Non-Extremal De Sitter Vacua

Zhi-Zhen Wang

Amsterdam, North-Holland, Netherlands

University of Amsterdam (2020)

A Thesis Presented to the Faculty of Science

of the University of Amsterdam in Candidacy for the Degree of

Master of Science in Theoretical Physics

Supervisor: Jan Pieter van der Schaar, Associate Professor

Examiner: Jácome Armas, Assistant Professor

Institute of Physics (IoP), Faculty of Science

University of Amsterdam

Aug, 2020

Abstract

This is the abstract section. You should replace this with your own abstract.

Acknowledgments

This is the acknowledgement section.

You should replace this with your own acknowledgements.

Contents

List of Figures

List of Tables

Chapter 1

Prologue

1.1 Introduction

1.2 Appetite

[]

Chapter 2

Flux Compactification

2.1 Klein–Kaluza Reduction

2.2 Compactification and Orbifolds

2.3 Superstring Compactification

2.3.1 IIA/IIB Type Superstring

2.3.2 IIA/IIB Type Supergravity

[]

Chapter 3

String Cosmology–Compactification to Four Dimensions

3.1 KPV Formalism

3.1.1 Anti-D3–NS5 Bound State

3.1.2 Kähler Potential and Superpotential

3.1.3 Flux and Metastability

3.2 KKLT Uplifting

3.2.1 AdS Vacua Construction

3.2.2 Anti-D3 Uplifting

3.3 $\mathcal{N} = 1$ SUSY Breaking

[]

Chapter 4

Blackfold Formalism

4.1 Blackfolds and Higher-Dimensional Black Holes

4.2 Far-Zone (Long Wavelength) VS. Near-Zone (Short-Wavelength)

4.3 Blackfold Equations

4.3.1 Extrinsic Equation

4.3.2 Intrinsic Equation

4.4 Charged Blackfolds

4.4.1 Charge Conservation Equation

4.5 Blackfold Equations for Anti-D3–NS5 Bound State

[]

Chapter 5

Blackfold Perturbation

5.1 Extremal Perturbation

We start from the basic information given by [?]. The metric near the tip of the throat of Klebanov–Strassler geometry can be written as

$$\begin{aligned} ds_{10}^2 &= g_s M\alpha' b_0^2(\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 + r^2 d\Omega_2^2 + d\Omega_3^2) \\ &= g_s M\alpha' b_0^2(\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 + r^2(d\tilde{\omega}^2 + \sin^2\tilde{\omega}d\tilde{\varphi}^2) + d\psi^2 + \sin^2\psi(d\omega^2 + \sin^2\omega d\varphi^2)), \end{aligned} \tag{5.1}$$

where $\eta_{\mu\nu}dx^\mu dx^\nu$ corresponds to the Minkowski spacetime, ψ, ω, φ, $\tilde{\omega}$ and $\tilde{\varphi}$ are spherical angles parametrize Ω_3 and Ω_2, respectively. $b_0^2 \simeq 0.93266$ is a constant induced by unit change from Einstein frame to string frame and M the RR charge. Besides, $\alpha' \equiv l_s^2$ will be set as 1 in the following discussion for convenience. According to this metric, the RR field strength is

$$\begin{aligned} F_3 &= 2M\mathrm{vol}(S^3) \\ &= 2M\sin^2\psi\sin\omega d\psi\wedge d\omega\wedge d\varphi \end{aligned} \tag{5.2}$$

Accordingly, the RR flux dual to NSNS field strength H_3 can be written as

$$\begin{aligned} H_7 &\equiv \frac{1}{g_s^2}(\star H_3) \\ &= -\frac{1}{g_s^2}[\star(2M^3 b_0^4 \sin^2\psi \sin\omega dr \wedge d\tilde{\omega} \wedge d\tilde{\varphi})] \\ &= -\frac{2M^3 b_0^4 \sin^2\psi \sin\omega}{g_s} dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\psi \wedge d\omega \wedge d\varphi \\ &= -\frac{M^2 b_0^4}{g_s} dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge F_3. \end{aligned} \tag{5.3}$$

To apply the blackfold formalism, we then need to treat the D3-NS5 system in a flat space [**?**] and extract the energy momentum tensor. The metric in this case is

$$ds^2 = D^{-1/2}\left(-f dt^2 + D\left(\left(dx^1\right)^2 + \left(dx^2\right)^2\right) + \sum_{i=3}^{5}\left(dx^i\right)^2\right) + HD^{-1/2}\left(f^{-1}dr^2 + r^2 d\Omega_3^2\right), \tag{5.4}$$

where

$$f = 1 - \frac{r_0}{r}, \quad D = \left(\sin^2\theta H^{-1} + \cos^2\theta\right)^{-1}, \quad H = 1 + \frac{r_0^2 \sinh^2\alpha}{r^2}. \tag{5.5}$$

r_0 is the Schwarzschilld radius of spherical part of KS geometry, θ is a parameter controls how much $\bar{D}3$ brane charge is dissolved inside the NS5 brane and α a parameter measures how extremal the horizon radius is. Besides, the IIB SUGRA action in Einstein frame is

$$\begin{aligned} I = \frac{1}{16\pi G}\int_{\mathcal{M}_{10}} [\star R - \frac{1}{2}d\phi \wedge \star d\phi - \frac{1}{2}e^{-\phi}H_3 \wedge \star H_3 - \frac{1}{2}\sum_{q=-1,1} e^{\frac{3-q}{2}\phi}\tilde{F}_{q+2} \wedge \star\tilde{F}_{q+2} \\ - \frac{1}{4}\tilde{F}_5 \wedge \star\tilde{F}_5 + \frac{1}{2}C_4 \wedge H_3 \wedge F_3], \end{aligned} \tag{5.6}$$

where $\tilde{F}_{q+2} = F_{q+2} - H_3 \wedge C_{q-1}$ (F_1 is self-dual), ϕ denotes the dilaton, which is a constant in our consideration. We can derive the Maxwell charges from the equation of motion of this action, with CS term considered as a source of the gauge field. The e.o.m for C_4, C_2 and B_2 fields are:

$$d\star\tilde{F}_3 - H_3\wedge F_5 = -16\pi G\star J_2, \tag{5.7}$$

$$d\star\tilde{F}_5 - H_3\wedge F_3 = -16\pi G\star J_4, \tag{5.8}$$

$$dH_3 = 16\pi G\star J_6. \tag{5.9}$$

According to the physical meaning of the e.o.ms, one can notice that J_2 is the C_2-field induced current, J_4, J_6 are currents corresponding to D3 brane charge and NS5 brane charge, respectively. Redefine Maxwell current as (we will set constants $16\pi G = 1$ later and $g_s = 1$ here.)

$$\begin{aligned}
d\star\tilde{F}_3 &= -\star J_2^{Max} = \star J_2 + H_3\wedge F_5,\\
d\star\tilde{F}_5 &= -\star J_4^{Max} = \star J_4 + H_3\wedge F_3,\\
d\star\tilde{H}_7 = d\star\frac{1}{g_s^2}\star H_3 &= \star J_6^{Max} = \star J_6.
\end{aligned} \tag{5.10}$$

