Complex variable function

> Analytic function, harmonic function, Cauchy-Riemann equations construction of analytic function, cauchy's Theorem, cauchy's integral formulae, Residue theorem and its application in evalution of real integrals.

Lecture 01

22-09-2022

2.0 → Every int is real number but every real number is not int.

$2+0i$ → also in complex form

2.5 — Real number → Not integer

$2.5+0i$ → Complex number

$2+3i$ → Complex number
- → not real
- → not integer

Complex number

A number of the form $a+ib$ where $i^2 = -1 \Rightarrow i = \sqrt{-1}$ is called complex number.

It is usually denoted by c. $c = a+ib$. Here a is called the real part of $c = Re(c) = a$ and b is called the imaginary part of $c = Im(c) = b$

$(x,y) = x\underline{i} + y\underline{j}$ $\quad (x,y)_{1\times 2}$ matrix

Complex Variable:

Any order pair of two variables (x,y) is called the complex variable. It is usually denoted by z.

$$z = (x,y) = x+iy$$

Here x is called the real part of z

$x = \text{Real of } z = Re(z) = x$ and y is called the imaginary part of z = Imaginary of $c = Im(c) = b$

Complex Conjugate:

Let, $z = x+iy$ be a complex number, the complex conjugate is $\bar{z} = (x+iy) = x-iy$

$ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$z = x+iy$ —— ①

$\bar{z} = x-iy$ —— ②

① + ② ⇒

$x = \frac{1}{2}(z+\bar{z})$

$Re(z) = \frac{1}{2}(z+\bar{z})$

① − ② ⇒

$y = \frac{1}{2i}(z-\bar{z})$

$Im(z) = \frac{1}{2i}(z-\bar{z})$

Absolute of z

$z = x+iy$ $\quad \bar{z} = x-iy$

$|z| = \sqrt{x^2+y^2} = |(x+iy)|$

$|\bar{z}| = |x-iy|$

$= \sqrt{x^2+y^2}$

$$\boxed{|z| = |\bar{z}|}$$

* $z \cdot \bar{z} = |z|^2 = |\bar{z}|^2$

$z \cdot \bar{z} = (x+iy)(x-iy)$

$= x^2 - (iy)^2$

$= x^2 + y^2$

$= |z|^2$

$|z|^2 = z \cdot \bar{z} = |\bar{z}|^2$

$|z| = \sqrt{x^2+y^2}$

$|z| \geq \sqrt{x^2}$

$|z| = \sqrt{x^2}$; $y = 0$

$\Rightarrow |z| \geq \sqrt{x^2}$; $y \neq 0$, $y = 0$

$\Rightarrow |z| \geq x$

$\Rightarrow |z| \geq Re(z)$

$\Rightarrow Re(z) \leq |z|$

$$\boxed{Re(z) \leq |z|}$$

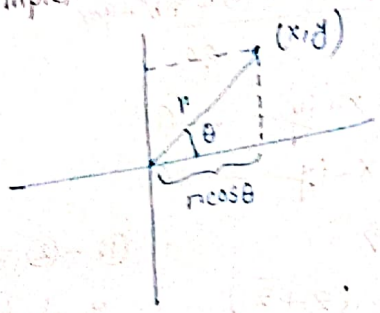
$$\boxed{Im(z) \leq |z|}$$

Polar form (Modulus and amplitude)

$z = x + iy$ be a complex number

Let $x = r\cos\theta$ —— ①

$y = r\sin\theta$ —— ②



①² + ②² ⇒

$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$

* $\boxed{r = \sqrt{x^2+y^2} = |z|}$

② ÷ ① ⇒

$\tan\theta = \frac{y}{x}$

* $\boxed{\theta = \tan^{-1} y/x}$

$z = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

* $\boxed{z = re^{i\theta}}$ [where, $r = \sqrt{x^2+y^2}$; $\theta = \tan^{-1} y/x$]

Express the following complex numbers in polar form

(i) $1+i$ (ii) $-1+i$ (iii) $-1-i$ (iv) $1-i$ (v) $-i$

(i) Sol^n: $1+i$

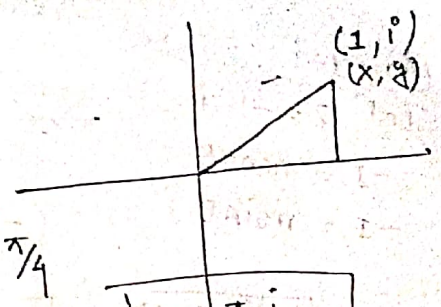
Let, $z = 1+i$

$1 = r\cos\theta$

$1 = r\sin\theta$

$r = \sqrt{x^2+y^2}$

$= \sqrt{1^2+1^2} = \sqrt{2}$



$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{1}{1} = \pi/4$

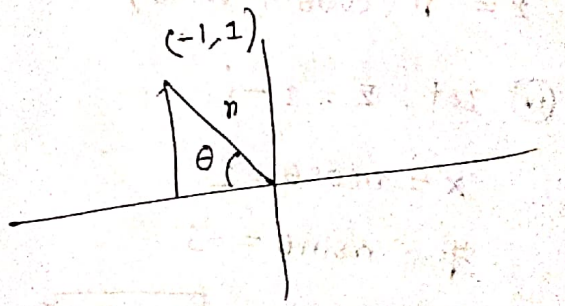
$z = r\cos\theta + ir\sin\theta = re^{i\theta} = \boxed{\sqrt{2}\, e^{\pi/4 i}}$

$= \sqrt{2}\,(\cos\pi/4 + i\sin\pi/4)$

$= \boxed{1+i}$

(ii) $-1+i$

Sol^n: Let, $z = -1+i$



$-1 = r\cos\theta$

$1 = r\sin\theta$

$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\theta = \pi - \tan^{-1}\frac{+1}{1}$

$= \frac{3\pi}{4}$

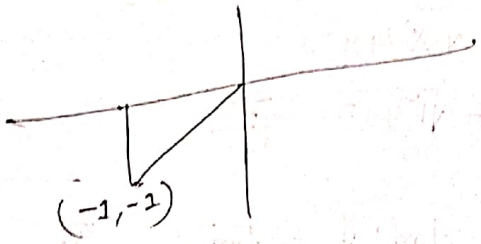
$z = r(\cos\theta + i\sin\theta) = re^{i\theta} = \boxed{\sqrt{2}\, e^{i\,3\pi/4}}$ (Ans)

(iii) $-1-i$

Let $z = -1-i$

$-1 = r\cos\theta$

$-1 = r\sin\theta$



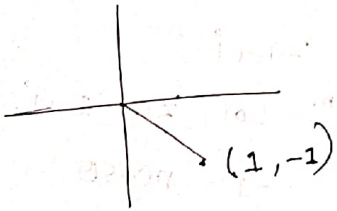
$r = \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$

$\theta = -\pi + \tan^{-1}\frac{1}{1} = \pi + \tan^{-1}\frac{1}{1} = \pi + \pi/4 = \frac{5\pi}{4}$

$= -\pi + \pi/4 = -\frac{3\pi}{4}$

$z = r(\cos\theta + i\sin\theta) = re^{i\theta} = \sqrt{2}\, e^{5\pi/4\, i}$

(iv) Let, $z = 1-i$



$x = r\cos\theta = 1$

$y = r\sin\theta = -1$

$r = \sqrt{x^2+y^2} = \sqrt{1+(-1)^2} = \sqrt{2}$

$\theta = 2\pi - \tan^{-1} x/y = 2\pi - \tan^{-1} 1 = 2\pi - \pi/4$

$= 7\pi/4$

$z = r(\cos\theta + i\sin\theta) = re^{i\theta} = \sqrt{2}\, e^{7\pi/4\, i}$

(v) $-i$

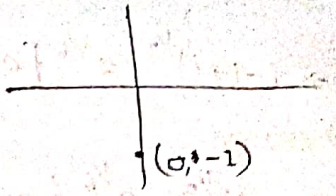
Let $z = -i$

$x = 0 = r\cos\theta$

$y = -1 = r\sin\theta$

$r = \sqrt{0^2+(-1)^2}$

$= 1$



$\theta = 2\pi + \tan\frac{1}{0}$

$= 2\pi + \frac{\pi}{2}$

$= \frac{3\pi}{2}$

$$\boxed{z = r(\cos\theta + i\sin\theta) = re^{i\theta} = e^{3\pi/2\, i}}$$

Lec-02 27/09/22

*important

Find each of the indicated roots and locate them graphically for $(-1+i)^{1/3}$

Sol^n:

Let, $z = -1+i$

$x = -1 = r\cos\theta$

$y = 1 = r\sin\theta$

$r = \sqrt{(-1)^2+(1)^2} = \sqrt{2}$

$\theta = \pi - \tan^{-1}\frac{1}{1} = \pi - \pi/4 = \frac{3\pi}{4}$

$z = r(\cos\theta + i\sin\theta)$

$z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$z^{1/3} = 2^{1/6}\left(\cos 3\pi/4 + i\sin 3\pi/4\right)^{1/3}$

$$\Rightarrow (-1+i)^{1/3} = 2^{1/6}\left[\cos\left(2n\pi + \frac{3\pi}{4}\right) + i\sin\left(2n\pi + \frac{3\pi}{4}\right)\right\}^{1/3}$$

; $n = 0, 1, 2 \ldots$

$$\Rightarrow (-1+i)^{1/3} = 2^{1/6}\left\{\cos\left(2n\pi + 3\pi/4\right)\tfrac{1}{3} + i\sin\left(2n\pi + 3\pi/4\right)\tfrac{1}{3}\right\}$$

[by euler's formulae]

when $n=0$, then,

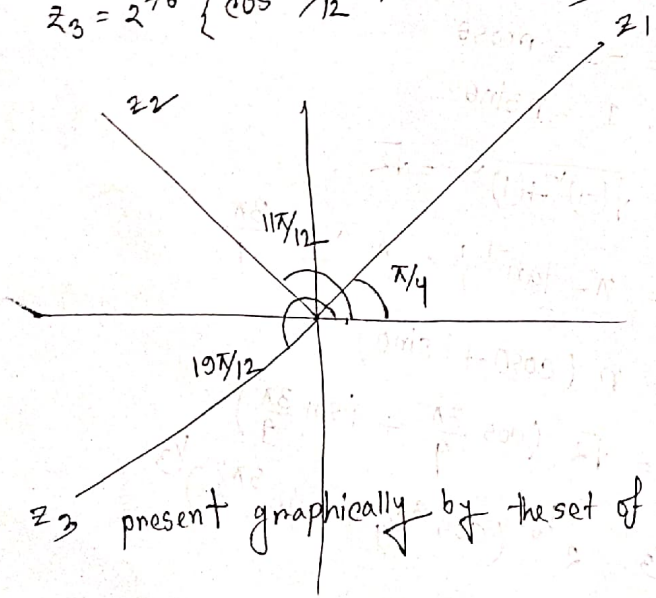
$$z_1 = 2^{1/6}\left\{\cos \pi/4 + i\sin \pi/4\right\}$$

$n=1$, then

$$z_2 = 2^{1/6}\left\{\cos 11\pi/12 + i\sin 11\pi/12\right\}$$

$n=2$, then

$$z_3 = 2^{1/6}\left\{\cos 19\pi/12 + i\sin 19\pi/12\right\}$$



present graphically by the set of z.

