بسم اللّه الرّحمن الرّحیم

تمرین سری هشتم

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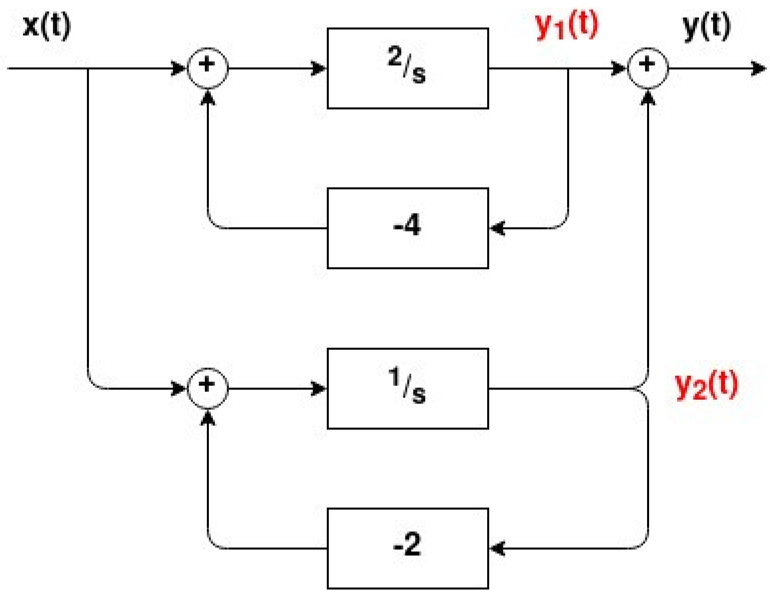

$$Y_1(s) = \frac{2}{s}\left(X(s) - 4Y_1(s)\right) \rightarrow sY_1(s) = 2X(s) - 8y_1(s)$$

$$\rightarrow (s+8)Y_1(s) = 2X(s) \rightarrow \frac{Y_1(s)}{X(s)} = \frac{2}{s+8}$$

$$Y_2(s) = \frac{1}{s}\left(X(s) - 2Y_2(s)\right) \rightarrow sY_2(s) = X(s) - 2Y_2(s)$$

$$\rightarrow (s+2)Y_2(s) = X(s) \rightarrow \frac{Y_2(s)}{X(s)} = \frac{1}{s+2}$$

$$\frac{Y(s)}{X(s)} = \frac{Y_1(s)}{X(s)} + \frac{Y_2(s)}{X(s)} = \frac{2}{s+8} + \frac{1}{s+2}$$

$$= \frac{2s+4+s+8}{(s+2)(s+8)} = \frac{3s+12}{(s+2)(s+8)}$$

$$s^2Y(s) + 10sY(s) + 16 = 3sX(s) + 12X(s)$$

$$\frac{d^2y(t)}{dt^2} + 10\frac{dy(t)}{dt} + 16 = 3\frac{dx(t)}{dt} + 12x(t)$$

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$$x(t) = te^{-2|t|} = t\left(e^{-2t}u(t) + e^{2t}u(-t)\right)$$

$$x_1(t) = e^{-2t}u(t) \rightarrow X_1(s) = \frac{1}{s+2} \qquad \Re e\{s\} > -2$$

$$x_2(t) = e^{2t}u(-t) \rightarrow X_2(s) = \frac{1}{s-2} \qquad \Re e\{s\} < 2$$

$$\begin{aligned} x(t) = t\left(x_1(t) + x_2(t)\right) \rightarrow X(s) &= -\frac{d}{ds}\left(X_1(s) + X_2(s)\right) \qquad -2 < \Re e\{s\} < 2 \\ &= -\frac{d}{ds}\frac{2s}{s^2-4} \qquad -2 < \Re e\{s\} < 2 \\ &= -\frac{2s^2+8}{\left(s^2-4\right)^2} \qquad -2 < \Re e\{s\} < 2 \end{aligned}$$

