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A Proposal for the Master Project

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Abstract: In this proposal, we will give a brief introduction to p-adic AdS/CFT and its insightful connection to the holographic tensor network, and then the summary of important features of p-adic CFT/tensor network correspondence which researchers have developed will be shown, with the advantages and disadvantages, open problems and defects exhibited. From these open questions, with explicit comments and results (such as the correlation function and Wilson line junction), we will try to conclude some potential directions we could pursue.

[1]This proposal is dedicated to the memory of Dr. Steven Gubser

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1 An Introduction to p-adic AdS/CFT

1.1 Axiomatic p-adic CFT

In 1989, Melzer presented a general formalism for non-Archimedean conformal field theory (CFT) [1], with the fields are defined by the p-adic variable $\mathbb{Q}_p$ (rather than R for usual

CFT) and the correlation functions are complex-valued[1]. Although their formalism can only have primary fields without any loop on the Witten diagram, it has been shown that the bulk geometry this CFT correspond to has strong connection to the tree version expression of tensor network—Bruhat–Tits tree [2–4]. In this subsection, we will mainly follow [2, 4] and give a brief for p-adic number analysis and the formalism of p-adic CFT.

1.1.1 p-adic analysis

p-adic number is a subfield of rational numbers $\mathbb{Q}$, which is the field of fractions of the commutative ring of integers $\mathbb{Z}$. We start from $\mathbb{Q}$:

Definition 1 The rational number field is the field of fractions of the commutative ring of integer number field equipped with Euclidean norm $|\cdot|$, which satisfies axioms:
(1) $|x| \geq 0$;
(2) $|x| = 0 \leftrightarrow x = 0$;
(3) $|xy| = |x||y| \quad \forall\, x, y \in \mathbb{R}$;
(4) $|x + y| \geq |x| + |y| \quad \forall\, x, y \in \mathbb{R}$ (triangle inequality).
Remark For a given prime number p, there exists an unique expansion for $x \in \mathbb{Q}$ such that

$$x_p = \sum_{n=-N}^{\infty} a_n p^n \quad \text{with} \quad 0 \leq a_n \leq p-1, a_n \in \mathbb{Z}. \tag{1.1}$$

According to this, one can alternatively define the rational number field as

$$\mathbb{Q} \equiv \{x = \sum_{n=-N}^{\infty} a_n p^n | a_n \in \mathbb{Z} \ \&\ 0 \leq a_n \leq p-1\} \tag{1.2}$$

with Euclidean norm equipped.

Now consider the similar expansion as showed above, the leading term of the p-adic expansion can be rewritten as:

$$x = p^{v_p(x)} \sum_{n=0}^{\infty} b_n p^n \quad \text{with} \quad b_0 \neq 0, \quad b_n \in \mathbb{Z} \ \&\ 0 \leq b_n \leq p-1, \quad v_p(x) \in \mathbb{Z}, \tag{1.3}$$

To normalize the expansion, it is natural to define the p-adic norm as:

$$|x|_p \equiv p^{-v_p(x)} \quad \text{with} \quad v_p(x) \in \mathbb{Z}. \tag{1.4}$$

Then the definition pf p-adic number field can be written:

Definition 2 p-adic number field (denoted by $\mathbb{Q}_p$) is one possible extension of $\mathbb{Q}$ equipping the p-adic norm $|x|_p$, which satisfies axioms:
(1) $|x|_p \geq 0$;
(2) $|x|_p = 0 \leftrightarrow x = 0$;
(3) $|xy|_p = |x|_p|y|_p \quad \forall\, x, y \in \mathbb{R}$;
(4)$|x + y|_p \leq \max(|x|_p, |y|_p)$ (strong triangle inequality).

[1]Only the complex-valued field has been covered in the previous literature, we will explain this explicitly in this subsection.

Analogous to (1.2), given a particular prime number p, the field $\mathbb{Q}_p$ consists of all possible formal expansions of the form:

$$\mathbb{Q}_p \equiv \{x_p = \sum_{n=-N}^{\infty} a_n p^n | a_n \in \mathbb{Z} \ \& \ 0 \le a_n \le p-1\}. \tag{1.5}$$

Remark
1. Here we have assumed that the addition and multiplication are continuous maps with respect to p-adic norm $|\cdot|_p$;
2. $|0|_p = 0$, and correspondingly $v_p(0) = \infty$;
3. According to the strong triangle inequality, $|p^n|_p = \frac{1}{p^n}$, and $|x+x|_p \le |x|_p$. Although counter-intuitive, that is what makes the physics in non-Archimedean geometry interesting.
Corollary *Tall isosceles property*: If $x+y+z=0$ and $x, y, z \in \mathbb{Q}_p$, then one always have $|x|_p = |y|_p \ge |z|_p$ (relabeling allowed). This property can be derived from the strong triangle inequality, which reveals an interesting fact that for a triangle in p-adic geometry, this triangle has to be an isosceles one.
Then we give some definitions constructed from $\mathbb{Q}_p$ which will be used frequently later.

Definition 3 Let $\mathbb{U}_p$ denotes the *unit sphere* of $\mathbb{Q}_p$, it satisfies:

$$\mathbb{U}_p = \{x_p \in \mathbb{Q}_p | |x|_p = 1\}. \tag{1.6}$$

Definition 4 $\mathbb{Z}_p$ denotes the *unit ball* of $\mathbb{Q}_p$,

$$\mathbb{Z}_p = \{x_p \in \mathbb{Q}_p | |x|_p \le 1\}, \tag{1.7}$$

note that $\mathbb{Z}_p$ is both open and closed in $\mathbb{Q}_p$.
Remark
1. $\mathbb{Z}_p$ is a ring without multiplication inverse, while the elements of $\mathbb{U}_p$ have multiplication inverses in $\mathbb{Z}_p$.
2. The set of all non-zero elements can be denoted as $\mathbb{Q}_p^* \equiv \mathbb{Q}_p/\{0\}$, where $\mathbb{Q}_p^* = \bigsqcup_{n\in\mathbb{Z}} p^n \mathbb{U}_p$.

1.1.2 p-adic CFT

Let $\mathbb{K}$ an unramified finite field extension of $\mathbb{Q}_p$, $\mathbb{K}$ forms a finite-dimensional vector space isomorphic to $(\mathbb{Q}_p)^n$ with multiplication with the elements of $\mathbb{Q}_p$. n is the degree of extension, which can be assumed as the dimension of $\mathbb{K}$[2]. For simplicity, all the norms $|\cdot|$ are attributed to $|\cdot|_{\mathbb{K}}$ unless specified.

Definition 5 On $\mathbb{P}^1(\mathbb{K})$[3], a p-adic CFT without carrying local derivatives (equivalently no descendants or spin structures) can be defined by the data as following:
(1) Primary fields $\mathcal{O}_a$ $(a \in \mathcal{P})$ and their conformal dimensions Δ_a;
(2) There exists a correlation function

$$\langle \mathcal{O}_{a_1}(x_1) \cdots \mathcal{O}_{a_m}(x_m)\rangle \in \mathbb{C}, \qquad x_1, \cdots, x_m \in \mathbb{K} \tag{1.8}$$

[2]There are some non-trivial stuff regarding this extension (ramified or unramified), to make this proposal more concrete, we will not explicitly cover that here, one can read [5] for more details.
[3]It [3] said that the p-adic projective space $\mathbb{P}^1(\mathbb{K})$ is the most natural space where p-adic CFT can live, but I don't know why, hope to receive some comments from you.

constructed by primary fields located on $\mathbb{K}$. Note that because of the vanish of spin structure, only bosonic fields will be considered, which means that the order of insertion can be freely exchanged;
(3) One can use operator product expansion (OPE) to construct a third-point primary field from two known primary fields, which means the identity

$$\mathcal{O}_a(x)\mathcal{O}_b(y) = \sum_{c\in\mathcal{P}} C^c{}_{ab}(x,y)\mathcal{O}_c(y) \tag{1.9}$$

is satisfied, with restriction that $|y-x| \leq |y-z|$ in the correlation function, where z denotes any other insertion except x and y.
(4) The CFT keeps $\mathrm{PGL}(2,\mathbb{K})$ covariance. Set μ a Möbius transformation, we have

$$\langle(\mu^*\mathcal{O}_{a_1})(x_1)\cdots(\mu^*\mathcal{O}_{a_m})(x_m)\rangle = \langle\mathcal{O}_{a_1}(\mu(x_1))\cdots\mathcal{O}_{a_m}(\mu(x_m))\rangle, \tag{1.10}$$

where $(\mu^*\mathcal{O}_a)(x) = |\frac{ad-bc}{(cx+d)^2}|^{-\Delta_a}\mathcal{O}_a(x)$.
We hope that the operator product algebra is consistent to the correlation function, which means that our definition should satisfy the crossing symmetry. As noted, the fields considered here are all locally constant without derivatives, thus the OPE takes the form:

$$\mathcal{O}_a(x_1)\mathcal{O}_b(x_2) = \sum_{c\in\mathcal{P}} C^c_{ab}|x_1-x_2|^{\Delta_c-\Delta_a-\Delta_b}\mathcal{O}_c(x_2). \tag{1.11}$$

As in usual CFT, there exists an unique identity operator $\mathbf{1}$ with conformal dimension $\Delta_{\mathbf{1}} = 0$. It is natural to define the associated OPE coefficients as $C^a_{b\mathbf{1}} = C^a_{\mathbf{1}b} = \delta^a_b$. Then we can insert the OPE above to obtain the 2-point correlation function as following:

$$\langle\mathcal{O}_a(x)\mathcal{O}_b(y)\rangle = C^{\mathbf{1}}_{ab}|x-y|^{-2\Delta_a} \tag{1.12}$$

This is consistent to the common CFT. Similarly, the 3-point function can be written as:

$$\langle\mathcal{O}_a(x)\mathcal{O}_b(y)\mathcal{O}_c(z)\rangle = \frac{C_{abc}}{|x-y|^{\Delta_a+\Delta_b-\Delta_c}|y-z|^{\Delta_b+\Delta_c-\Delta_a}|z-x|^{\Delta_c+\Delta_a-\Delta_b}}, \tag{1.13}$$

$$C_{abc} = \sum_d C^{\mathbf{1}}_{ad}C^d_{bc}. \tag{1.14}$$

C^a_{bc} is Abelian since only bosonic fields are contained in this p-adic CFT.

For 4-point function, one should first give $\mathcal{O}(\infty)$, we define it as

$$\mathcal{O}_a(\infty) = \lim_{|x|\to\infty}|x|^{2\Delta_a}\mathcal{O}_a(x). \tag{1.15}$$

One can then show that the result is still consistent to the usual CFT 4-point function (explicit results have been computed in [3]).

	upper half plane $\mathbb{H}$	BT tree $\mathbb{H}_p$	unramified extension $\mathbb{H}_{\mathbb{K}}$
Isometry group	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{PGL}(2,\mathbb{Q}_p)$	$\mathrm{PGL}(2,\mathbb{K})$
Isotopy subgroup	$\mathrm{SO}(2,\mathbb{R})$	$\mathrm{PGL}(2,\mathbb{Z}_p)$	$\mathrm{PGL}(2,\mathbb{Z}_{\mathbb{K}})$
Boundary	$\mathbb{R}$ (S^1)	$\mathbb{Q}_p$ $(\mathbb{P}^1(\mathbb{Q}_p))$	$\mathbb{K}$ $(\mathbb{P}^1(\mathbb{K}))$

Table 1. The parallel between $\mathbb{H}$, $\mathbb{H}_p$ and $\mathbb{H}_{\mathbb{K}}$.

1.2 Bruhat–Tits tree (BT tree) and p-adic AdS/CFT

Now we are ready to discuss how can p-adic CFT restore the bulk geometry and its tensor network representation. The first question one may have is that what kind of symmetry does the p-adic bulk could keep, which motivates us to take an analogy to the case of Archimedean geometry.

In $\mathrm{AdS}_3/\mathrm{CFT}_2$, the usual CFT gives the isometry group $\mathrm{SL}(2,\mathbb{R})$, and for vacuum case[4], the points of the dual AdS geometry have an one-to-one correspondence to the maximal compact subgroups of the conformal group, which is $\mathrm{SO}(2,\mathbb{R})$. And thus the quotient space is:

$$\mathbb{H} \equiv \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2,\mathbb{R})} = \mathbb{D}, \tag{1.16}$$

where $\mathbb{D}$ is the Poincaré disk, which can be equivalently deformed as an upper half plane, with coordinates $\mathbb{H} \equiv \{z = x + iy | x \in \mathbb{R}, y \in \mathbb{R}_+\}$ and $\mathbb{R}$ the boundary.

We can assume the same structure for p-adic case by changing the boundary from $\mathbb{R}$ to $\mathbb{Q}_p$. As [2] noted, the isometry group obtained from p-adic CFT is $\mathrm{PGL}(2,\mathbb{Q}_p)$, whose maximal compact subgroup is $\mathrm{PGL}(2,\mathbb{Z}_p)$. Therefore, one can construct a space similar to $\mathbb{H}$:

$$\mathbb{H}_p \equiv \frac{\mathrm{PGL}(2,\mathbb{Q}_p)}{\mathrm{PGL}(2,\mathbb{Z}_p)}. \tag{1.17}$$

One can notice that although $\mathrm{PGL}(2,\mathbb{Z}_p)$ is the maximal compact subgroup of $\mathrm{PGL}(2,\mathbb{Q}_p)$, it is both open and closed in it (as we pointed out in **Def. 4**). That means the bulk $\mathbb{H}_p$ have been forced to be discrete even though $\mathbb{Q}_p$ is a continuous field.

To generalize this relation to arbitrary dimensional case, one can recall the n-extension of $\mathbb{Q}_p$ which reproduces $\mathbb{Q}_p$ degrees of freedom per extension. This requires us to introduce $\mathbb{K}$, then the result is

$$\mathbb{H}_{\mathbb{K}} = \frac{\mathrm{PGL}(2,\mathbb{K})}{\mathrm{PGL}(2,\mathbb{Z}_{\mathbb{K}})}. \tag{1.18}$$

To summarize, see Table 1. In [2], Gubser *et al* have showed that the quotient space $\mathbb{H}_p$ is just the BT tree. One who interested in the details can check by himself, here we just give a brief comment. Since $\mathbb{H}_p$ has discrete topology, we would better take use of lattice description in $\otimes_n \mathbb{Q}_p$, rather than simply value $x_p \in \mathbb{Q}_p$ and $y_p \in \mathbb{Q}_{p^+}$. Thus we can construct a set of equivalence classes $\{\langle \vec{f}, \vec{g}\rangle\}$ for lattices $\langle \vec{f}, \vec{g}\rangle$ in $\otimes_n \mathbb{Q}_p$, two lattices $\langle \vec{f}, \vec{g}\rangle$ and $\langle \vec{f}', \vec{g}'\rangle$ are equivalent iff

$$(\vec{f}', \vec{g}') = (\Gamma \cdot \vec{f}, \Gamma \cdot \vec{g}) \quad \text{with} \quad \Gamma \in \mathrm{PGL}(2,\mathbb{Z}_p). \tag{1.19}$$

[4]Since there is no descendants in p-adic CFT, only pure AdS geometry will be considered.