* prove that $|z_1+z_2| \le |z_1| + |z_2|$ and give physical representation graphically.

Solⁿ: Let, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers

and, $\bar{z}_1 = x_1 - iy_1$ and $\bar{z}_2 = x_2 - iy_2$ be their conjugate

$$z \cdot \bar{z} = |z|^2$$

Now, $|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$ | $|z|^2 = z \cdot \bar{z}$

$$= (z_1+z_2)(\bar{z}_1 + \bar{z}_2)$$

$$\Rightarrow |z_1+z_2|^2 = z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$$

$$\Rightarrow |z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1$$

$$\Rightarrow |z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,Re(z_1\bar{z}_2)$$

| $Re(z) \le |z|$

$p \bar{z} Re(x+iy)$

p

$$\Rightarrow |z_1+z_2|^2 \le |z_1|^2 + |z_2|^2 + 2\,Re(z_1\bar{z}_2)$$

$$\Rightarrow |z_1+z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1||\bar{z}_2|$$

$$\Rightarrow |z_1+z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$$\Rightarrow |z_1+z_2|^2 \le \{|z_1| + |z_2|\}^2$$

$$\Rightarrow |z_1+z_2| \le |z_1| + |z_2|$$

[proved]

★★★ Tools:

① $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$

$$\overline{z_1+z_2} = \overline{x_1+iy_1+x_2+iy_2}$$

$$= \overline{(x_1+x_2)+i(y_1+y_2)}$$

$$= (x_1+x_2)-i(y_1+y_2)$$

$$= x_1-iy_1+x_2-iy_2$$

$$= \overline{x_1+iy_1}+\overline{x_2+iy_2}$$

$$= \overline{z_1}+\overline{z_2}$$

② $z_1z_2 = (x_1+iy_1)(x_2+iy_2)$

$$= x_1x_2+x_1iy_2+iy_1x_2-y_1y_2$$

$$= (x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)$$

$$\overline{z_1z_2} = x_1+iy_1+x_2-iy_2 \quad \text{—(i)}$$

$$\overline{z_2z_1} = x_2+iy_2+x_1-iy_1 \quad \text{—(ii)}$$

(i) + (ii)

$$z_1\overline{z_2}+z_2\overline{z_1} = 2x_1+2x_2$$

$$= 2(x_1+x_2)$$

$$= 2Re(z_1\overline{z_2})$$

$$z_1\overline{z_2} = (x_1+iy_1)(x_2-iy_2)$$

$$= x_1x_2-ix_1y_2+iy_1x_2-i^2y_1y_2$$

$$= x_1x_2+y_1y_2-ix_1y_2+ix_2y_1 \quad \text{—(i)}$$

$$\overline{z_2z_1} = (x_2+iy_2)(x_1-iy_1)$$

$$= x_1x_2-ix_2y_1+ix_1y_2-i^2y_1y_2$$

$$= x_1x_2+y_1y_2-ix_2y_1+ix_1y_2 \quad \text{—(ii)}$$

(i) + (ii)

$$z_1\overline{z_2}+z_2\overline{z_1} = 2(x_1x_2+y_1y_2)$$

$$= 2Re(z_1\overline{z_2})$$

absolute value of z

$z = x + iy$

$|z| = \sqrt{x^2+y^2} \Rightarrow |z^2| = x^2+y^2$

$|z| \geq \sqrt{x^2}$ | $2+0i$

$|z| \geq x$

$\Rightarrow Re(z) \leq |z|$

Assignment

(i) $|z_1 - z_2| \geq |z_1| - |z_2|$

(ii) $|z_1 - z_2| \leq |z_1| + |z_2|$

(iii) $|z_1 - z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$

(iv) $|z_1 + z_2 + z_3 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$

proof that: $|z_1 + z_2| \geq |z_1| - |z_2|$

Sol^n: we know, $|z_1 + z_2| \leq |z_1| + |z_2|$ —— (i)

Let,

$z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex number

and, $\bar{z}_1 = x_1 - iy_1$ and $\bar{z}_2 = x_2 - iy_2$ be thein conjugates

$z_1 = z_1 + z_2 + (-z_2)$

$|z_1| = |z_1 + z_2 + (-z_2)|$ —— (ii)

From (i) & (ii) we get $\Rightarrow$

$|z_1| \leq |z_1 + z_2| + |-z_2|$

$|z_1| \leq |z_1 + z_2| + |z_2|$

$\Rightarrow |z_1 + z_2| \geq |z_1| - |z_2|$ [proved]

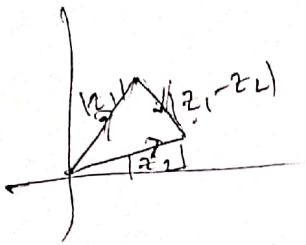
(2) proof that $|z_1 - z_2| \geq |z_1| - |z_2|$

Sol^n: Let, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers and $\bar{z}_1 = x_1 - iy_1$ and $\bar{z}_2 = x_2 - iy_2$ be their conjugates

Now,

$$|z_1 - z_2| = |x_1 + iy_1 - x_2 - iy_2|$$

$$= |(x_1 - x_2) + i(y_1 - y_2)|$$



$$= (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$= x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2(x_1x_2 + y_1y_2)$$

$$\Rightarrow |z_1 - z_2|^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2\,Re(z_1\bar{z}_2)$$

$$\Rightarrow |z_1 - z_2|^2 \geq x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2|z_1\bar{z}_2|$$

$$\Rightarrow |z_1 - z_2|^2 \geq x_1^2 + x_2^2 + y_2^2 + y_1^2 - 2|z_1||\bar{z}_2|$$

$$\Rightarrow |z_1 - z_2|^2 \geq |z_1| + |z_2| - 2|z_1||z_2|$$

$$\Rightarrow |z_1 - z_2|^2 \geq \{|z_1| - |z_2|\}^2$$

$$\Rightarrow |z_1 - z_2| \geq |z_1| - |z_2|$$

[proved

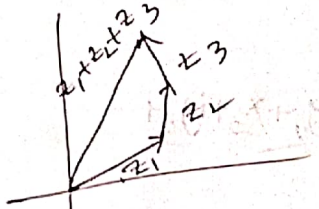
Question 3: prove that $|z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$

Let, $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $z_3 = x_3 + iy_3$ be three complex numbers

and $\bar{z}_1 = x_1 - iy_1$, $\bar{z}_2 = x_2 - iy_2$, $\bar{z}_3 = x_3 - iy_3$ be their conjugate

Now,

$$|z_1 + z_2 + z_3| = |x_1 + iy_1 + x_2 + iy_2 + x_3 + iy_3|$$



$$= |(x_1 + x_2 + x_3) + i(y_1 + y_2 + y_3)|$$

$$= (x_1 + x_2 + x_3)^2 + (y_1 + y_2 + y_3)^2$$

$$= x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_3x_1 + 2y_1y_2 + 2y_2y_3 + 2y_3y_1$$

$$|z_1 + z_2 + z_3|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\,Re(z_1\bar{z}_2) + 2\,Re(z_2\bar{z}_3) + 2\,Re(z_3\bar{z}_1)$$

$$\Rightarrow |z_1 + z_2 + z_3|^2 \leq |z_1|^2 + |z_2|^2 + |z_3|^2 + 2|z_1\bar{z}_2| + 2|z_2\bar{z}_3| + 2|z_3\bar{z}_1|$$

$$\Rightarrow |z_1 + z_2 + z_3|^2 \leq |z_1|^2 + |z_2|^2 + |z_3|^2 + 2|z_1||z_2| + 2|z_2||z_3| + 2|z_3||z_1|$$

$$\Rightarrow |z_1 + z_2 + z_3|^2 \leq \{|z_1| + |z_2| + |z_3|\}^2$$

$$\Rightarrow |z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$$

[proved

Q4. prove that

$$|z_1+z_2+z_3+\cdots+z_n| \le |z_1|+|z_2|+|z_3|+\cdots+|z_n|$$

Sol^n:

Let,

$z_1=x_1+iy_1,\ z_2=x_2+iy_2,\ z_3=x_3+iy_3 \cdots\ z_n=x_n+iy_n$

be complex numbers and,

$\bar{z}_1=x_1-iy_1,\ \bar{z}_2=x_2-iy_2,\ \bar{z}_3=x_3-iy_3 \cdots\ \bar{z}_n=x_n-iy_n$

be their conjugates.

Now,

$$|z_1+z_2+z_3+\cdots z_n|^2 = |x_1+iy_1+x_2+iy_2+x_3+iy_3+\cdots+x_n+iy_n|^2$$

$$= |(x_1+x_2+x_3+\cdots+x_n)+i(y_1+y_2+y_3+\cdots+y_n)|^2$$

$$= (x_1+x_2+x_3+\cdots+x_n)^2+(y_1+y_2+y_3+\cdots+y_n)^2$$

$$= x_1^2+x_2^2+x_3^2+\cdots x_n^2+2x_1x_2+2x_2x_3+2x_3x_1\cdots$$

$$+2x_1x_n+y_1^2+y_2^2+y_3^2\cdots+y_n^2+2y_1y_2+2y_2y_3$$

$$+2y_3y_1+\cdots+2y_1y_n$$

$$= |z_1|^2+|z_2|^2+|z_3|^2\cdots+|z_n|^2+2Re(z_1\bar{z}_2)$$

$$+2Re(z_2\bar{z}_3)+2Re(z_3\bar{z}_1)+\cdots+2Re(z_1\bar{z}_n)$$

$$|z_1+z_2+z_3+\cdots+z_n|^2 \le |z_1|^2+|z_2|^2+|z_3|^2+\cdots+|z_n|^2$$

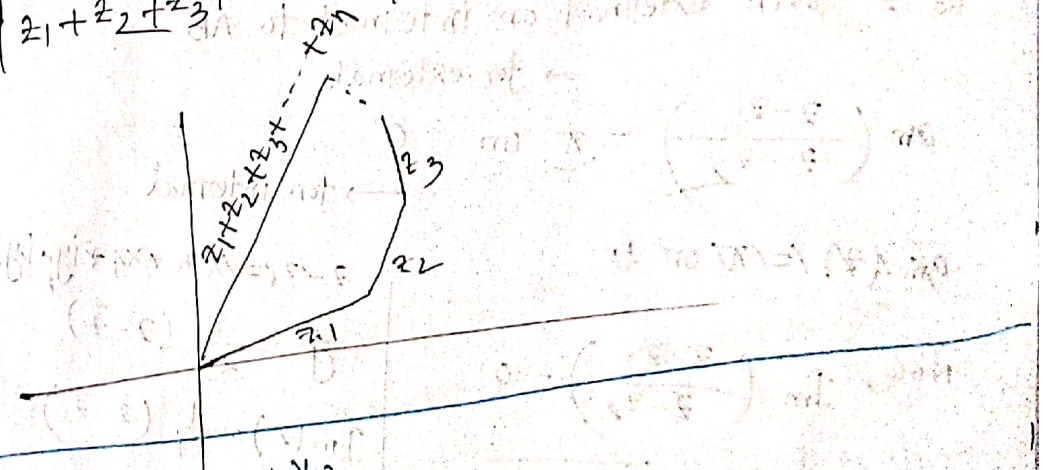
$$2|z_1\bar{z}_2|+2|z_2\bar{z}_3|+2|z_3\bar{z}_1|+\cdots+2|z_1\bar{z}_n|$$