$$x(t) = |t|e^{-2|t|} = te^{-2t}u(t) - te^{2t}u(-t)$$

$$x_1(t) = e^{-2t}u(t) \rightarrow X_1(s) = \frac{1}{s+2} \qquad \Re e\{s\} > -2$$

$$x_2(t) = e^{2t}u(-t) \rightarrow X_2(s) = \frac{1}{s-2} \qquad \Re e\{s\} < 2$$

$$\begin{aligned} X(s) &= -\frac{d}{ds}X_1(s) + \frac{d}{ds}X_2(s) \qquad -2 < \Re e\{s\} < 2 \\ &= -\frac{d}{ds}\frac{1}{s+2} + \frac{d}{ds}\frac{1}{s-2} \\ &= \frac{-4s}{\left(s^2-4\right)^2} \qquad -2 < \Re e\{s\} < 2 \end{aligned}$$

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$$X(s) = \frac{s+1}{s^2+5s+6}, \quad -3 < \Re e\{s\} < -2$$

$$X(s) = \frac{s+1}{(s+2)(s+3)} = -\frac{1}{s+2} + \frac{2}{s+3}, \quad -3 < \Re e\{s\} < -2$$

$$X_1(s) = -\frac{1}{s+2}, \quad \Re e\{s\} < -2 \to x_1(t) = -e^{-2t}u(-t)$$

$$X_2(s) = \frac{2}{s+3}, \quad \Re e\{s\} > -3 \to x_2(t) = 2e^{-3t}u(t)$$

$$x(t) = 2e^{-3t}u(t) - e^{-2t}u(-t)$$

$$X(s) = \frac{(s+1)^2}{s^2-s+1}, \quad \Re e\{s\} > \frac{1}{2}$$

$$= 1 + \frac{3s}{s^2-s+1} = 1 + \frac{3s}{\left(s-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= 1 + \frac{3\left(s-\frac{1}{2}\right)}{\left(s-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\frac{3}{2}}{\left(s-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$x(t) = \delta(t) + 3e^{-\frac{t}{2}} cos\left(\frac{\sqrt{3}}{2}t\right)u(t) + \sqrt{3}e^{-\frac{t}{2}} sin\left(\frac{\sqrt{3}}{2}t\right)u(t)$$

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+ سیگنال حقیقی است پس قطب‌های تبدیل لاپلاس آن، حقیقی یا مزدوج مختلط خواهند بود.

+ یک قطب در $s = -1 + j$ قرار دارد، پس قطب دیگر نیز در $s = -1 - j$ قرار خواهد داشت.

+ سیگنال دقیقا دو قطب دارد و صفر محدود ندارد، پس فرم کلی تبدیل لاپلاس آن:

$$X(s) = \frac{A}{(s+1-j)(s+1+j)} = \frac{A}{s^2+2s+2}$$

خواهد بود.

+ سیگنال مطلقا انتگرال‌پذیر نیست، پس ناحیه همگرایی آن شامل محور مختلط نخواهد بود.

+ برای پیدا کردن ضریب A از بند (هـ) استفاده می‌کنیم:

$$X(0) = \left.\frac{A}{s^2+2s+2}\right|_{s=0} = \frac{A}{2} = 8 \rightarrow A = 16$$

تبدیل لاپلاس سیگنال در نهایت به صورت زیر خواهد بود:

$$X(s) = \frac{16}{s^2+2s+2}, \quad \Re e\{s\} < -1$$

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$$H(s) = \frac{s+1}{s^2+2s+2}$$

$$x(t) = e^{-|t|} = e^{-t}u(t) + e^{t}u(-t)$$

$$X(s) = \frac{1}{s+1} + \frac{-1}{s-1} = \frac{-2}{(s+1)(s-1)}, \qquad -1 < \Re e\{s\} < 1$$

