Calabi–Yau Compactification

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Critical superstring theory requires the spacetime dimensions to be 10, or maybe 11 in strong coupling limit. in order to get 10 down to 4, the most straightforward possibility is that six or seven of the dimensions are compactified on an internal manifold, whose size is sufficiently small to have escaped detection. In the mid 1980s Calabi–-Yau manifolds were first considered for compactifying the six extra dimensions, and they were shown to be phenomenologically rather promising. In contrast to the circle, they do not have isometries, and part of their role is to break symmetries rather than to make them. The second way to deal with the extra dimension is the brane world scenario where the gauge fields as the ends of the open strings are confined on the D3 brane while gravity as the massless modes of closed strings could escape to higher dimensional space. In this note, we would focus on the former case.

1.1 Complex manifold

The definition of complex manifold is similar to the definition of real manifold except that there are some additional restrictions for complex manifolds. Briefly speaking, an n-dimensional complex manifold is locally $\mathbb{C}^n$ and the transition function of different coordinate charts is holomorphic.

One can also start from an almost complex manifold which is a real $2n$-dimensional manifold M with an almost complex structure $J : TM \to TM$ satisfying $J^2 = -I$.

A necessary and sufficient condition for J to be a complex structure is that the almost complex manifold is integrable, which is equivalent to the condition that the Nijenhuis tensor vanishes. Nijenhuis tensor $N : TM \otimes TM \to TM$ is given by

$$N(X,Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]. \tag{1}$$

In local coordinates it reads

$$N^a_{bc} = J^d_b \partial_{[d} J^a_{c]} - J^d_c \partial_{[d} J^a_{b]}. \tag{2}$$

An n-form in real manifold with complex structure could be decomposed as

$$\Omega^n(M) = \bigoplus_{p+q=m} \Omega^{p,q}(M), \tag{3}$$

where a (p,q) form $\omega^{p,q}$ in $\Omega^{p,q}$ is a complex differential form with p holomorphic pieces and q anti-holomorphic pieces. In local coordinates, we can write

$$\omega^{p,q} = \omega^{p,q}_{a_1\cdots a_p \bar{b}_1 \cdots \bar{b}_q} dz^{a_1} \wedge \cdots \wedge dz^{a_p} \wedge d\bar{z}^{\bar{b}_1} \wedge \cdots \wedge d\bar{z}^{\bar{b}_q}. \tag{4}$$

The exterior derivative $d : \Omega^n \to \Omega^{n+1}$ has a decomposition as well:

$$d = \partial + \bar{\partial} \tag{5}$$

where

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), \bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M). \tag{6}$$

Then $d^2 = 0$ gives $\partial^2 = \bar{\partial}^2 = \bar{\partial}\partial + \partial\bar{\partial} = 0$. The integrability condition is equivalent to $\bar{\partial}^2 = 0$. Since $\bar{\partial}^2 = 0$, we can define Dolbeault cohomology by

$$H^{p,q}_{\bar{\partial}}(M) = \frac{\text{Ker}\left(\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}\right)}{\text{Im}\left(\bar{\partial} : \Omega^{p,q-1} \to \Omega^{p,q}\right)}, \tag{7}$$

which is isomorphic to $H^q(\Omega^p(M))$ (Čech-Dolbeault).

1.2 Kähler Manifold

A Hermitian metric of a complex manifold is a positive definite inner product $g : TM \otimes \bar{T}M \to \mathbb{C}$ (here we are using the notation $T_{\mathbb{R}}M \otimes \mathbb{C} = TM \oplus \bar{T}M$). In local coordinates we can write $g = g_{a\bar{b}} dz^a \otimes d\bar{z}^{\bar{b}}$.

