An Introduction to Exceptional Field Theory

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1 Exceptional Field Theory

It is well-known that the M-theory, where the "M" refers to magic, mother, mystery, etc., could be a unified theory with both five types of ten-dimensioanl superstring (I, IIA, IIB, $E_8 \times E_8$, SO(32) $\times$ SO(32)) and eleven-dimensional supergravity (SUGRA) encoded. M-theory is an elegant unification while many features of it are to be revealed. There is a so-called U-duality which believes to be a combination of S- and T-duality emerged in common string theory. In particular, if one considers a low-energy effective version of M theory, namely a maximal SUGRA, this duality can further be understood as some hidden symmetries including S- and T-duality appear upon the toroidal compactification is taken.

More specifically, consider toroidal compactify the eleven-dimensional SUGRA on T^d (with d the internal dimension), the global symmetry exhibits the transforming way obeying exceptional group $E_{d(d)}(\mathbb{R})$, where $E_{d(d)}$ is a split form of E_d group[1], we will provide more details in **sec.XX**. At the same time, M-theory is believed to be invariant under $E_{d(d)}(\mathbb{Z})$ and this fact implies that the exceptional field theory is a natural candidate to construct a manifestly duality covariant formulation of maximal SUGRA and further develops a full string theory, or M-theory, includes duality-complete multiplets *prior* to any compactification. For example, consider both eleven-dimensional SUGRA and type II SUGRA, one can do toroidal compactifications on T^6 and T^5, respectively, to achieve a $D = 5$ SUGRA which keeps maximal supersymmetry (SUSY) and global $E_{6(6)}$ group symmetry. Then one can explore 5D SUGRA by analyzing the geometry and related spectrum from $E_{6(6)}$-based field theory, which is what we called exceptional field theory (ExFT).

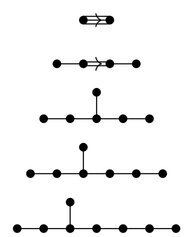
Before go further into the simplest example (5D $E_{6(6)}$ ExFT), we first provide a brief introduction to the exceptional group theory in mathematical perspective.

[1]For convenience, $E_{d(d)}$ denotes $E_{d(d)}(\mathbb{R})$ unless specified.

1.1 Exceptional Group

Exceptional group[2], usually denotes as E_n, is a compact[3] simple Lie group whose Dynkin diagram corresponds to one of the four fundamental rank-2 simple Lie algebra, namely G_2, when $n = 2$. More specifically, because of the severe restriction that the only possible angles between simple roots are $\pi/2$ (no line among two roots), $2\pi/3$ (single line between two roots), $3\pi/4$ (double line) and $5\pi/6$ (triple line) from Dynkin diagram, no loops of any kind are allowed and the number of lines emanating from any vertex must be less than 4. This implies that there are five diagrams are possible (including G_2) to consist an additional classical series of simple Lie algebras, which called E_n algebra. Their Dynkin diagrams are depicted as follow:

$E_2 = G_2$:
$E_4 = F_4$:
E_6:
E_7:
E_8:

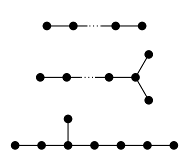


This type of Lie algebra, which has a bifurcating graph with three branches, is classified as E type (except G_2 and E_4 (also denotes as F_4), these two are classified as G- and F-type, respectively) in ADE classification.

1.1.1 ADE Classification

ADE classification is a classification for simply laced (single line between two simple roots) Dynkin diagrams of A-, D- and E-type. These three types of simple Lie algebra correspond to SU(n), SO($2n$) and $E_n (n = 6, 7, 8)$, respectively. Their Dynkin diagrams are depicted as follow:

A_n:
D_n:
E_n:



And the classification provides a correspondence between the Dynkin diagrams to the finite subgroups of $SO(3)$ and to the simple Lie group. The following table displays

[2]Actually, one should note that in many cases physicists concern, we are talking about the Lie algebra while refer as "Lie group". And as a physics student, I will not rigorously distinguish these two notions unless it is necessary.

[3]For our interests, we will always assume that the Lie groups we are talking about in this note is compact and omit this word unless specified.

some remarkable correspondences with all E_n groups included.

