بسم الله الرّحمن الرّحیم

تمرین سری ششم

۱.

$$x_1[n] = \left(\frac{1}{2}\right)^{|n|} cos\left(\frac{\pi}{8}(n-1)\right)$$

$$= \left(\frac{1}{2}\right)^{n} cos\left(\frac{\pi}{8}n-\frac{\pi}{8}\right)u[n] + \left(\frac{1}{2}\right)^{-n} cos\left(\frac{\pi}{8}n-\frac{\pi}{8}\right)u[-n-1]$$

$$X_1\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n}\left(\frac{e^{j\left(\frac{\pi}{8}n-\frac{\pi}{8}\right)}+e^{-j\left(\frac{\pi}{8}n-\frac{\pi}{8}\right)}}{2}\right)e^{-j\omega n}$$

$$+\sum_{-\infty}^{-1}\left(\frac{1}{2}\right)^{-n}\left(\frac{e^{j\left(\frac{\pi}{8}n-\frac{\pi}{8}\right)}+e^{-j\left(\frac{\pi}{8}n-\frac{\pi}{8}\right)}}{2}\right)e^{-j\omega n}$$

$$= \frac{e^{-j\frac{\pi}{8}}}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n}e^{j\frac{\pi}{8}n}e^{-j\omega n} + \frac{e^{j\frac{\pi}{8}}}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n}e^{-j\frac{\pi}{8}n}e^{-j\omega n}$$

$$+\frac{e^{-j\frac{\pi}{8}}}{2}\sum_{n=-\infty}^{-1}\left(\frac{1}{2}\right)^{-n}e^{j\frac{\pi}{8}n}e^{-j\omega n} + \frac{e^{j\frac{\pi}{8}}}{2}\sum_{n=-\infty}^{-1}\left(\frac{1}{2}\right)^{-n}e^{-j\frac{\pi}{8}n}e^{-j\omega n}$$

$$= \frac{e^{-j\frac{\pi}{8}}}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{j\frac{\pi}{8}}\right)^{n}e^{-j\omega n} + \frac{e^{j\frac{\pi}{8}}}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{-j\frac{\pi}{8}}\right)^{n}e^{-j\omega n}$$

$$+\frac{e^{-j\frac{\pi}{8}}}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2}e^{-j\frac{\pi}{8}}\right)^{n}e^{j\omega n} + \frac{e^{j\frac{\pi}{8}}}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2}e^{j\frac{\pi}{8}}\right)^{n}e^{j\omega n}$$

$$= \frac{e^{-j\frac{\pi}{8}}}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{j\frac{\pi}{8}}\right)^{n}e^{-j\omega n} + \frac{e^{j\frac{\pi}{8}}}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{-j\frac{\pi}{8}}\right)^{n}e^{-j\omega n}$$

$$+\frac{e^{-j\frac{\pi}{8}}}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{-j\frac{\pi}{8}}\right)^{n+1}e^{j\omega(n+1)} + \frac{e^{j\frac{\pi}{8}}}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{j\frac{\pi}{8}}\right)^{n+1}e^{j\omega(n+1)}$$

$$= \frac{e^{-j\frac{\pi}{8}}}{2}\frac{1}{1-\frac{1}{2}e^{j\frac{\pi}{8}}e^{-j\omega}} + \frac{e^{j\frac{\pi}{8}}}{2}\frac{1}{1-\frac{1}{2}e^{-j\frac{\pi}{8}}e^{-j\omega}}$$

$$+\frac{e^{-j\frac{\pi}{8}}}{2}\frac{\frac{1}{2}e^{j(\omega-\frac{\pi}{8})}}{1-\frac{1}{2}e^{-j\frac{\pi}{8}}e^{j\omega}} + \frac{e^{j\frac{\pi}{8}}}{2}\frac{\frac{1}{2}e^{j(\omega+\frac{\pi}{8})}}{1-\frac{1}{2}e^{j\frac{\pi}{8}}e^{j\omega}}$$

$$x_2[n] = \left(\frac{1}{2}\right)^{-n} u[-n-1]$$

$$X_2(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{-n} u[-n-1] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} \left(\frac{1}{2}e^{j\omega}\right)^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}e^{j\omega}\right)^{n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}e^{j\omega}\right)^{n+1} = \frac{1}{2}\frac{e^{j\omega}}{1-\frac{1}{2}e^{j\omega}}$$

