Problem Statement

Consider a tandem of 2 single server queues. Customers arrive according to a renewal process into server 1. After service completion they join the queue corresponding to the second server, and after service completion they leave the system. Interarrival and service are i.i.d. denoted generically by A (interarrival times), S1,S2 (service times in servers 1 and 2). At time t =0 there are L1(0) = q customers in the first queue (including those in service) and no customers in the second queue (L2(0) = 0). We observe the system during a finite time horizon [0,T].

Suppose arrival process is Poisson with rate λ (of the following values)= {1,5} , and the service times have exponential distributions with rates μ1 = {2,4}, μ2 = {3,4}.

A. Assume q = 0. Estimate the time average number of customers in the system

$$\frac{1}{T}\int_0^T (L_1(t) + L_2(t))dt$$

via simulation.

1. Namely, generate interarrival and service times according to their prescribed distribution.
2. Then generate the implied queue length process and compute the corresponding integral.
3. Perform the simulations several times N for T = 10, 50, 100 and 1000.
4. Report on the obtained values, averaged over N and compare your results with the prediction of a steady-state product form formulas for

$$E\pi[L1 + L2]$$

where π is the stationary distribution. Include the values of N.

B. Assume now q = 1000 and T = 2000. Perform the same experiment as above. Produce plots of L1 and L2 for one specific run of the simulation (it suffices to produce plots of L1,L2 only at times of arrivals or service completions). How do the two plots compare?

Objective:
This lab aims to simulate a tandem queueing system with two single-server queues to estimate the time-average number of customers in the system and compare the results with theoretical steady-state values.

System Description

Customers arrive at the first queue (Queue 1) according to a Poisson process with rate λ. After being served, they move to the second queue (Queue 2) and leave the system after service completion. The interarrival times and service times are independent and identically distributed (i.i.d.) with exponential distributions. The system is observed over a finite time horizon [0,T].

Parameters

- Arrival rate (λ) = {1, 5}
- Service rate at Queue 1 (μ1) = {2, 4}
- Service rate at Queue 2 (μ2) = {3, 4}
- Simulation time horizon (T) = {10, 50, 100, 1000, 2000}
- The initial number of customers in Queue 1 (q) = {0, 1000}

Methodology

1. Generate interarrival and service times according to their prescribed distributions.
2. Simulate the queue length process and compute the corresponding integral.
3. Perform simulations multiple times (N = 10) for various T values.
4. Compare the obtained values with the theoretical steady-state values

$$E\pi[L1 + L2] = \left(\frac{\lambda}{\mu_1 - \lambda}\right) + \left(\frac{\lambda}{\mu_2 - \lambda}\right)$$

where π is the stationary distribution.

