بسم اللّه الرّحمن الرّحيم

تمرين سری نهم

۱.

الف)

$$x[n] = \left(\frac{1}{5}\right)^n u[n-3]$$

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{5}\right)^n u[n-3] z^{-n} = \sum_{n=3}^{\infty} \left(\frac{1}{5}\right)^n z^{-n}$$

$$= \left(\frac{z^{-3}}{125}\right) \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n z^{-n}$$

$$= \frac{z^{-3}}{125} \frac{1}{1 - \frac{1}{5} z^{-1}}, \qquad |z| > \frac{1}{5}$$

قطب:

$$z = \frac{1}{5}$$

ب)

$$x[n] = \begin{cases} \left(\frac{1}{3}\right)^n cos\left(\frac{\pi}{4} n\right), & n \leq 0; \\ 0 & n > 0. \end{cases}$$

$$X(z) = \sum_{-\infty}^{0} \left(\frac{1}{3}\right)^n cos\left(\frac{\pi}{4} n\right) z^{-n}$$

$$= \frac{1}{2} \sum_{z=-\infty}^{n} \left(\frac{1}{3}\right)^n e^{j\frac{\pi}{4} n} z^{-n} + \frac{1}{2} \sum_{z=-\infty}^{n} \left(\frac{1}{3}\right)^n e^{-j\frac{\pi}{4} n} z^{-n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{-n} e^{-j\frac{\pi}{4} n} z^{n} + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{-n} e^{j\frac{\pi}{4} n} z^{n}$$

$$= \frac{1}{2} \frac{1}{1 - 3e^{-j\frac{\pi}{4}} z} + \frac{1}{2} \frac{1}{1 - 3e^{j\frac{\pi}{4}} z}, \qquad |z| < \frac{1}{3}$$

قطب‌ها:

$$z = \frac{1}{3} e^{j\frac{\pi}{4}}, \; \frac{1}{3} e^{-j\frac{\pi}{4}}$$

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$$X(z) = \frac{1 - \frac{1}{3}z^{-1}}{\left(1 - z^{-1}\right)\left(1 + 2z^{-1}\right)}, \qquad |z| > 2$$

$$= \frac{a}{1 - z^{-1}} + \frac{b}{1 + 2z^{-1}}$$

$$a = \left(1 - z^{-1}\right) X(z)\Big|_{z^{-1}=1} = \frac{\frac{2}{3}}{3} = \frac{2}{9}$$

$$b = \left(1 + 2z^{-1}\right) X(z)\Big|_{z^{-1}=-\frac{1}{2}} = \frac{\frac{7}{6}}{\frac{3}{2}} = \frac{7}{9}$$

$$= \frac{\frac{2}{9}}{1 - z^{-1}} + \frac{\frac{7}{9}}{1 + 2z^{-1}}$$

$$x[n] = \frac{2}{9}u[n] + \frac{7}{9}\left(-2\right)^{n} u[n]$$

.۳

الف)

$$\begin{array}{l|l} 1+z^{-1} & 1+\frac{1}{3}z^{-1} \\ \hline & 1+\frac{2}{3}z^{-1}-\frac{2}{9}z^{-2}+\dots \end{array}$$

$$\begin{array}{l} 1+z^{-1} \\ \underline{1+\frac{1}{3}z^{-1}} \\ \quad \frac{2}{3}z^{-1} \\ \quad \underline{\frac{2}{3}z^{-1}+\frac{2}{9}z^{-2}} \\ \qquad -\frac{2}{9}z^{-2} \\ \qquad \underline{-\frac{2}{9}z^{-2}-\frac{2}{27}z^{-3}} \\ \qquad\qquad \dots \end{array}$$

