.S. of (2)}=\frac{1}{12}[(U_{-3}+U_3)+23(U_{-1}+U_1)]$$

$$=\frac{1}{12}[2(a+9c)+23\times 2(a+c)]=\frac{1}{6}(24a+32c)=4a+\frac{16c}{3}=\text{L.H.S.}$$

Ex. 16. *If third differences are constant, prove that*

$$\int_{-1}^1 f(x)\,dx=\frac{2}{3}[f(0)+f(1/\sqrt{2})+f(-1/\sqrt{2})].$$

Sol. Since the third differences are given to be constant, we may take $f(x)$ as a polynomial of degree 3.

Let $\qquad f(x)=a+bx+cx^2+dx^3 \qquad ...(1)$

$$\text{L.H.S.}=\int_{-1}^1 f(x)\,dx=\int_{-1}^1 (a+bx+cx^2+dx^3)\,dx$$

$$=\left[ax+\frac{bx^2}{2}+\frac{cx^3}{3}+\frac{dx^4}{4}\right]_{-1}^1=2\left(a+\frac{c}{3}\right).$$

$$\text{R.H.S.}=\frac{2}{3}\left[a+\left(a+\frac{b}{\sqrt{2}}+\frac{c}{2}+\frac{d}{2\sqrt{2}}\right)+\left(a-\frac{b}{\sqrt{2}}+\frac{c}{2}-\frac{d}{2\sqrt{2}}\right)\right] \quad \text{[using (1)]}$$

$$=\frac{2}{3}(3a+c)=2\left(a+\frac{c}{3}\right)=\text{L.H.S.}$$

Ex. 17. *Prove Simpson's formula*

$$\int_a^b f(x)\,dx=\frac{b-a}{6n}[f(x_0)+4f(x_1)+2f(x_2)+...+f(x_{2n})],$$

where $\qquad x_0=a,\ x_{2n}=b$

and use it to evaluate $\int_1^2 \frac{dx}{x}$ *and give estimates of error for* $n=1$ *and 2, given that* $\log_e 2 = 0{\cdot}69315$.

Sol. We know that Simpson's '1/3' rule is

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\Big[f(x_0)+f(x_0+nh)+4\Big\{f(x_0+h)$$
$$+f(x_0+3h)+\ldots\Big\}+2\Big\{f(x_0+2h)+f(x_0+4h)+\ldots\Big\}\Big] \quad \ldots(1)$$

Putting $x_0=a$, $n=2n$, $x_0+2nh=x_{2n}=b$, $h=\frac{b-a}{2n}$ in (1), we get the required form of Simpson's rule.

When $n=1$, we should divide the whole range (1, 2) into 2 equal parts by three points x_0, x_1, x_2. Thus the above formula becomes

$$\int_1^2 \frac{dx}{x} = \int_{x_0}^{x_2} f(x)\,dx \text{ where } f(x)=\frac{1}{x},\ x_2=2,\ x_0=1$$
$$= \frac{b-a}{6n}\Big[f(x_0)+4f(x_1)+f(x_2)\Big]$$
$$= \frac{2-1}{6\times 1}\Big[\frac{1}{1}+\frac{4.1}{3/2}+\frac{1}{2}\Big] = {\cdot}69444.$$

When $n=2$, divide the whole range (1, 2) into 4 equal parts by the points x_0, x_1, x_2, x_3, x_4.

We have

$$\int_1^2 \frac{dx}{x} = \int_{x_0}^{x_4} f(x)\,dx = \frac{b-a}{6n}\Big[f(x_0)+4f(x_1)+2f(x_2)$$
$$+4f(x_3)+f(x_4)\Big]$$
$$= \frac{1}{12}\Big[\frac{1}{1}+4.\frac{1}{5/4}+2.\frac{1}{3/2}+4.\frac{1}{7/4}+\frac{1}{2}\Big] = {\cdot}69325.$$

The actual value of $\int_1^2 \frac{dx}{x} = \Big[\log_e x\Big]_1^2 = \log_e 2 = {\cdot}69315.$

Hence the error when $n=1$ is $= {\cdot}69315 - {\cdot}69444 = -{\cdot}00129.$

The error for $n=2$ is given by ${\cdot}69315 - {\cdot}69325 = -{\cdot}0001.$

Ex. 18. *If* $f(x)=a+bx+cx^2$, *prove that*

$$\int_1^3 f(x)\,dx = \frac{1}{12}\Big[f(0)+22f(2)+f(4)\Big].$$

Sol. Here $f(x)$ is a polynomial of degree two in x, so its third and higher order differences are zero.

In this case we have $h=2$.

Now $\quad f(x)=f\left(\frac{x}{2}\cdot 2\right)=f\left(0+\frac{x}{2}\,h\right)$

$$=E^{x/2}f(0)=(1+\Delta)^{x/2}f(0)$$
$$=f(0)+\frac{x}{2}\,\Delta f(0)+\frac{x}{2}\left(\frac{x}{2}-1\right)\frac{1}{2!}\,\Delta^2 f(0)$$

{neglecting other terms}

$$\therefore\ \int_1^3 f(x)\,dx = \int_1^3\Big[f(0)+\frac{x}{2}\,\Delta f(0)+\frac{1}{2}\left(\frac{x^2}{4}-\frac{x}{2}\right)\Delta^2 f(0)\Big]dx$$
$$=\Big[x\,f(0)+\frac{x^2}{4}\,\Delta f(0)+\frac{1}{2}\left(\frac{x^3}{12}-\frac{x^2}{4}\right)\Delta^2 f(0)\Big]_1^3$$
$$=2f(0)+2\Delta f(0)+\frac{1}{12}\,\Delta^2 f(0)$$
$$=2f(0)+2\,[f(2)-f(0)]+\frac{1}{12}\,[f(4)-2f(2)+f(0)]$$
$$=\frac{1}{12}\,[f(0)+22f(2)+f(4)].$$

Ex. 19. *Use the Euler-Maclaurin formula to prove that*

$$\sum_1^n x^2 = \frac{n(n+1)(2n+1)}{6}.$$

Sol. The Euler-Maclaurin formula is

$$\frac{1}{h}\int_{x_0}^{x_0+nh} f(x)\,dx = \sum_{i=0}^{n} f(x_i) - \frac{1}{2}\,[f(x_n)+f(x_0)]$$
$$-\frac{h}{12}\,[f'(x_0+nh)-f'(x_0)]+\frac{h^3}{720}\,[f'''(x_0+nh)-f'''(x_0)]-\ldots.$$

Putting $f(x)=x^2$, $f'(x)=2x$, $x_0=0$, $h=1$, $x_0+nh=n$, $x_i=x_0+ih=i$, we get

$$\int_0^n x^2\,dx = \sum_{i=0}^{n} i^2 - \frac{1}{2}\,[n^2+0]-\frac{1}{12}\,[2n-0],$$

because third and higher order derivatives of x^2 are zero

$$\therefore\ \frac{n^3}{3} = \sum_{i=0}^{n} i^2 - \frac{1}{2}\,n^2 - \frac{n}{6}.$$

$$\therefore\ \sum_0^n i^2 = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6} = \frac{2n^3+3n^2+n}{6} = \frac{n(2n^2+3n+1)}{6}$$
$$=\frac{n(n+1)(2n+1)}{6}.$$

$$\therefore\ \sum_1^n x^2 = \frac{n(n+1)(2n+1)}{6}.$$

Ex. 20. *Find the sum of the fourth powers of the first n natural numbers by means of the Euler-Maclaurin formula.*

Sol. The Euler-Maclaurin formula is

$$\frac{1}{h}\int_{x_0}^{x_0+nh} f(x)\,dx=\sum_{i=0}^{n} f(x_i)-\frac{1}{2}[f(x_n)+f(x_0)]-\frac{h}{12}[f'(x_0+nh)-f'(x_0)]+\frac{h^3}{720}[f'''(x_0+nh)-f'''(x_0)]-\dots$$

Putting $f(x)=x^4$, $h=1$, $x_0=0$, $x_0+nh=n$, $x_i=i$, we get

$$\int_0^n x^4\,dx=\sum_0^n i^4-\tfrac{1}{2}[n^4+0]-\frac{1}{12}[4n^3-0]+\frac{1}{720}[24n-0]$$

[$\because$ fifth and higher order derivatives of x^4 are zero]

or $$\frac{n^5}{5}=\sum_0^n i^4-\frac{n^4}{2}-\frac{n^3}{3}+\frac{n}{30}$$

or $$\sum_0^n i^4=\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}.$$

Ex. 21. (i) *Find the sum of*

$$\frac{1}{51^2}+\frac{1}{53^2}+\frac{1}{55^2}+\dots+\frac{1}{99^2}.$$

(ii) *Evaluate* $\sum_{m=0}^{\infty}\frac{1}{(10+m)^2}$. (Meerut M.Sc. Stat. 74)

Sol. (i) The Euler-Maclaurin formula is

$$\frac{1}{h}\int_{x_0}^{x_0+nh} f(x)\,dx=\sum_{i=0}^{n} f(x_0+ih)-\tfrac{1}{2}[f(x_0+nh)+f(x_0)]-\frac{h}{12}[f'(x_0+nh)-f'(x_0)]+\frac{h^3}{720}[f'''(x_0+nh)-f'''(x_0)]+\dots$$

Putting $f(x)=\frac{1}{x^2}$, $x_0=51$, $h=2$, $n=24$ in this formula, we get

$$\frac{1}{2}\int_{51}^{99}\frac{1}{x^2}\,dx=\sum_{i=0}^{24} f(51+2i)-\tfrac{1}{2}[f(99)+f(51)]-\frac{2}{12}[f'(99)-f'(51)]+\frac{8}{720}[f'''(99)-f'''(51)]$$

(neglecting higher order terms)

Now $f(x)=\frac{1}{x^2}$, $f(99)=\frac{1}{99^2}$, $f(51)=\frac{1}{51^2}$

$$f'(x)=-\frac{2}{x^3},\ f'(99)=-\frac{2}{99^3},\ f'(51)=-\frac{2}{51^3}$$

$$f'''(x)=-\frac{24}{x^5},\ f'''(99)=-\frac{24}{99^5},\ f'''(51)=-\frac{24}{51^5}.$$

$$\therefore\ \tfrac{1}{2}\int_{51}^{99}\frac{1}{x^2}\,dx=\sum_{i=0}^{24} f(51+2i)-\tfrac{1}{2}\left\{\frac{1}{99^2}+\frac{1}{51^2}\right\}-\frac{1}{6}\left\{-\frac{2}{99^3}+\frac{2}{51^3}\right\}+\frac{1}{90}\left\{-\frac{24}{99^5}+\frac{24}{51^5}\right\}$$

or $$\tfrac{1}{2}\left[-\frac{1}{x}\right]_{51}^{99}=\left\{\frac{1}{51^2}+\frac{1}{53^2}+\dots+\frac{1}{99^2}\right\}-\tfrac{1}{2}\left\{\frac{1}{99^2}+\frac{1}{51^2}\right\}+\tfrac{1}{3}\left\{\frac{1}{99^3}-\frac{1}{51^3}\right\}-\frac{24}{90}\left\{\frac{1}{99^5}-\frac{1}{51^5}\right\}$$

$$\therefore\ \frac{1}{51^2}+\frac{1}{53^2}+\dots+\frac{1}{99^2}=\tfrac{1}{2}\left\{-\frac{1}{99}+\frac{1}{51}\right\}+\tfrac{1}{2}\left\{\frac{1}{99^2}+\frac{1}{51^2}\right\}+\tfrac{1}{3}\left\{\frac{1}{51^3}-\frac{1}{99^3}\right\}+\frac{4}{15}\left\{\frac{1}{99^5}-\frac{1}{51^5}\right\}$$

$$=\cdot 00475342+\cdot 00024325+\cdot 00000022-\text{zero}=\cdot 00499689.$$

(ii) Put $f(x)=\frac{1}{(a+x)^2}$.

$$\therefore\ \int_0^{\infty} f(x)\,dx=\frac{1}{a},\ f'(0)=-\frac{2}{a^3},$$

$$f'''(0)=-\frac{24}{a^5},\ f^{v}(0)=-\frac{720}{a^7}.$$

Put $a=10$.

$$\therefore\ \sum_0^{\infty}\frac{1}{(10+m)^2}=\frac{1}{10}+\frac{1}{2}\cdot\frac{1}{100}+\frac{1}{12}\cdot\frac{2}{10^3}-\frac{1}{720}\cdot\frac{24}{10^5}$$

$$=\cdot 1+\cdot 005+\cdot 0001667-\cdot 0000003=\cdot 1051664.$$

Ex. 22. *Evaluate* $\int_0^1\frac{dx}{1+x}$ *correct upto five decimal places.*

Sol. Putting in Euler-Maclaurin's formula,

$x_0=0$, $x_0+nh=1$, $h=\cdot 1$, $n=10$, $f(x)=\frac{1}{1+x}$, $f'(x)=-\frac{1}{(1+x)^2}$,

$f'''(x)=-\frac{6}{(1+x)^4}$, we get

$$\frac{1}{\cdot 1}\int_0^1\frac{dx}{1+x}=\left[\frac{1}{1}+\frac{1}{1\cdot 1}+\frac{1}{1\cdot 2}+\dots+\frac{1}{1\cdot 9}+\frac{1}{2}\right]-\frac{1}{2}\left[\frac{1}{2}+\frac{1}{1}\right]-\frac{(\cdot 1)}{12}\left[-\frac{1}{2^2}-\frac{-1}{1^2}\right]+\frac{(\cdot 1)^3}{720}\left[\frac{-6}{2^4}-\frac{-6}{1^4}\right]$$

$= [1+ \cdot 90909 + \cdot 83333 + \ldots + \cdot 50000] - \cdot 75 - \cdot 00625 + \cdot 00001$
$= 6 \cdot 93773 - \cdot 00625 + \cdot 00001 = 6 \cdot 93149.$

$$\therefore \int_0^1 \frac{dx}{1+x} = 0 \cdot 693149 = 0 \cdot 69315.$$

Ex. 23. *Find* $I = \int_0^1 x\,dx$, *by Gauss's formula, with* $n=4$, *upto 5 decimal places.*

Sol. First we shall change the limits of integration by the relation $x = \frac{1}{2}(u+1)$.

This gives : $I = \frac{1}{4} \int_{-1}^{1} (u+1)\,du$

$$= \tfrac{1}{4} \sum_{i=1}^{n} w_i F(u_i) \text{ where } F(u_i) = u_i + 1.$$

Let us choose $n=4$ and using the abscissae and weights corresponding to $n=4$ in Table 6·1, we get

$$I \approx \tfrac{1}{4}[(-0 \cdot 86114+1)(0 \cdot 34785) + (-0 \cdot 33998+1)(0 \cdot 65214)$$
$$+ (0 \cdot 33998+1)(0 \cdot 65214) + (0 \cdot 86114+1)(0 \cdot 34785)]$$
$$= \tfrac{1}{4}[0 \cdot 483024 + \cdot 4304254 + \cdot 8738545 + \cdot 6473975] = \cdot 49999.$$

Ex. 24. *If* U_x *is a function whose fifth differences are constant and if* $\int_{-1}^{1} U_x\,dx$ *can be expressed in the form* $pU_{-\alpha} + qU_0 + pU_\alpha$, *find* p, q *and* α. *Use the formula to find* $\log_e 2$ *to four decimal places from the integral* $\int_0^1 \frac{dx}{1+x}$. (Meerut M.Sc. 1993)

Sol. Since fifth differences of U_x are given to be constant, so let us take

$$U_x = a + bx + cx^2 + dx^3 + ex^4 + fx^5 \qquad \ldots(1)$$

Let $\int_{-1}^{1} U_x\,dx = pU_{-\alpha} + qU_0 + pU_\alpha = p(U_{-\alpha} + U_\alpha) + qU_0.$

$$\therefore \int_{-1}^{1} (a + bx + cx^2 + dx^3 + ex^4 + fx^5)\,dx$$
$$= p[2(a + c\alpha^2 + e\alpha^4)] + qa$$

or $\left[ax + \frac{bx^2}{2} + \frac{cx^3}{3} + \frac{dx^4}{4} + \frac{ex^5}{5} + \frac{fx^6}{6}\right]_{-1}^{1} = (2p+q)a + 2pc\alpha^2 + 2pe\alpha^4$

or $2\left(a + \frac{c}{3} + \frac{e}{5}\right) = (2p+q)a + 2pc\alpha^2 + 2pe\alpha^4.$

Comparing coefficients of a, c and e on both sides, we get

$$2p + q = 2 \qquad \ldots(2)$$
$$2p\alpha^2 = 2/3 \qquad \ldots(3)$$
$$2p\alpha^4 = 2/5 \qquad \ldots(4)$$

Dividing (4) by (3), we get

$$\alpha^2 = 3/5 = 0 \cdot 6 \Rightarrow \alpha = \pm\sqrt{(0 \cdot 6)}.$$

Substituting in (3), we get

$$2p \cdot \frac{3}{5} = \frac{2}{3} \Rightarrow p = \frac{5}{9}.$$

From (2), $q = 2 - 2p = \frac{8}{9}$.

$$\therefore \int_{-1}^{1} U_x\,dx = \frac{5}{9}\left(U_{\sqrt{(0 \cdot 6)}} + U_{-\sqrt{(0 \cdot 6)}}\right) + \frac{8}{9} U_0 \qquad \ldots(5)$$

Changing the origin to $x = -1$, we get

$$\int_0^2 U_x\,dx = \frac{5}{9}\left(U_{1+\sqrt{(0 \cdot 6)}} + U_{1-\sqrt{(0 \cdot 6)}}\right) + \frac{8}{9} U_1.$$

Reducing the scale to $\frac{1}{2}$, we have

$$\int_0^1 U_x\,dx = \tfrac{1}{2}\left[\frac{5}{9}\left\{U_{\{1+\sqrt{(0 \cdot 6)}\}/2} + U_{\{1-\sqrt{(0 \cdot 6)}\}/2}\right\} + \frac{8}{9} U_{1/2}\right] \ldots(6)$$

Now $\int_0^1 \frac{dx}{1+x} = \left[\log_e (1+x)\right]_0^1 = \log_e 2.$

Putting $U_x = \frac{1}{1+x}$ in (6) and simplifying, we get

$$\log_e 2 = \int_0^1 \frac{dx}{1+x} = \frac{4}{9} U_{0 \cdot 5} + \frac{5}{18}\left(U_{\{1+\sqrt{(0.6)}\}/2} + U_{\{1-\sqrt{(0 \cdot 6)}\}/2}\right)$$
$$= 0 \cdot 694 \text{ (approx.)}$$

Exercises 6

1. Use Simpson's rule dividing the range into ten equal parts, to show that $\int_0^1 \frac{\log(1+x^2)}{(1+x^2)}\,dx = 0 \cdot 1730.$

2. Using Simpson's one-third rule, find $\int_0^6 \frac{dx}{(1+x)^2}$. (Kanpur B. Sc. 73)

3. Evaluate $\int_5^{12} \frac{dx}{x}$ by numerical methods. (Meerut M. Sc. Stat. 71)

4. Evaluate $\int_0^1 x^2\,dx$ by using a suitable formula. (Meerut B. Sc. 74)

5. Find approximate value of $\int_1^2 \frac{dx}{x}$ by using Simpson's rule. (Delhi Hons. 72 ; Meerut M. Sc. 73)

6. Calculate $\int_0^{\pi/2} e^{\sin x}\,dx$ correct to four decimal places. (Meerut M. Sc. Maths. 71)

8

Errors in Numerical Calculations and Remainder terms in Various Interpolation and Quadrature Formulae

§ 1. Introduction. In solving the problems of everyday life the numerical data used are usually not exact. Thus the numbers expressing such data are merely approximations, true to two, three or more figures. Sometimes the methods and processes used to find the desired result are also approximate. Therefore the error in a computed result may be due to the errors in the data, or the errors in the method, or both. In this chapter we shall discuss various kinds of errors and their determination in the case of various interpolation and quadrature formulae.

§ 2. Numbers and their accuracy. Numbers can be divided into two groups, one, *exact* and the other *approximate*. Exact numbers are 2, 5, 8, 11, $\frac{3}{8}$, $\frac{2}{3}$, ..., π, e, ..., etc. Approximate numbers are those numbers that represent the numbers to a certain degree of accuracy. An approximate value of π is 3·1416, or if we want a better approximation, it is 3·14159265. But we are unable to write the exact value of π. In a number the digits which carry real information as to the size of the number apart from the exponential portion are called significant digits or significant figures. Thus the numbers 3·1416, 0·55557 and 5·0786 each contains five significant digits. The number 0·00042 has only two significant digits, *viz*, 4 and 2, since the zeros serve only to fix the position of the decimal point. Often, we come across numbers with a large number of digits, so during the calculation work it will be necessary to cut them to a usable number of figures. This process of cutting off superfluous digits and retaining as many of them as desired is called *rounding off*. The numbers are rounded off according to the following rule :

To round off a number to n significant digits, all digits to the right of the nth digit are discarded and if this discarded number is

(i) less than half a unit in the nth place, the nth digit is left unaltered ;

(ii) greater than half a unit in the nth place, the nth digit is increased by unity ;

(iii) exactly half a unit in the nth place, the nth digit is increased by unity if it is odd, otherwise it is left unchanged.

Then we say that the number is correct to n significant figures. The following numbers are rounded off to four significant figures :

1·6473	to	1·647
20·0567	to	20·06
0·753378	to	0·7534
6·14159	to	6·142.

§ 3. Different Types of Errors. Usually errors can be classified in three categories : Inherent, Round-off and Truncation errors.

(i) **Inherent Errors.** These errors have their origin in human mistakes or they are committed in collection of data. In computations, inherent errors can be minimized by using better techniques for the collection of data. The main causes for this type of error are uncertainty im measurement, outright blunders and non-availability of exact expressions for certain numbers.

(ii) **Round-off Errors.** Due to the limitations of the computing aids, mathematical tables, desk calculators or the digital computers, numbers have to be rounded off, causing what are called rounding-off errors. Such errors can be minimized by using computing aids of higher precision. In hand computations to reduce the round-off error we should follow a useful rule :

At each step of the computation, retain at least one more significant figure than that given in the data, perform the last operation and finally round off.

(iii) **Truncation Errors.** The errors which arise when an infinite process is approximated by a finite one, are called truncation errors. For example, such errors arise when a definite integral is computed by Simpson's rule or when an infinite series is summed by any approximate formula, etc.

§ 4. Absolute, Relative and Percentage Errors. The numerical difference between the true value of a quantity and its approximate value is defined as Absolute Error. Let X be the true value of a quantity and X_1 be its approximate value, then the absolute error E_A is given by

$$E_A = X - X_1 = \delta X. \qquad \text{...(1)}$$

The ratio of the absolute error and the true value of the quantity is defined as Relative Error.

The relative error E_R is given by

$$E_R = \frac{E_A}{X} = \frac{\delta X}{X}. \qquad \ldots(2)$$

The 100 times the relative error is defined as the **Percentage Error.**

The percentage error E_P is given by

$$E_P = 100 E_R. \qquad \ldots(3)$$

Let ΔX be a number such that

$$| X_1 - X | \leqslant \Delta X. \qquad \ldots(4)$$

Then ΔX is an upper limit for the magnitude of the absolute error and measures *absolute accuracy*. Similarly, the quantity

$$\frac{\Delta X}{| X |} \approx \frac{\Delta X}{| X_1 |}$$

is the measure of the *relative accuracy*.

§ 5. A general Error Formula. Now we derive a general formula for the error committed in using a certain functional relation or a formula.

Let $y = f(x_1, x_2, \ldots, x_n)$ be a function of several variables x_i $(i = 1, 2, \ldots, n)$, and suppose for each i the error in x_i is Δx_i. Let Δy be the error in y; then we have

$$y + \Delta y = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n)$$

$$= f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \Delta x_i + \text{terms containing } (\Delta x_i)^2 \text{ etc.,}$$

expanding by Taylor's series.

Suppose that the errors in x_i are small and so we have $\frac{\Delta x_i}{x_i} \leqslant 1$. Hence the squares and higher powers of Δx_i can be omitted and the above relation gives

$$\Delta y \approx \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \Delta x_i$$

$$= \frac{\partial f}{\partial x_1} \Delta x_1 + \frac{\partial f}{\partial x_2} \Delta x_2 + \ldots + \frac{\partial f}{\partial x_n} \Delta x_n. \qquad \ldots(5)$$

The form of this formula is that of the total differential of y.

The relative error is given by

$$E_R = \frac{\Delta y}{y} = \frac{\partial f}{\partial x_1} \frac{\Delta x_1}{y} + \frac{\partial f}{\partial x_2} \frac{\Delta x_2}{y} + \ldots + \frac{\partial f}{\partial x_n} \frac{\Delta x_n}{y}. \qquad \ldots(6)$$

As an illustration let u be the function of 3 variables x, y and z. Take $u = \frac{5xy^2}{z^3}$.

Then $\frac{\partial u}{\partial x} = \frac{5y^2}{z^3}$, $\frac{\partial x}{\partial y} = \frac{10xy}{z^3}$, $\frac{\partial u}{\partial z} = -\frac{15xy^2}{z^4}$.

$$\therefore \quad \Delta u = \frac{5y^2}{z^3} \Delta x + \frac{10xy}{z^3} \Delta y - \frac{15xy^2}{z^4} \Delta z.$$

Since the errors Δx, Δy and Δz may be positive or negative so we take the absolute values. We have

$$| \Delta u | \leqslant \left| \frac{5y^2}{z^3} \Delta x \right| + \left| \frac{10xy}{z^3} \Delta y \right| + \left| \frac{15xy^2}{z^4} \Delta z \right|$$

or $$(\Delta u)_{max} \approx \left| \frac{5y^2}{z^3} \Delta x \right| + \left| \frac{10xy}{z^3} \Delta y \right| + \left| \frac{15xy^2}{z^4} \Delta z \right|.$$

If we take $\Delta x = \Delta y = \Delta z = 0{\cdot}001$ and $x = y = z = 1$, then the relative maximum error is given by

$$(E_R)_{max} = \frac{(\Delta u)_{max}}{u} = \frac{0{\cdot}03}{5} = 0{\cdot}006.$$

§ 6. Error committed in a series approximation.

Taylor's series for $f(x)$ at $x = a$ gives

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$$

$$+ \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + R_n(x),$$

where $R_n(x) = \frac{(x-a)^n}{n!} f^n(\theta)$, $a < \theta < x$.

As $n \to \infty$, $R_n(x) \to 0$ for a convergent series.

Suppose we approximate $f(x)$ by the first n terms of a series. Then in this approximation the maximum error committed is given by the remainder term. Conversely, if the accuracy desired is mentioned in advance, then it would be possible to obtain n (the number of terms) such that the finite series gives the desired accuracy.

Illustration. The expansion for e^x by Maclaurin's theorem is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} e^\theta, \; 0 < \theta < x.$$

Here, we shall obtain n, the number of terms, such that their sum gives the value of e^x correct to 8 decimal places at $x = 1$.

The error term *i.e.* the remainder term $= \frac{x^n}{n!} e^\theta$.

This gives the maximum absolute error at $\theta = x$, which is $\frac{x^n}{n!} e^x$. Hence the maximum relative error is given by

$$\frac{x^n e^x / n!}{e^x} = \frac{x^n}{n!}$$

$\therefore$ $(E_R)_{max}$ at $x = 1$ is $1/n!$.

We must have $\frac{1}{n!} < \frac{1}{2}.10^{-8}$, for an 8 decimal accuracy at $x=1$. This gives $n=12$.

Hence in order to make the sum of the exponential series at $x=1$ correct to 8 decimal places, we need to take its 12 terms.

In the representation of any function by an infinite Taylor's series, the remainder term gives us the measure of Truncation error.

In calculus of finite differences we make polynomial approximation of any function *e.g.*, if $(n+1)$ observations are given for any function $f(x)$, then the function can be approximated only by a polynomial of degree n. This gives rise to error in the final result. We shall derive expressions for the remainder terms in polynomial formulae. We shall need Rolle's theorem which is as given below.

Let $f(x)$ be any function of degree n defined in the interval $[a, b]$, satisfying the following properties :

(i) $f(x)$ is continuous in the closed interval $[a, b]$

(ii) $f(x)$ is differentiable in the open interval (a, b),

(iii) $f(a)=f(b)$.

Then $f'(x)$ vanishes at $(n-1)$ points in (a, b), $f''(x)$ vanishes at $(n-2)$ points in (a, b), ..., $f^{n-1}(x)$ vanishes at atleast one point in (a, b).

§ 7. Remainder Term in Lagrange's Interpolation Formula.

Let $f(x)$ be any function which is approximated by means of some polynomial $P_n(x)$ of degree n in x.

Let the values of the function $f(x)$ be given at the points $x=x_0, x_1, x_2, \ldots, x_n$ and suppose the function $f(x)$ satisfies all the conditions of Rolle's theorem.

Since $f(x)$ is approximated by $P_n(x)$, so $P_n(x)$ has the same values as $f(x)$ at the points $x_0, x_1, \ldots, x_n$.

Let $\qquad f(x)=P_n(x)+g(x)$

where $g(x)$ has the roots $x_0, x_1, \ldots, x_n$.

$$\therefore \quad f(x)=P_n(x)+k(x)\,(x-x_0)\,(x-x_1)\ldots(x-x_n), \qquad \ldots(1)$$

where $k(x)$ is to be determined.

To determine $k(x)$ we consider the function

$$\phi(t)=f(t)-P_n(t)-k(x)\,(t-x_0)\,(t-x_1)\ldots(t-x_n). \qquad \ldots(2)$$

Now the function $\phi(t)$ vanishes for $(n+1)$ values of t namely $t=x_0, x_1, \ldots, x_n$. Also the function $\phi(t)$ vanishes for $t=x$ by virtue of (1). Hence the function $\phi(t)$ vanishes for the $n+2$ real roots $x, x_0, x_1, \ldots, x_n$. It is also a continuous function of t possessing continuous derivatives of all orders within the interval

$[x_0, x_n]$. Hence by Rolle's theorem $\phi'(t)$ has at least $n+1$ roots lying between the smallest and greatest of the above roots. Similarly $\phi''(t)$ has at least n roots in the same interval and finally $\phi^{n+1}(t)$ has at least one root, say $t=\xi$ in the interval (x_0, x_n).

Now from (2), $\phi^{n+1}(t)=f^{n+1}(t)-k(x).\,\{(n+1)\,!\}$.

$$[\because \quad P_n^{n+1}(t)=0]$$

$$\therefore \quad \phi^{n+1}(\xi)=f^{n+1}(\xi)-k(x)\,\{(n+1)\,!\,\}.$$

Now $\phi^{n+1}(t)$ vanishes at the point $t=\xi$ where $x_0<\xi<x_n$, therefore

$$f^{n+1}(\xi)-k(x)\,\{(n+1)\,!\}=0 \quad , \qquad x_0<\xi<x_n$$

$$\Rightarrow \quad k(x)=\frac{1}{(n+1)\,!}\,f^{n+1}(\xi) \quad , \qquad x_0<\xi<x_n. \qquad \ldots(3)$$

Putting this value of $k(x)$ in (1), we have

$$f(x)=P_n(x)+f^{n+1}(\xi).\,\frac{(x-x_0)\,(x-x_1)\ldots(x-x_n)}{(n+1)\,!}.$$

Hence the required truncation error in approximating $f(x)$ by $P_n(x)=R_n=f(x)-P_n(x)$

$$=\frac{(x-x_0)\,(x-x_1)\ldots(x-x_n)}{(n+1)\,!}\,f^{n+1}(\xi)=\frac{f^{n+1}(\xi)}{(n+1)\,!}\prod_{i=0}^{n}(x-x_i). \qquad \ldots(4)$$

This is the remainder term in Lagrange's formula.

§ 8. Remainder Term in Newton's Forward Interpolation Formula.

Suppose the values $x_0, x_1, \ldots, x_n$ are given at equal interval h.

We have $\quad x-x_0=hu,\; x-x_1=x-(x_0+h)$

$$=hu-h=h\,(u-1),\;\ldots,\; x-x_n=h\,(u-n).$$

Putting the values of $x-x_0, x-x_1, \ldots, x-x_n$ in (4), we get

$$R_n=\frac{u\,(u-1)\ \ (u-n)}{(n+1)\,!}\,h^{n+1}f^{n+1}(\xi)$$

$$=\frac{u^{(n+1)}}{(n+1)\,!}\,h^{n+1}f^{n+1}(\xi),\; x_0<\xi<x_n. \qquad \ldots(5)$$

This is the remainder term in Newton's formula for forward interpolation.

From (5), we observe that the remainder term contains derivative of $(n+1)$th order of the function $f(x)$ at $x=\xi$ and we may be unable to say anything about the error when the form of the function is not known to us.

The only alternative is that if $f^{n+1}(\xi)$ can be expressed in terms of the differences of the function $f(x)$ then an approximate value of the error can be calculated.

From the relation between differences and derivatives of a function $f(x)$, we get

$$\Delta^n f(x)=(\Delta x)^n f^n(x+\theta n\Delta x),\ 0<\theta<1. \quad \ldots(6)$$

Putting $x=x_0$ and $\Delta x=h$ in (6), we have

$$f^n(x_0+\theta nh)=\frac{\Delta^n f(x_0)}{h^n} \quad \ldots(7)$$

Since $x_0+\theta nh$ and ξ are values of x at points within the interval of interpolation *i.e.*, (x_0, x_n) so for practical purposes, we may put $\xi=x_0+\theta nh$. Using this substitution in (7), we get

$$f^n(\xi)=\frac{\Delta^n f(x_0)}{h^n}.$$

This gives
$$f^{n+1}(\xi)=\frac{\Delta^{n+1} f(x_0)}{h^{n+1}} \quad \ldots(8)$$

Substituting the value of $f^{n+1}(\xi)$ from (8) in (5), we get

$$R_n=\frac{u^{(n+1)}}{(n+1)\,!}\ \Delta^{n+1} f(x_0).$$

§ 9. Remainder Term in Newton's Formula for Backward Interpolation.

We put $f(x)=P_n(x)+k(x)\,(x-x_n)(x-x_{n-1})\ldots(x-x_0)$, $\ldots(9)$
where $k(x)$ is to be determined in terms of the function $f(x)$.

Let us consider the function

$$\phi(t)=f(t)-P_n(t)-k(x)\,(t-x_n)\,(t-x_{n-1})\ldots(t-x_0).$$

Proceeding in the same way as in § 7, we get

$$k(x)=f^{n+1}(\xi)/\{(n+1)\,!\},\qquad x_0<\xi<x_n.$$

Hence error $R_n=f(x)-P_n(x)=k(x)\,(x-x_n)\,(x-x_{n-1})\ldots(x-x_0)$

$$=f^{n+1}(\xi)\,\frac{(x-x_n)\,(x-x_{n-1})\ldots(x-x_0)}{(n+1)\,!}. \quad \ldots(10)$$

To find the value of R_n in terms of u we recall that

$$\frac{x-x_n}{h}=u,\ \frac{x-x_{n-1}}{h}=u+1,\ \ldots,\ \frac{x-x_0}{h}=u+n.$$

Putting the values of $x-x_n$, $x-x_{n-1}$, ..., $x-x_0$ in (10), we get

$$R_n=u\,(u+1)\ldots(u+n)\,\frac{h^{n+1}}{(n+1)\,!}\,f^{n+1}(\xi). \quad \ldots(11)$$

This is the remainder term in Newton's Backward formula.

In case the analytical form of the given function is not known then to find a formula for R_n we put $\nabla^{n+1} f(x_n)/h^{n+1}$ for $f^{n+1}(\xi)$ in (11) and get the result

$$R_n=\frac{\nabla^{n+1} f(x_n)}{(n+1)\,!}\,u(u+1)\,(u+2)\ldots(u+n).$$

§ 10. Remainder Term in Stirling's Formula.

Suppose the function $f(x)$ is approximated by a polynomial $P_{2n}(x)$. In Stirling's formula there are $2n+1$ terms, so $P_{2n}(x)$ coincides with the function $f(x)$ at the $2n+1$ points

$$u=-n,\ -(n-1),\ -(n-2),\ \ldots,\ -2,\ -1,\ 0,\ 1,\ 2,\ \ldots,\ n-2,\ n-1,\ n$$

i.e., $x=x_0-nh,\ x_0-(n-1)\,h,\ \ldots,\ x_0-h,\ x_0,\ x_0+h,\ \ldots,\ x_0+(n-1)\,h,\ x_0+nh.$

Let
$$f(x)=P_{2n}(x)+k(x)\,(x-x_0)\,(x-x_1)\,(x-x_{-1})\ldots(x-x_n)(x-x_{-n}), \quad \ldots(12)$$
where $k(x)$ is to be determined in terms of $f(x)$.

We consider the function

$$\phi(t)=f(t)-P_{2n}(t)-k(x)\,(t-x_0)\,(t-x_1)\,(t-x_{-1})\ldots(t-x_n)\,(t-x_{-n}).$$

Since $\phi(t)$ vanishes for the $2n+2$ values

$t=x,\ x_0,\ x_1,\ x_{-1},\ \ldots,\ x_n,\ x_{-n}$, therefore by Rolle's theorem $\phi^{2n+1}(t)$ has at least one root $t=\xi$ in the interval x_{-n} to x_n.

Now $\phi^{2n+1}(t)=f^{2n+1}(t)-k(x)\,\{(2n+1)\,!\}$.

$\therefore\quad \phi^{2n+1}(\xi)=f^{2n+1}(\xi)-k(x)\,\{(2n+1)\,!\}$.

$\therefore\quad \phi^{2n+1}(\xi)=0\ \Rightarrow\ k(x)=f^{2n+1}(\xi)/\{(2n+1)\,!\}$,
$x_{-n}<\xi<x_n$.