Dynkin diagram/Dynkin Quiver	**Finite subgroups of** $\mathrm{SO}(3)$	**simple Lie group**
A_1	$\mathbb{Z}_2$	$\mathrm{SU}(2)$
$A_3 \simeq D3$	$\mathbb{Z}_4$	$\mathrm{SU}(4) \simeq \mathrm{SO}(6) \cong Spin(6)$
D_4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathrm{SO}(8) \cong Spin(8)$
$D_5 \simeq E_5$	order 6 dihedral group	$\mathrm{SO}(10) \cong Spin(10)$
E_6	tetrahedral group T	$E6$
E_7	octahedral group O	$E7$
E_8	icosahedral group I	$E8$

These structures are naturally aspects of the description of string theory KK-compactified on orbifolds with ADE singularities of the form $\mathbb{C}^n/\Gamma$ for Γ a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. Such description plays an important role in suiting standard model into M-theory symmetry breaking (from E_8 to $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$).

1.1.2 Cartan Matrix

With the Dynkin diagram of E_n has been explicitly displayed, one can easily write down its Cartan matrix and the related (simple) root basis. We first generally write down the Cartan matrix of E_n.

$$Cart(E_n):\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & -1 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 2 \end{pmatrix}, \tag{1}$$

Accordingly, a natural choice of simple roots is given by the rows of the following matrix:

$$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix} \tag{2}$$

1.1.3 Split Form and Representations

There is an unique complex Lie algebra of type E_8 with complex dimension 248. But for the sake of simplicity, we are accustomed to treat the real forms of complex Lie algebra, in which one called **split real form** plays the most important role in this note.

Definition 1.1.1: Given a field K and an algebraic group G over K, and given a field extension $k \hookrightarrow K$, then a k-form of G is an algebraic group G_k over k such that its base change to K yields G:

$$\mathrm{Spec}(K) \times_{\mathrm{Spec}(k)} G_k \simeq G. \tag{3}$$

Example: $\mathbb{R} \hookrightarrow \mathbb{C}$ and $G = \mathbb{C}^\times$. Then,

$$\mathfrak{g}_\mathbb{R} \times_\mathbb{R} \mathbb{C} \simeq \mathfrak{g}, \tag{4}$$

where $\mathfrak{g}$ is the Lie algebra of $G_\mathbb{R}$, it is a $\mathbb{R}$-form of G given by $\mathbb{R}^\times$, which is called *real form.*

Definition 1.1.2: A real form $\mathfrak{g}_\mathbb{R}$ of a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ is said to be *split* if in each Cartan decomposition $\mathfrak{g}_\mathbb{R} = k_\mathbb{R} \oplus p_\mathbb{R}$, where $p_\mathbb{R}$ the space contains a maximal abelian subalgebra of $\mathfrak{g}_\mathbb{R}$.

Remark Every complex semisimple Lie algebra $\mathfrak{g}$ has a split real form up to an isomorphism. [Cartan]

Remark Over an algebraically closed field, all semisimple Lie algebras have split real forms. Since the field we are considering, namely the complex number field, is algebraically closed, for all the E_n groups, we only need to consider its split real form and the related subgroups and representations (rep.).

The split real form of E_6, E_7 and E_8 are listed below:

Split real forms	Dimension	Real rank	Maximal compact group
$E_{6(6)}$	78	6	$Sp(4) \times \mathbb{Z}_2$
$E_{7(7)}$	133	7	$\mathrm{SU}(8) \times \mathbb{Z}_2$
$E_{8(8)}$	248	8	$Spin(16)/\mathbb{Z}_2$

Example $E_{6(6)}$. The Lie algebra of $E_{6(6)}$has dimension 78, so we can denote the generators of its Lie algebra as g_α with adjoint index $\alpha = 1, \cdots, 78$. There are two fundamental rep.[4] of dimension 27 of which we denote **27** for its vectors (upper indices) and $\overline{\mathbf{27}}$ for its covectors (lower indices). The Cartan–Killing form used to raise and lower indices is defined as $\kappa_{\alpha\beta} \equiv (g_\alpha)_M^{\ N}(t_\beta)_N^{\ M}$.

[4]Some authors may also refer these two as one fundamental and one anti-fundamental rep.