$$\Rightarrow |z_1+z_2+z_3+\cdots+z_n|^2 \le |z_1|^2+|z_2|^2+|z_3|^2+\cdots+|z_n|^2$$

$$+2|z_1||z_2|+2|z_2||z_3|+2|z_3||z_1|+\cdots+2|z_1||z_n|$$

$$\Rightarrow |z_1+z_2+z_3+\cdots+z_n|^2 \le \{|z_1|+|z_2|+|z_3|+\cdots+|z_n|\}^2$$

$$\Rightarrow |z_1+z_2+z_3+\cdots+z_n|^2 \le |z_1|+|z_2|+|z_3|+\cdots+|z_n|$$



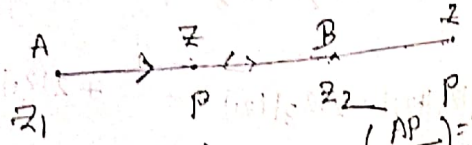
$$\text{amp}(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\text{amp}\left(\frac{z_1}{z_2}\right) = \text{amp}(z_1) - \text{amp}(z_2)$$

$$\text{arg}(z_1z_2) = \text{arg}\, z_1 + \text{arg}\, z_2$$

Find the equation of straight line joining two points z_1 and z_2

Sol^n:



$am\left(\frac{AP}{BP}\right) = 0$; $am\left(\frac{AP}{BP}\right) = \pi$

↳ internal ↳ external

Normal (Real)

(x_1, y_1) & (x_2, y_2)

$$\frac{(x-x_1)}{(x_2-x_1)} = \frac{(y-y_1)}{(y_2-y_1)}$$

$$\frac{x-x_1}{y-y_1} = (A)$$

Take any point z on the straight line joining two given points z_1 and z_2 as shown in fig 1 according as z taken external or internal to AB.

$$am\left(\frac{z-z_1}{z-z_2}\right) = \pi \text{ or } 0$$

→ for external

↳ for internal

$am(z) = \pi$ or 0

Now, $Im\left(\frac{z-z_1}{z-z_2}\right) = 0$

$$\Rightarrow \frac{1}{2i}\left\{\left(\frac{z-z_1}{z-z_2}\right) - \overline{\left(\frac{z-z_1}{z-z_2}\right)}\right\} = 0$$

$$\Rightarrow \frac{z-z_1}{z-z_2} = \frac{\bar{z}-\bar{z}_1}{\bar{z}-\bar{z}_2}$$

$$\Rightarrow (z-z_1)(\bar{z}-\bar{z}_2) = (z-z_2)(\bar{z}-\bar{z}_1)$$

$z - z_1 = x + x_1 + iy - iy_1$

$$y = \frac{1}{2i}(z-\bar{z})$$

$$Im(z) = \frac{1}{2i}(z-\bar{z})$$

$$\Rightarrow z\bar{z} - z\bar{z}_2 - z_1\bar{z} + z_1\bar{z}_2 = z\bar{z} - z\bar{z}_1 - z\bar{z}_2 + z_2\bar{z}_1$$

$$\Rightarrow z\bar{z} - z\bar{z}_2 - z_1\bar{z} + z_1\bar{z}_2 = z\bar{z} - z\bar{z}_1 - z\bar{z}_2 + z_2\bar{z}_1$$

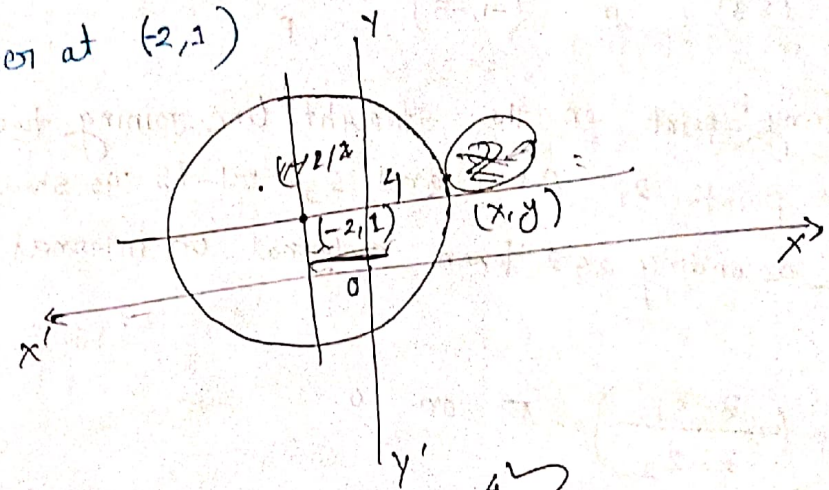
$$\Rightarrow z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_2 - z_1) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$$

$$\Rightarrow z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$$

$$\Rightarrow z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$$

$\boxed{L-03}$

Find an equation for a circle of radius 4 and center at $(-2, 1)$



$$(x+2)^2 + (y-1)^2 = 4^2$$

The center can be represented by complex number

$$c = (-2, 1) = -2 + i$$

If z be only point on the circle as shown in fig 1

the distance from z to $(-2, 1)$ is

$$|z - (-2+i)| = 4$$

$$\Rightarrow |x + iy + 2 - i| = 4$$

$$\Rightarrow |(x+2) + i(y-1)| = 4$$

$$\Rightarrow (x+2)^2 + (y-1)^2 = 4^2$$

(eqn)

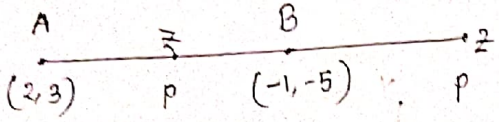
Home work

Question: Find the equation of straight line joining two points $z_1 = (2,3)$ and $z_2 = (-1,-5)$

Solⁿ:

$z_1 = 2+3i \quad z_2 = -1-5i$

$\bar{z}_1 = 2-3i \quad \bar{z}_2 = -1+5i$



Take any point on the straight line joining two given points $z_1 = 2+3i$ and $z_2 = -1-i5$ as shown in fig according as z taken external or internal to

Ans

$$\text{amp}\left(\frac{z-z_1}{z-z_2}\right) = \pi \text{ or } 0$$

$$\text{so, } \text{Im}\left(\frac{z-z_1}{z-z_2}\right) = 0$$

$$y = \frac{1}{2i}(z-\bar{z})$$

$$\text{Im}(z) = \frac{1}{2i}(z-\bar{z})$$

$$\Rightarrow \frac{1}{2i}\left(\frac{z-z_1}{z-z_2} - \overline{\left(\frac{z-z_1}{z-z_2}\right)}\right) = 0$$

$$\Rightarrow \left(\frac{z-z_1}{z-z_2} - \frac{\overline{z-z_1}}{\overline{z-z_2}}\right) = 0$$

$$\Rightarrow \left(\frac{z-z_1}{z-z_2} - \frac{\bar{z}-\bar{z}_1}{\bar{z}-\bar{z}_2}\right) = 0$$

$$\Rightarrow (z-z_1)(\bar{z}-\bar{z}_2) - (z-z_1)(\bar{z}-\bar{z}_1) = 0$$

$$\Rightarrow z(\bar{z}_1-\bar{z}_2) - \bar{z}(z_1-z_2) + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$$

$$\Rightarrow z(2-3i+1-5i) - \bar{z}(2+3i+1+5i) + (2+3i)(-1+5i) - (2-3i)(-1-5i) = 0$$

$$\Rightarrow \boxed{z(3-8i) - \bar{z}(3+8i) + 14i = 0}$$

$$\Rightarrow (x+iy)(3-8i) - (x-iy)(3+8i) + 14i = 0$$

$$\Rightarrow \cancel{3x} - 8xi + 3yi + \cancel{8y} - \cancel{3x} - 8xi + 3yi - \cancel{8y} + 14i = 0$$

$$\Rightarrow -16xi + 6yi + 14i = 0$$

$$\Rightarrow 16x - 6y + 14 = 0$$

$$\Rightarrow \boxed{8x - 3y + 7 = 0}$$

$$\boxed{8x - 3y - 7 = 0}$$

Find equation of a circle whose center at a and radius r. [a(2,3) r=5]

Sol^n:

Let z be any point the circle

$|z-a| = r$

$|z-(2+3i)| = r$

$|x-2+i(y-3)| = r$

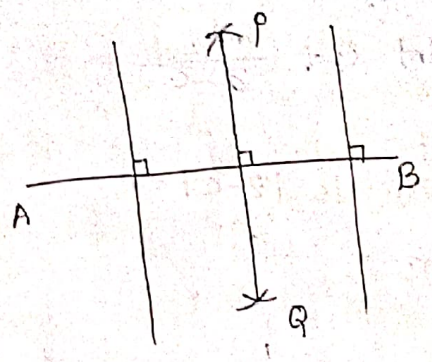
$\Rightarrow (x-2)^2 + (y-3)^2 = r^2$

$\Rightarrow \boxed{(x-2)^2+(y-3)^2=5^2}$ (Ans)

Lecture 03

Inverse point with respect to a straight line

Image point:

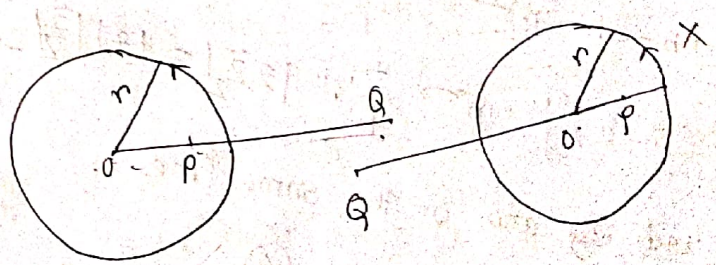


PQ image point.

Two points P and Q are said to be inverse point to AB if Q is the image point of P i.e if AB is the right bisection of PQ.

Inverse point with respect to circle

Two points P & Q are said to be inverse point with respect to a circle if they are colinear with the center O and on the same side of it and $OP \cdot OQ = r^2$ where r is the radius of the circle.



