SUSY and Superfields

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1 A Brief Introduction to Supersymmetry

Supersymmetry is a natural and unique extension of spacetime symmetry and gauge symmetry. Coleman and Mandula had a no-go theorem which states that there could not be any nontrivial extension of Poincare symmetry and internal symmetry. But they made a hidden assumption that the generators of the group must be bosonic and there's only commutation relation between these generators. Later Haag, Lopuszanski and Sohnius proved that by loosening this assumption by including fermionic generators, one can get a nontrivial extension of the Poincare algebra, namely the supersymmetry algebra. The consideration of supersymmetry leads to several theoretical highlights such as :

(1) gauge coupling unification;

(2) possible explanation of hierarchy problem and;

(3) milder UV behavior of the theory;

(4) possible resolution of fine-tuning problem, etc.

The SUSY algebra is given by

$$\left\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\right\} = 2\delta^{IJ}\sigma^\mu_{\alpha\dot{\alpha}}P_\mu, \tag{1}$$

$$\left\{Q^I_\alpha, Q^J_\beta\right\} = \epsilon_{\alpha\beta}Z^{IJ}, \tag{2}$$

$$\left\{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\right\} = \epsilon_{\alpha\beta}\left(Z^{IJ}\right)^*, \tag{3}$$

where σ_μ is identity matrix and Pauli matrices $(1, \sigma_i)$ and Z^{IJ} is the central charge which is antisymmetric in index I and J and commute with the other generators. The index α, $\dot{\alpha}$ takes the value 1, 2 to indicate the supercharge transform as the representation $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the Lorentz group, respectively. I, J takes the value $1, 2, \cdots, \mathcal{N}$. Normally, $\mathcal{N}$ cannot be bigger than 8 because the constraints that spin cannot exceed 2. Moreover, if we don't consider supergravity, $\mathcal{N}$ should be equal or smaller than 4.

Together with eq. (1)-(3), we also have the commutation relation

$$\left[Q^I_\alpha, P_\mu\right] = \left[\bar{Q}^I_{\dot\alpha}, P_\mu\right] = 0, \tag{4}$$

$$\left[J_{\mu\nu}, Q^I_\alpha\right] = \frac{1}{2}\left(\sigma_{\mu\nu}\right)_\alpha^{\ \beta} Q^I_\beta, \tag{5}$$

$$\left[J_{\mu\nu}, \bar{Q}^I_{\dot\alpha}\right] = -\frac{1}{2}\left(\sigma_{\mu\nu}\right)_{\dot\alpha}^{\ \dot\beta} Q^I_{\dot\beta}. \tag{6}$$

Eq. (1)-(6) with the normal Poincare algebra form the Super Poincare algebra. From eq (5) and (6) we conclude that the supercharges rise and lower the spin by half a unit. One can see that Super Poincare algebra is a $\mathbb{Z}_2$-graded algebra by taking the analogy that bosonic operators are even numbers and fermionic operators are like odd numbers.

For simplicity, we first discuss $\mathcal{N} = 1$ supersymmetry and its representations. Loot at massive representation, we go to the rest frame $P^\mu = (M, 0, 0, 0)$ which breaks $SO(3,1)$ to rotation $SO(3)$, the SUSY algebra simplifies to

$$\begin{aligned} &\left\{Q_\alpha, \bar{Q}_{\dot\alpha}\right\} = 2\delta_{\alpha\dot\alpha} M, \\ &\left\{Q_\alpha, Q_\beta\right\} = \left\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\right\} = 0. \end{aligned} \tag{7}$$

Define normalized generator $a_\alpha = \frac{1}{\sqrt{2M}} Q_\alpha$ and $a^\dagger_\alpha = \frac{1}{\sqrt{2M}} \bar{Q}_{\dot\alpha}$, the algebra becomes

$$\left\{a_\alpha, a^\dagger_\beta\right\} = \delta_{\alpha\beta}, \quad \left\{a_\alpha, a_\beta\right\} = \left\{a^\dagger_\alpha, a^\dagger_\beta\right\} = 0, \tag{8}$$