Note that we do not know the explicit expression of J_2, J_4 and J_6, in e.o.m they are expressed by higher-form fields, but we hope to obtain a more elegant form by describing them in the parameters of D3-NS5 bound states. It is not hard to see that the e.o.m we obtained above are just a generalized form of Maxwell equations. To write down an exact expression of these currents, we can take an analogy as equivalent currents setting as one learned in electromagnetism, where one supposes the effects of currents can be simulated by putting a charge far away. Since $Q = \int\star J$, those imaginary charges should satisfy

$$\begin{aligned}
Q_1^M &= \int_{\mathcal{M}_8}\star J_2^{Max} \sim \int_{\partial\mathcal{M}_8}\star\tilde{F}_3,\\
Q_3^M &= \int_{\mathcal{M}_6}\star J_4^{Max} \sim \int_{\partial NS_5}\star\tilde{F}_5,\\
Q_5^M &= \int_{\mathcal{M}_4}\star J_6^{Max} \sim \int_{\partial D3}\star\tilde{H}_7 \sim \int_{\partial D3} H_3,
\end{aligned} \tag{5.11}$$

where the J_2^{Max} is actually the electromagnetic dual current induced from non-zero C_2 field whose dual is the current corresponds to the D3 charge which keeps track of the number charge. In the D3-NS5 system, one can derive those higher-form fields from the metric, the RR fluxes are ($r_h \equiv r_0^2 \sinh \alpha^2$)

$$\tilde{F}_3 = -2\frac{r_h^2 D^2}{r^3 H^2} \sin\theta \cos\theta dx^1 \wedge dx^2 \wedge dr, \tag{5.12}$$

$$\tilde{F}_5 = 2\frac{r_h^2}{r^3 H^2} \sin\theta dt \wedge dx^3 \wedge dx^4 \wedge dx^5 \wedge dr, \tag{5.13}$$

and NSNS flux is

$$\tilde{H}_7 \sim \star H_3 = 2r_h^2 \cos\theta dt \wedge dx^1 \wedge \cdots \wedge dx^5 \wedge dr. \tag{5.14}$$

Insert into the current integration, we have[1]

$$\begin{aligned} Q_1^M &= vol_4 C r_h^2 \sin\theta \cos\theta, \\ Q_3^M &= vol_2 C r_h^2 \sin\theta, \\ Q_5^M &= C r_h^2 \cos\theta. \end{aligned} \tag{5.15}$$

$C = \frac{1}{(2\pi)^5 g_s^2}$ contains all constants mentioned above but refer to identity for convenience except G. Meanwhile, the vol_n are n-form volume which is a constant after integrated out n frames parametrizing the spacetime. And they have same effects at $r \to \infty$, then we can construct the equivalent currents as[2] ($dr = 0, r \to \infty$, $\theta \in (0, 2\pi)$):

$$J_2^{eq} = C r_h^2 \sin\theta \cos\theta v \wedge w, \tag{5.16}$$

$$J_4^{eq} = C r_h^2 \sin\theta * (v \wedge w), \tag{5.17}$$

$$J_6^{eq} = C r_h^2 \cos\theta * (1). \tag{5.18}$$

[1] All charges are 0-form, the subscript have number label just for distinction, they are decreased by one to their original currents because I want to make the convention of this note compatible to that of [**?**].

[2] One may notice that the results here is not exactly the same as Jay *et al.* originally derived ([**?**]'s result differ by $r_0^2 \sinh\alpha \cosh\alpha$ rather than $r_h^2 = r_0^2 \sinh\alpha^2$). But it does not really matter at far zone and extremal case, where one has $\alpha \to \infty$. I think this difference is induced from the fact that I directly use the result derived by [**?**] which defined the metric with only $\sinh\alpha$ dependence.

v, w are spacelike orthogonal vectors used to parametrize the induced metric of $NS5$ (can be regarded as a fivebrane), they can not only be regarded as a rotation transformation of $x^i, i = 1, \cdots, 5$, but also could describe the distribution of the dissolved $\overline{D3}$ charge, we will use them in deriving blackfold equation later. And $*$ is a Hodge dual on this fivebrane which acts as: $* : \Omega^r \to \Omega^{6-r}$.

Besides, the charge density explicitly carried by J_4 is

$$\tilde{Q}_3 = \int J_4 = -Cr_h^2 \sin\theta, \tag{5.19}$$

this is significantly different to the usual higher-form current defined by $\int \star J_4^{Max}$.

5.2 Blackfold Equations

Now consider KS background, according to the discussions above and the KS metric 5.1 implies that there are few variables which determine the physics near the tip of the throat:

$$\{r, \tilde{\omega}, \tilde{\varphi}, \psi, \alpha, \theta, v^a, w^a\}, \tag{5.20}$$

where the former four are variables parametrize the coordinates in string frame, the latter four are derived from the currents induced by e.o.m of higher-form fields, which denote the characteristic d.o.f describing the extremal event horizon, distribution and flow of the dissolved charges, respectively. The first order blackfold equations, which are taken for use in [?], describe the zeroth order terms in the derivative expansion of the metric and gauge fields. Consider the fact that we are only interested in the geometry near the tip of the throat, all the explicit coordinates can be set as fixed, which are corresponding to the position of the apex where metastable configuration appeared. Then all we need to worry about are just those terms which have no dependence on coordinates or basis: α, ψ and θ.

The blackfold equations consist of two parts:

1. Stress-tensor conservation equation[3]:

$$\nabla_a T^{a\mu} = \frac{g_s^{-1}}{6!} H_7^{\mu a_1\cdots a_6} J_{6a_1\cdots a_6} + \frac{1}{2!} F_3^{\mu a_1 a_2} J_{2a_1a_2} + \frac{3}{4!} H_3^{\mu a_1 a_2} C_2^{a_3a_4} J_{4a_1\cdots a_4} + \frac{1}{4!} \tilde{F}_5^{\mu a_1\ldots a_4} J_{4a_1\ldots a_4}, \tag{5.21}$$

and

2. Current conservation equations:

$$\begin{aligned} &d * J_4 + *J_6 \wedge F_3 = 0, \\ &d * J_2 + H_3 \wedge *J_4 = 0, \\ &d * J_6 = 0. \end{aligned} \tag{5.22}$$

One may note that these equations are very similar to that derived from e.o.m of higher-form fields, that is true for a trivial reason: all Maxwell equations are not complete unless one introduce the conservation condition.

From the conservation equations only, we can directly determine some charges as:

$$Q_5 = *J_6 = Cr_h^2 \cos\theta, \tag{5.23}$$

$$Q_3 = \int_{S^2} * \left(J_4 + * \left(*J_6 \wedge C_2\right)\right). \tag{5.24}$$

These charges are Page charges which denote the number charge of D3 and NS5, respectively. Insert (5.16),(5.17) and (5.18) into (5.24) we have

$$Q_3 = 4\pi[Cr_e^2 \sin\theta M b_0^2 \sin^2\psi + Cr_e^2 \cos\theta M(\psi - \frac{1}{2}\sin 2\psi)]. \tag{5.25}$$

According to an equation displayed in [?] which I have not understood how it comes: $Q_3/Q_5 = 4\pi^2 p g_s$, where p the number of $\overline{D3}$ brane, we can obtain an equation to

[3] $\tilde{F}_5$ term is vanishing at the tip so that [?] did not show it, but this term is non-trivial at the perturbative level.

restrict the relation between relative unfixed variables:

$$\tan\theta = \frac{1}{b_0^2 \sin^2\psi}\left[\frac{\pi p g_s}{M} - \left(\psi - \frac{1}{2}\sin 2\psi\right)\right]. \tag{5.26}$$

Besides, stress-tensor conservation gives

$$\cot\psi = \frac{1}{b_0^2}\sqrt{1-\psi'^2}\sqrt{1+\tan^2\theta} + \frac{1}{b_0^2}\tan\theta - \frac{1}{2}(1+\tan^2\theta)\frac{\psi''}{1-\psi'^2}. \tag{5.27}$$

Since the only interested case is the physics at the tip with metastable states now, we can set $\psi' = \psi'' = 0$ without any cost, then gives

$$\cot\psi - \frac{1}{b_0^2}\sqrt{1+\tan^2\theta} - \frac{1}{b_0^2}\tan\theta = 0. \tag{5.28}$$

Set ψ_0 as the solution of both two equations, one can then fix all three coordinate-independent variables at the tip of throat. We define the metastable states appear at:

$$\psi = \psi_0, \quad \tan\theta_0 = \frac{1}{b_0^2 \sin^2\psi_0}\left(\frac{\pi p g_s}{M} - \psi_0 - \frac{1}{2}\sin 2\psi_0\right), \quad r_e = \sqrt{\frac{Q_5}{C\cos\theta_0}}, \tag{5.29}$$

and coordinate-dependent variables are set as:

$$r = 0, \quad , v^a\partial_a = \frac{1}{\sqrt{g_s M} b_0 \sin\psi_0}\partial_\omega, \quad w^a\partial_a = \frac{1}{\sqrt{g_s M} b_0 \sin\psi_0 \sin\omega}\partial\varphi. \tag{5.30}$$

With the induced metric of the 6d worldvolume rewritten as:

$$\gamma_{ab} d\sigma^a d\sigma^b = M b_0^2\left(-dt^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 + \sin^2\psi_0\left(d\omega^2 + \sin^2\omega d\varphi\right)\right) \tag{5.31}$$

To here, we have successfully restore the metastability results at zeroth order extremal case which [?] have done. Since lots of details need to be changed in the derivation, we cannot treat the metastability near the tip with perturbative level until now.