Comments

1. A $\mathrm{PGL}(2,\mathbb{Q}_p)$ transformation acts on the lattice via

$$\langle \vec{f}, \vec{g}\rangle \to \langle \gamma\cdot\vec{f}, \gamma\cdot\vec{g}\rangle \quad \text{with} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2,\mathbb{Q}_p) \tag{1.20}$$

2. Two equivalence classes of lattices Λ and Λ' are directly connected iff

$$p\Lambda \subset \Lambda' \subset \Lambda, \tag{1.21}$$

and thus one can connect such a pair of nodes by an edge with distance $d(\Lambda,\Lambda') = 1$, so that the metric can be given. It is certainly that each node can have $(p+1)$ nearest neighbors, which is equivalent to say that $\mathbb{H}_p$ has the topology of an infinite $(p+1)$-valent tree (also called BT tree).

3. The nodes on the tree have the form

$$x^{(m)} = \sum_{n=-N}^{m-1} a_n p^n \quad 0 \le a_n \le p-1. \tag{1.22}$$

Because of the lattice representation, $x^{(m)}$ truncates at p^m, thus the node 1.22 represents the equivalence class

$$x^{(m)} + p^m \mathbb{Z}_p. \tag{1.23}$$

This comment will benefit our later discussion on introducing the renormalization group (RG) structure.

Figure 1 shows an example for $p=2$ distribution, and Figure 2 depicts an unramified extension $\mathbb{Q}_p \otimes \mathbb{Q}_p = \mathbb{Q}_{p^2}$ as a lattice representation, these two figures are cited from [3] Fig. 1 and [2] Fig. 2, respectively.

1.2.1 Bulk action principle

Since the dual bulk geometry of p-adic CFT can be identified as the BT tree, one hopes to calculate in this formalism should start from the correlators on the boundary, and so that the formula of propagators are required for computation. In [2], after that identification, Gubser *et al* proposed the p-adic AdS/CFT in which the generating function of the p-adic CFT is identified with the path integral of a dual theory living on the BT tree. While the path integral in the bulk is controlled by a classical local action with a set of scalar fields.

[2] gave an action with $\mathbb{Q}_p$ the boundary while [3] gave a more general case with $\mathbb{K}$ the boundary. Here we would like to introduce the latter one.

Consider a single real-valued scalar field ϕ on $\mathbb{H}_{\mathbb{K}}$, the bulk action can be written as:

$$S = S_2 + S_{\text{int}}, \tag{1.24}$$

where

$$S_2(\phi, J) = \sum_{\langle u,v\rangle} \frac{1}{2}(\phi(u)-\phi(v))^2 + \sum_u \left(\frac{1}{2}m^2\phi(u)^2 - J(u)\phi(u)\right), \tag{1.25}$$

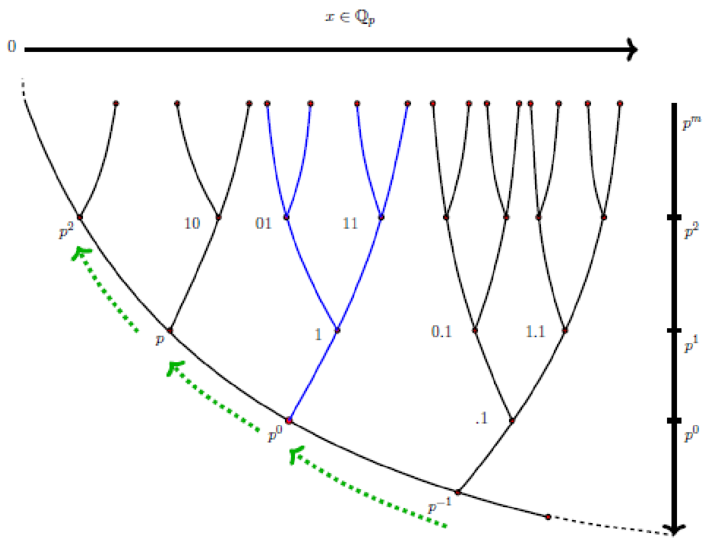

Figure 1. 2-adic BT tree (from [3] Fig. 1).

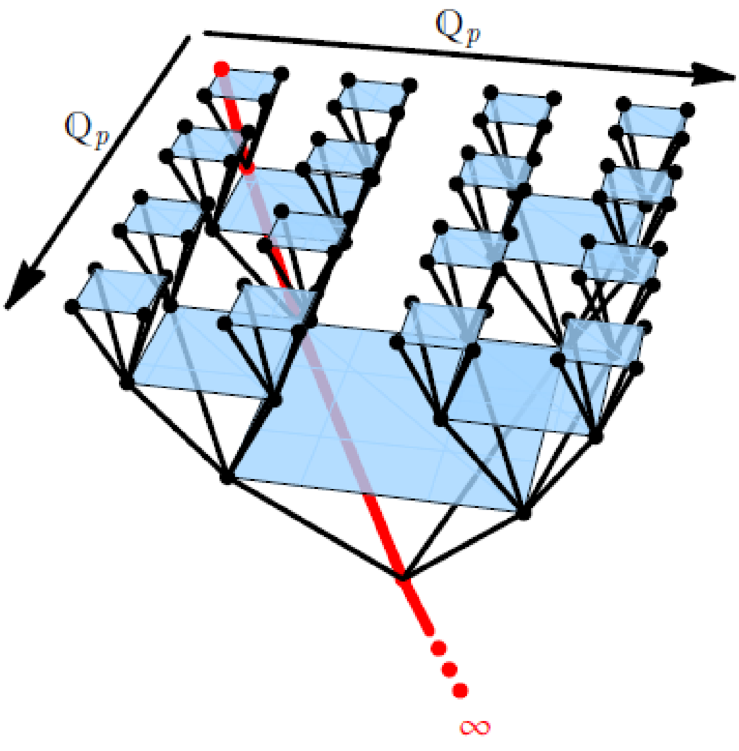

Figure 2. A variant of the BT tree for $p = 2$ $\mathbb{K}$ (from [2] Fig. 2).

and

$$S_{\text{int}} = \sum_{k=3}^{n} S_k \quad \text{with} \quad S_k = \frac{\lambda_k}{k!} \sum_u \phi(u)^k. \tag{1.26}$$

$\phi(u)$, $\phi(v)$ are the fields located on node u, v, and we take summation over nearest neighboring lattice sites.

The equation of motion for quadratic action S_2 can easily be written:

$$(\Box_p + m_p^2)\phi(u) = J(u). \tag{1.27}$$

It is certainly that p-adic CFT has unusual Laplacian and mass, we will denote $\Box_p$ and m_p without subscription later, unless specified. It is not hard to check that

$$\Box\phi_a = \sum_{\langle u,v\rangle} (\phi(u) - \phi(v)). \tag{1.28}$$

A solution of the equation of motion is

$$\phi(u) = \sum_w G(u,w)J(w), \quad \text{with} \quad (\Box + m^2)G(u,w) = \delta(u,w), \tag{1.29}$$

where

$$\delta(u,w) = \begin{cases} 1 \text{ if } & u = w \\ 0 & \text{otherwise} \end{cases} \tag{1.30}$$

Then one solution of Green function is

$$G(u,w) = \frac{\zeta_p(2\Delta)}{p^\Delta} p^{-\Delta d(u,w)}, \tag{1.31}$$

where Δ satisfies

$$m_p^2 = -\frac{1}{\zeta_p(\Delta - n)\zeta_p(-\Delta)} \quad \text{with} \quad \zeta_p(s) = \frac{1}{1 - p^{-s}}. \tag{1.32}$$

Note that $\zeta_p(s)$ is the p-adic zeta functions.

Similarly, the bulk-to-boundary propagator $K(u,x)$, where x is on the boundary $\mathbb{Q}_p$. $K(u,x)$ can be obtained by forcing the w of $G(u,w)$ to near boundary position, which requires a regularization to keep w always finite under the limitation. We assume that this can be realized by preserving the equation of motion

$$(\Box + m^2)K(u,x) = 0, \tag{1.33}$$

and keeping the translation invariance as [2] did, then we obtain the form

$$K(z_0, z; x) = \frac{\zeta_p(2\Delta)}{\zeta_p(2\Delta - n)} \frac{|z_0|_p^\Delta}{|(z_0, z - x)|_s^{2\Delta}}, \tag{1.34}$$

where z_0 is a normalization point which satisfies $\int_{\mathbb{Q}_p} \mathrm{d}x K(z_0, z; x) = |z_0|_p^{n-\Delta}$, the bulk point u can be denoted by (z_0, z) under such normalization. Besides,

$$|z_0, z - x|_s \equiv \sup\{|z_0|, |z - x|\}. \tag{1.35}$$

Thus, one can obtain the boundary correlation function by computing the appropriately regularized bulk path integral or Fourier transformed wave function.

Our setup has been set and it is the time for the next journey now, since the path integral formulation is not so clear with considering the regularization, we will calculate correlation functions in the next section from three aspects, of which two are regularized in tensor network configuration and one is discussed in Wilson line network.

2 Correlators

[4] take use of the tensor network language and generalized the results from perfect tensor [2] to the imperfect ones and identified BT tree as the bulk, which means the tensor network lives on the BT tree, so that showed the bulk geometry is just the tensor network on BT tree with the same space-time layer.

2.1 Wavelet transform

2.1.1 HKLL formula

In usual AdS/CFT, one can reconstruct the local bulk field $\phi^I(\vec{x}, z)$ by the boundary operators according to the HKLL formula [6, 7]:

$$\begin{aligned}\phi^I(\vec{x},z) = &\int \mathrm{d}^d y K_I(\vec{x},z|\vec{y})\mathcal{O}^I(\vec{y}) \\ &+ \sum_{\langle J,K\rangle} \frac{\lambda^I_{JK}}{N} \int \mathrm{d}^d x' \mathrm{d}z' G_I(\vec{x},z|\vec{x}',z') \int \mathrm{d}^d y_1 K_J(\vec{x}',z'|\vec{y}_1)\mathcal{O}^I(\vec{y}_1) \int \mathrm{d}^d y_2 K_K(\vec{x}',z|\vec{y}_2)\mathcal{O}^K(\vec{y}_2) \\ &+ o(\frac{\lambda^2}{N^2}) \int \cdots ,\end{aligned} \tag{2.1}$$

where $\vec{x}$, $\vec{y}$ are the boundary vector and z denotes the Poincaré coordinate with $\vec{x}$ by $ds^2 = \frac{1}{z^2}(dz^2 + d\vec{x}^2)$. $\mathcal{O}^I$ is the boundary operator dual to ϕ^I. The smearing function $K_I(\vec{x}, z|\vec{y})$ is the boundary-bulk kernel of $\mathcal{O}^I$:

$$K_I(\vec{x},z|\vec{y}) = \left(\frac{z}{z^2 - (\vec{x}-\vec{y})^2}\right)^{d-\Delta_I} \Theta(z^2 - (\vec{x}-\vec{y})^2). \tag{2.2}$$

And similar to our former discussion on the bulk-boundary correspondence of BT tree, $\lim_{z'\to 0} z'^{\Delta_I - d} G^I(\vec{x}, z|\vec{x}', \vec{z}') \sim K^I(\vec{x}, z|\vec{x}')$. To unify with the former convention, we will omit the vector arrow representation in the following.

It is certainly that we should have an analogous HKLL formula in the tensor network if p-adic AdS/CFT can be realized. Consider that the BT tree corresponds to the discrete topology of the isometry group, one accessible idea is to reconstruct $|\Psi_{bulk}\rangle$ by $|\psi_{boundary}\rangle$, which requires us to restore the causal structure (i.e. only a subregion is enough for reproducing the full reconstruction) and the fact that the linear term should dominate in the expansion with large N limit.[4] pointed out that the perfect tensor network is not applicable for such restorations, although it could naturally emergent the causal structure. After then, they give a more general version of tensor network analogue of HKLL formula. Here we will just show their results and give some comments.

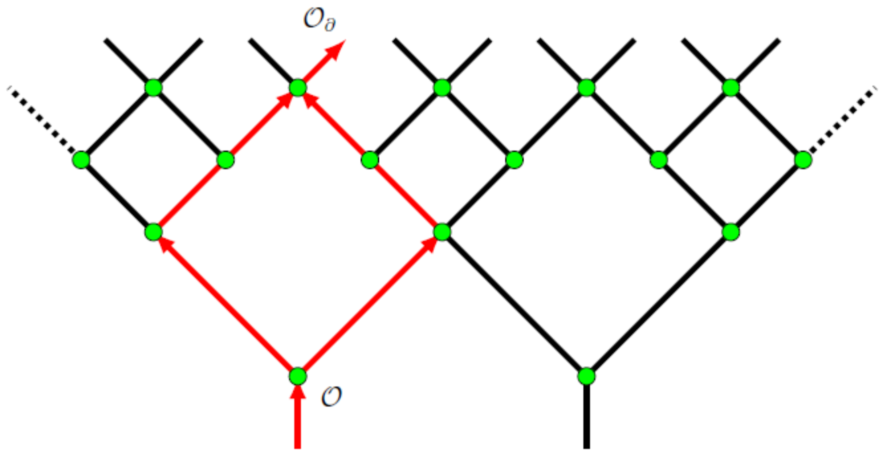

Figure 3. Linear operator pushing on a generic tensor network, where the red paths indicate all the paths joining the bulk operators and the boundary ones (from [4] Fig. 4).

After operator pushing (one can check details in Sec. 3.2 of [4]), one can move an operator acting in the bulk of the network $\mathcal{G}$ all the way to its boundary. Say $P^I(v,i)$ an operator acting on the state living on the i^{th} leg of the vertex v and located on I (where P letter used because we are treating the Pauli matrix basis $\{P^I\}$), we can get the effect of P^I on the boundary wave function:

$$\begin{aligned}
&(\otimes_{\text{all internal edges }\mu}|\mu\rangle\langle\mu|)P^I(v,i)|\Psi_{bulk}\rangle \\
=&\sum_{j=1}^{N}\sum_{J}\mathcal{A}^I_J(v,i|j)P^J(a)|\psi_{boundary}\rangle \\
+&\sum_{j\neq k}\sum_{J,K}\mathcal{A}^I_{JK}(v,i|j,k)P^J(j)P^K(k)|\psi_{boundary}\rangle \\
+&\sum_{j\neq k\neq l}\sum_{J,K,L}\mathcal{A}^I_{JKL}(v,i|j,k,l)P^J(j)P^K(k)P^L(l)|\psi_{boundary}\rangle+\cdots,
\end{aligned} \tag{2.3}$$

where the coefficients $\mathcal{A}$ are come from the local operator pushing coefficients, we now need to explicitly compute $\mathcal{A}^I_J$ and $\mathcal{A}^I_{JK}$ to check that the linear term can dominate.