In the fundamental rep., there are two cubic $E_{6(6)}$-invariant tensors d_{MNK} and d^{MNK} which contracted by $d_{MPQ}d^{NPQ} = \delta^N_M$. To smoothly reach the adjoint rep. from the fundamental rep., one can introduce a projector satisfies

$$\mathbb{P}^{M}{}_{N}{}^{K}{}_{L} \equiv (g_\alpha)_N{}^M (g^\alpha)_L{}^K = \frac{1}{18}\delta^M_N\delta^K_L + \frac{1}{6}\delta^K_N\delta^M_L - \frac{5}{3}d_{NLR}d^{MKR}, \tag{5}$$

and its self-contraction equals the dimension of $E_{6(6)}$algebra, namely $\mathbb{P}^{M}{}_{N}{}^{N}{}_{M} = 78$.

Hull and Townsend pointed out [1], it is the split real form that accounts for the gauge groups of SUGRA, as the epilogue of this lengthy voyage in group-theoretical land and the prologue of the funny gravitational sky-seeing, we attach the tables of [1] to show the relation between exceptional groups and supergravity and M-theory.

Groups	SUGRA gauge group (split real form)	T-duality group	U-duality (M-theory)	Maximal gauged supergravity
	$\mathrm{SL}(2,\mathbb{R})$	$\mathbf{1}$	$\mathrm{SL}(2,\mathbb{Z})$ S-duality	10D IIB SUGRA
	$\mathrm{SL}(2,\mathbb{R}) \times O(1,1)$	$\mathbb{Z}_2$	$\mathrm{SL}(2,\mathbb{Z}) \times \mathbb{Z}_2$	9D SUGRA
$\mathrm{SU}(3) \times \mathrm{SU}(2)$	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$	$O(2,2;\mathbb{Z})$	$\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$	8D SUGRA
$SU(5)$	$\mathrm{SL}(5,\mathbb{R})$	$O(3,3;\mathbb{Z})$	$\mathrm{SL}(5,\mathbb{Z})$	7D SUGRA
$Spin(10)$	$Spin(5,5)$	$O(4,4;\mathbb{Z})$	$O(5,5;\mathbb{Z})$	6D SUGRA
E_6	$E_{6(6)}$	$O(5,5;\mathbb{Z})$	$E_{6(6)}(\mathbb{Z})$	5D SUGRA
E_7	$E_{7(7)}$	$O(6,6;\mathbb{Z})$	$E_{7(7)}(\mathbb{Z})$	4D SUGRA
E_8	$E_{8(8)}$	$O(7,7;\mathbb{Z})$	$E_{8(8)}(\mathbb{Z})$	3D SUGRA

表 1: Duality symmetries for type II string compactified to D dimensions.

1.2 5D Maximal SUGRA

The maximal SUGRA stands for a supergravity preserving the maximal number of supercharges, which means has maximal SUSY. And such SUGRAs are usually obtained from torus reduction of 11D or IIB SUGRA. Cremmer *et al.* [2] first showed that the equations of motion of SO(8) SUGRA action inherits a hidden $E_{7(7)}$ symmetry when compactified to 4D. Motivated from this fact, many researchers conjectured [1,3-5] that the D-dimensional Lagrangian of SUGRA obtained from proper dimensional reduction can takes $E_{d(d)}$ (where $d = 11 - D$) invariant form, and the exceptional symmetry is a manifestation of U-duality of the full M-theory. Thus it is natural for us to consider higher reduction with higher rank exceptional group to reveal the mask of the M-theory/string theory. Here we start from the simplest non-trivial model—5D maximal

SUGRA with $E_{6(6)}$ from 11D SUGRA—and treat it as a warm-up.