The Hermitian condition for a real manifold with complex structure is

$$g(X,Y) = g(JX, JY). \tag{8}$$

In terms of the components J^a_b, we have $J_{ab} = -J_{ba}$. Therefore we can define a two form

$$\omega = \frac{1}{2} J_{ab} dx^a \wedge dx^b. \tag{9}$$

In complex coordinates we can build a $(1,1)$ form:

$$\omega = \frac{i}{2} g_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}. \tag{10}$$

If ω is closed, say, $d\omega = 0$, ω is called a Kähler form and the corresponding manifold M is a Kähler manifold. Locally one can find a Kähler potential K such that $\omega = i\partial\bar{\partial}K$.

One important consequence of Kählerian condition is found by calculating Laplacians. The adjoint of an operator is defined via inner products, i.e., $\langle \alpha, \partial\beta \rangle = \left\langle \partial^\dagger \alpha, \beta \right\rangle$, where α is a (p, q) form and β is $(n + 1 - p, n - q)$ form. The Laplacians are given by

$$\Delta_\partial = \partial\partial^\dagger + \partial^\dagger\partial, \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}. \tag{11}$$

For Kähler manifold, we have

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}. \tag{12}$$

Therefore we have a decomposition of de Rham cohomology into $\Delta_{\bar{\partial}}$- cohomology, which could be represented by Dolbeault cohomology:

$$H^r(M) = \bigoplus_{p+q=r} H^{p,q}(M). \tag{13}$$

The Hodge numbers and Betti numbers are given by

$$h^{p,q}(M) = \dim H^{p,q}(M), b_r(M) = \dim H^r(M) = \sum_{p+q=r} h^{p,q}(M). \tag{14}$$

Further, we have $h^{p,q} = h^{n-p,n-q}$ given by Hodge dual, and $h^{p,q} = h^{q,p}$ by complex conjugation.

1.3 Calabi-Yau manifold

The Calabi–Yau theorem was first conjectured by Calabi and then proven by Yau. The theorem is powerful in that it's quite hard to determine whether M admits a Ricci flat metric, while the first Chern class is relatively simpler to compute.

A Calabi–Yau n-fold M is a compact (sometimes compactness is not a requirement), complex, Kähler manifold with one of the following equivalent conditions:

1. The canonical bundle of M trivial (M has vanishing first Chern class);
2. M has nowhere-vanishing holomorphic n-form;
3. M is Ricci flat;
4. M has Kähler metric with global holonomy contained in $SU(n)$.

Now that we have learned some mathematical properties of Calabi–Yau manifold, one may ask how the Calabi–Yau manifold arise in string theory compactification. Consider heterotic string compactification as in [5]. Assuming the 10 dimensional spacetime decomposes as $M_{10} \cong M_4 \times M_{int}$ where M_4 is 4 dimensional non-compact spacetime and M_{int} be the internal compact space. We impose the following physical requirements:

(1) M_4 is maximally symmetric spacetime;

(2) There should be an unbroken $\mathcal{N}=1$ supersymmetry in four dimensions;

(3) The gauge group and fermion spectrum should be realistic.

The field content of heterotic supergravity is the graviton G_{MN}, the dilaton ϕ,the 2 form B_2, the gravitino Ψ_M, the dilatino λ and the gauge field transforming in the adjoint representation of $\mathrm{Spin}(32)/\mathbb{Z}_2$ or $E_8 \times E_8$.

The bosonic parts of the supersymmetry transformation of the fermi fields in heterotic supergravity is given by

$$
\begin{aligned}
\delta\Psi_M &= \nabla_M \varepsilon - \frac{1}{4}\tilde{H}^{(3)}_M \varepsilon, \\
\delta\lambda &= -\frac{1}{2}\Gamma^M \partial_M \Phi \varepsilon + \frac{1}{4}\tilde{H}^{(3)}\varepsilon, \\
\delta\chi &= -\frac{1}{2}F^{(2)}\varepsilon.
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
&\tilde{H}_3 = dB_2 + \frac{l_s^2}{4}\omega_3, \quad \omega_3 = \omega_L - \omega_{YM}, \\
&\omega_L = \mathrm{tr}\left(\omega \wedge d\omega + \frac{2}{3}\omega^3\right), \quad \omega_{YM} = \mathrm{tr}\left(A \wedge dA + \frac{2}{3}A^3\right),
\end{aligned}
\tag{16}
$$

