Numerical Methods

Question - 2018

5.a) **Liebmann's Iteration Method:**

We know that a diagonally dominant system of linear equations can be solved by iteration methods ~~sus~~ such as Gauss-Seidel method. When such an iteration is applied to Laplace's equation, the iterative method is called Liebmann's iterative method.

To obtain the pivotal values of f by Liebmann's iterative method, we solve for f_{ij}. The equation is,

$$f_{ij} = \frac{1}{4}\left(f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1}\right)$$

The value ~~of~~ f_{ij} at the point ij is the average of the values of f at the four adjoining points. If we know the "initial values" of the functions at the right hand side of the equation $f_1 - 4f_2 + f_4 = -100$; we can estimate the value f at the point ij. We can substitute the values thus obtained into the right-hand side to achieve improved approximations. This process may continue till the values f_{ij} converge to constant values.

Initial values may be obtained by either taking diagonal average or cross average of the adjoining four points.

Advantages: In the Liebmann method a correction process ~~it~~ is applied to each of the lattice points in succession in a regular pattern. The ϕ-value so corrected is used in all subsequent operations in that iteration step. It may thus be termed a "continuous substitution method". In its simplest form the lattice is scanned in the same direction along successive rows. Thus applied to the Laplace equation and boundary conditions described above, the Liebmann iteration process may be written as,

$$\phi_{j,k}^{n+1} = \phi_{j,k}^{n} + \alpha[\phi_{j-1,k}^{n+1} + \phi_{j+1,k}^{n} + \phi_{j,k-1}^{n+1} + \phi_{j,k+1}^{n} - 4\phi_{j,k}^{n}] \quad (1)$$

If α is ~~given~~ again given the value $1/4$ this expression becomes

$$\phi_{j,k}^{n+1} = \frac{1}{4}[\phi_{j-1,k}^{n+1} + \phi_{j+1,k}^{n} + \phi_{j,k-1}^{n+1} + \phi_{j,k+1}^{n}]$$

thus $L\phi_{j,k}$ is brought to zero (momentarily) at each of the lattice points succession. In this form the Liebmann procedure may be regarded as a very mechanical application of the relaxation method.

The iteration equation for the error

$$e_j^J \quad e_{j,k}^{n} \equiv \phi_{j,k}^{n} - \phi_{j,k}$$

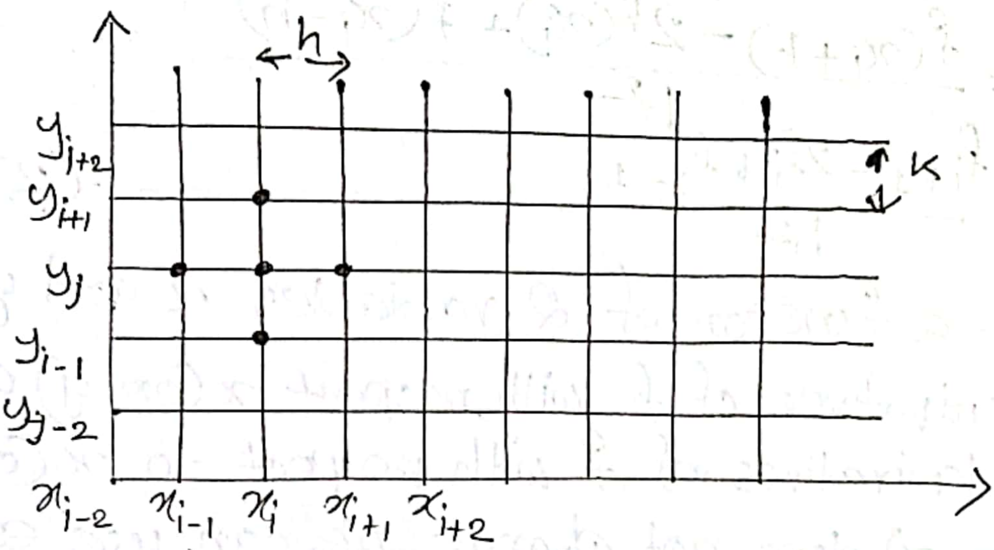
It is obtained by substitutions in eqn (1)