$\mathcal{A}^I_J(v,i|j)$ can be written by the sum over $\alpha^I_J(v_i)$ collected along all the paths $\mathcal{P}$ from the bulk vertex v to the boundary leg j,

$$\mathcal{A}^I_J(v,i|j)=\sum_{\mathcal{P}}\sum_{\{I_1,\cdots,I_{|\mathcal{P}|-1}\}}\prod_{v_i\in\mathcal{P}}\alpha^{I_{i-1}}_{I_i}(v_i), \tag{2.4}$$

where α^I_J are the local operator pushing coefficients whose valuation commonly depend on the value of the tensor state, $I_0\equiv I$, $v_0\equiv v$ and $I_{|\mathcal{P}|}\equiv J$ ($|\mathcal{P}|=\text{length}(\mathcal{P})$). One can view the interacting effect in Figure 3. Since we concern most the tensor network analogous of AdS/CFT correspondence here, we can assume that the bulk geometry is homogeneous so that all the vertex dependence can be extracted. Then the equation above simplifies to:

$$\mathcal{A}^I_J(v,i|j) = \sum_{\mathcal{P}}[(\alpha)^{\otimes|\mathcal{P}(v\to j)|}]^I_J, \tag{2.5}$$

where α denotes α^I_J which contains the local pushing coefficient from P^I to P^J. Then we can diagonalize α and take use of its eigenvectors as new basis, so that

$$\mathcal{A}^I_J(v,a|i) = \delta^I_J K_I(v,i) \quad \text{with} \quad K_I(v|i) \equiv \sum_{\mathcal{P}}(\lambda_I)^{|\mathcal{P}(v\to i)|}, \tag{2.6}$$

it is certainly that λ_I are the eigenvalues associated to the eigenvectors.

Inserting the result above into 2.3 one can easily obtain:

$$\mathcal{O}^I(v,1)|\Psi_{bulk}\rangle = \sum_i K_I(v|i)\mathcal{O}^I(i)|\psi_{boundary}\rangle. \tag{2.7}$$

Thus the linear term has been reconstructed.

Similarly, the non-linear term $\mathcal{A}^I_{JK}(v,i|j,k)$ can be computed as following:

$$\mathcal{A}^I_{JK}(v,i|j,k) = \sum_w G_I(v|w)\alpha^I_{JK}K_J(w|j)K_K(w|k), \tag{2.8}$$

where the bulk-bulk kernel is

$$G_I(v|w) \equiv \sum_{\mathcal{P}} \lambda_I^{|\mathcal{P}(v\to w)|}. \tag{2.9}$$

It is not hard to notice that we still take use of the eigenvectors as the basis. $\mathcal{A}^I_{JK}(v,i|j,k)$ denotes the propagation from $\mathcal{O}^I$ to $\mathcal{O}^J$ on leg j and $\mathcal{O}^K$ on leg k, where w is the node join both three ones (see Figure 4). Note that only tree network will be considered in this note, although a little idea regarding how to introduce loop diagrams will be discussed in the next section. For the BT tree, the linear term of global operator pushing becomes

$$\mathcal{O}^I(v,1)|\Psi_{bulk}\rangle = \sum_i(\lambda_I)^{|\mathcal{P}(v\to i)|}\mathcal{O}^I(i)|\psi_{boundary}\rangle, \tag{2.10}$$

with $\mathcal{P}(v \to i)$ labels the unique path from the bulk vertex v to the boundary edge i as before. Obviously, here the smearing function is

$$K_I(v,1|i) \equiv (\lambda_I)^{|\mathcal{P}(v\to i)|} = p^{-\sigma^I|\mathcal{P}(v\to i)|}, \tag{2.11}$$

where

$$p \equiv r - 1 \tag{2.12}$$

with $r = p + 1$ the valent of one vertex and

$$\sigma_I \equiv -\frac{\ln \lambda_I}{\ln p}. \tag{2.13}$$

Similarly, the non-linear pushing operator induces the bulk-bulk kernel as:

$$G_I(v|w) \equiv (\lambda_I)^{|\mathcal{P}(v\to w)|} = p^{-\sigma_I|\mathcal{P}(v\to w)|}, \tag{2.14}$$

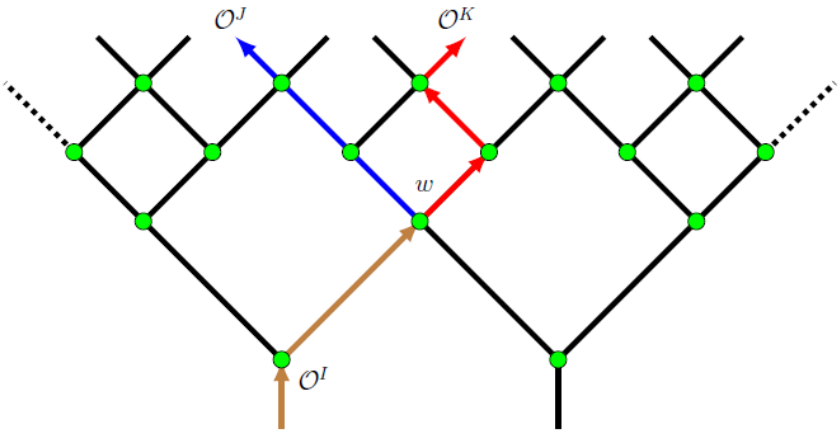

Figure 4. Non-linear contributions to operator pushing. (from [4] Fig. 5).

where w is the internal bifurcating point.

One may notice that both K and G are completely controlled by one parameter σ_I. To physically explain this parameter, let us recall the equation of motion with which both kernels satisfy:

$$(\Box + m^2)G_I(v|w) = \delta(v, w) \quad (\Box + m^2)K_I(v|i) = 0. \tag{2.15}$$

One can extract a relation between the mass square and σ_I,

$$m_I^2 = p^{1-\sigma_I} + p^{\sigma_I} - (p + 1). \tag{2.16}$$

Compare this formula to equation 1.32 with $n = 1$, it is explicitly that there are two solutions:

$$\sigma_I = \Delta_I \quad \text{or} \quad \sigma_I = 1 - \Delta_I. \tag{2.17}$$

To generalize the result with n introduced, we choose the latter solution so that

$$\sigma_I = d - \Delta_I \tag{2.18}$$

when the boundary dimension $d = n$. And if we want to identify the propagators derived in the last section, we just need to take a limit $\Delta_I \to \sigma_I$.

2.1.2 Wavelet review

If the tensor network is finite, the limit taken for bulk-boundary operator to the near boundary is enough to realize the correspondence, while for the infinite network, we have choose a certain slice to regularize or truncate. In [4], they tried the wavelet transform and propose a regularization natural to HKLL formula, which is effective for both real and p-adic case.

The wavelet basis contains $d + 1$ parameters. It start by defining a mother wavelet $\psi(x)$ which has d parameters x (here we assume that $x \equiv \vec{x}$), and then there are daughter wavelets which is generated from the mother wavelet by translation by a:

$$\psi_{a,s}(x) \equiv \frac{1}{s^{d/2}}\psi(\frac{x-a}{s}), \tag{2.19}$$

with s the rescaling parameter. These daughter wavelets form the wavelet basis.

Given a signal function $f(x)$, the wavelet transform gives

$$W_f(a,s) = \int \mathrm{d}^d x f(x) \psi^\dagger_{a,s}(x), \tag{2.20}$$

the wavelet basis $\{\psi_{a,s}(x)\}$ is over-complete for a mother wavelet. Nevertheless, in particular cases, the mother wavelet can allow an inverse transform:

$$\hat{\psi}(\vec{k}) = \hat{\psi}(k), \tag{2.21}$$

where $\hat{\psi}$ is the Fourier transform of $\psi(x)$. Then

$$f(x) = \frac{1}{C_\psi} \int_0^\infty \frac{\mathrm{d}s}{s^{d+1}} \int \mathrm{d}^d a W_f(a,s) \psi_{a,s}(x), \tag{2.22}$$

where

$$C_\psi \equiv \int_0^\infty \mathrm{d}k \frac{|\hat{\psi}(k)|^2}{k}. \tag{2.23}$$

If we want the inverse transform to be well-defined, the admissibility condition is $C_\psi < +\infty$.

We skip the arguments for real case, one just need to know that the holographic direction is identified with the scaling parameter of the wavelet transform while the inverse transform does not exist since $C_\psi \sim \Gamma(0) \to \infty$. However, this does not mean that we cannot use the wavelet transform, since the inverse transform is needless for us to obtain the boundary operator from the bulk. With a dressing factor introduced, we can normalize C_ψ to a finite value. We will explicitly show this in p-adic case.

For p-adic wavelet transform,

$$\psi_{a,s}(x) = \frac{1}{\sqrt{|s|_p}} \psi(\frac{x-a}{s}) \qquad x, a, s \in \mathbb{Q}_p, \tag{2.24}$$

and

$$W_f(a,s) = \int_{\mathbb{Q}_p} \mathrm{d}x \frac{1}{\sqrt{|s|_p}} \psi^\dagger(\frac{x-a}{s}) f(x) \tag{2.25}$$

with

$$f(x) = \frac{1}{C_\psi} \int_{\mathbb{Q}_p^*} \frac{\mathrm{d}s}{|s|_p^2} \int_{\mathbb{Q}_p} \mathrm{d}a W_f(a,s) \psi_{a,s}(x). \tag{2.26}$$

We can interpret the linear term of the bulk-boundary operator as a p-adic wavelet transform:

$$\phi_p(x,z) = \int_{\mathbb{Q}_p} \mathrm{d}y K_p(x,z|y) \mathcal{O}(y), \tag{2.27}$$

with

$$K_I(x,z|y) = \frac{\zeta_p(2\Delta)}{\zeta_p(2\Delta-1)} |z|_p^{\Delta_I - 1} \gamma(\frac{x-y}{z}), \tag{2.28}$$

where $\gamma(k) \equiv \int_{\mathbb{Z}_p} \mathrm{d}x e^{-2\pi i [kx]}$. For the p-adic case, the choice of the mother wavelet has no dependence on the conformal dimension Δ, which means that one can construct a mother wavelet as

$$\psi(x) = \frac{\zeta_p(2\Delta)}{\zeta_p(2\Delta-1)} \gamma(x). \tag{2.29}$$

The conformal dimension only appears in

$$K(x,z|y) = |z|_p^{\Delta-\frac{1}{2}} \psi^\dagger_{x,z}(y), \tag{2.30}$$

which in turn gives the mapping between the wavelet transform $W_{\mathcal{O}}(x,z)$ and the bulk field:

$$\phi_p(x,z) = |z|_p^{\Delta-\frac{1}{2}} W_{\mathcal{O}}(x,z). \tag{2.31}$$

Besides, one can show that the mother wavelet is still violate the admissibility condition,

$$C_\psi \sim \int_{\mathbb{Q}_p} \frac{\mathrm{d}k}{|k|_p} \gamma(k)^2 = \int_{\mathbb{Z}_p} \frac{\mathrm{d}k}{|k|_p} \to +\infty. \tag{2.32}$$

As noted in the real case discussion, this is not a problem for us since we can compute the boundary operator by

$$\int_{\mathbb{Q}_p^*} \frac{\mathrm{d}}{|z|_p^2} \int_{\mathbb{Q}_p} \mathrm{d}x \phi(x,z|y) K(x,z|y) \tag{2.33}$$

which has a dressing factor $|z|^{2\Delta-1}$. Now we can show that the normalized $\tilde{C}_p$ is finite. The smearing function is

$$K(x,z|y) = \frac{\zeta_p(2\Delta)}{\zeta_p(2\Delta-1)} |z|_p^\Delta \int_{\mathbb{Q}_p} \mathrm{d}k e^{-2\pi i [k(x-y)]} \gamma(kz). \tag{2.34}$$

Inserting this into the inverse transform, we get

$$\tilde{C}_p = \int_{\mathbb{Q}_p} \mathrm{d}z |z|_p^{2\nu-1} \gamma(z)^2 = \int_{\mathbb{Z}_p} \mathrm{d}z |z|_p^{2\nu-1} = \frac{p-1}{p(1-p^{-2\nu})} \quad \forall \nu > 0, \tag{2.35}$$

with $\nu = \Delta - \frac{1}{2}$. Therefore, normalized $\tilde{C}_p$ is finite and inverse transform is valid for all $\nu > 0$ in p-adic case.

In conclusion, the p-adic bulk reconstruction can be described by the continuous wavelet transform at least to linear term, while the inverse transform can only be valid with a dressing factor $z^{2\nu}$ with $\nu > 0$ considered.

2.1.3 Correlation functions

Now we are ready to compute correlation functions on a p-adic tree tensor network. As **Def. 5** (4) showed, we define conformal primaries satisfy $\mathcal{O}^I(i_2) = p^{-\Delta_I} \mathcal{O}^I(i_4)$ under the boundary scaling $x \to px$.

Besides, the local restriction is also needed to obtain the correlators. We would like to introduce the idea from MERA networks [8]. MERA proposes that one can moving operators through layers of tensors and so that the coarse graining can be realized. As Figure 1 shows, the operator on p-branch and p^2-branch are on different layer, they are related only if sandwiched between an extra layer of tensors. Therefore,

$$\mathcal{O}^I(i_2) = T_{a_1 b_2 b_3} \mathcal{O}^I_{b_2 \tilde{b}_2}(i_4) T^*_{\tilde{a}_1 \tilde{b}_2 b_3}. \tag{2.36}$$

Then $T_{a_1 b_2 b_3} \mathcal{O}^I_{b_2 \tilde{b}_2} T^*_{\tilde{a}_1 \tilde{b}_2 b_3} = p^{-\Delta_I} \mathcal{O}^I_{a_1 \tilde{a_1}}$, for one who wants to generalize this to r-valent tree, just add the subscript until b_r. This is illustrated in Figure 5 With Pauli basis, the

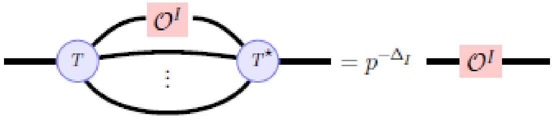

Figure 5. Illustration of equation 2.36. (from [4] Fig. 11).

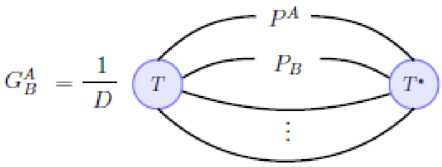

Figure 6. Illustration of equation 2.38. (from [4] Fig. 12).

conformal primaries can be constructed. The Pauli basis satisfy

$$T_{a_1 b_1 \cdots b_p} P^A_{b_1 \tilde{b}_1} T^*_{\tilde{a}_1 \tilde{b}_1 b_2 \cdots b_p} = G^A_B P^B_{a_1 \tilde{a}_1}. \tag{2.37}$$

Then

$$G^A_B = \frac{1}{D} T_{a_1 b_1 \cdots b_p} P^A_{b_1 \tilde{b}_1} P_{B\ \tilde{a}_1 a_1} T^*_{\tilde{a}_1 \tilde{b}_1 b_2 \cdots b_p}, \tag{2.38}$$

which is shown in Figure 6 This gives the spectrum of the theory defined by the tensor network.

