

# CE 311K: Linear System of Equations

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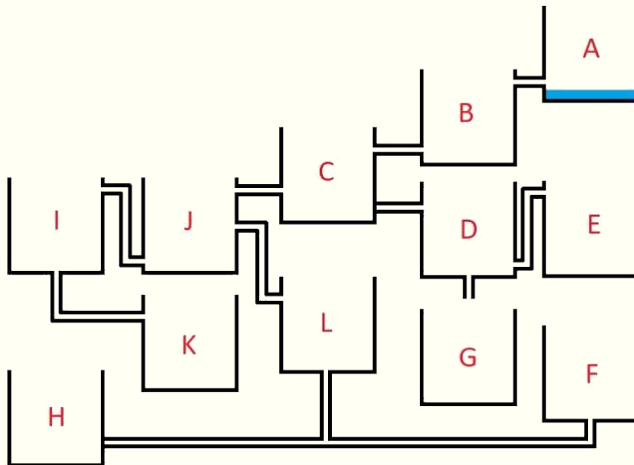
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# 1 Linear System of Equations

# Linear System of Equations

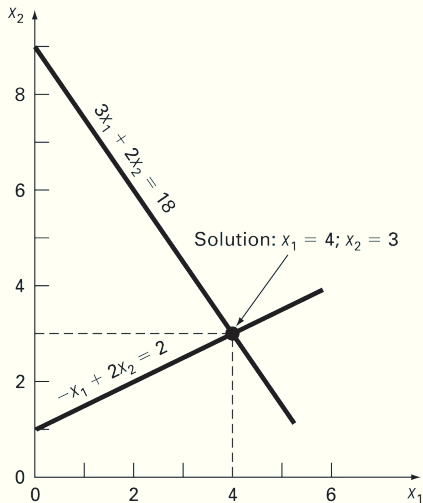
Which one will fill first?



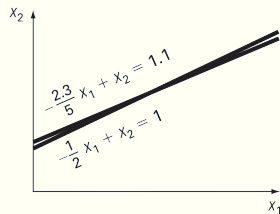
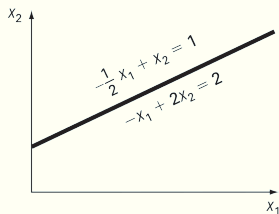
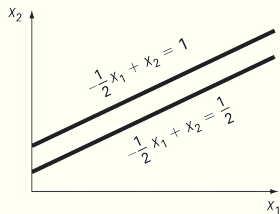
# Solving Linear System of Equations

$$\begin{aligned}3x_1 + 2x_2 &= 18 \\ -x_1 + 2x_2 &= 2\end{aligned}$$

# Solving Linear System of Equations



# Singularity and Ill-conditioned



# Solving Linear System of Equations

## ① Direct Methods

- ① Gauss Elimination
- ② Gauss-Jordan Elimination
- ③ LU decomposition

## ② Iterative Methods

- ① Jacobi iterative
- ② Gauss-Seidel

# Direct methods

Consider a system of 3 linear equations for simplicity:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Matrix form is:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Concise form:  $Ax = b$



# Systems that can be solved easily

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

## Solve by “back substitution” Upper triangle system (U)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Start with the last equation 3:  $x_3 = b_3/a_{33}$

Equation 2:  $a_{22}x_2 + a_{23}x_3 = b_2$  so

$$x_2 = \frac{b_2 - a_{23}x_3}{a_{22}}$$

Equation 1:  $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$  so

$$x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3)}{a_{11}}$$

General for ‘n’ systems:  $x_n = b_n/a_{nn}$   $x_i = \frac{b_i - \sum_{j=i+1}^n a_{ij}x_j}{a_{ii}}$

# Gauss Elimination

Consider a system of 3 linear equations:

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 11 & 21 \\ 6 & 21 & 52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 72 \\ 158 \end{bmatrix}$$

Divide row 2 by -2 and row 3 by -3

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 9 \\ 0 & 9 & 34 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \\ 86 \end{bmatrix}$$

Second reduction:

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \\ 14 \end{bmatrix}$$

$$x_3 = 2, x_2 = 2, x_1 = 2$$

# CE 311K: Linear Systems

## └ Linear System of Equations

### └ Gauss Elimination

Consider a system of 3 linear equations:

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 11 & 21 \\ 6 & 21 & 52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 72 \\ 158 \end{bmatrix}$$

Divide row 2 by -2 and row 3 by -3

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 9 \\ 0 & 9 & 34 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \\ 86 \end{bmatrix}$$

Second reduction:

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \\ 14 \end{bmatrix}$$

$$x_3 = 2, x_2 = 2, x_1 = 2$$

```
a = np.array([[2,4,6], [4,11,21], [6, 21, 52]])
b = np.array([24, 72, 158])
x = np.linalg.solve(a, b)
```

# Gauss Elimination: Limitations

- 1 Prone to round off errors, when we have many ( $> 100$ ) equations.
- 2 If coefficient matrix is sparse (lots of zeros), elimination methods are very inefficient.