After proper dualization of the degree of freedoms, the bosonic sector of the 5D Lagrangian takes the form

$$\begin{aligned}\mathcal{L} = R + \frac{1}{24}\partial_\mu \mathcal{M}_{MN}\partial^\mu \mathcal{M}^{MN} - \frac{1}{4}\mathcal{M}_{MN}F_{\mu\nu}{}^M F^{\mu\nu N} \\ + d_{KMN}F^M \wedge F^N \wedge A^K,\end{aligned} \tag{6}$$

where[5] $g_{\mu\nu}$ is the external metric field, $\mathcal{M}_{MN}$ is the internal metric field with all moduli and high-spin fields included, while $A_\mu{}^M$ stands for the vector fields which has dual two-form fields $B_{\mu\nu\ M}$, these fields are corresponding to the fluxes (field strength terms), and d_{KMN} is a symmetric $E_{6(6)}$ invariant tensor. Therefore, this Lagrangian is manifestly $E_{6(6)}$ invariant. The internal metric $\mathcal{M}_{MN}$ parametrizes the coset $E_{6(6)}/USp(8)$ and all internal coordinates are inherited in the Lagrangian so that the dependence on them are guaranteed. And this Lagrangian will accounts as our key element to understand SUGRA from exceptional field theory perspective.

To equip $E_{6(6)}$ symmetry on the whole manifold, there should have some fields which transform in rep. of $E_{6(6)}$. Note that, as we showed in sec. 1.1.3, the Lie algebra of $E_{6(6)}$ has dimension 78 and two fundamental rep. for vectors and covectors. Thus the 27 vector fields $A_\mu{}^M$ can transform in **27**, and decompose as

$$\begin{array}{ccccccc} \mathbf{27} & \to & \mathbf{6} & \oplus & \mathbf{15} & \oplus & \mathbf{6} \\ A_\mu{}^M & \to & A_\mu{}^m & \oplus & A_{\mu mn} & \oplus & A_{\mu klmnp} \\ & & \text{metric} & + & \text{3-form fields} & + & \text{6-form fields} \end{array}, \tag{7}$$

where the metric transforms under internal diffeomorphisms and 3-form, 6-form fields are controlled by tensor gauge transformations in internal spacetime. So that the gauge parameters affiliated to these decomposed fields infinitesimally transform as

$$\delta A_\mu^m = \partial_\mu \Lambda^m + \dots, \tag{8}$$

$$\delta A_{\mu mn} = \partial_\mu \Lambda_{mn} + \dots, \tag{9}$$

$$\delta A_{\mu klmnp} = \partial_\mu \Lambda_{klmnp} + \dots. \tag{10}$$

Similar to that of Kaluza–Klein (KK) compactification, one can combine all these gauge parameters into a single integrated coordinate

$$\left\{\begin{array}{c} \Lambda^m\left(x^\mu, y^m\right) \\ \Lambda_{mn}\left(x^\mu, y^m\right) \\ \Lambda_{klmnp}\left(x^\mu, y^m\right) \end{array}\right\} \Rightarrow \Lambda^M(x^\mu, Y^M), \tag{11}$$

[5]As a convention, all Greek indices stand for external components, run over 0 to 4, and all Roman indices denote the internal components, run over 5 to 10.

where the Y^M is just the collective coordinate which encodes the information of internal coordinates. It has rep. **27** of $E_{6(6)}$. Such coordinate is certainly invariant under usual diffeomorphisms and tensor gauge transformations, we combine these symmetries together to redefine a generalized diffeomorphism. Then with the perspective of differential geometry, as one usually comes across in general relativity and bosonic string theory, the field contents and relative symmetry-governed Lagrangian can be analyzed from a manifold (more precisely, for our example, the worldsheet) with generalized (principal) $E_{d(d)}$-bundle. Now let us treat the exceptional field theory in a geometrical view.

Standard diffeomorphisms have

$$D_\Lambda V^m = \Lambda^n \partial_n V^m - \partial_n \Lambda^m V^n. \tag{12}$$

Naively, one may conclude that the generalized diffeomorphisms have

$$D_\Lambda^g V^M {=} \Lambda^N \partial_N V^M - \partial_N \Lambda^M V^N, \tag{13}$$

where D_Λ and D_Λ^g denote the derivative transforming vector fields under usual diffeomorphism and generalized diffeomorphism, respectively. However, this guess is incorrect because $E_{6(6)}$ structure cannot be inherited in such derivative transformation. With the assistance of the concept of Lie derivative, it is not hard to conclude that the generalized derivative preserving $E_{6(6)}$ covariance and generalized diffeomorphisms is defined as

$$\begin{aligned}\mathcal{L}_\Lambda V^M {=} & \Lambda^N \partial_N V^M - \left[\partial_N \Lambda^M\right]_{\text{adj}} V^N \\ & = \Lambda^N \partial_N V^M - 6\left(\mathbb{P}\right)^{N\ \ M}_{\ \ P\ \ Q} \left(\partial_N \Lambda^P\right) V^Q + \lambda \partial_K \Lambda^K V^M.\end{aligned} \tag{14}$$