$$e_{j,k}^{n+1} = e_{j,k}^{n} + \alpha[e_{j-1,k}^{n+1} + e_{j+1,k}^{n} + e_{j,k-1}^{n+1} + e_{j,k+1}^{n} - 4e_{j,k}^{n}]$$

$$= 0$$

This error iteration process can be written briefly as $e^{n+1} = K(\alpha) e^{n}$; where $K(\alpha)$ is a linear operator depending on the parameter α.

2018 5.b)



$x_{i+1} = x_i + h$

$y_{j+1} = y_j + h$

fig: Two dimensional finitie difference grid

f_{ij} = A function of the two space variables x and y (The pivotal values at the points of intersection)

In the finite difference method, we replace derivatives that occur in the PDE by their finite difference equivalents. We then write the difference equation corresponding to each "grid point" (where derivative is required) using function values at the surrounding grid points. Solving these equations simultaneously gives values for the function at each grid point.

If the function $f(x)$ has a continuous 4th derivative, then its 1st and 2nd derivatives are given by the following central difference approxim...

$$f'(x_i) = \frac{f(x_i+h) - f(x_i-h)}{2h}$$

$$\Rightarrow f_i' = \frac{f_{i+1} - f_{i-1}}{2h} \quad \text{-------------- (1)}$$

$$f''(x_i) = \frac{f(x_i+h) - 2f(x_i) + f(x_i-h)}{h^2}$$

$$\Rightarrow f_i'' = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} \quad \text{-------------- (2)}$$

When f is a function of 2 variables x and y, the partial derivatives of f with respect x (or y) are the ordinary derivatives of f with respect to x (or y) when y (or x) does not change. We can use eqn (1) & (2) in the x-direction to determine derivatives with respect to y. Thus we have,

$$\frac{\partial f(x_i, y_j)}{\partial x} = f_x(x_i, y_j) = \frac{f(x_{i+1}, y_j) - f(x_{i-1}, y_j)}{2h}$$

$$\frac{\partial f(x_i, y_j)}{\partial y} = f_y(x_i, y_j) = \frac{f(x_i, y_{j+1}) - f(x_i, y_{j-1})}{\cancel{2h}\ 2k}$$

$$\frac{\partial^2 f(x_i, y_j)}{\partial x^2} = f_{xx}(x_i, y_j) = \frac{f(x_{i+1}, y_j) - 2f(x_i, y_j) + f(x_{i-1}, y_j)}{h^2}$$

$$\frac{\partial^2 f(x_i, y_j)}{\partial y^2} = f_{yy}(x_i, y_j) = \frac{f(x_i, y_{j+1}) - 2f(x_i, y_j) + f(x_i, y_{j+1})}{\cancel{k^2}\ k^2}$$

$$\frac{\partial^2 f(x_i, y_j)}{\partial x \partial y} = \frac{f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_{j-1}) - f(x_{i-1}, y_{j+1}) + f(x_{i-1}, y_{j-1})}{4hk}$$

It is convenient to use double subscripts i, j on f to indicate x and y values. Then, the above equations become, ~~becon~~

$$f_{x,ij} = \frac{f_{i+1,j} - f_{i-1,j}}{2h}$$

$$f_{y,ij} = \frac{f_{i,j+1} - f_{i,j-1}}{2k}$$

$$f_{xx,ij} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{k^2}$$

$$f_{y,ij} = \frac{f_{i,j+1} - 2f_{ij} + f_{i,j-1}}{k^2}$$

$$f_{xy,ij} = \frac{f_{i+1,j+1} - f_{i+1,j-1} - f_{i-1,j+1} + f_{i-1,j-1}}{4hk}$$

We know, we can write a second order equations involving two independent variables in general form as,

$$a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial x \partial y} + c\frac{\partial^2 f}{\partial y^2} = f\left(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

When $a = 1$, $b = 0$, $c = 1$ and $f\left(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = g(x, y)$ ~~a~~ the above equation becomes,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = g(x, y)$$

$$\Rightarrow \nabla^2 f = g(x, y) \text{ ------ } (3) \quad \left[\because \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \nabla^2 f\right]$$

Eqn (3) is called Poisson's Equation.

