Newton Raphson method ①

Prove:

Consider x_n a root of $f(x)$.

Now a small interval h is considered as below,

$$h = x_{n+1} - x_n$$

from Tylor series expansion, we can say

$$f(x_{n+1}) = f(x_n) + f'(x_n) \cdot h + f''(x_n) \cdot \frac{h^2}{2!} + f'''(x_n) \frac{h^3}{3!} + \cdots$$

As h is very small we can ignore 2 or higher power of h.

So $f(x_{n+1}) = f(x_n) + f'(x_n) h$.

As x_{n+1} is a root of the function,

$$f(x_{n+1}) = 0$$

$$\therefore f(x_n) + f'(x_n) \cdot h = 0$$

$$\therefore h = -\frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

[Proved]

Quadratic convergence Proof

Let, x_n an approximate root of the function $f(x)$. ~~And consider some small~~, And x_n and x_{n+1} is too close to each other.

So, from tylor series,

$$f(x_{n+1}) = \cancel{f(x_n) + f'(x_n) h + f''(x}$$

$$f(x_n) + f'(x_n)(x_{n+1}-x_n) + f''(x_n)\frac{(x_{n+1}-x_n)^2}{2} \quad \cdots (10)$$

$(x_{n+1}-x_n)^3$ and higher order are ~~have~~ removed ~~as~~ the difference is very small.

Let us assume absulute root x_n. So $x_{n+1} = x_n$.

$$f(x_n) = f(x_n) + f'(x_n)(x_n - x_n) + \frac{1}{2} f''(x_n)(x_n - x_n)^2$$

$$\text{or,} \quad 0 = f(x_n) + f'(x_n)(x_n - x_n) + \frac{1}{2} f''(x_n)(x_n - x_n)^2 \quad \cdots (11)$$

And from Newton Raphson formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore f(x_n) = (x_n - x_{n+1}) f'(x_n).$$

So substituting the value in equation (11)

$$0 = (x_n - x_{n+1}) f'(x_n) + f'(x_n)(x_n - x_n) + \frac{1}{2} f''(x_n)(x_n - x_n)^2$$

$$= f'(x_n)(x_n - x_{n+1}) + \frac{1}{2} f''(x_n)(x_n - x_n)^2 \cdots (111)$$

Again, we know the error for x_{n+1}

$$e_{n+1} = X_n - x_{n+1}$$

and $e_n = X_n - x_n$

So from (III)

$$\cancel{f}\; 0 = f'(x_n) \cdot e_{n+1} + \frac{f''(x_n)}{2} \cdot e_n^2$$

$$\text{or, } e_{n+1} = -\frac{f''(x_n)}{2 f'(x_n)} \cdot e_n^2$$

Here we see, e_{n+1} is roughly proportional to previous square of error and as the sign is negative the error is decreasing

Largest Possible Root

1. Consider a polynomial

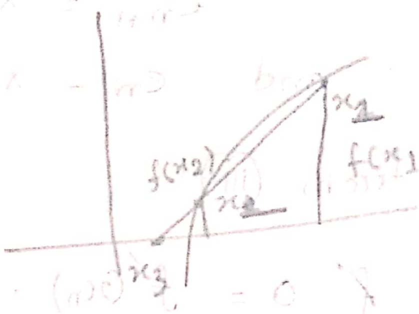
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$$

So the largest possible ~~co~~ root

$$X = \frac{a_{n-1}}{a_n}$$

Secant:

the line goes through x_2 and x_1 and cuts the x axis with x_3.



x_3 is the new estimation.