Putting the value of $k(x)$ in (12), we get

$$f(x)=P_{2n}(x)+f^{2n+1}(\xi)\,\frac{(x-x_0)\,(x-x_1)\,(x-x_{-1})\ \ldots\ (x-x_n)\,(x-x_{-n})}{(2n+1)\,!}.$$

Hence the error $R_n=f(x)-P_{2n}(x)$

$$=f^{2n+1}(\xi)\,\frac{(x-x_0)(x-x_1)\,(x-x_{-1})\ \ldots\ (x-x_n)\,(x-x_{-n})}{(2n+1)\,!}.$$

We write this formula in terms of u as follows:

Since $x-x_0=hu,\ x-x_1=h\,(u-1),\ \ldots,\ x-x_n=h\,(u-n)$,

and $x-x_{-1}=x-(x_0-h)=hu+h=h\,(u+1)$,

$x-x_{-2}=h\,(u+2),\ \ldots,\ x-x_{-n}=h\,(u+n)$,

so we have
$$R_n=\frac{h^{2n+1}\ f^{2n+1}(\xi)}{(2n+1)\,!}\ u(u^2-1^2)\,(u^2-2^2)\ldots(u^2-n^2), \quad \ldots(13)$$
where $x_{-n}<\xi<x_n$

To write the value of R_n in terms of differences we replace $f^{2n+1}(\xi)$ by $\dfrac{\Delta^{2n+1} f(x_{-n-1})+\Delta^{2n+1} f(x_{-n})}{2}$ in (13) and putting $h=1$, is omitted. This omitted quantity is the principal part of the inherent error in the formula and is $-\frac{h^7}{140} f^{vi}(x)$ in terms of derivatives.

$$\text{Thus} \quad E_W = -\frac{h}{140} \Delta^6 y = -\frac{h^7}{140} f^{vi}(x).$$

This shows that when $f(x)$ is a polynomial of fifth or lower degree, Weddle's rule gives an exact value.

Ex. *Find the maximum error in the interpolation of* $\sin 52°$ *by Newton formula for forward interpolation ; given*

$\sin 45° = 0{\cdot}7071$, $\sin 50° = 0{\cdot}7660$

$\sin 55° = 0{\cdot}8192$, $\sin 60° = 0{\cdot}8660$.

Sol. The difference table is as given below :

x	$y=f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
45	0·7071			
		0·0589		
50	0·7660		−0·0057	
		0·0532		−0·0007
55	0·8192		−0·0064	
		0·0468		
60	0·8660			

Using Newton's interpolation formula we have calculated $\sin 52°$ in one of the preceding chapters and thus we get

$$\sin 52° = 0{\cdot}7880032.$$

We have $R_n = \frac{\Delta^{n+1} y_0}{(n+1)!} u(u-1)(u-2)\ldots(u-n)$,

where $u = (x - x_0)/h$.

In this case $u = \frac{52-45}{5} = 1{\cdot}4$. Let us take $n = 2$.

Then, we have

$$R_n = \frac{\Delta^3 y_0}{3!} u(u-1)(u-2)$$

$$= -(0{\cdot}0007) \frac{(1{\cdot}4)(\cdot 4)(-\cdot 6)}{3!}$$

11

Numerical Solution of Ordinary Differential Equations

§ 1. Introduction.

We come across many problems in science and engineering which can be reduced to the problem of solving differential equations satisfying certain given conditions. An ordinary differential equation is an equation containing one independent and one dependent variable and at least one of its derivatives with respect to the independent variable *e.g.* a general equation of first order and first degree *i.e.*,

$$\frac{dy}{dx}=f(x, y). \qquad \ldots(1)$$

Many analytical methods exist to solve such equations. But these methods can be applied to solve only a selected class of differential equations. Sometimes a differential equation cannot be solved at all or gives solutions which are so difficult to obtain. For solving such differential equations numerical methods are used. In numerical methods we do not want to find a relation between x and y, but we find the numerical values of the dependent variable for certain values of independent variable.

It is well known that a differential equation of order n will have n arbitrary constants in its general solution. Hence solving (1) by an analytical method we get a relation

$$y=f(x)+c \qquad \ldots(2)$$

where c is an arbitrary constant. If along with the differential equation we are also given a point (x_0, y_0), then putting (x_0, y_0) in (2) we can get the value of the constant c. This condition of the point is called a boundary or initial condition. Problems in which all the conditions are specified at the initial point only are called initial value problems. For example, the problem defined by

$$\frac{dy}{dx}=f(x, y),\ y(x_0)=y_0 \qquad \ldots(3)$$

is an initial value problem. In the case of problems which involve second and higher order differential equations, we may

specify the conditions at two or more points. These problems are called boundary-value problems. There are methods by which numerical solution of an ordinary differential equation having numerical coefficients and given initial condition can be found to any desired degree of accuracy. The most important among the several methods for solving differential equations numerically will be discussed in this chapter.

§ 2. Picard's Method of Successive Approximations.

We consider the differential equation

$$\frac{dy}{dx}=f(x, y),\ y(x_0)=y_0. \qquad \ldots(1)$$

Integrating this between the corresponding limits for x and y, we get

$$\int_{y_0}^{y} dy=\int_{x_0}^{x} f(x, y)\, dx \quad \text{or} \quad y-y_0=\int_{x_0}^{x} f(x, y)\, dx$$

or

$$y=y_0+\int_{x_0}^{x} f(x, y)\, dx. \qquad \ldots(2)$$

In the equation (2) the unknown function y is present under the integral sign. An equation of this type is called an integral equation. Such an equation can be solved by the process of successive approximations or iteration.

To solve (1) by Picard's method of successive approximations, the first approximation to y is obtained by putting y_0 for y on the R.H.S. of (2).

We have

$$y^{(1)}=y_0+\int_{x_0}^{x} f(x, y_0)\, dx.$$

Now the integrand is a function of x only and the indicated integration can be performed. Hence we get $y^{(1)}$. Substituting it for y in the integrand of (2) and integrating again, we get the second approximation

$$y^{(2)}=y_0+\int_{x_0}^{x} f(x, y^{(1)})\, dx.$$

Proceeding in this way, we get $y^{(3)}, y^{(4)}, \ldots$

The nth approximation is given by

$$y^{(n)}=y_0+\int_{x_0}^{x} f(x, y^{(n-1)})\, dx. \qquad \ldots(3)$$

The process is repeated as many times as may be necessary or desirable. In fact the process is stopped when the two values of y, viz. $y^{(n-1)}$ and $y^{(n)}$ are same to the desired degree of accuracy.

Practically it is unsatisfactory because of the difficulties which arise in performing the necessary integrations. Each step gives a better approximation of the desired solution than the preceding one.

§ 3. Euler's Method.

It is one of the oldest and simplest methods but also the crudest.

Let the differential equation be

$$\frac{dy}{dx}=f(x, y),\ y(x_0)=y_0. \qquad \ldots(1)$$

Integrating (1), we get a relation between y and x which can be written in the form

$$y=F(x). \qquad \ldots(2)$$

In the xy-plane the equation (2) represents a curve. Practically a smooth curve is straight for a short distance from any point on it. Hence, we have the approximate relation (see Fig. 1)

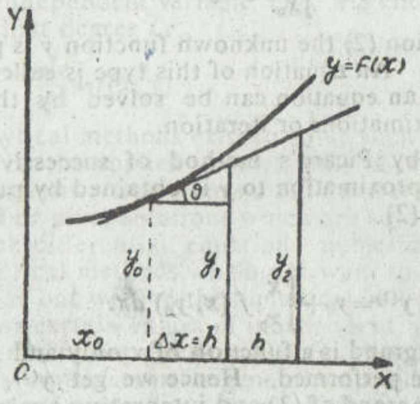

Fig. 1.

$$\Delta y \approx \Delta x \tan\theta$$

$$=\Delta x\left(\frac{dy}{dx}\right)_0=\Delta x . f(x_0, y_0)$$

$$\therefore\quad y_1 \approx y_0+\Delta x . f(x_0, y_0)$$

or

$$y_1 \approx y_0+h\,f(x_0, y_0).$$

This y_1 is the approximate value of y for $x=x_1$.

Similarly the values of y corresponding to

$$x_2=x_1+h,\ x_3=x_2+h, \text{ etc.}$$

are given by

$$y_2\approx y_1+h\,f(x_1, y_1),$$

$$y_3\approx y_2+h\,f(x_2, y_2), \text{ etc.}$$

In general, we obtain

$$y_{n+1}\approx y_n+hf(x_n, y_n),\ n=0, 1, 2, \ldots \qquad \ldots(3)$$

Taking h small enough and continuing in this way we could get the integral of (1) as a set of corresponding values of x and y.

This process is very slow. For practical use, the method is unsuitable because to get reasonable accuracy with this method we need to give a comparatively smaller value to h. If h is not small then the method is too inaccurate. In this method the actual solution curve is approximated by the sequence of short straight lines which sometimes deviates from the solution curve significantly. All these considerations have led to a modification of Euler's method.

Modified Euler's Method. Starting with the initial value y_0, an approximate value for y_1 is calculated from the relation

$$y_1^{(1)}\approx y_0+h\left(\frac{dy}{dx}\right)_0=y_0+h\,f(x_0, y_0).$$

$y_1^{(1)}$ is the first approximation of y_1 at $x=x_1$.

Substituting this value of y_1 into the given differential equation (1), we get an approximate value of $\frac{dy}{dx}$ at the end of the first interval

i.e.

$$\left(\frac{dy}{dx}\right)_1^{(1)}=f(x_1, y_1^{(1)}).$$

Now an improved value of Δy is obtained as

$$\Delta y \approx h.\left[\text{average of the values of } \frac{dy}{dx} \text{ at the ends of the interval } x_0 \text{ to } x_1\right]$$

i.e.

$$\Delta y\approx h.\ \frac{\left(\frac{dy}{dx}\right)_0+\left(\frac{dy}{dx}\right)_1^{(1)}}{2}.$$

The second approximation for y_1 is

$$y_1^{(2)}=y_0+h.\ \frac{\left(\frac{dy}{dx}\right)_0+\left(\frac{dy}{dx}\right)_1^{(1)}}{2}.$$

Substituting this improved value $y_1^{(2)}$ in the given equation, we get a second approximation for

$$\left(\frac{dy}{dx}\right)_1 \text{ viz. } \left(\frac{dy}{dx}\right)_1^{(2)}=f(x_1, y_1^{(2)}).$$

The third approximation for y_1 is given by

$$y_1^{(3)}=y_0+h\ \frac{\left(\frac{dy}{dx}\right)_0+\left(\frac{dy}{dx}\right)_1^{(2)}}{2}.$$

The process is applied until no change is produced in the value of y_1 to the desired degree of accuracy.

In the same manner we make computations for the next interval x_1 to $x_2 (=x_1+h)$ *i.e.* first by finding an approximate value of Δy and then using the averaging process until no improvement is made in the value of y_2.

First approximations to y_2, y_3,... etc. could be obtained by using the formula

$$y_{n+1} \approx y_n + h\left(\frac{dy}{dx}\right)_n,$$

but when two consecutive values of y are known, the first approximations to succeeding y's can be computed more accurately from the relation

$$y_{n+1} = y_{n-1} + 2hy_n'.$$

This relation can be derived by using Taylor's series in the neighbourhood of x_n.

The modified Euler's method gives a great improvement in accuracy over the original method. (See Fig. 2).

In this figure KM represents the Δy computed by the Euler method. If AN is drawn parallel to the tangent at B, then KN represents the Δy computed by using the slope at B. If we take the average of the slopes, we have

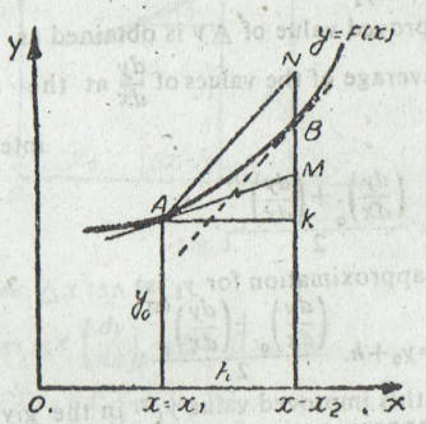

Fig. 2

$$\Delta y = h\,\frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1}{2} = \tfrac{1}{2}\left[h\left(\frac{dy}{dx}\right)_0 + h\left(\frac{dy}{dx}\right)_1\right]$$

$$= \tfrac{1}{2}[KM+KN] = \tfrac{1}{2}[KM+KM+MN] = KM + \tfrac{1}{2}MN,$$

which is much close to its true value KB.

§ 4. Solution by Taylor's Series. (Meerut M.Sc. 1987)

We consider the differential equation

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0. \qquad \ldots(1)$$

Let $y = F(x)$ be a solution of (1) such that $F(x_0) \neq 0$.

Then expanding it by Taylor's series about the point x_0, we get

$$y = y_0 + (x-x_0)\,y_0' + \frac{(x-x_0)^2}{2!}\,y_0'' + \ldots \qquad \ldots(2)$$

Putting $x = x_1 = x_0 + h$ in (2), we get

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}\,y_0'' + \ldots \qquad \ldots(3$$

If the values of y_0', y_0'',.... are known, then (3) gives a power series for y_1. The coefficients y_0', y_0'',... can be found from (1).

We can rewrite (1) as

$$y' = f(x, y) = F_1(x, y), \text{ say.} \qquad \ldots(4)$$

Differentiating (4) successively, we get

$$y'' = \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}\,y' = F_2(x, y, y'), \text{ say,}$$

$$y''' = \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}\,y' + \frac{\partial F_2}{\partial y'}\,y'' = F_3(x, y, y', y''), \text{ say,}$$

$$y^{(iv)} = \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}\,y' + \frac{\partial F_3}{\partial y'}\,y'' + \frac{\partial F_3}{\partial y''}\,y'''$$

$$= F_4(x, y, y', y'', y'''), \text{ say.}$$

Proceeding in this way, we have

$$y^{(n)} = \frac{\partial F_{n-1}}{\partial x} + \frac{\partial F_{n-1}}{\partial y}\,y' + \frac{\partial F_{n-1}}{\partial y'}\,y'' + \ldots + \frac{\partial F_{n-1}}{\partial y^{(n-2)}}\,y^{(n-1)}$$

$$= F_n(x, y, y', y'', \ldots, y^{(n-1)}), \text{ say.}$$

Substituting $x = x_0$, $y = y_0$ in the above results, we get

$$y_0', y_0'', y_0''', \ldots, y_0^{(n)}.$$

Practically this method is not of much importance because of its need of partial derivatives. It also suffers from the serious disadvantage that h should be small enough so that successive terms of the series diminish quite rapidly.

§ 5. Milne's Method. (Rohilkhand 1990; Meerut M.Sc. 72, 76, 78, 80, 86, 87, 88, 89, 90, 91, 92P, 93, 94)

It is a simple and reasonably accurate method of solving differential equations numerically. To solve the differential equation

$y'=f(x, y)$ by this method, first we get the approximate value of y_{n+1} by predictor formula and then improve this value using a corrector formula. These formulae are derived from Newton's formula.

Newton's forward interpolation formula in terms of y' and u is

$$y'=y_0'+u\Delta y_0'+\frac{u(u-1)}{2}\Delta^2 y_0'+\frac{u(u-1)(u-2)}{6}\Delta^3 y_0'$$
$$+\frac{u(u-1)(u-2)(u-3)}{24}\Delta^4 y_0'+\ldots, \quad \ldots(1)$$

where $u=\dfrac{x-x_0}{h}$ or $x=x_0+uh$.

Integrating (1) over the interval x_0 to x_0+4h or $u=0$ to 4, we get

$$\int_{x_0}^{x_0+4h} y'\,dx=h\int_0^4\Big[y_0'+u\Delta y_0'+\frac{u(u-1)}{2}\Delta^2 y_0'$$
$$+\frac{u(u-1)(u-2)}{6}\Delta^3 y_0'+\frac{u(u-1)(u-2)(u-3)}{24}\Delta^4 y_0'+\ldots\Big]du$$
$$[\because\ dx=h\,du]$$

or $$y_4-y_0=h\left(4y_0'+8\Delta y_0'+\frac{20}{3}\Delta^2 y_0'+\frac{8}{3}\Delta^3 y_0'+\frac{28}{90}\Delta^4 y_0'\right),$$

keeping upto fourth differences only.

Here y_0 and y_4 stand for the values of y at $x=x_0$ and $x=x_0+4h$ respectively.

Substituting the values of first, second and third differences, we get

$$y_4-y_0=h\Big[4y_0'+8(E-1)y_0'+\frac{20}{3}(E-1)^2 y_0'$$
$$+\frac{8}{3}(E-1)^3 y_0'+\frac{28}{90}\Delta^4 y_0'\Big]$$

or $$y_4-y_0=\frac{4h}{3}(2y_1'-y_2'+2y_3')+\frac{28}{90}h\Delta^4 y_0'$$

or $$y_4=y_0+\frac{4h}{3}(2y_1'-y_2'+2y_3')+\frac{28}{90}h\Delta^4 y_0. \quad \ldots(2)$$

This is Milne's extrapolation (predictor) formula.

It can be used to predict the value of y_4 when those of y_0, y_1, y_2 and y_3 are known.

To obtain the corrector formula we integrate (1) over the interval x_0 to x_0+2h or $u=0$ to 2 and get

$$y_2-y_0=h\left(2y_0'+2\Delta y_0'+\frac{1}{3}\Delta^2 y_0'-\frac{1}{90}\Delta^4 y_0'\right)$$

Expressing the first, second and third differences in terms of the function values by using $\Delta\equiv E-1$, we get

$$y_2-y_0=\frac{h}{3}(y_0'+4y_1'+y_2')-\frac{h}{90}\Delta^4 y_0'$$

or $$y_2=y_0+\frac{h}{3}(y_0'+4y_1'+y_2')-\frac{h}{90}\Delta^4 y_0'. \quad \ldots(3)$$

This is Milne's corrector formula. The value of y_4 obtained from (2) can be checked by using (3).

Since $x_0, \ldots, x_4$ are any five consecutive values of x, so in the general form the formulae (2) and (3) can be written as

$$y_{n+1}=y_{n-3}+\frac{4h}{3}(2y'_{n-2}-y'_{n-1}+2y'_n), \quad \ldots(4)$$

$$y_{n+1}^{(1)}=y_{n-1}+\frac{h}{3}(y'_{n-1}+4y'_n+y'_{n+1}). \quad \ldots(5)$$

It should be noted that we have considered the differences upto the third order, because we fit up a polynomial of degree four.

The terms involving $\Delta^4 y'$ are not used directly in the formulae, but they indicate the principal parts of the errors in the two values of y_{n+1} as computed from (4) and (5). We notice that this error in (5) is of opposite sign to that in (4), but it is very small in magnitude.

Since $\dfrac{28}{90}h\Delta^4 y'$ and $-\dfrac{h}{90}\Delta^4 y'$

are taken as the principal parts of the errors, we may take

$$(y_{n+1})_{exact}=y_{n+1}+\frac{28}{90}h\Delta^4 y'$$

and $$(y_{n+1})_{exact}=y_{n+1}^{(1)}-\frac{h}{90}\Delta^4 y'.$$

Here y_{n+1} and $y_{n+1}^{(1)}$ denote the predicted and first corrected values of y for $x=x_{n+1}$.

Equating these two values, we get

$$y_{n+1}+\frac{28}{90}h\Delta^4 y'=y_{n+1}^{(1)}-\frac{h}{90}\Delta^4 y'$$

or $$y_{n+1}-y_{n+1}^{(1)}=-\frac{29}{90}h\Delta^4 y'=29\left(-\frac{h}{90}\Delta^4 y'\right)=29E_2,$$

where E_2 denotes the principal part of the error in (5). From this we get

$$E_2=\frac{1}{29}(y_{n+1}-y_{n+1}^{(1)}). \quad \ldots(6)$$

Thus we observe that the error in (5) is (1/29)th of the difference between the predicted and corrected values.

We conclude from the above discussion that to tabulate the solution of the first order differential equation

$$\frac{dy}{dx}=y'=f(x, y), \qquad \ldots(7)$$

first we find three consecutive values of y and y' in addition to the initial values. Then we obtain the next value of y by using (4). Putting this new value of y in (7) we get the new y' *i.e.*, the value of y' at the new value of y. Then putting the new y' in (5) we get the corrected value of the new y. If the corrected value agrees closely with the predicted value then we proceed to the next interval.

If the corrected value differs significantly from the predicted value and no mistake can be found in the calculation work, then we compute E_3 by (6). If E_3 is too small then we proceed to the next interval. But if E_3 is large enough, the value of h is too large and must be reduced by taking its half etc.

§ 6. Runge-Kutta Method.

It is one of the most widely used methods and it is particularly suitable in cases when the computation of higher derivatives is complicated.

We consider the differential equation

$$y'=f(x, y)$$

with the initial condition $y(x_0)=y_0$.

Let h be the interval between equidistant values of x. Then the first increment in y is computed from the formulae

$$\left.\begin{aligned}
k_1&=hf(x_0, y_0),\\
k_2&=hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right),\\
k_3&=hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right),\\
k_4&=hf(x_0+h, y_0+k_3),\\
\Delta y&=\frac{1}{6}(k_1+2k_2+2k_3+k_4),
\end{aligned}\right\} \qquad \ldots(\cdot)$$

taken in the given order.

Then $x_1=x_0+h$ and $y_1=y_0+\Delta y$.

In a similar manner the increment in y for the second interval is computed by means of the formulae

$$k_1=hf(x_1, y_1),$$

$$k_2=hf\left(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}\right),$$

$$k_3=hf\left(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}\right),$$

$$k_4=hf(x_1+h, y_1+k_3),$$

$$\Delta y=\frac{1}{6}(k_1+2k_2+2k_3+k_4),$$

and similarly for the next intervals.

It is to be noted that the calculations for the first increment are exactly the same as for any other increment. The change in the formulae for the different intervals is only in the values of x and y to be substituted.

Hence, to obtain Δy for the nth interval we substitute x_{n-1}, y_{n-1}, in the expressions for k_1, k_2, etc.

Remark. Specially when dy/dx is independent of y, say $dy/dx=f(x)$, the Runge-Kutta method reduces to Simpson's rule.

We have $k_1=hf(x_0)$, $k_2=hf(x_0+\frac{1}{2}h)$,
$k_3=hf(x_0+\frac{1}{2}h)$, $k_4=hf(x_0+h)$,

and hence
$$\Delta y=\frac{h}{6}\left[f(x_0)+2f\left(x_0+\frac{h}{2}\right)+2f\left(x_0+\frac{h}{2}\right)+f(x_0+h)\right]$$
$$=\frac{\frac{1}{2}h}{3}\left[f(x_0)+4f\{x_0+(h/2)\}+f(x_0+h)\right],$$

which is the same result as would be obtained by using Simpson's rule to the interval x_0 to x_0+h if we take two equal subintervals of width $h/2$.

The inherent error in the Runge-Kutta method is of the order h^5. Therefore it is of the same order as that in Simpson's rule.

§ 7. A general approach to predictors and correctors.

Suppose we want to develop a formula which uses the information of the function $y(x)$ and its first derivative *i.e.*, of $y'(x)$ at the past three points together with one more old value of the derivative. The most general formula of such type can be written as

$$y_{n+1}=A_0y_n+A_1y_{n-1}+A_2y_{n-2}+h\,(B_0y'_n+B_1y'_{n-1}+B_2y'_{n-2}+B_3y'_{n-3}). \qquad \ldots(1)$$

It contains seven unknowns. Suppose it holds for polynomials upto degree four *i.e.*, we can take $y(x)$ as $1, x, x^2, x^3, x^4$. Let the space between the consecutive values of x be unity *i.e.*, $h=1$. Putting $h=1$ and $y(x)=1, x, x^2, x^3, x^4$ successively in (1), we get

$$\left.\begin{aligned}
1&=A_0+A_1+A_2,\\
1&=-A_1-2A_2+B_0+B_1+B_2+B_3,\\
1&=A_1+4A_2-2B_1-4B_2-6B_3,\\
1&=-A_1-8A_2+3B_1+12B_2+27B_3,\\
1&=A_1+16A_2-4B_1-32B_2-108B_3
\end{aligned}\right\} \qquad \ldots(2)$$

We have 5 equations in 7 unknowns so taking A_1 and A_2 as parameters and solving the equations (2), we get

$$A_0=1-A_1-A_2,\ B_0=\frac{1}{24}(55+9A_1+8A_2),$$

$$B_1=\frac{1}{24}(-59+19A_1+32A_2),\ B_2=\frac{1}{24}(37-5A_1+8A_2),$$

$$B_3=\frac{1}{24}(-9+A_1).$$

Since A_1 and A_2 are arbitrary so taking $A_1=A_2=0$, we obtain

$$A_0=1,\ B_0=\frac{55}{24},\ B_1=-\frac{59}{24},\ B_2=\frac{37}{24},\ B_3=-\frac{9}{24}.$$

Substituting these values in (1), we obtain

$$y_{n+1}=y_n+\frac{h}{24}[55y'_n-59y'_{n-1}+37y'_{n-2}-9y'_{n-3}]. \quad ...(3)$$

This is known as **Adam's Predictor Formula.**

We can find other predictor formulae by taking new suitable pairs of values of A_1 and A_2. The formulae obtained from equation (1) are called **Adam's Bashford type predictors.**

Now we shall find the local truncation error of Adam's predictor i.e., of (3).

We have

$$y_k=y_0+(kh)\,y_0'+\tfrac{1}{2}(kh)^2\,y_0^{(2)}+\frac{(kh)^3}{6}\,y_0^{(3)}+\frac{(kh)^4}{24}\,y_0^{(4)}+\frac{(kh)^5}{120}\,y_0^{(5)}+... \quad ...(4)$$

$$\therefore\ y_{n+1}=y_0+(n+1)\,hy_0'+\frac{(n+1)^2h^2}{2}\,y_0^{(2)}+\frac{(n+1)^3h^3}{6}\,y_0^{(3)}+\frac{(n+1)^4h^4}{24}\,y_0^{(4)}+\frac{(n+1)^5h^5}{120}\,y_0^{(5)}+... \quad ...(5)$$

and
$$y_n=y_0+nh\,y_0'+\frac{n^2h^2}{2}\,y_0^{(2)}+\frac{n^3h^3}{6}\,y_0^{(3)}+\frac{n^4h^4}{24}\,y_0^{(4)}+\frac{n^5h^5}{120}\,y_0^{(5)}+... \quad ...(6)$$

Also, we have

$$y_k'=y_0'+(kh)\,y_0^{(2)}+\frac{(kh)^2}{2}\,y_0^{(3)}+\frac{(kh)^3}{6}\,y_0^{(4)}+\frac{(kh)^4}{24}\,y_0^{(5)}+...$$

$$\therefore\ y_n'=y_0'+nhy_0^{(2)}+\frac{n^2h^2}{2}\,y_0^{(3)}+\frac{n^3h^3}{6}\,y_0^{(4)}+\frac{n^4h^4}{24}\,y_0^{(5)}+... \quad ...(7)$$

$$y'_{n-1}=y_0'+(n-1)\,h\,y_0^{(2)}+\frac{(n-1)^2h^2}{2}\,y_0^{(3)}+\frac{(n-1)^3h^3}{6}\,y_0^{(4)}+\frac{(n-1)^4h^4}{24}\,y_0^{(5)}+... \quad ...(8)$$

$$y'_{n-2}=y_0'+(n-2)\,h\,y_0^{(2)}+\frac{(n-2)^2h^2}{2}\,y_0^{(3)}+\frac{(n-2)^3h^3}{6}\,y_0^{(4)}+\frac{(n-2)^4h^4}{24}\,y_0^{(5)}+... \quad ...(9)$$

$$y'_{n-3}=y_0'+(n-3)\,h\,y_0^{(2)}+\frac{(n-3)^2h^2}{2}\,y_0^{(3)}+\frac{(n-3)^3h^3}{6}\,y_0^{(4)}+\frac{(n-3)^4h^4}{24}\,y_0^{(5)}+... \quad ...(10)$$

Substituting all these values in (3), we get

$$(y_{n+1}-y_n)-\frac{h}{24}[55\,y_n'-59\,y'_{n-1}+37\,y'_{n-2}-9y'_{n-3}]=\frac{251}{720}\,h^5\,y_0^{(5)}+... \quad ...(11)$$

This shows that the local truncation error is $\frac{251}{720}h^5y_0^{(5)}+...$

Using the first term of this error as an estimate the local truncation error is $\frac{251}{720}h^5y_0^{(5)}$.

In the same manner we can find the truncation error of the other predictors also.

To find correctors. The most general corrector formula which involves the information about the function and its first derivative at the past three points together with the value of the derivative at the point being computed is

$$y_{n+1}=a_0y_n+a_1y_{n-1}+a_2y_{n-2}+h\,(b_{-1}\,y'_{n+1}+b_0y'_n+b_1\,y'_{n-1}+b_2\,y'_{n-2}) \quad ...(12)$$

Again taking $h=1$ and making (12) exact successively for $y(x)=1,\ x,\ x^2,\ x^3,\ x^4$, we get

$$a_0+a_1+a_2=1,\ a_1+24b_{-1}=9$$
$$13\,a_1+8\,a_2-24\,b_0=-19,\ 13\,a_1+32\,a_2-24\,b_1=5$$
$$a_1-8\,a_2+24\,b_2=1.$$

Thus we have 5 equations in 7 unknowns. So taking a_1 and a_2 as parameters, we get

$$a_0=1-a_1-a_2,\ b_{-1}=\frac{1}{24}(9-a_1),$$

$$b_0=\frac{1}{24}(19+13\,a_1+8\,a_2),$$

$$b_1=\frac{1}{24}(-5+13\,a_1+32\,a_2),$$

$$b_2=\frac{1}{24}(1-a_1+8\,a_2).$$

Choosing $a_1=a_2=0$, we get

$$a_0=1,\ b_{-1}=\frac{9}{24},\ b_0=\frac{19}{24},\ b_1=-\frac{5}{24},\ b_2=\frac{1}{24}.$$

Substituting these values in (12), we get

$$y_{n+1}=y_n+\frac{h}{24}(9y'_{n+1}+19\,y_n'-5\,y'_{n-1}+y'_{n-2}). \quad \ldots(13)$$

By taking other suitable values of a_1 and a_2 various correctors can be found.

Proceeding as earlier in the case of (3), we can show that the corrector (13) has local truncation error $-\frac{19}{720}h^5y_0^{(5)}$.

We notice that the error in the corrector (13) is less enough than that of its predictor (3).

§ 8. Differential equations of second order.

We know that an equation involving derivatives upto second order is called a differential equation of second order. A second order differential equation can be put in the form

$$\frac{d^2x}{dt^2}=\phi\,[x,\,(dx/dt),\,t]$$

under the initial conditions $x\,(t_0)=x_0,\ x'(t_0)=x_0'$.

We get a system of simultaneous differential equations of first order by reducing this second order equation.

Taking $dx/dt=y$, we get an equivalent system

$$\left.\begin{aligned}\frac{dx}{dt}&=y, & x\,(t_0)&=x_0\\ \frac{dy}{dt}&=\phi\,(x,y,t), & y\,(t_0)&=x_0'\end{aligned}\right\}$$

which can be solved by usual methods.

Note. A differential equation of order n yields n simultaneous differential equations of first order.

§ 9. Numerov's Method.

This method is used for the differential equation of the form $y''=\phi\,(x,y)$ *i.e.* when there is a second order differential equation without the term of y'.

A formula is developed by using the method of undetermined coefficients.

Let $x_1, x_2, \ldots, x_k, \ldots$, be the equidistant values of x and the corresponding values of y be $y_1, y_2, \ldots, y_k, \ldots$. Let y_{k-1} and y_k be known and hence $\phi_{k-1}=\phi\,(x_{k-1}, y_{k-1})$ and $\phi_k=\phi\,(x_k, y_k)$ are also known. For integrating the differential equation we assume a corrector type formula of the form

$$y_{k+1}=Ay_k+By_{k-1}+h^2\,(C\phi_{k+1}+D\phi_k+E\phi_{k-1})+R, \quad \ldots(1)$$

where A, B, C, D, E are unknown coefficients to be determined and R denotes the error term. Let $y_k^{(p)}$ denote the pth derivative of y at $x=x_k$. By Taylor's series, we get

$$y_{k+1}=y_k+hy_k^{(1)}+\frac{h^2}{2!}y_k^{(2)}+\frac{h^3}{3!}y_k^{(3)}+\frac{h^4}{4!}y_k^{(4)}+\frac{h^5}{5!}y_k^{(5)}+\frac{h^6}{6!}y_k^{(6)},$$

$$y_{k-1}=y_k-hy_k^{(1)}+\frac{h^2}{2!}y_k^{(2)}-\frac{h^3}{3!}y_k^{(3)}+\frac{h^4}{4!}y_k^{(4)}-\frac{h^5}{5!}y_k^{(5)}+\frac{h^6}{6!}y_k^{(6)},$$

and $\phi_{k+1}=\phi_k+h\phi_k^{(1)}+\frac{h^2}{2!}\phi_k^{(2)}+\frac{h^3}{3!}\phi_k^{(3)}+\frac{h^4}{4!}\phi_k^{(4)}+\ldots$.

But, we have $\phi=y^{(2)}$.

$\therefore\ \phi^{(1)}=y^{(3)},\ \phi^{(2)}=y^{(4)},\ \phi^{(3)}=y^{(5)},\ \phi^{(4)}=y^{(6)}$, etc.

So $h^2\phi_{k+1}=h^2y_k^{(2)}+h^3y_k^{(3)}+\frac{h^4}{2!}y_k^{(4)}+\frac{h^5}{3!}y_k^{(5)}+\frac{h^6}{4!}y_k^{(6)}$

and $h^2\phi_{k-1}=h^2y_k^{(2)}-h^3y_k^{(3)}+\frac{h^4}{2!}y_k^{(4)}-\frac{h^5}{3!}y_k^{(5)}+\frac{h^6}{4!}y_k^{(6)}$,

The unknown coefficients are to be so determined that the formula (1) agrees with the Taylor's expansion of y_{k+1} upto fourth order term. Substituting the values of $y_{k+1}, y_k, y_{k-1}, \phi_{k+1}, \phi_k, \phi_{k-1}$ in (1) and then comparing the coefficients of various powers of h upto h^4 on both the sides, we have

$$1=A+B,\ 1=-B,\ \tfrac{1}{2}=(B/2)+C+D+E,$$
$$1/6=-(B/6)+C-E,\ 1/24=(B/24)+C/2+E/2.$$

Solving these equations, we get

$$A=2,\ B=-1,\ C=1/12,\ D=5/6,\ E=1/12.$$

It is to be noted that the coefficients of the fifth powers of h are also equal on both sides, though we have matched the terms only upto fourth powers.

Putting these values in (1), we get

$$y_{k+1}=2y_k-y_{k-1}+\frac{h^2}{12}(\phi_{k+1}+10\phi_k+\phi_{k-1})+R. \quad \ldots(2)$$

This is known as **Numerov's formula.**

This formula contains an error of order six and if we suppose all the terms of $y^{(6)}$ to be equal then the error estimate $= -h^6 y_x^{(6)}/240$. Now in (2), y_{k+1} appears on both the sides hence this formula seems to be a corrector formula. So in order to determine y_{k+1}, some previous approximation to y_{k+1} is necessary. For this purpose we may use (2) after ignoring the term ϕ_{k+1}. We readily get

$$y_{k+1} = 2y_k - y_{k-1} + (h^2/12)\,(10\phi_k + \phi_{k-1}). \qquad \text{...(3)}$$

The formula (3) may be used as a predictor formula and (2) as a corrector formula.

Solved Examples

Ex. 1. *Use Picard's method to approximate y when $x = 0{\cdot}2$, given that $y = 1$ when $x = 0$ and $\dfrac{dy}{dx} = x - y$.* **(Meerut M.Sc. 1992)**

Sol. Here $f(x, y) = x - y$, $x_0 = 0$, $y_0 = 1$.

We have first approximation,

$$y^{(1)} = y_0 + \int_0^x f(x, y_0)\, dx = 1 + \int_0^x (x-1)\, dx = 1 - x + \frac{x^2}{2}.$$

Second approximation,

$$y^{(2)} = y_0 + \int_0^x f(x, y^{(1)})\, dx = 1 + \int_0^x (x - y^{(1)})\, dx$$

$$= 1 + \int_0^x \left(x - 1 + x - \frac{x^2}{2}\right) dx = 1 - x + x^2 - \frac{x^3}{6}.$$

Third approximation,

$$y^{(3)} = y_0 + \int_0^x f(x, y^{(2)})\, dx = 1 + \int_0^x (x - y^{(2)})\, dx$$

$$= 1 + \int_0^x \left(x - 1 + x - x^2 + \frac{x^3}{6}\right) dx = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}.$$

Fourth approximation,

$$y^{(4)} = y_0 + \int_0^x f(x, y^{(3)})\, dx = 1 + \int_0^x (x - y^{(3)})\, dx$$

$$= 1 + \int_0^x \left(x - 1 + x - x^2 + \frac{x^3}{3} - \frac{x^4}{24}\right) dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}.$$

Fifth approximation,

$$y^{(5)} = y_0 + \int_0^x f(x, y^{(4)})\, dx = 1 + \int_0^x (x - y^{(4)})\, dx$$

$$= 1 + \int_0^x \left(x - 1 + x - x^2 + \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^5}{120}\right) dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{720}.$$

When $x = 0{\cdot}2$, we get

$y^{(1)} = 0{\cdot}82$, $y^{(2)} = 0{\cdot}83867$, $y^{(3)} = 0{\cdot}83740$,

$y^{(4)} = 0{\cdot}83746$ and $y^{(5)} = 0{\cdot}83746$.