Result

Summary of Simulation Results

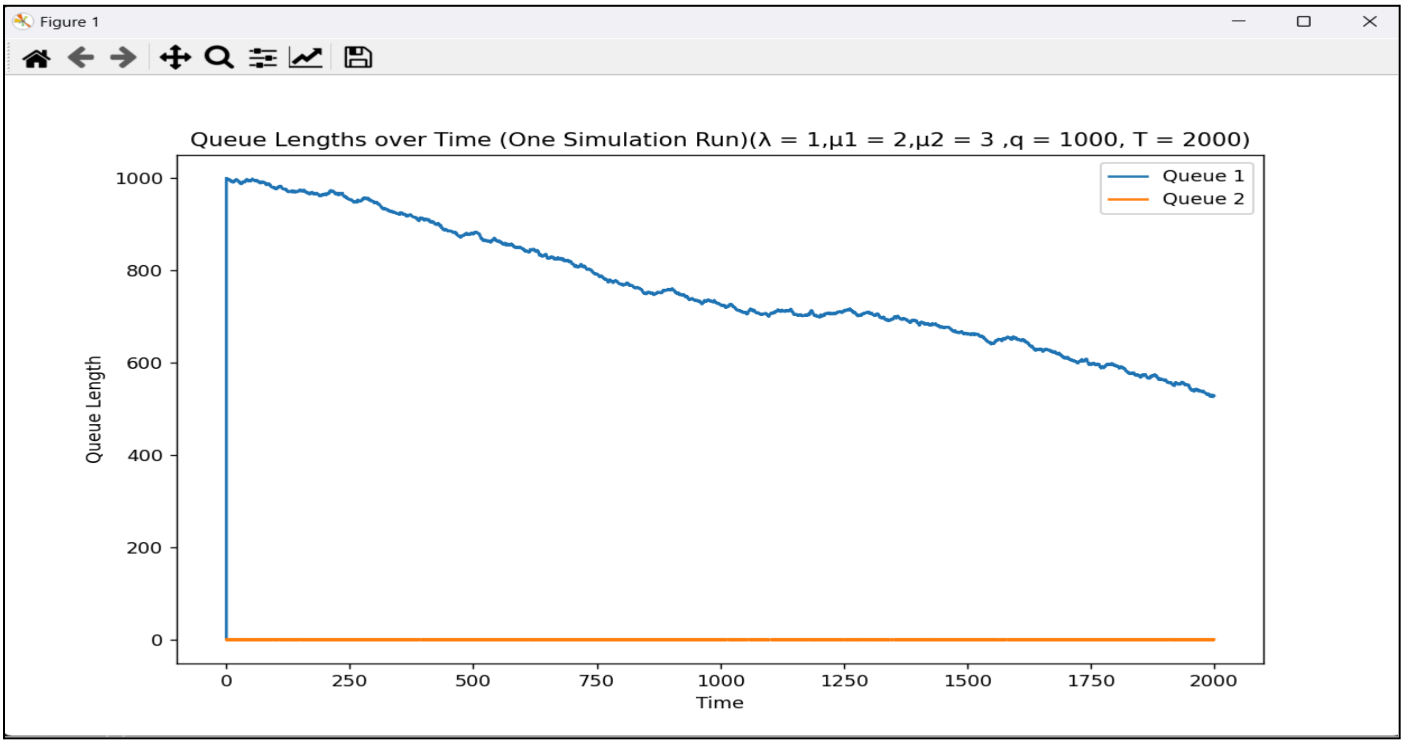
The table below summarizes the average time average number of customers in the system and the theoretical steady-state values for different combinations of λ, μ1, μ2, and T (N=10).