5.3 At Pertuebative Level

The blackfold equations are constraints for the gravitational theory, therefore, by perturbing the variables near the tip of the throat, we can obtain a new class of necessary condition for IIB SUGRA. Variate variables as following:

$$\text{coordinate-independent:} \tag{5.32}$$

$$\psi = \psi_0 + \delta\psi, \quad r_e = \sqrt{\frac{Q_5}{C\cos\theta_0}} + \delta r_e, \quad \tan\theta_0 = \frac{1}{b_0^2 \sin^2\psi_0}\left(\frac{\pi p g_s}{M} - \psi_0 - \frac{1}{2}\sin 2\psi_0\right) + \delta\tan\theta, \tag{5.33}$$

$$\text{coordinate-dependent:} \tag{5.34}$$

$$r = 0 + \delta r, \quad v^a\partial_a = \frac{1}{\sqrt{g_s M} b_0 \sin\psi_0}\partial_\omega + \delta v^a\partial_a, \quad w^a\partial_a = \frac{1}{\sqrt{g_s M} b_0 \sin\psi_0 \sin\omega}\partial\varphi + \delta w^a\partial_a, \tag{5.35}$$

index a runs over ω, φ and t. Before the derivation of the equations of first order variation, we can further minimize the number of free variables. According to the Lorentz symmetry (now we are actually treating in KPV background), the only interesting variables are those which contain time-dependence; besides, unitarity and orthogonality gives constraints: $v^a v_a = w^a w_a = 1$, $v^a w_a = 0$, then

$$\delta v^\omega = -\frac{\cos\psi}{\sqrt{g_s M} b_0 \sin^2\psi}\delta\psi, \tag{5.36}$$

$$\delta w^\varphi = -\frac{\cos\psi}{\sqrt{g_s M} b_0 \sin^2\psi \sin\omega}\delta\psi, \tag{5.37}$$

$$\delta v^\varphi = \delta w^\omega = 0. \tag{5.38}$$

Thus the variables remained in our later consideration are:

$$\begin{aligned}&\{\delta r(t,\omega,\varphi), \delta\psi(t,\omega,\varphi), \delta r_e(t,\omega,\varphi), \delta\tan\theta(t,\omega,\varphi),\\ &\delta v^t(t,\omega,\varphi), \delta v^\omega(t,\omega,\varphi) \equiv \delta v^\omega(\delta\psi), \delta w^t(t,\omega,\varphi), \delta w^\varphi(t,\omega,\varphi)\} \equiv \delta w^\varphi(\delta\psi).\end{aligned} \tag{5.39}$$

Now begin to display our main results, as the two parts of blackfold equation, there are two sets of equations of variation derived from those two parts. We start from the charge conservation part.

5.3.1 Charge Conservation

The conservation of J_6 is the easiest one to analyze, suppose the variation of perturbative variables force $Q_5 \to Q_5 + \delta Q_5(variable)$. Then the J_6 conservation equation can be rewritten as

$$\partial_a \delta Q_5 = 0 \Leftrightarrow \delta Q_5 \text{ is a constant of motion.} \tag{5.40}$$

Since the NS5 brane number would not variate near the tip of the throat, it is easy to show that $\delta Q_5 = 0$. Accordingly,

$$\begin{aligned} \partial_a \delta Q_5 = 0 \Leftrightarrow & -2Cr_e \cos\theta \delta r_e + Cr_e^2 \sin\theta \delta\theta \\ & \Rightarrow \delta r_e = \frac{1}{2} r_e \tan\theta \delta\theta = \frac{1}{2} r_e \sin\theta \cos\theta \delta \tan\theta. \end{aligned} \tag{5.41}$$

Similarly, for J_4,

$$\begin{aligned} &\partial_a \delta Q_3 = 0 \\ \Rightarrow & \sin\omega \left(M b_0^2 \sin^2\psi \partial_t \delta \tan\theta + 2M b_0^2 \tan\theta \cos\psi \sin\psi \partial_t \delta\psi + 2M \sin^2\psi \partial_t \delta\psi \right) \\ &= \left(\tan\theta \sqrt{g_s M} b_0 \sin\psi \partial_\varphi \delta w_t \right) + \left(\tan\theta \sqrt{g_s M} b_0 \sin\psi \partial_\omega \left(\sin\omega \delta v_t \right) \right), \end{aligned} \tag{5.42}$$

and

$$\delta Q_3 = -Q_5 M b_0^2 \sin^2\psi \int d\omega d\varphi \sin\omega \left(\delta \tan\theta + 2 \left(\tan\theta \cot\psi + \frac{1}{b_0^2} \right) \delta\psi \right). \tag{5.43}$$

Here the periodic boundary condition for transverse angles ω and φ have been applied. As the case for J_6, Q_3 is also a number charge of D3 brane near the metastability, thus we still have $\delta Q_3 = 0$, which gives

$$\frac{2}{b_0^2 \tan\theta}\delta\psi + 2\cot\psi\delta\psi + \frac{\delta\tan\theta}{\tan\theta} = 0. \tag{5.44}$$

This equation have a linear solution between $\sin\psi$ and $\tan\theta$ for large enough $\tan\theta$.

Then consider the last J_2 conservation, we have

$$\cot\theta\cos^2\theta\partial_\omega\delta\tan\theta + \sqrt{g_s M} b_0 \sin\psi\partial_t\delta v^t = 0, \tag{5.45}$$

$$\cot\theta\cos^2\theta\partial_\varphi\delta\tan\theta + \sqrt{g_s M} b_0 \sin\psi\sin\omega\partial_t\delta w^t = 0, \tag{5.46}$$

$$\partial_\varphi\delta v^t - \partial_\omega\left(\sin\omega\delta w^t\right) = 0 \tag{5.47}$$

5.3.2 Stress-tensor Conservation

The complete set of stress-tensor equation of blackfold formalism involves one intrinsic equation and one extrinsic equation. Inside the worldvolume,

$$\text{Intrinsic: } \nabla_a T^{ab} = \mathcal{F}^b, \tag{5.48}$$

while outside the worldvolume,

$$\text{extrinsic: } T^{ab} K_{ab}{}^{i} = \mathcal{F}^i, \tag{5.49}$$

where $K_{ab}{}^{i} = K_{ab}{}^{\rho} n^i_\rho$ with n^i_ρ the normal vector is defined as extrinsic curvature, $\mathcal{F}^b$ and $\mathcal{F}^i$ are projected force term which can be combined as the right side term of (5.21). Perturb them we can obtain the rest of differential equations for variations[4]:

[4]I have omitted most calculation details for this part since they are too annoying to appear in a simple note, some definition may be found in appendix A.