Here λ is an arbitrary density weight associated to a certain vector. Similarly, for covectors, the generalized derivative requires

$$\mathcal{L}_\Lambda W_M = \Lambda^N \partial_N W_M + 6\left(\mathbb{P}\right)^{N\ \ M}_{\ \ P\ \ Q} \left(\partial_N \Lambda^P\right) V^Q + \lambda' \partial_K \Lambda^K W_M. \tag{15}$$

To make the generalized derivative compatible with the $E_{6(6)}$algebra structure, the $E_{6(6)}$-invariant tensor should not be changed under $\mathcal{L}_\Lambda$, namely

$$\mathcal{L}_\Lambda d_{KMN} = 0 \Rightarrow \lambda_d = 0. \tag{16}$$

This implies that the weight of d-tensor is 0. Inserting the explicit expression of the adjoint projector, we have

$$\mathcal{L}_\Lambda V^M = \Lambda^N \partial_N V^M - \partial_N \Lambda^M V^N + \left(\lambda - \frac{1}{3}\right) \partial_K \Lambda^K V^M + 10 d_{PQR} d^{NMR} \partial_N \Lambda^P V^Q. \tag{17}$$

Thus the projector leads to an effective density weight as $\lambda_{eff} = (\lambda - \frac{1}{3})$ and will vanish once the weight of a vector is $\frac{1}{3}$. A remarkable observation of [3] is that, if W_M is a covariant tensor of weight $\frac{2}{3}$, the combination

$$d^{KMN}\partial_K W_M \equiv V^N \tag{18}$$

is a contravariant vector of weight $\frac{1}{3}$. This fact can be analogously understood as the covariance of generalized exterior derivative in p-form dynamics controlled by $E_{6(6)}$. Therefore, one can choose $\lambda = \frac{1}{3}$ as the weight of vectors so that no weight term will appear explicitly. This plays an important role in explicit analysis later.

On the other hand, in addition to the gauge invariance, the generalized derivative satisfies another constraint to keep its $E_{6(6)}$algebra to be closed. The generalized Lie derivatives form the algebra

$$[\mathcal{L}_{\Lambda_1}, \mathcal{L}_{\Lambda_2}] = \mathcal{L}_{[\Lambda_1,\Lambda_2]_E}, \tag{19}$$

where $[-,-]_E$ stands for E-bracket which is the M-theory analogue of the C-bracket in dual field theory [6], explicitly,

$$[\Lambda_1, \Lambda_2]^M_E = 2\Lambda^K_{[2}\partial_K\Lambda^M_{1]} - 10 d^{MNP} d_{KLP}\Lambda^K_{[2}\partial_N\Lambda^L_{1]}. \tag{20}$$

Furthermore, this E-bracket conforms to the generalized Jacobi identity where the Jacobiator is non-vanishing.

$$J(U,V,W) = [[U,V]_{\mathrm{E}}, W]_{\mathrm{E}} + [[V,W]_{\mathrm{E}}, U]_{\mathrm{E}} + [[W,U]_{\mathrm{E}}, V]_{\mathrm{E}} \neq 0 \tag{21}$$

This implies that the generalized Lie derivatives cannot form an associative algebra, but as [3] showed, this Jacobiator is a trivial gauge parameter like (18).And requirement (19) gives

$$Y^{MN}{}_{PQ}\partial_M \otimes \partial_N = 0, \tag{22}$$

here $Y^{MN}{}_{PQ} = d_{PQR}d^{RMN}$ is an $E_{d(d)}$ invariant tensor. This covariant consistency equation is called section constraint. With this constraint, the covariance can be rigorously formulated under the generalized diffeomorphism preserving $E_{6(6)}$. The section constraint is formulated generically. For certain sections (vector or tensor fields) with certain rep., the fiber bundle requires

$$Y^{MN}_{PQ}\partial_M\partial_N\Phi = 0$$
$$Y^{MN}_{PQ}\partial_M\Phi_1\partial_N\Phi_2 = 0$$