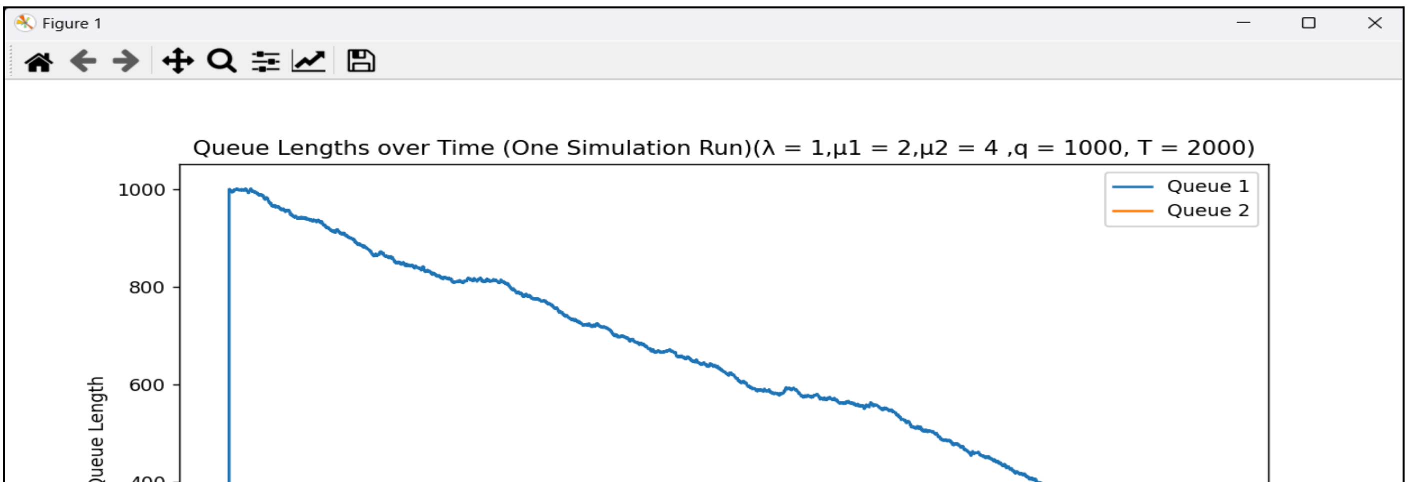
Values	Average time average customers	Standard deviation	Average customers using stationary distribution
$\lambda = 1, \mu_1 = 2, \mu_2 = 3$ $T = 10$	1.92	1.44	1.5
$\lambda = 1, \mu_1 = 2, \mu_2 = 3$ $T = 50$	2.01	0.93	1.5
$\lambda = 1, \mu_1 = 2, \mu_2 = 3$ $T = 100$	2.73	1.33	1.5
$\lambda = 1, \mu_1 = 2, \mu_2 = 3$ $T = 1000$	3.56	1.81	1.5
$\lambda = 1, \mu_1 = 2, \mu_2 = 4$ $T = 10$	1.37	1.46	1.333
$\lambda = 1, \mu_1 = 2, \mu_2 = 4$ $T = 50$	1.76	0.81	1.333
$\lambda = 1, \mu_1 = 2, \mu_2 = 4$ $T = 100$	2.37	0.95	1.333
$\lambda = 1, \mu_1 = 2, \mu_2 = 4$ $T = 1000$	2/./.24	0.18	1.333
$\lambda = 1, \mu_1 = 4, \mu_2 = 3$ $T = 10$	0.74	0.57	0.833
$\lambda = 1, \mu_1 = 4, \mu_2 = 3$ $T = 50$	0.77	0.16	0.833
$\lambda = 1, \mu_1 = 4, \mu_2 = 3$ $T = 100$	0.74	0.21	0.833
$\lambda = 1, \mu_1 = 4, \mu_2 = 3$ $T = 1000$	0.82	0.04	0.833

$\lambda = 1, \mu_1 = 4, \mu_2 = 4$ $T = 10$	0.44	0.48	0.666
$\lambda = 1, \mu_1 = 4, \mu_2 = 4$ $T = 50$	0.52	0.30	0.666
$\lambda = 1, \mu_1 = 4, \mu_2 = 4$ $T = 100$	0.56	0.19	0.666
$\lambda = 1, \mu_1 = 4, \mu_2 = 4$ $T = 1000$	0.64	0.07	0.666
$\lambda = 5, \mu_1 = 2, \mu_2 = 3$ $T = 10$	15.73	3.35	
$\lambda = 5, \mu_1 = 2, \mu_2 = 3$ $T = 50$	92.59	7.98	
$\lambda = 5, \mu_1 = 2, \mu_2 = 3$ $T = 100$	188.07	9.27	
$\lambda = 5, \mu_1 = 2, \mu_2 = 3$ $T = 1000$	1883.84	20.42	
$\lambda = 5, \mu_1 = 2, \mu_2 = 4$ $T = 10$	16.10	1.53	
$\lambda = 5, \mu_1 = 2, \mu_2 = 4$ $T = 50$	88.48	6.66	
$\lambda = 5, \mu_1 = 2, \mu_2 = 4$ $T = 100$	181.71	8.64	
$\lambda = 5, \mu_1 = 2, \mu_2 = 4$ $T = 1000$	1836.37	48.34	
$\lambda = 5, \mu_1 = 4, \mu_2 = 3$ $T = 10$	15.97	4.14	
$\lambda = 5, \mu_1 = 4, \mu_2 = 3$ $T = 50$	78.68	6.25	
$\lambda = 5, \mu_1 = 4, \mu_2 = 3$ $T = 100$	161.47	14.63	
$\lambda = 5, \mu_1 = 4, \mu_2 = 3$	1664.41	47.47	

