**بسم اللّه الرّحمن الرّحیم**

**تمرین فیلترهای زمان پیوسته و زمان گسسته**

۱.

$$x\left(t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X\left(j\omega\right)e^{j\omega t}d\omega$$

الف)

$$x\left(t\right)=\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}e^{j\omega t}d\omega=\frac{1}{2\pi jt}e^{j\omega t}\Big|_{-\omega_c}^{\omega_c}=\frac{1}{\pi t}sin\left(\omega_c t\right)$$

ب)

$$x\left(t\right)=\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}e^{j\omega T}e^{j\omega t}d\omega=\frac{1}{2\pi j\left(t+T\right)}e^{j\omega(t+T)}\Big|_{-\omega_c}^{\omega_c}=\frac{1}{\pi\left(t+T\right)}sin\left(\omega\left(t+T\right)\right)$$

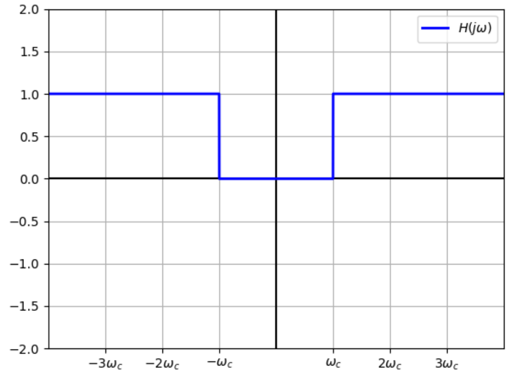
ج)

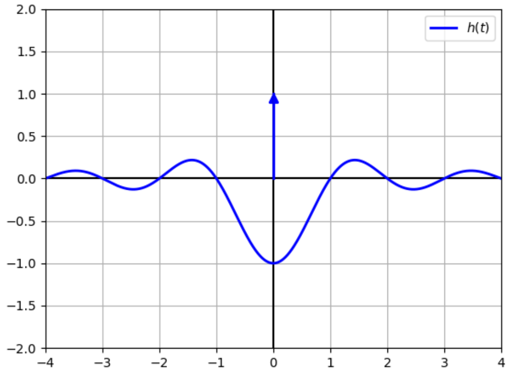
$$\begin{aligned}x\left(t\right)&=\frac{1}{2\pi}\left(\int_{-\omega_c}^{0}e^{-j\frac{\pi}{2}}e^{j\omega t}d\omega+\int_{0}^{\omega_c}e^{j\frac{\pi}{2}}e^{j\omega t}d\omega\right)\\&=\frac{1}{2\pi}\left(\frac{e^{-j\frac{\pi}{2}}}{jt}e^{j\omega t}\Big|_{-\omega_c}^{0}+\frac{e^{j\frac{\pi}{2}}}{jt}e^{j\omega t}\Big|_{0}^{\omega_c}\right)\\&=\frac{1}{2\pi}\left(\frac{-j}{jt}\left(1-e^{-j\omega_c t}\right)+\frac{j}{jt}\left(e^{j\omega_c t}-1\right)\right)\\&=\frac{1}{\pi t}\left(cos\left(\omega_c t\right)-1\right)\end{aligned}$$

۲.

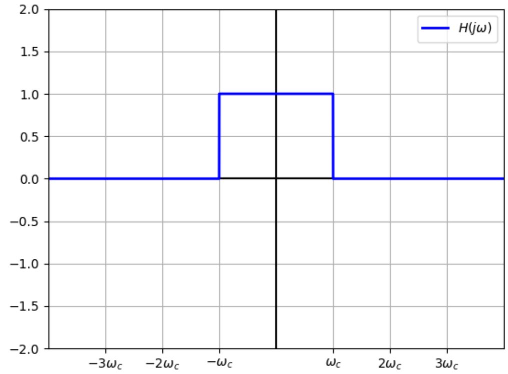
$$y(t) = x(t) - x(t) * h(t)$$
$$Y(j\omega) = X(j\omega) - X(j\omega) H(j\omega) = X(j\omega)(1 - H(j\omega))$$

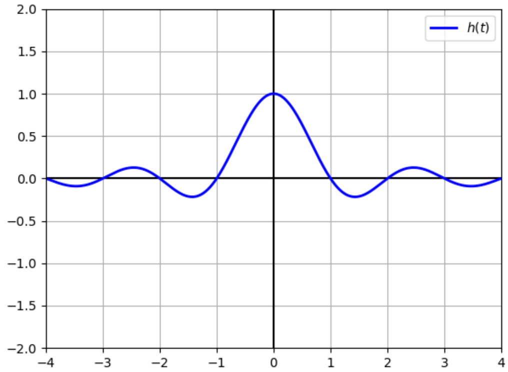
الف)




ب)