1.2.1 Breaking Branches

With the exceptional geometry constructed, now we are ready to uplift the gauged supergravity to the exceptional groups and related rep., for 5D SUGRA reduced from 11D SUGRA, $E_{6(6)}$break down as

$$E_{6(6)} \to \mathrm{SL}(6) \times \mathrm{SL}(2) \to \mathrm{SL}(6) \times \mathrm{GL}(1) = \mathrm{GL}(6),$$

with the fundamental rep. breaking as

$$\mathbf{27} \to \mathbf{6} \oplus \mathbf{15} \oplus \mathbf{6},$$

and the adjoint rep. breaking as

$$\mathbf{78} \to \mathbf{1} \oplus \mathbf{20} \oplus (\mathbf{1}+\mathbf{35}) \oplus \mathbf{20} \oplus \mathbf{1}.$$

While the coordinates explicitly split as

$$\{Y^M\} \to \{y^m, y_{mn}, y^{\bar{m}} = y_{klmnp}\}.$$

The section constraint can be solved by restricting the coordinate dependence of all fields to only y^m, namely

$$\{\partial_{\bar{m}} A = \partial^{mn} A = 0\}.$$

On the other hand, if we reduce from the type IIB SUGRA, there is another branch which can solve the section constraints.

$$E_{6(6)} \to \mathrm{SL}(6)\mathrm{SL}(2) \to \mathrm{GL}(5) \times \mathrm{SL}(2). \tag{23}$$

And the corresponding fundamental and adjoint rep. breaking as

$$\mathbf{27} \to (\mathbf{2},\mathbf{1}) \oplus (\mathbf{2},\mathbf{5}) \oplus (\mathbf{1},\mathbf{5}) \oplus (\mathbf{1},\mathbf{10}), \mathbf{78} \to (\mathbf{1},\mathbf{5}) \oplus (\mathbf{2},\mathbf{10}) \oplus (\mathbf{1}+\mathbf{15}+\mathbf{20}) \oplus (\mathbf{2},\mathbf{10}) \oplus (\mathbf{1},\mathbf{5}).$$

The coordinates explicitly split as

$$\{Y^M\} \to \{y^m, y_{m\alpha}, y^{mn}, y_\alpha\},$$

where $\alpha = 1, 2$ the SL(2) indices. Again, to preserve the invariance of d-tensor, the section constraints can be solved by restricting the dependence only on y^m

$$\{\partial^{m\alpha} A = \partial_{mn} A = \partial^\alpha A = 0\}.$$

The only rep. remained is $(\mathbf{1}, \mathbf{5})$.

Then we can take a glimpse on the field contents of ExFT. Define a covariant derivative on external space $\mathcal{D}_\mu = \partial_\mu - \mathcal{L}_{A_\mu}$. Usually, one obtains non-abelian fields by $F_{\mu\nu}{}^M = 2\partial_{[\mu} A_{\nu]}{}^M - [A_\mu, A_\nu]_E^M$ from the gauge connection A_μ^M, however, because of the non-vanishing Jacobiator of E-bracket, we know that this definition can hardly transform covariantly, it will introduce a term depends on the d-tensor. Thus we consider an additional correction term which has dependence on d^{MNK} and construct a covariant field strength as

$$\mathcal{F}_{\mu\nu}{}^M \equiv 2\partial_{[\mu} A_{\nu]}^M - [A_\mu, A_\nu]_{\mathrm{E}}^M + 10 d^{MNK} \partial_K B_{\mu\nu N}, \tag{24}$$

where $B_{\mu\nu N}$ is a two-form gauge connection dual to A_μ^M. The full p-form field content of $E_{6(6)}$field theory is thus given by

$$\left\{A_\mu^m, A_{\mu mn}, A_\mu^{\bar{m}}\right\}, \quad \left\{B_{\mu\nu\bar{m}}, B_{\mu\nu}^{mn}\right\}. \tag{25}$$

And this requires that there are 27 2-forms $B_{\mu\nu N}$, which is just what has presented in 5D SUGRA under usual KK reduction. The total field content can be conjectured to contain internal and external metric (or frame) fields and two dual p-form fields.