Does not at same side

HW

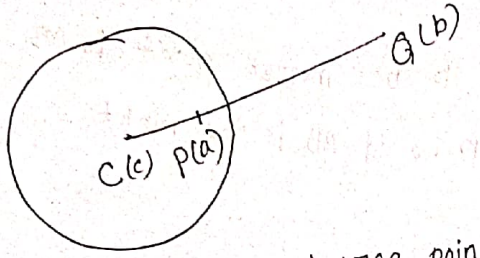
Show that the inverse of a point a ...

circle $|z-c| = r$ is point $c + \frac{r^2}{\bar{a}-\bar{c}}$

Sol^n:

Given eq^n of the circle is $|z-c| = r$

center c and radius $= r$



Let $Q(b)$ the inverse point of $P(a)$

$CP \cdot CQ = r^2$ —— (1)

$|(a-c)| \cdot |b-c| = r^2$ —— (2)

as we know,

$|a-c| = |\overline{a-c}| = |\bar{a}-\bar{c}|$

putting the value of (2) in (1)

$|\bar{a}-\bar{c}|\,|b-c| = r^2$

$\Rightarrow |(b-c)(\bar{a}-\bar{c})| = r^2$ —— (3) $\quad \because |z\bar{z}_2| = |z||\bar{z}_2|$

since CP and CQ are on the same line

$\text{Arg}(b-c) = \text{Arg}(a-c)$ —— (4)

we know,

$\text{Arg}(z) + \text{Arg}(\bar{z}) = 0$

$\Rightarrow \text{Arg}(z) = -\text{Arg}(\bar{z})$

$\Rightarrow \text{Arg}(\bar{a}-c) = -\text{Arg}(\overline{a-c})$

$\Rightarrow \text{Arg}(a-c) = -\text{Arg}(\bar{a}-\bar{c})$ —— (5)

putting this values in (4)

$\text{Arg}(b-c) = -\text{Arg}(\bar{a}-\bar{c})$

$\Rightarrow \text{Arg}(b-c) + \text{Arg}(\bar{a}-\bar{c}) = 0$

$\Rightarrow \text{Arg}[(b-c)(\bar{a}-\bar{c})] = 0$

As arg is zero, $(b-c)(\bar{a}-\bar{c})$ is purely real and positive

$|(b-c)(\bar{a}-\bar{c})| = (b-c)(\bar{a}-\bar{c})$ —— (6)

$\Rightarrow |(b-c)(\bar{a}-\bar{c})| = |25| = \sqrt{25^2} = 25$

putting this value in (3)

$(b-c)(\bar{a}-\bar{c}) = r^2$

$(b-c) = \frac{r^2}{\bar{a}-\bar{c}}$

$\Rightarrow b = c + \frac{r^2}{\bar{a}-\bar{c}}$ $\quad$ [Shown]

Limit

$$\lim_{x \to a-0} f(x) = \lim_{x \to a+0} f(x)$$

LHL $\downarrow$ ∞

RHL $\downarrow$ ∞ $\longrightarrow$ Limit doesnot exists

RHL $\rightarrow$ if equal, finite $\rightarrow$ limit exists

ϵ = functional value difference

δ = difference between point.

Limit: A function $f(z)$ is said to have a limit l at $z = z_0$ if for any $\epsilon > 0$ then there exists $\delta > 0$ such that $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$ and we can write $\lim_{z \to z_0} f(z) = l$

Does $\lim \frac{\bar{z}}{z}$ exists?

Sol^n: we have $\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{(x,y) \to (0,0)} \frac{x+iy}{x-iy}$

Suppose $(x,y) \to (0,0)$ along the x-axis then $y=0$ and $x \to 0$ and so the required limit

$$= \lim_{x \to 0} \frac{x}{x} = 1$$

suppose, $(x,y) \to 0$ along the y axis, then $x = 0$ and $y \to 0$, so the required limit $= \lim_{y \to 0} \frac{-iy}{iy} = -1$

Hence $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exist.

Continuity: A function $f(x)$ is said to be continuous at $z = z_0$ if for any $\epsilon > 0$ then there exsits $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$ and we write

$$\lim_{z \to z_0} f(z) = f(z_0)$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$x+h-x$

$$\lim_{x \to a} f(x) = f(a)$$

LHL = RHL = $f(a)$

$$\lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = f(a)$$

Differentiability :

A function $f(x)$ is said to be differentiable at $z = z_0$ is also continuous there but converse is not nece.

A function $f(x)$ is said to be differentiable at $z = z_0$ if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists and we write $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

$f(x+\Delta x, y+\Delta y) - f(x, y)$

Antiday f.

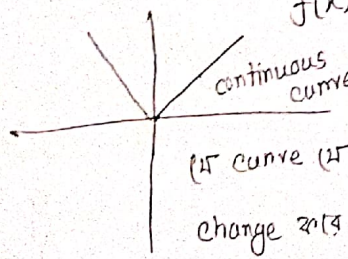
In Prove that, a function $f(z)$ which is differentiable at $z = z_0$ is also continuous there but converse is not necessary true.

Soln: suppose $f(z)$ is differentiable at $z = z_0$

$$\therefore f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{—— (1)}$$

Now, $f(z) - f(z_0) = \frac{f(z) - f(z_0)}{z - z_0} \times (z - z_0)$

$f(x) = |x|, \; x \neq 0$



continuous curve $= 0 \;|\; x = 0$

যে curve যে বিন্দুতে যতগুলো direction change করে তখনও বিন্দুতে কখনই differentiable হয় না।

$\rightarrow \lim_{z \to z_0}$

Taking Limit $z \to z_0$ on both sides we get

$$\lim_{z \to z_0} \{f(z) - f(z_0)\} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} (z - z_0)$$

$$\Rightarrow \lim_{z \to z_0} \{f(z) - f(z_0)\} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \times \lim_{z \to z_0} (z - z_0)$$

$$\Rightarrow \lim_{z \to z_0} f(z) - \lim_{z \to z_0} f(z_0) = f'(z) \times 0$$

$$\Rightarrow \lim_{z \to z_0} f(z) - f(z_0) = f'(z_0) \times 0$$

$$\boxed{\Rightarrow \lim_{z \to z_0} f(z) = f(z_0)}$$

So $f(z)$ is continuous at $z = z_0$

for conversely

Let, $f(z) = |z|$, $z = x + iy$

$= \sqrt{x^2 + y^2}$

$f(x)$ is continuous at $x = 0$

Now, $f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \to 0} \frac{|z|}{z} \Rightarrow \lim_{(x,y) \to 0} \frac{\sqrt{x^2 + y^2}}{x + iy}$$

Suppose, $(x,y) \to 0$ along the x axis ; then y and $x \to 0$, so the required limit $= \lim_{x\to 0} \frac{\sqrt{x^2}}{x} = 1$

Suppose, $(x,y) \to 0$ along the y-axis then $x = 0$ and $y \to 0$ so the required limit $= \lim_{y\to 0} \frac{\sqrt{y^2}}{iy} = \frac{1}{i}$

Lecture -04

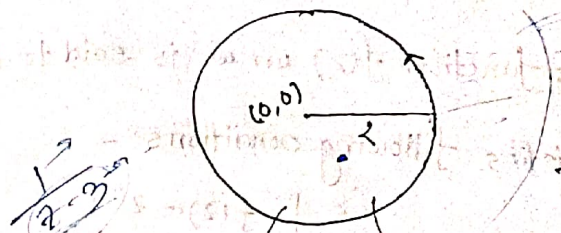
singular point: (যে সব মানের জন্য function এর value infinity হয়)

$$f(z) = \frac{z+1}{(z-1)(z+5)}$$

put $(z-1)(z+5) = 0$

$$\boxed{z = 1 \; ; \; z = -5}$$

Analytic function in a region: A function $f(z)$ is defined in a region R is called analytic if $f'(z)$ exist at every point of R. This is also known as a regular function.



$\sqrt{x^2+y^2}$

$|z| = 2$

$\Rightarrow \sqrt{x^2+y^2} = 2$

$$\boxed{x^2+y^2 = 2^2}$$

$f_1(z) = \frac{1}{z-1}$ ∞

$|z| = 1$ circle এর ভেতরে তাই analytic

$f_2(z) = \frac{1}{z-5}$

$|z| = 5$ circle এর বাইরে তাই analytic

$$f_3(z) = \frac{1}{(z-1)(z-5)}$$

for $z = 1$, the function $f(z)$ is not analytic but for $z = 5$, the function $f(z)$ is analytic

Analytic function at a point: A function $f(z)$ is said to be analytic at a point $z = z_0$ if there exist a neighbourhood N of z_0 such that $f'(z)$ exist at each point of N

$f(z) = \frac{1}{z-1}$ $\to z_0 = 1$ $\quad 0.999 \leftrightarrow 1 \to 10000.1$

$z = x+iy$ is complex variable / numbers

$f(z) = u(x,y) + iv(x,y)$ is the complex function.

Analytic / Regular / ~~Harmon~~ Holomorphic / monogenic function

A complex variable function $f(z)$ on ω is said to be Analytic iff it satisfies following conditions:-

① $f(z)$ should be single valued

$f(z) = z^2$

$f(z) = (4)^2 = 16$

$f(z) = \sqrt{z} = \sqrt{4} = \pm 2$

② $f(z)$ should have unique derivatives of it's Domain D

③ $f(z)$ should satisfy cauchy-Riemann (C-R) equations

① $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$ or $u_x = v_y$

② $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ or $u_y = -v_x$

$\delta \rightarrow$ delta (symbol)

$\partial \rightarrow$ del (operator)

if u & v are function of x & y

cauchy-Riemann equations in cartesian form

① Polynomials $(x^2-2x+7, x^{10}-7x)$

② Rationals with non-zero denominators $\frac{xy}{x^2+y^2}$ etc.

③ sine or cosine function e.g $\sin 3x, \cos 7x, x\sin x$ et.c.

④ Exponential functions $e^{2x}, e^{-x}\sin x, x^2 e^{10x}$ etc.

→ always have unique derivatives and single valued.

Entire function: A function $f(z)$ which is analytic Every where.

Cauchy-Riemann Equation in polar form

if u and v are the functions of r and θ

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Cauchy-Riemann equation

Let, $\omega = f(z) = u(x,y) + iv(x,y)$ be an analytic function

$u(x,y)$ and $v(x,y)$ being continuous partial derivatives

Then the cauchy-Riemann equations are:

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}}$$

$f(z) =$

$z = x + iy$

$u(x,y) + iv(x,y)$

(a) prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic

(b) find u such that $f(z) = u + iv$ is analytic where v is the harmonic conjugate:

(a) Soln: $u = e^{-x}(x\sin y - y\cos y)$ —— (1)

Differentiating (1) with respect to x we get:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} e^{-x}(x\sin y - y\cos y)$$

$$= e^{-x}\sin y + (x\sin y - y\cos y)(-e^{-x})$$

$$= e^{-x}\sin y - e^{-x}(x\sin y - y\cos y) \quad \text{—— (II)}$$

$$\frac{\partial^2 u}{\partial x^2} = -e^{-x}\sin y - (e^{-x}\sin y - e^{-x}(x\sin y - y\cos y))$$

$$= -2e^{-x}\sin y + e^{-x}(x\sin y - y\cos y) \quad \text{—— (III)}$$

Differentiating (1) w.r.to y ⇒

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} e^{-x}(x\sin y - y\cos y)$$

$$= e^{-x}(x\cos y + y\sin y - \cos y)$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}(-x\sin y + y\cos y + \sin y + \sin y)$$

$$= -e^{-x}(x\sin y - y\cos y) + 2e^{-x}\sin y \quad \text{—— (V)}$$

(iii) + (iv) →

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-x}(x\sin y - y\cos y) - 2e^{-x}\sin y - e^{-x}(x\sin y - y\cos y) + 2e^{-x}\sin y$$

$$= 0.$$

u is satisfied the laplace equation Here u is harmonic.