$T = 1000$			
$\lambda = 5,\ \mu_1 = 4,\ \mu_2 = 4$ $T = 10$	12.24	2.83	
$\lambda = 5,\ \mu_1 = 4,\ \mu_2 = 4$ $T = 50$	78.53	7.87	
$\lambda = 5,\ \mu_1 = 4,\ \mu_2 = 4$ $T = 100$	140.18	13.18	
$\lambda = 5,\ \mu_1 = 4,\ \mu_2 = 4$ $T = 1000$	1515.93	43.55	

Plots





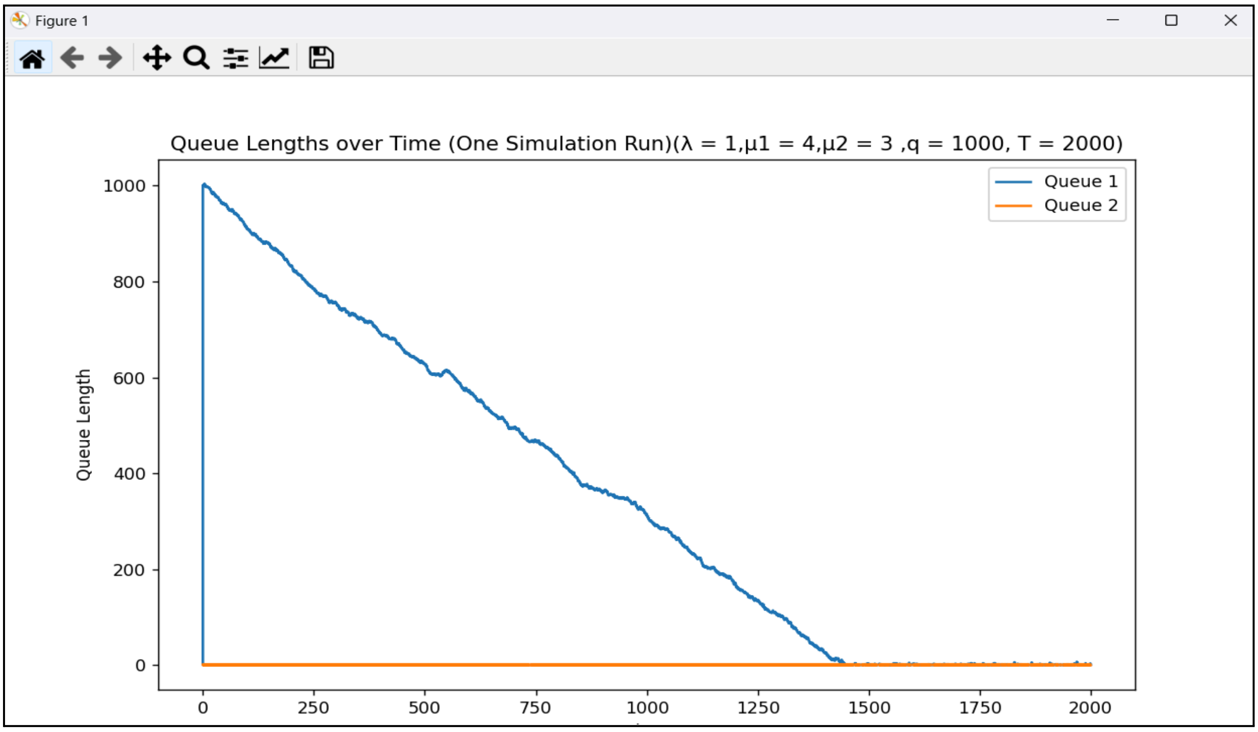


Figure 1
Queue Lengths over Time (One Simulation Run)(λ = 1,μ1 = 4,μ2 = 3 ,q = 1000, T = 2000)
Queue 1
Queue 2
Queue Length
1000
800
600
400
200
0
0
250
500
750
1000
1250
1500
1750
2000