We know, $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \nabla^2 f = 0$

$\Rightarrow f_{xx,ij} + f_{yy,ij} = \nabla^2 f$

$\Rightarrow \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{h^2} + \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{k^2} = \nabla^2 f_{ij} = 0$

If we assume, for simplicity $h=k$ then we get,

$\therefore \nabla^2 f_{ij} = \frac{1}{h^2}(f_{i+1,j} + f_{i-1,j} - 4f_{ij} + f_{i,j+1} + f_{i,j-1}) = 0$

$\Rightarrow h^2 \nabla^2 f_{ij} = f_{i+1,j} + f_{i-1,j} - 4f_{ij} + f_{i,j+1} + f_{i,j-1}$

$\Rightarrow h^2 g(x,y) = f_{i+1,j} + f_{i-1,j} - 4f_{ij} + f_{i,j+1} + f_{i,j-1}$ $[\because \nabla^2 f = g(x,y)]$

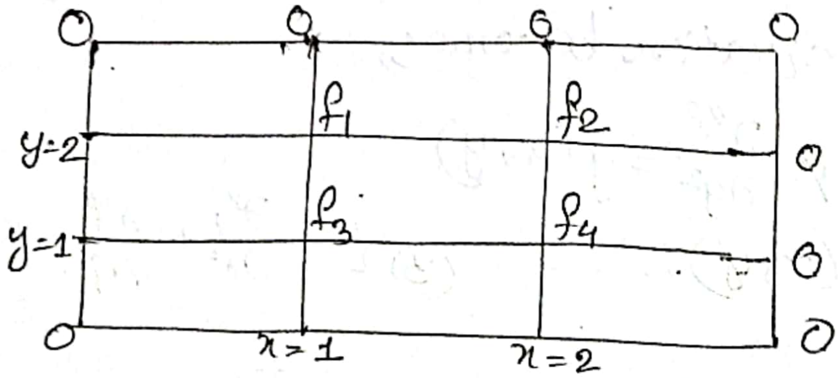
Using the notation $g_{ij} = g(x_i, y_j)$

$\therefore f_{i+1,j} + f_{i-1,j} - 4f_{ij} + f_{i,j+1} + f_{i,j-1} = h^2 g_{ij}$

This is the finite difference formula for Poisson's equation.

2013

5.c) The domain is divided into squares of one unit size as illustrated below:



We know, the finite difference formula is,

$$f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4f_{i,j} = h^2 g_{ij}$$

By applying the equation at each grid point, we get the following set of equations,

Point 1: $0 + 0 + f_2 + f_3 + \cancel{f} - 4f_1 = 2 \times 1^2 \times 2^2$

$$\Rightarrow f_2 + f_3 - 4f_1 = 8 \quad \text{———} \quad (\text{I})$$

Point 2: $0 + 0 + f_1 + f_4 - 4f_2 = 2 \times 2^2 \times 2^2$

$$\Rightarrow f_1 - 4f_2 + f_4 = 32 \quad \text{———} \quad (\text{II})$$

Point 3: $0 + 0 + f_1 + f_4 \cancel{4f_3} - 4f_3 = 2 \times 1^2 \times 1^2$

$$\Rightarrow f_1 - 4f_3 + f_4 = 2 \quad \text{———} \quad (\text{III})$$

Point 4: $0 + 0 + f_2 + f_3 - 4f_4 = 2 \times 2^2 \times 1^2$

$$\Rightarrow \cancel{4}\, f_2 + f_3 - f_4 = 8 \quad \text{———} \quad (\text{IV})$$