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$$x(t) = cos(\omega_0 t + \phi_0) = \frac{1}{2}\left(e^{j(\omega_0 t + \phi_0)} + e^{-j(\omega_0 t + \phi_0)}\right)$$

$$X(j\omega) = \frac{1}{2}e^{j\phi_0}\delta(\omega - \omega_0) + \frac{1}{2}e^{-j\phi_0}\delta(\omega + \omega_0)$$

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) \\ &= \left(\frac{1}{2}e^{j\phi_0}\delta(\omega - \omega_0) + \frac{1}{2}e^{-j\phi_0}\delta(\omega + \omega_0)\right)H(j\omega) \\ &= \frac{1}{2}e^{j\phi_0}H(j\omega_0)\delta(\omega - \omega_0) + \frac{1}{2}e^{-j\phi_0}H(-j\omega_0)\delta(\omega + \omega_0) \end{aligned}$$

$H(j\omega_0)$ در حالت کلی یک عدد مختلط ثابت است.

$$H(j\omega_0) = |H(j\omega_0)|\, e^{j\angle H(j\omega_0)}$$

از عبارت به‌دست آمده برای $Y(j\omega)$ عکس تبدیل فوریه می‌گیریم:

$$y(t) = \frac{1}{2}e^{j\phi_0}H(j\omega_0)e^{j\omega_0 t} + \frac{1}{2}e^{-j\phi_0}H(-j\omega_0)e^{-j\omega_0 t}$$

با توجه به حقیقی بودن پاسخ ضربه، پاسخ فرکانسی دارای تقارن هرمیتی می‌باشد:

$$H(-j\omega_0)^* = H(j\omega) \longrightarrow H(-j\omega_0) = |H(j\omega_0)|\, e^{-j\angle H(j\omega_0)}$$

از این رو خواهیم داشت:

$$\begin{aligned} y(t) &= \frac{1}{2}|H(j\omega_0)|\left(e^{j(\omega_0 t + \phi_0 + \angle H(j\omega_0))} + e^{-j(\omega_0 t + \phi_0 + \angle H(j\omega_0))}\right) \\ &= |H(j\omega_0)|\, cos(\omega_0 t + \phi_0 + \angle H(j\omega_0)) \end{aligned}$$

الف)

$$A = |H(j\omega_0)|$$

ب)

$$-\omega_0 t_0 = \angle H(j\omega) \longrightarrow t_0 = -\frac{\angle H(j\omega_0)}{\omega_0}$$

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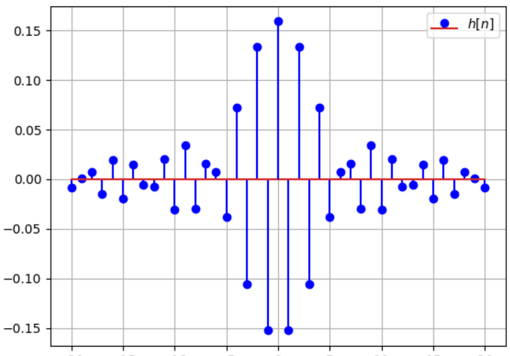
الف)

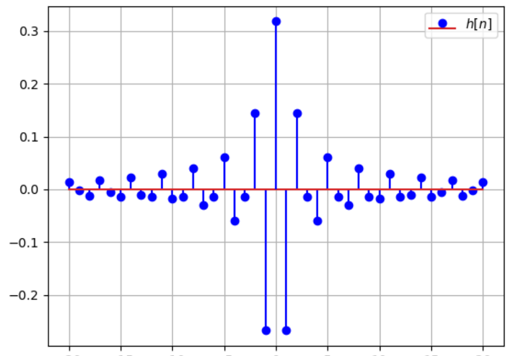
$$
\begin{aligned}
h[n] &= \frac{1}{2\pi}\int_{\omega=<2\pi>} H\left(e^{j\omega}\right)e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{\pi-\omega_c}^{\pi+\omega_c} e^{j\omega n}d\omega = \frac{1}{2j\pi n}e^{j\omega n}\Big|_{\pi-\omega_c}^{\pi+\omega_c} = \frac{e^{j\pi n}}{2j\pi n}\left(e^{j\omega_c n}-e^{-j\omega_c n}\right) \\
&= (-1)^n\,\frac{sin\left(\omega_c n\right)}{\pi n}
\end{aligned}
$$

$$
h[n] = \left(\frac{sin\left(\omega_c n\right)}{\pi n}\right)g[n] \rightarrow g[n] = (-1)^n
$$

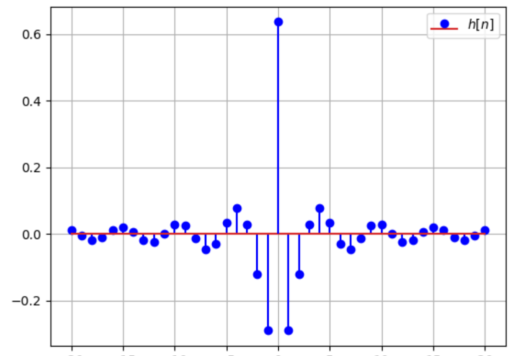
ب)