Thus $y = {\cdot}837$ when $x = 0{\cdot}2$.

Ex. 2. *Use Picard's method to approximate the value of y when $x = 0{\cdot}1$, given that $y = 1$ when $x = 0$ and $\dfrac{dy}{dx} = 3x + y^2$.*

(Baroda 1985; Meerut 93P)

Sol. Here $f(x, y) = 3x + y^2$, $x_0 = 0$, $y_0 = 1$.

We have first approximation,

$$y^{(1)} = y_0 + \int_{x_0}^x f(x, y_0)\, dx = 1 + \int_0^x (3x + y_0^2)\, dx$$

$$= 1 + \int_0^x (3x + 1)\, dx = 1 + x + 3\,\frac{x^2}{2}.$$

Second approximation,

$$y^{(2)} = y_0 + \int_0^x f(x, y^{(1)})\, dx = 1 + \int_0^x \left[3x + \{y^{(1)}\}^2\right] dx$$

$$= 1 + \int_0^x \left[3x + \left(\frac{3}{2}x^2 + x + 1\right)^2\right] dx$$

$$= 1 + \int_0^x \left(\frac{9}{4}x^4 + 3x^3 + 4x^2 + 5x + 1\right) dx$$

$$= 1 + x + \frac{5}{2}x^2 + \frac{4}{3}x^3 + \frac{3}{4}x^4 + \frac{9}{20}x^5.$$

Third approximation,

$$y^{(3)} = y_0 + \int_0^x f(x, y^{(2)})\, dx = 1 + \int_0^x \left[3x + \{y^{(2)}\}^2\right] dx$$

$$= 1 + \int_0^x \left(\frac{81}{400}x^{10} + \frac{27}{40}x^9 + \frac{141}{80}x^8 + \frac{17}{4}x^7 + \frac{1157}{180}x^6 + \frac{136}{15}x^5 + \frac{125}{12}x^4 + \frac{23}{3}x^3 + 6x^2 + 5x + 1\right) dx$$

$$= 1 + x + \frac{5}{2}x^2 + 2x^3 + \frac{23}{12}x^4 + \frac{25}{12}x^5 + \frac{68}{45}x^6 + \frac{1157}{1260}x^7 + \frac{17}{32}x^8 + \frac{47}{240}x^9 + \frac{27}{400}x^{10} + \frac{81}{4400}x^{11}.$$

When $x=0{\cdot}1$, we have

$y^{(1)}=1{\cdot}115$, $y^{(2)}=1{\cdot}1264$, $y^{(3)}=1{\cdot}12721$.

Thus $y=1{\cdot}127$ when $x=0{\cdot}1$.

Ex. 3. *Use Picard's method to approximate y when $x=0{\cdot}1$ given that $y=1$ when $x=0$ and $\dfrac{dy}{dx}=\dfrac{y-x}{y+x}$.* **(Meerut M.Sc. 1993; B.A. Hons. Delhi 77)**

Sol. Here $f(x, y)=\dfrac{y-x}{y+x}$, $x_0=0$, $y_0=1$.

We have first approximation,

$$y^{(1)}=y_0+\int_{x_0}^{x} f(x, y_0)\,dx=1+\int_0^x \frac{y_0-x}{y_0+x}\,dx=1+\int_0^x \frac{1-x}{1+x}\,dx$$

$$=1+\int_0^x \left[\frac{2}{1+x}-1\right] dx=1+\Big[2\log(1+x)-x\Big]_0^x$$

$$=1-x+2\log(1+x).$$

Second approximation,

$$y^{(2)}=y_0+\int_{x_0}^{x} f(x, y^{(1)})\,dx=1+\int_0^x \frac{y^{(1)}-x}{y^{(1)}+x}\,dx$$

$$=1+\int_0^x \frac{1-2x+2\log(1+x)}{1+2\log(1+x)}\,dx$$

$$=1+\int_0^x \left[1-\frac{2x}{1+2\log(1+x)}\right] dx$$

$$=1+x-2\int_0^x \frac{x}{1+2\log(1+x)}\,dx,$$

which is difficult to integrate.

Sometimes we get the complicated integrals and it is the practical difficulty associated with this method. Here in this example only first approximation can be obtained and so it gives the approximate value of y for $x=0{\cdot}1$.

Thus when $x=0{\cdot}1$, $y^{(1)}=1-0{\cdot}1+2\log(1{\cdot}1)=0{\cdot}9828$.

Ex. 4. *Approximate y and z by using Picard's method for the particular solution of*

$$\frac{dy}{dx}=x+z,\quad \frac{dz}{dx}=x-y^2,$$

given that $y=2$, $z=1$, when $x=0$. **(Delhi B.A. Hons. 72)**

Sol. Let $\phi(x, y, z)=x+z$ and $f(x, y, z)=x-y^2$.

Here $x_0=0$, $y_0=2$, $z_0=1$.

We have $\dfrac{dy}{dx}=\phi(x, y, z)$ $\Rightarrow$ $y=y_0+\displaystyle\int_{x_0}^{x} \phi(x, y, z)\,dx$. ...(1)

Also $\dfrac{dz}{dx}=f(x, y, z)$ $\Rightarrow$ $z=z_0+\displaystyle\int_{x_0}^{x} f(x, y, z)\,dx$ ...(2)

We have first approximation of y,

$$y^{(1)}=y_0+\int_{x_0}^{x} \phi(x, y_0, z_0)\,dx=2+\int_0^x (x+z_0)\,dx$$

$$=2+\int_0^x (x+1)\,dx=2+x+\frac{1}{2}x^2.$$

Again first approximation of z is,

$$z^{(1)}=z_0+\int_{x_0}^{x} f(x, y_0, z_0)\,dx=1+\int_0^x (x-y_0^2)\,dx$$

$$=1+\int_0^x (x-4)\,dx=1-4x+\frac{1}{2}x^2.$$

Second approximation of y,

$$y^{(2)}=y_0+\int_{x_0}^{x} \phi(x, y^{(1)}, z^{(1)})\,dx=2+\int_0^x (x+z^{(1)})\,dx$$

$$=2+\int_0^x \left(x+1-4x+\frac{1}{2}x^2\right)dx=2+x-(3/2)x^2+(1/6)x^3.$$

Second approximation of z,

$$z^{(2)}=z_0+\int_{x_0}^{x} f(x, y^{(1)}, z^{(1)})\,dx=1+\int_0^x [x-\{y^{(1)}\}^2]\,dx$$

$$=1+\int_0^x \left[x-\left(2+x+\frac{1}{2}x^2\right)^2\right]dx$$

$$=1+\int_0^x \left[x-\left(4+x^2+\frac{1}{4}x^4+4x+x^3+2x^2\right)\right]dx$$

$$=1-4x-(3/2)x^2-x^3-(x^4/4)-(x^5/20).$$

Ex. 5. *Given $\dfrac{dy}{dx}=\dfrac{y-x}{y+x}$, with $y=1$ for $x=0$. Find y approximately for $x=0{\cdot}1$ by Euler's method. (Five steps).*

(Delhi B.A. Hons. 73)

Sol. Here we want the value at $x=0{\cdot}1$ from $x=0$ in five steps. So we break up the interval 0 to $0{\cdot}1$ into five subintervals by introducing the points x_1, x_2, x_3, x_4, x_5. Let $h=0{\cdot}02$. We shall find the values of y at $x={\cdot}02$, ${\cdot}04$, ${\cdot}06$, ${\cdot}08$ and $0{\cdot}1$ successively. Thus we have

$$x_0=0,\ y_0=1,\ h={\cdot}02,\ f(x, y)=\frac{y-x}{y+x}.$$

Using $y_{n+1}=y_n+hf(x_n, y_n)$, we get

$$y_1 = y_0 + hf(x_0, y_0) = 1 + (\cdot 02)\left[\frac{1-0}{1+0}\right] = 1\cdot 02$$

$$y_2 = y_1 + hf(x_1, y_1) = 1\cdot 02 + (\cdot 02)\,\frac{1\cdot 02 - \cdot 02}{1\cdot 02 + \cdot 02} = 1\cdot 0392$$

$$y_3 = y_2 + hf(x_2, y_2) = 1\cdot 0392 + (\cdot 02)\,\frac{1\cdot 0392 - \cdot 04}{1\cdot 0392 + \cdot 04} = 1\cdot 0577$$

$$y_4 = y_3 + hf(x_3, y_3) = 1\cdot 0577 + (\cdot 02)\,\frac{1\cdot 0577 - \cdot 06}{1\cdot 0577 + \cdot 06} = 1\cdot 0756$$

$$y_5 = y_4 + hf(x_4, y_4) = 1\cdot 0756 + (\cdot 02)\,\frac{1\cdot 0756 - \cdot 08}{1\cdot 0756 + \cdot 08} = 1\cdot 0928.$$

Hence $y = 1\cdot 0928$ when $x = 0\cdot 1$.

Ex. 6. *Use Euler's modified method to compute y for* $x = 0\cdot 05$ *and* $x = 0\cdot 1$. *Given that* $\frac{dy}{dx} = x + y$, *with the initial condition* $x_0 = 0$, $y_0 = 1$. *Give the correct result upto four decimal places.*

Sol. The given differential equation is

$$\frac{dy}{dx} = f(x, y) = x + y. \qquad \ldots(1)$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_0 = f(x_0, y_0) = x_0 + y_0 = 0 + 1 = 1.$$

Take $h = 0\cdot 05$, then we have

$$y_1^{(1)} = y_0 + hf(x_0, y_0) = 1 + (\cdot 05).(1) = 1\cdot 05.$$

Now $\left(\frac{dy}{dx}\right)_1^{(1)} = x_1 + y_1^{(1)} = \cdot 05 + 1\cdot 05 = 1\cdot 10.$

The second approximation to y_1 is given by

$$y_1^{(2)} = y_0 + \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^{(1)}}{2}\, h = 1 + \frac{1 + 1\cdot 10}{2} \times 0\cdot 05$$

$$= 1\cdot 0525.$$

The second approximation to $\left(\frac{dy}{dx}\right)_1$ is

$$\left(\frac{dy}{dx}\right)_1^{(2)} = x_1 + y_1^{(2)} = \cdot 05 + 1\cdot 0525 = 1\cdot 1025.$$

Now the third approximation to y_1 is

$$y_1^{(3)} = y_0 + \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^{(2)}}{2}\, h = 1 + \frac{1 + 1\cdot 1025}{2} \times 0\cdot 05$$

$$= 1\cdot 05256.$$

The third approximation to $\left(\frac{dy}{dx}\right)_1$ is

$$\left(\frac{dy}{dx}\right)_1^{(3)} = x_1 + y_1^{(3)} = \cdot 05 + 1\cdot 05256 = 1\cdot 10256.$$

The fourth approximation to y_1 is

$$y_1^{(4)} = y_0 + \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^{(3)}}{2}\, h = 1 + \frac{1 + 1\cdot 10256}{2} \times 0\cdot 05$$

$$= 1\cdot 05256.$$

Since $y_1^{(3)} = y_1^{(4)}$ so we stop here.

Hence we take $y_1 = 1\cdot 0526$, $\left(\frac{dy}{dx}\right)_1 = 1\cdot 1026.$

As a first approximation to y_2, by using the relation

$$= y_{n-1} + 2h\, y_n',$$

we have

$$y_2^{(1)} = y_0 + 2h\left(\frac{dy}{dx}\right)_1 = 1 + 2\,(\cdot 05)\,(1\cdot 1026) = 1\cdot 1103.$$

Now $\left(\frac{dy}{dx}\right)_2^{(1)} = \cdot 1 + 1\cdot 1103 = 1\cdot 2103.$

The second approximation to y_2 is

$$y_2^{(2)} = y_1 + \frac{\left(\frac{dy}{dx}\right)_1 + \left(\frac{dy}{dx}\right)_2^{(1)}}{2}\, h$$

$$= 1\cdot 0526 + \left(\frac{1\cdot 1026 + 1\cdot 2103}{2}\right)(\cdot 05)$$

$$= 1\cdot 1104$$

and $\left(\frac{dy}{dx}\right)_2^{(2)} = 0\cdot 1 + 1\cdot 1104 = 1\cdot 2104.$

Then $y_2^{(3)} = y_1 + \frac{\left(\frac{dy}{dx}\right)_1 + \left(\frac{dy}{dx}\right)_2^{(2)}}{2}\, h$

$$= 1\cdot 0526 + \frac{(1\cdot 1026 + 1\cdot 2104)}{2} \times \cdot 05 = 1\cdot 1104.$$

Since $y_2^{(2)} = y_2^{(3)}$ so we take

$$y_2 = 1\cdot 1104, \quad \left(\frac{dy}{dx}\right)_2 = 1\cdot 2104.$$

The following table shows the results in tabular f

x	y	dy/dx
0·00	1·0000	1·0000
0·05	1·0526	1·1026
0·10	1·1104	1·2104

Ex. 7. *Using Euler's modified method, find a solution of the equation*

$$dy/dx = x + |\sqrt{y}| = f(x, y)$$

with initial condition $y=1$ at $x=0$ for the range $0 \leqslant x \leqslant 0{\cdot}6$ in steps of 0·2. **(Rohilkhand M.Sc. 1990; Delhi B.A. Hons. 71)**

Sol. Here $f(x, y) = x + |\sqrt{y}|$, $x_0 = 0$, $y_0 = 1$, $h = 0{\cdot}2$.

$\therefore\ f(x_0, y_0) = x_0 + |\sqrt{y_0}| = 1.$

We have $y_1^{(1)} = y_0 + h\, f(x_0, y_0) = 1 + (0{\cdot}2).1 = 1{\cdot}2.$

Now $\left(\dfrac{dy}{dx}\right)_1^{(1)} = f(x_1, y_1^{(1)}) = x_1 + |\sqrt{y_1^{(1)}}| = 0{\cdot}2 + |\sqrt{(1{\cdot}2)}|$

$= 1{\cdot}2954.$

The second approximation to y_1 is

$$y_1^{(2)} = y_0 + \frac{f(x_0, y_0) + f(x_1, y_1^{(1)})}{2}.h = 1 + \frac{1 + 1{\cdot}2954}{2} \times 0{\cdot}2$$

$= 1{\cdot}2295.$

Again $f(x_1, y_1^{(2)}) = x_1 + |\sqrt{y_1^{(2)}}| = 0{\cdot}2 + \sqrt{(1{\cdot}2295)}$

$= 1{\cdot}3088.$

So $y_1^{(3)} = y_0 + \dfrac{f(x_0, y_0) + f(x_1, y_1^{(2)})}{2}.h = 1 + \dfrac{1 + 1{\cdot}3088}{2} \times 0{\cdot}2$

$= 1{\cdot}2309.$

We have $f(x_1, y_1^{(3)}) = 0{\cdot}2 + \sqrt{(1{\cdot}2309)} = 1{\cdot}309.$

Then $y_1^{(4)} = y_0 + \dfrac{f(x_0, y_0) + f(x_1, y_1^{(3)})}{2}.h = 1{\cdot}2309.$

Since $y_1^{(4)} = y_1^{(3)}$, so we take $y_1 = 1{\cdot}2309$.

Now $y_2^{(1)} = y_1 + h\, f(x_1, y_1) = 1{\cdot}2309 + (0{\cdot}2)\,[0{\cdot}2 + \sqrt{(1{\cdot}2309)}]$

$= 1{\cdot}4927.$

and $f(x_2, y_2^{(1)}) = 0{\cdot}4 + \sqrt{(1{\cdot}4927)} = 1{\cdot}622.$

Then $y_2^{(2)} = y_1 + \dfrac{f(x_1, y_1) + f(x_2, y_2^{(1)})}{2}.h$

$$= 1{\cdot}2309 + \frac{[0{\cdot}2 + \sqrt{(1{\cdot}2309)}] + 1{\cdot}622}{2} \times 0{\cdot}2$$

$= 1{\cdot}5240.$

Now $y_2^{(3)} = y_1 + \dfrac{f(x_1, y_1) + f(x_2, y_2^{(2)})}{2}.h$

$$= 1{\cdot}2309 + \frac{[\{0{\cdot}2 + \sqrt{(1{\cdot}2309)}\} + \{0{\cdot}4 + \sqrt{(1{\cdot}5240)}\}]}{2} \times 0{\cdot}2$$

$= 1{\cdot}5253.$

$$y_2^{(4)} = 1{\cdot}2309 + \frac{[\{0{\cdot}2 + \sqrt{(1{\cdot}2309)}\} + \{0{\cdot}4 + \sqrt{(1{\cdot}5253)}\}]}{2} \times 0{\cdot}2$$

$= 1{\cdot}5253.$

Since $y_2^{(3)} = y_2^{(4)}$, so we take $y_2 = 1{\cdot}5253$.

We have $y_3^{(1)} = y_2 + h\, f(x_2, y_2) = 1{\cdot}5253 + (0{\cdot}2)\,[0{\cdot}4 + \sqrt{(1{\cdot}5253)}]$

$= 1{\cdot}8523.$

$y_3^{(2)} = y_2 + \frac{1}{2}\,[\,f(x_2, y_2) + f(x_3, y_3^{(1)})].h$

$= 1{\cdot}5253 + \frac{1}{2} \times 0{\cdot}2\,[\{0{\cdot}4 + \sqrt{(1{\cdot}5253)}\} + \{0{\cdot}6 + \sqrt{(1{\cdot}8523)}\}]$

$= 1{\cdot}8849.$

Now $y_3^{(3)} = y_2 + \frac{1}{2}.h\,[\,f(x_2, y_2) + f(x_3, y_3^{(2)})]$

$= 1{\cdot}5253 + \frac{1}{2} \times 0{\cdot}2\,[\{0{\cdot}4 + \sqrt{(1{\cdot}5253)}\} + \{0{\cdot}6 + \sqrt{(1{\cdot}8849)}\}]$

$= 1{\cdot}8861.$

$y_3^{(4)} = y_2 + \frac{1}{2}.h\,[\,f(x_2, y_2) + f(x_3, y_3^{(3)})]$

$= 1{\cdot}5253 + \frac{1}{2}\,(0{\cdot}2)\,[\{0{\cdot}4 + \sqrt{(1{\cdot}5253)}\} + \{0{\cdot}6 + \sqrt{(1{\cdot}8861)}\}]$

$= 1{\cdot}8861.$

Since $y_3^{(4)} = y_3^{(3)}$, so we take $y_3 = 1{\cdot}8861$.

Ex. 8. *Use Taylor's series method to solve $(dy/dx) = x + y$: $y(1) = 0$ numerically upto $x = 1{\cdot}2$ with $h = 0{\cdot}1$. Compare the final result with the value of the explicit solution.* **(Delhi B.A. Hons. 77)**

Sol. Here we have $x_0 = 1$, $y_0 = 0$.

Also $y' = x + y$, $y'' = 1 + y'$, $y''' = y''$, $y^{(iv)} = y'''$, $y^{(v)} = y^{(iv)}$.

$\therefore\ y_0' = 1$, $y_0'' = 2$, $y_0''' = 2$, $y_0^{(iv)} = 2$, $y_0^{(v)} = 2$.

Now, we have by Taylor's series

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{(iv)} + \frac{h^5}{5!} y_0^{(v)}.$$

Substituting the values in this, we get

$$y_1 = 0 + (\cdot 1) + \frac{(\cdot 1)^2}{2} \times 2 + \frac{(\cdot 1)^3}{3!} \times 2 + \frac{(\cdot 1)^4}{4!} \times 2 + \frac{(\cdot 1)^5}{5!} \times 2$$

$= 0{\cdot}1103081 = {\cdot}110$ (approx.)

Also $x_1 = x_0 + h = 1 + {\cdot}1 = 1{\cdot}1.$

Again $y_1'=x_1+y_1=1\cdot1+\cdot11=1\cdot21$

$$y_1''=1+y_1'=1+1\cdot21=2\cdot21$$

$$y_1'''=y_1''=2\cdot21,\ y_1^{(iv)}=2\cdot21,\ y_1^{(v)}=2\cdot21.$$

Substituting the values in

$$y_2=y_1+h\,y_1'+\frac{h^2}{2!}y_1''+\frac{h^3}{3!}y_1'''+\frac{h^4}{4!}y_1^{(iv)}+\frac{h^5}{5!}y^{(v)},\text{ we get}$$

$$y_2=\cdot11+(\cdot1)(1\cdot21)+\frac{(\cdot1)^2}{2!}(2\cdot21)+\frac{(\cdot1)^3}{3!}(2\cdot21)+\frac{(\cdot1)^4}{4!}(2\cdot21)+\frac{(\cdot1)^5}{5!}(2\cdot21)$$

$$=0\cdot232\text{ (approx).}$$

The analytical solution of the given differential equation is

$$y=-x-1+2e^{x-1}.$$

[Solve it as a linear differential equation of first order]

When $x=1\cdot2$, we get $y=-1\cdot2-1+2e^{\cdot2}=0\cdot242$.

Ex. 9. *Tabulate by Milne's method the numerical solution of* $(dy/dx)=x+y$ *with* $x_0=0$, $y_0=1$ *from* $x=\cdot20$ *to* $x=\cdot30$.

Sol. Since the analytical explicit solution of the differential equation $dy/dx=x+y$, with the initial conditions

$$x_0=0,\ y_0=1,\text{ is } y=2e^x-x-1,$$

we can compute the exact value of y corresponding to any value of x. We shall find three consecutive values of y and y' corresponding to $x=\cdot05$, $\cdot10$ and $\cdot15$ *i.e.*, taking $h=\cdot05$.

x	y	$y'=(dy/dx)$
0·00	1·0000	1·0000
0·05	1·0525	1·1025
0·10	1·1103	1·2103
0·15	1·1736	1·3236

In general form Milne's predictor and corrector formulae are

$$y_{n+1}=y_{n-3}+\frac{4h}{3}[2y'_{n-2}-y'_{n-1}+2y'_n] \quad \ldots(1)$$

and $$y_{n+1}=y_{n-1}+\frac{h}{3}[y'_{n-1}+4y'_n+y'_{n+1}] \quad \ldots(2)$$

We are given y_0, y_1, y_2, y_3 and we are to find y_4 corresponding to $x_4=\cdot20$.

Putting $n=3$, $h=\cdot05$ in (1), we have

$$y_4=y_0+\frac{4h}{3}[2y_1'-y_2'+2y_3']$$

$$=1+\frac{4(\cdot05)}{3}[2\cdot205-1\cdot2103+2\cdot6472]=1\cdot2428,$$

which is the predicted value.

Putting $x=0\cdot20$ and $y=1\cdot2428$, in $(dy/dx)=x+y$, we get

$$y_4'=1\cdot4428.$$

Again putting $n=3$, $h=\cdot05$ in (2), we get

$$y_4=y_2+\frac{\cdot05}{3}[y_2'+4y_3'+y_4']$$

$$=1\cdot1103+\frac{\cdot05}{3}[1\cdot2103+5\cdot2944+1\cdot4428]=1\cdot2428,$$

which is the corrected value and is the same as the predicted value.

Hence $y=1\cdot2428$ when $x=\cdot20$ and $y'=1\cdot4428$.

Now putting $n=4$, $h=\cdot05$ in (1), we have

$$y_5=y_1+\frac{4h}{3}[2y_2'-y_3'+2y_4']$$

$$=1\cdot0525+\frac{4(\cdot05)}{3}[2\cdot4206-1\cdot3236+2\cdot8856]=1\cdot3180$$

which is corrected by

$$y_5=y_3+\frac{h}{3}[y_3'+4y_4'+y_5']$$

$$=1\cdot1736+\frac{\cdot05}{3}[1\cdot3236+5\cdot7712+1\cdot568]=1\cdot3180$$

which is the same as the predicted value.

Thus $y_5=1\cdot3180$ and $y_5'=1\cdot5680$.

Putting $n=5$, $h=\cdot05$, we get

$$y_6=y_2+\frac{4h}{3}[2y_3'-y_4'+2y_5']$$

$$=1\cdot1103+\frac{4(\cdot05)}{3}[2\cdot6472-1\cdot4428+3\cdot1360]=1\cdot3997$$

and it is corrected by

$$y_6=y_4+\frac{h}{3}[y_4'+4y_5'+y_6']$$

$$=1\cdot2428+\frac{\cdot05}{3}[1\cdot4428+6\cdot2720+1\cdot6997]=1\cdot3997$$

which is the same as the predicted value.

Thus $y=1\cdot3997$ when $x=\cdot30$ and $y'=1\cdot6997$.

Collecting our results in tabular form, we have the following table :

x	y	$y'=dy/dx$
$x_4=\cdot 20$	$y_4=1\cdot 2428$	$y'_4=1\cdot 4428$
$x_5=\cdot 25$	$y_5=1\cdot 3180$	$y'_5=1\cdot 5680$
$x_6=\cdot 30$	$y_6=1\cdot 3997$	$y'_6=1\cdot 6997$

Ex. 10. *Find $y(2)$, if $y(x)$ is the solution of $\frac{dy}{dx}=\frac{1}{2}(x+y)$ assuming $y(0)=2$, $y(0\cdot 5)=2\cdot 636$, $y(1\cdot 0)=3\cdot 595$ and $y(1\cdot 5)=4\cdot 968$.*

(Kurukshetra M.A. 76; Meerut M.Sc. 72, 91)

Sol. Here we shall use Milne's method. Let $x_0=0$, $x_1=0\cdot 5$, $x_2=1\cdot 0$, $x_3=1\cdot 5$. Then we are given y_0, y_1, y_2, y_3 and we require y_4 corresponding to $x_4=2$.

Using the predictor formula, we get

$$y_4=y_0+(4h/3)\,[2y'_1-y'_2+2y'_3].$$

We have $y'=\frac{1}{2}(x+y)$. Therefore

$$y_1'=\tfrac{1}{2}(x_1+y_1)=\tfrac{1}{2}(\cdot 5+2\cdot 636)=1\cdot 568,$$
$$y_2'=\tfrac{1}{2}(x_2+y_2)=\tfrac{1}{2}(1+3\cdot 595)=2\cdot 2975,$$
$$y_3'=\tfrac{1}{2}(x_3+y_3)=\tfrac{1}{2}(1\cdot 5+4\cdot 968)=3\cdot 234.$$

Substituting the values, we get the predicted value

$$y_4=2+\frac{4\,(0\cdot 5)}{3}\,[3\cdot 136-2\cdot 2975+6\cdot 468]=6\cdot 8710.$$

By the corrector formula, the first corrected value

$$y_4^{(1)}=y_2+(h/3)\,[y_2'+4y_3'+y_4']$$
$$=3\cdot 595+\frac{0\cdot 5}{3}\,[2\cdot 2975+12\cdot 936+4\cdot 4355]$$
$$=6\cdot 87317.$$

Now $(y_4^{(1)})'=\frac{1}{2}\{x_4+y_4^{(1)}\}=\frac{1}{2}\{2+6\cdot 87317\}=4\cdot 43659.$

Again by the corrector formula we get the second corrected value

$$y_4^{(2)}=y_2+\frac{h}{3}\,[y_2'+4y_3'+(y_4^{(1)})']$$

$$=3\cdot 595+\frac{0\cdot 5}{3}\,[2\cdot 2975+12\cdot 936+4\cdot 43659]=6\cdot 87335.$$

Ex. 11. *Use Runge-Kutta method to approximate y, when $x=0\cdot 1$ and $x=0\cdot 2$, given that $x=0$ when $y=1$ and $(dy/dx)=x+y$.*

(Meerut M.Sc. 1988, 89; Rohilkhand 88)

Sol. Let us take $h=0\cdot 1$. Here $f(x,y)=x+y$.

Now $k_1=hf(x_0,y_0)=0\cdot 1\times 1=0\cdot 1$,

$$k_2=hf\left(x_0+\frac{h}{2},\,y_0+\frac{k_1}{2}\right)=0\cdot 1\,\{0\cdot 05+1\cdot 05\}=0\cdot 11,$$

$$k_3=hf\left(x_0+\frac{h}{2},\,y_0+\frac{k_2}{2}\right)=0\cdot 1\,\{0\cdot 05+1\cdot 055\}=0\cdot 1105,$$

$$k_4=hf\,(x_0+h,\,y_0+k_3)=0\cdot 1\,\{0\cdot 1+1\cdot 1105\}=0\cdot 12105.$$

$$\therefore\quad \Delta y=\frac{1}{6}\,\{k_1+2k_2+2k_3+k_4\}$$

$$=\frac{1}{6}\,\{0\cdot 1+0\cdot 22+0\cdot 221+0\cdot 12105\}=0\cdot 11034.$$

Thus $x_1=x_0+h=0\cdot 1$ and $y_1=y_0+\Delta y=1+0\cdot 1103=1\cdot 1103.$

Now for the second interval, we have

$$k_1=hf\,(x_1,y_1)=0\cdot 1\,(0\cdot 1+1\cdot 1103)=0\cdot 12103,$$

$$k_2=hf\left(x_1+\frac{h}{2},\,y_1+\frac{k_1}{2}\right)$$
$$=0\cdot 1\,(0\cdot 1+0\cdot 05+1\cdot 1103+0\cdot 06051)=0\cdot 13208,$$

$$k_3=hf\left(x_1+\frac{h}{2},\,y_1+\frac{k_2}{2}\right)$$
$$=0\cdot 1\,(0\cdot 1+0\cdot 05+1\cdot 1103+0\cdot 06604)=0\cdot 13263,$$

$$k_4=hf\,(x_1+h,\,y_1+k_3)$$
$$=0\cdot 1\,(0\cdot 1+0\cdot 1+1\cdot 1103+0\cdot 13262)=0\cdot 14429.$$

$$\therefore\quad \Delta y=\frac{1}{6}\,\{0\cdot 12103+0\cdot 26416+0\cdot 26526+0\cdot 14429\}$$
$$=0\cdot 13246.$$

Hence $x_2=0\cdot 2$ and $y_2=y_1+\Delta y=1\cdot 1103+\cdot 13246=1\cdot 2428.$

Ex. 12. *For the solution of the first order differential equation $y'=f(x,y)$, obtain the 'predictor' formula of the form*

$$y_{n+1}=a_1y_n+h\,(a_2y'_n+a_3y'_{n-1}+a_4y'_{n-2})$$

and its corresponding corrector formula. Also find their local truncation error. **(Meerut M.Sc. Maths. 1971)**

Sol. It is Adam's type predictor involving the information at the past points x_n, x_{n-1}, together with one more old derivative y'_{n-2}. Making it exact for $y=1, x, x^2, x^3$ respectively, we have

$$1=a_1 \qquad \text{when } y=1$$
$$1=a_2+a_3+a_4 \qquad y=x,\ y'=1$$
$$1=-2a_3-4a_4 \qquad y=x^2,\ y'=2x$$
$$1=3a_3+12a_4 \qquad y=x^3,\ y'=3x^2.$$

These equations give $a_1=1$, $a_2=1/6$, $a_3=13/6$, $a_4=-4/3$.

Putting these values, we obtain the required predictor

$$y_{n+1}=y_n+h\left(\frac{1}{6}\,y'_n-\frac{13}{6}\,y'_{n-1}-\frac{4}{3}\,y'_{n-2}\right)$$

or $\quad y_{n+1}=y_n+(h/6)\ (y'_n-13y'_{n-1}-8y'_{n-2}).$ $\quad$ ...(1)

Let the corresponding corrector formula be

$$y_{n+1}=a_0y_n+h\ (b_{-1}y_{n+1}+b_0y'_n+a_1y'_{n-1}).$$

By usual method, making it exact for $y=1, x, x^2, x^3$, we get

$$a_0=1,\ 2b_{-1}-2b_1=1,\ b_{-1}+b_0+b_1=1,\ 3b_{-1}+3b_1=1.$$

Solving these, we get $a_0=1,\ b_{-1}=\frac{5}{12},\ b_0=\frac{8}{12},\ b_1=-\frac{1}{12}.$

Hence the corresponding corrector formula is

$$y_{n+1}=y_n+(h/12)\ (5y'_{n+1}+8y'_n-y'_{n-1}). \qquad ...(2)$$

The truncation errors of (1) and (2) can be computed as in the case of Adam's predictor.

Ex. 13. *Find a solution of the set of simultaneous equations*

$$\frac{dx}{dt}=xy+2t \quad ...(1) \qquad \frac{dy}{dt}=2ty+x \quad ..(2)$$

with the initial conditions $x=1, y=-1, t=0.$ (Meerut M.Sc. 91)

Sol. We shall use Taylor's series method. Here x and y both are functions of t *i.e.*, x, y are dependent and t is independent variable. By Taylor's series, we get

$$\left.\begin{aligned} x(t)&=x_0+tx_0'+\frac{t^2}{2!}x_0''+\frac{t^3}{3!}x_0'''+\ldots+\frac{t^r}{r!}x_0^{(r)}+\ldots\\ \text{and } y(t)&=y_0+t.y_0'+\frac{t^2}{2!}y_0''+\frac{t^3}{3!}y_0'''+\ldots+\frac{t^r}{r!}y_0^{(r)}+\ldots\end{aligned}\right\} \quad ...(3)$$

where dashes represent derivatives w.r.t. 't'.

Thus in order to determine x and y we require the higher order derivatives of x and y at $t=0$. Differentiating the given equations successively w.r.t. 't', we have

$$\left.\begin{aligned} x'&=xy'+x'y+2, & y''&=2ty'+2y+x'\\ x''&=xy''+x'y'+x''y+x'y', & y'''&=2y'+2ty''+2y'+x'' \text{ etc.}\end{aligned}\right\} \quad ...(4)$$

Putting $x=1, y=-1, t=0$ in (1), (2) and (4), we get

$x_0'=-1,\ y_0'=1,\ x_0''=1\cdot1+(-1).(-1)+2=4,$

$y_0''=0+2\ (-1)+(-1)=-3,\ x_0'''=-3-1-4-1=-9,$

$y_0'''=2+0+2+4=8$ etc.

Substituting these values in (3), we have

$$x(t)=1-t+2t^2-\frac{3}{2}\,t^3+\ldots\ldots$$

and $$y(t)=-1+t-\frac{3}{2}\,t^2+\frac{4}{3}\,t^3+\ldots\ldots$$

Ex. 14. *Use Taylor's series method to obtain the power series in t for x and y satisfying the differential equations*

$$(dx/dt)=x+y+t, \quad (d^2y/dt^2)=x-t$$

with the initial conditions

$$x=0,\ y=1,\ dy/dt=-1, \text{ at } t=0. \quad \textbf{(Meerut 87, 89)}$$

Sol. Let $dy/dt=z$. Then the given equations reduce to a new system of first order equations

$$dx/dt=x+y+t, \quad dz/dt=x-t \text{ and } dy/dt=z,$$

under the initial conditions $x=0, y=1, z=-1$ at $t=0$.

By Taylor's series, we have

$$\left.\begin{aligned} x(t)&=x_0+tx_0'+(t^2/2!)\ x_0''+(t^3/3!)\ x_0'''+\ldots\ldots\\ y(t)&=y_0+ty_0'+(t^2/2!)\ y_0''+(t^3/3!)\ y_0'''+\ldots\ldots\\ z(t)&=z_0+tz_0'+(t^2/2!)\ z_0''+(t^3/3!)\ z_0'''+\ldots\ldots\end{aligned}\right\}. \quad ...(1)$$

From the differential equations, we have

$x'=x+y+t,\ y'=z,\ z'=x-t,$

$x''=x'+y'+1=x+y+z+t+1,$

$y''=z'=x-t,\ z''=x'-1=x+y+t-1,$

$x'''=x''+y''=2x+y+z+1,\ y'''=z''=x+y+t-1,$

$z'''=x''=x+y+z+t+1$ etc.

Putting $x=0, y=1, z=-1$ and $t=0$, we get

$$x_0'=1,\ y_0'=-1,\ z_0'=0$$
$$x_0''=1,\ y_0''=0,\quad z_0''=0$$
$$x_0'''=1,\ y_0'''=0,\ z_0'''=1 \text{ etc.}$$

Substituting these values in (1), we get

$$x=t+t^2/2!+t^3/3!+t^4/4!+2t^5/5!+\ldots\ldots$$
$$y=1-t+t^4/4!+t^5/5!+\ldots\ldots$$
and $$z=-1+t^3/3!+t^4/4!+t^5/5!+\ldots\ldots$$

Ex. 15. *Find $y(1)$ using Numerov's formula, where y satisfies $y''=y$, given that $y(0)=1, y'(0)=-1$.*

Sol. We take $h=0{\cdot}5$ and hence $x_0=0, x_1=0{\cdot}5, x_2=1$. Then the corresponding values of y are denoted by y_0, y_1, y_2 respectively. We want to find the value of y_2 which can be obtained by putting $k=1$ in equations (2) and (3) of § ᵕ. For this we require the values y_0 and y_1. But y_0 is given. Using Taylor's series method we can obtain y_1.

We have $y_0=1,\ y_0'=-1,\ y_0''=1,\ y_0'''=-1$ etc.

Putting these values in

$$y_1=y_0+hy_0'+\frac{h^2}{2!}y_0''+\frac{h^3}{3!}y_0'''+\ldots, \text{ we get}$$

$$y_1 = 1 - 0{\cdot}5 + 0{\cdot}125 - 0{\cdot}0208 + 0{\cdot}0026 - 0{\cdot}0003 = {\cdot}6065.$$

The predicted value of y_2 is given by

$$y_2 = 2y_1 - y_0 + (h^2/12)\,(10\phi_1 + \phi_0)$$
$$= 1{\cdot}2130 - 1 + \{(0{\cdot}5)^2/12\}\,\{10 \times {\cdot}6065 + 1\} = {\cdot}3602.$$

Now $\phi_2 = \phi(x_2, y_2) = y_2 = 0{\cdot}3602$.