$x[0] = 1,\ x[1] = \dfrac{2}{3},\ x[2] = -\dfrac{2}{9}$

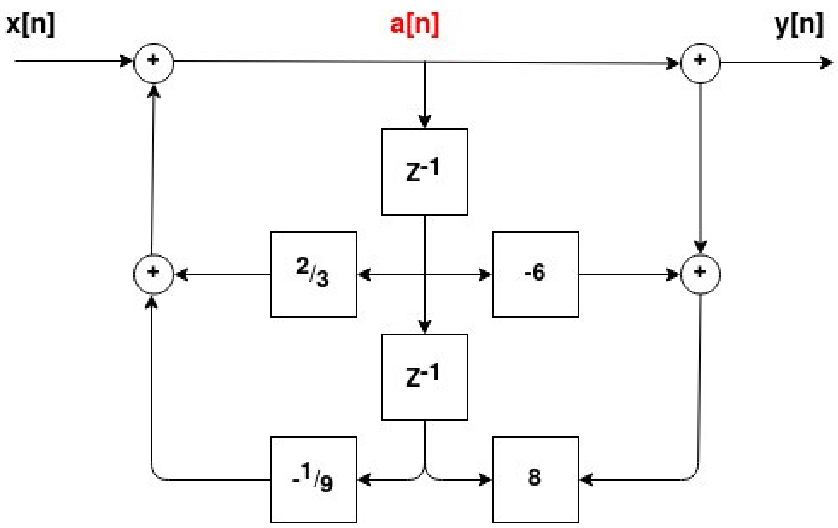
ب)

$$\begin{array}{r|l} 1+z^{-1} & 1+\frac{1}{3}z^{-1} \\ 3+z^{-1} & \overline{3-6z+18z^{2}\dots} \end{array}$$

$$\begin{array}{r} 1+z^{-1} \\ \underline{3+z^{-1}} \\ -2 \\ \underline{-6z-2} \\ 6z \\ \underline{18z^{2}+6z} \\ \dots \end{array}$$

$x[0] = 3,\ x[1] = -6,\ x[2] = 18$

۴.



$$a[n] = x[n] + \frac{2}{3}a[n-1] - \frac{1}{9}a[n-2]$$

$$A(z) - \frac{2}{3}z^{-1}A(z) + \frac{1}{9}z^{-2}A(z) = X(z) \to \frac{A(z)}{X(z)} = \frac{1}{1 - \frac{2}{3}z^{-1} + \frac{1}{9}z^{-2}}$$

$$y[n] = a[n] + 8a[n-2] - 6a[n-1]$$

$$Y(z) = A(z)\left(1 - 6z^{-1} + 8z^{-2}\right) \to \frac{Y(z)}{A(z)} = 1 - 6z^{-1} + 8z^{-2}$$

$$\frac{Y(z)}{X(z)} = \frac{A(z)}{X(z)}\frac{Y(z)}{A(z)} = \frac{1 - 6z^{-1} + 8z^{-2}}{1 - \frac{2}{3}z^{-1} + \frac{1}{9}z^{-2}}$$

$$Y(z) - \frac{2}{3}z^{-1}Y(z) + \frac{1}{9}z^{-2}Y(z) = X(z) - 6z^{-1}X(z) + 8z^{-2}X(z)$$

$$y[n] - \frac{2}{3}y[n-1] + \frac{1}{9}y[n-2] = x[n] - 6x[n-1] + 8x[n-2]$$

قطب‌ها:

$$z^2 - \frac{2}{3}z + \frac{1}{9} = \left(z - \frac{1}{3}\right)^2 \to z = \frac{1}{3}$$ قطب مضاعف

سیستم علی بوده و ناحیه همگرایی $|z| > \frac{1}{3}$ است. چون ناحیه همگرایی شامل دایره واحد می‌باشد، لذا سیستم پایدار است.

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الف)

$$x_1[n] = \left(\frac{1}{4}\right)^n u[n+5]$$

$$X_1(z) = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n u[n+5] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n z^{-n} = \frac{1}{1-\frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{4}$$

ب)

$$x_2[n] = \delta[n+3] + \delta[n] + 2^n u[-n]$$

$$X_2(z) = \sum_{n=0}^{\infty} \left(\delta[n+3] + \delta[n] + 2^n u[-n]\right) z^{-n}$$

با توجه به اینکه

$$n = -3 \to \delta[n+3] \neq 0$$

$$n = 0 \to \delta[n] \neq 0$$

$$n \leq 0 \to 2^n u[-n] \neq 0 \to \sum_{n=0}^{\infty} 2^n u[-n] = \sum_{n=0}^{\infty} \delta[n]$$

$$X_2(z) = \sum_{n=0}^{\infty} \left(\delta[n+3] + \delta[n] + \delta[n]\right) z^{-n}$$

$$= 2z^0 = 2$$

ج)

$$x_3[n] = \left(\frac{1}{2}\right)^{|n|}$$

$$X_3(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{|n|} z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}$$