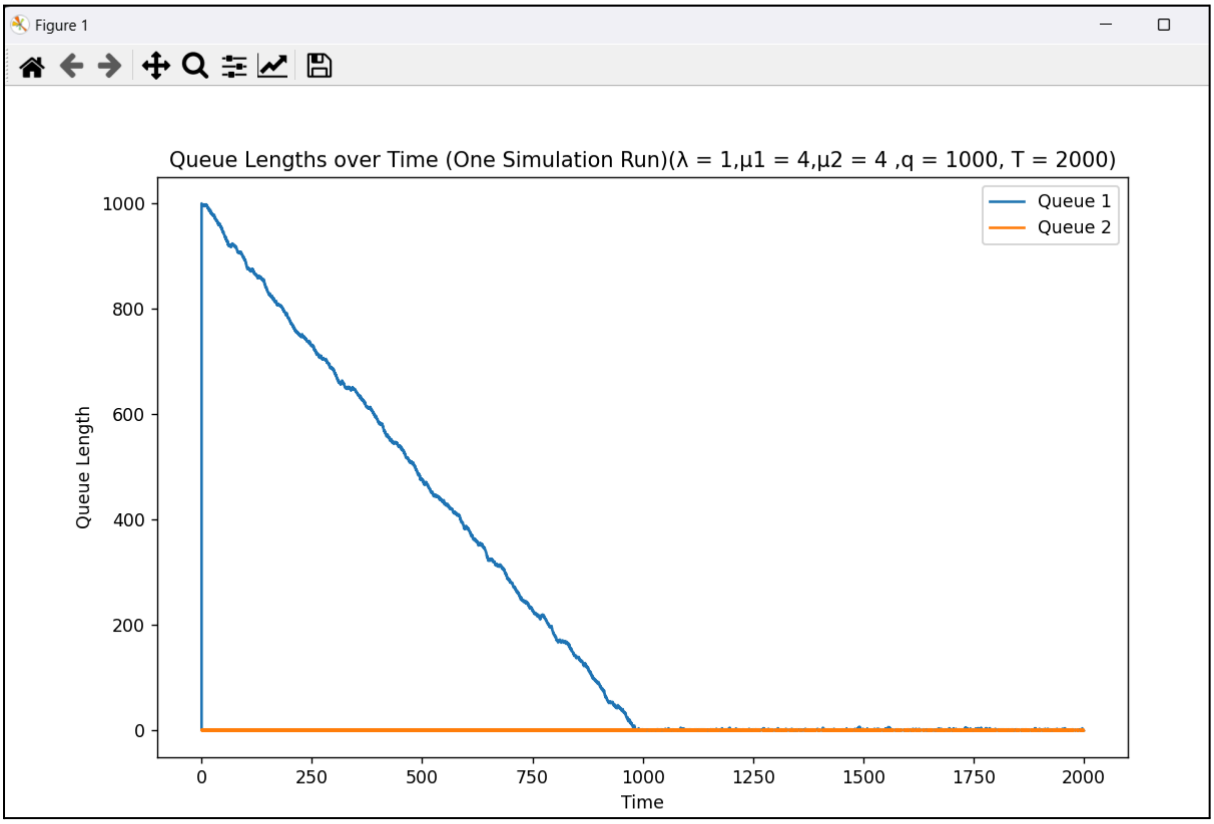


Figure 1
Queue Lengths over Time (One Simulation Run)(λ = 1,μ1 = 4,μ2 = 4 ,q = 1000, T = 2000)
Queue 1
Queue 2
Queue Length
1000
800
600
400
200
0
0
250
500
750
1000
1250
1500
1750
2000
Time