Hence the first corrected value of y_2 is

$$y_2 = 2y_1 - y_0 + (h^2/12)\,(\phi_2 + 10\phi_1 + \phi_0)$$
$$= 1{\cdot}2130 - 1 + \{(0{\cdot}5)^2/12\}\,(0{\cdot}3602 + 6{\cdot}065 + 1\} = 0{\cdot}3677.$$

Again using the corrector formula, the next corrected value of y_2 is given by

$$y_2 = {\cdot}2130 + \{(0{\cdot}5)^2/12\}\,\{0{\cdot}3677 + 6{\cdot}065 + 1\} = 0{\cdot}3678.$$

Thus $y = 0{\cdot}3678$ when $x = 1$.

Exercises 11

1. Use Picard's method to solve $\dfrac{dy}{dx} = 1 + xy$, with $x_0 = 2$, $y_0 = 0$.

2. Use Picard's method to approximate y when $x = 0{\cdot}1$, $x = 0{\cdot}2$, given that $y = 1$ when $x = 0$, $dy/dx = x + y$. Check the result with the exact value. **(Gorakhpur 1990)**

3. Use Euler's method to find $y(1)$ from the differential equation

$$\frac{dy}{dx} = \frac{-y}{1+x},\ y(3) = 2.$$

Take for each step $h = 0{\cdot}1$. **(Meerut M. Sc. 72)**

4. Given that $(dy/dx) = \log(x + y)$, with the initial conditions that $y = 1$ when $x = 0$, find y for $x = 0{\cdot}2$ and $x = 0{\cdot}5$, by using Euler's modified method. **(Bombay 1983)**

5. Solve $y' = x - y^2$ by series expansion for $x = 0{\cdot}2$ to $0{\cdot}6$ with $h = 0{\cdot}2$. Initially $x = 0$, $y = 1$.

6. Obtain by Taylor's series five consecutive starting values for the numerical solution of

$$dy/dx = 2y + 3e^x$$

where $x_0 = 0$, $y_0 = 0$. Check the values.

7. Solve $(dy/dx) = y^2 + 1$, assuming $y = 0$ at $x = 0$ in the range $0 \leqslant x \leqslant 1$. Obtain y as a series in powers of x and check your result with the exact value.

8. Apply Taylor's algorithm to $y' = x^2 + y^2$, $y(0) = 1$. Take $h = 0{\cdot}5$ and determine approximations to $y(0{\cdot}5)$ and $y({\cdot}1)$. Carry the calculations to 3 decimals.

9. Solve numerically $(dy/dx) = 2e^x - y$, at $x = 0{\cdot}4$, $0{\cdot}5$ by Milne's method given their values at the four points $x = 0$, $0{\cdot}1$, $0{\cdot}2$, $0{\cdot}3$.

12

Solution of Algebraic and Transcendental Equations

§ 1. Introduction.

In applied mathematics, the most frequently occurring problem is to find the roots of equations of the form

$$f(x)=0. \qquad ...(1)$$

The equation $f(x)=0$ is called *Algebraic* or *Transcendental* according as $f(x)$ is purely a polynomial in x or contains some other functions such as logarithmic, exponential and trigonometric functions etc., *e.g.*,

$$1+\cos x-5x,\ x\tan x-\cosh x,\ e^{-x}-\sin x, \text{ etc.}$$

A polynomial in x of degree n is an expression of the form $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$, where a's are constants and n is a positive integer. The *zeros* or the *roots* of the polynomial $f(x)$ are those values of x for which $f(x)$ is zero. Geometrically, if the graph of $f(x)$ crosses the x-axis at the point $x=a$ then $x=a$ is a root of the equation $f(x)=0$. We conclude that a is a root of the equation $f(x)=0$ if and only if $f(a)=0$.

By finding the solution of an equation $f(x)=0$, we mean to find zeros of $f(x)$.

In this chapter, we shall discuss some numerical methods for the solution of equations of the form (1). Here $f(x)$ may be algebraic or transcendental or a combination of both. Before finding the solutions of algebraic and transcendental equations we give some properties of algebraic equations which help for locating the roots.

(i) An algebraic equation of degree n, where n is a positive integer, has n and only n roots.

(ii) If $a+ib$ is a root of $f(x)=0$, then $a-ib$ is also a root of the equation *i.e.*, complex roots occur in conjugate pairs.

(iii) **Descarte's rule of signs.** In an algebraic equation $f(x)=0$, with real coefficients the number of positive roots cannot exceed the number of changes of signs from positive to negative and from negative to positive in $f(x)$. Also the number of negative roots in $f(x)=0$, cannot exceed the number of variations in $f(-x)$.

(iv) If the values $f(a)$ and $f(b)$ are of opposite signs, when we substitute two real quantities a and b for x in any polynomial $f(x)$ then at least one or an odd number of real roots of the equation $f(x)=0$ lie between a and b. If $f(a)$ and $f(b)$ are of the same sign, then either there is no real root or an even number of real roots of $f(x)=0$ lie between a and b.

(v) Every equation of odd degree has at least one real root. Every equation of an even degree with last term negative has at least two real roots, one positive and the other negative. Here it is understood that the coefficient of the leading term is positive.

(vi) The largest root of the polynomial equation $a_0x^n+a_1x^{n-1}+\ldots\ldots+a_{n-1}x+a_n=0$, $a_0\neq 0$ may be approximated by the root of the equation $a_0x+a_1\neq 0$, or by that root of the quadratic equation $a_0x^2+a_1x+a_2=0$ which is larger in absolute value. Similarly the smallest root can be obtained approximately by the root of the equation $a_{n-1}x+a_n=0$ or by that root of $a_{n-2}x^2+a_{n-1}x+a_n=0$, which is smaller in absolute value.

(vii) If $x=a$ is a root of the equation $f(x)=0$, then $f(x)$ is exactly divisible by $(x-a)$ and conversely, if $f(x)$ is exactly divisible by $(x-a)$, a is a root of the equation $f(x)=0$.

Synthetic Division. *To find the quotient and the remainder when a polynomial is divided by a binomial.*

Let $f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n$ be the polynomial of nth degree and let it be divided by the binomial $x-a$. If $Q=b_0x^{n-1}+b_1x^{n-2}+\ldots+b_{n-2}x+b_{n-1}$ be the quotient and R the remainder, then the coefficients of Q and R can be exhibited in the following manner:

a	a_0	a_1	a_2	$a_3\ldots\ldots a_{n-1}$	a_n
		ab_0	ab_1	$ab_2\ldots\ldots ab_{n-2}$	ab_{n-1}
	b_0	b_1	b_2	$b_3\ldots\ldots b_{n-1}$	R

Rule for synthetic division. First we make the polynomial $f(x)$ complete (if it is not so) by supplying the missing terms with zero coefficients.

In the first row we write the successive coefficients $a_0, a_1, a_2, \ldots, a_n$ of the polynomial $f(x)$.

If we are to divide the polynomial by $x-a$, we write a in the corner as shown above. In the third row write b_0 below a_0 (note that $b_0=a_0$). The first term in the second row is obtained by multiplying b_0 (or a_0) by a. The product ab_0 is written under a_1. Adding ab_0 to a_1, we get b_1, which is the second term in the third row. Multiplying b_1 by a the product ab_1 is written under a_2. Adding ab_1 to a_2 we get b_2, which is the third term in the third row. We continue this process. The last term in the third row is the value of the remainder R while the last but one term in the third row is the value of b_{n-1}.

Note. In case $R=0$, we can say that a is a root of the equation $f(x)=0$ and the equation $f(x)=0$ can be depressed by one dimension.

In case R is not equal to zero, we get $R=f(a)$ by remainder theorem.

§ 2. Methods for finding the initial approximate value of the root.

To find the real roots of a numerical equation by any method except that of Graeffe, it is necessary first to find an approximate value of the root from any method. The general technique is that we begin with an initial approximate value say x_0 and then find the better approximations $x_1, x_2, \ldots, x_n$ successively by repeating the same method. If at each step of a method the successive approximations approach the root more and more closely, then we say that the method converges.

(i) Graphical Method. Suppose we are to find the roots of the equation $f(x)=0$. Taking a set of rectangular coordinate axes we plot the graph of $y=f(x)$. Then the real roots of the given equation are the abscissae of the points where the graph crosses the x-axis. Because at these points y is zero and so the equation $f(x)=0$ is satisfied. Hence from the graph of the given equation, approximate values for the real roots of the equation can be found. Sometimes when $f(x)$ involves difference of two functions, the approximate values of the real roots of $f(x)=0$ can be found by writing the equation in the form $f_1(x)=f_2(x)$ where $f_1(x)$ and $f_2(x)$ are both functions of x. Then we plot the two equations $y_1=f_1(x)$ and $y_2=f_2(x)$ on the same axes. The real roots of the given equation are the abscissae of the points of intersection of these two curves because at these points $y_1=y_2$ and so $f_1(x)=f_2(x)$.

(ii) Bisection Method. We know that if a function $f(x)$ is continuous between a and b, and $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one root between a and b. Let $f(a)$ be negative and $f(b)$ be positive. Also let the approximate value of the root be given by $x_0=(a+b)/2$. Now if $f(x_0)=0$ then it ensures that x_0 is a root of the equation $f(x)=0$. If $f(x_0)\neq 0$ then the root either lies between x_0 and b or between x_0 and a. It depends on whether $f(x_0)$ is negative or positive. Then again we bisect the interval and repeat the process until the root is obtained to the desired accuracy.

(iii) The Method of False Position (Regula-Falsi Method). It is the oldest method for computing the real root of a numerical equation $f(x)=0$. It is closely similar to the bisection method.

In this method we find a sufficiently small interval (x_1, x_2) in which the root lies. Since the root lies between x_1 and x_2, the graph of $y=f(x)$ must cross the x-axis between $x=x_1$ and $x=x_2$, and hence y_1 and y_2 must be of opposite signs.

This method is based upon the principle that any portion of a smooth curve is practically straight for a short distance. Hence we assume that the graph of $y=f(x)$ is a straight line between the points (x_1, y_1) and (x_2, y_2). The points are on opposite sides of the x-axis.

The x-coordinate of the point of intersection of the straight line joining (x_1, y_1) and (x_2, y_2) and the axis of x gives an approximate value of the desired root. The Fig. 1 represents a magnified view of that part of the graph which lies between (x_1, y_1) and (x_2, y_2).

Now from the similar triangles PAR and PSQ we have

$$\frac{AR}{AP}=\frac{SQ}{SP}$$

or

$$\frac{h}{|y_1|}=\frac{x_2-x_1}{|y_1|+|y_2|}$$

$$h=\frac{(x_2-x_1)\,|y_1|}{|y_1|+|y_2|}.$$

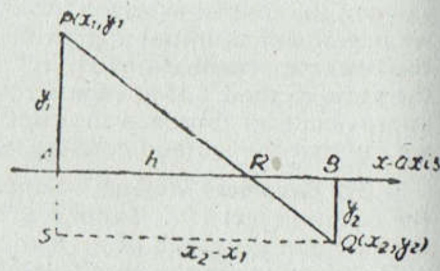

Fig. 1

Hence the approximate value of the desired root

$$=x_1+AR=x_1+h=x_1+\frac{(x_2-x_1)\,|y_1|}{|y_1|+|y_2|}.$$

Let this value be denoted by $x^{(1)}$. Then to find the better approximation, we find $y^{(1)}$ by $y^{(1)}=f(x^{(1)})$. Now either $y^{(1)}$ and y_1 or $y^{(1)}$ and y_2 will be of opposite signs. If $y^{(1)}$ and y_1 are of opposite signs then one root lies in the interval $(x_1, x^{(1)})$. We again apply the method of false position to this interval and get the second approximation.

If $y^{(1)}$ and y_2 are of opposite signs, then second approximation can be obtained by using the method of false position to the interval $(x^{(1)}, x_2)$. Continuing this process we can obtain the root to the desired degree of accuracy.

(iv) The Secant Method. This method is similar to that of Regula-Falsi method except the condition that $f(x_1).f(x_2)<0$. In this method the graph of the function $y=f(x)$ in the neighbourhood of the root is approximated by a secant line (chord). Here it is not necessary that the interval at each iteration should contain the root. Let x_1 and x_2 be the limits of interval initially, then the first approximation is given by

$$x_3=x_2+\frac{(x_2-x_1)\,f(x_2)}{f(x_1)-f(x_2)}.$$

The formula for successive approximations in general form is given by

$$x_{n+1}=x_n+\frac{(x_n-x_{n-1}).\,f(x_n)}{\{f(x_{n-1})-f(x_n)\}},\ n\geqslant 2.$$

If at any iteration we have $f(x_n)=f(x_{n-1})$, then the secant method fails. Hence the method does not converge always while the Regula-Falsi method converges surely. But if the secant method converges then it converges more rapidly than the Regula-Falsi method.

(v) Iteration Method. In this method for finding the roots of the equation $f(x)=0$ it is expressed in the form $x=\phi(x)$.

Let x_0 be an initial approximation to the solution of $x=\phi(x)$. Substituting it in $\phi(x)$ the next approximation x_1 is given by $x_1=\phi(x_0)$. Again, substituting $x=x_1$ in $\phi(x)$, we get the next approximation as $x_2=\phi(x_1)$.

Continuing in this way, we get

$$x_n=\phi(x_{n-1}) \text{ or } x_{n+1}=\phi(x_n).$$

Thus we get a sequence of successive approximations which may converge to the desired root.

For using this method, the equation $f(x)=0$ can be put as $x=\phi(x)$ in many different ways.

Let $f(x)=x^2-x-1=0$. It can be written as

(i) $x=x^2-1$,

(ii) $x^2=x+1$ or $x=\sqrt{(x+1)}$,

(iii) $x^2=x+1$ or $x=1+(1/x)$,

(iv) $x=x-(x^2-x-1)$ or $x=2x-x^2+1$.

There can be other arrangements also of this equation.

In order to find the root of $f(x)=0$, *i.e.*, $x=\phi(x)$, we are to find the abscissa of the point of intersection of the line $y=x$ and the curve $y=\phi(x)$. These two may or may not intersect. If these two curves do not intersect, then the equation $f(x)=0$ has no real root.

Now we suppose that the two curves intersect and both $\phi(x)$ and $\phi'(x)$ are continuous in the neighbourhood of the real root.

This method is convergent conditionally and the condition is that $|\phi'(x)|<1$ in the neighbourhood of the real root $x=a$.

We can observe this fact from the figures given below :

Let $x_0, x_1, x_2, \ldots$, be the successive approximations and let $0<\phi'(x)<1$. From the fig. 2 we observe that y-coordinate of A is $\phi(x_0)$ and the horizontal line AB meets the line $y=x$ at B (x_1, x_1).

Further C is the point on $y=\phi(x)$ corresponding to $x=x_1$. Thus the y-coordinate of C is $\phi(x_1)=x_2$. Continuing in this way we observe that the sequence $x_1, x_2, \ldots, x_n$ converges to $x=a$.

Similarly from the fig. 3 it is obvious that the sequence $x_1, x_2, \ldots, x_n$ converges to $x=a$ whereas in Fig. 4 and Fig. 5, it diverges.

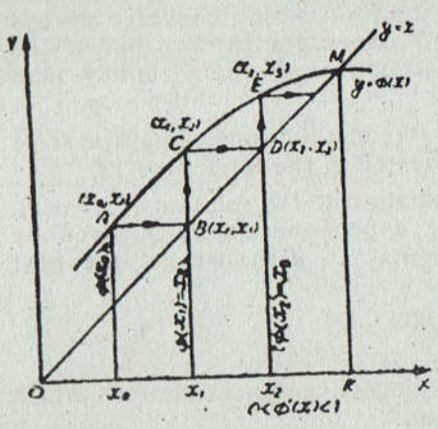

Fig. 2.

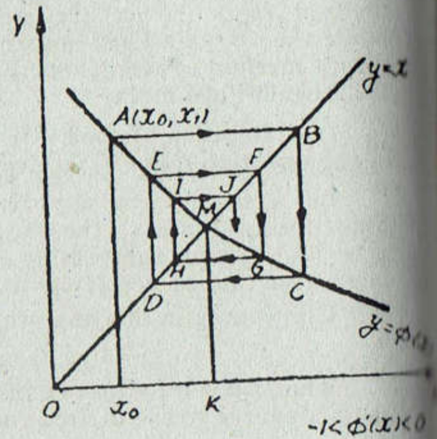

Fig. 3.

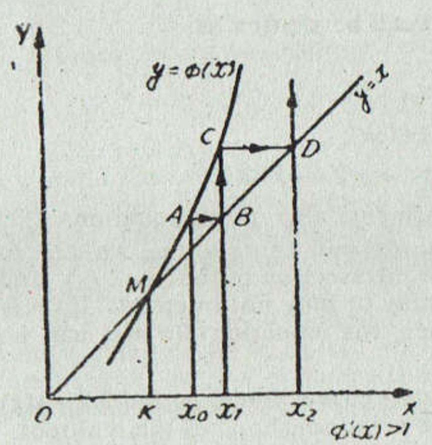

Fig. 4.

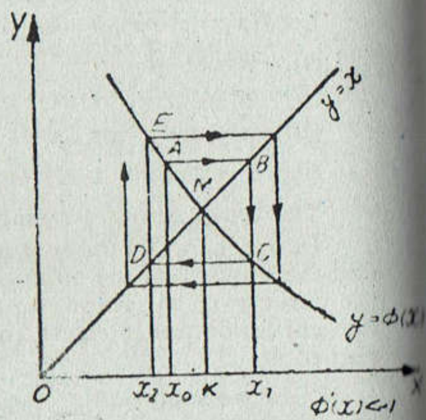

Fig. 5.

Theorem. *Let $x=a$ be a root of $f(x)=0$ and let I be an interval containing the point $x=a$. Let $\phi(x)$ and $\phi'(x)$ be continuous in I where $\phi(x)$ is defined by the equation $x=\phi(x)$ which is equivalent to $f(x)=0$. Then if $|\phi'(x)|<1$ for all x in I, the sequence of approximations $x_0, x_1, x_2, \ldots, x_n$ defined by $x_{n+1}=\phi(x_n)$ converges to the root a, provided that the initial approximation x_0 is chosen in I.*

Proof. Let $x=a$ be a root of the equation $x=\phi(x)$. Then we have,

$$a=\phi(a). \qquad \ldots(1)$$

The first approximation satisfies

$$x_1=\phi(x_0). \qquad \ldots(2)$$

Subtracting the equation (2) from (1), we get

$$a-x_1=\phi(a)-\phi(x_0). \qquad \ldots(3)$$

By the mean value theorem of differential calculus, the right hand side can be written as

$$\phi(a)-\phi(x_0)=(a-x_0)\,\phi'(a_0),\ x_0<a_0<a.$$

Hence (3) becomes

$$a-x_1=(a-x_0)\,\phi'(a_0),\ x_0<a_0<a. \qquad \ldots(4)$$

Similarly, we get

$$(a-x_2)=(a-x_1)\,\phi'(a_1),\ x_1<a_1<a \qquad \ldots(5)$$

$$(a-x_3)=(a-x_2)\,\phi'(a_2),\ x_2<a_2<a \qquad \ldots(6)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

and

$$(a-x_{n+1})=(a-x_n)\,\phi'(a_n),\ x_n<a_n<a. \qquad \ldots(7)$$

Multiplying together all these equations, member by member, and dividing the resulting equation throughout by the common factor $(a-x_1)(a-x_2)\ldots(a-x_n)$, we get

$$(a-x_{n+1})=(a-x_0)\,\phi'(a_0)\,\phi'(a_1)\ldots\phi'(a_n). \qquad \ldots(8)$$

Let $|\phi'(a_i)|\leqslant K<1$, for all i. Then from the equation (8) it follows that

$$|a-x_{n+1}|\leqslant|a-x_0|\,K^{n+1}. \qquad \ldots(9)$$

When $n\to\infty$, the R.H.S. of (9) tends to zero and it implies that the sequence of approximations $x_0, x_1, \ldots$ converges to the root a if $K<1$.

Acceleration of Convergence: Aitken's Δ^2-Method.

We have the relation

$$|a-x_{n+1}|=|\phi(a)-\phi(x_n)|\leqslant K\,|a-x_n|,\ K<1.$$

It is obvious from this relation that the iteration method is linearly convergent. This slow rate of convergence can be accelerated as follows:

Let the three successive approximations to the desired root $x=a$ of the equation $x=\phi(x)$ be x_{i-1}, x_i, x_{i+1}. Then we know that

$$a-x_i\simeq K(a-x_{i-1})$$

and

$$a-x_{i+1}\simeq K(a-x_i).$$

Dividing, we get

$$\frac{a-x_i}{a-x_{i+1}}\simeq\frac{a-x_{i-1}}{a-x_i},$$

or $$a \simeq x_{i+1} - \frac{(x_{i+1}-x_i)^2}{x_{i+1}-2x_i+x_{i-1}}. \quad \ldots(1)$$

But in the sequence of successive approximations, we hav

$$\Delta x_i = x_{i+1} - x_i$$

and $$\Delta^2 x_i = \Delta(\Delta x_i).$$

Now $\Delta^2 x_{i-1} = \Delta(\Delta x_{i-1}) = \Delta(x_i - x_{i-1}) = \Delta x_i - \Delta x_{i-1}$

$$= (x_{i+1}-x_i)-(x_i-x_{i-1}) = x_{i+1}-2x_i+x_{i-1}.$$

Hence (1) can be written as

$$a \simeq x_{i+1} - \frac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}. \quad \ldots(2)$$

This gives the new approximate value of the root in term of three previously known successive approximations. Hen we can obtain the next approximation by substituting the valu of x_{i-1}, x_i, x_{i+1} in the formula

$$x_{i+2} = x_{i+1} - \frac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}.$$

This modified iterative scheme is known as Aitken's Δ^2-method or Aitken's extrapolation method.

(vi) **Newton-Raphson Method.** **(Kanpur 1986, Meerut M. 90, B.Sc. 92, 92M; Roh. B.Sc.**

Let x_0 denote an approximate value of the desired root the equation $f(x)=0$ and let h be the correction which must applied to x_0 to get the exact value of the root. Then x_0+h i root of the equation $f(x)=0$, so that $f(x_0+h)=0$ ...(1)

Expanding $f(x_0+h)$ by Taylor's theorem, we get

$$f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\ldots=0.$$

Now if h is sufficiently small, we may neglect the terms co taining second and higher powers of h and get simple rel tion

$$f(x_0)+hf'(x_0)=0.$$

This gives $h=-\frac{f(x_0)}{f'(x_0)}$, provided $f'(x_0)\neq 0$. The impro value of the root is

$$x_1 = x_0+h = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \ldots(2)$$

Successive approximations are given by $x_2, x_3, \ldots, x_{n+1}$

where $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad \ldots(3)$$

Formula (3) is known as Newton-Raphson formula.

In this method we have assumed that h is a small quantity which is so if the derivative $f'(x)$ is large. In other words the correct value of the root can be obtained more rapidly and with very little labour when the graph is nearly vertical where it crosses the x-axis. If $f'(x)$ is small in the neighbourhood of the root then the value of h is large and by this method the computation of the root will be a slow process or might even fail altogether. Hence this method is not suitable in cases when the graph of $f(x)$ is nearly horizontal where it crosses the x-axis. In such cases the regula-falsi method should be used.

Geometric Significance of the Newton-Raphson Method.

A magnified view of the graph of $y=f(x)$, where it crosses the x-axis is represented in Fig. 6.

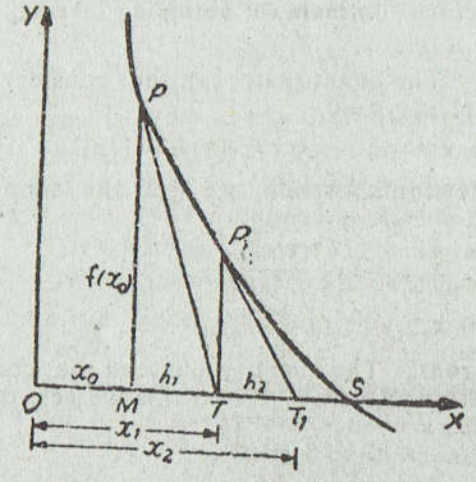

Fig. 6

Let us draw a tangent at the point P whose x-coordinate is x_0. It intersects the x-axis at some point T. Then we draw a tangent at the point P_1 whose abscissa is OT. Suppose it meets the x-axis at some point T_1 which lies between T and S. Further we draw a tangent at the point P_2 whose abscissa is OT_1. This tangent intersects the x-axis at a point T_2 which lies between T_1 and S. We continue this process. Let the curvature of the graph do not change sign between P and S. Then the points $T, T_1, T_2, \ldots$ will approach the point S as a limit or in other words the intercepts $OT, OT_1, OT_2, \ldots$ will tend to the intercept OS as a limit. But OS denotes the real root of the equation $f(x)=0$. So $OT, OT_1, OT_2, \ldots$ denote successive approximations to the desired root.

From this figure we derive the fundamental formula. Let $MT=h_1$ and $TT_1=h_2$, etc.

We have $PM=f(x_0)$, slope at $P=\tan \angle XTP = -\frac{f(x_0)}{h_1}$

Also the slope of the graph at P is $f'(x_0)$.

Thus we get $f'(x_0)=-\frac{f(x_0)}{h_1} \Rightarrow h_1=-\frac{f(x_0)}{f'(x_0)}$.

Similarly, we find from the $\Delta\ P_1 TT_1$ that

$$h_2=-\frac{f(x_1)}{f'(x_1)}.$$

From this discussion we conclude that in this method we replace the graph of the given function by a tangent at each successive stop in the approximation process

It can be used for solving both algebraic and transcendental equations and it can also be used when the roots are complex.

Newton's iterative formula for obtaining inverse, square root, cube root etc.

1. **Inverse.** The quantity a^{-1} can be considered as a root of the equation $(1/x)-a=0$.

Here $f(x)=x^{-1}-a$. $\therefore\ f'(x)=-1/x^2$.

Hence by Newton's formula, we get the simple recursion formula

$$x_{n+1}=x_n+\frac{(1/x_n)-a}{1/x_n^2}$$

or $$x_{n+1}=x_n(2-a\,x_n).$$

2. **Square root.** The quantity $\sqrt{a}$ can be considered as a root of the equation $x^2-a=0$. From this we get the recursion formula

$$x_{n+1}=\tfrac{1}{2}\left(x_n+\frac{a}{x_n}\right).$$

3. **Inverse Square root.** The inverse square root of a is the root of the equation $\frac{1}{x^2}-a=0$. From this we get the iterative formula

$$x_{n+1}=\tfrac{1}{2}x_n(3-ax_n^2).$$

4. **Formula of pth root and reciprocal pth root.** For computing pth root of a we can solve the equation $x^p-a=0$.

Here $f(x)=x^p-a$. $\therefore\ f'(x)=px^{p-1}$.

Hence by Newton's formula, we obtain the recursion formula

$$x_{n+1}=x_n-\frac{(x_n^p-a)}{px_n^{p-1}}=\frac{(p-1)\,x_n^p+a}{px_n^{p-1}}.$$

The reciprocal of pth root of a can be obtained by solving

the equation $\frac{1}{x^p}-a=0$ by Newton's method. We get the iterative formula

$$x_{n+1}=x_n\,\frac{(p+1-ax_n^p)}{p}.$$

(vii) **Bairstow's Method.** Suppose we are dealing with an algebraic equation with real coefficients and we are to find its complex roots. All complex roots appear in pairs $a\pm ib$. Each such pair corresponds to a quadratic factor x^2+px+q with real coefficients. Let the given polynomial equation be

$$f(x)=x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0. \qquad \ldots(1)$$

In this method we start with an approximate quadratic expression, say x^2+px+q. Then using an iterative process, we get such values of p and q that x^2+px+q becomes a factor of $f(x)$. In this way the roots of this quadratic factor are the roots of the equation (1).

If we divide $f(x)$ by x^2+px+q, we get a quotient $x^{n-2}+b_1x^{n-3}+\ldots+b_{n-2}$ and a remainder $Rx+S$. Thus, we have $x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$

$$\equiv(x^2+px+q)(x^{n-2}+b_1x^{n-3}+\ldots+b_{n-2})+Rx+S \qquad \ldots(2)$$

If x^2+px+q is a factor of $f(x)$, the remainder $Rx+S=0$ *i.e.* $R=0$ and $S=0$.

Obviously R and S both are functions of p and q. Our problem is then to find p and q, so that

$$R(p,q)=0\ ;\ S(p,q)=0. \qquad \ldots(3)$$

These relations are not satisfied in general for arbitrary values p and q and we try to find small corrections Δp and Δq such that

$$R(p+\Delta p,\ q+\Delta q)=0 : S(p+\Delta p,\ q+\Delta q)=0. \qquad \ldots(4)$$

Expanding (4) by Taylor's series and neglecting the second and higher order terms of Δp and Δq, we obtain

$$\left.\begin{aligned} R(p,q)+\frac{\partial R}{\partial p}\Delta p+\frac{\partial R}{\partial q}\Delta q=0 \\ S(p,q)+\frac{\partial S}{\partial p}\Delta p+\frac{\partial S}{\partial q}\Delta q=0 \end{aligned}\right\} \qquad \ldots(5)$$

Regarding (5) as a linear system of equations for Δp and Δq, we get the values of Δp and Δq. The procedure is repeated with the corrected values for p and q.

In order to obtain the coefficients $b_1, b_2, \ldots, b_{n-2}$, R and S, we use the identity (2). Comparing the coefficients of like powers of x in (2), we get

$$\left.\begin{aligned}
a_1&=b_1+p\\
a_2&=b_2+pb_1+q\\
a_3&=b_3+pb_2+qb_1\\
&\;\;\dots\\
a_k&=b_k+pb_{k-1}+qb_{k-2}\\
&\;\;\dots\\
a_{n-2}&=b_{n-2}+pb_{n-3}+qb_{n-4}\\
a_{n-1}&=R+pb_{n-2}+qb_{n-3}\\
a_n&=S+qb_{n-2}.
\end{aligned}\right\} \quad \dots(6)$$

We can find the quantities $b_1, b_2, \dots, b_{n-2}$, R and S recursively. Conveniently, we introduce two other quantities b_{n-1} and b_n, and define :

$$b_k=a_k-pb_{k-1}-qb_{k-2} \; (k=1, 2, \dots, n), \quad \dots(7)$$

with $b_0=1$, $b_{-1}=0$ and $b_{-2}=0$. This gives

$$b_{n-1}=a_{n-1}-pb_{n-2}-qb_{n-3}=R$$

and
$$b_n=a_n-pb_{n-1}-qb_{n-2}=S-pb_{n-1}.$$

Thus
$$\left.\begin{aligned} R&=b_{n-1}\\ S&=b_n+pb_{n-1}\end{aligned}\right\} \quad \dots(8)$$

Putting these values in (5), we get

$$b_{n-1}+\frac{\partial b_{n-1}}{\partial p}\,\Delta p+\frac{\partial b_{n-1}}{\partial q}\,\Delta q=0,$$

$$b_n+pb_{n-1}+\left(\frac{\partial b_n}{\partial p}+p\,\frac{\partial b_{n-1}}{\partial p}+b_{n-1}\right)\Delta p$$
$$+\left(\frac{\partial b_n}{\partial q}+p\,\frac{\partial b_{n-1}}{\partial q}\right)\Delta q=0.$$

Multiplying the first of these equations by p and subtracting it from the second, we get

$$\left.\begin{aligned}
&\frac{\partial b_{n-1}}{\partial p}\,\Delta p+\frac{\partial b_{n-1}}{\partial q}\,\Delta q+b_{n-1}=0\\
&\left(\frac{\partial b_n}{\partial p}+b_{n-1}\right)\Delta p+\frac{\partial b_n}{\partial q}\,\Delta q+b_n=0
\end{aligned}\right\} \quad \dots(9)$$

Differentiating (7) w.r.t. p and q, noting that all a_k's are constants and all b_k's are functions of p and q, we get

$$\left.\begin{aligned}
-\frac{\partial b_k}{\partial p}&=b_{k-1}+p\,\frac{\partial b_{k-1}}{\partial p}+q\,\frac{\partial b_{k-2}}{\partial p}\;;\;\frac{\partial b_0}{\partial p}=\frac{\partial b_{-1}}{\partial p}=0\\
-\frac{\partial b_k}{\partial q}&=b_{k-2}+p\,\frac{\partial b_{k-1}}{\partial q}+q\,\frac{\partial b_{k-2}}{\partial q}\;;\;\frac{\partial b_0}{\partial q}=\frac{\partial b_{-1}}{\partial q}=0
\end{aligned}\right\} \quad \dots(10)$$

From (10), we have

$$\frac{\partial b_0}{\partial p}=0,\;\frac{\partial b_1}{\partial q}=0$$

$$\frac{\partial b_1}{\partial p}=-b_0,\;\frac{\partial b_2}{\partial q}=-b_0-p\,\frac{\partial b_1}{\partial q}=-b_0$$

$$\frac{\partial b_2}{\partial p}=-b_1-p\,\frac{\partial b_1}{\partial p}=-b_1+pb_0$$

$$\frac{\partial b_3}{\partial q}=-b_1-p\,\frac{\partial b_2}{\partial q}-q\,\frac{\partial b_1}{\partial q}=-b_1+pb_0.$$

Thus $\dfrac{\partial b_1}{\partial q}=\dfrac{\partial b_0}{\partial p}$, $\dfrac{\partial b_2}{\partial q}=\dfrac{\partial b_1}{\partial p}$ and $\dfrac{\partial b_3}{\partial q}=\dfrac{\partial b_2}{\partial p}$.

By mathematical induction we shall prove that

$$\frac{\partial b_{k+1}}{\partial q}=\frac{\partial b_k}{\partial p},\text{ for all values of } k.$$

Let the result be true for $k=r$ *i.e.* we have

$$\frac{\partial b_{r+1}}{\partial q}=\frac{\partial b_r}{\partial p} \quad \dots(11)$$

Now $\dfrac{\partial b_{r+2}}{\partial q}=-b_r-\dfrac{\partial b_{r+1}}{\partial q}-q\,\dfrac{\partial b_r}{\partial q}$.

and $$\frac{\partial b_{r+1}}{\partial p}=-b_r-p\,\frac{\partial b_r}{\partial p}-q\,\frac{\partial b_{r-1}}{\partial p}$$
$$=-b_r-p\,\frac{\partial b_{r+1}}{\partial q}-q\,\frac{\partial b_r}{\partial q}. \quad \text{using (11).}$$

This shows that $\dfrac{\partial b_{r+2}}{\partial q}=\dfrac{\partial b_{r+1}}{\partial p}$ *i.e.* the result is true for $k=r+1$. But the result is true for $k=1$ and 2. Hence by mathematical induction it is true for all values of k.

We set $$\frac{\partial b_{k+1}}{\partial q}=\frac{\partial b_k}{\partial p}=-c_{k-1},\; k=0, 1, 2, \dots, n-1. \quad \dots(12)$$

Using (12) the two equations of (10) can be expressed as

$$c_{k-1}=b_{k-1}-p\,c_{k-2}-q\,c_{k-3},$$
$$c_{k-2}=b_{k-2}-p\,c_{k-3}-q\,c_{k-4}.$$

These can be expressed in one single equation :

$$c_k=b_k-p\,c_{k-1}-q\,c_{k-2}\;;\;c_0=1\;;\;c_{-1}=0\;;\;(k=1, 2, \dots, n-1). \quad \dots(13)$$

Hence c_k is computed from b_k in exactly the same way as b_k from a_k.

Differentiating the relations in (8) and using the relation (12), we get

$$\frac{\partial R}{\partial p}=\frac{\partial b_{n-1}}{\partial p}=-c_{n-2},\ \frac{\partial R}{\partial q}=\frac{\partial b_{n-1}}{\partial q}=-c_{n-3}$$

and
$$\frac{\partial S}{\partial p}=\frac{\partial b_n}{\partial p}+b_{n-1}+p\,\frac{\partial b_{n-1}}{\partial p}=-c_{n-1}-p\,c_{n-2}+b_{n-1}$$

$$\frac{\partial S}{\partial q}=\frac{\partial b_n}{\partial q}+p\,\frac{\partial b_{n-1}}{\partial q}=-c_{n-2}-p\,c_{n-3}.$$

Substituting these values in (5), we get

$$b_{n-1}-c_{n-2}\Delta p-c_{n-3}\Delta q=0$$

and
$$b_n+p\,b_{n-1}+(-c_{n-1}-p\,c_{n-2}+b_{n-1})\,\Delta p+(-c_{n-2}-p\,c_{n-3})\,\Delta q=0$$

These two equations can now be written as

$$\left.\begin{aligned}c_{n-2}\Delta p+c_{n-3}\Delta q&=b_{n-1}\\ \text{and}\quad (c_{n-1}-b_{n-1})\Delta p+c_{n-2}\Delta q&=b_n\end{aligned}\right\} \qquad \ldots(14)$$

Obtaining the values of c_k's and b_k's from (13) and (7) and putting in (14) we can find the approximate values of Δp and Δq say Δp^* and Δq^*. These values are approximate because we have neglected higher powers of Δp and Δq in (5). Taking the new values $p+\Delta p^*$ and $q+\Delta q^*$ as the initial values and repeating the same process again, we can get better values of p and q.

Thus we can find the desired values of p and q by this iterative process.

(viii) Graeffe's root squaring method. This is a method which does not require any prior information of the roots such as approximate value of the root etc. But this method is applicable to polynomial equations only. Sometimes it gives all the roots of the polynomial equation.