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$$y[n-1]+2y[n]=x[n]$$

الف)

$$z^{-1}Y(z)+y[-1]+2Y(z)=X(z), \qquad X(z)=0$$

$$Y(z)\left(2+z^{-1}\right)=-2 \rightarrow Y(z)=-\frac{2}{2+z^{-1}}=-\frac{1}{1+\frac{1}{2}z^{-1}}$$

$$y[n]=-\left(-\frac{1}{2}\right)^{n}u[n]$$

ب)

$$z^{-1}Y(z)+2Y(z)=X(z) \rightarrow Y(z)=\frac{X(z)}{2+z^{-1}}$$

$$X(z)=\frac{1}{1-\frac{1}{4}z^{-1}}, \qquad |z|>\frac{1}{4}$$

$$Y(z)=\frac{1}{1-\frac{1}{4}z^{-1}}\ \frac{\frac{1}{2}}{1+\frac{1}{2}z^{-1}}=\frac{a}{1-\frac{1}{4}z^{-1}}+\frac{b}{1+\frac{1}{2}z^{-1}}$$

$$a=\left(1-\frac{1}{4}z^{-1}\right)Y(z)\Big|_{z^{-1}=4}=\frac{1}{6}$$

$$b=\left(1+\frac{1}{2}z^{-1}\right)Y(z)\Big|_{z^{-1}=-2}=\frac{1}{3}$$

$$Y(z)=\frac{\frac{1}{6}}{1-\frac{1}{4}z^{-1}}+\frac{\frac{1}{3}}{1+\frac{1}{2}z^{-1}}$$

$$y[n]=\frac{1}{6}\left(\frac{1}{4}\right)^{n}u[n]+\frac{1}{3}\left(-\frac{1}{2}\right)^{n}u[n]$$

ج)

با توجه به خاصیت جمع آثار در سیستم‌های LTI حاصل مجموع خروجی بند الف و ب خواهد بود:

$$\begin{aligned} y[n]&=\frac{1}{6}\left(\frac{1}{4}\right)^{n}u[n]+\frac{1}{3}\left(-\frac{1}{2}\right)^{n}u[n]-\left(-\frac{1}{2}\right)^{n}u[n] \\ &=\frac{1}{6}\left(\frac{1}{4}\right)^{n}u[n]-\frac{2}{3}\left(-\frac{1}{2}\right)^{n}u[n] \end{aligned}$$

.۷

الف)

$$x[n] = 2^n u[-n] + \left(\frac{1}{4}\right)^n u[n-1]$$

$$x_1[n] = 2^n u[-n]$$

$$X_1(z) = \sum_{-\infty}^{\infty} x_1[n] z^{-n} = \sum_{-\infty}^{0} 2^n z^{-n}$$

$$= \sum_{n} = 0^{\infty} 2^{-n} z^n = \frac{1}{1-\frac{1}{2}z}, \qquad |z| < 2$$

$$= \frac{-2z^{-1}}{1-2z^{-1}}, \qquad |z| < 2$$

$$x_2[n] = \left(\frac{1}{4}\right)^n u[n-1]$$

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2[n] z^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n+1} z^{-n-1}$$

$$= \frac{z^{-1}}{4} \left(\frac{1}{1-\frac{1}{4}z^{-1}}\right), \qquad |z| > \frac{1}{4}$$

$$X(z) = X_1(z) + X_2(z) = \frac{-2z^{-1}}{1-2z^{-1}} + \frac{z^{-1}}{4} \left(\frac{1}{1-\frac{1}{4}z^{-1}}\right), \qquad \frac{1}{4} < |z| < 2$$

ب)

$$x[n] = n \left(\frac{1}{2}\right)^{|n|} = n \left(\left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1]\right) = n \left(x_1[n] + x_2[n]\right)$$

$$X_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}$$

$$X_2(z) = -\frac{1}{1-2z^{-1}}, \qquad |z| < 2$$

$$X(z) = -z \frac{d}{dz} \left(X_1(z) + X_2(z)\right) = -\frac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} + \frac{2z^{-1}}{\left(1-2z^{-1}\right)^2}$$

.۸

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{1}{2}z^{-1}}$$

$$\begin{array}{l|l}
1 - \frac{1}{2}z^{-1} & 1 + \frac{1}{2}z^{-1} \\ \cline{2-2}
\underline{1 + \frac{1}{2}z^{-1}} & 1 - z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{4}z^{-3} + \frac{1}{8}z^{-4} - \cdots \\
-z^{-1} & \\
\underline{-z^{-1} - \frac{1}{2}z^{-2}} & \\
\frac{1}{2}z^{-2} & \\
\underline{\frac{1}{2}z^{-2} + \frac{1}{4}z^{-3}} & \\
-\frac{1}{4}z^{-3} & \\
\underline{-\frac{1}{4}z^{-3} - \frac{1}{8}z^{-4}} & \\
\frac{1}{8}z^{-4} & \\
\underline{\frac{1}{8}z^{-4} + \frac{1}{16}z^{-5}} & \\
-\frac{1}{16}z^{-5} & \\
\cdots &
\end{array}$$