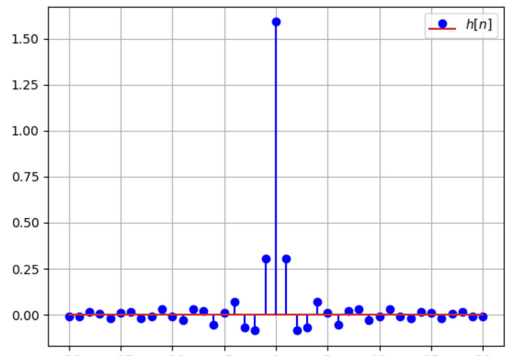

$\omega_c = 0.5$




$\omega_c = 1.0$




$\omega_c = 2.0$




$\omega_c = 5.0$

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الف)

$$x_2[n] = (-1)^n x[n], \quad y_2[n] = x_2[n] * h_{lp}[n], \quad y[n] = (-1)^n y_2[n]$$

$$y[n]\Big|_{x[n]} = (-1)^n \left(((-1)^n x[n]) * h_{lp}[n]\ \right)$$

$$= \sum_{k=-\infty}^{\infty} (-1)^{n+k} x[k]\, h_{lp}[n-k]$$

$$y[n-n_0] = \sum_{k=\infty}^{\infty} (-1)^{n-n_0+k} x[k]\, h_{lp}[n-n_0-k]$$

$$y[n]\Big|_{x[n-n_0]} = \sum_{k=-\infty}^{\infty} (-1)^{n+k} x[k-n_0]\, h_{lp}[n-k]$$

تغییر متغیر $l = k - n_0$ را انجام می‌دهیم:

$$y[n]\Big|_{x[n-n_0]} = \sum_{l=-\infty}^{\infty} (-1)^{n+l+n_0} x[l]\, h_{lp}[n-n_0-l]$$

با توجه به اینکه $(-1)^{n+k+n_0} = (-1)^{n+k-n_0}$ لذا

$$y[n-n_0] = y[n]\Big|_{x[n-n_0]}$$

پس سیستم فوق یک سیستم تغییر ناپذیر با زمان است.

ب)

$$x_2[n] = (-1)^n x[n] = e^{j\pi n} x[n] \to X_2\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)\Big|_{\omega\to\omega-\pi}$$

$$Y_2\left(e^{j\omega}\right) = H_{lp}\left(e^{j\omega}\right) X_2\left(e^{j\omega}\right)$$

$$y[n] = (-1)^n y_2[n] = e^{j\pi n} y_2[n] \to Y\left(e^{j\omega}\right) = Y_2\left(e^{j\omega}\right)\Big|_{\omega\to\omega-\pi}$$

$$Y\left(e^{j\omega}\right) = H_{lp}\left(e^{j\omega}\right)\Big|_{\omega\to\omega-\pi} X\left(e^{j\omega}\right)\Big|_{\omega\to\omega-2\pi}$$

$$= H_{lp}\left(e^{j\omega}\right)\Big|_{\omega\to\omega-\pi} X\left(e^{j\omega}\right)$$

همان طور که مشاهده می‌شود پاسخ فرکانسی سیستم اولیه به اندازه $\pi$ واحد شیفت داده شده است. یعنی اگر در فرکانس‌های بالا تضعیف داشتیم و فرکانس‌های پایین از فیلتر عبور می‌کردند، حالا تضعیف در فرکانس‌های پایین بوده و فرکانس‌های بالا عبور می‌کنند.

۶.

الف)

$$
\begin{aligned}
Y\left(e^{j\omega}\right) &= b\left\{ae^{-j\omega}X\left(e^{j\omega}\right)+X\left(e^{j\omega}\right)+ae^{j\omega}X\left(e^{j\omega}\right)\right\}\\
&= b\left(1+2acos\left(\omega\right)\right)X\left(e^{j\omega}\right)\\
H\left(e^{j\omega}\right) &= b\left(1+2acos\left(\omega\right)\right)
\end{aligned}
$$

ب)

$$H\left(e^{j\omega}\right)|_{\omega=0}=b\left(1+2acos\left(0\right)\right)=b\left(1+2a\right)$$

$$H\left(e^{j\omega}\right)|_{\omega=0}\equiv 1\rightarrow b=\frac{1}{2a+1}$$

ج)

$$a=\frac{1}{2}\rightarrow b=\frac{1}{2}$$

$$H\left(e^{j\omega}\right)=\frac{1}{2}\left(1+cos\left(\omega\right)\right)$$