We consider the equation $x^n+a_1x^{n-1}+\ldots+a_n=0.$ ...(1)

For the sake of simplicity we assume all roots to be real and different from each other. Keeping all even terms on one side and all odd terms on the other, we have

$$(x^n+a_2x^{n-2}+a_4x^{n-4}+\ldots)^2=(a_1x^{n-1}+a_3x^{n-3}+a_5x^{n-5}+\ldots)^2.$$

Putting $x^2=y$, we get the new equation

$$y^n+b_1y^{n-1}+b_2y^{n-2}+\ldots+b_n=0, \qquad \ldots(2)$$

where
$$\left.\begin{aligned}b_1&=-a_1^2+2a_2\\ b_2&=a_2^2-2a_1a_3+2a_4\\ b_3&=-a_3^2+2a_2a_4-2a_1a_5+2a_6\\ &\ldots\ \ldots\ \ldots\ \ldots\\ b_n&=(-1)^na_n^2\end{aligned}\right\}$$

or
$$(-1)^rb_r=a_r^2-2a_{r-1}a_{r+1}+2a_{r-2}a_{r+2}\ldots\ldots \qquad \ldots(3)$$

The roots of the equation (2) are the squares of the roots of the equation (1). Suppose that, after m squarings, we have obtained the equation

$$z^n+\lambda_1z^{n-1}+\ldots+\lambda_n=0, \qquad \ldots(4)$$

with the roots $q_1, q_2, \ldots, q_n$, while the roots of the original equation are $p_1, p_2, \ldots, p_n$.

Then $q_i=p_i^{2^m}$, $i=1, 2, \ldots, n$. Also, we assume that

$$|p_1|>|p_2|>\ \ >|p_n| \text{ and } |q_1|\gg|q_2|\gg\ldots\gg|q_n|.$$

where the symbol $\gg$ indicates "much greater than". Hence $\frac{|q_2|}{|q_1|}=\frac{q_2}{q_1}$ etc. can be neglected in comparison with unity. Since q_i is an even power of p_i so it is always positive.

From the theory of equations the relations between the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ and the coefficients $a_1, a_2, \ldots, a_n$ of the equation

$$x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0 \text{ are}$$

$$\Sigma\alpha_1=-a_1=(-1)\,a_1$$

$$\Sigma\alpha_1\alpha_2=(-1)^2\,a_2,\ \Sigma\alpha_1\alpha_2\alpha_3=(-1)^3\,a_3$$

$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

$$\alpha_1\alpha_2\ldots\alpha_n=(-1)^n\,a_n.$$

Hence from the equation (4), we get

$$\left.\begin{aligned}\Sigma q_1&=q_1+q_2+\ldots+q_n=(-1)^1\,\lambda_1\\ \Sigma q_1q_2&=q_1q_2+q_2q_3+\ldots=(-1)^2\,\lambda_2\\ \Sigma q_1q_2q_3&=q_1q_2q_3+q_1q_2q_4+\ldots=(-1)^3\,\lambda_3\\ &\ldots\ \ldots\ \ldots\ \ldots\\ q_1q_2q_3\ldots q_n&=(-1)^n\,\lambda_n\end{aligned}\right\}.$$

or
$$\lambda_1=-(q_1+q_2+\ldots+q_n)=-q_1\left(1+\frac{q_2}{q_1}+\ldots\right)$$

$$\lambda_2=(q_1q_2+q_2q_3+\ldots)=q_1q_2\left(1+\frac{q_3}{q_1}+\ldots\right)$$

$$\lambda_3=-(q_1q_2q_3+q_1q_2q_4+\ldots)=-q_1q_2q_3\left(1+\frac{q_4}{q_3}+\ldots\right)$$

$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

$$\lambda_n=(-1)^n\,q_1q_2\ldots q_n.$$

Since the quantities q_2/q_1 etc. are negligible in comparison with unity so we get

$$\left.\begin{array}{l}\lambda_1 \simeq -q_1 \\ \lambda_2 \simeq q_1 q_2 \\ \lambda_3 \simeq -q_1 q_2 q_3 \\ \dots \quad \dots \quad \dots \\ \lambda_n \simeq (-1)^n q_1 q_2 \dots q_n\end{array}\right\} \qquad \dots(5)$$

Dividing each equation after first by its preceding equation, we obtain

$$\left.\begin{array}{l}q_1 \simeq -\lambda_1 \\ q_2 \simeq -\lambda_2/\lambda_1 \\ q_3 \simeq -\lambda_3/\lambda_2 \\ \dots \quad \dots \quad \dots \\ q_n \simeq -\lambda_n/\lambda_{n-1}\end{array}\right\} \qquad \dots(6)$$

Now since $q_i = p_i^{2^m}$, so we get p_i by m successive square-root extractions of q_i and the sign of p_i has to be determined by inserting the root into the equation. **(Meerut B.Sc. 93, 94)**

§ 3. Nearly Equal Roots.

We know that Newton's method can be used when $f'(x) \neq 0$ in the neighbourhood of the actual root $x = a$ *i.e.*, in the interval $(a-h, a+h)$. But this restriction may not be satisfied in practice or the quantity h satisfying this condition may be too small. This situation happens when the roots are very close to each other. We proceed as follows to remove this difficulty.

We know by Rolle's theorem that $f'(x)$ vanishes at least once between two roots of $f(x)=0$ and so we can find a sufficiently small quantity h to satisfy the above said interval $(a-h, a+h)$ in case of nearly equal roots.

In the case of equal roots at $x=a$, both $f(x)$ and $f'(x)$ vanish at $x=a$. Hence while applying Newton's method if x_i is very close to the root of $f(x)$ and $f'(x)$ *i.e.*, both $f(x_i)$ and $f'(x_i)$ are very small then we proceed to find two new starting values for the two nearly equal roots.

Now we first apply Newton's method to the equation $f'(x)=0$ in order to find these two values *i.e.*, we apply the iteration formula

$$x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)}. \qquad \dots(1)$$

Suppose $x=c$ is the solution obtained by (1). (If there does not exist a solution like $x=c$ of (1) or $f'(x)$ is small throughout in a small neighbourhood of $x=c$ then further modification is needed).

By Taylor's theorem, we get

$$f(x) = f(c + \overline{x-c}) = f(c) + (x-c)\, f'(c) + \frac{(x-c)^2}{2!} f''(c) + \dots\dots$$

$$= f(c) + \tfrac{1}{2}(x-c)^2 f''(c) + \dots\dots \qquad [\because\ f'(c)=0]$$

$$= f(c) + \tfrac{1}{2}(x-c)^2 f''(c), \quad \text{neglecting higher power terms.}$$

Thus the zeros of $f(x)$ near $x=c$ are approximately given by

$$f(x)=0$$

i.e., $\quad f(c) + \frac{1}{2}(x-c)^2 f''(c) = 0$

i.e., $\quad x = c \pm \sqrt{\{-2f(c)/f''(c)\}}. \qquad \dots(2)$

Now taking these two values as the starting values we can obtain the two nearly equal roots by Newton's method.

§ 4. Rate of convergence of Newton's method.

Let a be the exact value of the root. Suppose x_n differs from a by the small quantity ϵ_n.

Then $\quad x_n = a + \epsilon_n \Rightarrow x_{n+1} = a + \epsilon_{n+1}$.

By Newton-Raphson method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

i.e. $$a + \epsilon_{n+1} = a + \epsilon_n - \frac{f(a+\epsilon_n)}{f'(a+\epsilon_n)}$$

or $$\epsilon_{n+1} = \epsilon_n - \frac{f(a+\epsilon_n)}{f'(a+\epsilon_n)} \qquad \dots(1)$$

By Taylor's theorem, we have

$$f(a+\epsilon_n) = f(a) + \epsilon_n f'(a) + \frac{1}{2!}\epsilon_n^2 f''(a) + \dots$$

$$= \epsilon_n f'(a) + \tfrac{1}{2}\epsilon_n^2 f''(a) + \dots \qquad [\because\ f(a)=0]$$

and $$f'(a+\epsilon_n) = f'(a) + \epsilon_n f''(a) + \dots$$

Substituting these values in (1), we get

$$\epsilon_{n+1} = \epsilon_n - \epsilon_n \left[\frac{f'(a) + \frac{1}{2}\epsilon_n f''(a) + \dots}{f'(a) + \epsilon_n f''(a) + \dots}\right]$$

$$= \frac{\epsilon_n^2 f''(a)}{2[f'(a) + \epsilon_n f''(a)]} = \frac{\epsilon_n^2 f''(a)}{2f'(a)\left[1 + \epsilon_n \dfrac{f''(a)}{f'(a)}\right]}$$

$$= \frac{\epsilon_n^2 f''(a)}{2f'(a)}\left[1 + \epsilon_n \frac{f''(a)}{f'(a)}\right]^{-1} \approx \frac{\epsilon_n^2 f''(a)}{2f'(a)} \qquad \dots(2)$$

From (2) we observe that the subsequent error is proportional to the square of the previous error so that the Newton-Raphson method has a second-order or quadratic convergence.

§ 5. Rate of Convergence of Newton's method when there exist double roots.

Proceeding as earlier, we get

$$\epsilon_{n+1}=\epsilon_n-\frac{f(a+\epsilon_n)}{f'(a+\epsilon_n)}$$

$$=\epsilon_n-\frac{f(a)+\epsilon_n f'(a)+\frac{\epsilon_n^2}{2!}f''(a)+\frac{\epsilon_n^3}{3!}f'''(a)+\dots}{f'(a)+\epsilon_n f''(a)+\frac{\epsilon_n^2}{2!}f'''(a)+\frac{\epsilon_n^3}{3!}f^{iv}(a)+\dots}$$

$$=\epsilon_n-\frac{\frac{\epsilon_n^2}{2!}f''(a)+\frac{\epsilon_n^3}{3!}f'''(a)+\dots}{\epsilon_n f''(a)+\frac{\epsilon_n^2}{2!}f'''(a)+\dots}\quad [\because\ f(a)=0=f'(a)]$$

$$=\epsilon_n-\epsilon_n\frac{\frac{1}{2}f''(a)+\frac{\epsilon_n}{6}f'''(a)}{f''(a)+\frac{\epsilon_n}{2}f'''(a)}, \text{ neglecting higher powers}$$

$$=\epsilon_n\frac{\frac{1}{2}f''(a)+\frac{\epsilon_n}{3}f'''(a)}{f''(a)+\frac{\epsilon_n}{2}f'''(a)}=\epsilon_n\cdot\frac{1}{2}\frac{f''(a)}{f''(a)}$$

$$=\tfrac{1}{2}\epsilon_n=\frac{2-1}{2}\epsilon_n.$$

This implies that the convergence is linear. In general we can prove that if $x=a$ is a root of multiplicity m, where $m>1$ then the speed of convergence is given by

$$\epsilon_{n+1}\approx\frac{m-1}{m}\epsilon_n, \text{ which is also linear.}$$

Solved Examples

Ex. 1. *Find the approximate value of the root of*

$$x-\sin x-1=0.$$

Sol. Since the L.H.S. of the equation is the difference of two functions, we can put it in the form

$$x-1=\sin x.$$

Then we plot separately on the same set of coordinate axes the two equations

$$y_1=x-1 \text{ and } y_2=\sin x.$$

The abscissa of the point of intersection of the graphs of these equations is seen to be about 1·9 in the diagram. Hence the approximate value of the root is 1·9.

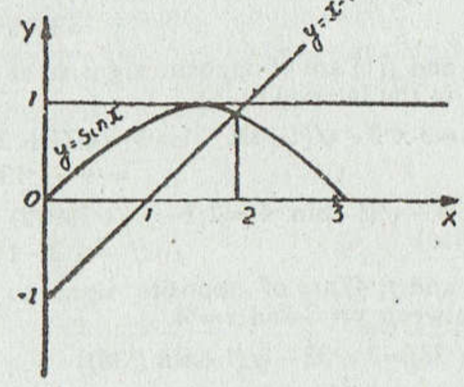

Fig. 7

Ex. 2. *Find the approximate value of the smallest real root of* $e^{-x}=\sin x$.

Sol. We plot separately on the same set of axes the two equations $y_1=e^{-x}$ and $y_2=\sin x$ as shown in the following diagram.

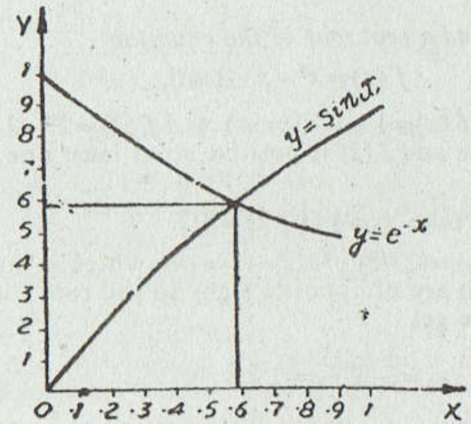

Fig. 8

The abscissa of the point of intersection of the graphs of these equations is seen to be about 0·6. Hence the approximate value of the root is 0·6.

Ex. 3. *Find the approximate value of the root of the equation*

$$3x-\sqrt{(1+\sin x)}=0.$$

Sol. Let $f(x)=3x-\sqrt{(1+\sin x)}$. Then the given equation is $f(x)=0$.

We have $f(0)=3\times 0-\sqrt{(1+\sin 0)}=-1$

and $f(1)=3\times 1-\sqrt{(1+\sin 1)}=3-(1+0\cdot 8147)^{1/2}$

$$=3-1\cdot 3570=1\cdot 6430.$$

Since $f(0)$ and $f(1)$ are of opposite signs so at least one root of $f(x)=0$ lies in the interval $(0, 1)$.

Now $f(\cdot 3)=3\times\cdot 3-\sqrt{(1+\sin\cdot 3)}=\cdot 9-\sqrt{(1+\cdot 29552)}$

$$=\cdot 9-1\cdot 1381=-0\cdot 2381,$$

and $f(\cdot 4)=3\times\cdot 4-\sqrt{(1+\sin\cdot 4)}=1\cdot 2-\sqrt{(1\cdot 38912)}$

$$=1\cdot 2-1\cdot 1787=0\cdot 0213.$$

Since $f(\cdot 3)$ and $f(\cdot 4)$ are of opposite signs so at least one real root lies between $x=\cdot 3$ and $x=\cdot 4$.

Further $f(\cdot 38)=3\times\cdot 38-\sqrt{\{1+\sin(\cdot 38)\}}$

$$=1\cdot 14-\sqrt{\{1+\cdot 37092\}}=1\cdot 14-1\cdot 1708$$

$$=-0\cdot 0808,$$

and $f(\cdot 40)=0\cdot 0213.$

Hence at least one root lies between $x=\cdot 38$ and $x=\cdot 40$. So $x=\cdot 38$ can be taken as an approximate value of the root which lies in the interval $(\cdot 38, \cdot 40)$.

Ex. 4. *Find a real root of the equation*

$$f(x)=x^3-x-1=0.$$

Sol. Here $f(1)=1-1-1=-1$ and $f(2)=2^3-2-1=5$. Since $f(1)$ is negative and $f(2)$ is positive so at least one root lies between 1 and 2.

Hence we take $x_0=\frac{1}{2}(1+2)=3/2$.

Then $f(x_0)=(27/8)-(3/2)-1=7/8$, which is positive. Since $f(1)$ and $f(1\cdot 5)$ are of opposite signs so the root lies between 1 and $1\cdot 5$ and we get

$$x_1=\frac{1+1\cdot 5}{2}=1\cdot 25.$$

We have $f(x_1)=-19/64$, which is negative. Hence we conclude that the root lies between $1\cdot 25$ and $1\cdot 5$. It gives that

$$x_2=\frac{1\cdot 25+1\cdot 5}{2}=1\cdot 375.$$

Repeating the procedure, we obtain the successive approximations as

$$x_3=1\cdot 3125,\ x_4=1\cdot 34375,\ x_5=1\cdot 328125, \text{ etc.}$$

Ex. 5. *Compute the real root of*

$$x\log_{10}x-1\cdot 2=0$$

correct to five decimal places.

Sol. Let $f(x)=x\log_{10}x-1\cdot 2$.

We have $f(2)=-0\cdot 60$ and $f(3)=0\cdot 23$.

It shows that the root lies between 2 and 3 and it is nearer to 3. The following table shows successive approximations where corrections are computed by

$$h=\frac{(x_2-x_1)\,|\,y_1\,|}{|\,y_1\,|+|\,y_2\,|}$$

	x	y	
First	2	$-0\cdot 6$	$h_1=\frac{1\times 0\cdot 6}{0\cdot 83}=0\cdot 72$
approx.	3	$+0\cdot 23$	
Diff	1	$0\cdot 83$	$x^{(1)}=2+0\cdot 72=2\cdot 72.$
Second	$2\cdot 7$	$-0\cdot 04$	$h_2=\frac{0\cdot 1\times 0\cdot 04}{0\cdot 09}=0\cdot 044$
approx.	$2\cdot 8$	$0\cdot 05$	
	$0\cdot 1$	$0\cdot 09$	$x^{(2)}=2\cdot 7+0\cdot 044=2\cdot 744.$
Third	$2\cdot 74$	$-0\cdot 0006$	$h_3=\frac{0\cdot 01\times 0\cdot 0006}{0\cdot 0087}=0\cdot 0007$
approx.	$2\cdot 75$	$+0\cdot 0081$	$x^{(3)}=2\cdot 74+0\cdot 0007=2\cdot 7407.$
	$0\cdot 01$	$0\cdot 0087$	
Fourth	$2\cdot 7406$	$-0\cdot 000039$	$h_4=\frac{0\cdot 0001\times 0\cdot 000039}{0\cdot 000084}$
approx.	$2\cdot 7407$	$+0\cdot 000045$	$=0\cdot 000046$
	$0\cdot 0001$	$0\cdot 000084$	$x^{(4)}=2\cdot 7406+0\cdot 000046=2\cdot 74065.$

Hence the required value of the root is $2\cdot 74065$.

Ex. 6. *The equation $x^6-x^4-x^3-1=0$ has one real root between $1\cdot 4$ and $1\cdot 5$. Find the root to four decimals by false position method.* **(Agra 1984, Jiwaji 84, Udaipur 86, Jabalpur 84)**

Sol. The successive approximations are given below :

	x	y	
1st	$1\cdot 4$	$-\cdot 056064$	$h_1=\frac{\cdot 1\times\cdot 056064}{2\cdot 009189}=\cdot 003$
approx.	$1\cdot 5$	$1\cdot 953125$	$x^{(1)}=1\cdot 4+\cdot 003=1\cdot 403$
Diff.	$\cdot 1$	$2\cdot 009189$	

Second	1·403	−·0167	$h_2=\dfrac{\cdot 001\times\cdot 0167}{\cdot 0223}=\cdot 0007$
approx.	1·404	·0056	
	·001	·0223	$x^{(2)}=1\cdot 403+\cdot 0007=1\cdot 4037.$
Third	1·4036	−·0001	$h_3=\dfrac{\cdot 0001\times\cdot 0001}{\cdot 0005}=\cdot 00002$
approx.	1·4037	·0004	
	·0001	·0005	$x^{(3)}=1\cdot 4036+\cdot 00002=1\cdot 4036.$

Hence the value of the root is 1·4036.

Ex. 7. *Use Newton's method to find a root of the equation*

$$x^3-3x-5=0.$$

Sol. Here we have $f(x)=x^3-3x-5 \Rightarrow f'(x)=3x^2-3.$

Also $f(2)=-3$ and $f(3)=13$.

Hence a root lies between 2 and 3.

We take $x_0=3$ and obtain successive approximations using the formula $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$.

It gives $x_{n+1}=x_n-\dfrac{x_n^3-3x_n-5}{3x_n^2-3}$.

$$\therefore \quad x_1=x_0-\frac{x_0^3-3x_0-5}{3x_0^2-3}=3-\frac{13}{24}=2\cdot 46,$$

$$x_2=2\cdot 295,\ x_3=2\cdot 279,\ x_4=2\cdot 279.$$

Hence the value of the required root is 2·279.

Ex. 8. *Find the real root of the equation $x^2+4\sin x=0$ correct to four places of decimals by using Newton-Raphson method.*

(Rohilkhand B.Sc. 1991; Rajasthan 85)

Sol. Since the term x^2 is positive for all real values of x so it is obvious that the equation will be satisfied only by a negative value of x. By the graph we observe that an approximate value of the root is $-1\cdot 9$.

Here $f(x)=x^2+4\sin x$. $\quad\therefore\ f'(x)=2x+4\cos x$.

We have $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$, $n=0, 1, 2, \ldots$

$$\Rightarrow \quad x_{n+1}=x_n-\frac{x_n^2+4\sin x_n}{2x_n+4\cos x_n},\ n=0, 1, 2, \ldots \qquad \ldots(1)$$

Taking $x_0=-1\cdot 9$, the relation (1) gives

$$x_1=x-\frac{x_0^2+4\sin x_0}{2x_0+4\cos x_0}=-1\cdot 9-\frac{(-1\cdot 9)^2+4\sin(-1\cdot 9)}{2(-1\cdot 9)+4\cos(-1\cdot 9)}$$

$$=-1\cdot 9-\frac{3\cdot 61-3\cdot 78}{-3\cdot 8-1\cdot 293}=-1\cdot 93.$$

The second approximation

$$x_2=x_1-\frac{x_1^2+4\sin x_1}{2x_1+4\cos x_1}$$

$$=-1\cdot 93-\frac{(-1\cdot 93)^2+4\sin(-1\cdot 93)}{2(-1\cdot 93)+4\cos(-1\cdot 93)}=-1\cdot 93-\frac{0\cdot 0198}{5\cdot 266}$$

$$=-1\cdot 93-0\cdot 0038=-1\cdot 9338.$$

Hence the value of the root is $-1\cdot 9338$.

Ex. 9. *Find $\sqrt{(12)}$ to five places of decimal by Newton-Raphson method.* **(Rohilkhand 1990; Meerut B.Sc. 91, 91P, 91S)**

Sol. Let $x=\sqrt{(12)}$. Then $x^2=12 \Rightarrow x^2-12=0$.

We take $x^2-12=f(x)$. So the given equation is $f(x)=0$. Here $f(3)=-3$ and $f(4)=4$.

Hence a root lies between 3 and 4. Taking $x_0=3\cdot 5$ and using the relation

$$x_{n+1}=x_n-\frac{x_n^2-12}{2x_n},\ n=0, 1, 2, \ldots$$

we get $x_1=x_0-\dfrac{x_0^2-12}{2x_0}=\dfrac{1}{2}\left(x_0+\dfrac{12}{x_0}\right)$

$$=\frac{1}{2}\left(3\cdot 5+\frac{12}{3\cdot 5}\right)=3\cdot 464.$$

Now $x_2=x_1-\dfrac{x_1^2-12}{2x_1}=\dfrac{1}{2}\left(x_1+\dfrac{12}{x_1}\right)$

$$=\frac{1}{2}\left(3\cdot 464+\frac{12}{3\cdot 464}\right)=3\cdot 4641$$

and $x_3=x_2-\dfrac{x_2^2-12}{2x_2}=\dfrac{1}{2}\left(x_2+\dfrac{12}{x_2}\right)$

$$=\frac{1}{2}\left(3\cdot 4641+\frac{12}{3\cdot 4641}\right)=3\cdot 4641.$$

Since $x_2=x_3$ upto four decimal places so we take

$$\sqrt{(12)}=3\cdot 4641.$$

Ex. 10. *Find the approximate value for the real root of $x\log_{10}x-1\cdot 2=0$ correct to five decimal places by Newton-Raphson method.*

Sol. Here $f(x)=x\log_{10}x-1\cdot 2$.

$\therefore\ f'(x)=\log_{10}x+x.(1/x).\log_{10}e=\log_{10}x+\cdot 43429.$

We have $\quad x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$, $n=0, 1, 2, \ldots$

$$\therefore \quad x_{n+1}=x_n-\frac{x_n\log_{10}(x_n)-1\cdot 2}{\log_{10}x_n+\cdot 43429}=\frac{\cdot 43429\,x_n+1\cdot 2}{\log_{10}x_n+\cdot 43429}$$

Taking initial value $x_0=2$, we get

$$x_1=\frac{\cdot 43429\,(2)+1\cdot 2}{\log_{10} 2+\cdot 43429}=2\cdot 81$$

Now $$x_2=\frac{\cdot 43429\,(2\cdot 81)+1\cdot 2}{\log_{10} 2\cdot 81+\cdot 43429}=2\cdot 741$$

$$x_3=\frac{\cdot 43429\,(2\cdot 741)+1\cdot 2}{\log_{10} 2\cdot 741+\cdot 43429}=2\cdot 7406$$

$$x_4=\frac{\cdot 43429\,(2\cdot 7406)+1\cdot 2}{\log_{10} 2\cdot 7406+\cdot 43429}=2\cdot 74065$$

$$x_5=\frac{\cdot 43429\,(2\cdot 74065)+1\cdot 2}{\log_{10} 2\cdot 74065+\cdot 43429}=2\cdot 74065.$$

Hence the required value of the root is 2·74065.

Ex. 11. *By using Newton-Raphson method, find the root of $x^4-x-10=0$, which is nearer to $x=2$, correct to three places of decimals.* **(Meerut 1986, 88, 91, B.Sc. 91)**

Sol. Here $f(x)=x^4-x-10$. $\therefore$ $f'(x)=4x^3-1$.

$$\therefore \quad x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x_n^4-x_n-10}{4x_n^3-1}=\frac{3x_n^4+10}{4x_n^3-1}.$$

The approximate value of the root is given to be 2. Taking $x_0=2$, we get

$$x_1=\frac{3x_0^4+10}{4x_0^3-1}=\frac{3\cdot 2^4+10}{4\cdot 2^3-1}=1\cdot 871$$

$$x_2=\frac{3x_1^4+10}{4x_1^3-1}=\frac{3\,(1\cdot 871)^4+10}{4\,(1\cdot 871)^3-1}=1\cdot 856$$

$$x_3=\frac{3x_2^4+10}{4x_2^3-1}=\frac{3\,(1\cdot 856)^4+10}{4\,(1\cdot 856)^3-1}=1\cdot 856.$$

Since $x_2=x_3$, so the required root is 1·856.

Ex. 12. *Using the starting value $x_0=i$, find a zero of*

$$x^4+x^3+5x^2+4x+4=0.$$

Sol. Using Newton's formula $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$, we have

First approx., $$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=i-\frac{f(i)}{f'(i)}=i-\frac{3i}{1+6i}$$
$$=0\cdot 486+0\cdot 919i,$$

second approx., $$x_2=x_1-\frac{f(x_1)}{f'(x_1)}=0\cdot 486+0\cdot 919\,i$$
$$-\frac{f(0\cdot 486+0\cdot 919i)}{f'(0\cdot 486+0\cdot 919i)}$$

$$=0\cdot 486+0\cdot 919\,i-\frac{-0\cdot 292+0\cdot 174\,i}{1\cdot 780+6\cdot 005\,i}=-0\cdot 499+0\cdot 866\,i.$$

The actual root is $x=(-1+i\sqrt{3})/2$.

Note. To evaluate $f\,(0\cdot 486+0\cdot 919\,i)$, the various powers can be obtained by using the De-moivre's theorem. For this first $(\cdot 486+0\cdot 919\,i)$ is expressed in the form $r\,(\cos\theta+i\sin\theta)$.

Ex. 13. *Solve the equation $3x-\cos x-1=0$ by*

(i) *false position method* (ii) *Newton-Raphson method.*

Sol. The given equation is $f(x)=3x-\cos x-1=0$.

Here $f(\cdot 60)=-0\cdot 025, f(\cdot 61)=\cdot 010$.

Hence a real root lies in the interval $(\cdot 60, \cdot 61)$.

By false position method the successive approximations are given below :

	x	y	
First approx.	·60	−·025	$h_1=\dfrac{\cdot 01\times\cdot 025}{\cdot 035}=\cdot 007$
	·61	·010	
Diff.	·01	·035	$x^{(1)}=\cdot 60+\cdot 007=\cdot 607$
Second approx.	·607	−·00036	$h_2=\dfrac{\cdot 001\times\cdot 00036}{\cdot 00357}=\cdot 0001$
	·608	·00321	
	·001	·00357	$x^{(2)}=\cdot 607+\cdot 0001=\cdot 6071$
Third approx.	·6071	·00000	$h_3=\dfrac{\cdot 0001\times(\cdot 00000)}{\cdot 00035}=0$
	·6072	·00035	
	·0001	·00035	$x^{(3)}=\cdot 6071.$

Since $x^{(2)}=x^{(3)}$ so the value of the root is ·6071.

Newton-Raphson method. Here $f(x)=3x-\cos x-1$.

$\therefore$ $f'(x)=3+\sin x$.

Now $$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{3x_n-\cos x_n-1}{3+\sin x_n}.$$

Taking $x_0=\cdot 60$, we get

$$x_1=\cdot 60-\frac{3\,(\cdot 60)-\cos\cdot 60-1}{3+\sin\cdot 60}=\cdot 60-\frac{-\cdot 025}{3\cdot 56464}=\cdot 60701$$

$$x_2=\cdot 60701-\frac{3\,(\cdot 60701)-\cos\,(\cdot 60701)-1}{3+\sin\,(\cdot 60701)}=\cdot 60710.$$

Ex. 14 *Solve $x^4-5x^3+20x^2-40x+60=0$, by Newton-Raphson method given that all the roots of the given equation are complex.* **(Rohilkhand 1988)**

Sol. Here $f(x)=x^4-5x^3+20x^2-40x+60$.

$$\therefore \quad f'(x)=4x^3-15x^2+40x-40.$$

We have $\quad x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$

$$=x_n-\frac{x_n^4-5x_n^3+20x_n^2-40x_n+60}{4x_n^3-15x_n^2+40x_n-40}=\frac{3x_n^4-10x_n^3+20x_n^2-60}{4x_n^3-15x_n^2+40x_n-40}$$

Taking $x_0=2(1+i)$, we get

$$x_1=\frac{3(2+2i)^4-10(2+2i)^3+20(2+2i)^2-60}{4(2+2i)^3-15(2+2i)^2+40(2+2i)-40}=\frac{-92}{24i-24}$$

$$=\frac{23}{12}(1+i)=1{\cdot}92+1{\cdot}92i, \quad \ldots(1)$$

$$x_2=\frac{3(1{\cdot}92+1{\cdot}92i)^4-10(1{\cdot}92+1{\cdot}92i)^3+20(1{\cdot}92+1{\cdot}92i)^2-60}{4(1{\cdot}92+1{\cdot}92i)^3-15(1{\cdot}92+1{\cdot}92i)^2+40(1{\cdot}92+1{\cdot}92i)-40}$$

$$=1{\cdot}915+1{\cdot}908i. \quad \ldots(2)$$

Since complex roots occur in conjugate pairs so the roots of $f(x)=0$ are $1{\cdot}915 \pm 1{\cdot}908i$ upto three places of decimals. Suppose the other pair of roots of the given equation is $\alpha \pm i\beta$.

Now the sum of the roots

$$=(\alpha+i\beta)+(\alpha-i\beta)+(1{\cdot}915+1{\cdot}908i)+(1{\cdot}915-1{\cdot}908i)\equiv-(-5)=5 \quad \ldots(3)$$

The product of roots

$$=(\alpha^2+\beta^2)\{(1{\cdot}915)^2+(1{\cdot}908)^2\}\equiv 60 \quad \ldots(4)$$

Solving the equations (3) and (4), we obtain

$$\alpha={\cdot}585 \text{ and } \beta=2{\cdot}805.$$

So the other two roots are ${\cdot}585 \pm 2{\cdot}805i$.

Ex. 15. *Obtain the Newton-Raphson extended formula*

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}-\frac{1}{2}\frac{\{f(x_0)\}^2 f''(x_0)}{\{f'(x_0)\}^3}$$

for the root of the equation $f(x)=0$.

Sol. Expanding $f(x)$ in the neighbourhood of x_0 by Taylor's series, we get

$$0=f(x)=f(x_0+x-x_0)=f(x_0)+(x-x_0)f'(x_0)$$

$$\Rightarrow \quad x=x_0-\frac{f(x_0)}{f'(x_0)}.$$

Hence the first approximation to the root is given by

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}.$$

Again by Taylor's series, we obtain

$$f(x)=f(x_0)+(x-x_0)f'(x_0)+\tfrac{1}{2}(x-x_0)^2 f''(x_0).$$

$$\therefore \quad f(x_1)=f(x_0)+(x_1-x_0)f'(x_0)+\tfrac{1}{2}(x_1-x_0)^2 f''(x_0).$$

Since x_1 is an approximation to the root so $f(x_1)=0$.

Hence $\quad f(x_0)+(x_1-x_0)f'(x_0)+\tfrac{1}{2}(x_1-x_0)^2 f''(x_0)=0$

$$\Rightarrow \quad f(x_0)+(x_1-x_0)f'(x_0)+\frac{1}{2}\frac{\{f(x_0)\}^2 f''(x_0)}{\{f'(x_0)\}^2}=0$$

$$\Rightarrow \quad x_1=x_0-\frac{f(x_0)}{f'(x_0)}-\frac{1}{2}\frac{\{f(x_0)\}^2 f''(x_0)}{\{f'(x_0)\}^3}.$$

This formula is known as Chebyshev formula of third order.

Ex. 16. *Show that the modified Newton-Raphson's method*

$$x_{n+1}=x_n-\frac{2f(x_n)}{f'(x_n)}$$

gives a quadratic convergence when the equation $f(x)=0$ *has a pair of double roots in the neighbourhood of* $x=x_n$.

Sol. Suppose $x=a$ is a double root near $x=x_n$.

Then $\quad f(a)=0, f'(a)=0.$

We have $\quad \epsilon_{n+1}=\epsilon_n-\dfrac{2f(a+\epsilon_n)}{f'(a+\epsilon_n)},$

where ϵ_n, ϵ_{n+1} have their usual meanings.

Expanding in powers of ϵ_n and using the results $f(a)=0$, $f'(a)=0$, we get

$$\epsilon_{n+1}=\epsilon_n-\frac{2\left[\frac{\epsilon_n^2}{2!}f''(a)+\ldots\right]}{\left[\epsilon_n f''(a)+\frac{\epsilon_n^2}{2!}f'''(a)+\ldots\right]}$$

$$=\epsilon_n-\frac{2.\epsilon_n^2\,\frac{1}{2!}\left[f''(a)+\frac{1}{3}\epsilon_n f'''(a)+\ldots\right]}{\epsilon_n\left[f''(a)+\frac{\epsilon_n}{2!}f'''(a)+\ldots\right]}$$

$$\simeq \epsilon_n-\frac{\epsilon_n\left[f''(a)+\frac{1}{3}\epsilon_n f'''(a)\right]}{\left[f''(a)+\frac{\epsilon_n}{2!}f'''(a)\right]}$$

$$\simeq \frac{1}{6}\epsilon_n^2\frac{f'''(a)}{\left[f''(a)+\frac{\epsilon_n f'''(a)}{2!}\right]}\simeq\frac{1}{6}\epsilon_n^2\frac{f'''(a)}{f''(a)}.$$

which shows $\epsilon_{n+1} \propto \epsilon_n^2$ and so the convergence is quadratic.

Ex. 17. (i) *Show that the following two sequences, both have convergence of the second order with the same limit* $\sqrt{a}$.

$$x_{n+1}=\tfrac{1}{2}\,x_n\left(1+\frac{a}{x_n^2}\right) \quad \text{and} \quad x_{n+1}=\tfrac{1}{2}\,x_n\left(3-\frac{x_n^2}{a}\right).$$

(Kurubshetra 1987)

(ii) *If x_n is suitable close approximation to $\sqrt{a}$, show that error in the first formula for x_{n+1} is about one third in the second formula and deduce that*

$$x_{n+1}=(x_n/8)\,[6+(3a/x_n^2)-(x_n^2/a)],$$

gives a sequence with third order convergence.

Sol. (i) We have $x_{n+1}=\tfrac{1}{2}\,x_n\left(1+\dfrac{a}{x_n^2}\right)$

$$\Rightarrow \quad x_{n+1}-\sqrt{a}=\tfrac{1}{2}\,x_n\left(1+\frac{a}{x_n^2}\right)-\sqrt{a}$$

$$=\tfrac{1}{2}\left(x_n+\frac{a}{x_n}-2\sqrt{a}\right)=\tfrac{1}{2}\left(\sqrt{x_n}-\frac{\sqrt{a}}{\sqrt{x_n}}\right)^2$$

$$=\frac{1}{2x_n}\,(x_n-\sqrt{a})^2$$

$$\Rightarrow \quad \epsilon_{n+1}=\frac{1}{2x_n}\,\epsilon_n^2. \qquad \ldots(1)$$

which shows the quadratic convergence.

Similarly for the second sequence, we have

$$x_{n+1}-\sqrt{a}=\tfrac{1}{2}x_n\left(3-\frac{x_n^2}{a}\right)-\sqrt{a}$$

$$=\tfrac{1}{2}\,x_n\left(1-\frac{x_n^2}{a}\right)+(x_n-\sqrt{a})$$

$$=(x_n-\sqrt{a})\left[1-\frac{x_n}{2a}\,(x_n+\sqrt{a})\right]$$

$$=\frac{(x_n-\sqrt{a})}{2a}\,\{2a-x_n^2-\sqrt{a}\;x_n\}$$

$$=\frac{(x_n-\sqrt{a})}{2a}\,\{(a-x_n^2)+(a-\sqrt{a}\;x_n)\}$$

$$=-\frac{(x_n-\sqrt{a})}{2a}\,(x_n-\sqrt{a})\,(x_n+2\sqrt{a})$$

$$=-\frac{(x_n+2\sqrt{a})}{2a}\,(x_n-\sqrt{a})^2.$$

Thus $$\epsilon_{n+1}=-\frac{(x_n+2\sqrt{a})}{2a}\,\epsilon_n^2, \qquad \ldots(2)$$

which shows that the convergence is quadratic.