$$Y(s) = H(s)X(s) = \frac{s+1}{s^2+2s+2}\frac{-2}{(s+1)(s-1)} = \frac{-2}{(s-1)(s^2+2s+2)}$$

$$= -\frac{0.4}{s-1} + \frac{0.4s+1.2}{s^2+2s+2}, \qquad -1 < \Re e\{s\} < 1$$

$$= -\frac{2}{5}\frac{1}{s-1} + \frac{2}{5}\frac{s+1}{(s+1)^2+1} + \frac{4}{5}\frac{1}{(s+1)^2+1}, \qquad -1 < \Re e\{s\} < 1$$

$$y(t) = -0.4e^{t}u(-t) + 0.4e^{-t}cos(t)u(t) + 0.8e^{-t}sin(t)u(t)$$

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$$\frac{d^3y(t)}{dt^3}+6\frac{d^2y(t)}{dt^2}+11\frac{dy(t)}{dt}+6y(t)=x(t)$$

الف)

$$s^3Y(s)+6s^2Y(s)+11sY(s)+6Y(s)=X(s)$$

$$\frac{Y(s)}{X(s)}=\frac{1}{s^3+6s^2+11s+6},\ X(s)=\frac{1}{s+4}$$

$$Y(s)=\frac{1}{(s+4)\left(s^3+6s^2+11s+6\right)}=\frac{1}{(s+4)(s+1)(s+2)(s+3)}$$

$$=-\frac{1}{6}\frac{1}{s+4}+\frac{1}{6}\frac{1}{s+1}+\frac{1}{2}\frac{1}{s+3}-\frac{1}{2}\frac{1}{s+2}$$

$$y(t)=-\frac{1}{6}e^{-4t}u(t)+\frac{1}{6}e^{-t}u(t)+\frac{1}{2}e^{-3t}u(t)-\frac{1}{2}e^{-2t}u(t)$$

ب)

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=0^-}=1,\ \left.\frac{dy(t)}{dt}\right|_{t=0^-}=-1,\ y(0^-)=1$$

$$s^3Y(s)-s^2y(0^-)-s\frac{dy(0^-)}{dt}-\frac{d^2y(0^-)}{dt^2}+6s^2Y(s)-6sy(0^-)-6\frac{dy(0^-)}{dt}$$

$$+11sY(s)-11y(0^-)+6Y(s)=X(s),\qquad X(s)=0$$

$$s^3Y(s)+6s^2Y(s)+11sY(s)+6Y(s)=s^2-s+1+6s-6+11$$

$$Y(s)=\frac{s^2+5s+6}{s^3+6s^2+11s+6}=\frac{(s+2)(s+3)}{(s+1)(s+2)(s+3)}$$

$$=\frac{1}{s+1}$$

$$y(t)=e^{-t}u(t)$$

ج) خروجی کلی ،مجموع خروجی‌ها در دو حالت الف و ب خواهد بود:

$$y(t)=-\frac{1}{6}e^{-4t}u(t)+\frac{1}{6}e^{-t}u(t)+\frac{1}{2}e^{-3t}u(t)-\frac{1}{2}e^{-2t}u(t)+e^{-t}u(t)$$

$$=-\frac{1}{6}e^{-4t}u(t)+\frac{7}{6}e^{-t}u(t)+\frac{1}{2}e^{-3t}u(t)-\frac{1}{2}e^{-2t}u(t)$$

$$F(s) = \frac{10(s+2)(s+4)}{(s+1)(s+3)(s+5)^2}$$

$$= \frac{a}{s+1} + \frac{b}{s+3} + \frac{c}{s+5} + \frac{d}{(s+5)^2}$$

$$a = (s+1)F(s)\Big|_{s=-1} = \frac{10(1)(3)}{(2)(4)^2} = \frac{30}{32} = \frac{15}{16}$$

$$b = (s+3)F(s)\Big|_{s=-3} = \frac{10(-1)(1)}{(-2)(2)^2} = \frac{10}{8} = \frac{5}{4}$$