the slope is given by

$$\frac{f(x_1)}{x_1 - x_3} = \frac{f(x_2)}{x_2 - x_3}$$

$$f(x_1)\,x_2 - f(x_1)\cdot x_3 = f(x_2)\cdot x_1 - f(x_2)\cdot x_3$$

$$\therefore x_3\,(f(x_2) - f(x_1)) = f(x_2)\cdot x_1 - f(x_1)\cdot x_2$$

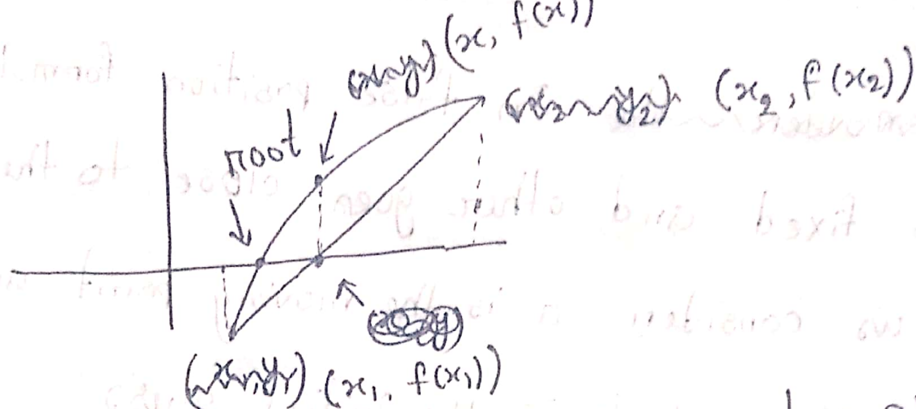
$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{x_1 f(x_2) - x_2 f(x_1) + x_2 f(x_2) - x_2 f(x_2)}{f(x_2) - f(x_1)}$$

$$= \frac{x_2(f(x_2) - f(x_1)) - f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$$= x_2 - f(x_2)\,\frac{x_2 - x_1}{f(x_2) - f(x_1)}$$

False position



In false position method we connect the two given point with a straight line. This straight line intercept x axis with $(x, f(x))$ point.

This X is our new estimation for root

So, from the graph we get,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x) - f(x_1)}{X - x_1}$$

when $x =$ root

$x = x_0$

$f(x_0) = 0$.

$$\therefore \frac{-f(x_1)}{x_0 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$x_0 - x_1 = \frac{-f(x_1)}{\frac{f(x_2) - f(x_1)}{x_2 - x_1}} = f(x_1) \frac{x_2 - x_1}{f(x_2) - f(x_1)}$$

$$x_0 = x_1 - f(x_1) \frac{x_2 - x_1}{f(x_2) - f(x_1)}$$

Convergence of False position

~~Consider the~~ In false position formula one point is fixed and other goes close to the solution.

Let us consider a is the moving point and b is fixed

where a and b is the initial guess.

And root is x_n and Let $x_1 = a$

$$\therefore \; e_1 = e_n - e_1$$

$$e_2 = e_n - e_2$$

So total error of a process is

$$e_{i+1} = e_n \frac{(x_n - b)\, f''(R)}{f'(R)}$$

where R is any value with in the interval x_i and b.

So e_{i+1} is roughly proportional to e_n

So we can say it convert linearly.

b Langrance Interpolation:

Let $x_0, x_1 \ldots x_n$ denote n real number (distinct) and value of them. The point $f_1, f_2, f_3 \ldots f_n$ be the functional value of them. The point

$(x_0, f_0), (x_1, f_1) \ldots (x_n, f_n)$ are the points on curve.

where $l_i(x) = \prod_{j=0\ j\neq i}^{2} \frac{(x-x_j)}{(x_i-x_j)}$

এই L কে $\boxed{\text{Lagrance basis polynomial}}$ বলে

$P(x) \rightarrow \boxed{\text{lagrance interpolation polynomial.}}$

in general, $P_n(x) = \sum_{i=0}^{n} f_i l_i(x)$.

$$L_i = \prod_{j=0, j\neq i}^{n} \frac{(x-x_j)}{(x_i-x_j)}.$$

Geting linear interpolation:

if ~~we~~ $n = 1$;

$$P_1(x) = \sum_{i=0}^{1} f_i l_i(x).$$

so, for $i=0$, $l_0 = \frac{(x-x_1)}{(x_0-x_1)}$.