Intrinsic:

t:

$$\partial_t \delta \tan\theta + \frac{\sqrt{g_s M} b_0}{\sin\psi} \tan\theta (\partial_\omega \delta v^t + \frac{1}{\sin\omega} \partial_\varphi \delta w^t + \cot\omega \delta v^t) \\ + 2\left(\cot\psi \tan\theta + \frac{1}{b_0^2}\right) \partial_t \delta\psi = 0. \tag{5.50}$$

ω:

$$\sqrt{g_s M} b_0 \sin\psi \tan^2\theta \partial_t \delta v^t + \sin\theta \cos\theta \partial_\omega \delta \tan\theta = 0. \tag{5.51}$$

φ:

$$\sqrt{g_s M} b_0 \sin\psi \sin\omega \tan^2\theta \partial_t \delta w^t + \sin\theta \cos\theta \partial_\varphi \delta \tan\theta = 0. \tag{5.52}$$

Extrinsic:

ψ:

$$-\frac{1}{\sin^2\psi}\delta\psi - \frac{1}{b_0^2}(1+\sin\theta)\delta\tan\theta + \frac{1}{2}\left(1+\tan^2\theta\right)(\partial_t)^2\delta\psi - \frac{1}{2\sin^2\psi}\nabla^2\delta\psi = 0. \tag{5.53}$$

r:

$$(\partial_t)^2 \delta r - \frac{\cos^2\theta}{\sin^2\psi}\nabla^2 \delta r = -\frac{g_s a_0}{b_0^2} M^2 \sin\theta \delta r. \tag{5.54}$$

$a_0 = 0.71805$ is a constant related to the frame change. The Laplacian is defined by the transverse angles: $\nabla^2 = (\partial_\omega)^2 + 1/\sin^2\omega\,(\partial_\varphi)^2 + \cot\omega\partial_\omega$.

Over all the equations above, the easiest one for analyzing and solving is the last one (5.54). In this harmonic-wave-like equation, if $\sin\theta > 0$, the solution for δr would be an sinusoidal-like function which is suppressed; however, if $\sin\theta < 0$, the solution for δr would be an exponential-like function and boundless, which will induce a tachyonic δr deformation and prevent the presence of perturbation near the metastability.

5.4 Discussion

From (5.54) we notice that the sign of $\sin\theta$ determines the possibility of instabilities at the perturbative level, which is corresponding to the stability near the tip of the throat. Inspect the variables and conserved charges used for derivation, it is natural to conclude that the $\sin\theta$ dependence is induced from J_4. And there was a current defined directly by integrating this current: $\tilde{Q}_3$. This charge is obviously not localized by definition and have different properties to the Page charge Q_3. Now we give some remarks for discussing the difference between $\tilde{Q}_3$ and Q_3.

$\tilde{Q}_3$ is a Maxwell charge and Q_3 is a Page charge [**?**].

Maxwell charge is a conserved and gauge invariant charge which is carried by the gauge fields without any external source, thus it is not localized. As we did in deriving the equivalent currents, there was an assumption that we have put the charge at $r \to \infty$. For a theory with bulk fields like gravity, it is usual that the conservation law of such an un-localized charge can give no helpful information for us. This is because the charge measured in this way is unchanged when we deform the surface $\partial\mathcal{M}$ (after applying Gauss theorem in integration) so long as this surface does not pass through any charge. Since Maxwell charge is carried by the bulk fields, such charge-preserving deformations may not exist at all. Therefore, Maxwell charge is usually not a well-defined charge in brane source case, but can be useful in our discussion. According to the fact that $\tilde{Q}_3$ is not localized but conserved, it can be regarded as the source for the signature of $\sin\theta$, which means $\tilde{Q}_3$ describes the flow of dissolved D3 brane charge density and is not proportional to p.

On the other hand, the Page charge is a charge both conserved and localized but not rigorously gauge invariant, the "rigorously" means although it can be invariant under some local gauge transformations like $B_2 \to B_2 + d\Lambda_1$ with $\Lambda_1 \in \Omega^1(x^\mu)$ an arbitrary smooth one-form, it will still become a variant under general gauge transformations. This charge is the one physicists often use when they come across brane sources since they can denote a conserved quantity characterized on the embedding branes of higher-dimensional configuration, which in our case the D3 and NS5 embedded on 6d worldvolume of 10d KS geometry. Thus Q_3 and Q_5 are the characteristic charges correspond to the relative branes, with Q_3 denotes D3 number charge and Q_5 denotes NS5 number charge. So they should have some proportionality to p, as $Q_3/Q_5 = -4\pi^2 p$ we mentioned above showed.

From the discussion above, the sign of $\sin\theta$ actually not only depends on the range of θ, but also depends on the sign of $\tilde{Q}_3$. In our assumption, this Maxwell charge can be varied with the induced equivalent currents conserved, on the other hand, these equivalent currents are conserved thus all Page charges are kept conserved. According to (5.26), once the value of p/M changes, the sign of $\tan\theta$ may change and so as for $\tilde{Q}_3$. For $\sin\theta > 0$, $\tilde{Q}_3$ is negative and no instabilities will appear in the perturbative level (for radial variation equation at least), while for $\sin\theta < 0$, $\tilde{Q}_3$ transit to positive and the radial perturbation becomes unstable.

For Q_3/Q_5, we took the quotient of (5.23) and (5.25), then used $Q_3/Q_5 = -4\pi^2 p$ successfully derived (5.26). The latter condition seems like a topological condition for describing the relation between branes. If we can know such equation for $\tilde{Q}_3$ case, then by similar procedure, take the quotient $\tilde{Q}_3/Q_5$ which is also depending on $\tan\theta$ and accordingly the value of p/M, there is another equation of perturbative level determining the metastability around the apex of conifold. The solution of this equation will give another critical p value if we fix M. Suppose this critical value as p^c, if $p^c < p^{KPV}$ [5], there is a slightly[6] stronger bound than in KPV. Let us briefly sketch what would happen if this idea is true.

For $p/M \in (0, p^c)$, θ is small enough so that $\sin\theta > 0$ and $\tilde{Q}_3$ is negative. Now

[5] p^{KPV} denotes the critical value derived from the zeroth order equation of KPV background, of which the value is 0.0805 from [**?**].

[6] I say slightly because $\tilde{Q}_3$ does not differ a lot to Q_3 in the explicit expression, although I do not know how to obtain or even estimate the range of value yet.

nothing new happen, there is no instabilities. However, once the value of p exceed p^c and $p/M \in (p^c, p^{KPV})$, there is a transition from negative charge to a positive charge which dynamically unstably perturbed the metastabilities near the tip of the throat since the radial coordinates become unstable. The KPV metastability is lost before arriving p^{KPV}, p^c brings a stronger bound. Of course, if $p^c > p^{KPV}$, then it is a trivial result, just a slightly change of critical value when higher order contribution considered[7].

Last but not least, the instability appear at (p^c, p^{KPV}) can trigger an interesting idea to reconsider Buchel's paper , although they are certainly not the same background (one is extremal while one is non-extremal). In Buchel, the KS black hole emerge with positive charge and admit a perturbative instability of the chiral symmetric phase near the critical value, which induced the second bound of Bena *et al.*, of energy density. One available approach (if my statement above are correct and we can find a $p^c < p^{KPV}$) is to treat perturbative expansion for non-extremal case of equation (12) of [**?**], which may reveal more physics regarding the relation of bounds behind these two pictures. This point still need more discussion with you.

Appendix A

We give some intermediate results of section 3 and some definitions of worldvolume geometry in this appendix. The conventions are mainly based on [**?**].

A.1

Given a manifold $\mathcal{M}$ and a submanifold $\mathcal{W}$ defined by the embedding $X^\mu(\sigma^a)$, then the induced metric is

$$\gamma_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}, \tag{5.55}$$

[7] Actually, in my opinion, it must be less than p^{KPV}, because $\tilde{Q}_3$ do not receive the contribution from $\star J_6 \wedge C_2$.

with the tangent projector $h^{\mu\nu} = \gamma^{ab}\partial_a X^\mu \partial_b X^\nu$ and the orthogonal projector $\perp_{\mu\nu}= g_{\mu\nu} - h_{\mu\nu}$.