which is just the Clifford algebra. The vacuum $|\Omega\rangle$ is defined by $a_\alpha|\Omega\rangle = 0$. The other states could be constructed by applying creation operator on the vacua: $a^\dagger_\alpha|\Omega\rangle$, $a^\dagger_1 a^\dagger_2|\Omega\rangle$. If the vacuum has spin j, the other states has spin $j - \frac{1}{2}$, $j + \frac{1}{2}$, j. For example, If $j = 0$, we have 4 states $(-\frac{1}{2}, 0, 0, \frac{1}{2})$ and their CPT conjugate; If $j = \frac{1}{2}$, we have four states $(0, \frac{1}{2}, \frac{1}{2}, 1)$ and their CPT conjugate $(-1, -\frac{1}{2}, -\frac{1}{2}, 0)$.

For massless states, we take $P^\mu = (E, 0, 0, E)$, say the particle is moving in the z-direction, thus the SUSY algebra becomes

$$\left\{Q_\alpha, \bar{Q}_{\dot\alpha}\right\} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{9}$$

Define $a_\alpha = \frac{1}{2\sqrt{E}} Q_\alpha$, $a^\dagger_\alpha = \frac{1}{2\sqrt{E}} \bar{Q}_{\dot\alpha}$. The only nontrivial algebra is

$$\left\{a_1, a^\dagger_1\right\} = 1. \tag{10}$$

So there are only two states with helicity j, $j + \frac{1}{2}$. When $j = 0$, together with its CPT conjugate, there are a total of four states with $(-\frac{1}{2}, 0, 0, \frac{1}{2})$, these are represented by a Weyl spinor and a complex scalar. This multiplet is called chiral multiplet. When $j = \frac{1}{2}$, together with its CPT conjugate, these are states $(-1, -\frac{1}{2}, \frac{1}{2}, 1)$, which are represented by a massless gauge field and a Weyl spinor, this multiplet is called vector multiplet. Also, if we consider supergravity we have gravitino multiplet $(-\frac{3}{2},-1,1,\frac{3}{2})$ and graviton multiplet $(-2,-\frac{3}{2},\frac{3}{2},2)$. Since we do not want helicity larger than 2, we must stop here.

Now we turn to $\mathcal{N}=2$ SUSY, so the I, J in eq. (1)-(3) takes the value 1, 2. There is $SU(2)\times U(1)$ R-symmetry and the supercharge transforms as a doublet under $SU(2)_R$. For massless representation, the central charge is set to 0 due to unitary considerations. The creation and annihilation operators are doubled compare to the case of $\mathcal{N}=1$, so there are 4 states with helicity j, $j+\frac{1}{2}$, $j+\frac{1}{2}$, $j+1$. We have the following multiplets:

$$\begin{aligned}&\text{vector multiplet: } \left(-1,-\frac{1}{2}^2,0\right)\oplus\left(0,\frac{1}{2}^2,1\right),\\ &\text{hypermultiplet: } \left(-\frac{1}{2},0^2,\frac{1}{2}\right)\oplus\left(-\frac{1}{2},0^2,\frac{1}{2}\right).\end{aligned}\tag{11}$$

Notice that if the hypermultiplet is self CFT conjugate, we do not need two copies of the states.

For massive particles, we go to the rest frame as usual. The SUSY algebra becomes

$$\{Q^I_\alpha,\bar{Q}^J_{\dot{\alpha}}\}=2\delta^{IJ}\delta_{\alpha\dot{\alpha}}M,\{Q^I_\alpha,Q^J_\beta\}=\epsilon_{\alpha\beta}\epsilon^{IJ}Z,\left\{\bar{Q}^I_{\dot{\alpha}},\bar{Q}^J_{\dot{\beta}}\right\}=\epsilon_{\alpha\beta}\epsilon^{IJ}Z^*.\tag{12}$$

Define $b^{(\varphi)}_\alpha=\frac{1}{\sqrt{2}}(Q^1_\alpha+e^{i\varphi}\epsilon_{\alpha\dot{\beta}}\bar{Q}^2_{\dot{\beta}})$, we have

$$\left\{b^{(\varphi)}_\alpha,\left(b^{(\varphi)}_\beta\right)^\dagger\right\}=\delta_{\alpha\beta}\left(2M-\operatorname{Re}\left(e^{i\varphi}Z\right)\right).\tag{13}$$