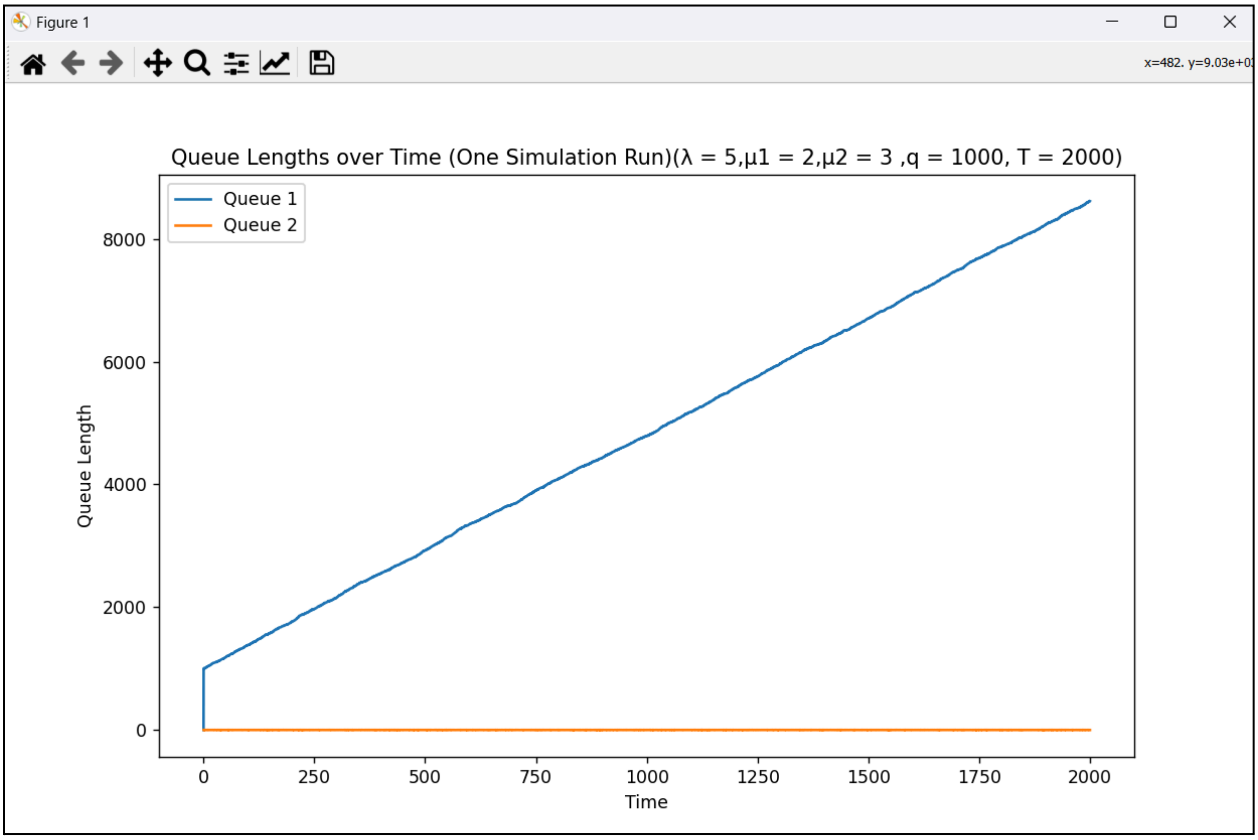


Figure 1
x=482. y=9.03e+0
Queue Lengths over Time (One Simulation Run)(λ = 5,μ1 = 2,μ2 = 3 ,q = 1000, T = 2000)
Queue 1
Queue 2
Queue Length
0
2000
4000
6000
8000
0
250
500
750
1000
1250
1500
1750
2000
Time