۲. عبارت داده شده برای $X_1(e^{j\omega})$ را با رابطه تبدیل فوریه گسسته سیگنال‌های متناوب مطابقت می‌دهیم:

$$\omega_0 = \frac{\pi}{4},\ a_k = \frac{1}{2\pi}(-1)^k$$

$$x[n] = \sum_{k=0}^{7} (-1)^k e^{jk\frac{\pi}{4}n}$$

$$= \frac{1}{2\pi}\left(1 - e^{j\frac{\pi}{4}n} + e^{j\frac{\pi}{2}n} - e^{j\frac{3\pi}{4}n} + e^{j\pi n} - e^{j\frac{5\pi}{4}n} + e^{j\frac{3\pi}{2}n} - e^{j\frac{7\pi}{4}n}\right)$$

$$= \frac{1}{2\pi}\left(-1\right)^n \left(1 + \left(-1\right)^n - 2cos\left(\frac{3\pi}{4}n\right) + 2cos\left(\frac{\pi}{2}n\right) - 2cos\left(\frac{\pi}{4}n\right)\right)$$

$$X_2(e^{j\omega}) = \frac{1-\frac{1}{3}e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega}}$$

$$= \frac{1-\frac{1}{3}e^{-j\omega}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1+\frac{1}{4}e^{-j\omega}\right)}$$

$$= \frac{a}{1-\frac{1}{2}e^{-j\omega}} + \frac{b}{1+\frac{1}{4}e^{-j\omega}}$$

$$a = \left(1-\frac{1}{2}e^{-j\omega}\right) X_2(e^{j\omega})\Big|_{e^{-j\omega}=2} = \frac{1-\frac{2}{3}}{1+\frac{1}{2}} = \frac{2}{9}$$

$$b = \left(1+\frac{1}{4}e^{-j\omega}\right) X_2(e^{j\omega})\Big|_{e^{-j\omega}=-4} = \frac{1+\frac{4}{3}}{1+2} = \frac{7}{9}$$

$$X_2(e^{j\omega}) = \frac{\frac{2}{9}}{1-\frac{1}{2}e^{-j\omega}} + \frac{\frac{7}{9}}{1+\frac{1}{4}e^{-j\omega}}$$

$$x_2[n] = \frac{2}{9}\left(\frac{1}{2}\right)^n u[n] + \frac{7}{9}\left(-\frac{1}{4}\right)^n u[n]$$

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$$
\begin{aligned}
H(e^{j\omega}) &= H_1(e^{j\omega})H_2(e^{j\omega}) \\
&= \frac{2-e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}\frac{1}{-\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-2j\omega}} \\
&= \frac{2-e^{-j\omega}}{1+\frac{1}{8}e^{-j3\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}
\end{aligned}
$$

$$
2X(e^{j\omega}) - e^{-j\omega}X(e^{j\omega}) = Y(e^{j\omega}) + \frac{1}{8}e^{-j3\omega}Y(e^{j\omega})
$$

با گرفتن عکس تبدیل فوریه خواهیم داشت:

$$
2x[n] - x[n-1] = y[n] + \frac{1}{8}y[n-3]
$$

برای پیدا کردن پاسخ ضربه سیستم با بسط به کسرهای جزئی:

$$
H(e^{j\omega}) = \frac{2-e^{-j\omega}}{1+\frac{1}{8}e^{-j3\omega}} = \frac{\frac{4}{3}}{1+\frac{1}{2}e^{-j\omega}} + \frac{\frac{1+j\sqrt(3)}{3}}{1-\frac{1}{2}e^{j120}e^{-j\omega}} + \frac{\frac{1-j\sqrt(3)}{3}}{1-\frac{1}{2}e^{-j120}e^{-j\omega}}
$$