The pullback of a general tensor from $\mathcal{M}$ to $\mathcal{W}$ is given by

$$T^{a_1 a_2 \ldots a_n}{}_{b_1 b_2 \ldots b_m} \equiv \partial^{a_1} X_{\mu_1} \ldots \partial_{b_1} X^{\nu_1} \ldots T^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_m}, \tag{5.56}$$

and the extrinsic curvature is defined as

$$K_{\mu\nu}{}^{\rho} \equiv h^\sigma_\nu \bar{\nabla}_\mu h^\rho_\sigma = -h^\sigma_\nu \bar{\nabla}_\mu \perp^\rho_\sigma, \tag{5.57}$$

where $\overline{\nabla}_\mu = h^\rho_\mu \nabla_\rho$. It satisfies

$$K_{ab}{}^{\rho} = \partial_a X^\mu \partial_b X^\nu K_{\mu\nu}{}^{\rho} = \nabla_a \left(\partial_b X^\rho\right) + \Gamma^\rho{}_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \tag{5.58}$$

when written in terms of Christoffel connection.

A.2

Now write down the necessary variations for successfully variate stress-tensor.

1. Variation of induced metric:

$$\delta\gamma_{ab} = -2K^\rho_{ab} \left(\delta X^\alpha_\perp g_{\alpha\rho}\right), \tag{5.59}$$

where $\delta X^\mu_\perp$ is the variation of transverse scalars, which satisfies $\partial^a X_\mu \delta X^\mu_\perp = 0$.

2. Variation of normal vectors:

$$\delta\left(n^{\alpha i}\right) = -h^{\alpha\mu}\partial_\mu \delta X^\rho_\perp n^i_\rho - h^{\rho\alpha} n^{\sigma i} \partial_\gamma g_{\rho\sigma} \delta X^\gamma_\perp - \frac{1}{2}\partial_\gamma g_{\rho\sigma} \delta X^\gamma_\perp \perp^{\sigma\alpha} n^{\rho i}, \tag{5.60}$$

where the Greek indices denote 10d spacetime, while the Roman indices denote embedding (6d in our case) worldvolume spacetime.

3. Variation of extrinsic curvature:

$$\delta\left(K_{ab}{}^{i}\right)=n_{\mu}^{i}\nabla_{a}\left(\partial_{b}\delta X^{\mu}\right)+n_{\sigma}^{i}\Gamma^{\sigma}{}_{\mu\nu}\delta\left(\partial_{a}X^{\mu}\partial_{b}X^{\nu}\right)+n_{\sigma}^{i}\delta X^{\alpha}\partial_{\alpha}\Gamma^{\sigma}{}_{\mu\nu}\partial_{a}X^{\mu}\partial_{b}X^{\nu} \quad (5.61)$$
$$+\frac{1}{2}K_{ab}{}^{\rho}\left(n^{\alpha i}\delta X^{\gamma}\partial_{\gamma}g_{\alpha\rho}\right). \quad (5.62)$$

4. Variation of stress-tensor:

$$\delta T^{ab}=-Q_{5}(\sin\theta\delta\tan\theta)\gamma^{ab}-Q_{5}\frac{1}{\cos\theta}\left(2K_{\rho}^{ab}\delta X^{\rho}\right) \quad (5.63)$$
$$+Q_{5}\left(\delta\left(v^{a}\right)v^{b}+v^{a}\delta\left(v^{b}\right)+\delta\left(w^{a}\right)w^{b}+w^{a}\delta\left(w^{b}\right)\right)\tan\theta\sin\theta \quad (5.64)$$
$$+Q_{5}\left(v^{a}v^{b}+w^{a}w^{b}\right)\sin\theta\delta(\tan\theta)+Q_{5}\left(v^{a}v^{b}+w^{a}w^{b}\right)\sin\theta\cos^{2}\theta\delta(\tan\theta). \quad (5.65)$$

A.3

Then from (5.48) and (5.49) we can obtain the equation of perturbative variables. For the intrinsic case,

$$\delta\left(\nabla_{a}T^{ab}\right)=\delta\mathcal{F}^{b}$$
$$\Leftrightarrow\nabla_{a}\delta T^{ab}-T^{cb}\nabla_{c}\left(K^{\rho}\left(\delta X_{\perp}^{\alpha}g_{\alpha\rho}\right)\right)-2T^{ac}\nabla_{a}\left(K^{b}{}_{c}{}^{\rho}\left(\delta X_{\perp}^{\alpha}g_{\alpha\rho}\right)\right) \quad (5.66)$$
$$+T^{ac}\nabla^{b}\left(K_{ac}{}^{\rho}\left(\delta X_{\perp}^{\alpha}g_{\alpha\rho}\right)\right)=\delta\mathcal{F}^{b}, \quad (5.67)$$

insert $\delta\left(\mathcal{F}^{b}\right)=\delta\left(\gamma^{ba}\partial_{a}X^{\nu}g_{\mu\nu}\mathcal{F}^{\mu}\right)$ and express all the terms in dependence of variables, we can finally derive (5.50), (5.51) and (5.52).

For extrinsic case,

$$\delta\left(T^{ab}K_{ab}^{i}\right)=\delta\mathcal{F}^{i} \quad (5.68)$$
$$\Leftrightarrow\delta T^{ab}K_{ab}{}^{i}+n_{\mu}^{i}T^{ab}\nabla_{a}\left(\partial_{b}\delta X_{\perp}^{\mu}\right)+n_{\sigma}^{i}T^{ab}\Gamma^{\sigma}{}_{\mu\nu}\delta\left(\partial_{a}X^{\mu}\partial_{b}X^{\nu}\right) \quad (5.69)$$
$$+n_{\sigma}^{i}T^{ab}\partial_{a}X^{\mu}\partial_{b}X^{\nu}\delta X_{\perp}^{\alpha}\partial_{\alpha}\Gamma^{\sigma}{}_{\mu\nu} \quad (5.70)$$
$$+\frac{1}{2}T^{ab}K_{ab}{}^{\rho}n^{\alpha i}\delta X_{\perp}^{\gamma}\partial_{\alpha\rho}=\delta\mathcal{F}^{i}. \quad (5.71)$$

Here $\delta\mathcal{F}^{i}=\delta\left(\mathcal{F}^{\mu}n_{\mu}^{i}\right)=\delta\mathcal{F}^{\mu}n_{\mu}^{i}+\mathcal{F}^{\mu}\delta n_{\mu}^{i}$. Set $n^{1}=d\psi$ and $n^{2}=dr$, we can derive

(5.53) and (5.54).

5.4.1 Radial Instability

5.4.2 Instability Range

5.5 Numerical KS Black Hole

KS black hole (KSBH) is a black hole solution on warped deformed conifold (e.g. KS geometry) in type IIB SUGRA with fluxes and global $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathbb{Z}_2$ symmetry[8]. The ten-dimensional type IIB supergravity action, which can be extracted from the gravitational approximation of IIB superstring action, takes the form

$$S_{10} = \frac{1}{16\pi G_{10}}\int_{\mathcal{M}_{10}}[\star R_{10} - \frac{1}{2}d\phi\wedge\star d\phi - \frac{1}{2}e^{-\phi}H_3\wedge\star H_3 - \frac{1}{2}e^{\phi}\tilde{F}_3\wedge\star\tilde{F}_3 \\ - \frac{1}{4}\tilde{F}_5\wedge\star\tilde{F}_5 + \frac{1}{2}C_4\wedge H_3\wedge F_3], \tag{5.72}$$

one can notice that this action is different to that used in **4.1 Equ(TBA)** only by a F_1 contribution. This is because we have set the axion C_0 to zero. Besides, we have $\tilde{F}_3 = F_3 = dC_2$, $\tilde{F}_5 = F_5 - C_2\wedge H_3 = dC_4 - H_3\wedge C_2$. With the aid of the self-duality condition

$$\star F_5 = F_5, \tag{5.73}$$

we can obtain the equations of motion by variating RR fields C_2, C_4 again, but this time we do not have the one obtained from C_0 field.