2-point functions We start with the 2-point function:

$$\langle \mathcal{O}^I(i) \mathcal{O}^J(j) \rangle \equiv \langle \psi_{boundary} | \mathcal{O}^I(i) \mathcal{O}^J(j) | \psi_{boundary} \rangle. \tag{2.39}$$

To compute this function, one should move operators from boundary to bulk until they coincide on a node (say v), then we have a factor regarding the length: $p^{-\Delta_I d(i \to v) - \Delta_J d(j \to v)}$. This is shown in Figure 7. Then according to 2.38, we get

$$\langle \mathcal{O}^I(i) \mathcal{O}^J(j) \rangle = \delta^{IJ} A p^{-\Delta_I d(i \to j)}, \tag{2.40}$$

where

$$A \equiv T_{a_1 a_2 b_3 \cdots b_{p+1}} \mathcal{O}^I_{a_1 \tilde{a}_1} \mathcal{O}^I_{a_2 \tilde{a}_2} T^*_{\tilde{a}_1 \tilde{a}_2 b_3 \cdots b_{p+1}}, \tag{2.41}$$

and $d(i \to j)$ counts the number of edges in the path that connects the boundary leg i and j[5]. In the BT tree, this distance will diverge. By comparison between equation 1.34 and 2.11, it is not hard to observe that

$$p^{-\Delta_I d(i \to j)} \to \frac{1}{|x_{ij}|_p^{2\Delta_I}}, \tag{2.42}$$

[5]This path is unique according to the causal structure

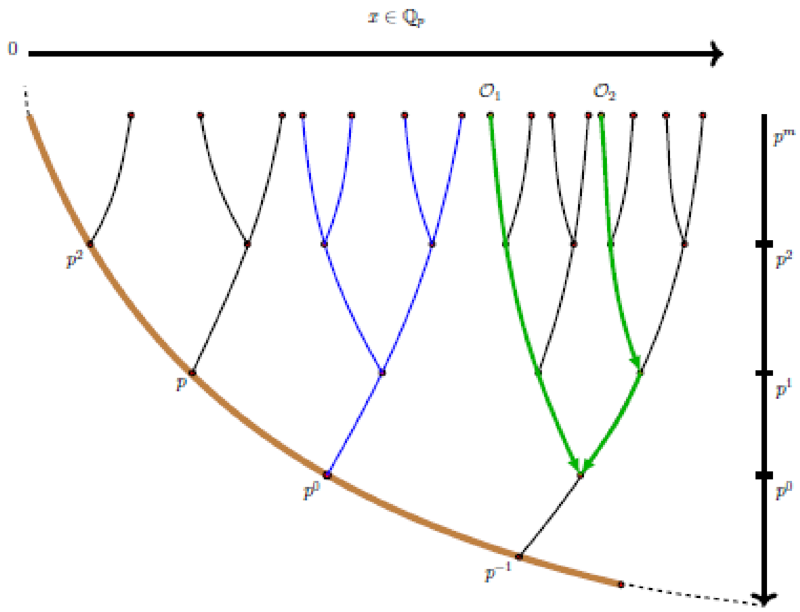

Figure 7. Pushing down the boundary operators to the joint point to obtain the 2-point correlation function, with $p = 2$. (from [4] Fig. 13).

where $|x_{ij}|_p = |x_i - x_j|_p$. The rigorous version of such regularization has been proposed in [5]. Denote $A \subset C_I$ up to a constant, we obtain the remarkable result

$$\langle \mathcal{O}^I(x_i)\mathcal{O}^J(x_j)\rangle = \delta^{IJ}\frac{C_I}{|x_i - x_j|_p^{2\Delta_I}}, \tag{2.43}$$

which perfectly coincide that in usual AdS/CFT.

Higher point functions Now it is easy to generalize our results to 3- or more points correlation function. For instance, the three point vertex is given by

$$\lambda^I_{JK} = T_{a_1a_2a_3b_4\cdots b_{p+1}}\mathcal{O}^I_{a_1\tilde{a}_1}\mathcal{O}_{J\ a_2\tilde{a}_2}\mathcal{O}_{K\ a_3\tilde{a}_3}T^*_{\tilde{a}_1\tilde{a}_2\tilde{a}_3b_4\cdots b_{p+1}}. \tag{2.44}$$

They coincide with α^I_{JK} up to an overall normalization depend on D.

Last but not least, we should note that the primary basis obtained here coincide with the HKLL basis obtained by diagonalization. Readers may check the proof in Sec. 6.4 of [4]. And so that they concluded, in the weakly interacting limit, $\sigma_I = d - \Delta_I$. This relation emerges automatically in the tensor network.

2.2 Path integral

This approach is induced from state-operator correspondence, which associates to the radial quantization of p-adic CFT. In usual CFT, radial quantization equips this CFT a ball-like physical Hilbert space $\mathcal{H}$ with boundary ∂B. And state-operator correspondence allows us to associate $\mathcal{H}$ with the space of local operators acting at the center of the ball, then the OPE will iteratively reproduce the general correlation function.

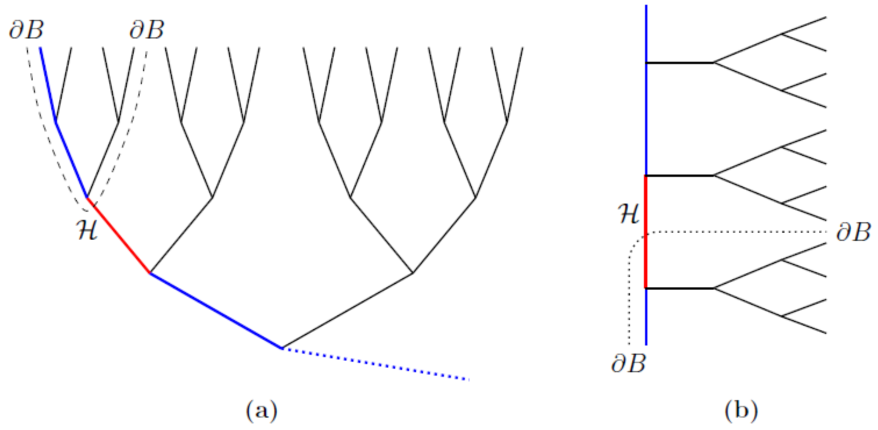

Figure 8. (a) The Hilbert space $\mathcal{H}$ associated to ∂B. (b) A representation of the BT tree adapted to the p-adic cylinder, corresponding to radial quantization. The central branch is the bulk line between the ball's center and ∞, while the boundary of each branching from the central branch is a sphere ($|x|$ =constant).

Now we should treat such situation on BT tree. A p-adic ball B consists of all points lying above some bulk vertex (z, x), and we associate ∂B the boundary of $\mathcal{H}$ which spanned by the set of local operators. We now wish to associate the same state space to the bulk minimal surface ending on ∂B; alternatively, we wish to separate the ball from its complement by cutting the smallest number of bulk legs possible, and associate $\mathcal{H}$ with the cut legs. Since the bulk geometry is a tree, the minimal surface cuts precisely one leg (see Figure 8). This suggests that each bulk leg is naturally associated a copy of the space of local operators.

We consider the holographic regularization as truncating the BT tree at some finite level, so that CFT insertions are mapped to insertion of states in $\mathcal{H}$ on the cut edges. As noted, in the tensor network, we hope that the tensor at the vertex could implement the OPE. And this suggests constructing the OPE as an action of $\mathcal{H}$ on itself as a subalgebra of its own operator algebra, which means an injective complex linear map $\mathcal{C} : \mathcal{H} \to \mathrm{End}(\mathcal{H})$ determined by the OPE. In particular, $\mathcal{H}$ should be a C^* algebra. Let $\{|a\rangle\}$ be a basis of $\mathcal{H}$. We rewritten $\mathcal{C}$ in components form with indices:

$$\mathcal{C} : |c\rangle \mapsto \sum_{a,b} |a\rangle C^a_{cb} \langle b|. \tag{2.45}$$

Theorem 1 $(\mathcal{C}, \mathcal{H})$ can reproduce the complete OPE structure when they satisfy the following properties:

1. *Commutativity*: $\mathcal{C}(a)\mathcal{C}(b) = \mathcal{C}(b)\mathcal{C}(a)\ \forall\ a, b \in \mathcal{H}$, which is equivalent to say that $\mathcal{C}^c_{ab}$ is symmetric.

2. *Conjugation*: $\exists$ an anti-unitary map $\imath : \mathcal{H} \to \mathcal{H}$ such that $\imath^2 = \mathrm{id}_{\mathcal{H}}$ and $\mathcal{C}(\imath(a)) = \mathcal{C}^\dagger(a)$ $\forall |a\rangle \in \mathcal{H}$.

3. *Identity*: $\exists$ an element of $\mathcal{H}$, which is denoted by $|\mathbf{1}\rangle$, such that $\mathcal{C} = \mathbf{1}_{\mathcal{H}}$, $\langle \mathbf{1}|\mathbf{1}\rangle = 1$ and $\imath|\mathbf{1}\rangle = |\mathbf{1}\rangle$.
4. *Associativity*: $\sum_e C^a_{be} C^e_{cd} = \sum_e C^a_{ce} C^e_{bd}$.
5. *Propagator*: $\exists$ a symmetric bivector $G \in \mathcal{H} \otimes \mathcal{H}$ such that $G(\langle \mathbf{1}|, \langle \mathbf{1}|) = \mathbf{1}$.

Remark

1. Defining $C_{abc} \equiv \sum_e C^{\mathbf{1}}_{ae} C^e_{bc}$, it is totally symmetric.
2. One can define
$$C_{ab} \equiv C^{\mathbf{1}}_{ab} = \langle \mathbf{1}|\mathcal{C}(a)\mathcal{C}(b)|\mathbf{1}\rangle = \langle \bar{a}|b\rangle = \langle \bar{b}|a\rangle. \tag{2.46}$$
This is a symmetric non-degenerate bilinear form, which acts as a metric on the operator space.
3. The component of G is $G^{ab} = G(\langle a|, \langle b|)$.
4. Defining the operator $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ such that $\langle a|\mathcal{G}|b\rangle = \mathcal{G}^a_b = G^{ac} C_{cb}$, and the scaling dimension operator can be constructed as $\Delta = -\log_p \mathcal{G}$.
5. If unitary condition is required, $\mathcal{G}$ has to be Hermitian.
6. The basis is similar to the former sections, $C_{ab} = \delta_{ab}$ and G^{ab} is diagonal. In this case, $\mathcal{G}^a_b$ takes the natural form
$$\mathcal{G}^a_b = \delta^a_b p^{-\Delta_a}. \tag{2.47}$$
Thus $C_{ab} = 0$ unless $\Delta_a = \Delta_b$. This coincides with what correlation function requires.

2.2.1 Regularization and generating functional

In path integral formalism, one need to write down the generating functional to compute the explicit path integral and extract correlators. We will start from "global patch" of p-adic AdS (which is $\mathbb{P}^1(\mathbb{K})$ as specified before) to give the generating functional.

Suppose the regularized space of BT tree is denoted by $\mathbb{H}^\Lambda_\mathbb{K}$ with $\Lambda \in \mathbb{Z}$ and $\Lambda > d(o, v)$, where o an arbitrary point inside the bulk and v a vertex. Then the number of boundary edges is
$$N = q^\Lambda (1 + \frac{1}{q}), \tag{2.48}$$
and the resulting tensor network has the same number of slots. Now we begin to construct the generating functional.

Consider the tensor network as a map
$$\mathcal{Z}_\Lambda : \mathcal{H}_\Lambda \to \mathbb{C}, \quad \mathcal{H}_\Lambda = \bigotimes_{e \in \partial \mathbb{H}^\Lambda_\mathbb{K}} \mathcal{H}_e, \tag{2.49}$$
and we just want to persuade this map to be identified with the path integral. As $\Lambda \to \infty$, $\mathcal{Z}_\Lambda$ will encode all correlation functions.

First, the identity operator can be freely inserted,

$$|\mathbf{1}\rangle_N = \bigotimes_{e\in\partial\mathbb{H}_\Lambda} |\mathbf{1}\rangle_e. \tag{2.50}$$

And $Z_\Lambda = \mathcal{Z}_\Lambda(\mathbf{1}_\Lambda)$ on $\mathbb{P}^1(\mathbb{K})$, where Z_Λ the regularized partition function. This corresponds to the trivial value. To generalize, we can perturb the partition function by $|a\rangle$ at a boundary point X, which is equivalent to

$$|1\rangle_X + \lambda_X^{(a)}|a\rangle_X, \tag{2.51}$$

where $\lambda_X^{(a)} \in \mathbb{C}$, it can be regarded as the source for state a at X.

The regularized generating state is an element of $\mathcal{H}_\Lambda$,

$$|J\rangle \equiv \bigotimes_{x\in\partial\mathcal{T}_\Lambda} |J\rangle_X \quad \text{with} \quad |J\rangle_X \equiv \sum_a J_a(X)|a\rangle_X, \tag{2.52}$$

of which the regularized generating functional can be defined as $Z_\Lambda(J) = \mathcal{Z}_\Lambda \cdot |J\rangle$. A general expression for regularized correlation function can be written as

$$\langle \mathcal{O}_{a_1}(X_1)\cdots\mathcal{O}_{a_N}(X_N)\rangle_\Lambda = \mathcal{Z}_\Lambda(|a_1\rangle \otimes |a_2\rangle \otimes \cdots \otimes |a_N\rangle). \tag{2.53}$$

For point-like sources, this formalism has been enough while it will be ineffective when we come across the case with smeared sources. We can circumvent this problem in two cases: when the sourced operator is relevant or when there is a parameter with respect to which all non-trivial n-point vertices ($n \geq 3$) are small, as happens in the large N expansion.
Poincaré patch regularization Now we analyze the mostly used Poincaré patch which has infinite volume as an affine space. Thus both UV and IR regulators should be considered. Given a cutoff z_ϵ which satisfies $|z_\epsilon| = \epsilon < 1$ (note that we have mentioned, all the norms without subscript are the p-adic norm in $\mathbb{K}$), the set of bulk points is:

$$\mathbb{H}_\mathbb{K}^{(\epsilon)} = \{(z,x) : |z| > \epsilon \quad \text{and} \quad |x| < \epsilon^{-1}\}, \tag{2.54}$$

which is the truncation of the tree spanning $z_\epsilon^{-1}\mathbb{Z}_\mathbb{K}$ to those points with $|z| > \epsilon$[6]. This corresponds to choosing $z_\epsilon^{\mathrm{IR}} = 1/z_\epsilon^{\mathrm{UV}}$. See Figure 9.

The number of hanging edges is $N_\epsilon = q^{-1}\epsilon^{-2n} + 1$, with the regularized Hilbert space $\mathcal{H}_\epsilon = \mathcal{H}^{\otimes N_\epsilon}$.