(ii) Since x_n is very close to $\sqrt{a}$, from (2) we have

$$\epsilon_{n+1}\simeq-\frac{x_n+2x_n}{2x_n^2}\,\epsilon_n^2=-\frac{3}{2x_n}\,\epsilon_n^2. \qquad \ldots(3)$$

If we look at the equations (1) and (3), we conclude that the error in the first formula for x_{n+1} is about one third of that in the second formula.

Now we shall find the rate of convergence of the third formula.

We have

$$x_{n+1}-\sqrt{a}=\frac{1}{8}\,x_n\left(6+\frac{3a}{x_n^2}-\frac{x_n^2}{a}\right)-\sqrt{a}$$

$$=\frac{x_n\,(6x_n^2a+3a^2-x_n^4)}{8x_n^2a}-\sqrt{a}=\frac{6x_n^2a+3a^2-x_n^4-8x_na\sqrt{a}}{8x_na}$$

$$=-\frac{(x_n+3\sqrt{a})\,(x^3_n-3x_n^2\sqrt{a}+3x_na-a\sqrt{a})}{8x_na}$$

$$=-(x_n+3\sqrt{a})\,\frac{(x_n-\sqrt{a})^3}{8x_na}$$

Thus $$\epsilon_{n+1}=-\frac{(x_n+3\sqrt{a})}{8x_na}\,\epsilon_n^3,$$

which shows that the convergence is of third order.

Ex. 18 *Find a real root of equation $f(x)=x^3+x^2-1=0$, by using iteration method.*

Sol. Here $f(0)=-1$ and $f(1)=1$, so a root lies between 0 and 1. To find this root, we put the equation in the form $x=\phi(x)$.

$$\therefore \qquad x^3+x^2-1=0 \;\Rightarrow\; x=\frac{1}{\sqrt{(1+x)}},$$

so that $\phi(x)=\dfrac{1}{\sqrt{(1+x)}}$, $\phi'(x)=-\dfrac{1}{2\,(1+x)^{3/2}}$

We have $|\phi'(x)|<1$, for $x<1$.

Hence the iterative method can be applied.

Taking $x_0=\cdot5$, we get

$$x_1=\phi(x_0)=\frac{1}{\sqrt{(1\cdot5)}}=\cdot81649,$$

$$x_2=\phi(x_1)=\frac{1}{\sqrt{(1\cdot81649)}}=\cdot74196,$$

$$x_3=\phi(x_2)=\frac{1}{\sqrt{(1\cdot74196)}}=\cdot75767.$$

Similarly we get other approximations as

$$x_4=\cdot75427,\; x_5=\cdot75500,\; x_6=\cdot75485,$$

$$x_7=\cdot75488,\; x_8=\cdot75487, \text{ etc.}$$

Ex. 19. (i) *Find the root of the equation $2x=\cos x+3$ correct to three decimal places by using iteration method.*

Sol. The given equation can be put in the form

$$x=\tfrac{1}{2}(\cos x+3).$$

Here $\phi(x)=\frac{1}{2}(\cos x+3)$. $\therefore$ $\phi'(x)=\frac{1}{2}(-\sin x)$.

We have $|\phi'(x)|=\left|\frac{\sin x}{2}\right|<1$.

Hence the iterative method is applicable and starting with $x_0=\frac{\pi}{2}$, we obtain successive approximations as follows :

$$x_1=1\cdot5,\ x_2=1\cdot535,\ x_3=1\cdot518,\ x_4=1\cdot526,$$
$$x_5=1\cdot522,\ x_6=1\cdot524,\ x_7=1\cdot523,\ x_8=1\cdot524.$$

Hence we take the value of the root as 1·524 correct to three decimal places.

Ex. 19. (ii) *Solve the above equation by Aitken's Δ^2-method.*

Sol. As before, we get 3 approximations x_1, x_2, x_3.

x_i	Δx_i	$\Delta^2 x_i$
$x_1=1\cdot5$		
	0·035	
$x_2=1\cdot535$		$-0\cdot052$
	$-0\cdot017$	
$x_3=1\cdot518$		

Hence we get $\quad x_4=1\cdot518-\dfrac{(0\cdot017)^2}{-0\cdot052}$

$$=1\cdot524,$$

this value is corresponding to six normal iterations.

Ex. 20. *Starting with $x=\cdot12$, solve $x=\cdot21\sin(\cdot5+x)$ by using iteration method.* **(Meerut 1987 ; Rohilkhand 90)**

Sol. The given equation is $x=\cdot21\sin(\cdot5+x)$ so that

$$\phi(x)=\cdot21\sin(\cdot5+x)$$

and $\quad |\phi'(x)|<1.$

Hence the iterative method is applicable.

First approximation $x_1=\cdot21\sin(\cdot5+\cdot12)=\cdot21\sin(\cdot62)$
$=\cdot21\times\cdot58104=\cdot1220.$

Second approximation $x_2=\cdot21\sin(\cdot5+\cdot122)$
$=\cdot21\sin(\cdot622)=\cdot21\times\cdot58267=\cdot1224.$

Third approximation $x_3=\cdot21\sin(\cdot5+\cdot1224)$
$=\cdot21\sin(\cdot6224)=\cdot21\times\cdot58299=\cdot12243.$

Fourth approx. $x_4=\cdot21\sin(\cdot5+\cdot12243)$
$=\cdot21\sin(\cdot62243)=\cdot21\times\cdot58301=\cdot12243.$

Since $x_3=x_4$, so the required root is ·12243.

Ex. 21. *Use synthetic division to solve*

$$f(x)=x^3-x^2-1\cdot0001\,x+0\cdot9999=0,$$

in the neighbourhood of $x=1$. **(Garhwal 1986)**

Sol. First we shall find $f(1)$ and $f'(1)$ by synthetic division.

1	1	-1	$-1\cdot0001$	$0\cdot9999$
		1	0	$-1\cdot0001$
1	1	0	$-1\cdot0001$	$-0\cdot0002=f(1)$
		1	1	
1	1	1	$-0\cdot0001=f'(1)$	
		1		
	1	$2=\frac{1}{2}f''(1)$		

We observe that $f(1)$ and $f'(1)$ are very small. Hence two nearly equal roots exist. Taking $x_0=1$, we shall use the formula $x_{i+1}=x_i-\dfrac{f'(x_i)}{f''(x_i)}$ to modify the root.

$\therefore$ First approx. $x_1=x_0-\dfrac{f'(x_0)}{f''(x_0)}=1-\dfrac{f'(1)}{f''(1)}$

$$=1-\frac{(-0\cdot0001)}{4}=1\cdot000025=c,\text{ say.}$$

Again we have $\quad x=c\pm\sqrt{\left\{\dfrac{-2f(c)}{f''(c)}\right\}}$

$$=1\cdot000025\pm\sqrt{\left\{\frac{-2f(1\cdot000025)}{f''(1\cdot000025)}\right\}}. \quad ...(1)$$

Now we calculate $f(1\cdot000025)$ and $f''(1\cdot000025)$ by synthetic division.

1 000025	1	−1	−1·0001	0·9999	
		1·000025	·000025	−1·0001	
1·000025	1	·000025	−1·000075	$-0{\cdot}0002=f(1{\cdot}000025)$	
		1·000025	1·000075		
1·000025	1	1·000050	$0=f'(1{\cdot}000025)$		
		1·000025			
	1	$2{\cdot}000075=\frac{1}{2}f''(1{\cdot}000025)$			

Putting the values $f(1{\cdot}000025)=-0{\cdot}0002$ and $f''(1{\cdot}000025)=2\times 2{\cdot}000075$ in (1), we get

$$x=1{\cdot}000025\pm\sqrt{\left(\frac{{\cdot}000200}{2{\cdot}000075}\right)}=1{\cdot}000025\pm{\cdot}009$$
$$=1{\cdot}009025,\ 0{\cdot}991025.$$

Ex. 22. *Find a real root of the polynomial equation*

$$f(x)=x^5-0{\cdot}346284\,x^4+x^3+3{\cdot}768x+10=0,$$

correct to six decimal places by using Newton-Raphson's formula and the method of synthetic division. **(Agra 1987)**

Sol. Here the values $f(-1)$ and $f(-2)$ are of opposite signs, so a real root lies between $x=-1$ and $x=-2$.

Taking $x_0=-1$, we get by Newton's method, first approximation $\quad x_1=-1-\dfrac{f(-1)}{f'(-1)}$

We calculate $f(-1)$ by synthetic division.

−1	1	−0·346284	1	0	3 768	10	
		−1·000000	1·346284	−2 346284	2·346284	−6·114284	
	1	−1·346284	2·346284	−2·346284	6·114284	3·885716	

$$\therefore\quad f(-1)=3{\cdot}885716.$$

We have $\quad f'(x)=5x^4-1{\cdot}385136\,x^3+3x^2+3{\cdot}768.$

$$\therefore\quad f'(-1)=5+1{\cdot}385136+3+3{\cdot}768=13{\cdot}153136.$$

$$\therefore\quad x_1=-1-\frac{3\ 885716}{13{\cdot}153136}=-1{\cdot}295.$$

We shall require $f(-1{\cdot}295)$ and $f'(-1{\cdot}295)$ to find the second approximation x_2 by Newton's method. Now we calculate these by synthetic division.

−1·295	1	−0·346284	1	0	3·768	10	
		−1·295	2·12546	−4·04747	5·24147	−11·66726	
	1	−1·641284	3·12546	−4·04747	9·00947	$-1{\cdot}66726=f(-1{\cdot}295)$	

The value of $f'(-1{\cdot}295)$ is calculated by a new synthetic division scheme as below :

−1·295	5	−1·385136	3	0	3·768	
		−6·475	10·17888	−17 06665	22·10131	
	5	−7·86014	13·17888	−17·06665	$25{\cdot}86931=f'(-1{\cdot}295)$	

Now second approximation x_2 is given by

$$x_2=-1{\cdot}295-\frac{f(-1{\cdot}295)}{f'(-1{\cdot}295)}$$
$$=-1{\cdot}295-\frac{(-1{\cdot}66726)}{25{\cdot}86931}=-1{\cdot}231.$$

Repeating the same process we obtain

$$x_3=-1{\cdot}22539 \text{ and } x_4=-1{\cdot}225386.$$

Ex. 23. *Find a quadratic factor of the polynomial*

$$x^4+5x^3+3x^2-5x-9=0$$

starting with $p_0=3$, $q_0=-5$ by using Bairstow's method.

Sol. We put all the calculations in the form of a table which is self explanatory.

(a_k)	1	5	3	−5	−9	
		−3	−6	−6	3	−3
			5	10	−5	5
(b_k)	1	2	2	$-1=b_{n-1}$	$4=b_n$	
		−3	3	−30		
			5	−5		
(c_k)	1	−1	10	−36		
		↓	↓	↓		
		c_{n-3}	c_{n-2}	c_{n-1}		

The corrections Δp_0 and Δq_0 are given by

$$\left.\begin{aligned} c_{n-2}\Delta p_0+c_{n-3}\Delta q_0&=b_{n-1} \\ (c_{n-1}-b_{n-1})\,\Delta p_0+c_{n-2}\Delta q_0&=b_n \end{aligned}\right\}$$

or
$$\left.\begin{aligned} 10\Delta p_0-\Delta q_0&=-1 \\ -35\Delta p_0+10\Delta q_0&=4 \end{aligned}\right\} \Rightarrow \Delta p_0=-0{\cdot}09,\ \Delta q_0=0{\cdot}08.$$

Thus p_1, q_1, the first approximations of p and q are given by $p_0+\Delta p_0$ and $q_0+\Delta q_0$

i.e., 2·91 and −4·92.

Now, the computation is repeated with the new values of p and q.

1	5	3	−5	−9	
	−2·91	−6·08	−5·35	0·20	−2·91
		4·92	10·28	9·05	4·92
1	2·09	1·84	−0·07	0·25	
	−2·91	2·37	−26·57		
		4·92	−4·03		
1	−0·82	9·13	−30·67		

At this step, the corrections Δp_1 and Δq_1 are given by

$$\left.\begin{aligned} 9{\cdot}13\Delta p_1-0{\cdot}82\Delta q_1&=-0{\cdot}07 \\ -30{\cdot}60\Delta p_1+9{\cdot}13\Delta q_1&=0{\cdot}25 \end{aligned}\right\}$$

$$\Rightarrow \Delta p_1=-0{\cdot}00745 \text{ and } \Delta q_1=0{\cdot}00241.$$

Hence the second approximations of p and q are given by

$$p_2=p_1+\Delta p_1 \text{ and } q_2=q_1+\Delta q_1$$

i.e., 2·90255 and −4·91759.

Thus a quadratic factor is $x^2+p_2x+q_2$

i.e., $x^2+2{\cdot}90255x-4{\cdot}91759$.

Dividing the given equation by this factor we can obtain the other quadratic factor.

Ex. 24. *Solve $x^3-8x^2+17x-10=0$ by Graeffe's method.* **(Rohilkhand B.Sc. 1990, M.Sc. 90; Meerut B.Sc. 91, 93, 94, M.Sc. 88, 91)**

Sol. The given equation can be written as

$$x^3+17x=8x^2+10.$$

Squaring, we get

$$x^2(x^2+17)^2=(8x^2+10)^2.$$

Putting $x^2=y$, we have

$$y(y+17)^2=(8y+10)^2$$

or
$$y^3-30y^2+129y-100=0. \qquad \ldots(1)$$

The roots of (1) are the squares of the roots of the given equation.

Squaring twice again, we obtain

$$z^3-642z^2+10641z-10^4=0, \qquad \ldots(2)$$

and
$$u^3-390882u^2+100390881u-10^8=0. \qquad \ldots(3)$$

The roots of (3) are 8th powers of the roots of the given equation. Let the roots of (3) be u_1, u_2, u_3. Then we have

$$\left.\begin{aligned} u_1&\simeq 390882 \\ u_1u_2&\simeq 100390881 \\ u_1u_2u_3&\simeq 10^8 \end{aligned}\right\} \Rightarrow \begin{aligned} u_1&\simeq 390882 \\ u_2&\simeq (100390881/390882) \\ u_3&\simeq (10^8/100390881). \end{aligned}$$

Let the roots of the original equation be p_1, p_2, p_3. Hence

$$|p_1|=(390882)^{1/8}=5{\cdot}00041$$
$$|p_2|=(100390881/390882)^{1/8}=2{\cdot}00081$$
$$|p_3|=(10^8/100390881)^{1/8}=0{\cdot}999512.$$

The exact roots of the given equation are 5, 2 and 1.

Exercises 12

1. Find the root of $x^2-5x+2=0$ correct to five decimal places by Newton-Raphson method. **(Meerut 1987, B.Sc. 92, 92M)**
2. Solve the equation $\log_e x=\cos x$ to five decimals by Newton-Raphson method.
3. Find a root of $x=\frac{1}{2}+\sin x$ near $x=1{\cdot}5$.
4. The equation $x^4+5x^3-12x^2+76x-79=0$ has two roots close to $x=2$. Find the roots to four decimals.
5. Find the real root to four decimals of the equations (i) $x^5-x^4-x^3-1=0$ (ii) $x^6-x^4-x^3-1=0$.
6. Solve the equation $x^4-8x^3+39x^2-62x+50=0$, starting with $p=q=0$. **(Rohilkhand 1988)**
7. Solve $x^4-5x^3+20x^2-40x+60=0$, given that all the roots of $f(x)=0$ are complex by using Bairstow's method.
8. Find the quadratic factor of $x^3-3{\cdot}7x^2+6{\cdot}25x-4{\cdot}069$, after two iterations. Use $p_0=-2{\cdot}5$, $q_0=0$.
9. Determine the real root of $\tan x=x$ by iteration method.
10. Find the roots of $x^3-2x^2-5x+6=0$, squaring three times.
11. Solve the equation $x^3-5x^2-17x+20=0$ by Graeffe's method (squaring three times). **(Rohilkhand 1988)**

14

Simultaneous Linear Algebraic Equations

§ 1. Introduction.

The solution of systems of simultaneous linear equations is one of the most important and most frequently encountered problems of numerical mathematics.

We consider the following m first degree equations in n unknowns $x_1, x_2, x_3, \ldots, x_n$:

$$\left.\begin{array}{l}
a_{11}x_1+a_{12}x_2+\ldots+a_{1j}x_j+\ldots+a_{1n}x_n=b_1 \\
a_{21}x_1+a_{22}x_2+\ldots+a_{2j}x_j+\ldots+a_{2n}x_n=b_2 \\
\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \\
a_{i1}x_1+a_{i2}x_2+\ldots+a_{ij}x_j+\ldots+a_{in}x_n=b_i \\
\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \\
a_{m1}x_1+a_{m2}x_2+\ldots+a_{mj}x_j+\ldots+a_{mn}x_n=b_m
\end{array}\right\} \quad \ldots(1)$$

The above system can be written in a single matrix equation

as $$AX=B,$$

where $$A=\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix},$$

$$X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } B=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

By finding a solution of a system of equations we mean to find a set of values of $x_1, x_2, \ldots, x_n$ which satisfies all these equations simultaneously. The above system of equations is said to be homogeneous if all the b_i $(i=1, 2, \ldots, m)$ vanish otherwise the system is called non-homogeneous system.

Various methods have been devised for the numerical solution of simultaneous linear equations. Perhaps no single method is best in all cases. Some methods are of general applicability while some are restricted in their application. We shall discuss some of the best general methods in the present chapter.

A general restriction in this chapter is that $m=n$. Suppose the system of equations (1) (taking $m=n$) can be written as

$$AX=B \qquad ...(2)$$

where $A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{nm} \end{bmatrix}$, $X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$, $B=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$.

Here **A** is the coefficient matrix, **X** the solution vector and **B** the known right-hand-side vector.

The necessary and sufficient condition for the existence of a solution of the system $AX=B$ *is that* $r(A)=r(A\ B)$ i.e., the rank of the coefficient matrix is the same as the rank of the augmented matrix.

Below we give some rules which give the information about the existence of a solution of the system $AX=B$.

(i) If $B=0$, det $A=0$, then apart from the trivial solution $X=0$, we have an infinity of non-trivial solutions. In this case $r(A)<m$.

(ii) If $B=0$, det $A\neq 0$, then the system has the only unique trivial solution $X=0$. In this case $r(A)=m$.

(iii) If $B\neq 0$, det $A\neq 0$, then we have exactly one solution. In this case $r(A)=m$.

(iv) If $B\neq 0$, det $A=0$, then there exist infinite number of solutions provided the equations are consistent. In this case $r(A)<m$.

§ 2. Different Methods of obtaining the solutions.

(a) Matrix Inversion Method. Let **A** be non-singular *i.e.*, det $A\neq 0$. Then A^{-1} exists. Now premultiplying both sides of $AX=B$ by A^{-1}, we get

$$A^{-1}AX=A^{-1}B \text{ i.e., } X=A^{-1}B. \qquad [\because A^{-1}A=I \text{ and } IX=X]$$

Here $$A^{-1}=\frac{1}{\det A}\text{ Adj } A.$$

Thus if A^{-1} is known, then the solution vector **X** can be found from the above matrix relation.

(b) Cramer's Rule. Suppose that det $A=D\neq 0$ and $B\neq 0$. According to Cramer's rule, the system $AX=B$ has the solution

$$x_r=D_r/D,\ r=1, 2, \dots, m$$

where D_r is the determinant obtained by replacing the rth column in D by **B**.

Giving different values to r, we can find $x_1, x_2, \dots, x_m$ and hence the solution is obtained. This method is true from a purely theoretical point of view but from a calculation point of view it is not good. It is feasible when $m=3$ or 4.

(c) The elimination method by Gauss. This is an elementary elimination method which reduces the system of equations to an equivalent upper triangular system which can be solved by back substitution.

Suppose we have a system of n equations in n unknowns as given below :

$$\left.\begin{aligned} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n&=b_1 \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n&=b_2 \\ \dots\dots \quad \dots\dots \quad \dots\dots \quad &\dots\dots \\ a_{n1}x_1+a_{n2}x_2+\dots+a_{nn}x_n&=b_n \end{aligned}\right\} \qquad ...(1)$$

This system can be put in the form $AX=B$. Let det $A\neq 0$ and $B\neq 0$.

We begin by dividing the first equation of (1) by a_{11} (if $a_{11}=0$, the equations are arranged in a suitable way) and then we subtract this equation multiplied by $a_{21}, a_{31}, \dots, a_{n1}$ from the second, third, ..., nth equations. Again the second equation is divided by the new coefficient a_{22}' of the variable x_2 (this element is known as the pivot element), and then in a similar way x_2 is eliminated from the third, fourth, ..., nth equations. We continue this procedure as far as possible, and finally we obtain $x_n, x_{n-1}, \dots, x_1$ by back substitution.

Suppose the system takes the following form, when the elimination is completed :

$$\left.\begin{aligned} c_{11}x_1+c_{12}x_2+\dots+c_{1n}x_n&=d_1 \\ c_{22}x_2+\dots+c_{2n}x_n&=d_2 \\ \dots\dots \quad \dots\dots \quad \dots\dots \quad &\dots\dots \\ c_{nn}x_n&=d_n \end{aligned}\right\} \qquad ...(2)$$

The new coefficient matrix is an upper triangular matrix ; the diagonal elements c_{ii} are usually equal to 1.

To find a better result by this method it is desirable to select the largest element of the row as the pivotal element. For this purpose, we reorder the columns and the equations, if necessary.

(d) Jordan's method. It is a modification of the method due to Gauss. In this method the elimination is performed not only in the equations below but also in the equations above. Thus, finally we get a diagonal (even unit) coefficient matrix and without further computation we have the solution.

(e) Crout's method. This method is superior to the Gauss elimination method because it requires less calculations. We shall describe this method by considering a system of 3 equations. Let the system be

$$\left.\begin{aligned} a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1\\ a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2\\ a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3 \end{aligned}\right\} \quad \ldots(1)$$

The system (1) can be written as $AX=B$ ...(2)

where $A=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$, $X=\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}$, $B=\begin{bmatrix} b_1\\ b_2\\ b_3\end{bmatrix}$.

The augmented matrix of (2) is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1\\ a_{21} & a_{22} & a_{23} & b_2\\ a_{31} & a_{32} & a_{33} & b_3\end{bmatrix}. \quad \ldots(3)$$

Now we consider a derived matrix

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & b'_1\\ a'_{21} & a'_{22} & a'_{23} & b'_2\\ a'_{31} & a'_{32} & a'_{33} & b'_3\end{bmatrix}, \quad \ldots(4)$$

which is to be calculated as follows:

(i) To determine the first column : $a'_{i1}=a_{i1}\ \forall\ i$, *i.e.*, the first column is identical with the first column of the coefficient matrix A.

(ii) To determine the first row to the right of the first column :

$$a'_{1j}=\frac{a_{1j}}{a_{11}},\ j=2,3\ ;\ b'_1=\frac{b_1}{a_{11}}$$

i.e., the first row except the first element is obtained by dividing the corresponding elements of the first row of the matrix (3) by the first element of that row.

(iii) To determine the remaining elements of the second column :

$$a'_{i2}=a_{i2}-a'_{12}.a'_{i1},\ i=2,3.$$

(iv) To determine the remaining second row :

$$a'_{2j}=\frac{a_{2j}-a'_{1j}a'_{21}}{a'_{22}},\ j=3$$

and
$$b'_2=(b_2-b'_1.a'_{21})/a'_{22}.$$

(v) To determine the remaining third column :

$$a'_{33}=a_{33}-a'_{23}a'_{32}-a'_{13}a'_{31}.$$

(vi) To determine the remaining third row :

$$b'_3=\frac{b_3-b'_2a'_{32}-b'_1a'_{31}}{a'_{33}}.$$

Now the solution is given by

$$x_3=b'_3,\ x_2=b'_2-a'_{23}x_3,\ x_1=b'_1-a'_{13}x_3-a'_{12}x_2.$$

We observe that various elements of the derived matrix are determined in the following order.

First of all first column ; then elements of the first row to the right of the first column ; then elements of the second column below the first row ; then elements of the second row to the right of the second column, etc.

(c) **Method of Factorization.** In this method we use the fact that a square matrix A can be factorized into the form LU where L is unit lower triangular matrix and U is upper triangular, if all the principal minors of A are non-singular, *i.e.*, if

$$a_{11}\neq 0,\ \begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}\neq 0,\ \begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}\neq 0, \text{ etc.}$$

Let us consider a system of linear equations :

$$\left.\begin{aligned} a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1\\ a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2\\ a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3 \end{aligned}\right\}. \quad \ldots(1)$$

This can be put in the form

$$AX=B. \quad \ldots(2)$$

Let $A=LU$

where $L=\begin{bmatrix} 1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}$ and $U=\begin{bmatrix} u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$.

Writing $A=LU$ in (2), we get

$$LUX=B. \quad \ldots(3)$$

Setting $UX=Y$, the equation (3) becomes

$$\boxed{LY=B.} \quad \ldots(4)$$

The eqn. (4) is equivalent to the system

$$\left.\begin{aligned} y_1=b_1\\ l_{21}y_1+y_2=b_2\\ l_{31}y_1+l_{32}y_2+y_3=b_3 \end{aligned}\right\}. \quad \ldots(5)$$

By the forward substitution we get the values of y_1, y_2, y_3.

When we know Y, the system $UX=Y$ gives :

$$\left.\begin{aligned} u_{11}x_1+u_{12}x_2+u_{13}x_3=y_1\\ u_{22}x_2+u_{23}x_3=y_2\\ u_{33}x_3=y_3 \end{aligned}\right\}. \quad \ldots(6)$$

By the backward substitution we get the values of x_1, x_2 and x_3.

Now we shall discuss the procedure of computing the matrices L and U. From the relation $A=LU$, we get

$$\begin{bmatrix} 1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}.$$

Multiplying the matrices on the left hand side and then equating the corresponding elements on both sides, we have

$$u_{11}=a_{11},\ u_{12}=a_{12},\ u_{13}=a_{13},$$
$$l_{21}u_{11}=a_{21} \Rightarrow l_{21}=a_{21}/a_{11},$$
$$l_{21}u_{12}+u_{22}=a_{22} \Rightarrow u_{22}=a_{22}-(a_{21}/a_{11})\,a_{12},$$
$$l_{21}u_{13}+u_{23}=a_{23} \Rightarrow u_{23}=a_{23}-(a_{21}/a_{11})\,a_{13},$$
$$l_{31}u_{11}=a_{31} \Rightarrow l_{31}=a_{31}/a_{11},$$
$$l_{31}u_{12}+l_{32}u_{22}=a_{32} \Rightarrow l_{32}=\frac{a_{32}-(a_{31}/a_{11})\,a_{12}}{u_{22}},$$

and $\quad l_{31}u_{13}+l_{32}u_{23}+u_{33}=a_{33}$ which gives u_{33}.

Hence in a systematic way the elements of **L** and **U** can be evaluated.

(g) Iterative Methods. So far we have discussed some direct methods for the solution of simultaneous linear equations. Now we shall discuss the iterative or indirect methods. In these methods we start from an approximation to the true solution and, if convergent, derive a sequence of closer approximations. We repeat the cycle of computations till the required accuracy is obtained. Thus in an iterative method the amount of computation depends on the accuracy required, while in a direct method the amount of computation is fixed.

(i) Jacobi Iterative method. Let the system of simultaneous linear equations be

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots+a_{2n}x_n=b_2 \\ a_{31}x_1+a_{32}x_2+a_{33}x_3+\ldots+a_{3n}x_n=b_3 \\ \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \\ a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\ldots+a_{nn}x_n=b_n \end{array}\right\} \quad \ldots(1)$$

Suppose the diagonal coefficients a_{ii} in (1) do not vanish. If it is not so then we can rearrange the equations so that this condition is satisfied.

Now the system (1) can be written as :

$$\left.\begin{array}{l} x_1=\dfrac{b_1}{a_{11}}-\dfrac{a_{12}}{a_{11}}x_2-\dfrac{a_{13}}{a_{11}}x_3-\ldots-\dfrac{a_{1n}}{a_{11}}x_n \\ x_2=\dfrac{b_2}{a_{22}}-\dfrac{a_{21}}{a_{22}}x_1-\dfrac{a_{23}}{a_{22}}x_3-\ldots-\dfrac{a_{2n}}{a_{22}}x_n \\ x_3=\dfrac{b_3}{a_{33}}-\dfrac{a_{31}}{a_{33}}x_1-\dfrac{a_{32}}{a_{33}}x_2-\ldots-\dfrac{a_{3n}}{a_{33}}x_n \\ \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \\ x_n=\dfrac{b_n}{a_{nn}}-\dfrac{a_{n1}}{a_{nn}}x_1-\dfrac{a_{n2}}{a_{nn}}x_2-\ldots-\dfrac{a_{n,n-1}}{a_{nn}}x_{n-1} \end{array}\right\} \quad \ldots(2)$$

Let $x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$ be any first approximations to the unknowns $x_1, x_2, \ldots, x_n$. Putting these in the R.H.S. of (2), we get a system of second approximations :

$$\left.\begin{array}{l} x_1^{(2)}=\dfrac{b_1}{a_{11}}-\dfrac{a_{12}}{a_{11}}x_2^{(1)}-\ldots-\dfrac{a_{1n}}{a_{11}}x_n^{(1)} \\ x_2^{(2)}=\dfrac{b_2}{a_{22}}-\dfrac{a_{21}}{a_{22}}x_1^{(1)}-\ldots-\dfrac{a_{2n}}{a_{22}}x_n^{(1)} \\ x_3^{(2)}=\dfrac{b_3}{a_{33}}-\dfrac{a_{31}}{a_{33}}x_1^{(1)}-\ldots-\dfrac{a_{3n}}{a_{33}}x_n^{(1)} \\ \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \\ x_n^{(2)}=\dfrac{b_n}{a_{nn}}-\dfrac{a_{n1}}{a_{nn}}x_1^{(1)}-\ldots-\dfrac{a_{n,n-1}}{a_{nn}}x_{n-1}^{(1)} \end{array}\right\} \quad \ldots(3)$$

In the same way if $x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}$ are nth approximations, then the system of next approximations is given by :

$$\left.\begin{array}{l} x_1^{(n+1)}=\dfrac{b_1}{a_{11}}-\dfrac{a_{12}}{a_{11}}x_2^{(n)}-\ldots-\dfrac{a_{1n}}{a_{11}}x_n^{(n)} \\ x_2^{(n+1)}=\dfrac{b_2}{a_{22}}-\dfrac{a_{21}}{a_{22}}x_1^{(n)}-\ldots-\dfrac{a_{2n}}{a_{22}}x_n^{(n)} \\ \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \\ x_n^{(n+1)}=\dfrac{b_n}{a_{nn}}-\dfrac{a_{n1}}{a_{nn}}x_1^{(n)}-\ldots-\dfrac{a_{n,n-1}}{a_{nn}}x_{n-1}^{(n)} \end{array}\right\} \quad \ldots(4)$$

Writing (2) in the matrix form

$$\mathbf{X}=\mathbf{BX}+\mathbf{C}, \quad \ldots(5)$$

we can write the iteration formula (4) as

$$\mathbf{X}^{(n+1)}=\mathbf{BX}^{(n)}+\mathbf{C}. \quad \ldots(6)$$

(ii) Gauss-Seidel Iterative method. It is a simple modification of Jacobi's method. **(Meerut M.Sc. 1989)**

Substitute the first approximations $(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_n^{(1)})$ into the R.H.S. of the first equation of (2) and denote the result by $x_1^{(2)}$. Now we substitute $(x_1^{(2)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_n^{(1)})$ in the second equation and the result is denoted by $x_2^{(2)}$. We put $(x_1^{(2)}, x_2^{(2)}, x_3^{(1)}, \ldots, x_n^{(1)})$ in the third equation and denote the result by $x_3^{(2)}$. Continuing this process we put $(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots, x_{n-1}^{(2)}, x_n^{(1)})$ in the last equation and denote the result by $x_n^{(2)}$. Thus first stage of iteration is completed. We repeat the entire process till the values of $x_1, x_2, \ldots, x_n$ are obtained to the desired accuracy. In this method we use an improved component as soon as it is available.

Gauss-Seidel method is called the method of *successive displacements* and Jacobi method is called the method of *simultaneous displacements.*

For any choice of the first approximation $x_j^{(1)}$ $(j=1, 2, \ldots, n)$ the Jacobi and Gauss-Seidel methods converge, if the following condition is satisfied by every equation of the system (2).

The sum of the absolute values of the coefficients a_{ij}/a_{ii} is atmost equal to, or in atleast one equation less than unity,

i.e., if $$\sum_{\substack{j=1\\ j\neq i}}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| \leqslant 1, \; (i=1, 2, \ldots, n) \qquad \ldots(7)$$

where the sign $<$ is valid in the case of atleast one equation.

(iii) Relaxation method. In this iterative method initially assumed values of the unknowns are improved by reducing the so called residuals, denoted by $R_i^{(m)}$, to zero. The residual of ith equation at mth iteration is given by

$$R_i^{(m)}=b_i-a_{i1}x_1^{(m)}-a_{i2}x_2^{(m)}-\ldots-a_{in}x_n^{(m)},$$

where $x_j^{(m)}$ denotes the value of x_j at mth iteration. Thus we get $R_i^{(m)}$ by putting $x_j^{(m)}$ in the L.H.S. of ith equation and then subtracting it from the R.H.S. We assume the initial values of unknowns at each iteration and then the residuals corresponding to all the equations are calculated. Then we reduce the largest residual to zero at that iteration. We continue the process of reducing the residuals till all the residuals become zero or negligible.

Solved Examples

Ex. 1. *Solve the equations*

$$\begin{aligned} 3x+y+2z&=3\\ 2x-3y-z&=-3\\ x+2y+z&=4. \end{aligned}$$

Sol. Here $A=\begin{bmatrix} 3 & 1 & 2\\ 2 & -3 & -1\\ 1 & 2 & 1\end{bmatrix}$ and so $|A|=\begin{vmatrix} 3 & 1 & 2\\ 2 & -3 & -1\\ 1 & 2 & 1\end{vmatrix}=8.$

Now $A'=\begin{bmatrix} 3 & 2 & 1\\ 1 & -3 & 2\\ 2 & -1 & 1\end{bmatrix}$ and adj $A=\begin{bmatrix} -1 & 3 & 5\\ -3 & 1 & 7\\ 7 & -5 & -11\end{bmatrix}$.

Hence $A^{-1}=\dfrac{1}{|A|}$ Adj $A=\dfrac{1}{8}\begin{bmatrix} -1 & 3 & 5\\ -3 & 1 & 7\\ 7 & -5 & -11\end{bmatrix}$.

Now $X=A^{-1}B \Rightarrow \begin{bmatrix} x\\ y\\ z\end{bmatrix}=\dfrac{1}{8}\begin{bmatrix} -1 & 3 & 5\\ -3 & 1 & 7\\ 7 & -5 & -11\end{bmatrix}\begin{bmatrix} 3\\ -3\\ 4\end{bmatrix}$

$$=\frac{1}{8}\begin{bmatrix} 8\\ 16\\ -8\end{bmatrix}=\begin{bmatrix} 1\\ 2\\ -1\end{bmatrix}$$

which gives $x=1, y=2, z=-1$.

Ex. 2. *Solve the given system by Cramer's rule* :

$$\left.\begin{aligned} 3x+2y-z+t&=1\\ x-y-2z+4t&=3\\ 2x-3y+z-2t&=-2\\ 5x-2y+3z+2t&=0.\end{aligned}\right\} \qquad \ldots(1)$$

Sol. Here we have

$$D=\begin{vmatrix} 3 & 2 & -1 & 1\\ 1 & -1 & -2 & 4\\ 2 & -3 & 1 & -2\\ 5 & -2 & 3 & 2\end{vmatrix}=50,$$ which is the denominator of the fractions for all the unknowns.

$$D_1=\begin{vmatrix} 1 & 2 & -1 & 1\\ 3 & -1 & -2 & 4\\ -2 & -3 & 1 & -2\\ 0 & -2 & 3 & 2\end{vmatrix}=-19,\quad D_2=\begin{vmatrix} 3 & 1 & -1 & 1\\ 1 & 3 & -2 & 4\\ 2 & -2 & 1 & -2\\ 5 & 0 & 3 & 2\end{vmatrix}=-29,$$

$$D_3=\begin{vmatrix} 3 & 2 & 1 & 1\\ 1 & -1 & 3 & 4\\ 2 & -3 & -2 & -2\\ 5 & -2 & 0 & 2\end{vmatrix}=-51,\quad D_4=\begin{vmatrix} 3 & 2 & -1 & 1\\ 1 & -1 & -2 & 3\\ 2 & -3 & 1 & -2\\ 5 & -2 & 3 & 0\end{vmatrix}=0.$$

Now $x=\dfrac{D_1}{D}=\dfrac{19}{50}$, $y=\dfrac{D_2}{D}=-\dfrac{29}{50}$, $z=\dfrac{D_3}{D}=-\dfrac{51}{50}$, $t=\dfrac{D_4}{D}=0$.