و گرفتن عکس فوریه خواهیم داشت:

$$
h[n] = \frac{4}{3}\left(-\frac{1}{2}\right)^n u[n] + \frac{1+j\sqrt(3)}{3}\left(\frac{1}{2}e^{j120}\right)^n u[n] + \frac{1-j\sqrt{3}}{3}\left(\frac{1}{2}e^{-j120}\right)^n u[n]
$$

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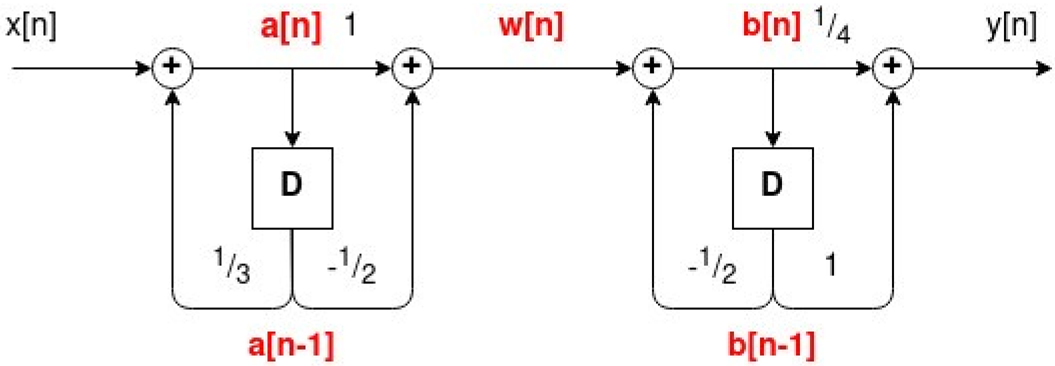

$$a[n] = x[n] + \frac{1}{3}a[n-1] \rightarrow a[n] - \frac{1}{3}a[n-1] = x[n]$$

$$w[n] = a[n] - \frac{1}{2}a[n-1]$$

$$\frac{A(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-\frac{1}{3}e^{-j\omega}} \qquad \frac{W(e^{j\omega})}{A(e^{j\omega})} = 1 - \frac{1}{2}e^{-j\omega}$$

$$\frac{W(e^{j\omega})}{X(e^{j\omega})} = \frac{A(e^{j\omega})}{X(e^{j\omega})}\frac{W(e^{j\omega})}{A(e^{j\omega})} = \frac{1-\frac{1}{2}e^{-j\omega}}{1-\frac{1}{3}e^{-j\omega}}$$

$$b[n] = w[n] - \frac{1}{2}b[n-1] \rightarrow b[n] + \frac{1}{2}b[n-1] = w[n]$$

$$y[n] = \frac{1}{4}b[n] + b[n-1]$$

$$\frac{B(e^{j\omega})}{W(e^{j\omega})} = \frac{1}{1+\frac{1}{2}e^{-j\omega}} \qquad \frac{Y(e^{j\omega})}{B(e^{j\omega})} = \frac{1}{4} + e^{-j\omega}$$

$$\frac{Y(e^{j\omega})}{W(e^{j\omega})} = \frac{B(e^{j\omega})}{W(e^{j\omega})}\frac{Y(e^{j\omega})}{B(e^{j\omega})} = \frac{\frac{1}{4}+e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}$$

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{Y(e^{j\omega})}{W(e^{j\omega})}\frac{W(e^{j\omega})}{X(e^{j\omega})} = \frac{\left(\frac{1}{4}+e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)}{\left(1+\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)}$$

$$H(e^{j\omega}) = \frac{\left(\frac{1}{4}+e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)}{\left(1+\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)} = 3 - \frac{2.1}{1+\frac{1}{2}e^{-j\omega}} - \frac{0.65}{1-\frac{1}{3}e^{-j\omega}}$$

$$h[n] = 3\delta[n] - 2.1\left(-\frac{1}{2}\right)^n u[n] - 0.65\left(\frac{1}{3}\right)^n u[n]$$

$$\frac{1}{4}x[n] + \frac{7}{8}x[n-1] - \frac{1}{2}x[n-2] = y[n] + \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2]$$