and $\tilde{H}^{(3)}_M$, $\tilde{H}^{(3)}$ and $F^{(2)}$ are constructed by contracting the field strength with 10 dimensioanl Dirac matrices:

$$
\begin{aligned}
\tilde{H}^{(3)}_M &= \frac{1}{2}\tilde{H}_{MNP}\Gamma^{NP} \\
\tilde{H}^{(3)} &= \frac{1}{3!}\tilde{H}_{MNP}\Gamma^{MNP} \\
F^{(2)} &= \frac{1}{2}F_{MN}\Gamma^{MN}
\end{aligned}
\tag{17}
$$

Also χ is the gaugino in the super Yang Mills multiplet.

The condition for unbroken $\mathcal{N}=1$ supersymmetry in 4 dimensions requires that for every field α, $\delta_\epsilon \alpha = 0$. This is automatically true for bosonic fields since the background fermi field vanish. That's why we only write the transformation rule for fermi fields. $\delta\ ambda = 0$ implies $\tilde{H}^{(3)} = 0$. Therefore,

$$
\nabla_M \varepsilon = 0. \tag{18}
$$

This is a Killing spinor equation and the solution ϵ is a covariant constant spinor.

Since the ten dimensional spacetime is a direct product, we can decompose the spinor into a product

$$
\varepsilon(x, y) = \zeta(x) \otimes \eta(y). \tag{19}
$$

We make the convention that the spacetime indicies take the value $M = (\mu, m)$. So the Killing equation in 4 dimensions takes the form

$$\nabla_\mu \zeta = 0. \tag{20}$$

Since $[\nabla_\mu, \nabla_\nu]\,\zeta = \frac{1}{4}R_{\mu\nu\rho\sigma}\gamma^{\rho\sigma} = 0$, along with the assumption that the 4 dimensional spacetime is maximally symmetric, we conclude that M_4 is just Minkowskian.

The existence of the covariant constant spinor in internal manifold implies that the internal space is Ricci flat, namely, $R_{mn} = 0$.

Now we got the Ricci flat condition, the next step is to show the internal manifold is complex and Kähler.

A 6 dimensional spinor has 8 real components which can be decomposed into two $SU(4)$ irreducible representations $\mathbf{8} = \mathbf{4} \oplus \bar{\mathbf{4}}$ where $\mathbf{4}$ and $\bar{\mathbf{4}}$ represents spinors with opposite chirality. Since a covariant constant spinor remains unchanged after parallel transported around the loop, we argue that the internal manifold has $SU(3)$ holonomy. The group theoretic explanation is that $\mathbf{4}$ has an $SU(3)$ decomposition $\mathbf{4} = \mathbf{3} \oplus \mathbf{1}$. We denote the singlet pieces of $\mathbf{4}$ and $\bar{\mathbf{4}}$ by $\eta_\pm(y)$ and they are properly normalized.

The almost complex structure could be constructed by

$$J_m^n = i\eta_+^\dagger \gamma_m^n \eta_+ = -i\eta_-^\dagger \gamma_m^n \eta_- \tag{21}$$

satisfying $J_m^n J_n^p = -\delta_m^p$, and the almost complex structure is also convariantly constant following the property of convariant constant spinor and the metric. Therefore the Nijenhuis tensor vanishes and the internal manifold is complex. We can introduce the local coordinates z^a and $\bar{z}^{\bar{a}}$ in terms of which

$$J_b^a = i\delta_b^a, \quad J_{\bar{b}}^{\bar{a}} = -i\delta_{\bar{b}}^{\bar{a}}, \quad J_{\bar{b}}^a = J_b^{\bar{a}} = 0 \tag{22}$$

We find $J_{a\bar{b}} = ig_{a\bar{b}}$.