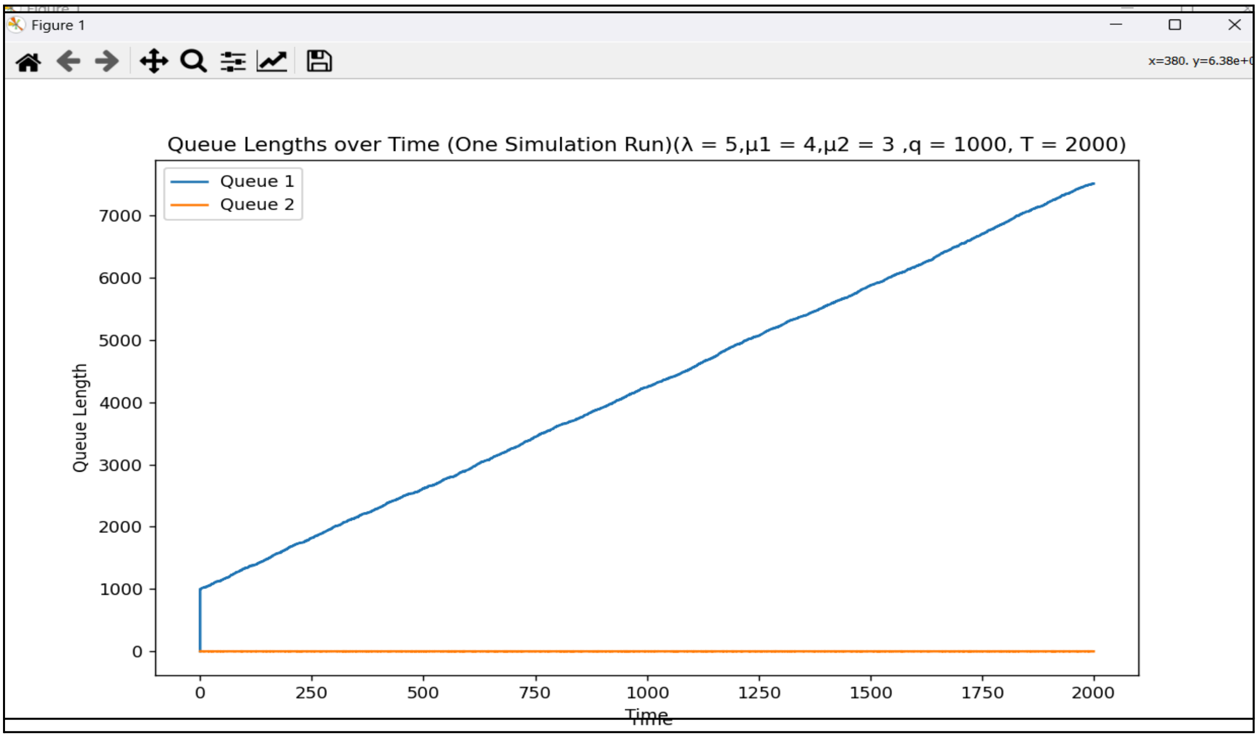
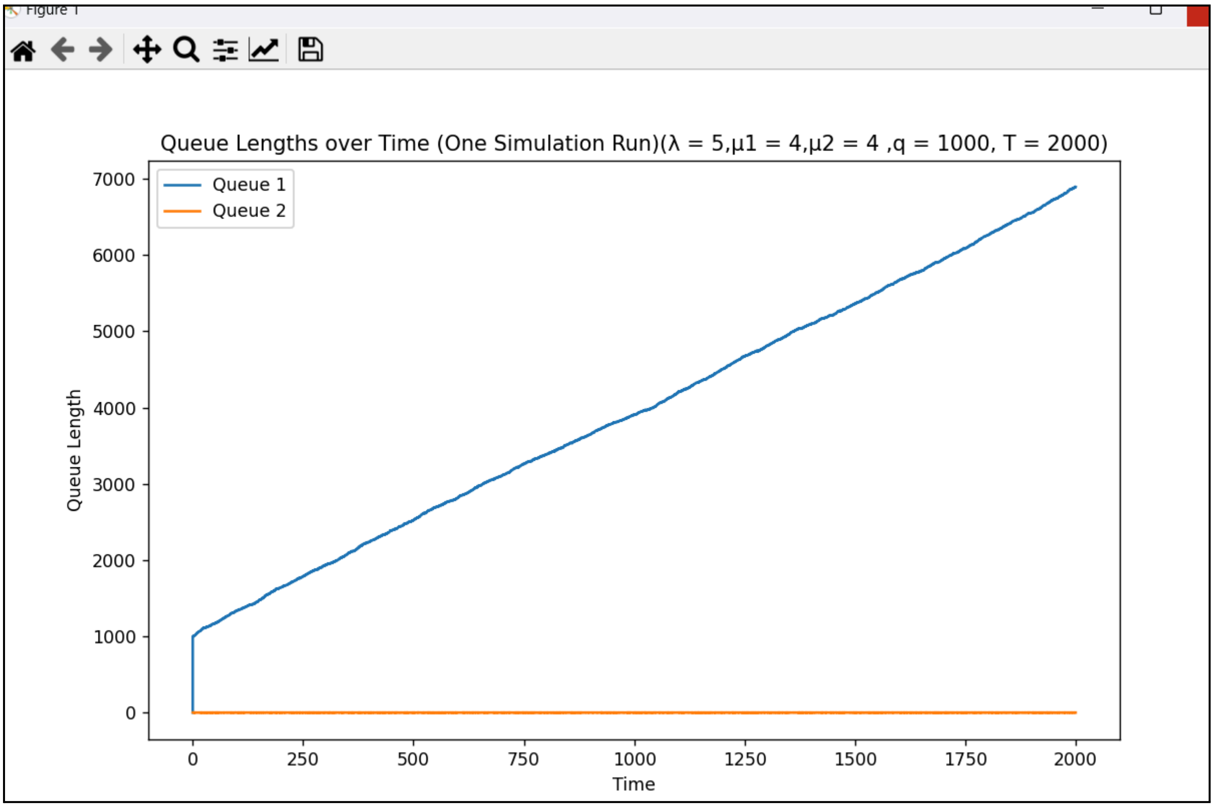