Rearranging the equations (I), (II), (III), (IV) we get

$$-4f_1 + f_2 + f_3 = 8$$
$$f_1 - 4f_2 + f_4 = 32$$
$$f_1 - 4f_3 + f_4 = 2$$
$$f_2 + f_3 - 4f_1 = 8$$

Solving these equations by elimination method, we get the answers,

$f_1 = -\frac{22}{4}$ $\qquad$ $f_2 = -43/4$

$f_3 = \cancel{11} - 13/4$ $\qquad$ $f_4 = -22/4$

2018

6. a) Dependent Variable:

A variable (often denoted by y) whose value depends on that anothero is called dependent variable. The dependent variable is the effect. Its value depends on changes in the independent variable. Dependant variable is the outcome that we measure.

Let $y = 3x + 5$

Here y is the dependent variable.

Independent variable:

A variable (often denoted by x) whose value doesn't depend on that of anothero. The independent variable is the cause, its value is independent of othero variables. ~~in~~ The independent variable is usually applied at different levels to see how the outcome differs.

Let $y = 3x+5$; Here x is the independent variable.

2018

6. b) In Eulero's method, consider the equation,

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0 = 1 \quad \text{------} (1)$$

$$\Delta y = \Delta x \tan\theta = \Delta x \left(\frac{dy}{dx}\right)_0 = \Delta x \, f(x_0, y_0)$$

$$\therefore y_1 = y_0 + \Delta x \cdot f(x_0, y_0)$$

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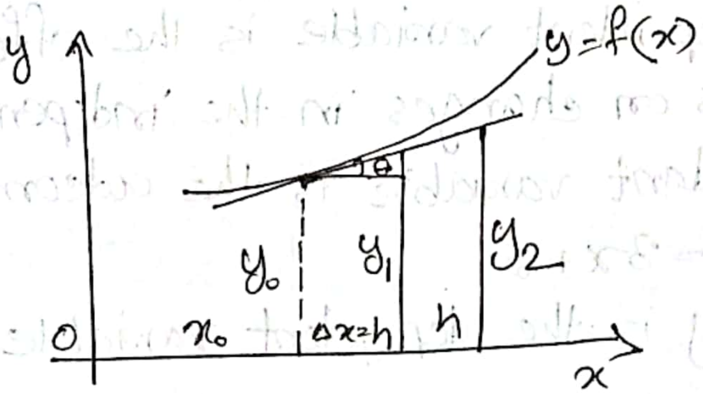
In general we obtain,

$$y_{n+1} = y_n + hf(x_n, y_n)$$

[where $n = 0, 1, 2, \ldots$]



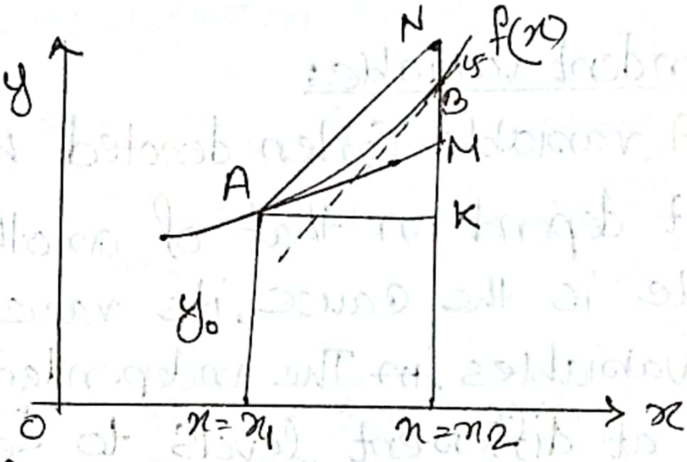


fig: Modified Euler's Method

Starting with the initial value y_0, an approximate value for y_1 is calculated from the relation,

$$y_1^{(1)} \approx y_0 + h\left(\frac{dy}{dx}\right)_0 = y_0 + hf(x_0, y_0)$$

Substituting this value of y_1 into the given differential equation (1), we get an approximate value of dy/dx at the end of the first interval;