$$d\star\tilde{F}_3 - H_3\wedge F_5 = -16\pi G\star J_2, \tag{5.74}$$

$$d\star\tilde{F}_5 - H_3\wedge F_3 = -16\pi G\star J_4, \tag{5.75}$$

[8]For general warped deformed conifold, the global symmetry is $SU(2)\times SU(2)\times\mathbb{Z}_{2M}\times\mathbb{Z}_2$, while the appendix **C.1.1** tells us the $\mathbb{Z}_{2M}$ will break the R-symmetry of SU(M) SYM theory to $\mathbb{Z}_2$ and the symmetry of KS metric becomes $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathbb{Z}_2$

the equation for B_2 is not displayed above, it is insignificant for the discussion in this section. Still, although we do not know the exact expressions of these Maxwell currents, we can mimic their effects far away by using Maxwell charges to construct a set of equivalent currents. The D3-NS5 bound state metric, which is the seed metric of KS geometry with $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{Z}_{2M}$ symmetry, is still (5.4) and the RR fields are

$$\begin{aligned} \tilde{F}_3 &= F_3 = -2r_h^2 r^{-3} D^2 H^{-2} \sin\theta \cos\theta dx^1 \wedge dx^2 \wedge dr \\ \tilde{F}_5 &= 2r_h^2 r^{-3} H^{-2} \sin\theta dt \wedge dx^3 \wedge dx^4 \wedge dx^5 \wedge dr \end{aligned} \tag{5.76}$$

By integrating over the appropriate manifolds, the Maxwell charges correspond to these fluxes can be obtained. Here we are only interested in D3 charge which is the integration of J_4 source which induced by $\tilde{F}_5$.

$$\begin{aligned} Q_3 = \int_{\mathcal{M}_6} \star J_4 &= -\frac{1}{16\pi G_{10}} \int_{\mathcal{M}_6} d \star \tilde{F}_5 \\ &= -\frac{1}{16\pi G_{10}} \int_{\Sigma_5 \equiv \partial \mathcal{M}_6} \star \tilde{F}_5 \\ &= -\frac{1}{16\pi G_{10}} \int_{\Sigma_5} \frac{2rr_h^2}{(r^2+r_h^2)^2} \sin\theta \; dx^1 dx^2 \wedge d\Omega_3 \\ &= C \int_{\Sigma_2} \frac{rr_h^2}{(r_h^2+r^2)^2} \sin\theta \, dx^1 \wedge dx^2 \\ &= C vol_2 r_h^2 \sin\theta. \end{aligned} \tag{5.77}$$

The last step takes use of the fact that $r/(r^2+r_h^2)^2$ is just the Jacobian factor of the volume form constructed by $dx^1 \wedge dx^2$. This D3 charge is the same as previously derived result (5.15). Therefore, we have shown that the ten-dimensional KS geometry, once the equivalent current technique has been applied, provide consistent Maxwell D3 charge to that obtained from charged branes.

5.5.1 Truncated Fine-Dimensional Effective KS Action

From the cascade gauge theory at strong coupling, as C.1 showed, the IIB string theory on the warped conifold with fluxes can be decribed by $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{Z}_2$

symmetry. As the full ten-dimensional KS action cannot provide us any new information about the D3 charges, we would like to follow the steps Buchel [**?**] did to obtain a five-dimensional effective action. We will take Einstein frame to accommodate with the original notation of [**?**].

The effective action is obtained from KK reduction of ten-dimensional full action with few constraints taken into account (e.g. self-duality equation of $\tilde{F}_5$, Bianchi identity, etc.). With assistance of **Sec. 2.1**, to smoothly explain how this effective action comes, let us start from analyzing the type IIB SUGRA equations of motion to leading order in M/N (suppose the cascade duality originates from $\mathrm{SU}(N+M)\times\mathrm{SU}(N)$). The background has to contain M units of RR three-form flux through the three-cycle of $T_{1,1}$,

$$\int_{3-cycle} H_3 = M. \tag{5.78}$$

If M is fixed as $N \to \infty$ then the backreaction of H_3 on the metric and the $\tilde{F}_5$ background may be ignored to leading order of N. So it is reasonable to guess that there is no backreaction effect to the order of M/N. On the other hand, the NSNS potential should be

$$B_2 = e^{\Phi} f(r)\omega_2, \tag{5.79}$$

where ω_2 is a closed two-form corresponding to the two-cycle dual to the three-cycle. The RG flow can be expressed as [**?**]

$$\frac{1}{g_1^2} - \frac{1}{g_2^2} \sim e^{-\phi}\left(\int_{C^2} B_2 - \frac{1}{2}\right), \tag{5.80}$$

where g_1 and g_2 are the gauge coupling constants of $\mathrm{SU}(N+M)\times\mathrm{SU}(N)$, respectively. Now consider the corresponding β-function in field theory perspective, then

$$\frac{1}{g_1^2} - \frac{1}{g_2^2} \sim M \ln\frac{\Lambda}{\mu} \tag{5.81}$$

if one treats $M/N \to 0$ near the fixed point. Λ corresponds to a characteristic scale (e.g. r).

For IIB SUGRA, we hope that we can reproduce (5.81) by inserting explicit H_3, F_3 into (5.80). Consider still $AdS_5 \times T^{1,1}$ as the background and set $\tau = C_0 + ie^{-\Phi}$ as a combined scalar field, the equations of motion can be rewritten as

$$d(\star G) = iF_5 \wedge G, \tag{5.82}$$

where $G_3 \equiv F_3 + \tau H_3$ satisfies the Bianchi identity $dG = 0$. Note that we have absorbed the Chern–Simons term $C_4 \wedge H_3 \wedge F_3$ into the right hand side of this equation.

Since the fractional D3 branes create RR flux through $T^{1,1}$, F_3 should be proportional to the closed three-form

$$F_3 = Pe_\psi \wedge (e_{\theta_1} \wedge e_{\phi_1} - e_{\theta_2} \wedge e_{\phi_2}) \sim Me_\psi \wedge (e_{\theta_1} \wedge e_{\phi_1} - e_{\theta_2} \wedge e_{\phi_2}) \tag{5.83}$$

with

$$e_\psi = \frac{1}{3}(d\psi + \sum_{i=1}^{2} \cos\theta_i d\phi_i), \quad e_{\theta_i} = \frac{1}{\sqrt{6}} d\theta_i, \quad e_{\phi_i} = \frac{1}{\sqrt{6}} \sin\theta_i d\phi_i. \tag{5.84}$$

In this coordinates, ω_2 can be written as

$$\omega_2 = \frac{1}{\sqrt{2}}(e_{\theta_1} \wedge e_{\phi_1} - e_{\theta_2} \wedge e_{\phi_2}), \tag{5.85}$$

and then we can get

$$e^{-\Phi} H_3 \sim \frac{df(r)}{dr} dr \wedge (e_{\theta_1} \wedge e_{\phi_1} - e_{\theta_2} \wedge e_{\phi_2}). \tag{5.86}$$

Since $\tilde{F}_5 = vol(AdS_5) + vol(T^{1,1})$, $F_5 \wedge H_3 = 0$. Let us force $C_0 = 0$, then the real part of (5.82) can easily be satisfied, and from the imaginary part of (5.82) we have

$$\frac{1}{r^3}\frac{d}{dr}\left(r^5 \frac{df(r)}{dr}\right) \sim M \tag{5.87}$$

which is equivalent to

$$f(r) \sim M \ln r \tag{5.88}$$

Thus, IIB SUGRA equations reproduced the field theoretic β-function for $\frac{1}{g_1^2} - \frac{1}{g_2^2}$ to the order of M/N. This establishes the gravity dual of the logarithmic RG flow in the $\mathcal{N} = 1$ supersymmetric SU$(M+N) \times$ SU(N) gauge theory on N regular and M fractional D3 branes placed at the conifold singularity.

