CE 311K: Linear System of Equations

Krishna Kumar

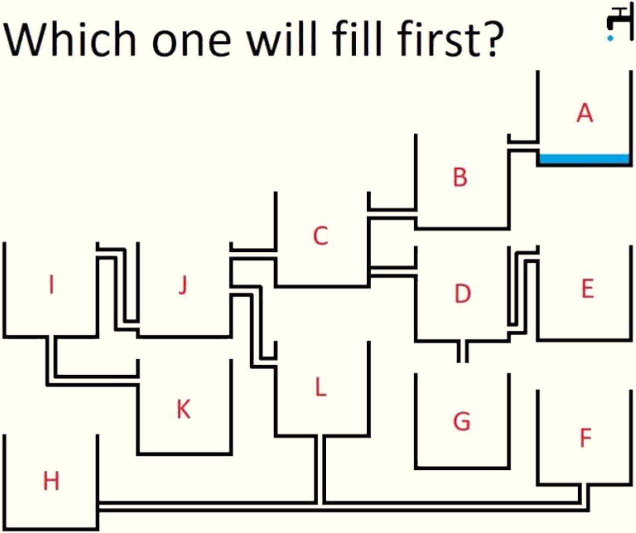
University of Texas at Austin

krishnak@utexas.edu

December 2, 2019

1. Linear System of Equations

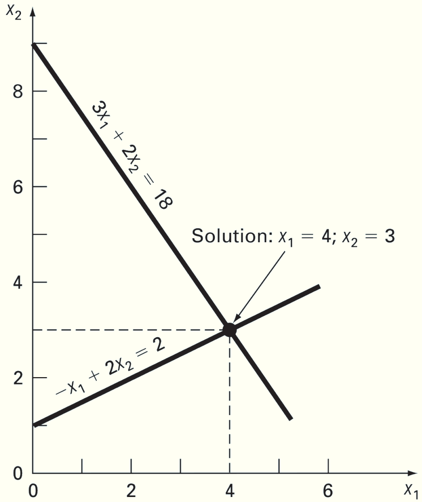
Linear System of Equations



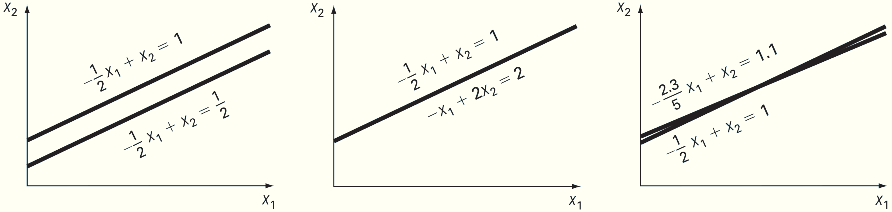
Solving Linear System of Equations

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$

Solving Linear System of Equations



Singularity and Ill-conditioned



Solving Linear System of Equations

1. Direct Methods
   1. Gauss Elimination
   2. Gauss-Jordan Elimination
   3. LU decomposition
2. Iterative Methods
   1. Jacobi iterative
   2. Gauss-Seidel

Direct methods

Consider a system of 3 linear equations for simplicity:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
$$

Matrix form is:

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
$$

Concise form: $Ax = b$

Systems that can be solved easily

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Solve by "back substitution' Upper triangle system (U)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Start with the last equation 3: $x_3 = b_3/a_{33}$
Equation 2: $a_{22}x_2 + a_{23}x_3 = b_2$ so

$$x_2 = \frac{b_2 - a_{23}x_3}{a_{22}}$$

Equation 1: $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$ so

$$x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3)}{a_{11}}$$

General for 'n' systems: $x_n = b_n/a_{nn}$ $x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}$

Consider a system of 3 linear equations:

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 11 & 21 \\ 6 & 21 & 52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 24 \\ 72 \\ 158 \end{bmatrix}$$

Divide row 2 by -2 and row 3 by -3

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 9 \\ 0 & 9 & 34 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 24 \\ 24 \\ 86 \end{bmatrix}$$

Second reduction:

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 24 \\ 24 \\ 14 \end{bmatrix}$$

$x_3 = 2, x_2 = 2, x_1 = 2$

Gauss Elimination

Consider a system of 3 linear equations:

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 11 & 21 \\ 6 & 21 & 52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 24 \\ 72 \\ 158 \end{bmatrix}$$

Divide row 2 by -2 and row 3 by -3

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 9 \\ 0 & 9 & 34 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 24 \\ 24 \\ 86 \end{bmatrix}$$

Second reduction:

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 24 \\ 24 \\ 14 \end{bmatrix}$$

$x_3 = 2, x_2 = 2, x_1 = 2$

```
a = np.array([[2,4,6], [4,11,21], [6, 21, 52]])
b = np.array([24, 72, 158])
x = np.linalg.solve(a, b)
```

Gauss Elimination: Limitations

1. Prone to round off errors, when we have many (> 100) equations.
2. If coefficient matrix is sparse (lots of zeros), elimination methods are very inefficient.

Gauss Seidel Iterative approach

For conciseness, we limit to 3×3 equations. If diagonal elements are all non-zero, then the equations can be solved as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$
$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$
$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

Gauss Seidel Iterative approach

Using Gauss-Seidel solve for [x]

$$
\begin{aligned}
4x_1 + x_2 + 2x_3 &= 4 \\
3x_1 + 5x_2 + x_3 &= 7 \\
x_1 + x_2 + 3x_3 &= 3
\end{aligned}
$$

Assuming an initial guess of $x_1, x_2, x_3 = 0$.

$$
\begin{aligned}
x_1 &= \frac{4 - x_2 - 2x_3}{4} = \frac{4 - 0 - 0}{4} = 1 \\
x_2 &= \frac{7 - 3x_1 - x_3}{5} = \frac{7 - 3 * 1 - 0}{5} = 0.8 \\
x_3 &= \frac{3 - x_1 - x_2}{3} = \frac{3 - 1 - 0.8}{3} = 0.4
\end{aligned}
$$

Gauss Seidel Convergence criteria

Convergence can be checked using the relative error.

$$|\varepsilon_{a,i}| = \left| \frac{x_i^k - x_i^{k-1}}{x_i^k} * 100\% \right| < \varepsilon_s$$

where k, and $k-1$ represents the current and previous iterations