$\left(\frac{dy}{dx}\right)_1^{(1)} = f(x_1, y_1^{(1)})$

Now an improved value of Δy is obtained as
$\Delta y = h$ [avg of values $\frac{dy}{dx}$ at the ends of the interval x_0 to x_1]

$$\therefore \Delta y \approx h \, \frac{\left(\frac{dy}{dx}\right)^0 + \left(\frac{dy}{dx}\right)_1^{(1)}}{2}$$

The second approximation for y_1 is $y_1^{(2)} = y_0 + h \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^{(1)}}{2}$
Substituting this improved value of $y_1^{(2)}$ in the given equation (1), we get a second approximation for

$$\left(\frac{dy}{dx}\right)_1^{(2)} = f(x_1, y_1^{(2)})$$

The third approximation for y_1 is given by,

$$y_1^{(3)} = y_0 + h \, \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^{(2)}}{2}$$

The process is applied until no changes is produced in the value of y_1 to the desired degree of accuracy.

In the same manner we make computations for the next interval x_1 to $x_2 (= x_1 + h)$ first by finding an approximate value of Δy and then using the averaging process until no improvement is made in the value of y_2

First approximation to y_2, y_3, y_4 .... etc could be obtained by using formula:

$$y_{n+1} = y_n + h\left(\frac{dy}{dx}\right)_n$$

2018

6.c) $\frac{dy}{dx} = \frac{y-x}{y+x}$

Here we want the value at $x = 0.1$ from $x = 0$.
We can split value in 5 parts. So the points will be x_1, x_2, x_3, x_4, x_5.

$h = \frac{0.1-0}{5} = 0.02$

We shall find the values of y at $x = 0.02, 0.04, 0.06, 0.08$ and 0.1 successively.

Thus we have $x_0 = 0$, $y_0 = 1$, $h = 0.02$; $f(x,y) = \frac{y-x}{y+x}$

Using $y_{n+1} = y_n + hf(x_n, y_n)$ we get,

$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.02 \times \frac{1-0}{1+0} = 1.02$

$y_2 = y_1 + hf(x_1, y_1) = 1.02 + 0.02 \times \frac{1.02 - 0.02}{1.02 + 0.02} = 1.0396$

$y_3 = y_2 + hf(x_2, y_2) = 1.0396 + 0.02 \times \frac{1.0392 - 0.04}{1.0392 + 0.04} = 1.0577$

$y_4 = y_3 + hf(x_3, y_3) = 1.0577 + 0.02 \times \dfrac{1.0577 - 0.06}{1.0577 + 0.06}$

$= 1.0756$

$y_5 = y_4 + hf(x_4, y_4) = 1.0756 + 0.02 \times \dfrac{1.0756 - 0.08}{1.0756 + 0.08}$

$= 1.0928$

Hence $y = 1.0928$ when $x = 0.1$ (Ans)

2018

7. a) Differences between homogeneous and non-homogeneous system of equations are given below:

Homogeneous	Non homogeneous
1) A homogeneous system of linear equations is one in which all constant terms are zero.	1) A non homogeneous system of linear equations is one in which all constant terms are not zero.
2) This system always has at least one solution, namely the zero vector.	2) A $n \times n$ nonhomogeneous system of linear equations has a unique non-trivial solution if and only if its determinant is non zero. If this determinant is zero, then the system has either no non-trivial solutions or an infinite numbers of solutions.

Homogeneous	Non Homogeneous
3) When a row operation is applied to a homogeneous system, the new system is still homogeneous.	3) A non homogeneous system has an associated homogeneous system, which you get by replacing the constant terms in each equation with zero.
4) When ~~a~~ we represent a homogeneous system as a matrix, we often leave ~~the~~ off the final column of constant terms, since applying row operations would not modify that column. So we use ~~an~~ a regular matrix instead of an augmented matrix.	~~3~~ 4) We represent a non-homogeneous system as a matrix in augmented matrix form.