Following the property of anticommutation relation, we have

$$2M-\operatorname{Re}\left(e^{i\varphi}Z\right)\geq 0.\tag{14}$$

Choosing $\varphi=-\operatorname{Arg}Z$, we have the inequality

$$2M\geq|Z|.\tag{15}$$

Generally there are $2^4=16$ states in the supermultiplet, say $(-1,-\frac{1}{2}^4,0^6,\frac{1}{2}^4,1)$. But when the inequality is saturated, $\left\{b^{(-\operatorname{Arg}Z)}_\alpha,\left(b^{(-\operatorname{Arg}Z)}_\beta\right)^\dagger\right\}=0$, only 4 states are generated (with CPT the number of states are doubled to 8). Such multiplets are called BPS(Bogomolny-Prasad-Sommerfeld) and eq. (7) is called BPS bound. A BPS state is rather robust: under a generic perturbation, the number of states in a multiplet can not jump. Therefore the BPS state will generically stay BPS. We see that the number of BPS states equals the number of massless states. So if we change the coupling constant of a massless $\mathcal{N} = 2$ supersymmetric theory a little, the states could become massive and also preserving supersymmetry, but the deformed theory must have a central charge.

2 Superspace and Superfields

Superspace and superfields provide a convenient and compact way to construct supersymmetric field theories. Here I will first restrict ourselves to $\mathcal{N} = 1$ superspace and superfields because extended supersymmetric theories could constructed out of $\mathcal{N} = 1$ supersymmetric theories by (a) dimensional reduction of higher dimensional $\mathcal{N} = 1$ theories; (b) combining the supersymmetry manifest in the superfield formalism and an R-symmetry. The ordinary spacetime coordinates x^μ are enlarged to superspace coordinates $\left(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\right)$ where the Grassmann number has the odd property $\{\theta_\alpha, \theta_\beta\} = \left\{\theta_\alpha, \bar{\theta}_{\dot{\beta}}\right\} = \left\{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\right\} = 0$. One can understand this weird property by viewing them as classical analogs of fermion fields. Recall the anticommutation relation of fermion fields:

$$\{\psi, \psi\} = \left\{\psi^\dagger, \psi^\dagger\right\} = 0, \left\{\psi, \psi^\dagger\right\} = \hbar. \tag{16}$$

By taking the classical limit $\hbar \to 0$ and trading Hermitian conjugation for complex conjugation, we see the classical fermion field take Grassmann number values.

Define the group element of the supergroup by exponential mapping of the generators:

$$G(x, \theta, \bar{\theta}) = \exp i\left(-x_\mu P^\mu + \theta Q + \bar{\theta}\bar{Q}\right) \tag{17}$$

The multiplication of the group element is given by Baker-Campbell-Hausdorff (BCH) formula:

$$G(0, \eta, \bar{\eta}) G\left(x^\mu, \theta, \bar{\theta}\right) = G\left(x^\mu + i\theta\sigma^\mu\bar{\eta} - i\eta\sigma^\mu\bar{\theta}, \theta + \eta, \bar{\theta} + \bar{\eta}\right). \tag{18}$$

So the trasformation

$$\begin{aligned} & x^\mu \to x^\mu + i\theta\sigma^\mu\bar{\eta} - i\eta\sigma^\mu\bar{\theta}, \\ & \theta \to \theta + \eta, \\ & \bar{\theta} \to \bar{\theta} + \bar{\eta} \end{aligned} \tag{19}$$

is generated by

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \bar{Q}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu. \tag{20}$$

One can show that these generators satisfy the SUSY algebra. A superfield $F(x, \theta, \bar{\theta})$ is defined as a function of superspace. An infinitesimal variation of the superfield is given by

$$\delta_{\epsilon,\bar{\epsilon}} F = (i\epsilon Q + i\bar{\epsilon}\bar{Q}) F. \tag{21}$$