$i = 1$, $l_1 = \frac{(x-x_0)}{(x_1-x_0)}$.

$$\therefore P_1(x) = f_0 \frac{(x-x_1)}{(x_0-x_1)} + f_1 \frac{(x-x_0)}{(x_1-x_0)}.$$

$$= f_0 \frac{(x-x_1)}{(x_0-x_1)} - \frac{f_1(x-x_0)}{(x_0-x_1)}$$

$$= \frac{f_0(x-x_1) - f_1(x-x_0)}{(x_0-x_1)}.$$

$$= f_0 + \frac{f_0(x-x_1) - f_1(x-x_0)}{x_0-x_1} - f_0$$

$\cancel{f_0x - f_0x_1 - f_1x + f_1x_0}$

$$= f_0 + \frac{f_0 x - \cancel{f_0 x_1} - f_1 x + f_1 x_0 - f_0 x_0 + \cancel{f_0 x_1}}{x_0 - x_1}$$

$$= f_0 + \frac{f_1(x_0 - x) - f_0(x_0 - x)}{x_0 - x_1}$$

$$= f_0 + \frac{f_1 - f_0}{x_0 - x_1}(x_0 - x)$$

$$= f_0 + \frac{f_1 - f_0}{x_1 - x_0}(x - x_0)$$

[Proved]

~~Newton Forward difference interpolation~~

Newton interpolation

Consider a polynomial

$$p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots \quad a_n(x-x_n)$$

Now suppose the interpolating points are $x_0, f(x_0)), (x_1, f(x_1))\ldots$

if we can get the coefficient $a_0, a_1 \ldots a_n$ we can build the bolynomial.

As, (x_0, f_0) is a interpolating point,

$$\therefore f_0 = p(x_0) = a_0$$

again (x_1, f_1) is also an interpolating point,

$$\therefore f_1 = a_0 + a_1(x_1 - x_0)$$

$$-a_1(x_0-x_1) = f_1 - a_0 = f_1 - f_0$$

$$\therefore a_1 = \frac{f_1 - f_0}{x_1 - x_0}$$

Again (x_2, f_2) is also interpolating point,

$$\therefore f_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2-x_0)(x_2-x_1)$$

$$a_2(x_2-x_0)(x_2-x_1) = f_2 - f_0 - \frac{f_1-f_0}{x_1-x_0}\cdot(x_2-x_0)$$

$$= \frac{(f_2-f_0)(x_1-x_0) - (f_1-f_0)(x_2-x_0)}{(x_1-x_0)}$$

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$$(f_2-f_0)(x_1-x_0) - (f_1-f_0)(x_2-x_0)$$

$$= f_2x_1 - f_2x_0 - f_0x_1 + \cancel{f_0x_0} - f_1x_2 + f_1x_0 + f_0x_2 - \cancel{f_0x_0}$$

$$= f_2x_1 - f_2x_0 - f_0x_1 - f_1x_2 + f_1x_0 + f_0x_2 + f_1x_1 - f_1x_1$$

$$= f_2x_1 - f_2x_0 - f_0x_1$$

[adding f_1x_1 and subtract it]

$$= x_1(f_2-f_1) - x_0(f_2-f_1) + (f_1-f_0)x_1 - x_2(f_1-f_0)$$

$$= (f_2-f_1)(x_1-x_0) + (f_1-f_0)(x_1-x_2)$$

$$= (f_2-f_1)(x_1-x_0) - (f_1-f_0)(x_2-x_1)$$

So, $$a_2(x_2-x_0)(x_2-x_1) = \frac{(f_2-f_1)(x_2-x_0) - (f_1-f_0)(x_2-x_1)}{(x_1-x_0)}$$

$$\therefore a = \frac{(f_2-f_1)(x_1-x_0) - (f_1-f_0)(x_2-x_1)}{(x_1-x_0)(x_2-x_0)(x_2-x_1)}$$

$$= \frac{\frac{(f_2-f_1)}{(x_2-x_1)} - \frac{f_1-f_0}{x_1-x_0}}{x_2-x_0}$$

Now denote,

$$f[x_0] = f_0$$

$$f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0} = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$\vdots$$

$$f[x_0, x_1, x_2, \ldots x_n] = \frac{f[x_1, x_2 \ldots x_n] - f[x_0, x_1 \ldots x_{n-1}]}{x_n - x_0}$$

So, $P(x) = f[x_0] + f[x_0, x_1] \cdot (x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \ldots$

$f[x_0, x_1, \ldots x_n](x - x_0)(x - x_1) \ldots (x - x_n)$.

$$P(x) = \sum_{i=0}^{n} f[x_0 \ldots x_i] \prod_{j=0}^{i-1} (x - x_j)$$

Newton forward difference formula

As the interpolation is with equidistant points, we consider some equal divided points with the difference of h.