7.b) Runge-Kutta method is an effective and widely used method for solving the initial value ~~va~~ problems of differential equations. ~~It~~ Runge-Kutta method can be used to construct high order accurate numerical

method functions' self without needing the high order derivatives of the functions.
The Runge - Kutta method attempts to overcome the problem of the Euler's method, as far as the choice of a sufficiently small step size is concerned, to reach a reasonable accuracy in the problem resolution.
In Euler's method if the interval size is not small then result can be inaccurate. The R-K method produces a better result in fewer steps.

The R-K method is used to solve the Ordinary differential equations. In some case it is also used in solving a partial differential equations. For solving partial differential equation, we have to discretize the space of PDE. The idea is to apply Fourier transform w.r.to space, doing this removes the derivative (w r to x) terms and thus converts the system in the form of $y' = f(y)$ which can be solved using R-K method.
Now let's consider a particular example:

$$\frac{dy}{dx} = \frac{5y}{1+x}, \; y = 1 \text{ when } x = 0$$

In each case take $h = 0.1$ and carry the computation to $x = 1$.

The results obtained by applying this method are given below and compared with those obtain from the actual solution.

x	Runge Kutta Method	Error	Actual solution $(1+x)^5$
0	1.0000	-----	1.0000
0.1	1.6103	0.0002	1.6105
0.2	2.4878	0.0005	2.4883
0.3	3.7119	0.0010	3.7129
0.4	5.3765	0.0017	~~3~~ 5.3782
0.5	7.5911	0.0027	7.5938
0.6	10.4819	0.0039	10.4858
0.7	14.1931	0.0055	11.1986
0.8	18.8882	0.0075	18.8957
0.9	24.~~5~~7509	0.0101	24.7610
1.0	31.9867	0.0133	32.0000

The main advantages of R-K methods are that they are easy to implement, very stable, and they are self starting" (unlike multisteps methods, we don't have to treat the first few steps taken by a single step integration method as special cases). This method is sometimes referred to as single-step methods, since they evolve the solution from x_n to x_{n+1} without needing to know the solutions at x_{n-1}, x_{n-2} etc.

2018

7.e) $\frac{dy}{dx} = \frac{1}{x+y}$ for $x=0.5$ to $x=2$ assume $h=0.5$ by using R-K method, with $x_0 = 0$, $y=1$

Here $f(x,y) = \frac{1}{x+y}$; $h = 0.5$

Now $k_1 = hf(x_0, y_0) = 0.5 \times \frac{1}{0+1} = 0.5$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.5 \times \frac{1}{0.25 + 1.25}$

$= 0.33$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.5 \times \frac{1}{0.25 + 1 + \frac{0.33}{2}}$

$= 0.353$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.5 \times \frac{1}{0.5 + 1.353} = 0.2698$

Thus $x_1 = x_0 + h = 0 + 0.5 = 0.5$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.35597$

$\therefore y_1 = y_0 + \Delta y = 1 + 0.35597 = 1.35597$

Now for the second interval,

$k_1 = hf(x_1, y_1) = 0.5 \times \frac{1}{0.5 + 1.35597} = 0.2694$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.5 \frac{1}{0.5 + \frac{0.5}{2} + 1.35597 + \frac{0.2694}{2}}$

$= 0.2231$

$k_3 = hf\left(x_1 + \frac{h}{2}, \cancel{x} y_1 + \frac{k_2}{2}\right) = 0.5 \times \frac{1}{0.5 + \frac{0.5}{2} + 1.35597 + \frac{0.2231}{2}}$

$= 0.22547$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.5\left(\frac{1}{0.5 + 0.5 + 1.35597 + 0.22547}\right)$