Similar to the global case, one can easily construct the regularized generating functional, the source takes the form:

$$|J_\epsilon\rangle = \bigotimes_{(z,x)\in\partial\mathbb{H}_\mathbb{K}^{(\epsilon)}} |J_\epsilon(z,x)\rangle_{(z,x)}. \tag{2.55}$$

We hope this source can be encoded in the 1-point function of bulk operators as usual AdS/CFT has, alternatively, we expect that the source can reproduce couplings with boundary operators in a exponential expansion behavior. Let us cutoff at $z = \zeta$, with

[6] not so clear, need more comments.

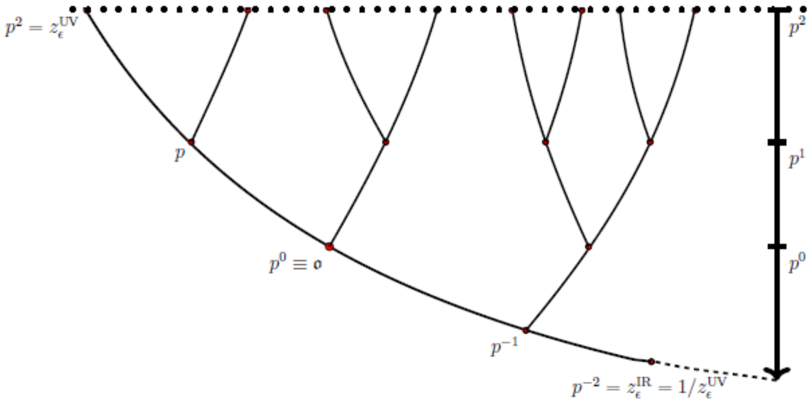

Figure 9. Diagram illustrating the UV and IR cutoff. UV and IR cutoff have been picked at p^2 and p^{-2}, respectively (from [3] Fig. 5).

$\epsilon = |\zeta|$ small. As before, we can insert $|\mathbf{1}\rangle_{(\zeta,x)} + J_\epsilon(\zeta,x)|\phi\rangle_{(\zeta,x)}$ at each branch and perform one renormalization step toward the IR region, which bring us from level ζ to level $p^{-1}\zeta$. And the insertion at $(p^{-1}\zeta, x)$ is

$$[\prod_{(\zeta,y)}(1 + J_\epsilon(\zeta,y)p^{-\Delta\phi})]|\mathbf{1}\rangle = |\mathbf{1}\rangle + p^{-\Delta}\sum_{(\zeta,y)} J(\zeta,y)|\phi\rangle + o(J^2), \tag{2.56}$$

where the product over (ζ, y) runs over $q = p^n$ boundary points lying on $(p^{-1}\zeta, x)$, which suggests that the value of the source function should be defined by p-adic integration of $J(x)$ over the ball $x + \zeta\mathbb{Z}_\mathbb{K}$. Therefore,

$$J(\zeta,x) = |\zeta|^{-\Delta}\int_{x+\zeta\mathbb{Z}_\mathbb{K}} J(x)\mathrm{d}x = |\zeta|^{n-\Delta}[J]_{x+\zeta\mathbb{Z}_\mathbb{K}}, \tag{2.57}$$

where $[J]_S = \frac{1}{\mathrm{vol}(S)}\int_S J(x)dx$ denotes the average value of J. If $|z| \to 0$, which means that the current function will become continuous and so that $J(z,x)$ is very small. To achieve this, it is not hard to note that we just need to force $n - \Delta > 0 \Leftrightarrow \Delta < n$, which is just the definition for relevant operator.

One remarkable fact is that the discussion above can only be applied to the case with relevant or marginal operators(in perturbation), the stuff regarding irrelevant operators are under control from the IR CFT perspective, we will cover this in the next section.

Another remarkable thing is that in evolving the boundary state into the interior, we have considered the vertex to be a map $\mathcal{H}^{\otimes q} \to \mathcal{H}$, which differs from our definition as a map $\mathcal{H}^{\otimes(q+1)} \to \mathbb{C}$. Therefore, we have to convert the leg lying below the vertex into a vector. And the safe way to do this is using the C^{ab} as the 2-point coefficient. Then G can be contracted with C_{ab} as a bivector and result in the propagator map $\mathcal{G}$. In this way, we can split the tensor network into layers, each of which maps the Hilbert space at one renormalization step to that of the next. The later computations in this subsection will all follow this picture.

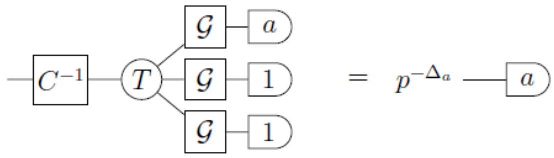

Figure 10. The contribution from the insertion of a single non-trivial operator a on the boundary gets renormalized by $p^{-\Delta_a}$ as we move one step into the interior, which is exactly the same as in Fig. 5 (from [3] Fig. 6).

For the boundary correlation functions, one can easily extract the same rule as in wavelet transform (see Figure). The regularized current is

$$J(z,x) = |z^{-\Delta}| \begin{cases} 1 \ |x-y| < |z|, \\ 0 \quad \text{otherwise.} \end{cases} \tag{2.58}$$

And

$$\langle \mathcal{O}_b(z,x) \rangle_{J_a(x)=\delta(x-y)} = C_{ab} \begin{cases} |z|^{-\Delta} & |x-y| \leq |z|, \\ \frac{|z|^{\Delta}}{|x-y|^{2\Delta}} & \text{otherwise,} \end{cases} \tag{2.59}$$

where the bulk-boundary propagator is reproduced as before, up to a normalization factor.

2.2.2 2-point function

We still take use of the Poincaré patch regularization, the 2-point function is equal to the partition function $Z(J)$, with the current

$$J_a(x) = \delta(x-x_1) \quad J_b(x) = \delta(x-x_2). \tag{2.60}$$

The partition function of the regularized network can be computed by inserting the states $|z_\epsilon|^{-\Delta_a}|a\rangle$ and $|z_\epsilon|^{-\Delta_b}|b\rangle$ at (z_ϵ, x_1), (z_ϵ, x_2), respectively, with $|\mathbf{1}\rangle$ inserted on all the other legs.

The maximal value of the length between x_1 and x_2 is $|x_1 - x_2|$, and any branch of the tree disjoint from the path connecting x_1 and x_2 collapses to $|\mathbf{1}\rangle$. The propagation from the boundary to the bottom of bulk gives

$${\frac{|z_\epsilon|}{|x_{12}|}}^{\Delta_a+\Delta_b}, \tag{2.61}$$

while the fusion gives a factor of C_{ab}. Since $C_{ab} = 0$ unless $\Delta_a = \Delta_b$ in the diagonalized basis, we can then obtain

$$\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2) \rangle = {\frac{C_{ab}}{|x_1 - x_2|}}^{2\Delta_a}. \tag{2.62}$$

One can note that this result coincide with both the usual CFT and the computation showed in the last subsection with only a minor difference on the interpretation (in wavelet formalism, we have restricted the insertions on the same spatial slice). This procedure is illustrated in Figure 11

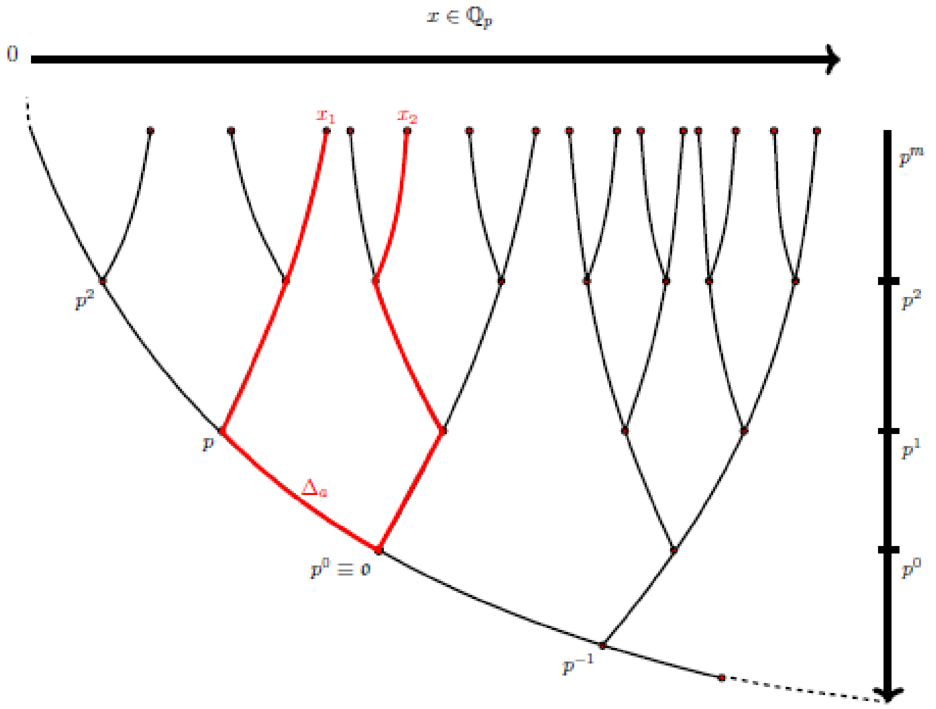

Figure 11. Computation procedure of the 2-point function. (from [3] Fig. 7).

2.2.3 3-point function

Since we should calculate the length between two insertions. It is important to notice that the property of p-adic geometry forces that

$$|x_{12}| \leq |x_{13}| = |x_{23}|, \tag{2.63}$$

as the corollary of **Def. 2** showed. Illustrated in Figure 12.

Non-trivial states will propagate along downward-directed paths with weight $p^{-\Delta}$ at each step until two non-trivial states are joined at a single vertex. As Figure 12 showed, x_1 and x_2 fuse at A and gives an intermediate operator $\sum_d |d\rangle \mathcal{C}_{ab}^d$. This operator continue to propagate to B where it meets the propagation of x_3. The last combination contributes a C_{cd}. Therefore, the cutoff-dependent normalization factors from the current gives

$$\begin{aligned}
\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\mathcal{O}_c(x_3)\rangle =& \sum_d C_{cd}C_{ab}^d |z_\epsilon|^{-\Delta_a-\Delta_b-\Delta_c} \left(\frac{|z_\epsilon|}{|x_{12}|}\right)^{\Delta_a+\Delta_b+\Delta_c} \left(\frac{|x_{12}|}{|x_{32}|}\right)^{\Delta_d+\Delta_c} \\
=& \frac{C_{abc}}{|x_{12}|^{\Delta_a+\Delta_b-\Delta_c}} \cdot \frac{1}{|x_{32}|^{2\Delta_c}} \\
=& \frac{C_{abc}}{|x_{12}|^{\Delta_a+\Delta_b-\Delta_c}} \cdot \frac{1}{|x_{23}|^{\Delta_b+\Delta_c-\Delta_a}} \cdot \frac{1}{|x_{13}|^{\Delta_a+\Delta_c-\Delta_b}} \\
=& \frac{C_{abc}}{|x_{12}|^{\Delta_a+\Delta_b-\Delta_c}|x_{23}|^{\Delta_b+\Delta_c-\Delta_a}|x_{13}|^{\Delta_a+\Delta_c-\Delta_b}}.
\end{aligned} \tag{2.64}$$

This also confirms that C_{abc} is in fact the CFT structure constants.

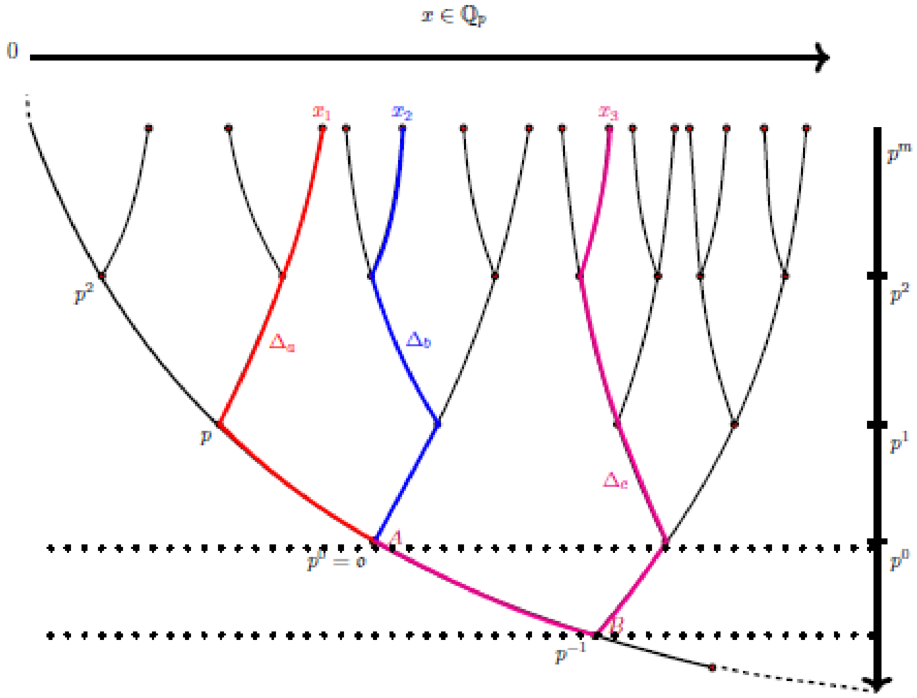

Figure 12. Computation procedure of the 3-point function. (from [3] Fig. 8).