In order to construct a Lagragian which is invatiant under supersymmetry transformation, we introduce the covariant derivative:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu, \bar D_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \tag{22}$$

which satisfy

$$\begin{aligned} &\left\{D_\alpha, \bar D_{\dot\alpha}\right\} = -2\sigma^\mu_{\alpha\dot\alpha}P_\mu \\ &\{D_\alpha, D_\beta\} = \left\{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\right\} = \{D_\alpha, Q_\beta\} = \left\{\bar D_{\dot\alpha}, \bar Q_{\dot\beta}\right\} = \left\{D_\alpha, \bar Q_{\dot\beta}\right\} = \left\{\bar D_{\dot\alpha}, Q_\beta\right\} = 0. \end{aligned} \tag{23}$$

Now let us consider various field-worlds.

a. **Chiral Superfields**

A chiral superfield Φ is defined by imposing

$$\bar D_{\dot\alpha}\Phi = 0, \tag{24}$$

which is an analogy to holomorphic functions, say, the functions which satisfy

$$\bar\partial_{\bar z} f(z, \bar z) = 0 \tag{25}$$

So Φ depends only on θ and $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$, expanding on θ we have

$$\Phi(y, \theta) = \phi(y) + \theta\psi(y) + \theta\theta F(y), \tag{26}$$

where we absorbed the usual convention $\sqrt{2}$ into ψ.

Expanding in terms of x and θ, we have

$$\Phi(x, \theta, \bar\theta) = \phi(x) + \theta\psi(x) + \theta\theta F(x) + i\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) - \frac{i}{2}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar\theta - \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi(x). \tag{27}$$

The infinitesimal transformation rule of ϕ, ψ, F could be read off from eq. (21) and (27).

b. **Vector Superfields**

The vector superfield is expanded in Wess-Zumino gauge as

$$V_\mu = -\theta\sigma^\mu\bar\theta A_\mu + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda + \frac{1}{2}\theta^2\bar\theta^2 D, \tag{28}$$

where F and D are both auxiliary fields which help supersymmetry manifested explicitly.

The general $\mathcal{N} = 1$ supersymmetric Lagrangian has the following form:

$$L = \int d^4\theta K(\bar\Phi, \Phi) + \int d^2\theta \left[W(\Phi) + \frac{1}{8\pi i}\tau(\Phi)\,\mathrm{Tr}\,(W_\alpha W^\alpha)\right] + \text{h. c.}, \tag{29}$$

where K is the Kähler potential, $W(\Phi)$ is a holomorphic function of chiral superfield and W_α is the field strength built from V:

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}e^{-V}D_\alpha e^V, \tag{30}$$

also the trace is taken in the fundamental representation of the gauge group and $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$ transforming under $SL(2,\mathbb{Z})$ equals constant for UV renormalizable lagragian.

Now we are ready to construct theories with more supersymmetry. The simplest case is that the Lagrangian (29) has $\mathcal{N}=2$ SUSY if $W(\Phi)=0$. We only need to make sure that the λ in $\mathcal{N}=1$ vector multiplet and ψ in $\mathcal{N}=1$ chiral multiplet transform as a doublet under $SU(2)$ R-symmetry. Since the $\mathcal{N}=2$ vector multiplet is consisted of the following $\mathcal{N}=1$ multiplets

$$\begin{array}{c} \phi \leftrightarrow \psi \\ \updownarrow \nearrow \updownarrow \\ \lambda \leftrightarrow A_\mu \end{array} \tag{31}$$

The horizontal arrow indicates the $\mathcal{N}=1$ sub-supersymmetry and the vertical arrow shows the second $\mathcal{N}=1$ sub-supersymmetry. The slanted arrow shows the R-symmetry rotating ψ and λ.

Also, 4D $\mathcal{N}=2$ SYM and 4D $\mathcal{N}=4$ SYM could be constructed from dimensional reduction of 6D (0,1) $\mathcal{N}=1$ SYM and 10D $\mathcal{N}=1$ SYM.

The above construction applies to the theory with lagrangian description, however, these are only a very small subset of all possible supersymmetric theories. There are other interesting theories for which no lagrangian description is known. Such theories can be found as the IR limit of those theories with lagrangian description, etc. Thanks to the work of Nekrasov, Gaiotto, Tachikawa etc., we now know how to understand the property of some supersymmetric theories, especially $\mathcal{N}=2$ ones by bypassing the lagragian formalism.

References

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