1.2.2 Lagrangian and Dynamics

To extract more information about the dynamics of ExFT, it is necessary to construct a $E_{6(6)}$-preserving covariant Lagrangian. Recall the 5D SUGRA Lagrangian (6). In [3], the authors have improved this to a generalized diffeomorphism invariant action as

$$\mathcal{L} = \widehat{R} + \frac{1}{24}\mathcal{D}_\mu \mathcal{M}_{MN} \mathcal{D}^\mu \mathcal{M}^{MN} - \frac{1}{4}\mathcal{M}_{MN}\mathcal{F}_{\mu\nu}^M \mathcal{F}^{\mu\nu N} + e^{-1}\mathcal{L}_{\text{top}} - V_{\text{pot}}\left(\mathcal{M}_{MN}, g_{\mu\nu}\right), \tag{26}$$

where $\hat{R}_{\mu\nu}^{ab} = R_{\mu\nu}{}^{ab} + \mathcal{F}_{\mu\nu}{}^M e^{a\rho}\partial_M e_\rho^b$,

$$S_{\text{top}} = \int d^{27}Y \int_{\mathcal{M}_0} \left(d_{MNK}\mathcal{F}^M \wedge \mathcal{F}^N \wedge \mathcal{F}^K - 40 d^{MNK}\mathcal{H}_M \wedge \partial_N \mathcal{H}_K\right) \tag{27}$$

and

$$\begin{aligned} V_{\text{pot}} =& \frac{1}{24}\mathcal{M}^{MN}\partial_M \mathcal{M}^{KL}\left(12\partial_L \mathcal{M}_{NK} - \partial_N \mathcal{M}_{KL}\right) \\ &- \frac{1}{2}g^{-1}\partial_M g \partial_N \mathcal{M}^{MN} - \frac{1}{4}\mathcal{M}^{MN} g^{-1}\partial_M g g^{-1}\partial_N g - \frac{1}{4}\mathcal{M}^{MN}\partial_M g^{\mu\nu}\partial_N g_{\mu\nu} \end{aligned}. \tag{28}$$

In this section, we will briefly introduce the spirit for constructing such Lagrangian.

Kinetic term As we have explained, the field content involves e_μ^a, $\mathcal{M}_{MN}$, A_μ^M and $B_{\mu\nu M}$. Here e_μ^a is the external five-dimensional frame field, which has weight $\lambda = \frac{1}{3}$. It is invariant under general gauge transformation. Since

$$\mathcal{D}_\mu e^a_\nu = \partial_\mu e^a_\nu - A^M_\mu \partial_M e^a_\nu - \frac{1}{3}\partial_M A^M_\mu e^a_\nu, \tag{29}$$

the transformation gives e^a_ν a weight-dependent term, so as to remove this dependence, the Riemann tensor must also be covariantized, we have

$$\hat{R}^{ab}_{\mu\nu} = R_{\mu\nu}{}^{ab} + \mathcal{F}_{\mu\nu}{}^{M} e^{a\rho}\partial_M e^b_\rho. \tag{30}$$

And we generalized to a new Einstein–Hilbert kinetic term

$$S_{EH} = \int d^5x d^{27}Y e\hat{R}, \tag{31}$$

where e is the determinant of volume form of five-dimensional external frame, it is a generic term inherited in all kinetic terms. Next consider that for internal metric $\mathcal{M}_{MN}$. This matrix parametrizes the coset space $E_{6(6)}/\mathrm{USp}(8)$ whose 42 coordinates describe the scalar fields of the theory. It transforms as a rank-2 symmetric tensor of weight $\lambda' = 0$. Consider also the fact that $\det \mathcal{M} = 1$, we can define the gauge invariant kinetic term

$$\mathcal{L}_{scalar} = \frac{1}{24} e g^{\mu\nu}\mathcal{D}_\mu \mathcal{M}_{MN}\mathcal{D}_\nu \mathcal{M}^{MN} = \frac{1}{24} e\mathcal{D}_\mu \mathcal{M}_{MN}\mathcal{D}^\mu \mathcal{M}^{MN}. \tag{32}$$

And the Yang–Mills kinetic term $-\frac{1}{4}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu M}\mathcal{F}^N_{\mu\nu}$. One can easily check that all terms have weight 1 in total, which also implies that $\mathcal{F}^M_{\mu\nu}$ carry a weight of 1/3 and consistent with our argument before.