However, one can notice that there are no other backreaction effects induced from three-form field strengths, dilaton and the five-form field strengths. To add these terms, let us now generalize the formalism above to the order of $\left(\frac{M}{N}\right)^2$. We first introduce the following ansatz in which the metric is spanned over the five-dimensional spacetime metric and the internal five-manifold metric (these two metrics hold the same symmetry):

$$ds_{10}^2 = L^2 \left[e^{-\frac{2}{3}(A+4B)} ds_5^2 + ds_{5'}^2 \right], \tag{5.89}$$

$$ds_{5'}^2 = \frac{1}{9} e^{2A} \left(d\psi + \sum_{i=1}^{2} \cos\theta_i d\phi_i \right)^2 + \frac{1}{6} e^{2B} \sum_{i=1}^{2} \left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2 \right), \tag{5.90}$$

where B, C are functions of spacetime coordinates used to preserve the Einstein frame under compactification to five dimensions. L is the scale related to the AdS_5 radius which we will set it to 1 in our discussion.

Again, C_0 is treated as zero, consider the spacetime metric we have

$$ds_5^2 = dr^2 + e^{2C(r)} dx_i^2 = g_{\mu\nu} dy^\mu dy^\nu, \tag{5.91}$$

then we can assume that B, C and ten-dimensional dilaton Φ are all functions of r. Following the same reasoning as in the order M/N,

$$F_3 = P e_\psi \wedge \omega_2. \tag{5.92}$$

P is a constant proportional to M. On the other hand, the NSNS potential is

$$B_2 = T(r)\omega_2 \sim f(r)\omega_2 \quad \Rightarrow \quad H_3 = \frac{dT(r)}{dr} dr \wedge \omega_2, \tag{5.93}$$

where $T(r)$ is a function proportional to $f(r)$ which enters as a scalar field. Furthermore,

$$\begin{aligned}\tilde{F}(5) &= K(r)[vol(AdS_5) + vol(T^{1,1})] = K(r)vol(T^{1,1}) + K(r)[\star vol(T^{1,1})] \\ &= K(r)e_\psi \wedge e_{\theta_1} \wedge e_{\phi_1} \wedge e_{\theta_2} \wedge e_{\phi_2} + e^{4C-\frac{8}{3}A+4B} K(r) dr \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3.\end{aligned} \tag{5.94}$$

Note that $\tilde{F}_5 \wedge H_3 = 0$.

The ansatz preserves the symmetries between the S^2 factors of (5.90). It is possible that we have a solution which keeps these two factors equal even if there are different warp factors for the two S^2.

From the self-duality condition of $\tilde{F}_5$, one induces a relation between K and T,

$$\partial_r K = P \partial_r T \quad \Rightarrow \quad K(r) = Q + PT(r). \tag{5.95}$$

Here the constant Q plays the role of the five-brane charge (e.g. NS5 charge) Q_5 if one takes $P = 0$. With backreaction of which P has a non-zero value, it is also possible to redefine C_4 to absorb Q into the function $T(r)$. For instance, we redefine

$$C_4' \rightarrow C_4 + 5B_2C_2. \tag{5.96}$$

Now $d^2C_4 = 0$ so that C_4 must contain the five-form volume $vol(T^{1,1})$ with constant Q. Moreover, the vanished running coupling equation of the dilaton requires

$$\partial_r T = P e^{\Phi} e^{-\frac{4}{3}(A+B)}, \tag{5.97}$$

which means that T is dependent to A and B. Then by taking the equation of motion for $\tilde{F}_5$ one can obtain the five-dimensional effective action in terms of warp factors depend on the radial coordinate and the dilaton after KK reduction:

$$S_5 = -\frac{2}{\kappa_5^2}\int d^5x\sqrt{g_5}\left[\frac{1}{4}R_5 - \frac{1}{2}G_{ab}(\varphi)\partial\varphi^a\partial\varphi^b - V(\varphi)\right], \tag{5.98}$$

where $\varphi^a = (p, q, \Phi, T)$ denotes a collection of the scalar fields with

$$q = \frac{2}{15}(A + 4B), \quad f = -\frac{1}{5}(A - B), \tag{5.99}$$

these two functions measure the volume and the ratio of scales of the internal manifold. They have dimension $\pm(\text{mass})^2$, where the sign depends on that of cosmological constant. And

$$G_{ab}(\varphi)\partial\varphi^a\partial\varphi^b = 15(\partial q)^2 + 10(\partial f)^2 + \frac{1}{4}(\partial\Phi)^2 + \frac{1}{4}e^{-\Phi-4f-6q}(\partial T)^2, \tag{5.100}$$

$$V(\varphi) = e^{-8q}\left(e^{-12f} - 6e^{-2f}\right) + \frac{1}{8}P^2e^{\Phi+4f-14q} + \frac{1}{8}(Q + PT)^2e^{-20q}. \tag{5.101}$$

The red term comes from H_3^2, while the first term of the potential has its origin from R_{10} which reflects the curvature of the internal space. This also implies that one can express S_5 in terms of R_{10} rather than R_5 if one determines the curvature recursive relation between them. The green and blue term are obtained from F_3^2 and F_5^2, respectively. Meanwhile, one should note that the Q part of the blue term can be absorbed into T once we redefine C_4. Therefore, we have obtained an effective action which only has dependence on Φ and some warp factors. This is the spirit of the truncated effective KS geometry. Now we consider a variant of the Einsten frame metric as

$$ds_{10}^2 = g_{\mu\nu}(y)dy^\mu dy^\nu + \Omega_1^2(y)g_5^2 + \Omega_2^2(y)\left[g_3^2 + g_4^2\right] + \Omega_3^2(y)\left[g_1^2 + g_2^2\right], \tag{5.102}$$

where

$$
\begin{aligned}
g_1 &= \frac{\alpha^1 - \alpha^3}{\sqrt{2}}, \quad g_2 = \frac{\alpha^2 - \alpha^4}{\sqrt{2}}, \\
g_3 &= \frac{\alpha^1 + \alpha^3}{\sqrt{2}}, \quad g_4 = \frac{\alpha^2 + \alpha^4}{\sqrt{2}}, \\
g_5 &= \alpha^5,
\end{aligned} \tag{5.103}
$$

and

$$
\begin{aligned}
\alpha^1 &= -\sin\theta_1 d\phi_1, \quad \alpha^2 = d\theta_1, \\
\alpha^3 &= \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2, \\
\alpha^4 &= \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2, \\
\alpha^5 &= d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2.
\end{aligned} \tag{5.104}
$$

And indices μ, ν runs over 0 to 4 with y^μ denotes the coordinate parametrizes $\mathcal{M}_5$. What we should do now is just to rewrite the ansatz in terms of g^n (n runs over 1 to 5) with the warp factors kept.

$$
\begin{aligned}
B_2 =& h_1(y) g_1 \wedge g_2 + h_3(y) g_3 \wedge g_4, \\
F_3 =& \frac{1}{9} P g_5 \wedge g_3 \wedge g_4 + h_2(y) \left(g_1 \wedge g_2 - g_3 \wedge g_4\right) \wedge g_5, \\
& + \left(g_1 \wedge g_3 + g_2 \wedge g_4\right) \wedge d\left(h_2(y)\right) \\
\Phi =& \Phi(y),
\end{aligned} \tag{5.105}
$$

and

$$
F_5 = dC_4 + \left(h_2(y)\left(h_3(y) - h_1(y)\right) + \frac{1}{9} P h_1(y)\right) g_5 \wedge g_3 \wedge g_4 \wedge g_1 \wedge g_2. \tag{5.106}
$$