So $dJ = (\partial + \bar{\partial})J = 0$ and we conclude that the manifold is Kähler.

To summarize, the three physical requirements we listed above implies that the internal manifold for heterotic string theory compactified to 4 dimensions is a Calabi-Yau 3-fold.

The holomorphic 3-form could be constructed from fermion bilinears:

$$\begin{aligned} \Omega_{abc} &= \eta_-^T \gamma_{abc} \eta_-, \\ \Omega &= \frac{1}{3!}\Omega_{abc} dz^a \wedge dz^b \wedge dz^c. \end{aligned} \tag{23}$$

we can check that $d\Omega = \bar{\partial}\Omega = 0$ thus Ω is closed and holomorphic.

$$\mathcal{R} = i\partial\bar{\partial}\log\sqrt{g} = -i\partial\bar{\partial}\log\|\Omega\|^2. \tag{24}$$

The curvature form is given by $\mathcal{R} = i\partial\bar{\partial}\log\sqrt{g} = -i\partial\bar{\partial}\log\|\Omega\|^2$, and the holomorphic 3-form is non-vanishing everywhere, so $\mathcal{R}$ is exact, thus

$$c_1 = \frac{1}{2\pi}[\mathcal{R}] = 0. \tag{25}$$

Now we can consider an explicit spectrum from such setting. The heterotic string compactification gives a massless spectrum.

The 3-form field strength satisfies

$$dH_3 = \frac{\alpha'}{4}(\mathrm{tr}\, R \wedge R - \mathrm{tr}\, F \wedge F). \tag{26}$$

The left hand side is exact therefore $\mathrm{tr}\, R\wedge R$ and $\mathrm{tr}\, F\wedge F$ belong to the same cohomology class.In the case of Calabi–Yau compactification of heterotic $E_8 \times E_8$, the curvature 2-form takes values in the Lie algebra of the holonomy group $SU(3)$ while F takes values in the $E_8 \times E_8$ Lie algebra. The simplest way to obtain this constraint is by embedding the spin connection in the gauge group. We consider

$$E_8 \times E_8 \to SU(3) \times E_6 \times E_8. \tag{27}$$

The 248 adjoint representation of E_8 has the following decomposition

$$\mathbf{248} = (\mathbf{1}, \mathbf{78}) \oplus (\mathbf{3}, \mathbf{27}) \oplus (\overline{\mathbf{3}}, \overline{\mathbf{27}}) \oplus (\mathbf{8}, \mathbf{1}). \tag{28}$$

The full spectrum in 4 dimensions is given by

supergravity multiplet	$G_{\mu\nu}$ and Ψ_μ
axion-dilaton chiral superfields	S
gauge bosons and gauginos	adjoint of $E_6 \times E_8$
$h^{2,1}$chiral superfields containing complex structure moduli	$\mathbf{27}$ of E_6
$h^{1,1}$chiral superfields containing Kähler moduli	$\overline{\mathbf{27}}$ of E_6

(29)

It's not so straightforward to see the $(2,1)$-form fields by dimensional reduction of 10 dimensional fields. Take metric for an example, $G_{MN} \to (G_{\mu\nu}, G_{\mu a}, G_{\mu\bar{a}}, G_{a\bar{b}}, G_{ab})$.

$G_{\mu\nu}$ and $G_{\mu\bar{a}}$ have no zero modes because $h^{1,0} = 0$, $G_{a\bar{b}}$ belongs to the $h^{1,1}$ chiral supermultiplet. G_{ab} is symmetric so it's not a form field but we can form a $(1,2)$-form by

$$G_{a\bar{b}\bar{c}} = G_{ab}G^{b\bar{a}}\Omega_{\bar{a}\bar{b}\bar{c}}, \tag{30}$$

which belongs to the $h^{2,1}$ chiral supermultiplet.

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