$\therefore x_1 = x_0 + h$.

$x_2 = x_0 + 2h$

In general $x_k = x_0 + kh$.

we also define

$$\Delta f_0 = f_1 - f_0$$

$$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$$

in general, $\Delta^j f_i = \Delta^{j-1} f_{i+1} - \Delta^{j-1} f_i$

From newton interpolation polynomial,

$$f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0} = \frac{\Delta f_0}{h}.$$

$$\Delta f_0 = h \cdot f[x_0, x_1].$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{\frac{\Delta f_1}{h} - \frac{\Delta f_0}{h}}{2h}$$

$$= \frac{\Delta f_1 - \Delta f_0}{2h^2}.$$

$$\Delta^2 f_0 = 2h^2 f[x_0, x_1, x_2].$$

in general,

$$\Delta^j f_i = j!\, h^j f[x_0, x_1 \ldots\ldots x_{i+j}]$$

$$f[x_0, x_1 \ldots x_{i+j}] = \frac{\Delta^j f_i}{j!\, h^j}$$

General Q... ! !

5-1

$$P_n(x) = \sum_{j=0}^{n} \frac{\Delta^j f_0}{j!\, h^j} \cdot \prod_{K=0}^{j-1} (x - x_K)$$

Now,

$$x = x_0 + sh$$

$$x_K = x_0 + Kh$$

$$x - x_K = (s-K)\,h$$

$$P(x) = \sum_{j=0}^{n} \frac{\Delta^j f_0}{j!\, h^j} \prod_{K=0}^{j-1} (x - x_K)$$

$$= \sum_{j=0}^{n} \frac{\Delta^j f_0}{j!\, \cancel{h^j}} \cdot \left[s(s-1)(s-2)(s-3) \ldots\ldots (s-j+1) \right] \cancel{h^j}$$

$$= \sum_{j=0}^{n} \frac{\Delta^j f_0}{j!} \left[s(s-1)(s-2)(s-3) \ldots\ldots (s-j+1) \right]$$

$$= \sum_{j=0}^{n} \binom{s}{j} \Delta^j f_0$$

Sreeanth

General Quadrature formula

Let $I = \int_a^b y\,dx$

Now the interval a to b is divided into n equal parts. Lets assume length of each part is 'h'.

So, $x_0,\ x_0+h,\ x_0+2h \ldots\ldots\ x_0+nh$

where $x_0 = a$ and $x_0 + nh = b$

$$h = \frac{b-a}{n}.$$

So, $I = \int_a^b y\,dx = \int_{x_0}^{x_0+nh} y\,dx$

$$= \int_0^n h\,y_{x_0+uh}\,du$$

let
$x = x_0 + uh$.
$\therefore dx = h\,du$.

limit:
when,
$x = x_0 \quad u = 0$
$x = x_0 + nh \quad u = n$

and

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that

From Newton Forward interpolation formula:

$$\int_0^n y_{x_0+uh} = \int_0^n h\left[y_0 + \Delta y_0 (u) + \frac{\Delta^2 y_0}{2!}(u)(u-1) + \frac{\Delta^3 y_0}{3!}(u)(u-1)(u-2) + \ldots\right] du.$$

$$= h\left(n y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{6} + \ldots\right]$$

[Proved]

Trapezoidal

putting $n = 1$,

$$\int_{x_0}^{x_0+h} y\,dx = h\left(n y_0 + \frac{n^2}{2}\Delta y_0\right)$$

$$= h\left(y_0 + \frac{y_1 - y_0}{2}\right)$$

$$= \frac{h(y_1 + y_0)}{2}$$

Again,

$$\int_{x_0+h}^{x_0+2h} y\,dx = \frac{h(y_1 + y_2)}{2}$$

$$\vdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} y\,dx = h\left(\frac{y_{n-1} + y_n}{2}\right)$$