Ex. 3. *Solve* $$\left.\begin{aligned} 10x-7y+3z+5u&=6,\\ -6x+8y-z-4u&=5,\\ 3x+y+4z+11u&=2,\\ 5x-9y-2z+4u&=7\end{aligned}\right\}$$

by Gauss's elimination method. **(Jodhpur 1986)**

Sol. We can rewrite the given system as

$$\left.\begin{aligned} x-0{\cdot}7y+0{\cdot}3z+0{\cdot}5u&=0{\cdot}6\\ -6x+8y-z-4u&=5\\ 3x+y+4z+11u&=2\\ 5x-9y-2z+4u&=7\end{aligned}\right\} \qquad \ldots(1)$$

First, we eliminate x from the second, third and fourth equations, using the first equation. Subtracting (-6) times the first equation from the second equation, 3 times the first equation from the third equation and 5 times the first equation from the fourth equation, we get the new system as

$$\left.\begin{aligned} x-0{\cdot}7y+0{\cdot}3y+0{\cdot}5u&=0{\cdot}6\\ 3{\cdot}8y+0{\cdot}8z-\quad u&=8{\cdot}6\\ 3{\cdot}1y+3{\cdot}1z+9{\cdot}5u&=0{\cdot}2\\ -5{\cdot}5y-3{\cdot}5z+1{\cdot}5u&=\ 4 \end{aligned}\right\}. \qquad \ldots(2)$$

Since the numerically largest y-coefficient is in the fourth equation of system (2) so we permute the second and fourth equations. After that, y is eliminated from the third and fourth equations by using the second equation. The new system becomes

$$\left.\begin{aligned} x-0{\cdot}7y+0{\cdot}3z+0{\cdot}5u&=0{\cdot}6\\ y+0{\cdot}63636z-0{\cdot}27273u&=-0{\cdot}72727\\ -1{\cdot}61818z+0{\cdot}03636u&=11{\cdot}36364\\ 1{\cdot}12727z+10{\cdot}34545u&=2{\cdot}45455 \end{aligned}\right\}. \qquad \ldots(3)$$

Now elimination of z gives

$$\left.\begin{aligned} x-0{\cdot}7y+\ 0{\cdot}3\ z+\ 0{\cdot}5u\quad&=0{\cdot}6\\ y+0{\cdot}63636z-\ 0{\cdot}27273u&=-\ 0{\cdot}72727\\ z-\ 0{\cdot}02247u&=-\ 7{\cdot}02247\\ 10{\cdot}37079u&=\ 10{\cdot}37079 \end{aligned}\right\}. \qquad \ldots(4)$$

Thus the final solution is given by :

$$u=1,\ z=-7,\ y=4 \text{ and } x=5.$$

Ex. 4. *Solve the system of equations given in Ex. 3. by Jordan's method.*

Sol. First, we eliminate x, using the first equation :

$$\left.\begin{aligned} x-0{\cdot}7y+0{\cdot}3z+0{\cdot}5u&=0{\cdot}6\\ 3{\cdot}8y+0{\cdot}8z-\quad u&=8{\cdot}6\\ 3{\cdot}1y+3{\cdot}1z+9{\cdot}5u&=0{\cdot}2\\ -5{\cdot}5y-3{\cdot}5z+1{\cdot}5u&=\ 4 \end{aligned}\right\}. \qquad \ldots(1)$$

Now we permute the second and fourth equations and then y is eliminated from the first, third and fourth equations by using the second equation. We get the new system as :

$$\left.\begin{aligned} x\qquad +0{\cdot}74545z+\ 0{\cdot}30909u&=\ \ 0{\cdot}09091\\ y+0{\cdot}63636z-\ 0{\cdot}27273u&=-\ 0{\cdot}72727\\ -1{\cdot}61818z+\ 0{\cdot}03636u&=\ 11{\cdot}36364\\ 1{\cdot}12727z+10{\cdot}34545u&=\ \ 2{\cdot}45455 \end{aligned}\right\}. \qquad \ldots(2)$$

Further elimination of z from the first, second and fourth equations, by using the third equation gives :

$$\left.\begin{aligned} x\qquad\qquad +\ 0{\cdot}32584u&=\ \ 5{\cdot}32582\\ y\qquad -\ 0{\cdot}25843u&=\ \ 3{\cdot}74156\\ z\quad -\ 0{\cdot}02447u&=-\ 7{\cdot}02248\\ 10{\cdot}37078u&=\ 10{\cdot}37078 \end{aligned}\right\} \qquad \ldots(3)$$

In the last, elimination of u from the first three equations gives : $x=5,\ y=4,\ z=-7,\ u=1.$

Ex. 5. *Apply Jordan's method to solve*

$$\left.\begin{aligned} x+2y+\ z&=\ 8\\ 2x+3y+4z&=20\\ 4x+3y+2z&=16 \end{aligned}\right\}.$$

(Jaipur 1984)

Sol. First, we eliminate x, from the last two equations using the first equation :

$$\left.\begin{aligned} x+2y+\ z&=\ \ 8\\ -\ y+2z&=\ \ 4\\ -5y-2z&=-16 \end{aligned}\right\}. \qquad \ldots(1)$$

Now y is eliminated from the first and third equations, which gives

$$\left.\begin{aligned} x\quad+\ 5z&=16\\ -y+\ 2z&=\ 4\\ 12z&=36 \end{aligned}\right\}. \qquad \ldots(2)$$

Eliminating z from the first two equations with the help of the third equation, we get

$$x=1,\ y=2,\ z=3.$$

Ex. 6. *Solve the system by Crout's method :*

$$\left.\begin{aligned} x_1+\ x_2+\ x_3&=\ 1\\ 3x_1+\ x_2-3x_3&=\ 5\\ x_1-2x_2-5x_3&=10 \end{aligned}\right\}.$$

(Meerut M.Sc. 1989)

Sol. The augmented matrix is given by

$$\begin{bmatrix} 1 & 1 & 1 & 1\\ 3 & 1 & -3 & 5\\ 1 & -2 & -5 & 10 \end{bmatrix}.$$

Let the derived matrix of the augmented matrix be

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & b'_1\\ a'_{21} & a'_{22} & a'_{23} & b'_2\\ a'_{31} & a'_{32} & a'_{33} & b'_3 \end{bmatrix}.$$

Then $a'_{11}=1,\ a'_{21}=3,\ a'_{31}=1$ (first column)

$$a'_{12}=\frac{1}{1}=1,\ a'_{13}=\frac{1}{1}=1,\ b_1'=\frac{1}{1}=1$$

(remaining first row)

$$a'_{22}=1-1.3=-2,\ a'_{32}=-2-1.1=-3$$

(remaining second column)

$$a'_{23}=\frac{-3-1.3}{-2}=3,\ b'_2=\frac{5-1.3}{-2}=-1$$

(remaining second row)

$$a'_{33} = -5-(3).(-3)-(1).(1) = 3$$

(remaining third column)

$$b'_3 = \frac{10-(-1).(-3)-(1).(1)}{3} = 2$$

(remaining third row)

$\therefore$ the derived matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & -2 & 3 & -1 \\ 1 & -3 & 3 & 2 \end{bmatrix}.$$

Thus $x_3 = b'_3 = 2$, $x_2 = b'_2 - a'_{23} x_3 = -7$

and $x_1 = b'_1 - a'_{13} x_3 - a'_{12} x_2 = 6.$

Ex. 7. *Solve the equations*

$$\begin{aligned} 2x+3y+z &= 9 \\ x+2y+3z &= 6 \\ 3x+y+2z &= 8 \end{aligned}$$

by the factorization method. **(Meerut B.Sc. 1993, M.Sc. 89, 90; Udaipur 1986)**

Sol. Here $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$.

Let $A = LU$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

Then $u_{11} = 2$, $u_{12} = 3$, $u_{13} = 1$,

$l_{21} u_{11} = 1 \Rightarrow l_{21} = 1/2$, $l_{31} u_{11} = 3 \Rightarrow l_{31} = 3/2$,

$l_{21} u_{12} + u_{22} = 2 \Rightarrow u_{22} = 1/2$, $l_{21} u_{13} + u_{23} = 3 \Rightarrow u_{23} = 5/2$,

$l_{31} u_{12} + l_{32} u_{22} = 1 \Rightarrow l_{32} = -7$,

$l_{31} u_{13} + l_{32} u_{23} + u_{33} = 2 \Rightarrow u_{33} = 18.$

Thus

L U

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$$

Now the given system can be written as

L U X B

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}.$$

L Y B

LUX = B

$\boxed{\text{LUX} = \text{Y}}$

UX = B

This system is equivalent to

$$\begin{aligned} y_1 &= 9 \\ \tfrac{1}{2} y_1 + y_2 &= 6 \quad \text{or} \quad y_2 = \tfrac{3}{2} \\ \tfrac{3}{2} y_1 - 7 y_2 + y_3 &= 8 \quad \text{or} \quad y_3 = 5. \end{aligned}$$

Now the solution of the original system is given by

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}.$$

Solving this system by back substitution, we get

$$x = 35/18,\ y = 29/18 \text{ and } z = 5/18.$$

Ex. 8. *Solve the following system by iteration methods :*

$$\left.\begin{aligned} 27x + 6y - z &= 85 \\ 6x + 15y + 2z &= 72 \\ x + y + 54z &= 110 \end{aligned}\right\} \quad \ldots(1)$$

(Meerut M.Sc. 1988, 89; Bombay 85)

Sol. Since the requirement for iteration is satisfied by these equations, we solve each equation for the unknown having the largest coefficient and the new equations are

$$x = \frac{1}{27}(85 - 6y + z) \quad \ldots(2)$$

$$y = \frac{1}{15}(72 - 6x - 2z) \quad \ldots(3)$$

$$z = \frac{1}{54}(110 - x - y). \quad \ldots(4)$$

Gauss-Seidel iteration method. Starting with $y = 0$, $z = 0$, we get $x = x^{(1)} = 85/27 = 3{\cdot}15$ = first approximation.

Putting $x = 3{\cdot}15$, $z = 0$ in (3), we get

$$y^{(1)} = \frac{1}{15}(72 - 18{\cdot}90) = 3{\cdot}54 = \text{first approx.}$$

Now putting $x = 3{\cdot}15$, $y = 3{\cdot}54$ in (4), we get

$$z^{(1)} = \frac{1}{54}(110 - 3{\cdot}15 - 3{\cdot}54) = 1{\cdot}91 = \text{first approx.}$$

Now we obtain the second approximations

$$x^{(2)} = \frac{1}{27}(85 - 6y^{(1)} + z^{(1)}) = \frac{1}{27}(85 - 21{\cdot}24 + 1{\cdot}91) = 2{\cdot}43,$$

$$y^{(2)} = \frac{1}{15}(72 - 6x^{(2)} - 2z^{(1)}) = \frac{1}{15}(72 - 14{\cdot}58 - 3{\cdot}82) = 3{\cdot}57,$$

$$z^{(2)} = \frac{1}{54}(110 - x^{(2)} - y^{(2)}) = \frac{1}{54}(110 - 2{\cdot}43 - 3{\cdot}57) = 1{\cdot}92.$$

Similarly, we get

$$x^{(3)}=\frac{1}{27}(85-6y^{(2)}+z^{(2)})=\frac{1}{27}(85-21\cdot42+1\cdot92)=2\cdot426,$$

$$y^{(3)}=\frac{1}{15}(72-6x^{(3)}-2z^{(2)})=3\cdot572,$$

$$z^{(3)}=\frac{1}{54}(110-x^{(3)}-y^{(3)})=1\cdot926.$$

Since $x^{(3)}$, $y^{(3)}$, $z^{(3)}$ are sufficiently close to $x^{(2)}$, $y^{(2)}$, $z^{(2)}$ respectively, so the values 2·426, 3·572, 1·926 can be taken as the solution of the given system.

Jacobi Iterative method. Starting with $x=0, y=0, z=0$, we get

$$x^{(1)}=\frac{85}{27}=3\cdot15,\ y^{(1)}=\frac{72}{15}=4\cdot8,\ z^{(1)}=\frac{110}{54}=2\cdot04.$$

These are the first approximations. We proceed to obtain the second approximations as follows:

$$x^{(2)}=\frac{1}{27}(85-6y^{(1)}+z^{(1)})=\frac{1}{27}(85-28\cdot8+2\cdot04)=2\cdot16,$$

$$y^{(2)}=\frac{1}{15}(72-6x^{(1)}-2z^{(1)})=\frac{1}{15}(72-18\cdot9-4\cdot08)=3\cdot27,$$

$$z^{(2)}=\frac{1}{54}(110-x^{(1)}-y^{(1)})=\frac{1}{54}(110-3\cdot15-4\cdot8)=1\cdot89.$$

Similarly the next approximations are given by

$$x^{(3)}=\frac{1}{27}(85-6y^{(2)}-z^{(2)}),\ y^{(3)}=\frac{1}{15}(72-6x^{(2)}-2z^{(2)}),$$

$$z^{(3)}=\frac{1}{54}(110-x^{(2)}-y^{(2)}).$$

Continuing in this way further iterations can be obtained.

Exercises 14

1. Solve the following equations for x, y, z and w, with the help of matrices:

$$\left.\begin{aligned} x-3y+z&=a\\ 2x+y-w&=b\\ 3x-2y-z-2w&=c\\ 4x-y+3w&=d. \end{aligned}\right\}.$$

2. Compute the inverse of the matrix

$$\begin{bmatrix} 3 & 2 & 4\\ 2 & 1 & 1\\ 1 & 3 & 5 \end{bmatrix}.$$

and use the result to solve the system of equations:

$$\begin{aligned} 3x+2y+4z&=7\\ 2x+y+z&=4\\ x+3y+5z&=2. \end{aligned}$$

3. Solve the system by Cramer's rule:

$$\left.\begin{aligned} 2x_1+3x_2+x_3&=9\\ x_1+2x_2+3x_3&=6\\ 3x_1+x_2+2x_3&=8. \end{aligned}\right\}$$

4. Solve the system

$$\left.\begin{aligned} x+\tfrac{1}{2}y+\tfrac{1}{3}z&=1\\ \tfrac{1}{2}x+\tfrac{1}{3}y+\tfrac{1}{4}z&=0\\ \tfrac{1}{3}x+\tfrac{1}{4}y+\tfrac{1}{5}z&=0 \end{aligned}\right\}$$

by Gauss's elimination method.

5. Factorize the matrix

$$A=\begin{bmatrix} 5 & -2 & 1\\ 7 & 1 & -5\\ 3 & 7 & 4 \end{bmatrix}$$

into the form LU, where L is unit lower triangular matrix and U is upper triangular and hence solve the system of equations

$$\begin{aligned} 5x-2y+z&=4\\ 7x+y-5z&=8\\ 3x+7y+4z&=10. \end{aligned}$$

6. Solve the above system of equations of Exercise 5 by

(i) Gaussian elimination method (ii) Gauss-Jordan method.

7. Solve the system by Crout's method

$$\left.\begin{aligned} 2x-6y+8z&=24\\ 5x+4y-3z&=2\\ 3x+y+2z&=16. \end{aligned}\right\}$$ **(Meerut B.Sc. 1994)**

8. Solve the system of equations of Ex. 3. by Crout's method.

9. Solve the system of equations

$2x-3y+10z=3,\ -x+4y+2z=20,\ 5x+2y+z=-12$

by factorization method.

10. Solve the following systems

(i)
$$\begin{aligned} 10x+2y+z&=9\\ 2x+20y-2z&=-44\\ -2x+3y+10z&=22 \end{aligned}$$

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Matrix Inversion

§ 1. Introduction.

Suppose A is a square non-singular matrix of order n ($|A|\neq 0$). Then a square matrix B of order n is said to be the inverse of A, if $AB=BA=I_n$, where I_n is the unit matrix of order n. Inverse of A is denoted by A^{-1} and it satisfies the relation $A^{-1}A=I_n=AA^{-1}$. Inverse of a matrix is always unique. Also we note that a matrix A is invertible iff it is non-singular *i.e.* det $A\neq 0$. Thus a singular matrix (for which det $A=0$) has no inverse. By computing the adjoint of a matrix, we can find its inverse. But this requires a lot of calculation work. In the present chapter we shall discuss some methods for finding A^{-1} which require less computational work.

§ 2. Gauss elimination method. (Meerut M.Sc. 1990)

In this method we take an identity matrix of the same order as that of A and place it with A. Then our aim is to convert A into an upper triangular matrix by the procedure discussed in the chapter 14 in Gauss elimination method. This upper triangular matrix is then also transformed into an identity matrix by applying row transformations. All the row transformations performed on A are also performed on the identity matrix placed adjacent to A. In the end this adjacent matrix gives the inverse of A when A is transformed into an identity matrix. In order to increase the accuracy, it is necessary to fix the largest element as the pivot element by performing row transformations only.

§ 3. Gauss-Jordan method.

It is similar to the Gauss elimination method. We put an identity matrix by the side of A and convert the matrix A into an identity matrix. The only difference is that the matrix A is directly converted into the identity matrix. Here the intermediate triangular matrix is not required.

§ 4. Crout's method.

In this method we factorize the given matrix as

$$A=LU$$

where L is a lower triangular matrix and U is an upper triangular matrix with diagonal elements unity.

Now $\quad A=LU \Rightarrow A^{-1}=(LU)^{-1}=U^{-1}L^{-1}.$

The inverses of L and U can be obtained easily. Let A be a matrix of order 3. Then

$$L=\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U=\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $A=LU$, we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}=\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

or $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}=\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}.$$

Equating the corresponding elements, we obtain

$$l_{11}=a_{11},\quad l_{21}=a_{21};\quad l_{31}=a_{31}, \qquad \text{...(1)}$$

$$l_{11}u_{12}=a_{12},\quad l_{11}u_{13}=a_{13} \qquad \text{...(2)}$$

$$l_{21}u_{12}+l_{22}=a_{22},\quad l_{31}u_{12}+l_{32}=a_{32} \qquad \text{...(3)}$$

$$l_{21}u_{13}+l_{22}u_{23}=a_{23} \qquad \text{...(4)}$$

$$l_{31}u_{13}+l_{32}u_{23}+l_{33}=a_{33}. \qquad \text{...(5)}$$

From (1) it is clear that the elements of the first column of the matrix L are the same as those of A.

From (2), we get

$$u_{12}=a_{12}/l_{11}=a_{12}/a_{11},\ u_{13}=a_{13}/a_{11}. \qquad \text{...(6)}$$

From (3), we get

$$l_{22}=a_{22}-l_{21}u_{12},\quad l_{32}=a_{32}-l_{31}u_{12}. \qquad \text{...(7)}$$

From (4), we get

$$u_{23}=(a_{23}-l_{21}u_{13})/l_{22}. \qquad \text{...(8)}$$

From (5), we get

$$l_{33}=a_{33}-l_{31}u_{13}-l_{32}u_{23}. \qquad \text{...(9)}$$

Thus we can determine the matrices L and U completely.

To find L^{-1} and U^{-1}. Let $L^{-1}=X$ where X is a lower triangular matrix.

Then we have $\qquad LX=I$

or $$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying the matrices on the L.H.S. and then equating the corresponding elements, we have

$$l_{11}x_{11}=1,\quad l_{22}x_{22}=1,\quad l_{33}x_{33}=1 \qquad \text{...(10)}$$

and $$\left.\begin{matrix} l_{21}x_{11}+l_{22}x_{21}=0,\quad l_{31}x_{11}+l_{32}x_{21}+l_{33}x_{31}=0 \\ l_{32}x_{22}+l_{33}x_{32}=0. \end{matrix}\right\} \qquad \text{...(11)}$$

Relation (10) gives $x_{11}=1/l_{11}$, $x_{22}=1/l_{22}$, $x_{33}=1/l_{33}$.

Relation (11) gives $x_{21}=-(l_{21}x_{11})/l_{22}$,

$$x_{31}=-\frac{l_{31}x_{11}+l_{32}x_{21}}{l_{33}} \quad \text{and} \quad x_{32}=-\frac{l_{32}x_{22}}{l_{33}}$$

Thus $X=L^{-1}$ is determined completely.

To find the inverse of U, we use the following method. Let V be an upper triangular matrix and Y be an upper triangular matrix which is the inverse of V. Then $YV=I$

$$\Rightarrow \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & 0 & y_{33} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ 0 & v_{22} & v_{23} \\ 0 & 0 & v_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying the matrices on the L.H.S. and then equating the corresponding elements, we have

$$y_{11}v_{11}=1,\ y_{22}v_{22}=1,\ y_{33}v_{33}=1, \qquad ...(12)$$

and
$$\left.\begin{aligned} y_{11}v_{12}+y_{12}v_{22}=0,\ y_{11}v_{13}+y_{12}v_{23}+y_{13}v_{33}=0 \\ y_{22}v_{23}+y_{23}v_{33}=0. \end{aligned}\right\} \qquad ...(13)$$

From (12), we get

$$y_{11}=1/v_{11},\ y_{22}=1/v_{22},\ y_{33}=1/v_{33}.$$

From (13), we get

$$y_{12}=-\frac{y_{11}v_{12}}{v_{22}},\ y_{13}=-\frac{y_{11}v_{13}+y_{12}v_{23}}{v_{33}},\ y_{23}=-\frac{y_{22}v_{23}}{v_{33}}.$$

Thus we can determine $Y=V^{-1}$ completely.

§ 5. Doolittle method.

In this method, we decompose the given matrix A as $A=LU$, where

$$L=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad \text{and} \quad U=\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

We notice that in Crout's method the diagonal elements of U were taken as unity while in this method we take the diagonal elements of L to be unity. Now repeating the procedure of the previous method, we can find the inverse of A.

§ 6. Choleski' method. (Meerut B.Sc. 1993; Agra 87)

Suppose A is a symmetric square matrix *i.e.* $A=A'$. Here we decompose the matrix A as $A=TT'$ where T is a lower triangular matrix and T' is its transpose. Then

$$A^{-1}=(TT')^{-1}=(T')^{-1}\,T^{-1}=(T^{-1})'\,T^{-1}.$$

Thus, here we have to obtain the inverse of a lower triangular matrix.

Let A be a matrix of order 3. Then $A=TT'$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} t_{11} & 0 & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} t_{11} & t_{21} & t_{31} \\ 0 & t_{22} & t_{32} \\ 0 & 0 & t_{33} \end{bmatrix}.$$

Multiplying the matrices on the R.H.S. and then equating the corresponding elements, we get

$$\begin{aligned} t^2_{11}=a_{11} &\Rightarrow t_{11}=\sqrt{a_{11}} \\ t_{11}t_{21}=a_{12} &\Rightarrow t_{21}=a_{12}/t_{11} \\ t_{11}t_{31}=a_{13} &\Rightarrow t_{31}=a_{13}/t_{11} \\ t^2_{21}+t^2_{22}=a_{22} &\Rightarrow t_{22}=\sqrt{(a_{22}-t^2_{21})} \\ t_{21}t_{31}+t_{22}t_{32}=a_{23} &\Rightarrow t_{32}=(a_{23}-t_{21}t_{31})/t_{22} \\ t^2_{31}+t^2_{32}+t^2_{33}=a_{33} &\Rightarrow t_{33}=\sqrt{(a_{33}-t^2_{31}-t^2_{32})}. \end{aligned}$$

Thus T is determined completely.

Since the diagonal elements are to be determined by taking square roots, it may give some imaginary elements of T.

In general, the formulae for determining T are

$$t_{ii}=\sqrt{\left(a_{ii}-\sum_{k=1}^{i-1} t^2_{ik}\right)},\ i=1,\ldots,n$$

$$t_{ij}=\left(a_{ij}-\sum_{k=1}^{j-1} t_{ik}t_{jk}\right)\Big/ t_{jj},\quad \begin{matrix} i=2,\ldots,n \\ j<i \end{matrix}$$

$$t_{ij}=0,\quad i<j. \qquad \text{(Agra 87)}$$

§ 7. The Escalator method.

This method uses the fact that if the inverse of a matrix A_n of order n is known, then we can easily obtain the inverse of the matrix A_{n+1}, where one extra row has been added at the bottom and one extra column to the right.

Let
$$A=\left[\begin{array}{c:c} A_1 & A_2 \\ \hdashline A_3^T & a \end{array}\right] \quad \text{and} \quad A^{-1}=\left[\begin{array}{c:c} X_1 & X_2 \\ \hdashline X_3^T & x \end{array}\right]$$

where A_2 and X_2 are column vectors, A_3^T and X_3^T row vectors, and a and x ordinary numbers.

Also we assume A_1^{-1} to be known.

From $AA^{-1}=I_{n+1}$, we get

$$A_1X_1+A_2X_3^T=I, \qquad ...(1)$$
$$A_1X_2+A_2x=0, \qquad ...(2)$$
$$A_3^TX_1+aX_3^T=0, \qquad ...(3)$$
$$A_3^TX_2+ax=1. \qquad ...(4)$$

From (2), we get $\quad X_2=-A_1^{-1}A_2x$

and using this (4) gives $\quad (a-A_3^T\,A_1^{-1}A_2)\,x=1.$

Hence x and then also X_2 can be obtained. Further we get from (1): $\quad X_1=A_1^{-1}\,(I-A_2X_3^T)$ and using this (3) gives

$$(a-A_3^T\,A_1^{-1}A_2)\,X_3^T=-A_3^T\,A_1^{-1}.$$

and hence X_3^r is determined. In the end, we get X_1 from (1) and hence A^{-1} has been computed.

Thus we can increase the dimension number successively.

§ 8. Iterative method.

Suppose we want to compute A^{-1} and we know that B is an approximate inverse of A. Let $AB-I=E$, so that we get

$$AB=I+E \Rightarrow (AB)^{-1}=(I+E)^{-1}$$
$$\Rightarrow B^{-1}A^{-1}=(I+E)^{-1}$$
$$\Rightarrow A^{-1}=B\,(I+E)^{-1}=B\,(I-E+E^2-\ldots)$$

if the series converges.

Thus we can find further approximations of A^{-1}, *i.e.* of B by using the relation

$$A^{-1}=B\,(I-E+E^2-\ldots).$$

§ 9. The Eigenvalue Problem.

Let A be a square matrix of order n with elements a_{ij}. If there exists a number λ and a non-zero vector $\mathbf{u}$ such that

$$A\mathbf{u}=\lambda\mathbf{u}, \qquad \ldots(1)$$

then λ is called an *eigenvalue* or *latent root* or *characteristic value* and $\mathbf{u}$ is called the corresponding *eigenvector* or *characteristic vector* of the matrix A.

The matrix equation (1) can be written as

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{bmatrix} \mathbf{u}=\mathbf{0}. \qquad \ldots(2)$$

It represents a set of homogeneous linear equations. A non-trivial solution exists only when $|A-\lambda I|=0$. Hence, we have

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix}=0. \qquad \ldots(3)$$

The equation (3) is a polynomial equation of degree n in λ. The polynomial on the L.H.S. of (3) is called the *characteristic polynomial* of A and the equation (3) is called the *characteristic equation* of the matrix A.

Let the roots of (3) be given by λ_i $(i=1, 2, \ldots, n)$, then for each value of λ_i there exists a corresponding $\mathbf{u}_i$ such that

$$A\mathbf{u}_i=\lambda_i\mathbf{u}_i.$$

The eigenvalues λ_i may be either all different or repeated.

§ 10. Iterative method for dominant latent root. (Meerut 85, 94)

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be latent roots of the matrix $A=[a_{ij}]_{n\times n}$ such that $|\lambda_1| > |\lambda_2| \geqslant |\lambda_3| \geqslant \ldots \geqslant |\lambda_n|$. Then λ_1 is called the *dominant* latent root.

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be the corresponding eigenvectors. Then any vector $\mathbf{Y}_0$ can be expressed as a linear combination of eigenvectors *i.e.*, $\mathbf{Y}_0=x_1\mathbf{u}_1+x_2\mathbf{u}_2+\ldots+x_n\mathbf{u}_n$, where $x_1, \ldots\ldots, x_n$ are scalars.

Pre-multiplying by A, we have

$$\mathbf{Y}_1=A\mathbf{Y}_0=x_1A\mathbf{u}_1+x_2A\mathbf{u}_2+\ldots+x_nA\mathbf{u}_n$$
$$=x_1\lambda_1\mathbf{u}_1+x_2\lambda_2\mathbf{u}_2+\ldots+x_n\lambda_n\mathbf{u}_n.$$

Pre-multiplying again by A, we get

$$\mathbf{Y}_2=A\mathbf{Y}_1=x_1\lambda_1^2\mathbf{u}_1+x_2\lambda_2^2\mathbf{u}_2+\ldots+x_n\lambda_n^2\mathbf{u}_n.$$

Continuing the process of pre-multiplication by A, we get

$$\mathbf{Y}_r=A^r\mathbf{Y}_0=x_1\lambda_1^r\mathbf{u}_1+x_2\lambda_2^r\mathbf{u}_2+\ldots+x_n\lambda_n^r\mathbf{u}_n$$
$$=x_1\lambda_1^r\mathbf{u}_1+\lambda_1^r\sum_{i=2}^{n} x_i\,(\lambda_i/\lambda_1)^r\mathbf{u}_i.$$

Now $(\lambda_i/\lambda_1)^r\to 0$ when $r\to\infty$ because $\lambda_i<\lambda_1 \Rightarrow (\lambda_i/\lambda_1)<1$. So we get

$\mathbf{Y}_r=(x_1\lambda_1^r)\,\mathbf{u}_1=$ a scalar multiple of the eigenvector $\mathbf{u}_1$ corresponding to the eigenvalue λ_1.

This shows that $\mathbf{Y}_r$ is also an eigenvector corresponding to the largest eigenvalue λ_1.

Practically, we take

$$\mathbf{Y}_1=A\mathbf{Y}_0=k_1\mathbf{Z}_1 \quad\text{or}\quad \mathbf{Z}_1=(1/k_1)\,\mathbf{Y}_1,$$

where we get $\mathbf{Z}_1$ by dividing each element of $\mathbf{Y}_1$ by its numerically largest element.

Repeating the process, we get

$$\mathbf{Y}_{r+1}=A\mathbf{Z}_r=A^{r+1}\mathbf{Y}_0.$$

If we have $\mathbf{Y}_{r+1}=k_{r+1}\mathbf{Z}_{r+1}$, then k_{r+1} is the eigenvalue and $\mathbf{Z}_{r+1}$ is the eigenvector.

This method is also known as power series method. **(Meerut M.Sc. 1988; Rohilkhand 88)**

Solved Examples

Ex. 1. *Find the inverse of* $A=\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ *by Gauss elimination method.* **(Meerut M.Sc. 1989)**

Sol. We have $A=AI$

or
$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

interchanging the first and third rows to bring the maximum element 3 at the place a_{11}

$$\sim\begin{pmatrix}1 & 1/3 & 1/3\\ 1 & 2 & 3\\ 0 & 1 & 2\end{pmatrix}\begin{pmatrix}0 & 0 & 1/3\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}, \text{ dividing the first row by 3}$$

$$\sim\begin{pmatrix}1 & 1/3 & 1/3\\ 0 & 5/3 & 8/3\\ 0 & 1 & 2\end{pmatrix}\begin{pmatrix}0 & 0 & 1/3\\ 0 & 1 & -1/3\\ 1 & 0 & 0\end{pmatrix}; R_2-R_1$$

$$\sim\begin{pmatrix}1 & 1/3 & 1/3\\ 0 & 1 & 8/5\\ 0 & 1 & 2\end{pmatrix}\begin{pmatrix}0 & 0 & 1/3\\ 0 & 3/5 & -1/5\\ 1 & 0 & 0\end{pmatrix}, \frac{3}{5}R_2$$

$$\sim\begin{pmatrix}1 & 1/3 & 1/3\\ 0 & 1 & 8/5\\ 0 & 0 & 2/5\end{pmatrix}\begin{pmatrix}0 & 0 & 1/3\\ 0 & 3/5 & -1/5\\ 1 & -3/5 & 1/5\end{pmatrix}, R_3-R_2$$

$$\sim\begin{pmatrix}1 & 1/3 & 1/3\\ 0 & 1 & 8/5\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 1/3\\ 0 & 3/5 & -1/5\\ 5/2 & -3/2 & 1/2\end{pmatrix}, \frac{5}{2}R_3.$$

Thus, A has been converted into an upper triangular matrix. Now by making row transformations, A will be converted into an identity matrix.

$$\sim\begin{pmatrix}1 & 1/3 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-5/6 & 1/2 & 1/6\\ -4 & 3 & -1\\ 5/2 & -3/2 & 1/2\end{pmatrix}, R_1-\frac{1}{3}R_3$$

$$R_2-\frac{8}{5}R_3$$

$$\sim\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1/2 & -1/2 & 1/2\\ -4 & 3 & -1\\ 5/2 & -3/2 & 1/2\end{pmatrix}, R_1-\frac{1}{3}R_2.$$

$$\text{Hence } A^{-1}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\ -4 & 3 & -1\\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2}\end{pmatrix}$$

Ex. 2. *Find the inverse of* $A=\begin{pmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{pmatrix}$ *by Gauss-Jordan method.* **(Meerut B.Sc. 1991, 91S)**

Sol. We have $\begin{pmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

$$\sim\begin{pmatrix}1 & 3 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}, \begin{matrix}R_2-R_1\\ R_3-R_1\end{matrix}$$

$$\sim\begin{pmatrix}1 & 0 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}4 & -3 & 0\\ -1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}, R_1-3R_2$$

$$\sim\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}7 & -3 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}, R_1-3R_3.$$

$$\text{Hence } A^{-1}=\begin{bmatrix}7 & -3 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$$

Ex. 3. *Find the inverse of* $A=\begin{bmatrix}2 & -2 & 4\\ 2 & 3 & 2\\ -1 & 1 & -1\end{bmatrix}$ *by Crout's method.* **(Rohilkhand 88)**

Sol. Let $A=LU$. Then $A^{-1}=U^{-1}L^{-1}$, where L is a lower triangular matrix and U is an upper triangular matrix with diagonal elements as unity.

Let $L=\begin{bmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}$ and $U=\begin{bmatrix}1 & u_{12} & u_{13}\\ 0 & 1 & u_{23}\\ 0 & 0 & 1\end{bmatrix}$

Now $A=LU$ gives

$$\begin{bmatrix}2 & -2 & 4\\ 2 & 3 & 2\\ -1 & 1 & -1\end{bmatrix}=\begin{bmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}\begin{bmatrix}1 & u_{12} & u_{13}\\ 0 & 1 & u_{23}\\ 0 & 0 & 1\end{bmatrix}$$

Multiplying the matrices on the R.H.S. and then equating the corresponding elements, we get

$$l_{11}=2,\ l_{21}=2,\ l_{31}=-1,\ l_{22}=5,\ l_{32}=0,\ l_{33}=1$$

and $$u_{12}=-1,\ u_{13}=2,\ u_{23}=-2/5.$$

Thus we have

$$L=\begin{bmatrix}2 & 0 & 0\\ 2 & 5 & 0\\ -1 & 0 & 1\end{bmatrix} \text{ and } U=\begin{bmatrix}1 & -1 & 2\\ 0 & 1 & -2/5\\ 0 & 0 & 1\end{bmatrix}.$$

To find L^{-1}. Let $L^{-1}=X$, where X is also a lower triangular matrix. Then we have $LX=I$

$$i.e.\ \begin{bmatrix}2 & 0 & 0\\ 2 & 5 & 0\\ -1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_{11} & 0 & 0\\ x_{21} & x_{22} & 0\\ x_{31} & x_{32} & x_{33}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

By equality principle of matrices, we get

$$x_{11}=1/2,\ x_{21}=-1/5,\ x_{22}=1/5,\ x_{31}=1/2,\ x_{32}=0,\ x_{33}=1.$$

$$\text{Hence } L^{-1}=X=\begin{bmatrix}1/2 & 0 & 0\\ -1/5 & 1/5 & 0\\ 1/2 & 0 & 1\end{bmatrix}.$$

To find U^{-1}. Let $U^{-1}=Y$, where Y is an upper triangular matrix with diagonal elements as unity. Then $UY=I$

$$\Rightarrow\begin{bmatrix}1 & -1 & 2\\ 0 & 1 & -2/5\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & y_{12} & y_{13}\\ 0 & 1 & y_{23}\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Multiplying the matrices on the L.H.S. and then equating the corresponding elements, we get

$$y_{12}=1,\ y_{13}=-8/5,\ y_{23}=2/5.$$

Hence $U^{-1}=Y=\begin{bmatrix}1&1&-8/5\\0&1&2/5\\0&0&1\end{bmatrix}$.

Now $A^{-1}=U^{-1}L^{-1}$

$$=\begin{bmatrix}1&1&-8/5\\0&1&2/5\\0&0&1\end{bmatrix}\begin{bmatrix}1/2&0&0\\-1/5&1/5&0\\1/2&0&1\end{bmatrix}=\begin{bmatrix}-1/2&1/5&-8/5\\0&1/5&2/5\\1/2&0&1\end{bmatrix}.$$

Ex. 4. *Find the inverse of*

$$A=\begin{bmatrix}1&2&6\\2&5&15\\6&15&46\end{bmatrix},$$ *using Choleski's method.* **(Meerut B.Sc. 1991, 93, M.Sc. 89, 91; Baroda 86)**

Sol. Since $A=A'$, so A is a symmetric matrix.

Let $A=TT'$ where T is a lower triangular matrix.

Now $A=TT' \Rightarrow A^{-1}=(T^{-1})'\,T^{-1}$.

Proceeding in the same way as in § 6, we get

$$t_{11}=1,\ t_{21}=2,\ t_{31}=6,\ t_{22}=1,\ t_{32}=3,\ t_{33}=1.$$

Hence the matrix $T=\begin{bmatrix}1&0&0\\2&1&0\\6&3&1\end{bmatrix}$.

To find T^{-1}. Let $T^{-1}=X$ where X is a lower triangular matrix. Then $TX=I$. Now proceeding as in Crout's method, we get

$$x_{11}=1,\ x_{22}=1,\ x_{33}=1,\ x_{21}=-2,\ x_{31}=0,\ x_{32}=-3.$$

Hence $T^{-1}=X=\begin{bmatrix}1&0&0\\-2&1&0\\0&-3&1\end{bmatrix}\Rightarrow (T^{-1})'=\begin{bmatrix}1&-2&0\\0&1&-3\\0&0&1\end{bmatrix}$.

$$\therefore\quad A^{-1}=(T^{-1})'\,T^{-1}=\begin{bmatrix}1&-2&0\\0&1&-3\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&-3&1\end{bmatrix}$$

$$=\begin{bmatrix}5&-2&0\\-2&10&-3\\0&-3&1\end{bmatrix}.$$

Ex. 5. *Find the inverse of* $A=\begin{bmatrix}13&14&6&4\\8&-1&13&9\\6&7&3&2\\9&5&16&11\end{bmatrix}$, *using the escalator method.* **(Meerut M.Sc. 1988)**

Sol. Here $A=\left[\begin{array}{ccc:c}13&14&6&4\\8&-1&13&9\\6&7&3&2\\ \hdashline 9&5&16&11\end{array}\right]=\left[\begin{array}{c:c}A_1&A_2\\ \hdashline A_3^T&a\end{array}\right]$

where $A_1=\begin{bmatrix}13&14&6\\8&-1&13\\6&7&3\end{bmatrix},\ A_2=\begin{bmatrix}4\\9\\2\end{bmatrix},$

$$A_3^T=[9\ 5\ 16] \text{ and } a=11.$$

We have $A_1^{-1}=\dfrac{1}{94}\begin{bmatrix}94&0&-188\\-54&-3&121\\-62&7&125\end{bmatrix}$.