$$c = \frac{d}{ds}(s+5)^2 F(s)\Big|_{s=-5} = \frac{d}{ds}\frac{10(s+2)(s+4)}{(s+1)(s+3)}\Big|_{s=-5}$$

$$= \frac{10(2s+6)(s+1)(s+3) - 10(s+2)(s+4)(2s+4)}{(s+1)^2(s+3)^2}\Big|_{s=-5}$$

$$= \frac{10(-4)(-4)(-2) - 10(-3)(-1)(-6)}{(-4)^2(-2)^2} = \frac{-320+180}{4 \times 16} = -\frac{35}{16}$$

$$d = (s+5)^2 F(s)\Big|_{s=-5} = \frac{10(-3)(-1)}{(-4)(-2)} = \frac{15}{4}$$

$$F(s) = \frac{15}{16}\frac{1}{s+1} + \frac{5}{4}\frac{1}{s+3} - \frac{35}{16}\frac{1}{s+5} + \frac{15}{4}\frac{1}{(s+5)^2}$$

$$f(t) = \frac{15}{16}e^{-t}u(t) + \frac{5}{4}e^{-3t}u(t) - \frac{35}{16}e^{-5t}u(t) + \frac{15}{4}L^{-1}\left\{\frac{1}{(s+5)^2} = \frac{d}{ds}\frac{-1}{s+5}\right\}$$

$$f(t) = \frac{15}{16}e^{-t}u(t) + \frac{5}{4}e^{-3t}u(t) - \frac{35}{16}e^{-5t}u(t) + \frac{15}{4}te^{-5t}u(t)$$

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$$f_1(t) = te^{-t}sin(5t)u(t)$$

$$L\left\{sin(5t)u(t)\right\} = \frac{5}{s^2+25}$$

$$L\left\{e^{-t}sin(5t)u(t)\right\} = \frac{5}{(s+1)^2+25}$$

$$L\left\{f_1(t)\right\} = -\frac{d}{ds}\frac{5}{(s+1)^2+25}$$

$$= \frac{10(s+1)}{\left((s+1)^2+25\right)^2}$$

$$f_2(t) = cos(2\omega t)cos(3\omega t)u(t)$$

$$= \frac{1}{2}\left(cos(5\omega t)+cos(\omega t)\right)u(t)$$

$$F_2(s) = \frac{1}{2}\left(\frac{s}{s^2+25\omega^2}+\frac{s}{s^2+\omega^2}\right)$$

$$= \frac{s(s^2+13\omega^2)}{(s^2+25\omega^2)(s^2+\omega^2)}$$

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$$
\begin{aligned}
L\left\{f(t)\right\} &= \int_{t=0}^{\infty} f(t)e^{-st}dt \\
&= \sum_{n=0}^{\infty} \int_{t=nT}^{(n+1)T} f(t)e^{-st}dt
\end{aligned}
$$

تغییر متغیر $\tau = t - nT$ را انجام می‌دهیم و دقت می‌کنیم که تابع $f\left(t\right)$ با دوره تناوب T متناوب است:

$$
= \sum_{n=0}^{\infty} e^{-nTs} \int_{\tau=0}^{T} f(\tau)e^{-s\tau}d\tau
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty} e^{-nTs} &= 1 + e^{-Ts} + e^{-2Ts} + \cdots \\
&= 1 + e^{-Ts}\left(1 + e^{-Ts} + e^{-2Ts} + \cdots\right) \\
&= 1 + e^{-Ts}\left(\sum_{n=0}^{\infty} e^{-nTs}\right)
\end{aligned}
$$

$$
\sum_{n=0}^{\infty} e^{-nTs} = \frac{1}{1 - e^{-Ts}}
$$

$$
L\left\{f(t)\right\} = \frac{\int_{t=0}^{T} f(t)e^{-st}dt}{1 - e^{-Ts}}
$$