Topological term This term requires a bunch of sandwiches which is gauge invariant only up to boundary terms and have no dependence on the metric. As usual, we start from the flux viewpoint with dual p-forms as the essential components. As an analogy to that of Chern–Simons theory, the topological term can be written as an integral of an exact 6-form as

$$\begin{aligned} S_{\text{top}} &= \int d^5x d^{27}Y \mathcal{L}_{\text{top}} \\ &= \kappa \int d^{27}Y \int_{\mathcal{M}_6} \left(d_{MNK}\mathcal{F}^M \wedge \mathcal{F}^N \wedge \mathcal{F}^K - 40 d^{MNK}\mathcal{H}_M \wedge \partial_N \mathcal{H}_K \right) \end{aligned}, \tag{33}$$

where $\mathcal{F}^M \equiv \frac{1}{2}\mathcal{F}^M_{\mu\nu}dx^\mu \wedge dx^\nu$, $\mathcal{H}_M \equiv \frac{1}{3!}\mathcal{H}_{\mu\nu\rho M}dx^\mu \wedge dx^\nu \wedge dx^\rho$ and κ is a constant which satisfies $\kappa^2 = \frac{5}{32}$.

Potential term The potential term is given without explicit reasoning in [3]. The only remarkable conclusion is that this potential is gauge invariant under Λ^M transformation and certainly also the generalized diffeomorphism.

The full action is invariant under diffeomorphism transformation on various fields:

$$\delta e_\mu^a = \xi^\nu \mathcal{D}_\nu e_\mu^a + \mathcal{D}_\mu \xi^\nu e_\nu^a \tag{34}$$

$$\delta \mathcal{M}_{MN} = \xi^\mu \mathcal{D}_\mu \mathcal{M}_{MN} \tag{35}$$

$$\delta A_\mu^M = \xi^\nu \mathcal{F}_{\nu\mu}^M + \mathcal{M}^{MN} g_{\mu\nu} \partial_N \xi^\nu \tag{36}$$

$$\Delta B_{\mu\nu M} = \frac{1}{16\kappa} \xi^\rho e \varepsilon_{\mu\nu\rho\sigma\tau} \mathcal{F}^{\sigma\tau N} \mathcal{M}_{MN}, \Delta_\Lambda = \delta_\Lambda - \mathcal{L}_\Lambda. \tag{37}$$

Every term of the full action is separately invariant under internal diffeomorphism and there are some relative coefficients uniquely fixed by external diffeomorphism $\xi^\mu(x, Y)$. Then re-embed the solutions of section constraints, the theory encoded by this action coincides with the full 11D/IIB SUGRA[6].

Therefore, a manifestly duality covariant formulation of maximal 5D SUGRA in ExFT perspective has been successfully achieved. With the exceptional group algebra (d-tensors, gauge connections, curvatures), one can build dictionaries between ExFT controlled by the Lagrangian (26) and 11D and IIB SUGRA. Furthermore, as Cremmer showed and we have discussed at the beginning of this note, a 5D maximal gauged SUGRA can be obtained from 11D/IIB SUGRA by toroidal compactification over T^6/T^5. Now that we can intuitively describe 5D maximal SUGRA from ExFT by Scherk–Schwarz (SS) reduction (also over a warped six-dimensional manifold), it means one can make the symmetry enhancement after toroidal reduction become manifest directly from field theory perspective. It is pointed out that a IIB SUGRA can spherically compactify as $AdS_5 \times S^5$ to reach a 5D maximal SUGRA with gauge group SO(6) and global $E_{6(6)}$[7], which allows researchers to demystify a more compact description of complicated reductions. This is called consistent truncation in literatures. Nonetheless, because of the time and knowledge limit, the author has not understood all of the key spirits of this theory. The story of exceptional field theory will stop here and the author hope this excursion to the frontier of field-theoretically described SUGRA does not disappointed you.

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[6] One can examine this fact by extracting the equation of motion of each term. The details are pretty cumbersome and we left as exercises for both readers and the author himself, interested but lazy ones can directly go to section 3 and 4 of [3] for having fun.

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