2.2.4 4-point function

For four p-adic numbers, there are six possible coordinate differences which satisfy either:

$$|x_{12}| \leq |x_{13}| = |x_{23}| \leq |x_{14}| = |x_{24}| = |x_{34}|, \tag{2.65}$$

or

$$|x_{12}|, |x_{34}| \leq |x_{13}| = |x_{14}| = |x_{23}| = |x_{24}|. \tag{2.66}$$

These two situations are connected by a Möbius transformation, the result can be extracted from each one to another by a different permutation, so here we just focused on the second case. The propagation procedures are the same as above, with one more intermediate vertex B added, so that

$$\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\mathcal{O}_c(x_3)\mathcal{O}_d(x_4)\rangle = \sum_{e,f} C_{ef} C^e_{ab} C^f_{cd} |x_{12}|^{-\Delta_a-\Delta_b} |x_{34}|^{-\Delta_c-\Delta_d} |\frac{x_{12}}{x_{13}}|^{\Delta_e} |\frac{x_{34}}{x_{13}}|^{\Delta_f}, \tag{2.67}$$

this result is a little complicated, we define

$$u = \frac{|x_{12}||x_{34}|}{|x_{13}||x_{24}|} \quad \text{and} \quad v = \frac{|x_{14}||x_{23}|}{|x_{13}||x_{24}|}, \tag{2.68}$$

giving

$$\sum_e C_{abe} C^e_{cd} \frac{u^{\Delta_e}}{|x_{12}|^{\Delta_a+\Delta_b}} |x_{34}|^{\Delta_c+\Delta_d}. \tag{2.69}$$

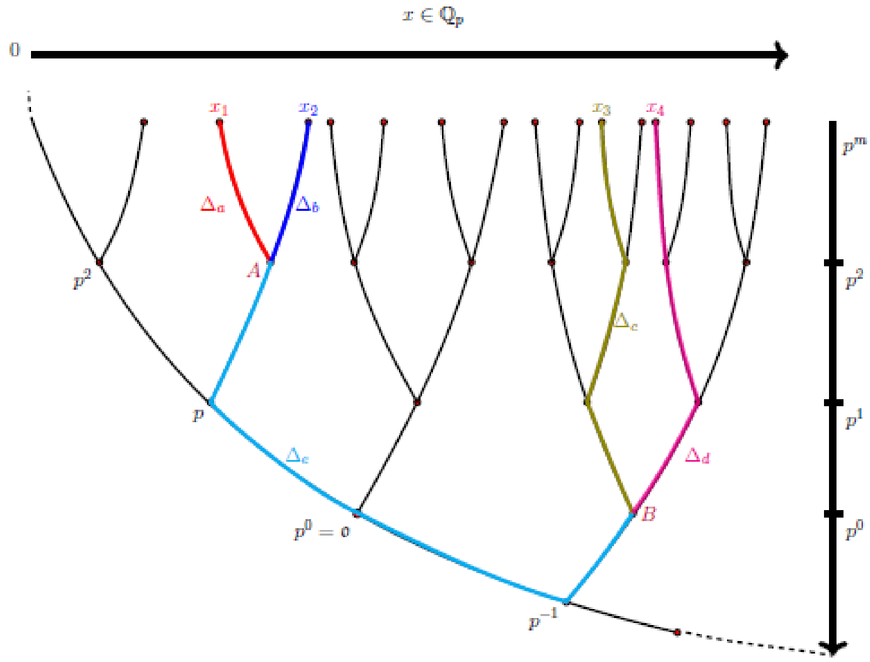

Figure 13. Computation procedure of the 4-point function. (from [3] Fig. 9).

Note that $-\log_p u$ is equal to the distance between A and B. $v = 1$ in this case. Computation procedures are illustrated in Figure 13. For more point case, one just need to recall the associativity condition of p-adic CFT and the structure constants satisfy:

$$C_{a_1 \cdots a_k} = \sum_{\{e_i\}} C_{a_1 a_2 e_1} C^{e_1}_{a_3 e_2} \cdots C^{e_{k-3}}_{a_{k-1} a_k}, \tag{2.70}$$

and the associated k-point function can be written.

2.3 Wilson line network

In the p-adic AdS/CFT discussed so far, the boundary correlation functions are reproduced by bulk Witten diagrams. However, we know that the bulk gravity dual can be redormulated as an $SL(2,\mathbb{C})$ Chern–Simons (CS) theory and thus the boundary correlation functions can be reproduced by Wilson line networks of the CS theory in usual AdS_3/CFT_2 correspondence [9–11]. In this network, there is no need to integrate the bulk point of the Witten diagram and the $1/c$ expansion is more transparent. Here we will start from this perspective to discuss the CS-like theory of p-adic version. The main reference for this subsection is from [13].

Although we know nothing about the CS theory on BT tree, the $PGL(2,\mathbb{Q}_p)$ is clear as a gauge theory on BT tree. Analogous to that of usual AdS/CFT, in which one can construct the CS theory with gauge group $SL(2,\mathbb{C})$, we can construct the p-adic CS theory from $PGL(2,\mathbb{Q}_p)$ gauge theory.

As in general formulation of lattice gauge theories, gauge connections are attached to links of the lattice. Namely, the gauge connection at each link takes values from the gauge group $G = \mathrm{PGL}(2, \mathbb{Q}_p)$. To each edge of the tree, we associate a connection

$$U(v, i) = U_{v \to v+i}, \tag{2.71}$$

where $v+i$ is v's nearest neighbor along the direction i. The orientation of the link has also be attached, so that an inverse gauge connection exists. Finally, the link variable $U(v, i)$ in the lattice gauge theory is the direct analogue of the Wilson line in the continuous gauge theory:

$$U(v, i) \sim W(v, i) = e^{(i \int_{\mathrm{link}(v,i)} A)}. \tag{2.72}$$

Under a gauge transformation, the connection in the specified orientation transforms as:

$$U_{v \to v+i} \to g^{-1}(v) U_{v \to v+i} g(v + i), \tag{2.73}$$

where $g(v)$ is a $\mathrm{PGL}(2, \mathbb{Q}_p)$-valued function on the vertex v of the tree. A pure gauge is

$$U_{v \to v+i} = g^{-1}(v) g(v + i) \quad \text{with} \quad g(v) \in \mathrm{PGL}(2, \mathbb{Q}_p). \tag{2.74}$$

We have specified that the complex-valued p-adic CFT considered here has no descendant, thus no stress tensor will appear, and so that no propagating gravitational degrees of freedom are allowed either in the bulk or on the boundary. Then the only permitted configuration is the pure AdS configuration. It would be very fascinating to discuss the p-adic valued p-adic CFT (which is resembling ordinary 2D meromorphic CFT), in which spin structure and local derivative and graviton degrees of freedom can be considered and more information inside the bulk of p-adic AdS can be extracted. We will put this down until the next section.

2.3.1 Pure AdS$_3$ configuration

First, we give a brief review for pure AdS solution. It has metric (in Poincaré coordinates)

$$\mathrm{d}s^2_{\mathrm{AdS}_3} = \frac{1}{\mathrm{tr}[(L_0^2)]} \mathrm{tr}(e \otimes e) = l^2 \frac{\mathrm{d}\rho^2 + \mathrm{d}z \mathrm{d}\bar{z}}{\rho^2}, \tag{2.75}$$

with gauge field

$$A = \mathfrak{g}^{-1} \mathrm{d}\mathfrak{g} \quad \text{with} \quad \mathfrak{g} = e^{z L_{-1}} e^{\ln \rho L_0}, \tag{2.76}$$

and $\tilde{A} = -A^\dagger$. With the choice for the basis of the $\mathfrak{sl}(2)$ algebra, we have

$$\mathfrak{g} = \mathfrak{g}(\rho, z) = \frac{1}{\sqrt{\rho}} \begin{pmatrix} \rho & z \\ 0 & 1 \end{pmatrix}. \tag{2.77}$$

The element $\mathfrak{g}$ can be used to parametrize the AdS$_3$ space. The metric above can be rewritten as

$$\mathrm{d}s^2_{\mathrm{AdS}_3} = \frac{l^2}{2} \mathrm{tr}(G^{-1} \mathrm{d}G G^{-1} \mathrm{d}G) \quad \text{with} \quad G = \mathfrak{g}\mathfrak{g}^\dagger. \tag{2.78}$$

Note that G is invariant under the maximal compact subgroup of global symmetry.

For the BT tree, we can similarly first parametrize it using $\mathrm{PGL}(2,\mathbb{Q}_p)/\mathrm{PGL}(2,\mathbb{Z}_p)$ elements (without considering extension), and then use them to build the $\mathrm{PGL}(2,\mathbb{Q}_p)$ connections living on the edges of the tree. The element of $\mathrm{PGL}(2,\mathbb{Q}_p)$ can be written as

$$\mathfrak{g}(v) = \prod_{a_i \in \mathcal{P}(o\to v)} \Gamma[a_i \to a_{i+1}], \tag{2.79}$$

where o an arbitrary point and v a vertex, Γ satisfy

$$\Gamma_{-1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \quad \text{and} \quad \Gamma_n = \begin{pmatrix} p & n \\ 0 & 1 \end{pmatrix} \quad n = 0, 1, \cdots, p-1. \tag{2.80}$$

Obviously, the group element at the origin is just the identity and the points on the main branch all have

$$\mathfrak{g}(\text{main branch}) = \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.81}$$

More generally, we have a map between a vertex on the tree to a $\mathrm{PGL}(2,\mathbb{Q}_p)$ element

$$\{\langle \vec{f}, \vec{g} \rangle\}_v = \{\langle \begin{pmatrix} p^n \\ 0 \end{pmatrix}, \begin{pmatrix} x^{(n)} \\ 1 \end{pmatrix} \rangle\} \leftrightarrow \mathfrak{g}(v) = \begin{pmatrix} p^n & x^{(n)} \\ 0 & 1 \end{pmatrix}. \tag{2.82}$$

Similarly, the gauge redundancy is controlled by the isotropy group $\mathrm{PGL}(2,\mathbb{Z}_p)$. If one needs to add cutoff in this picture, just give a cutoff surface end on the line of constant p^N and replace N to n above. The vacuum solution can be denoted as

$$U_{v\to v+i} = \mathfrak{g}^{-1}(v)\mathfrak{g}(v+i) \tag{2.83}$$

as usual.

2.3.2 p-adic Wilson lines

Now we can construct the Wilson line structure on the BT tree. The link variables are denoted by $U(v,i)$ while a Wilson line is just the ordered product of all the link variables connect v to $v+i$. Thus the Wilson line from v_1 to v_2 can be written as:

$$\mathfrak{W}(v_{1\to v_2}) = \prod_{(v,i)\in\mathcal{P}(v_1\to v_2)} U(v,i). \tag{2.84}$$

Since the configuration is a pure one, the Wilson line corresponds to the vacuum $\mathfrak{W}$ is

$$\mathfrak{W}_{\text{fund}}(v_1 \to v_2) = \mathfrak{g}(v_1)^{-1}\mathfrak{g}(v_2), \tag{2.85}$$

where the fund denotes the fundamental representation. To reproduce correlation functions in p-adic CFT using Wilson lines, we should now project the lines to representations that transform as conformal primaries. The essential procedures are the same as for usual 2D CFT, the only thing we should note for p-adic case is that we can only define the highest weight states corresponding to primary fields because of the vanishing of local derivatives.

These highest weight states will be used as projectors that relate the Wilson line operator in the fundamental representation to correlation functions of primary fields.

We define

$$\langle\Delta| \equiv \langle\text{vac}|\mathcal{O}_{\Delta(Z)}, \quad |\Delta\rangle \equiv \mathcal{O}_\Delta(0)|\text{vac}\rangle, \tag{2.86}$$

where

$$\langle\Delta|\Delta\rangle = |Z|^{-2\Delta}, \tag{2.87}$$

with Z a free parameter[7], and

$$\langle X;\Delta| \equiv \langle\text{vac}|\mathcal{O}_\Delta(X), \quad X \in \mathbb{Q}_p \tag{2.88}$$

since $\mathcal{O}_\Delta(X)$ transforms non-trivially under $\mathrm{PGL}(2,\mathbb{Q}_p)$. These states form another representation of $\mathrm{PGL}(2,\mathbb{Q}_p)$, transforming as

$$\langle X;\Delta| \to |\frac{ad-bc}{(cX+d)^2}|_p^\Delta \langle\frac{aX+b}{cX+d};\Delta|, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2,\mathbb{Q}_p). \tag{2.89}$$

Then

$$\langle X;\Delta_i|\Delta_j\rangle = \frac{\delta_{ij}}{|X|_p^{2\Delta}}. \tag{2.90}$$

The discussion for the ket expression is a little complicated, one can view [12, 13] for details. The ket state can be written as:

$$|X;\Delta\rangle \equiv \tilde{\mathcal{O}}_\Delta(X)|\text{vac}\rangle, \tag{2.91}$$

where $\tilde{\mathcal{O}}_\Delta$ is the shadow operator of $\mathcal{O}_\Delta$, which is defined as:

$$\tilde{\mathcal{O}}_\Delta(X) \equiv \mathcal{N}(d,\Delta)\int_{\mathbb{Q}_p} \mathrm{d}Y |X-Y|_p^{2\Delta-2d}\mathcal{O}_\Delta(Y), \tag{2.92}$$

with

$$\mathcal{N}(d,\Delta) \equiv \frac{\zeta_p(2d-2\Delta)\zeta_p(2\Delta)}{\zeta_p(d-2\Delta)\zeta_p(2\Delta-d)} \tag{2.93}$$

the normalization constant, $d \equiv n$ the dimension. For simplicity, we will usually take $d = 1$ in this subsection. The shadow operator is orthogonal to the ordinary operator, so that

$$\langle\Delta_i|X;\Delta_j\rangle = \delta_{ij}\delta(X-Z). \tag{2.94}$$

The projection operator $\mathcal{P}_\Delta$ construct a complete set and

$$\mathbf{1}_\Delta = \int_{\mathbb{Q}_p} \mathrm{d}X|X;\Delta\rangle\langle X;\Delta|. \tag{2.95}$$

Then we can obtain a primary representation for Wilson line. We use equation 2.95 to project the Wilson line from fundamental representation to the primary representation:

$$\hat{\mathfrak{W}}_\Delta(v_1 \to v_2) = \int_{\mathbb{Q}_p} \mathrm{d}X|X;\Delta\rangle\hat{\mathfrak{W}}_{\text{fund}}(v_1 \to v_2)\langle X;\Delta|. \tag{2.96}$$

[7] We will see that different choices of Z could lead to sifferent results if the open ends of the Wilson lines are lying in the bulk.

With the transformation property 2.89,

$$\hat{\mathfrak{W}}_\Delta(v_1 \to v_2) = \int_{\mathbb{Q}_p} \mathrm{d}X |\frac{ad-bc}{(cX+d)^2}|_p^\Delta |X;\Delta|\rangle\langle\frac{aX+b}{cX+d};\Delta|, \tag{2.97}$$

then the orthogonality condition 2.94 forces the Wilson line sandwiched by primary states to be

$$\langle\Delta|\hat{\mathfrak{W}}_\Delta(v_1 \to v_2)|\Delta\rangle = \frac{|(ad-bc)|_p^\Delta}{|aZ+b|_p^{2\Delta}}. \tag{2.98}$$

2.3.3 2-point function

The Wilson line for the vacuum configuration with fundamental representation is

$$\hat{W}_{\text{fund}}(v_1 \to v_2) = \begin{pmatrix} p^{n_2-n_1} & p^{-n_1}(x_2^{(n_2)} - x_1^{(n_1)}) \\ 0 & 1 \end{pmatrix}, \tag{2.99}$$

Inserting this into 2.98, we have

$$\langle\Delta|\hat{\mathfrak{W}}_\Delta(v_1 \to v_2)|\Delta\rangle = \frac{p^{-(n_1+n_2)\Delta}}{|p^{n_2}Z + (x_2^{(n_2)} - x_1^{(n_1)})|_p^{2\Delta}}. \tag{2.100}$$

According to the definition, the boundary 2-point function can be assembled by pushing the vertices to the boundary:

$$\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\rangle = \lim_{v_i \to boundary}\langle\Delta|\hat{\mathfrak{W}}_\Delta(v_1 \to v_2)|\Delta\rangle. \tag{2.101}$$

As v_i approaches the boundary, both n_1 and n_2 will limit to infinity, the term $p^{n_2}Z$ in the denominator thus drops out, then

$$\lim_{N\to\infty}\langle\Delta|\hat{\mathfrak{W}}_\Delta(v_1 \to v_2)|\Delta\rangle = \frac{p^{-2N\Delta}}{|x_2 - x_1|_p^{2\Delta}}. \tag{2.102}$$

The $p^{-N\Delta}$ term can always be assumed as an overall normalization factor which will be absorbed in the normalization of $\langle\Delta_i|$ and $|\Delta_i\rangle$, the following subsubsections will have the same effect. Therefore, the Wilson line segments between two ends at the boundary correctly reproduces the 2-point function of the boundary p-adic CFT.