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$$x[n] = \left(\frac{4}{5}\right)^n u[n] \rightarrow X(e^{j\omega}) = \frac{1}{1-\frac{4}{5}e^{-j\omega}}$$

$$y[n] = n\left(\frac{4}{5}\right)^n u[n] \rightarrow Y(e^{j\omega}) = j\frac{dX(e^{j\omega})}{d\omega} = \frac{\frac{4}{5}e^{-j\omega}}{\left(1-\frac{4}{5}e^{-j\omega}\right)^2}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\frac{4}{5}e^{-j\omega}}{1-\frac{4}{5}e^{-j\omega}}$$

$$Y(e^{j\omega})\left(1-\frac{4}{5}e^{-j\omega}\right) = X(e^{j\omega})\left(\frac{4}{5}e^{-j\omega}\right)$$

$$y[n] - \frac{4}{5}y[n-1] = \frac{4}{5}x[n-1]$$

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$$Y(e^{j\omega}) - \frac{1}{6}e^{-j\omega}Y(e^{j\omega}) - \frac{1}{6}e^{-j2\omega}Y(e^{j\omega}) = X(e^{j\omega})$$

$$Y(e^{j\omega})\left(1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-j2\omega}\right) = X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-j2\omega}}$$

$$= \frac{\frac{3}{5}}{1-\frac{1}{2}e^{-j\omega}} + \frac{\frac{2}{5}}{1+\frac{1}{3}e^{-j\omega}}$$

$$h[n] = \frac{3}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{5}\left(-\frac{1}{3}\right)^n u[n]$$

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$$x_1[n] = (n-1)^2\, x[n]$$

$$F\left\{nx[n]\right\} = j\frac{dX(e^{j\omega})}{d\omega}$$

$$F\left\{n^2x[n]\right\} = (-1)\frac{d^2X(e^{j\omega})}{d\omega^2}$$

$$x_1[n] = n^2x[n] - 2nx[n] + x[n]$$

$$X_1(e^{j\omega}) = -\frac{d^2X(e^{j\omega})}{d\omega^2} - 2j\frac{dX(e^{j\omega})}{d\omega} + X(e^{j\omega})$$

$$x_2[n] = x[1-n] + x[-1-n]$$
$$F\{x[-n]\} = X(e^{-j\omega})$$
$$F\{x[-n+1]\} = e^{-j\omega}X(e^{-j\omega})$$
$$F\{x[-n-1]\} = e^{j\omega}X(e^{-j\omega})$$

$$X_2(e^{j\omega}) = e^{-j\omega}X(e^{-j\omega}) + e^{j\omega}X(e^{-j\omega}) = 2X(e^{-j\omega})cos\omega$$

.۸

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$
$$H_1(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$
$$H_2(e^{j\omega}) = \frac{-12+5e^{-j\omega}}{12-7e^{-j\omega}+e^{-j2\omega}} - \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$
$$= -\frac{2}{1-\frac{1}{4}e^{-j\omega}} - \frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$
$$h_2[n] = -2\left(\frac{1}{4}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$$

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$$N = 6$$
$$x[n] = \frac{1}{2j}e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} - \frac{1}{2j}e^{-j(\frac{\pi}{3}n+\frac{\pi}{4})}$$
$$= \frac{1}{2j}e^{j\frac{\pi}{4}}e^{j\frac{2\pi}{6}n} - \frac{1}{2j}e^{-j\frac{\pi}{4}}e^{-j\frac{2\pi}{6}n}$$
$$a_1 = \frac{1}{2j}e^{j\frac{\pi}{4}},\ a_{-1} = -\frac{1}{2j}e^{-j\frac{\pi}{4}}$$
$$X(e^{j\omega}) = 2\pi a_1\delta(\omega - \frac{2\pi}{6}) + 2\pi a_{-1}\delta(\omega + \frac{2\pi}{6})$$
$$= \frac{\pi}{j}\left(e^{j\frac{\pi}{4}}\delta(\omega - \frac{2\pi}{6}) - e^{-j\frac{\pi}{4}}\delta(\omega + \frac{2\pi}{6})\right)$$