$= 0.19369$

~~Hen~~ $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2267$

Hence $x_2 = x_1 + h = 0.5 + 0.5 = 1$

$\cancel{y_3}\ y_2 = y_1 + \Delta y = 1.35597 + 0.2267$

$= 1.5827$

Now for the third interval,

$K_1 = hf(x_2, y_2) = 0.5 \times \frac{1}{1 + 1.5827} = 0.19359$

$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.5 \times \frac{1}{125 + 1.5827 + \frac{0.19359}{2}}$

$= 0.17068$

$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0.5 \times \frac{1}{1.25 + 1.5827 + \frac{0.17068}{2}}$

$= 0.1713$

$k_4 = hf(x_1 + h, y_2 + k_3) = 0.5 \frac{1}{1.50 + 1.5827 + 0.1713}$

$= 0.1537$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1718$

Hence $x_3 = x_2 + h = 1 + 0.5 = 1.5$

$y_3 = y_2 + \Delta y = 1.5827 + 0.1718 = 1.7545$

Now for the fourth interval,

$k_1 = hf(x_3, y_3) = 0.5 \times \frac{1}{1.5 + 1.7545} = 0.1536$

$k_2 = hf(x_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}) = 0.5 \times \frac{1}{1.5 + \frac{0.5}{2} + 1.7545 + \frac{0.1536}{2}}$

$= 0.1396$

$k_3 = hf(x_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}) = 0.5 \times \frac{1}{1.5 + \frac{0.5}{2} + 1.7545 + \frac{0.1396}{2}}$

$= 0.1399$

$k_4 = hf(x_3 + h, y_3 + k_3) = 0.5 \times \frac{1}{1.5 + 0.5 + 1.7545 + 0.1399}$

$= \cancel{1} = 0.1284$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1402$

Hence $x_4 = x_3 + h = 1.5 + 0.5 = 2$

$y_4 = y_3 + \Delta y = 1.7545 + 0.1402 = 1.8947$

2018

8.a) $27x + 6y - z = 85$

$6x + 15y + 2z = 72$

$x + y + 54z = 110$

Starting with $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ we get

$x^{(1)} = \frac{85}{27} = 3.15 \quad y^{(1)} = \frac{72}{15} = 4.8 \quad z^{(1)} = \frac{110}{54} = 2.04$

These are the first approximations, we proceed to obtain the second approximations as follows:

$$x^{(2)} = \frac{1}{27}(85 - 6y^{(1)} + z^{(1)}) = \frac{1}{27}(85 - 28.8 + 2.04)$$
$$= 2.16$$

$$y^{(2)} = \frac{1}{15}(72 - 6x^{(1)} - 2z^{(1)}) = \frac{1}{15}(72 - \cancel{6}\ 18.9 - 4.08) = 3.27$$

$$z^{(2)} = \frac{1}{54}(110 - x^{(1)} - y^{(1)}) = \frac{1}{54}(110 - 3.15 - 4.8) = 1.89$$

Similarly the next approximations are given by.

$$x^{(3)} = \frac{1}{27}(85 - 6y^{(2)} - z^{(2)}) = \frac{1}{27}(85 - 6 \times 3.27 - 1.89)$$
$$= 2.3515$$

$$y^{(3)} = \frac{1}{15}(72 - 6x^{(2)} - 2z^{(2)}) = \frac{1}{15}(72 - 6 \times 2.16 - 2 \times 1.89)$$
$$= 3.684$$

$$z^{(3)} = \frac{1}{54}(110 - x^{(2)} - y^{(2)}) = \frac{1}{54}(110 - 2.16 - 3.27) = 1.936$$

The values of 2nd & 3rd approximations are almost same.

So, $x = 2.3515$, $y = 3.684$, $z = 1.936$ (Ans)

We can ~~f~~ continue the iteration method to get more accurate values.