(a) Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic we have $u = e^{-x}(x\sin y - y\cos y)$ — (1)

Differentiating (1) partially w.r.t. x

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} e^{-x}(x\sin y - y\cos y)$$

$$= -\boxed{e^{-x}(x\sin y - y\cos y)} + e^{-x}(\sin y)$$

$$\frac{\partial^2 u}{\partial x^2} = -\{-e^{-x}(x\sin y - y\cos y) + e^{-x}\sin y\} + e^{-x}\sin y$$

$$= e^{-x}(x\sin y - y\cos y) - 2e^{-x}\sin y.$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} e^{-x}(x\sin y - y\cos y)$$

$$= +e^{-x}(x\cos y + y\sin y - \cos y) \quad \text{— (1)}$$

$$\frac{\partial^2 u}{\partial y^2} = +e^{-x}(-x\sin y + \cancel{\sin y} + y\cos y + \cancel{\sin y})$$

$$= e^{-x}(-x\sin y + y\cos y + 2\sin y)$$

now, $\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$ harmonic.

we know that the cauchy riemann equation.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f(z) = u(x,y) + i\,v(x,y)$$

Differentiating (8) w.r.t. x

$$f'(z) = \boxed{\frac{\partial u(x,y)}{\partial x}} + i\frac{\partial v(x,y)}{\partial x}$$

$$= \frac{\partial u(x,y)}{\partial x} - i\frac{\partial u(x,y)}{\partial y}$$

~~Now put~~.

$$f'(z) = u_1(x,y) - i\,u_2(x,y) \quad \text{--- (2)}$$

by Milton's theorem,

$$f'(z) = u_1(z,0) + i\,u_2(z,0)$$

put $x=z$ and $y=0$ in (2) we get,

$u_1 = 0$

put in (3)

$u_2 = e^{-z}(z-1)$

placing in eqn (10) we get,

$$f'(z) = 0 - i\,e^{-z}(z-1)$$

Now integrating,

$$\int f'(z)\,dz = -i\int e^{-z}(z-1)\,dz$$

$$= -i\left[(z-1)\int e^{-z}dz - \int \frac{d(z-1)}{dz}\int e^{-z}dz\,dz\right]$$

$$= -i\left[-(z-1)e^{-z} - \int -e^{-z}\right]$$

$$= -i\left[-(z-1)e^{-z} - e^{-z} + c\right]$$

$$= -i\left[\cancel{-ze^{-z}} + \cancel{e^{-z}} - \cancel{e^{-z}} + c\right]$$

$$= -i\left[-ze^{-z} + c\right]$$

$$= +ze^{-z}i + ci$$

$$= +(x+iy)e^{-(x+iy)} - ic$$

$$= +ie^{-(x+iy)} + yie^{-(x+iy)} - ic$$

$$= +e^{-x}i\left(e^{-iy} + (x+iy)\right) - ic$$

$$= ie^{-x}\{(x+iy)e^{-iy}\} - ic$$

$$= ie^{-x}\{(x+iy)(\cos y - i\sin y)\} - ic$$

$$= +ie^{-x}\{x\cos y - ix\sin y + iy\cos y + y\sin y\} + ic$$

$$= ie^{-x}x\cos y + x\sin y\,e^{-x} - ye^{-x}\cos y + ie^{-x}y\sin y + ic$$

$$= \underbrace{e^{-x}(x\sin y - y\cos y)}_{u} + \underbrace{ie^{-x}(x\cos y + y\sin y)}_{v} - ic$$

$-ic = 0$

$f(z_0) = 0$

$\frac{1}{(z-1)(z-4)(z-5)}$

poles

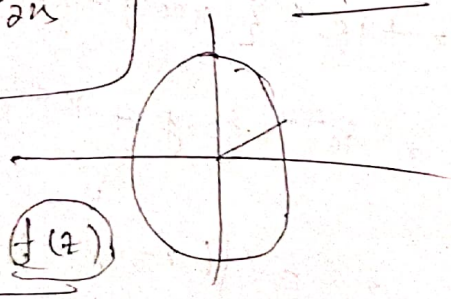
branch point

$xe^{-x}(x\sin y - y\cos y)$

necessary

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$



$f(z+\Delta z) - f(z)$

$- f$

* Sufficient condition for a function to be analytic

prove that (a) necessary and (b) sufficient condition that $\boxed{w = f(z) = u(x,y) + iv(x,y)}$ be analytic in a region R is that the cauchy - Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\left|\frac{\partial v}{\partial x}\right|$ are satisfied) in R where it is supposed that these partial derivatives are continuous in R

Soln:

Sufficient condition for a function to be analytic:

since $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ supposed to be continuous, we have

$$\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$$

$$= \left\{ \frac{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)}{\Delta x} \Delta x \right\} + \left\{ \frac{u(x+y+\Delta y) - u(x,y)}{\Delta y} \Delta y \right\}$$

$$= \left(\frac{\partial u}{\partial x} + \varepsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y$$

$$= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \varepsilon_1 \Delta x + \eta_1 \Delta y$$

where $\varepsilon_1 \to 0$ and $\eta_1 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$

Similarly $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are supposed to be continuous so, we have,

$$\Delta v = v(x+\Delta x; y+\Delta y) - v(x,y)$$

$$= \{v(x+\Delta x, y+\Delta y) - v(x, y+\Delta y)\}$$

$$+ \{v(x, y+\Delta y) - v(x,y)\}$$

$$= \left(\frac{\partial v}{\partial x} + \varepsilon_2\right)\Delta x + \left(\frac{\partial v}{\partial y} + \eta_2\right)\Delta y$$

$$= \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \varepsilon_2 \Delta x + \eta_2 \Delta y$$

$\varepsilon_2 \to 0$ and $\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$

<u>Then</u>,

$$\Delta w = \Delta u + i\Delta v$$

$$= \left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \varepsilon_1 \Delta x + \eta_1 \Delta y\right)$$

$$+ i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \varepsilon_2 \Delta x + \eta_2 \Delta y\right)$$

$$= \Delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \Delta y\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$+ (\varepsilon_1 + i\varepsilon_2)\Delta x + \Delta y(\eta_1 + i\eta_2)$$

$$= \Delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \Delta y\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) + \varepsilon \Delta x + \eta$$

Necessary Condition

Let $f(z)$ be analytic in R then for any point $z \in R$,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(\Delta z)}{\Delta z}$$

$$u(z) + iv(z) = \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{f\{(x+\Delta x) + i(y+\Delta y)\} - f(x+iy)}{(\Delta x + i\Delta y)}$$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta x + i\Delta y}$$

must exist independent of the manner in which Δx and Δy approach zero.

Suppose $(\Delta x, \Delta y) \to (0,0)$ along the path $\Delta x \to 0$, $\Delta y = 0$

So the limit

$$\lim_{\Delta x \to 0} \frac{u(x+\Delta x; y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$\frac{\partial u}{\partial x} \qquad i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{—— (1)}$$

Suppose $(\Delta x, \Delta y) \to (0,0)$ along the path y;

So, $\Delta y \to 0$ and $\Delta x = 0$

So, the limit

$$\lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y) + i\{v(x, y+\Delta y) - v(x,y)\}}{i\Delta y}$$

$$= \frac{\partial u}{i\,\partial y} + \frac{i}{i}\frac{\partial v}{\partial y}$$

$$= \lim_{\Delta y \to 0} \frac{i\,u(x, y+\Delta y) - u(x,y)}{i \times i\,\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= \frac{i}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \boxed{-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}} \quad \text{—— (2)}$$

Since, $f(z)$ is analytic then the above limits must equal,

[analytic → continuous
↓
প্রতি point এ differentiable
হওয়া লাগবে]

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

equating Real and imaginary part,

we get

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\left|\frac{\partial u}{\partial y}\right| \end{aligned}}$$

that is the cauchy Riemann equation.

Sufficient condition for a function to be analytic

since $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are supposed to be continuous, we have,

$$\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$$

$$= \left\{\frac{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)}{\Delta x}\right\}\Delta x + \left\{\frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}\right\}\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + \varepsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y$$

$$= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \varepsilon_1\Delta x + \eta_1\Delta y$$

where, $\varepsilon_1 \to 0$ and $\eta_1 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$

again $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are supposed to be continuous

$\Delta z = \Delta u + i\Delta v$

we have,

$$\Delta v = v(x+\Delta x, y+\Delta y) - v(x,y)$$

$$= \left\{\frac{v(x+\Delta x, y+\Delta y) - v(x, y+\Delta y)}{\Delta x}\right\}\Delta x + \left\{\frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}\right\}\Delta y$$

$$= \left(\frac{\partial v}{\partial x} + \varepsilon_2\right)\Delta x + \left(\frac{\partial v}{\partial y} + \eta_2\right)\Delta y$$

$$= \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \varepsilon_2\Delta x + \eta_2\Delta y$$

where $\varepsilon_2 \to 0$ and $\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$

now,

$$\Delta w = \Delta u + i\Delta v$$

$$= \left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \varepsilon_1 \Delta x + \eta_1 \Delta y\right)$$

$$+ i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \varepsilon_2 \Delta x + \eta_2 \Delta y\right)$$

$$= \Delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \Delta y\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) + (\varepsilon_1 + \varepsilon_2)\Delta x$$

$$+ (\eta_1 + \eta_2)\Delta y$$

$$= \Delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \Delta y\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) + \varepsilon\Delta x + \eta\Delta y \quad (1)$$

where $\varepsilon = \varepsilon_1 + \varepsilon_2 \to 0$ and $\eta = \eta_1 + \eta_2 \to 0$

as $\Delta x \to 0$ and $\Delta y \to 0$

By Cauchy Riemann equation (1) can be written as

$$= \Delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \Delta y\left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)$$

$$+ \varepsilon\Delta x + \eta\Delta y$$

$$= \Delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + i\Delta y\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$+ \varepsilon\Delta x + \eta\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + \varepsilon\Delta x + \eta\Delta y$$

now, dividing by $\Delta z = \Delta x + i\Delta y$ and taking the limit as $\Delta z \to 0$ we get

$$\lim_{z \to 0} \frac{\Delta w}{\Delta z} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{\Delta x + i\Delta y}{\Delta z}$$

$$\frac{dw}{dz} = f'(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$\frac{dw}{dz} = f'(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

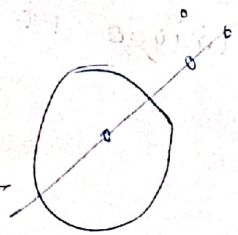
so that the derivative exists and is unique i.e $f(z)$ is analytic in R.