Let $A^{-1}=\left[\begin{array}{c:c}X_1&X_2\\ \hdashline X_3^T&x\end{array}\right]$. Then $AA^{-1}=I$.

Using the scheme of § 7, we get

$$x=-94,\ X_2=\begin{bmatrix}0\\-1\\65\end{bmatrix},\ X_3^T=[-416,\ 97,\ 913],$$

$$X_1=\begin{bmatrix}1&0&-2\\-5&1&11\\287&-67&-630\end{bmatrix}.$$

Finally, we get $A^{-1}=\begin{bmatrix}1&0&-2&0\\-5&1&11&-1\\287&-67&-630&65\\-416&97&913&-94\end{bmatrix}$.

Ex. 6. *Obtain a more accurate inverse of A, given*

$$A=\begin{bmatrix}5&2\\3&-1\end{bmatrix},\ A^{-1}=B=\begin{bmatrix}0{\cdot}1&0{\cdot}2\\0{\cdot}3&-0{\cdot}4\end{bmatrix}$$

Sol. We have $AB=\begin{bmatrix}5&2\\3&-1\end{bmatrix}\begin{bmatrix}0{\cdot}1&0{\cdot}2\\0{\cdot}3&-0{\cdot}4\end{bmatrix}=\begin{bmatrix}1{\cdot}1&{\cdot}2\\0&1\end{bmatrix}$.

Hence $E=AB-I=\begin{bmatrix}1{\cdot}1&{\cdot}2\\0&1\end{bmatrix}-\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}0{\cdot}1&0{\cdot}2\\0&0\end{bmatrix}$

Taking $B_0=B$, the next approximation is given by

$$B(I-E)=\begin{bmatrix}0{\cdot}1&0{\cdot}2\\0{\cdot}3&-0{\cdot}4\end{bmatrix}\begin{bmatrix}{\cdot}9&-{\cdot}2\\0&1\end{bmatrix}=\begin{bmatrix}{\cdot}09&{\cdot}18\\{\cdot}27&-{\cdot}46\end{bmatrix}.$$

Now $E^2=\begin{bmatrix}0{\cdot}1&0{\cdot}2\\0&0\end{bmatrix}\begin{bmatrix}0{\cdot}1&0{\cdot}2\\0&0\end{bmatrix}=\begin{bmatrix}0{\cdot}01&0{\cdot}02\\0&0\end{bmatrix}$.

Hence the next approximation is given by

$$B(I-E+E^2)=\begin{bmatrix}0\cdot1 & 0\cdot2\\ 0\cdot3 & -0\cdot4\end{bmatrix}\begin{bmatrix}0\cdot91 & -0\cdot18\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0\cdot091 & 0\cdot182\\ 0\cdot273 & -0\cdot454\end{bmatrix}$$

Here the exact inverse of A is given by

$$A^{-1}=\begin{bmatrix}\frac{1}{11} & \frac{2}{11}\\ \frac{3}{11} & -\frac{5}{11}\end{bmatrix}.$$

Ex. 7. *Find the eigenvalues and eigenvectors of the matrix*

$$A=\begin{bmatrix}5 & 0 & 1\\ 0 & -2 & 0\\ 1 & 0 & 5\end{bmatrix}.$$ **(Meerut M.Sc. 1990)**

Sol. The characteristic equation of the matrix A is given by $|A-\lambda I|=0$

i.e.
$$\begin{vmatrix}5-\lambda & 0 & 1\\ 0 & -2-\lambda & 0\\ 1 & 0 & 5-\lambda\end{vmatrix}=0.$$

Solving this, we get $\lambda=-2, 4, 6$, which are eigenvalues.

Now we obtain the corresponding eigenvectors.

(i) $\lambda_1=-2$. Let the corresponding eigenvector be

$$\mathbf{u}_1=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}.$$ Then $A\mathbf{u}_1=\lambda_1\mathbf{u}_1$ gives

$$(A-\lambda_1 I)\,\mathbf{u}_1=0 \Rightarrow \begin{bmatrix}7 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

which gives the equations

$$7x_1+x_3=0 \text{ and } x_1+7x_3=0.$$

Solving these, we get $x_1=x_3=0$ with x_2 arbitrary (say k).

Hence the corresponding eigenvectors are given by $\begin{bmatrix}0\\ k\\ 0\end{bmatrix}$.

(ii) $\lambda_2=4$. Let the corresponding eigenvector be

$$\mathbf{u}_2=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}.$$ Then $A\mathbf{u}_2=\lambda_2\mathbf{u}_2 \Rightarrow$

$$(A-\lambda_2 I)\,\mathbf{u}_2=0 \Rightarrow \begin{bmatrix}1 & 0 & 1\\ 0 & -6 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}.$$

The equations are $x_1+x_3=0$ and $-6x_2=0$, from which we get $x_1=-x_3$ and $x_2=0$.

Taking $x_1=k$, $x_3=-k$, the corresponding eigenvectors are given by

$$\begin{bmatrix}k\\ 0\\ -k\end{bmatrix}.$$

(iii) $\lambda_3=6$. Let $\mathbf{u}_3=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ be the required eigenvector.

Then we have $A\mathbf{u}_3=\lambda_3\mathbf{u}_3$ or $(A-\lambda_3 I)\,\mathbf{u}_3=0$

i.e.
$$\begin{bmatrix}-1 & 0 & 1\\ 0 & -8 & 0\\ 1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix},$$

which gives the equations

$$-x_1+x_3=0,\ -8x_2=0 \text{ and } x_1-x_3=0.$$

The solution is $x_1=x_3$ and $x_2=0$.

Taking $x_1=x_3=k$, the corresponding eigenvectors are given by

$$\begin{bmatrix}k\\ 0\\ k\end{bmatrix}.$$

Note. The eigenvector $\mathbf{u}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ is called normalized or orthonormal vector if $x_1^2+x_2^2+x_3^2=1$. The eigenvector corresponding to $\lambda=-2$ is $\begin{bmatrix}0\\ k\\ 0\end{bmatrix}$. For this vector to be orthonormal, we must have $0+k^2+0=1 \Rightarrow k=1$. Hence the corresponding normalized eigenvector is $\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$.

Ex. 8. *Find the dominant latent root and the corresponding eigenvector of the matrix*

$$A=\begin{bmatrix}1 & 6 & 1\\ 1 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}, \textit{ taking } \mathbf{Y}_0=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}.$$ **(Rohilkhand 88)**

Sol. Here the initial eigenvector is $\mathbf{Y}_0=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$. Then we have

$$\mathbf{Y}_1=A\mathbf{Y}_0=\begin{bmatrix}1 & 6 & 1\\ 1 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}=1.\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}=k_1\mathbf{Z}_1.$$

Now $Y_2 = AZ_1 = \begin{bmatrix}1&6&1\\1&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}7\\3\\0\end{bmatrix} = 7\begin{bmatrix}1\\ \cdot 4\\0\end{bmatrix}$

$= k_2\, Z_2.$

Further $Y_3 = A\, Z_2 = \begin{bmatrix}1&6&1\\1&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1\\ \cdot 4\\0\end{bmatrix} = \begin{bmatrix}3\cdot 4\\1\cdot 8\\0\end{bmatrix} = 3\cdot 4\begin{bmatrix}1\\ \cdot 52\\0\end{bmatrix}$

$= (3\cdot 4)\, Z_3$

$Y_4 = AZ_3 = \begin{bmatrix}1&6&1\\1&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1\\ \cdot 52\\0\end{bmatrix} = \begin{bmatrix}4\cdot 12\\2\cdot 04\\0\end{bmatrix} = (4\cdot 12)\begin{bmatrix}1\\ \cdot 49\\0\end{bmatrix}$

$= (4\cdot 12)\, Z_4$

$Y_5 = AZ_4 = \begin{bmatrix}1&6&1\\1&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1\\ \cdot 49\\0\end{bmatrix} = \begin{bmatrix}3\cdot 94\\1\cdot 98\\0\end{bmatrix} = (3\cdot 94)\begin{bmatrix}1\\ \cdot 50\\0\end{bmatrix}$

$= (3\cdot 94)\, Z_5$

$Y_6 = AZ_5 = \begin{bmatrix}1&6&1\\1&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1\\ \cdot 50\\0\end{bmatrix} = \begin{bmatrix}4\\2\\0\end{bmatrix} = 4\begin{bmatrix}1\\ \cdot 5\\0\end{bmatrix} = 4Z_6.$

Hence the largest eigenvalue is 4 and the corresponding eigenvector is $\begin{bmatrix}1\\ \cdot 5\\0\end{bmatrix}$.

Exercises 15

1. Find the inverse of the matrix

$$A = \begin{bmatrix}1 & 1/3 & 1/5\\ 1/3 & 1/5 & 1/7\\ 1/5 & 1/7 & 1/9\end{bmatrix}, \text{ by Jordan's method.}$$

2. Find the inverse of the matrix

$$A = \begin{bmatrix}-2 & 4 & 8\\ -4 & 18 & -16\\ -6 & 2 & -20\end{bmatrix}, \text{ by using}$$

(i) Crout's method (ii) Doolittle method.

3. Using Choleski's method, find the inverse of the matrix

$$A = \begin{bmatrix}1 & 2 & 0\cdot 5 & 1\\ 2 & 5 & 0 & -2\\ 0\cdot 5 & 0 & 2\cdot 25 & 7\cdot 5\\ 1 & -2 & 7\cdot 5 & 27\end{bmatrix}. \quad \text{(Meerut B.Sc. 92)}$$

4. Find the inverse of

$$A = \begin{bmatrix}3 & -1 & 10 & 2\\ 5 & 1 & 20 & 3\\ 9 & 7 & 39 & 4\\ 1 & -2 & 2 & 1\end{bmatrix}$$

by repeated use of the escalator method.

5. Find the inverse of the matrix

$$A = \begin{bmatrix}2 & 1 & 1\\ 3 & 2 & 3\\ 1 & 4 & 9\end{bmatrix}, \text{ using Gauss-elimination method.}$$

6. Obtain a more accurate inverse, given

$$A = \begin{bmatrix}2 & 4\\ 3 & 5\end{bmatrix},\ A^{-1} = B = \begin{bmatrix}-2\cdot 45 & 2\cdot 05\\ 1\cdot 45 & -0\cdot 95\end{bmatrix}. \quad \text{(Agra 87)}$$

7. Find the eigenvalues and the corresponding eigenvectors for the following matrices :

(i) $\begin{bmatrix}1 & 2 & 3\\ 0 & -4 & 2\\ 0 & 0 & 7\end{bmatrix}$; (ii) $\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$.

8. Determine the largest eigenvalue and the corresponding eigenvector of the matrices :

(i) $\begin{bmatrix}10 & -2 & 1\\ -2 & 10 & -2\\ 1 & -2 & 10\end{bmatrix}$; (ii) $\begin{bmatrix}1 & 3 & -1\\ 3 & 2 & 4\\ -1 & 4 & 10\end{bmatrix}$.

Answers

1. $\dfrac{15}{64}\begin{bmatrix}15 & -70 & 63\\ -70 & 588 & -630\\ 63 & -630 & 735\end{bmatrix}$. 2. $\begin{bmatrix}4\cdot 1 & -0\cdot 8 & -1\\ -0\cdot 2 & 0\cdot 1 & 0\\ -1\cdot 25 & 2\cdot 25 & 0\cdot 25\end{bmatrix}$.

3. $\begin{bmatrix}13\cdot 5 & -6 & 2 & -1\cdot 5\\ -6 & 3 & -2 & 1\\ 2 & -2 & 10 & -3\\ -1\cdot 5 & 1 & -3 & 1\end{bmatrix}$

4. $\begin{bmatrix}7 & -3 & 0 & -5\\ 8 & 1 & -2 & -11\\ -5 & 0 & 1 & 6\\ 19 & 5 & -6 & -28\end{bmatrix}$

5. $\begin{bmatrix}-3 & 5/2 & -1/2\\ 12 & -17/2 & 3/2\\ -5 & 7/2 & -1/2\end{bmatrix}$.

7. (i) $\lambda_1 = 1,\ \lambda_2 = -4,\ \lambda_3 = 7$;

$$u_1 = \begin{bmatrix}k\\0\\0\end{bmatrix},\ u_2 = \begin{bmatrix}-2k/5\\k\\0\end{bmatrix},\ u_3 = \begin{bmatrix}37k/66\\2k/11\\k\end{bmatrix}.$$

(ii) $\lambda_1=8$, $\lambda_2=\lambda_3=2$.

Characteristic vectors corresponding to the characteristic root 8 are given by $\begin{bmatrix} 2k \\ -k \\ k \end{bmatrix}$

and the characteristic vectors corresponding to the characteristic root 2 are given by

$$\begin{bmatrix} k_1 \\ k_2 \\ k_2-2k_1 \end{bmatrix}.$$

8. (i) $\lambda=9$; $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, (ii) $11\cdot66$; $\begin{bmatrix} 0\cdot025 \\ 0\cdot422 \\ 1\cdot0 \end{bmatrix}$

16
Curve Fitting

§ 1. Introduction.

Let (x_i, y_i) ; $i=1, 2, \ldots, n$ be a given set of n pairs of values, x being independent variable and y the dependent variable. In curve fitting the general problem is to find, if possible, an analytic expression of the form $y=f(x)$, for the functional relationship suggested by the given data. The curve fitting to a set of numerical data is considered very important both from the point of view of theoretical and practical Statistics. Theoretically, it is useful in the study of correlation and regression *e.g.*, the lines of regression can be regarded as the fitting of linear curves to the given bivariate values. Practically it enables us to represent the relationship between two variables by simple algebraic expressions, *e.g.*, polynomials, exponential or logarithmic functions. It is also used to estimate the values of one variable corresponding to the specified values of the other variable.

§ 2. The method of Least Squares.

Suppose we have m observations (x_1, y_1), (x_2, y_2), ..., (x_m, y_m) of two variables x and y and it is required to fit a curve of the type

$$y=a+bx+cx^2+ \ldots\ldots +kx^n \qquad \ldots(1)$$

to these values. We have to determine the constants a, b, c,..., k, such that it represents the curve of best fit of that degree. In case $m=n$, we get in general a unique set of values satisfying the given system of equations. But if $m > n$, then by substituting the different values of x and y in equation (1) we get m equations and we want to find only n constants. Hence there may be no such solution to satisfy all m equations. So we try to obtain those values of a, b, ..., k which may give the best fit *i.e.* which may satisfy all the equations as nearly as possible.

In such cases the principle of least squares asserts a suitable method.

Putting $x_1, x_2, \ldots, x_m$ for x in eqn. (1), we have

$$\left.\begin{aligned} Y_1&=a+b\,x_1+c\,x_1^2+\ldots+kx_1^n \\ Y_2&=a+b\,x_2+c\,x_2^2+\ldots+kx_2^n \\ &\ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots \\ Y_m&=a+b\,x_m+c\,x_m^2+\ldots+kx_m^n \end{aligned}\right\}$$

The quantities $Y_1, Y_2, \ldots, Y_m$ are the *expected values* of y corresponding to $x=x_1, x_2, \ldots$ respectively. The values $y_1, y_2, \ldots, y_m$ are the *observed values* of y corresponding to the values $x_1, x_2, \ldots, x_m$ of x. In general the expected values are different from the observed values.

Let $R_r=y_r-Y_r$. For different values of r these differences are called *residuals*.

The quantity $R_1^2+R_2^2+\ldots+R_n^2$ provides a measure of the 'goodness of fit' of the curve to the given data.

If it is small, the fit is good, if it is large the fit is bad.

Definition. Among all the curves approximating a given set of points, the curve for which $R_1^2+R_2^2+\ldots+R_n^2$ is minimum, is called the curve of best fitting.

Now set
$$U=R_1^2+R_2^2+\ldots+R_n^2=\Sigma\,(y_r-Y_r)^2$$
$$=\Sigma\,(y_r-a-bx_r-cx_r^2-\ldots)^2.$$

By the principle of least squares the constants $a, b, c, \ldots, k$ are chosen in such a manner that the sum of squares of residuals is minimum. For maximum or minimum of U, we must have
$$\frac{\partial U}{\partial a}=0=\frac{\partial U}{\partial b}=\frac{\partial U}{\partial c}=\ldots=\frac{\partial U}{\partial k}.$$

After simplifying these relations, we have
$$\left.\begin{aligned}
\Sigma y&=ma+b\Sigma x+\ldots+k\,\Sigma x^n\\
\Sigma xy&=a\Sigma x+b\Sigma x^2+\ldots+k\Sigma x^{n+1}\\
\Sigma x^2y&=a\Sigma x^2+b\Sigma x^3+\ldots+k\Sigma x^{n+2}\\
\ldots&\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\
\Sigma x^ny&=a\Sigma x^n+b\Sigma x^{n+1}+\ldots+k\Sigma x^{2n}
\end{aligned}\right\}$$

These equations are $(n+1)$ in number and can be solved as simultaneous equations to give the values of the constants $a, b, c, \ldots, k$. These equations are called the *normal equations.*

If we calculate the second order partial derivatives and the values of $a, b, \ldots, k$ are put in these derivatives, they give a positive value of the function. So U is minimum.

§ 3. Particular cases.

If $n=1$, then the curve to be fitted is a straight line $y=a+bx$ and the normal equations are
$$\left.\begin{aligned}\Sigma y&=ma+b\Sigma x\\ \Sigma xy&=a\Sigma x+b\Sigma x^2\end{aligned}\right\}.$$

If $n=2$, then the curve to be fitted is a second degree parabola $y=a+bx+cx^2$ and the normal equations are
$$\left.\begin{aligned}\Sigma y&=ma+b\Sigma x+c\Sigma x^2\\ \Sigma xy&=a\Sigma x+b\Sigma x^2+c\Sigma x^3\\ \Sigma x^2y&=a\Sigma x^2+b\Sigma x^3+c\Sigma x^4\end{aligned}\right\}.$$

§ 4. Change of Origin.

Let us suppose that the values of x are given to be equidistant at an interval of h *i.e.*, x takes the values (say) $a, a+h, a+2h, \ldots$. If m is odd, *i.e.*, $m=2n+1$ (say), we take
$$u=\frac{x-(\text{middle term})}{\text{interval}}=\frac{x-(a+nh)}{h}.$$

Now u takes the values $-n, -(n-1), \ldots, -1, 0, 1, \ldots, (n-1), n$ and we get $\qquad \Sigma u=0=\Sigma u^3=\Sigma u^5=\ldots$

If m is even *i.e.*, $m=2n$ (say), then two terms are in middle, viz., nth and $(n+1)$th terms which are $a+(n-1)\,h$ and $a+nh$. Here, we take
$$u=\frac{x-(\text{mean of two middle terms})}{\frac{1}{2}\,(\text{interval})}=\frac{x-\{a+\frac{1}{2}\,(2n-1)\,h\}}{\frac{1}{2}h}.$$

The values of u are $-(2n-1), -(2n-3), \ldots, -3, -1, 1, 3, \ldots (2n-3), (2n-1)$ for $x=a, a+h, \ldots, a+(2n-1)\,h$. So we get
$$\Sigma u=0=\Sigma u^3=\ldots$$

§ 5. Conversion of Data to Linear Form.

Sometimes the original data is not in a linear form but it can be reduced to linear form by some simple transformation of variables.

(i) *Fitting of the curve of the type $y=ax^b$ to a set of n points.*

We have
$$\log y=\log a+b\log x$$
or
$$u=A+bv$$
where $u=\log y$, $A=\log a$ and $v=\log x$.

This is a linear equation in u and v. For estimating A and b the normal equations are
$$\Sigma u=nA+b\Sigma v\quad\text{and}\quad\Sigma uv=A\Sigma v+b\Sigma v^2.$$

Solving these equations we get A and b and consequently, we have $a=\text{antilog}\,(A)$.

(ii) *Fitting of the curve of the type $y=ab^x$ to a set of n points.*

We have
$$\log y=\log a+x\log b$$
or
$$u=A+Bx$$
where $u=\log y$, $A=\log a$, $B=\log b$.

This is a linear equation in u and x. For estimating A and B the normal equations are
$$\Sigma u=nA+B\Sigma x\quad\text{and}\quad\Sigma\,xu=A\Sigma x+B\Sigma x^2.$$

These equations can be solved for A and B.

Finally, we get $a=\text{antilog}\,(A)$ and $b=\text{antilog}\,(B)$.

(iii) *Fitting of the curve of the type $y=ae^{bx}$ to a set of n points.*

We have $\log y=\log a+bx \log e=\log a+(b \log e)\, x$

or $u=A+Bx$

where $u=\log y$, $A=\log a$, $B=b \log e$.

This is a linear equation in u and x. The normal equations for estimating A and B are

$$\Sigma u=nA+B\Sigma x \quad \text{and} \quad \Sigma xu=A\Sigma x+B\Sigma x^2.$$

Solving these equations we get A and B and so

$$a=\text{antilog}\,(A) \quad \text{and} \quad b=B/\log e.$$

§ 6. Most Plausible Solution of a System of Linear Equations.

We consider the following set of m equations in n variables $x, y, z, \ldots, t$:

$$\left.\begin{aligned} a_1x+b_1y+c_1z+\ldots+k_1t&=l_1\\ a_2x+b_2y+c_2z+\ldots+k_2t&=l_2\\ \ldots\quad \ldots\quad &\ldots\\ a_mx+b_my+c_mz+\ldots+k_mt&=l_m \end{aligned}\right\} \quad \ldots(1)$$

where $a_i, b_i, \ldots, l_i$; $i=1, 2, \ldots, m$ are constants.

In case $m=n$, the system of equations (1) can be solved uniquely by using algebra. In case $m > n$, it may not be possible to find a unique solution $x, y, z, \ldots, t$ satisfying the system (1). In this case we find the values of $x, y, z, \ldots, t$ which will satisfy the system (1) as nearly as possible.

The principle of least squares asserts that these values are those which minimise the sum of the squares of the 'residuals' or the 'errors'.

Let $E_i=a_ix+b_iy+c_iz+\ldots+k_it-l_i$; $i=1, 2, \ldots, m$ be the residual for the ith equation. Then we have to determine $x, y, z, \ldots, t$ so that

$$U=\sum_{i=1}^{m} E_i^2=\sum_{i=1}^{m} (a_ix+b_iy+c_iz+\ldots+k_it-l_i)^2$$

is minimum.

Using the conditions of maxima and minima of differential calculus, we must have

$$\frac{\partial U}{\partial x}=0=\sum_{i=1}^{m} a_i\,(a_ix+b_iy+c_iz+\ldots+k_it-l_i)$$

$$\frac{\partial U}{\partial y}=0=\sum_{i=1}^{m} b_i\,(a_ix+b_iy+c_iz+\ldots+k_it-l_i)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{\partial U}{\partial t}=0=\sum_{i=1}^{m} k_i\,(a_ix+b_iy+c_iz+\ldots+k_it-l_i).$$

These are called the *normal equations* for $x, y, z, \ldots, t$ respectively. We have n normal equations in n unknowns $x, y, z, \ldots, t$. Solving these we get a unique solution which is the best or the most plausible solution of the system of equations (1).

We observe that *to obtain the normal equation for any variable first multiply each equation by the coefficient of the variable in that equation and then add all the resulting equations.*

Solved Examples

Ex. 1. *Fit a straight line to the following data regarding x as the independent variable* :

x :	0	1	2	3	4
y :	1	1·8	3·3	4·5	6·3

Sol. Let the straight line to be fitted to the given data be $y=a+bx$. Then the normal equations are

$$\Sigma y=ma+b\Sigma x \quad \text{and} \quad \Sigma xy=a\Sigma x+b\Sigma x^2.$$

	x	y	xy	x^2
	0	1	0	0
	1	1·8	1·8	1
	2	3·3	6·6	4
	3	4·5	13·5	9
	4	6·3	25·2	16
Total	10	16·9	47·1	30

In this case $m=5$, $\Sigma x=10$, $\Sigma y=16{\cdot}9$, $\Sigma xy=47{\cdot}1$, $\Sigma x^2=30$.

Substituting these values in the normal equations, we have

$$\left.\begin{aligned} 16{\cdot}9&=5a+10\,b\\ \text{and} \quad 47{\cdot}1&=10a+30b \end{aligned}\right\}$$

Solving the above equations, we obtain $a=0{\cdot}72$, $b=1{\cdot}33$.

Hence the fitted line is $y=0{\cdot}72+1{\cdot}33x$.

Ex. 2. *Fit a second degree parabola to the following data* :

x :	0	1	2	3	4
y :	1	5	10	22	38.

Sol. Let the parabola to be fitted to the given data be $y=a+bx+cx^2$. Then the normal equations are

$$\left.\begin{aligned} \Sigma y&=ma+b\Sigma x+c\Sigma x^2\\ \Sigma xy&=a\Sigma x+b\Sigma x^2+c\Sigma x^3\\ \Sigma x^2y&=a\Sigma x^2+b\Sigma x^3+c\Sigma x^4 \end{aligned}\right\} \quad \ldots(1)$$

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	5	1	1	1	5	5
2	10	4	8	16	20	40
3	22	9	27	81	66	198
4	38	16	64	256	152	608
Total 10	76	30	100	354	243	851

Substituting the values of Σx, Σy, ... etc. in the normal equations (1), we get

$$76=5a+10b+30c$$
$$243=10a+30b+100c$$
$$851=30a+100b+354c.$$

Solving these equations simultaneously, we get

$$a=1{\cdot}43,\ b=0{\cdot}24,\ c=2{\cdot}21.$$

Hence the fitted parabola is

$$y=1{\cdot}43+0{\cdot}24\,x+2{\cdot}21\,x^2.$$

Ex. 3. *The weights of a calf taken at weekly intervals are given below. Fit a straight line using the method of least squares and calculate the average rate of growth per week.*

Age (x) :	1	2	3	4	5	6	7	8	9	10
Weight (y):	52·5	58·7	65·0	70·2	75·4	81·1	87·2	95·5	102·2	108·4

Sol. Let x and y denote the variables age and weight respectively. Here the number of values of x is 10 *i.e.*, even and the values of x are equidistant at an interval of unity, *i.e.*, $h=1$. So we take

$$u=\frac{x-\{(5+6)/2\}}{1/2}=2x-11.$$

Let the least-square line of y on u be $y=a+bu$. Then the normal equations are

$$\Sigma y=na+b\Sigma u \quad \text{and} \quad \Sigma uy=a\Sigma u+b\Sigma u^2.$$

x	y	u	u^2	uy
1	52·5	−9	81	−472·5
2	58·7	−7	49	−410·9
3	65·0	−5	25	−325·0
4	70·2	−3	9	−210·6
5	75·4	−1	1	−75·4
6	81·1	1	1	81·1
7	87·2	3	9	261·6
8	95·5	5	25	477·5
9	102·2	7	49	715·4
10	108·4	9	81	975·6
Total	796·2	0	330	1016·8

Substituting the values in the normal equations, we get

$$796{\cdot}2=10a+0.b \quad \text{and} \quad 1016{\cdot}8=a.0+330\,b.$$

These give $a=79{\cdot}62$ and $b=3{\cdot}08$ (approx.)
Hence the least square line of y on u is

$$y=79{\cdot}62+3{\cdot}08\,u.$$

$\therefore$ the line of best fit of y on x is

$$y=79{\cdot}62+3{\cdot}08\,(2x-11) \quad \text{or} \quad y=45{\cdot}74+6{\cdot}16x.$$

By the line of best fit $y=a+bx$ the weights of the calf after 1, 2, 3, ... weeks are $a+b$, $a+2b$, $a+3b$, ..., respectively. Hence the average rate of growth per week is b units, *i.e.*, 6 16 units.

Ex. 4. *Fit a second degree parabola to the following :*

x :	0	1	2	3	4
y :	1	1·8	1·3	2·5	6·3.

Sol. Here the number of values of x is odd and the values of x are equi-distant. So we take the origin at the middle value 2 for the x series.

Let us take $X=x-2$ and $Y=y$. Let the curve of best fit be $Y=a+bX+cX^2$. Then the normal equations are

$$\left.\begin{aligned}\Sigma Y&=na+b\Sigma X+c\Sigma X^2\\ \Sigma XY&=a\Sigma X+b\Sigma X^2+c\Sigma X^3\\ \Sigma X^2Y&=a\Sigma X^2+b\Sigma X^3+c\Sigma X^4\end{aligned}\right\}$$

x	y	X	Y	XY	X^2	X^2Y	X^3	X^4
0	1	−2	1	−2	4	4	−8	16
1	1·8	−1	1·8	−1·8	1	1·8	−1	1
2	1·3	0	1·3	0	0	0	0	0
3	2·5	1	2·5	2·5	1	2·5	1	1
4	6·3	2	6·3	12·6	4	25·2	8	16
Total		0	12·9	11·3	10	33·5	0	34

Substituting the values from the table in the normal equations, we get

$$12{\cdot}9=5a+10c,\ 11{\cdot}3=10b \quad\text{and}\quad 33{\cdot}5=10a+34c.$$

Solving these equations simultaneously, we get

$$a=1{\cdot}48,\ b=1{\cdot}13,\ c=0{\cdot}55.$$

Hence the curve of best fit is

$$Y=1{\cdot}48+1{\cdot}13X+0{\cdot}55X^2.$$

Changing the origin *i.e.* putting $X=x-2$, $Y=y$, we have

$$y=1{\cdot}48+1{\cdot}13\,(x-2)+0{\cdot}55\,(x-2)^2$$

or $$y=1{\cdot}42-1{\cdot}07x+0{\cdot}55x^2.$$

Ex. 5. *Fit a parabolic curve to the following data.*

x:	1	2	3	4	5
y:	1090	1220	1390	1625	1915

Sol. Changing the origin by $u=x-3$, $v=(y-1450)/5$, let the curve of best fit be $v=a+bu+cu^2$.

Then the normal equations are

$$\left.\begin{aligned}\Sigma v&=na+b\Sigma u+c\Sigma u^2\\ \Sigma uv&=a\Sigma u+b\Sigma u^2+c\Sigma u^3\\ \Sigma u^2v&=a\Sigma u^2+b\Sigma u^3+c\Sigma u^4\end{aligned}\right\}$$

x	y	u	v	u^2	u^4	uv	u^2v
1	1090	−2	−72	4	16	144	−288
2	1220	−1	−46	1	1	46	−46
3	1390	0	−12	0	0	0	0
4	1625	1	35	1	1	35	35
5	1915	2	93	4	16	186	372
Total		0	−2	10	34	411	73

Putting the values in the normal equations, we get

$$-2=5a+10c,\ 411=10b,\ 73=10a+34c.$$

Solving these, we get $a=-11{\cdot}4$, $b=41{\cdot}1$, $c=5{\cdot}5$.

Hence the curve of best fit is

$$v=-11{\cdot}4+41{\cdot}1\,u+5{\cdot}5\,u^2.$$

Changing the origin back *i.e.*, putting $u=x-3$ and $v=(y-1450)/5$, the required curve of best fit is

$$\frac{1}{5}\,(y-1450)=-11{\cdot}4+41{\cdot}1\,(x-3)+5{\cdot}5\,(x-3)^2$$

or $$y=1024+40{\cdot}5\,x+27{\cdot}5\,x^2.$$

Ex. 6. *Fit an exponential curve of the form* $y=ab^x$ *to the following data*:

x:	1	2	3	4	5	6	7	8
y:	1·0	1·2	1·8	2·5	3·6	4·7	6·6	9·1

Sol. $y=ab^x$ takes the form $Y=A+Bx$, where $Y=\log y$, $A=\log a$ and $B=\log b$.

The normal equations are

$$\Sigma Y=nA+B\Sigma x \quad\text{and}\quad \Sigma xY=A\Sigma x+B\Sigma x^2.$$

x	y	$Y=\log y$	xY	x^2
1	1·0	0·0000	0·0000	1
2	1·2	0·0792	0·1584	4
3	1·8	0·2553	0·7659	9
4	2·5	0·3979	1·5916	16
5	3·6	0·5563	2·7815	25
6	4·7	0·6721	4·0326	36
7	6·6	0·8195	5·7365	49
8	9·1	0·9590	7·6720	64
Total 36	30·5	3·7393	22·7385	204

Putting the values in the normal equations, we get

$$3{\cdot}7393=8A+36B \text{ and } 22{\cdot}7385=36A+204B.$$

Solving these, we get $B=0{\cdot}1406$, $A=\bar{1}{\cdot}8336$.

$\therefore$ $b=\text{antilog } B=1{\cdot}38$ and $a=\text{antilog } A=0{\cdot}68$.

Hence the required curve of best fit is $y=(0{\cdot}68)\,(1{\cdot}38)^x$.

Ex. 7. *Find the most plausible values of x, y and z from the following equations*:

$$x-y+2z=3,\ 3x+2y-5z=5,\ 4x+y+4z=21,\ -x+3y+3z=14.$$

Sol. Normal equation for x is

$$(x-y+2z-3)+3\,(3x+2y-5z-5)+4\,(4x+y+4z-21)$$
$$-1.(-x+3y+3z-14)=0$$

or $$27x+6y=88. \qquad \ldots(1)$$

Normal equation for y is

$$-(x-y+2z-3)+2\,(3x+2y-5z-5)+(4x+y+4z-21)$$
$$+3\,(-x+3y+3z-14)=0$$

or $$6x+15y+z=70. \qquad \ldots(2)$$

Normal equation for z is

$$2\,(x-y+2z-3)-5\,(3x+2y-5z-5)+4\,(4x+y+4z-21)$$
$$+3\,(-x+3y+3z-14)=0$$

or $$y+54z=107. \qquad \ldots(3)$$

Solving the equations (1), (2) and (3), we get

$$x=2{\cdot}47,\ y=3{\cdot}55,\ z=1{\cdot}92.$$

Hence the most plausible values of x, y and z are 2·47, 3·55, 1·92 respectively.

Ex. 8. *Derive the least square equations for fitting a curve of the type $y=ax^2+(b/x)$ to a set of n points. Hence fit a curve of this type to the data :*

x :	1	2	3	4
y :	−1·51	0·99	3·88	7·66.

Sol. Let the n points be $(x_1, y_1), (x_2, y_2) \ldots, (x_n, y_n)$. The error of estimate for the ith point (x_i, y_i) is given by

$$E_i=\{y_i-ax_i^2-(b/x_i)\}.$$

By the principle of least squares, we have to find the values of a and b so that the sum of the squares of errors U, viz,

$$U=\sum_{i=1}^{n} E_i^2=\sum_{i=1}^{n}\left(y_i-ax_i^2-\frac{b}{x_i}\right)^2 \text{ is minimum}$$

Hence the normal equations are given by

$$\frac{\partial U}{\partial a}=0 \text{ and } \frac{\partial U}{\partial b}=0$$

or $$\sum_{i=1}^{n} y_i x_i^2=a\sum_{i=1}^{n} x_i^4+b\sum_{i=1}^{n} x_i,\quad \sum_{i=1}^{n}\frac{y_i}{x_i}=a\sum_{i=1}^{n} x_i+b\sum_{i=1}^{n}\frac{1}{x_i^2}.$$

These are the required least square equations.

x	y	x^2	x^4	$\frac{1}{x}$	$\frac{1}{x^2}$	yx^2	$\frac{y}{x}$
1	−1·51	1	1	1	1	−1·51	−1·51
2	0·99	4	16	·5	·25	3·96	·495
3	3·88	9	81	·3333	·1111	34·92	1·2933
4	7·66	16	256	·25	·0625	122·56	2·9150
10			354		1·4236	159·93	1·1933

Putting the values in the above least square equations, we get

$$159{\cdot}93=354a+10b \text{ and } 2{\cdot}1933=10a+1{\cdot}4236b.$$

Solving these, we get $a=0{\cdot}509$, $b=-2{\cdot}04$. Hence the equation of the curve fitted to the given data is $y=0{\cdot}509\ x^2-\dfrac{2{\cdot}04}{x}$.

Exercises 16

1. Find the line of fit to the following data :

x :	0	5	10	15	20	25
y :	12	15	17	22	24	30.

2. Fit a straight line to the following data treating y as the dependent variable :

x :	1	2	3	4	5
y :	2	7	9	10	11.

3. Show that the line of fit to the following data is given by $y=-{\cdot}5x+8$.

x :	6	7	7	8	8	8	9	9	10
y :	5	5	4	5	4	3	4	3	3.

4. Fit a second degree parabola to the following data taking x as the independent variable :

x :	1	2	3	4	5	6	7	8	9
y :	2	6	7	8	10	11	11	10	9.

5. The profit of a certain company in the xth year of its life are given by

x :	1	2	3	4	5
y :	1250	1400	1650	1950	2300.

Taking $u=x-3$ and $v=\dfrac{y-1650}{50}$, show that the parabola of the second degree of v on u is

$$v+0{\cdot}086=5{\cdot}30u+0{\cdot}643u^2,$$

and deduce that the parabola of second degree of y on x is

$$y=1140+72x+32{\cdot}15x^2.$$