2013

8.b)
$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

Here
$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Let $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$\therefore u_{11} = a_{11} = 2 \qquad \therefore u_{12} = a_{12} = 3 \qquad \therefore u_{13} = a_{13} = 1$

$$\therefore l_{21} = \frac{a_{21}}{u_{11}} = \frac{1}{2}$$

$$\therefore u_{22} = a_{22} - \frac{a_{21}}{a_{11}} \times a_{12} = 2 - \frac{1}{2} \times 3 = \frac{1}{2}$$

$$\therefore u_{23} = a_{23} - \frac{a_{21}}{a_{11}} \times a_{13} = 3 - \frac{1}{2} \times 1 = \frac{5}{2}$$

$$\therefore l_{31} = \frac{a_{31}}{a_{11}} = \frac{3}{2}$$

$$\therefore l_{32} = \frac{a_{32} - (a_{31}/a_{11})\, a_{12}}{a_{22} - (a_{21}/a_{11})\, a_{12}} = -7$$

$$\therefore u_{33} = a_{33} - u_{13} l_{31} - l_{32} u_{23} = 18$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$$

Now the system can be written as.

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1/2 & 0 & 0 \\ 3/2 & -7 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}}_{B} \qquad [\because LUX = B]$$

$$\begin{array}{ccc} L & Y & B \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$ $[\because UX = Y]$ ~~$[\because UX = B]$~~

This system is equivalent to $y_1 = 9$

$\frac{1}{2} y_1 + y_2 = 6 \Rightarrow y_2 = 3/2$

$3/2\, y_1 - 7y_2 + y_3 = 8 \quad \therefore y_3 = 5$

Now the solution of the original system is given by.

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$ $[\because UX = B]$

$\therefore z = 5/18$

$\Rightarrow 1/2\, y + \frac{5}{2} z = 3/2 \quad \therefore y = \frac{29}{18}$

$\Rightarrow 2x + 3y + z = 9 \quad$ ~~$\therefore z =$~~ $\therefore x = \frac{35}{18}$

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8.c) The bisection method is used to find the roots of a polynomial equation. It seperates the intervals and subdivides the interval in which the root of the equation lies. The principle behind this method is the intermediate theorem for continuous functions. It works by ~~continuining~~ narrowing the gap between the positive and negative until it closes in ~~the~~ on the correct answer. This method narrows the gap by taking the average of the positive and negative intervals. It is a simple method and it is relatively slow. The bisection method is also known as interval halving method, root finding method, binary search method or dichotomy method.

The fundamental mathematical principle ~~using~~ underlying the Bisection method is the Intermediate Value theorem.

Theorem 1.1: Let $f:[a,b]\rightarrow[a,b]$ be a continuous function. Suppose that d is any value between $f(a)$ and $f(b)$. Then there is a ~~ca~~ c, $a<c<b$, such that $f(c)=d$.

In particular, the ~~Intermiate~~ Intermediate Value Theorem implies that if $f(a)f(b)<0$, then there is a point c, $a<c<b$ such that $f(c)=0$. Thus if we have a continuous function f ~~an~~ on an interval such that $f(a)\cdot f(b)<0$, then $f(x)=0$ has a solution in the interval.

The Intermediate Value Theorem not only guarantees a solution to the equation, but it also provides a means of numerically approximating a solution to arbitrary accuracy.

Proceed as follows:

1. Let $e>0$ be the upper bound for the error required of the answer.

2. Compute $c=\frac{a+b}{2}$ and ~~f~~ $d=f(c)f(a)$.

3. If $d<0$, then let ~~be~~ $b=c$ and $a=a$. If $d>0$, then let $a=c$ and $b=b$. If $d=0$ then c is a solution of $f(x)=0$ and a solution has been found to the required accuracy.

4. The new interval $[a,b]$ will then be half the length of the original $[a,b]$ and will contain a point $x\in[a,b]$ such that $f(x)=0$

Repeat 2, 3 and 4 until either an exact solution is found in 3 or until at step 4 half the length of $[a,b]$ is less than e, $\frac{b-a}{2}<e$