So adding them,

$$\int_{x_0}^{x_0+nh} y\,dx = h\left(\frac{y_1 + y_0}{2} + \frac{y_1 + y_2}{2} + \frac{y_2 + y_3}{2} + \cdots + \frac{y_{n-1} + y_n}{2}\right)$$

$$= h\left(\frac{(y_0 + y_n) + 2(y_1 + y_2 + y_3 \cdots y_{n-1})}{2}\right)$$

$$= h\left(\frac{1}{2}(y_0 + y_n) + (y_1 + y_2 + y_3 + \cdots y_{n-1})\right)$$

[Proved].

Simnons 1/3

n=2 in trapezoidal Rule:

$$\int_{x_0}^{x_0+2h} y\,dx = h\left[\cancel{\Delta y_0}\; n y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right) \cdot \frac{\Delta^2 y_0}{2} \right].$$

$$= h\left(2y_0 + 2\cancel{\Delta}(y_1 - y_0) + \left(\frac{8}{3} - 2\right)\frac{\Delta y_1 - \Delta y_0}{2}\right]$$

$$= h\left(2y_1 + \frac{1}{\cancel{6}3}(y_2 - y_1 - y_1 + y_0)\right)$$

$$= \frac{h}{3}\left(6y_1 + y_2 - 2y_1 + y_0\right).$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

$$\int_{x_0+2h}^{x_0+4h} y\,dx = \frac{h}{3}\left(y_2 + 4y_3 + y_4\right)$$

$$\vdots$$

$$\int_{x_0+(n-2)h}^{x_0+nh} y\,dx = \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right).$$

$$\therefore \int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left(y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + \cdots y_{n-2} + 4y_{n-1} + y_n\right).$$

$$= \frac{h}{3}\left((y_0 + y_n) + 2(y_2 + y_4 + \cdots y_{n-2}) + 4(y_1 + y_3 + \cdots y_{n-1})\right)$$

[Proved]

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Simsons 3/8

n= 3 in General formula

$$\int_{x_0}^{x_0+nh} y\,dx = h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{6}\right].$$

$$\int_{x_0}^{x_0+3h} y\,dx = h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \left(\frac{27}{3} - \frac{9}{2}\right)\frac{(y_2 - y_1 - y_1 + y_0)}{2} + \left(\frac{81}{4} - 27 + 9\right)\frac{(y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)}{6}\right].$$

$$= h\left(3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \left(\frac{\cancel{9}\,3}{4}\right)\left(\frac{y_3 - 3y_2 + 3y_1 - y_0}{\cancel{6}\,2}\right)\right).$$

$$= \frac{3h}{8}\left(8y_0 + 12(y_1 - y_0) + \cancel{12}\,6\,(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0)\right).$$

$$= \frac{3h}{8}\left(8y_0 + \cancel{12y_1} - 12y_0 + 6y_2 - \cancel{12y_1} + 6y_0 + y_3 - 3y_2 + 3y_1 - y_0\right).$$

$$= \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3\right).$$

$$\int_{x_0+3h}^{x_0+6h} y\,dx = \frac{3h}{8}\left(y_3 + 3y_4 + 3y_5 + y_6\right)$$

$\vdots$

$$\int_{x_0+(n-3)h}^{x_0+nh} y\,dx = \frac{3h}{8}\left(y_{n-3} + 3y_{n-2} + 3(y_{n-1}) + y_n\right)$$

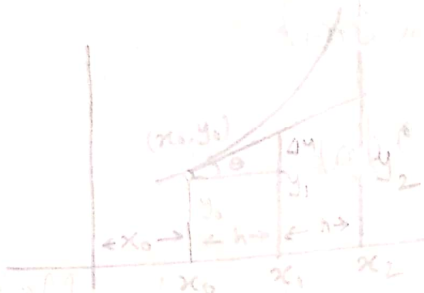
$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3 + y_3 + 3y_4 + 3y_5 + y_6 + \cdots + y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right)$$

$$= \frac{3h}{8}\left((y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 \ldots) + 2(y_3 + y_6 \cdots y_{n-3})\right).$$

Eular Method:

Let A differential equation be

$$f(x,y) = \frac{dy}{dx} \quad \text{and} \quad f(x_0) = y_0$$



So we draw a tangent line from (x_0, y_0) point. and we draw a perpendicular line over x_1 point. This tangent line cuts the perpendicular line in a certain point. Eular method considers this point as ~~point of the~~ the fonctional vaule of that point.