Singular point

A point at which $f(z)$ fails to be analytic i.e functional value is α is called a singular point or singularity of $f(z)$

Example: $f(z) = \dfrac{(z+1)}{(z-1)(z+5)}$

at, $z-1=0$ $\quad z=1$
$z+5=0$ $\quad z=-5$ $\Big\}$ singular point of $f(z)$



Types of singular point

① Isolated Singular point:

Isolated Singularities:

The point $z=z_0$ is called an isolated singularity or isolated singular point of $f(z)$ if we can find $\delta>0$ such that the circle $|z-z_0|=\delta$ encloses no ~~sig~~ singular point other than z_0.

Example: If no such δ can be found we call z_0 a non isolated singularity.

Example: $f(z) = \dfrac{z+1}{z(z^2+2)}$ have isolated singularity at $0, \sqrt{2}i, -\sqrt{2}i$ ~~Expect~~ Except these points there is no singularity within the neighbourhood of these points.

② Removable ~~sing~~ singularity: An isolated singular point z_0 is called removable of $f(z)$ if $\lim_{z\to z_0} f(z)$ exists finitely.

Example: $\boxed{f(z) = \dfrac{\sin z}{z}}$ singular at $z=0$

But;

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]$$

$$= \lim_{z\to 0}\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right]$$

$$= 1$$

Hence $z=0$ is a removable singularity of $f(z)$

③ Essential Singular Point : $(z-3)(z-4)$

A singular point that is not pole or removable singularity is called an essential singular point.

Example : $f(z) = e^{\frac{1}{z-3}}$ has an essential singularity at $z=3$.

④ poles: If z_0 is an isolated singularity and we can find a positive integer n such that

$$\lim_{z \to z_0} (z-z_0)^n f(z) = A \neq 0$$ then $z = z_0$ is called a pole of order n.

Example: $\boxed{f(z) = \frac{1}{(z-2)^3}}$ has a pole of order 3 at $z=2$

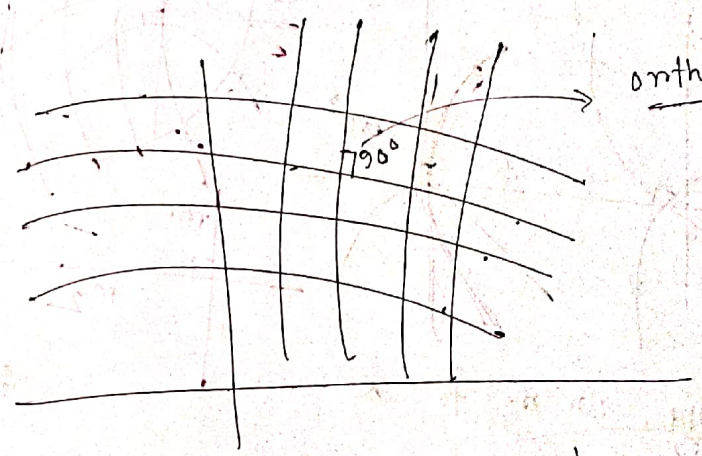
$f(z) = \frac{3z-2}{(z-1)^2 (z+1)(z-4)}$ has poles at $z=1$ of order 2 at $z=-1, 4$ of order 1.

⑤ Branch points : Branch points of multiple valued functions are non isolated singular points, since a multiple valued function is not continuous and not analytic in a deleted neighbourhood of branch point.

Skip

Example: $f(z) = (z-3)^{1/2}$ has branch point at $z=3$

$f(z) = \ln(z^2+z-2)$ has branch points where $(z^2+z-2)=0$ $(z=1,-2)$



orthogonal family

$$\frac{\partial u}{\partial x} \times \frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$

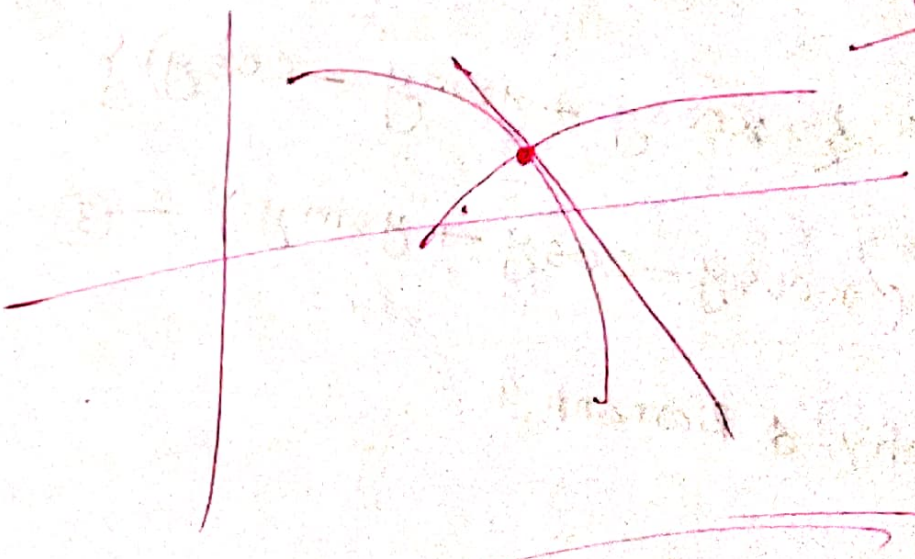
$$\frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} = m_1$$

Now, $\frac{\partial v}{\partial x} \cdot \frac{dx}{dy} + \frac{\partial v}{\partial y} \cdot \frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial v/\partial x}{\partial v/\partial y} = m_2$$

Now, if $m_1 \times m_2 = -1$ then (I) & (II) are orthogonal.

$$\boxed{m = \frac{dy}{dx}}$$



$$\frac{\partial u(x,y)}{\partial x}$$

$$\boxed{\frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}} = m_1$$

$$\boxed{m_1 \times m_2 = -1}$$

$$\frac{dy}{dx} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = m_2$$

* Find the orthogonal trajectories of the family of curves in the xy plane defined by $e^{-x}(x\sin y - y\cos y) = \alpha$ where α is real constant.

Soln:

let, $u = e^{-x}(x\sin y - y\cos y)$ —— (1)

$$\frac{\partial u}{\partial x} = e^{-x}(\sin y) - e^{-x}(x\sin y - y\cos y) \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial y} = e^{-x}\{(x\cos y + y\sin y - \cos y)\}$$

$$= e^{-x}(x\cos y - \cos y + y\sin y) \quad \text{—— (3)}$$

according to cauchy & Riemann

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

we know that

$$f(z) = u(x,y) + i\,v(x,y)$$

Diff partially w.r.to x

$$f'(z) = \frac{\partial u(x,y)}{\partial x} + i\,\frac{\partial v(x,y)}{\partial x}$$

$$= \frac{\partial u(x,y)}{\partial x} - i\,\frac{\partial u(x,y)}{\partial y}$$

$$= u_1(x,y) - i\,u_2(x,y) \quad \text{—— (4)}$$

By millon's theorem:

put, $x = z$, $y = 0$ put these in (4)

$$f'(z) = u_1(z,0) - i\,u_2(z,0)$$

put. $u_1(z,0) = e^{-z}(0) - e^{-z}(0) = 0$

$$\boxed{u_2(z,0) = e^{-z}(z-1)}$$

$$f'(z) = 0 - i\,e^{-z}(z-1) \quad \text{—— (5)}$$

$$\Rightarrow f'(z) = -i\,e^{-z}(z-1)$$

$$\int uv\,dx = u\int v\,dx - \int \frac{du}{dx}\int v\,dx\,dx$$

Integrating (5)

$$f(z) = -i\int (z-1)\,e^{-z}\,dz$$

$$= -i\left\{\int e^{-z}dz\,(z-1) - \int \frac{d(z-1)}{dz}\int e^{-z}dz\,dz\right\}$$

$$= -i\left\{-e^{-z}(z-1) - \int -e^{-z}dz\right\}$$

$$= -i\left\{-e^{-z}(z-1) - e^{-z} + c\right\}$$

$$= +i\left\{e^{-z}(z)\right\} - ic$$

$$= i(x+iy) * e^{-x-iy} - ic$$

$$= i(x+iy) * e^{-x}\cdot e^{-iy} - ic$$

$$= (ix-y)\,e^{-x}(\cos y - i\sin y) - ic$$

$$= e^{-x}(ix\cos y + x\sin y - y\cos y + iy\sin y) - ic$$

$$= e^{-x}(x\sin y - y\cos y) + i\,e^{-x}(x\cos y + y\sin y) - ic$$

Equating the imaginary parts

$$u = x\sin y - y\cos y$$

$$v = x\cos y + y\sin y$$

now,

$$m_1 = \frac{\partial u/\partial x}{\partial u/\partial y}$$

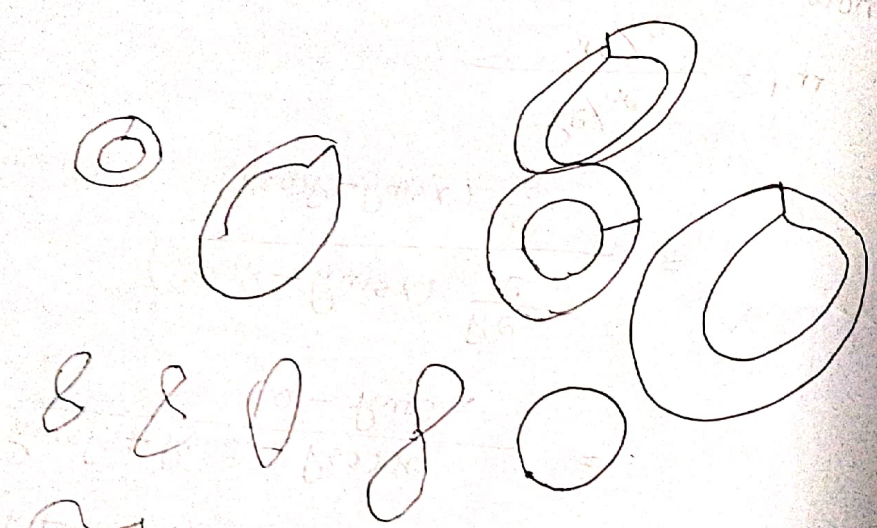
$$= \frac{\frac{\partial}{\partial x}(x\sin y - y\cos y)}{\frac{\partial}{\partial y}(x\sin y - y\cos y)}$$

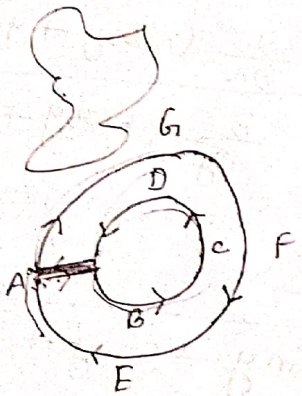
$$= \frac{\sin y - 0}{x\cos y - \cos y + y\sin y}$$

$$m_2 = \frac{\partial v/\partial x}{\partial v/\partial y} = \frac{\frac{\partial}{\partial x}(x\cos y + y\sin y)}{\frac{\partial}{\partial y}(x\cos y + y\sin y)}$$