# Gauss Seidel Iterative approach

For conciseness, we limit to  $3 \times 3$  equations. If diagonal elements are all non-zero, then the equations can be solved as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

# Gauss Seidel Iterative approach

Using Gauss-Seidel solve for  $[x]$

$$4x_1 + x_2 + 2x_3 = 4$$

$$3x_1 + 5x_2 + x_3 = 7$$

$$x_1 + x_2 + 3x_3 = 3$$

Assuming an initial guess of  $x_1, x_2, x_3 = 0$ . End of first iteration.

$$x_1 = \frac{4 - x_2 - 2x_3}{4} = \frac{4 - 0 - 0}{4} = 1$$

$$x_2 = \frac{7 - 3x_1 - x_3}{5} = \frac{7 - 3 * 1 - 0}{5} = 0.8$$

$$x_3 = \frac{3 - x_1 - x_2}{3} = \frac{3 - 1 - 0.8}{3} = 0.4$$

# Gauss Seidel Convergence criteria

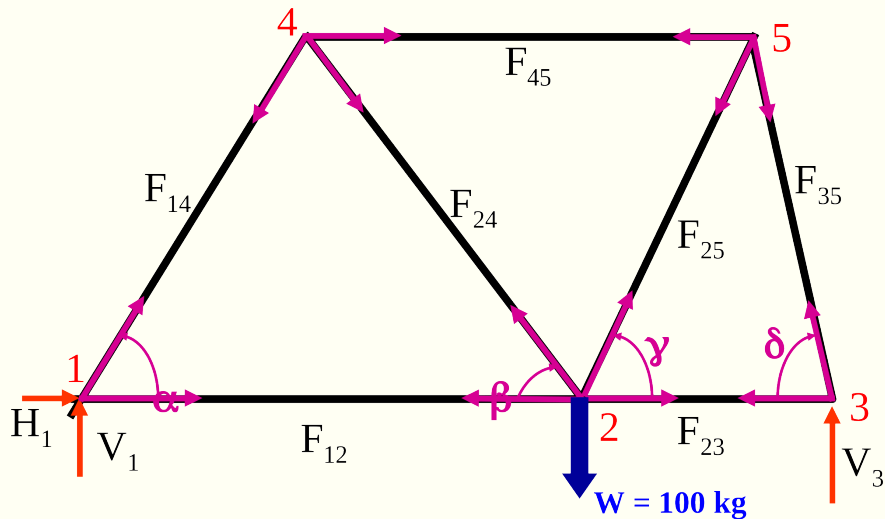
Convergence can be checked using the relative error.

$$|\varepsilon_{a,i}| = \left| \frac{x_i^k - x_i^{k-1}}{x_i^k} * 100\% \right| < \varepsilon_s$$

where  $k$ , and  $k - 1$  represents the current and previous iterations



# Truss analysis



# Truss analysis: Force balance

$$\text{Node 1} \quad \begin{cases} \sum F_{y,1} = V_1 + F_{14} \sin \alpha = 0 \\ \sum F_{x,1} = H_1 + F_{12} + F_{14} \sin \alpha = 0 \end{cases}$$

$$\text{Node 2} \quad \begin{cases} \sum F_{y,2} = F_{24} \sin \beta + F_{25} \sin \gamma = 100 \\ \sum F_{x,2} = -F_{12} + F_{23} - F_{24} \cos \beta + F_{25} \cos \gamma = 0 \end{cases}$$

$$\text{Node 3} \quad \begin{cases} \sum F_{y,3} = V_3 + F_{35} \sin \delta = 0 \\ \sum F_{x,3} = -F_{23} - F_{35} \cos \delta = 0 \end{cases}$$

$$\text{Node 4} \quad \begin{cases} \sum F_{y,4} = -F_{14} \sin \alpha - F_{24} \sin \beta = 0 \\ \sum F_{x,4} = -F_{14} \cos \alpha + F_{24} \cos \beta + F_{45} = 0 \end{cases}$$

$$\text{Node 5} \quad \begin{cases} \sum F_{y,5} = -F_{25} \sin \gamma - F_{35} \sin \delta = 0 \\ \sum F_{x,5} = -F_{25} \cos \gamma + F_{35} \cos \delta - F_{45} = 0 \end{cases}$$

# Truss analysis: Matrix formulation

$$\begin{bmatrix}
 1 & 0 & 0 & 0 & \sin \alpha & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 1 & \cos \alpha & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & \sin \beta & \sin \gamma & 0 & 0 \\
 0 & 0 & 0 & -1 & 0 & 1 & \cos \beta & \cos \gamma & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \sin \delta & 0 \\
 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -\cos \delta & 0 \\
 0 & 0 & 0 & 0 & -\sin \alpha & 0 & -\sin \beta & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & -\cos \alpha & 0 & \cos \beta & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin \gamma & \sin \delta & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos \gamma & \cos \delta & -1
 \end{bmatrix}
 \begin{Bmatrix}
 V_1 \\
 H_1 \\
 V_3 \\
 F_{12} \\
 F_{14} \\
 F_{23} \\
 F_{24} \\
 F_{25} \\
 F_{35} \\
 F_{45}
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 0 \\
 0 \\
 100 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{Bmatrix}$$