From the figure,

$$\tan\theta = \frac{\Delta y}{\Delta x}$$

$$\text{or, } \frac{dy_0}{dx_0} = \frac{\Delta y}{x_1 - x_0}$$

$$\text{or, } f(x_0, y_0) = \frac{\Delta y}{h}$$

$$\therefore \Delta y = f(x_0, y_0) \cdot h$$

$$\therefore y = y_0 + \Delta y = y_0 + h f(x_0, y_0) \quad \text{and } x_1 = x_0 + h.$$

similarly for (x_2, y_2) points

$X_2 = x_1 + \Delta h = \Delta_0 +$

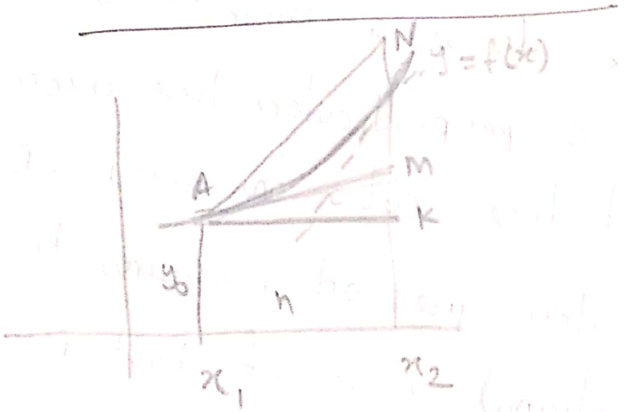
$\Delta y_2 = h f(x_1, y_1)$

$\therefore y_2 = y_1 + h f(x_1, y_1)$

$\therefore$

$y_{n+1} = y_n + h f(x_n, y_n)$.

Modified Eular Method



Here x_1 is the first value of x. And x_2 is the second Approximation of X.

we draw a tangent from the (x_0, y_0) point which cuts the perpendicular line drawn in x_2. This is the first appriaximation of y_1

from eular method,

$$y_1 = y_0 + h f(x_0, y_0).$$

Formula:

so in modified eular method we will draw a tangent line in (x_1, y_1) point and improved Δy_1 would be the average of this tangent and the previous tangent ~~by~~ line drawn from (x_0, y_0).

$$\therefore \Delta y_1 = h \cdot \frac{\left(\frac{dy_0}{dx_0}\right) + \frac{dy_1^{(1)}}{dx_1}}{2}$$

$$\therefore y_1^{(2)} = y_0 + h \cdot \frac{\left(\frac{dy_0}{dx_0}\right) + \left(\frac{dy_1}{dx_1}\right)^{(1)}}{2}$$

$$= y_0 + h \cdot \frac{f(x_0, y_0) + f(x_1, y_1^{(1)})}{2}$$

So the next approximation for y would be

$$y_2^{(3)} = y_0 + h \cdot \frac{\left(\frac{dy_0}{dx_0}\right) + \left(\frac{dy_2^{(2)}}{dx_1}\right)}{2}$$

$$= y_0 + h \frac{f(x_0, y_0) + f(x_1, y_2^{(2)})}{2}$$

This process will continue untill same approximation comes twice.

From the figure we see,

$$\cancel{y} = \cancel{\frac{dy_1}{dx} + \frac{dy_2}{dx}}$$

$$\Delta y = \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1}{2} \times h = \frac{\frac{MK}{h} + \frac{KN}{h}}{2} \times h$$

$$= (MK + MK + MN)\frac{1}{2}$$

$$= MK + \frac{1}{2} MN$$

which is very close to real point.