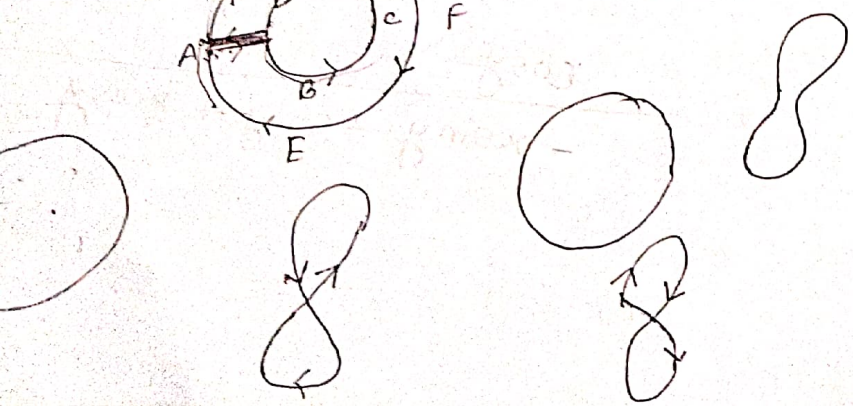
$$= \frac{\cos y}{-x\sin y + \sin y + y\cos y}$$

$$m_1 \times m_2 = \frac{\sin y}{x\cos y - \cos y + y\sin y} \times \frac{\cos y}{-x\sin y + \sin y + y\cos y}$$

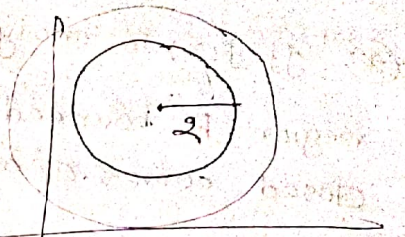




+ directive
→ ant



$$f(z) = \frac{1}{z-3}$$



$C \to |z| = 2$

$$\oint_C f(z)\,dz = 0$$

circle এর respect
এ $f(z)$ analytic
Analytic হলে
অবশ্যই closed curve = 0

Simple Close Curve:

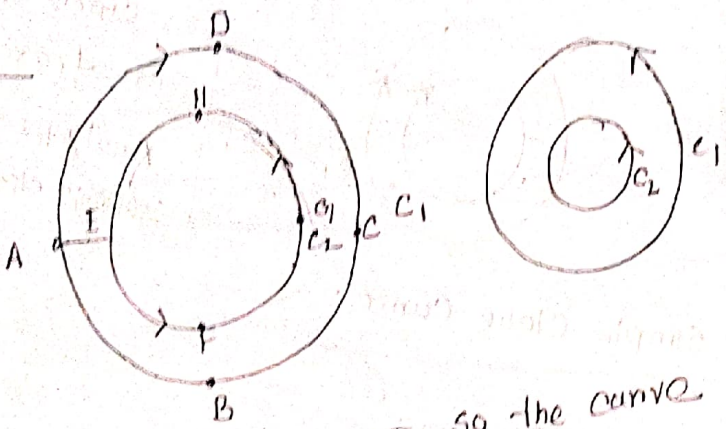
Cauchy's Theorem:

Statement:

If $f(z)$ is analytic and inside and on a region R and bounded by a simple closed curve C and $f'(z)$ is continuous there then $\oint_C f(z)\,dz = 0$

(*) If $f(z)$ is analytic inside and on a region R bounded by two simple closed curves C_1 and C_2 $\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$ where C_1 and C_2 are both traversed in positive sense relative to their interiors.

Sol^n:



Draw a cross section AE so the curve AEFGHEDCBA is a simple close close curve. Then by Cauchy's Theorem.

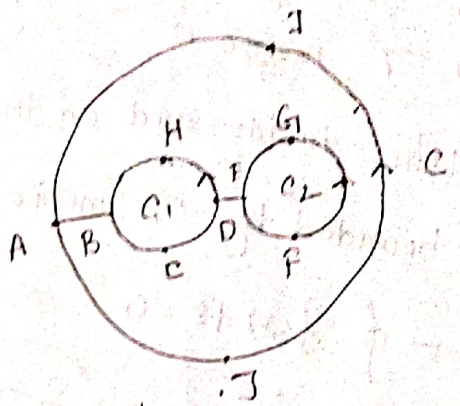
$$\oint_{AEFGHEDCBA} f(z)\,dz = 0$$

$$\Rightarrow \int_{AE} f(z)dz + \oint_{EFGH} f(z)dz + \oint_{EA} f(z)dz + \oint_{ADCBA} f(z)dz = 0$$

$$\int_{AE} f(z)dz - \int_{AE} f(z)dz + \int_{EFGH} f(z)dz - \int_{ABCDA} f(z)dz = 0$$

$$\Rightarrow \oint_{C_1} f(z)dz = \oint_{C_2} f(z)\,dz$$

[proved]



$$\oint_{ABCDEFGEDHBAIJA} f(z\,dz = 0$$

$$\oint_{AB} f(z)dz + \oint_{BCD} f(z)dz + \oint_{DE} f(z)dz + \oint_{EFGE} f(z)dz + \oint_{ED} f(z)dz$$

$$+ \int_{DHB} f(z)\,dz + \int_{BA} f(z)dz + \int_{AIJA} f(z)\,dz = 0$$

$$= \int_{AB} f(z)dz - \int_{AB} f(z)dz + \int_{BCD} f(z)dz + \int_{DHB} f(z)dz + \int_{DE} f(z)dz$$

$$- \oint_{DE} f(z)dz + \int_{EFGE} f(z)dz - \int_{AJCIA} f(z)dz$$

$$\Rightarrow \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz = \oint_{C} f(z)dz$$

Evaluate $\oint_C \frac{dz}{z-a}$ where C is any closed curve and $z=a$ (i) inside C (ii) outside C.

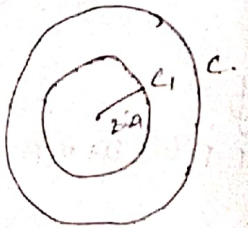
Soln:

$z=a$ is inside C, let $f(z) = \frac{1}{z-a}$

Construct a circle C_1 which center at a and radius r inside C.

so, $f(z)$ is analytic inside and on the closed curve C bounded by C_1 (inside C)

so, $\oint_C f(z)\,dz - \oint_{C_1} f(z)\,dz = 0$



$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz = 0$

$\oint_{C_1} \frac{dz}{z-a} = \oint_C \frac{dz}{z-a}$ $\boxed{z-a = x+iy}$

On C_1; $|z-a| = r$

$z-a = re^{i\theta}$

$z = x+iy$

$z = re^{i\theta}$

$|z| = \sqrt{x^2+y^2}$

$= r$

$\boxed{|x+iy-a| = \sqrt{(x-a)^2+y^2} = r}$

$x+iy =$

$|x+iy-a| = re^{i\theta}$

$\boxed{(x-a)+iy = re^{i\theta}}$

$\boxed{((x-a)+iy) = \sqrt{(x-a)^2+y^2} = r}$

$dz = rie^{i\theta}\,d\theta$

(1) $\Rightarrow \int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{\cancel{rie^{i\theta}}\,d\theta}{\cancel{re^{i\theta}}}$

$= i[2\pi - 0] = 2\pi i$

(2) $z=a$ is outside C, $f(z) = \frac{1}{z-a}$

so, $f(z)$ is analytic inside and on C then by cauchy's theorem

$\oint_C f(z)\,dz = 0$, $\oint_{C_1} \frac{dz}{f(z)-a}\,dz = 0$

Vector Differential Operator

$$\nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k} = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}$$

is called the vector differential operator.

∇ it is called nebula operator

Gradient: Let $\varphi = \varphi(x,y,z)$ be a function of x, y, z. The gradient of φ = grad $\varphi = \nabla\varphi$

$$= \left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right)(\varphi(x,y,z))$$

$$= \frac{\partial\varphi}{\partial x}\underline{i} + \frac{\partial\varphi}{\partial y}\underline{j} + \frac{\partial\varphi}{\partial k}\underline{k}$$

Divergence $\underline{V} = \nabla \cdot V$

$$\underline{V} = V_1\underline{i} + V_2\underline{j} + V_3\underline{k}$$

$$= \left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right) \cdot (V_1\underline{i} + V_2\underline{j} + V_3\underline{k})$$

$$= \frac{\partial V_1}{\partial x}(\underline{i}\cdot\underline{i}) + \frac{\partial V_2}{\partial y}(\underline{j}\cdot\underline{j}) + \frac{\partial V_3}{\partial z}(\underline{k}\cdot\underline{k})$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Directional derivative

(কোন একটা surface এর direction কোন দিকে হবে)

$$\varphi = \varphi(x,y,z)$$

$$\text{Quad } \varphi = \nabla\varphi = \frac{\partial\varphi}{\partial x}\underline{i} + \frac{\partial\varphi}{\partial y}\underline{j} + \frac{\partial\varphi}{\partial z}\underline{k}$$

In the direction $\rightarrow \underline{A} = A_1\underline{i} + A_2\underline{j} + A_3\underline{k}$

The unit vector of $\underline{A}$ is

$$\underline{n} = \frac{\underline{A}}{|A|} = \frac{A_1\underline{i} + A_2\underline{j} + A_3\underline{k}}{\sqrt{A_1^2 + A_2^2 + A_3^2}}$$

The directional derivative in the direction $\underline{A}$ is

$$\nabla\varphi \cdot n = \text{constant}$$

$$\text{curl } \underline{A} = \nabla \times \underline{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \underline{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \underline{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right)$$

$$+ \underline{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

If $\varphi(x,y,z) = 3x^2y - y^3z^2$, find $\nabla\varphi$ (grad φ) of the point $(1,-2,-1)$

$$\nabla\varphi = \left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right)(3x^2y - y^3z^2)$$

$$= 6xy\,\underline{i} + 0 + (3x^2 - 3y^2z^2)\,\underline{j} + 0 - (2y^3z)\,\underline{k}$$

3 – 12

At $(1,-2,-1)$

$$\nabla\varphi = -12\,\underline{i} - 9\,\underline{j} - 16\,\underline{k}$$

(Ans)

Find the value of the constants a, b, c so that directional derivative of $\varphi = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a maximum of magnitude 64 in the direction parallel to z axis.

Step 1 (grad φ) and put value $(1,2,-1)$

z axis $\rightarrow$ $\underline{k}$ is unit vector.