Figure 1
x=380. y=6.38e+0
Queue Lengths over Time (One Simulation Run)(λ = 5,μ1 = 4,μ2 = 3 ,q = 1000, T = 2000)
Queue 1
Queue 2
Queue Length
0
1000
2000
3000
4000
5000
6000
7000
0
250
500
750
1000
1250
1500
1750
2000
Time



Analysis and Discussion

The results demonstrate the simulation's capability to approximate the theoretical steady-state values under varying parameters. The discrepancies observed in some scenarios can be attributed to the simulation's stochastic nature and the finite time horizon.

- For $\lambda=1$, the simulation results are close to the theoretical values, particularly for larger T values.
- For $\lambda=5$, the system becomes unstable when the service rates are insufficient to handle the arrival rate, leading to negative theoretical steady-state values, which are not physically meaningful. The simulation results reflect this instability with high average numbers of customers in the system.

The initialization of the system with a large number of customers ($q=1000$) significantly impacts the average time average number of customers in the system for smaller T values. As T increases, the impact of initial conditions diminishes, and the results converge toward the theoretical steady-state values.

Conclusion

The simulation successfully approximates the time-average number of customers in a tandem queueing system for different parameter combinations. The comparison with theoretical steady-state values highlights the importance of system stability and the impact of initial conditions on the simulation outcomes. Further studies could explore more complex queueing networks and consider alternative service time distributions to enhance the model's applicability to real-world scenarios.