2.3.4 3-point function

We still limit our discussion in $p = 2$ case.For three Wilson lines, they can meet at a vertex v_a on the BT tree. Suppose their end points are $v_{1,2,3}$, respectively with representation $\Delta_{1,2,3}$, we have

$$\prod_{i=1}^{3}\hat{\mathfrak{W}}_{\Delta_i}(v_a \to v_i) = \left[\prod_{i=1}^{3}\int_{\mathbb{Q}_p}\mathrm{d}X_i|X_i;\Delta_i\rangle\right]\left[\prod_{i=1}^{3}\hat{\mathfrak{W}}_{\text{fund}}(v_a \to v_i)\right]\left[\prod_{i=1}^{3}\langle X_i;\Delta_i|\right]. \tag{2.103}$$

The three end points should be contracted with $\langle\Delta_i|$. Besides, at the point where three Wilson lines coincide, the gauge invariance requires that the product of $\langle X_i;\Delta_i|$ be projected to the singlet $|\mathcal{S}\rangle$ which arises from the tensor product of the three representations. Thus the final expression should be

$$\left[\prod_{i=1}^{3}\langle\Delta_i|\right]\left[\prod_{i=1}^{3}\hat{\mathfrak{W}}_{\Delta_i}(v_a\to v_i)\right]|\mathcal{S}\rangle, \tag{2.104}$$

which can be split into three parts:

$$\left[\prod_{i=1}^{3}\int_{\mathbb{Q}_p}\mathrm{d}X_i\langle\Delta_i|X_i;\Delta_i\rangle\right],\left[\prod_{i=1}^{3}\hat{\mathfrak{W}}_{\text{fund}}(v_a\to v_i)\right],\left[\prod_{i=1}^{3}\langle X_i;\Delta_i|\right]|\mathcal{S}\rangle. \tag{2.105}$$

First we should note that

$$\left[\prod_{i=1}^{3}\langle X_i;\Delta_i|\right]|\mathcal{S}\rangle\equiv P_{123}(X_1,X_2,X_3), \tag{2.106}$$

where

$$P_{123}(X_1,X_2,X_3)=\frac{C_{123}}{|X_{12}|_p^{\Delta_1+\Delta_2-\Delta_3}|X_{13}|_p^{\Delta_1+\Delta_3-\Delta_2}|X_{23}|_p^{\Delta_2+\Delta_3-\Delta_1}}. \tag{2.107}$$

Again, this is completely coincide with the usual CFT.

Besides,

$$\left[\prod_{i=1}^{3}\hat{\mathfrak{W}}_{\text{fund}}(v_a\to v_i)\right]P_{123}(X_1,X_2,X_3)=\left[\prod_{i}p^{(n_a-n_i)\Delta_i}\right]P_{123}(X_1',X_2',X_3'), \tag{2.108}$$

where $X_i'\equiv p^{n_i-n_a}X_i+p^{-n_a}(x_i^{(n_i)}-x_a)$. After then, just need to contract 2.108 with $\langle\Delta_i|$. We then obtain

$$\left[\prod_{i=1}^{3}\langle\Delta_i|\right]\hat{\mathfrak{W}}_{\Delta_i}(v_a\to v_i)|\mathcal{S}\rangle$$
$$=\frac{p^{-n_1\Delta_1-n_2\Delta_2-n_3\Delta_3}C_{123}}{|(p^{n_1}-p^{n_2})Z+x_{12}|_p^{\Delta_1+\Delta_2-\Delta_3}|(p^{n_2}-p^{n_3})Z+x_{23}|_p^{\Delta_3+\Delta_2-\Delta_1}|(p^{n_3}-p^{n_1})Z+x_{13}|_p^{\Delta_1+\Delta_3-\Delta_2}}. \tag{2.109}$$

Then we can show that this is equivalent to the boundary 3-point function. Still, we can pushing the operators to the near boundary surface and thus $|p^{n_i}Z|_p\to 0$ for $i=1,2,3$. Since Z terms are dropped out, the 3-point Wilson line network reduces to

$$\lim_{N\to\infty}\left[\prod_{i=1}^{3}\langle\Delta_i|\right]\hat{\mathfrak{W}}_{\Delta_i}(v_a\to v_i)|\mathcal{S}\rangle=\frac{p^{-N(\Delta_1+\Delta_2+\Delta_3)}C_{123}}{|x_{12}|_p^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|_p^{\Delta_3+\Delta_2-\Delta_1}|x_{13}|_p^{\Delta_1+\Delta_3-\Delta_2}}. \tag{2.110}$$

The Wilson line junction and the correlation on BT tree are illustrated in Figure 14 and 15, respectively.

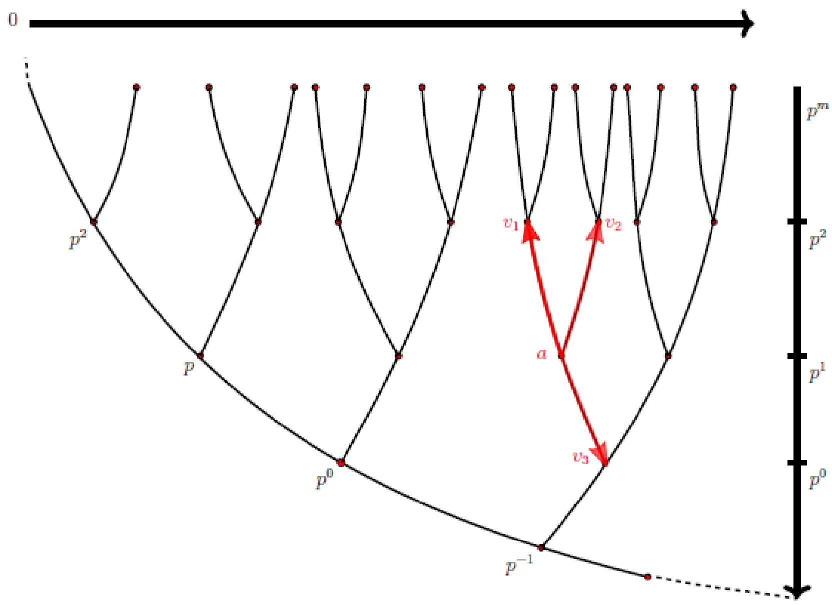

Figure 14. A junction of three Wilson lines. (from [13] Fig. 3).

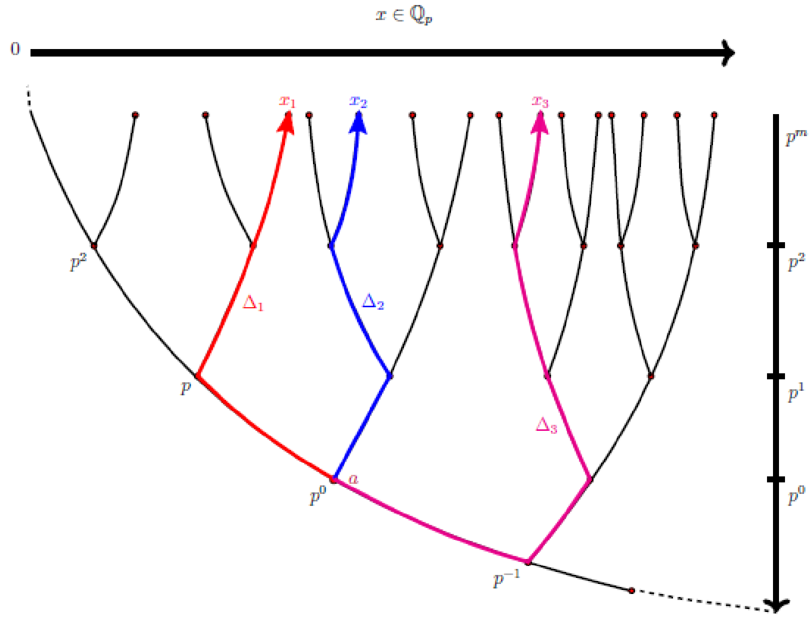

Figure 15. A network of three Wilson lines connecting three boundary points and joining at a. (from [13] Fig. 4).

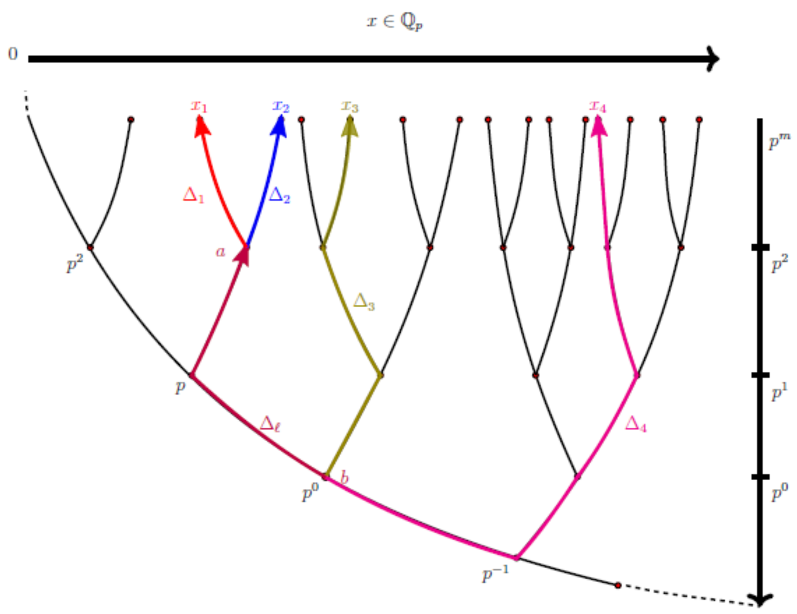

Figure 16. A network of four Wilson lines with two intermediate junctions. (from [13] Fig. 5).

2.3.5 4-point function

For more points, there will have different connecting ways, while the other procedures are exactly the same as above, here we will briefly list the possible permutations and the results for 4-point Wilson line junction. One figure is cited for illustration (Figure 16). Recall the corollary of **Def. 2**, only two configurations are possible:

1. (1,2,3):$|x_{12}|_p \leq |x_{13}|_p = |x_{23}|_p \leq |x_{14}|_p = |x_{24}|_p = |x_{34}|_p$,
2. (1,1,4): $|x_{12}|_p$, $|x_{34}|_p \leq |x_{13}|_p = |x_{14}|_p = |x_{23}|_p = |x_{24}|_p$.

We focus on the first situation here. To point out, for this configuration, we still have three channels according to the permutation between 1, 2 and 3. But we only need to analyze one channel, say s-channel:

$$|x_{12}|_p < |x_{13}|_p = |x_{23}|_p < |x_{14}|_p = |x_{24}|_p = |x_{34}|_p. \tag{2.111}$$

Then the junction configuration can be expressed as:

$$\left[\prod_{i=1}^{2} \hat{\mathfrak{W}}_{\Delta_i}(v_a \to v_i)\right] \left[\sum_l \hat{\mathfrak{W}}_{\Delta_l}(v_b \to v_a)\right] \left[\prod_{i=3}^{4} \hat{\mathfrak{W}}_{\Delta_i}(v_b \to v_i)\right], \tag{2.112}$$

where l runs over all the primary fields that connect from $\mathcal{O}_1$ to $\mathcal{O}_2$. Since the intermediate propagation has no boundary state, it has its own basis rather than X_i, say Y basis. We have

$$\left[\prod_{i=1}^{2} \hat{\mathfrak{W}}_{\Delta_i}(v_a \to v_i)\right] = \left[\prod_{i=1}^{2} \int_{\mathbb{Q}_p} \mathrm{d}X_i |X_i; \Delta_i\rangle\right] \left[\prod_{i=1}^{2} \hat{\mathfrak{W}}_{\text{fund}}(v_a \to v_i)\right] \left[\prod_{i=1}^{2} \langle X_i; \Delta_i|\right], \left[\sum_l \hat{\mathfrak{W}}_{\Delta_l}(v_b \to v_a)\right] = \sum_l \tag{2.113}$$

As before, $|X_i;\Delta_i\rangle$ can be contracted with $\langle\Delta_i|$. But there are still four remaining $\langle X_i;\Delta_i|$, one $\langle Y;\Delta_l|$ and one $|Y;\Delta_l\rangle$. They can contract with the singlet $|\mathcal{S}\rangle$ as 3-point functions:

$$\langle Y;\Delta_l|\langle X_3;\Delta_3|\langle X_4;\Delta_4||\mathcal{S}\rangle\rangle = P_{l34}(Y,X_3,X_4) \equiv \langle \mathcal{O}_l(Y)\mathcal{O}_3(X_3)\mathcal{O}_4(X_4)\rangle, \tag{2.114}$$

and

$$\langle X_1;\Delta_1|\langle X_2;\Delta_2|Y;\Delta_l\rangle = \langle \mathcal{O}_1(X_1)\mathcal{O}_2(X_2)\tilde{\mathcal{O}}_l(Y)\rangle, \tag{2.115}$$

where $|Y;\Delta_l\rangle \equiv \tilde{\mathcal{O}}_l(Y)|\mathrm{vac}\rangle$. Thus the expectation value of the Wilson line network is

$$\begin{aligned} W_4 \equiv & \left[\prod_{i=1}^{4}\langle\Delta_i|\right]\left[\prod_{i=1}^{2}\hat{\mathfrak{W}}_{\Delta_i}(v_a\to v_i)\right]\hat{\mathfrak{W}}_{\Delta_l}(v_b\to v_a)\left[\prod_{i=3}^{4}\hat{\mathfrak{W}}_{\Delta_i}(v_b\to v_i)\right]|\mathcal{S}\rangle \\ =&\left[\prod_{i=1}^{4}\int_{\mathbb{Q}_p}\mathrm{d}X_i\langle\Delta_i|X_i;\Delta_i\rangle\right]\int_{\mathbb{Q}_p}\mathrm{d}Y\left[\prod_{i=1}^{2}\hat{\mathfrak{W}}_{\text{fund}}(v_a\to v_i)\right]P_{12\tilde{l}}(X_1,X_2,Y) \\ &\hat{\mathfrak{W}}_{\text{fund}}(v_b\to v_a)\left[\prod_{i=3}^{4}\hat{W}_{\text{fund}}(v_b\to v_i)\right]P_{l34}(Y,X_3,X_4) \\ =&c\sum_l\int_{\mathbb{Q}_p}\mathrm{d}Y P_{12\tilde{l}}(X'_1,X'_2,Y)P_{l34}(Y',X'_3,X'_4) \end{aligned} \tag{2.116}$$

with

$$X'_i \equiv \begin{cases} p^{n_i-n_a}Z + p^{-n_a}(x_i-x_a) & i=1,2 \\ p^{n_i-n_b}Z + p^{-n_b}(x_i-x_b) & i=3,4 \end{cases} \tag{2.117}$$

And

$$Y' \equiv p^{n_a-n_b}Y + p^{-n_b}(x_a-x_b), \tag{2.118}$$

with c the constant expressed by p,

$$c = p^{-\sum_{i=1}^{4}n_i\Delta_i}p^{n_a(\Delta_1+\Delta_2-\Delta_l)}p^{n_b(\Delta_3+\Delta_4+\Delta_l)}. \tag{2.119}$$

From equation 2.94,

$$P_{l34}(y,x_3,x_4) = \frac{C_{l34}}{|y-x_3|_p^{\Delta_l+\Delta_3-\Delta_4}|x_3-x_4|_p^{\Delta_3+\Delta_4-\Delta_l}|x_4-y|^{\Delta_4+\Delta_l-\Delta_3}}, \tag{2.120}$$

and

$$P_{12\tilde{l}}(x_1,x_2,y) = \frac{C_{12l}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_l}}\delta(x_2-y). \tag{2.121}$$

Therefore,

$$W_4 = p^{-N\sum_{i=1}^{4}\Delta_i}\sum_l\frac{C_{12l}C_{l34}}{|x_{12}|_p^{\Delta_1+\Delta_2-\Delta_l}|x_{13}|^{\Delta_l+\Delta_3-\Delta_4}|x_{14}|_p^{2\Delta_4}}. \tag{2.122}$$

Again, one can note that the result is independent to the positions of the vertices, it is natural since our background configuration is pure without any dynamics.