$$\begin{aligned}
X(z) &= 1 - z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{4}z^{-3} + \cdots \\
&= 1 - z^{-1}\left(1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \cdots\right) \\
&= 1 - z^{-1}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n} z^{-n}
\end{aligned}$$

$$x[n] = \delta[n] - \left(-\frac{1}{2}\right)^{n-1} u[n-1]$$

.۹

$$y[n] = x_1[n+3] * x_2[-n+1]$$

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n],\ x_2[n] = \left(\frac{1}{3}\right)^n u[n]$$

$$Y(z) = z^3 X_1(z)\ .\ z^{-1} X_2\left(z^{-1}\right)$$

$$= z^2 X_1(z) X_2\left(z^{-1}\right)$$

$$X_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$X_2(z) = \frac{1}{1-\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$X_2\left(z^{-1}\right) = \left.\frac{1}{1-\frac{1}{3}z^{-1}}\right|_{z\to z^{-1}} = \frac{1}{1-\frac{1}{3}z}$$

$$Y(z) = z^2 \frac{1}{1-\frac{1}{2}z^{-1}} \frac{1}{1-\frac{1}{3}z}$$

$$= \frac{z^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z\right)}$$

.۱۰

با توجه به حقیقی بودن سیگنال قطب‌ها مزدوج مختلط خواهند بود:

$$z_{1,2} = \frac{1}{2} e^{\pm j\frac{\pi}{3}}$$

با توجه به اینکه تبدیل z سیگنال دو صفر در مبدا دارد، حالت کلی آن به شکل زیر خواهد بود:

$$X(z) = \frac{Az^2}{\left(z-\frac{1}{2}e^{j\frac{\pi}{3}}\right)\left(z-\frac{1}{2}e^{-j\frac{\pi}{3}}\right)}$$

با توجه به دست راستی بودن سیگنال ناحیه همگرایی خارج دایره است:

$$|z| > \frac{1}{2}$$

برای پیدا کردن ضریب ثابت نیز به این شکل عمل می‌نماییم:

$$\left.X(z)\right|_{z=1} = \frac{8}{3} \to \left.\frac{Az^2}{z^2 - zcos\left(\frac{\pi}{3}\right)+\frac{1}{4}}\right|_{z=1} = \frac{4A}{3} \to A = 2$$

.۱۱

$$\phi_{xx}[n] = \sum_{k=-\infty}^{\infty} x[k]x[n+k]$$
$$= x[n] * x[-n]$$
$$\Phi_{xx}(z) = X(z)\underset{z\to\frac{1}{z}}{X(z)} = X(z)X\left(\frac{1}{z}\right)$$

.۱۲

$$X_1(z) = log\left(1-2z\right), \qquad |z| < \frac{1}{2}$$
$$= -\sum_{n=1}^{\infty}\frac{2^n z^n}{n} = -\sum_{n=-\infty}^{-1} -\frac{2^{-n}z^{-n}}{n}$$
$$x[n] = \frac{2^{-n}}{n}u[-n-1]$$

$$X_2(z) = log\left(1-\frac{1}{2}z^{-1}\right), \qquad |z| > \frac{1}{2}$$
$$= -\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n z^{-n}}{n}, \qquad |z| > \frac{1}{2}$$
$$x[n] = -\frac{2^{-n}}{n}u[n-1]$$

.۱۳

الف)

$$Y_1(z) = -z\frac{dX_1(z)}{dz} \to Y_1(z) = -z\frac{-2}{1-2z} = \frac{2z}{1-2z}$$
$$= -\frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| < \frac{1}{2}$$
$$y_1[n] = \left(\frac{1}{2}\right)^n u[-n-1] \to x_1[n] = \frac{1}{n}\left(\frac{1}{2}\right)^n u[-n-1]$$
$$= \frac{2^{-n}}{n}u[-n-1]$$

ب)

$$Y_2(z) = -z\frac{dX_2(z)}{dz} = \frac{\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}$$

$$y_2[n] = -\frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$x[n] = -\frac{1}{n}\left(\frac{1}{2}\right)^{n} u[n-1] = -\frac{2^{-n}}{n}u[n-1]$$