$\boxed{\nabla\varphi \cdot \underline{k}} = 64$ then determine

grad $\varphi = \nabla\varphi$

$$= \left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right)(axy^2 + byz + czx^3)$$

$$= (ay^2 + 3czx^2)\,\underline{i} + (2axy + bz)\,\underline{j} + (by + 2czx^3)\,\underline{k}$$

at point $(1,2,-1)$

$$\nabla\varphi = (4a + 3c)\,\underline{i} + (4a - b)\,\underline{j} + (2b - 2c)\,\underline{k}$$

Directional derivative $\nabla\varphi \cdot \underline{k} = 64$

$$\{(4a + 3c)\,\underline{i} + (4a - b)\,\underline{j} + (2b - 2c)\,\underline{k}\} \cdot \underline{k} = 64$$

$$4a + 3c = 0$$
$$4a - b = 0$$
$$2b - 2c = 64$$
$$b - c = 32$$

Solving $a = 6$, $b = 24$, $c = -8$

(Ans)

* Find a unit normal to the surface $x^2y+2xz=1$ at the point $(2,-2,3)$

Let, $\phi(x,y,z) = x^2y+2xz$

grad $\phi = \nabla\phi = \left(\frac{\partial}{\partial x}\underline{i}+\frac{\partial}{\partial y}\underline{j}+\frac{\partial}{\partial z}\underline{k}\right)(x^2y+2xz)$

$= (2yx+2z)\underline{i} + x^2\underline{j} + 2x\underline{k}$

At $(2,-2,3)$

$\underline{N} = \nabla\phi = -2\underline{i}+4\underline{j}+4\underline{k}$

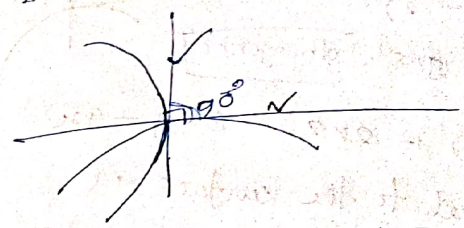
The unit normal $n = \frac{\underline{N}}{|N|}$

$= \frac{-2\underline{i}+4\underline{j}+4\underline{k}}{\sqrt{(-2)^2+4^2+4^2}}$

$= -\frac{1}{3}\underline{i}+\frac{2}{3}\underline{j}+\frac{2}{3}\underline{k}$

Another unit normal is $\frac{1}{3}\underline{i}-\frac{2}{3}\underline{j}-\frac{2}{3}\underline{k}$ having direct opposite of the above.

* Find the angle between the surface $x^2+y^2+z^2=9$ and $z = x^2+y^2-z$ at the point $(2,-1,2)$



To get normal we grad ϕ

Let, $\phi_1 = x^2+y^2+z^2$

$\phi_2 = x^2+y^2-z$

$\nabla\phi_1 = \left(\frac{\partial}{\partial x}\underline{i}+\frac{\partial}{\partial y}\underline{j}+\frac{\partial}{\partial z}\underline{k}\right)(x^2+y^2+z^2)$

$= 2x\underline{i}+2y\underline{j}+2z\underline{k}$

$\nabla\phi_2 = \left(\frac{\partial}{\partial x}\underline{i}+\frac{\partial}{\partial y}\underline{j}+\frac{\partial}{\partial z}\underline{k}\right)(x^2+y^2-z)$

$= 2x\underline{i}+2y\underline{j}-\underline{k}$

at point, $(2,-1,2)$

$\underline{\theta_1} = \nabla\phi_1 = 4\underline{i}-2\underline{j}+4\underline{k}$ $\quad \underline{\theta_2} = \nabla\phi_2 = 4\underline{i}-2\underline{j}-\underline{k}$

$\underline{\theta_1}\cdot\underline{\theta_2} = \theta_1\theta_2\cos\theta \Rightarrow \theta = \cos^{-1}\frac{(4\underline{i}-2\underline{j}+4\underline{k})\cdot(4\underline{i}-2\underline{j}-\underline{k})}{\sqrt{4^2+2^2+4^2}\sqrt{4^2+2^2+1^2}}$

$= \cos^{-1}\frac{8}{2\sqrt{21}} = 54°25'$ $\quad = \cos^{-1}\frac{16+4-4}{6\sqrt{21}}$

* Find the equation of the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1,-1,2)$

Sol^n. / Just $\boxed{\text{divergence}}$

Let, $\varphi(x,y,z) = 2xz^2 - 3xy - 4x$

Now, The Normal to the surface is.

$\underline{N} = \text{grad}\ \varphi = \underline{\nabla}\varphi$

$= \left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right)(2xz^2 - 3xy - 4x)$

$= (2z^2 - 3y - 4)\underline{i} - \underline{j}\ 3x + 4xz\ \underline{k}$

at $(1,-1,2)$

$\underline{N} = 7\underline{i} - 3\underline{j} + 8\underline{k}$

Now the eq^n of the tangent plane passing through a point $(1,-1,2)$ whose position vector is $r_0 = \underline{i} - \underline{j} + 2\underline{k}$ and perpendicular to the normal $\underline{N}$ is $(r - r_0)\cdot\underline{N} = 0$

$\{(x\underline{i} + y\underline{j} + \underline{k}z) - (\underline{i} - \underline{j} + 2\underline{k})\}\cdot(7\underline{i} - 3\underline{j} + 8\underline{k}) = 0$

$\Rightarrow \{(x-1)\underline{i} + (y-1)\underline{j} + (z-2)\underline{k}\}\cdot(7\underline{i} - 3\underline{j} + 8\underline{k}) = 0$

$\Rightarrow 7x - 7 - 3y + 3 + 8z + 16 = 0$

$\Rightarrow 7x - 3y + 8z + 12 = 0$

H.W. Find the constant a, b, c so that the force $\underline{F} = (x + 2y + az)\underline{i} + (bx - 3y - z)\underline{j} + (4x + cy + 2z)\underline{k}$ is irrotational.

(ii) Show, that $\underline{F}$ can be expressed as the gradient of scalar function.

Sol^n:

given $\underline{F}$ is irrotational.

so, curl $\underline{F} = \underline{\nabla} \times \underline{F}$

$$\Rightarrow \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (4x+cy+2z) & (bx-3y-z) \end{vmatrix} = 0$$

$$\Rightarrow \underline{i}\left\{\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right\} = 0$$

$$- \underline{j}\left\{\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right\}$$

$$+ \underline{z}\left\{\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right\}$$

$$\Rightarrow \underline{i}(c+1) - \underline{j}(4-a) + \underline{z}(b-2) = 0$$

$$\boxed{\begin{aligned} c &= -1 \\ a &= 4 \\ b &= 2 \end{aligned}}$$ (Ans)

$\underline{F} = \underline{\nabla} \cdot \varphi$

$$\Rightarrow (x+2y+4z)\,\underline{i} + (2x-3y-z)\,\underline{j} + (4x-y+2z)\,\underline{k}$$

$$= \underline{\nabla}\varphi$$

$$= \frac{\partial \varphi}{\partial x}\underline{i} + \frac{\partial \varphi}{\partial y}\underline{j} + \frac{\partial \varphi}{\partial z}\underline{k}$$

$\Rightarrow$ equating the equation.

$\Rightarrow$ so,

$$\frac{\partial \varphi}{\partial x} = (x+2y+4z) \quad \text{--- (1)}$$

$$\frac{\partial \varphi}{\partial y} = (2x-3y-z) \quad \text{--- (2)}$$

$$\frac{\partial \varphi}{\partial z} = (4x-y+2z) \quad \text{--- (3)}$$

integrating (1) partially w.r.to x taking y & z constants:

$$\varphi = \frac{x^2}{2} + 2xy + 4xz + h(y,z) \quad \text{--- (4)}$$

integrating (2) partially w.r.to y + taking x, z const

$$\varphi = 2yx - \frac{3y^2}{2} - yz + g(x,z) \quad \text{--- (5)}$$

integrating (3) partially w.r.to z taking x & y constant

$$\varphi = 4xz - yz + z^2 + \omega(x,y) \quad \text{--- (6)}$$

Here $h(y,z)$, $g(x,z)$ and $w(x,y)$ are constant functions

Hence $\varphi = \frac{x^2}{2} + 4xz + 2yx - \frac{3y^2}{2} - yz + 4xz + z^2 + \text{constant}$

$= \frac{x^2}{2} + 4xz + 2yx - \frac{3y^2}{2} - yz + z^2 + \text{constant}$

(Ans)

Vector integration

* If $\underline{A} = (3x^2+6y)\underline{i} - 14yz\underline{j} + 20x^2z\underline{k}$

Evaluate $\oint_C A \cdot dr$ from $(0,0,0)$ to $(1,1,1)$ along the following path C

(a) $x = t,\ y = t^2,\ z = t^3$

(b) The straight line from $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ and then to $(1,1,1)$

(c) The straight line from $(0,0,0)$ to $(1,1,1)$

Soln,

Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ be position vector of a point.

$d\underline{r} = dx\underline{i} + dy\underline{j} + dz\underline{k}$

Now, $\oint_C A\, d\underline{r} = \oint_C \{(3x^2+6y)\underline{i} - 14yz\underline{j} + 20x^2z\underline{k}\} \cdot (dx\underline{i} + dy\underline{j} + dz\underline{k})$

$= \oint_C (3x^2+6y)\,dx - \oint_C 14yz\,dy + \oint_C 20x^2z\,dz$ —(1)

(a) $x = t, \quad y = t^2, \quad z = t^3$

$dx = dt \; ; \Rightarrow dy = 2t\,dt \Rightarrow dz = 3t^2 dt$

when, $x = 0, y = 0, z = 0$ then $t = 0$

when, $x = 1, y = 1, z = 1$ then $t = 1$

$$① \Rightarrow \oint_C \underline{A}\, dr = \int_{t=0}^{1} \{ (3t^2 + 6t^2)\,dt - 14t^2 t^3\, 2t\,dt + 20t^2 t^3\, 3t^2 dt \}$$

(b) The straight line passing through the points $(0,0,0)$ to $(1,0,0)$ is

$$\frac{x-0}{0-1} = \frac{y-0}{0-0} = \frac{z-0}{0-0} = -t$$

$\Rightarrow \quad x = +t \qquad y = 0 \qquad z = 0$

$dx = dt \qquad dy = 0 \qquad dz = 0$

$$① \Rightarrow \oint_0^1 (3t^2 + 0)\,dt + 0$$

$$= 3\left[\frac{t^3}{3}\right]_0^1$$

$$= 1$$

The straight line passing through the points $(1,0,0)$ to $(1,1,0)$ is

$$\frac{x-1}{1-1} = \frac{y-0}{0-1} = \frac{z-0}{0-0} = -t$$

$\Rightarrow \quad x - 1 = 0 \qquad \Rightarrow y = t \qquad \Rightarrow z = 0$

$x = 1 \qquad dy = dt \qquad dz = 0$

$dx = 0$

Now

$$t \Rightarrow \oint_0^1 0 + 0 + 0 = 0$$

Now, part 3:

The straight line passing through $(1,1,0)$ to $(1,1,1)$ is

$$\frac{x-1}{1-1} = \frac{y-1}{1-1} = \frac{z-0}{0-1} = -t$$

$x = 1, \quad y = 1, \quad z = t$

$\therefore \; dx = 0, \; dy = 0 \quad dz = dt$

$$\text{So } ① \Rightarrow \int_{t=0}^{1} 20t\, dt = \left[\frac{20t^2}{2}\right]_0^1 = 10$$

(c) The straight line joining the point $(0,0,0)$ to $(1,1,1)$ is

$$\frac{x-0}{0-1} = \frac{y-0}{0-1} = \frac{z-0}{0-1} = -t$$

$\Rightarrow x = t \Rightarrow y = t \quad \Rightarrow z = t$

$dx = dt \quad \therefore dy = dt \quad dz = dt$

(1) $\Rightarrow \oint \underline{A}\, dr = \int_{t=0}^{2}$

$=$