3 Discussion and outlook

In this section, we will give some comments for the results obtained above, with both their advantages and disadvantages introduced. After then, the suggestive and potential direction for future research will be displayed and the author would try to provide some explicit calculations for the next step research, with a few recent developments referenced.

There are three important core problems underlying the constructions we have obtained, which is also a main source for the following introduction of new directions.

1. As noted many times before, the p-adic CFT prohibits the common local derivatives and so that only pure gravity dual and tree structure can be allowed in our consideration. However, to equip spin or graviton degree of freedom, it is necessary to introduce a new formalism in which the descendants can be restored and the Virasoro algebra can be written down to explore the bulk dynamics. Since the symmetry group of correlators is $\mathrm{PGL}(2,\mathbb{Q}_p)$, it is then natural to ask if there exists a larger symmetry group and associated algebra to obtain a more general p-adic CFT? Namely, we hope to construct a more general formalism for p-adic CFT with spin structure.

2. A tree structure is not enough to compute the entanglement entropy, since the bulk length has no dependence on the position. One may ask how to add the loop corrections beyond the field theory living on BT tree. Equivalently, what physical effects would appear if we consider a dynamical edge with its length fluctuating in the bulk?

3. Although the p-adic CFT has been constructed for a general extension with even the case that $\mathbb{K}$ replaced to $\mathbb{Q}_p$ for the symmetry group, such extensions seem to be difficult in the corresponding bulk. [2] discussed the chordal distance on the fixed time slice and [4] showed the causal structure with fixed space-time layer. Nevertheless, none of them exactly specified the correlator behaviors on the BT tree when p^n extension considered. We can propose such a question that what sorts of correlators are sensitive to the particular extension we pick?

To answer these questions, we may first recall some developments from the recent literature [3, 4, 13–16].

3.1 Problem I

Although none of papers until know have tried the mathematical way to formulate a p-adic CFT with spin from OPE, it has been believed that a larger gauge group can be constructed with the fermion contained [13, 16]. [13] takes an analogous way, which is reconstruct the bulk Witten diagrams by Chern–Simons theory, as that in usual AdS/CFT duality. After the lattice gauge theory formulation, the same gauge invariance restored in p-adic AdS configuration, with only the gauge group replacement of $\mathrm{PGL}(2,\mathbb{Q}_p)$. This gives us another way (Wilson line network) to interpret the correspondence between the tensor network and p-adic CFT. While in [16], the authors directly equip a non-dynamical U(1) on the BT tree and construct the associated actions for bosons and fermions, with the nearest neighbor interaction considered. The bosonic action and fermionic action is

$$S_\phi = \sum_e |D\phi_e|^2 + \sum_a m^2|\phi_a|^2, \tag{3.1}$$

and

$$S_\psi = \sum_e [i\chi_e^* D\psi_e + i\chi_e D^*\psi_e^* + m\chi_e^*\chi_e] - \sum_a M\psi_a^*\psi_a, \tag{3.2}$$

respectively, where D is the covariant derivative conform to U(1) gauge transformation. Then again apply this in to our p-adic bulk-boundary propagators and the associated correlation function. One can obtain the result in which both bosonic and fermionic degrees of freedom are manifested. With additional kinematic term added, they can even discuss with which condition can they consider the dynamics of p-adic AdS/CFT. In this way, even though we still know nothing about the p-adic valued p-adic CFT, we can view what would happen when the p-adic AdS/CFT equipped with spin in another perspective, as [16] noted, we can compute the sign character to understand the interaction behavior of $\mathbb{Q}_p$. And so that the first new direction can be put for demystifying the first problem:
Idea I: As an analogous to [16], although a little radical, I have tried the introduction of a supersymmetric action which is a simplified WZW action:

$$S = \sum_e |DA_e|^2 + |DB_e^2| + \bar{\chi}_e \not{D}\chi_e + \text{mass term}, \tag{3.3}$$

where A a scalar and B a pseudo-scalar, with $A + iB$ a free complex scalar, χ a free Majorana fermion. In this way, even if the p-adic CFT does not have spin structure in itself, the supersymmetrization can introduce the weight representation with non-zero spin particles. Personally, it seems like an inspiring idea while the application to the bulk reconstruction suffers a lot.

1. Although this action is not very complicated, the equation of motion on the BT tree is very weird because of the strange supersymmetric correspondence under p-adic value. The mass formula which could be used to connect $1 - \Delta$ and σ, as (2.18) showed, have dependence to Δ term, I do not know why this could happen, or is that unsafe to apply supersymmetry in this discrete topology? Then if there is any more convincing improving way (e.g. non-dynamical SU(2))?

2. It seems it is necessary to obtain the sign characters to treat the correlation function, but I do not know what and how many sign characters this kind of symmetry could have.

Another way to reconcile the conflict between p-adic CFT and descendants is to mathematically formulate a more general p-adic CFT by considering a much more complicated p-adic valued p-adic CFT. In this way, it is natural to endow the metric $g_{\mu\nu}$ with dynamics and the graviton degrees of freedom, and also allow us to generalize the gauge group to a larger one in a Chern–Simons-like theory view.
Idea II This road seems full of thorns since only a very few references have covered such

topic. One of the most promising reference is [17] (actually, the p-adic valued CFT is just a meromorphic field theory), in which they noted, if we want to generalize the p-adic CFT with complex valued to a p-adic valued one, it is necessary to require the fields to be an analytic functions. While in this case, the common local derivative is still prohibited and if one wants to introduce descendants in, another type of derivative—"Vladimirov derivative" is required, which denotes as

$$D_p f(x) = \int_{\mathbb{Q}_p} \mathrm{d}y \frac{f(y) - f(x)}{|y - x|_p^2}. \tag{3.4}$$

This derivative is written in integration form, so that the computation can be done by analogy of Cauchy's contour integral. I am still learning this part and thus have not done any explicit calculation yet. My thought is that we can try to construct a CFT with Vladimirov derivative and extract some "p-adic Ward identity" and "p-adic quasi-parimary field". After then, since the bulk geometry has been generalized with dynamics, we can compute the mutual information and entanglement entropy in the p-adic bulk, which could give many insightful results compared to that of [14].

3.2 Problem II

In [15], the authors explicitly considered the edge length dynamics by varying the action with the length of the edge:

$$S_\phi = \sum_{\langle xy \rangle} \frac{(\phi_x - \phi_y)^2}{2a_{xy}^2} + \sum_x V(\phi_x), \tag{3.5}$$

where a_{xy} is the length of the edge xy, in previous discussion, we have assumed that $a_{xy} = 1$. With this new parameter added, it is natural to obtain a dependence on the edge length. Besides, they also considered a geometrical term in the total action so that the relation between the gravitational effect and edge length can also be explored. Still, this formalism is from the correction of the action rather than a general treatment on the BT tree. It just tell us what would happen if we consider such variation, while give no explanation why such new terms are allowed or how can they be equipped in the p-adic AdS/CFT. Therefore, [5, 14] rigorously analyzed the introduction of loop structure and entanglement entropy with various type of holographic quantity displayed (e.g. BTZ black hole). [5] even restored the RT formula of p-adic version. Both loop structure and the entropy measures of p-adic AdS/CFT are fascinating to be explored, we divide them to two subsubsections.

3.2.1 Higher genus p-adic AdS/CFT

Any local 2D CFT has consistent sorrelation functions, not only on the $\mathbb{P}^1(\mathbb{C})$, but also on any Riemann surface Σ_n of genus n. This motivates the exploration for higher genus p-adic effect of CFT, according to [5], for $\mathrm{SL}(2,\mathbb{C})$, which corresponds to the usual CFT, there is a subgroup $\mathcal{S}_\mathbb{C} \subset \mathrm{SL}(2,\mathbb{C})$, which called Schottky geoup. If $P_\mathcal{S}$ is the closure of the set of fixed points of $\mathcal{S}_\mathbb{C}$, then $\mathcal{S}_\mathbb{C}$ can freely act on the complement $\Omega_\mathcal{S} \equiv \mathbb{P}^1(\mathbb{C})\P_\mathcal{S}$. And therefore, the quotient

$$\Sigma_\mathcal{S}^\mathbb{C} \equiv \Omega_\mathcal{S}/\mathcal{S}_\mathbb{C} \tag{3.6}$$

is a compact complex manifold of genus n. If we replace $\mathbb{C}$ by $\mathbb{K}$, the resulting manifold is an algebraic curve and has a simple visualization with BT tree. Now we can define $\mathcal{S} \in \mathrm{PGL}(2, \mathbb{K})$. For any non-trivial $\gamma \in \mathcal{S}$, there is a unique $\mathbb{C}_\gamma$ connecting its fixed points. Let $\Delta_\mathcal{S}$ be the minimal tree containing all these $\mathbb{C}_\gamma$. Then the quotient graph is denoted by

$$\Sigma_\mathcal{S} \equiv \Delta_\mathcal{S}/\mathcal{S}. \tag{3.7}$$

This graph represents that the Mumford curve is the unique $(p^n + 1)$-valent topological graph that contracts to Σ. If we assign these graphs with the tensor network, we can explicitly construct the partition function and correlation functions, and so that the higher genus graph can be considered with some restrictions on tensor contracting and the form of correlation function.

Idea III We can compute the partition function with genus. I have tried the simplest $g = 1$ case. Defining

$$q \in \mathbb{K}^* \quad \text{with} \quad |q| < 1 \tag{3.8}$$

which is the element multiply with $\mathbb{K}^*$ as $q\mathbb{K}^*$ with $\mathcal{S}_q$ generated, where $\mathbb{K}^* \equiv \mathbb{K}\ \{0, \infty\}$. Then $E_q(\mathbb{K}) = \mathbb{K}^*/\mathcal{S}_q$ is an elliptic curve. Since all non-trivial elements of $\mathcal{S}_q$ have the same endpoints $\{0, \infty\}$, $\Delta_\mathcal{S}$ is just the geodesic $\mathcal{C}$ connecting 0 and ∞. Therefore, the geometry will contract to $\mathcal{C}/\mathcal{S}$, which is a loop with $k = -\log_p |q|$ vertices.

Now we assume that we can insert $|\mathbf{1}\rangle$ on all boundary legs, such legs will collapse and only the network with $\mathcal{C}/\mathcal{S}$ will be left. Thus

$$\mathcal{Z}_{\mathcal{C}/\mathcal{S}} = \sum_a \langle a|\mathcal{G}^k|a\rangle = \sum_a |q|^{\Delta_a}. \tag{3.9}$$

This is just the simplified version of usual partition function of complex 2d CFT with all descendants vanished[8]. And then the correlation function can be computed (I have not done), but for two or more genus, maybe the torus sewing and cutting technology should be applied like in string theory.

3.2.2 Entanglement entropy of p-adic AdS/CFT

We can do many things if the entropy and RT formula can be defined in the p-adic AdS/CFT, such as the purification and entanglement distillation [20, 21], entanglement wedge cross section [18, 19], the measures of entanglement for mixed states [22] and the kinematic space on the BT tree of p-adic AdS [23], etc. However, as [14] themselves and [4] noted, the behavior of entropy described by [5, 14] is a function which relies on the choice of region in the dual tensor network, which is in contrast to the natural case where the entropy is a function of boundary points and configurations of points. Consider that there is no dynamics inside the bulk, it is convincing that their results can be at most effective to to investigate the information under sub-AdS scale. To conclude, I have two ideas for reconciling this conflict:

[8] Also, the central charge c has be set to zero, but this is a point I have not convinced myself why it could happen. Hope to receive some comments.

Idea IV

1. Still take the way as described in **Idea II**, to contain dynamics and non-trivial AdS configuration in the CFT construction, the p-adic valued CFT should be considered first and then the relatively correct computation of entanglement entropy can be naturally given, with a much richer field theory exhibited.

2. Choose to believe the entropy have been derived, but with application range (sub-AdS). Then we can follow [13] in which the Wilson line network have been argued as another perspective for explaining the reconstruction and boundary correlation functions. And in usual AdS/CFT situation, one also usually meets the computation of holographic quantities such as minimal surface and relative entropy with Wilson lines. Thus it is natural to develop this network to calculate the entropy and then compare to the results obtained from sub-AdS scale [20]. I am recently computing the direct results from 2-point function and 3-point function, which corresponds to a single line connect two endpoints and a junction of three endpoints, respectively, as an analogy of [5, 14] in the Wilson line network. The computation is a little complicated and I have not done all of them, I will try to report these results in two weeks.

As for the problem 3, I am too tired to explore and it may be put for further consideration. This proposal will end with other minor ideas for future direction.

1. It is pointed out that we can explore to what extent the error correcting properties of tensor network can directly manifest themselves in the Wilson line network [13].

2. As a realization of p-adic CFT, tensor network have also been discovered as a product induced from kinematic space. Although there are many differences between such picture to p-adic case here (such as the tensor network induced from kinematic space is commonly the MERA [23] which has fixed time slice, while in p-adic AdS/CFT the non-trivial correlation function can only be reproduced on the BT tree with fixed space-time layer other than MERA [4], besides, kinematic space requires the network has a dynamical length, which is also a defect of p-adic AdS/CFT), the perspective from Wilson line network may provide us some insights about the internal connections between different networks. I am also interested in this question, if possible, I do hope that I can clearly discuss this with Dr. Zukowski. I wonder how the spin structures emerged in kinematic space.

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