Again, with a slightly change over the volume form of internal manifold, we have

$$
dC_4 = \frac{K(y)}{\Omega_1 \Omega_2^2 \Omega_3^2} \operatorname{vol}_{\mathcal{M}_5} \equiv \frac{K(y)}{\Omega_1 \Omega_2^2 \Omega_3^2} \sqrt{-\det\left(g_{\mu\nu}\right)} dy^1 \wedge \cdots \wedge dy^5 \tag{5.107}
$$

and then the self-duality condition requires

$$K(y) = h_2(y)\left(h_3(y) - h_1(y)\right) + \frac{1}{9}Ph_1(y). \tag{5.108}$$

Here we generalize from the warp factors to have dependence on general y rather than r. From the five-dimensional perspective, let us allow fluctuations in the metric and scalar fields (warp factors) $\{\Omega_1, \Omega_2, \Omega_3, h_1, h_2, h_3, \Phi\}$. This implies that we have more parameters than the case with order M/N to constrain the parameter space and related truncation. Now perform the KK reduction we have, term by term, first,

$$\int_{\mathcal{M}_{10}} \star R_{10} = 108\, vol_{T^{1,1}} \int_{\mathcal{M}_5} \Omega_1\Omega_2^2\Omega_3^2\, vol_{\mathcal{M}_5}. \tag{5.109}$$

Besides, the Ricci scalar of different dimensions can be matched by

$$\begin{aligned} R_{10} = R_5 &+ \left(\frac{1}{2\Omega_1^2} + \frac{2}{\Omega_2^2} + \frac{2}{\Omega_3^2} - \frac{\Omega_2^2}{4\Omega_1^2\Omega_3^2} - \frac{\Omega_3^2}{4\Omega_1^2\Omega_2^2} - \frac{\Omega_1^2}{\Omega_2^2\Omega_3^2}\right) - 2\Box\ln\left(\Omega_1\Omega_2^2\Omega_3^2\right) \\ &- \left\{\left(\nabla\ln\Omega_1\right)^2 + 2\left(\nabla\ln\Omega_2\right)^2 + 2\left(\nabla\ln\Omega_3\right)^2 + \left(\nabla\ln\left(\Omega_1\Omega_2^2\Omega_3^2\right)\right)^2\right\}, \end{aligned} \tag{5.110}$$

where the covariant derivative construct a covariant vector by itself when acting on scalar Ω_i,

$$\begin{aligned} \nabla_\lambda\Omega_i &= \partial_\lambda\Omega_i, \\ \nabla_\lambda\nabla_\nu\Omega_i &= \partial_\lambda\partial_\nu\Omega_i - \Gamma^\rho_{\lambda\nu}\partial_\rho\Omega_i. \end{aligned}$$

Then the five-dimensional effective action can be written as

$$\begin{aligned} S_5 =& \frac{108}{16\pi G_5}\int_{\mathcal{M}_5} \mathrm{vol}_{\mathcal{M}_5}\, \Omega_1\Omega_2^2\Omega_3^2 \left\{ R_{10} - \frac{1}{2}(\nabla\Phi)^2 \right. \\ & - \frac{1}{2}e^{-\Phi}\left(\frac{\left(h_1 - h_3\right)^2}{2\Omega_1^2\Omega_2^2\Omega_3^2} + \frac{1}{\Omega_3^4}\left(\nabla h_1\right)^2 + \frac{1}{\Omega_2^4}\left(\nabla h_3\right)^2\right) \\ & - \frac{1}{2}e^{\Phi}\left(\frac{2}{\Omega_2^2\Omega_3^2}\left(\nabla h_2\right)^2 + \frac{1}{\Omega_1^2\Omega_2^4}\left(h_2 - \frac{P}{9}\right)^2 + \frac{1}{\Omega_1^2\Omega_3^4}h_2^2\right) \\ & \left. - \frac{1}{2\Omega_1^2\Omega_2^4\Omega_3^4}\left(4\Omega_0 + h_2\left(h_3 - h_1\right) + \frac{1}{9}Ph_1\right)^2 \right\}. \end{aligned} \tag{5.111}$$

5.5.2 Maxwell D3 Charge of KSBH

5.5.3 Instability of KSBH

5.5.4 Deterministic Topological Term

In this subsection, we would like to explore the interconnection between the KSBH and blackfold described KPV. It seems that the near-metastability of ANNOR has a D3 charge which possess an opposite value to that of KSBH Buchel constructed, however, one should note again the difference between the Page charge and the Maxwell charge. In ANNOR, the negative D3 charge near metastability if the Page charge. While in Buchel, the D3 charge which keeps a positive value is the Maxwell charge. It is possible for us to examine if the KSBH is somewhat a particular case can be described by ANNOR, this may reveal us how the qualitative non-extremal perturbative behavior of blackfold described KPV would have without explicitly compute the perturbation equations of blackfold equations.

The difference of the Page charge and the Maxwell charge has an interesting expression which is topologically related, it is this fact that supports our analysis. We can calculate as follow:

$$\begin{aligned} Q_3 - \mathcal{Q}_3 &= \int_{T_{\partial KS}} \tilde{F}_5 - \int_{T_H} F_5 - \int_{KS} H_3 \wedge F_3 \\ &= -\int_{T_{\partial KS|_H}} H_3 \wedge C_2 - \int_{KS} H_3 \wedge F_3 \\ &= -2 \int_{KS} H_3 \wedge F_3. \end{aligned} \tag{5.112}$$

In the second equality, we have taken a cut-off at the horizon geometry limit for $T_{\partial KS}$ so that the first two integrations can be combined together. This can be reasonable because what we care about is only the region near the metastability where the horizon can be raised to cloak the singularity. The third equality comes from Gauss's theorem. This implies that the difference of Page charge and Maxwell charge can be controlled by a topological term related to Euler characteristic number.

We first consider the specific example, i.e. Buchel's KSBH, assume this black hole solution is one particular solution which can be obtained from blackfold described KPV, its Page D3 charge near metastability will keeps in a negative value, meanwhile, the Maxwell D3 charge is positive. Thus if our conjecture is true (at least for KSBH), we should expect that the topological term can be positive so that the difference between Page and Maxwell (we call this PM difference later on) charges is negative. Now we are going to check this,

$$\begin{aligned}\int_{KS} H_3 \wedge F_3 &\sim \int_{\mathcal{M}_6} (4M^4 b_0^4 \sin^4\psi \sin^2\omega) dr \wedge d\tilde{\omega} \wedge d\tilde{\varphi} \wedge d\psi \wedge d\omega \wedge d\varphi \\ &= 6\pi^2 M^4 b_0^4 vol_4, \end{aligned} \tag{5.113}$$

where vol_4 is the integration over the volume form $dr \wedge d\tilde{\omega} \wedge d\tilde{\varphi} \wedge d\varphi$, this is a positive volume. The second equality comes from insertion of (5.2) and (5.3). Therefore, we obtained that for Buchel's KSBH, the PM difference

$$Q_3 - \mathcal{Q}_3 \propto -2 \int_{KS} H_3 \wedge F_3 < 0 \tag{5.114}$$

This shows that our conjecture makes sense, one can take use of the PM difference to detect the interconnection between KSBH and blackfold described KPV, also some of the non-extremal perturbative behaviors can be extracted from this perspective. One should note that, to this stage, we have assumed that KSBH is a particular case decribed by ANNOR. Actually, the only fact we can confirm is that the Maxwell charge of KSBH is positive and the PM difference is negative, it is not necessary for the Page D3 charge of KSBH to be negative. And the positivity of this Page charge will reveal us how the Buchel's KSBH related to the blackfold formalism. We list the few possibilities below:

1. The Page charge is negative $\Rightarrow$ The KSBH solution is located at a metastability of blackfold analysis, which is the case we expected in previous analysis. Nevertheless, because of the positivity of Maxwell D3 charge, this solution, although has negative Page charge, should be very close to the merger point so that admit a perturbative instability.

2. The Page charge is positive $\Rightarrow$ The KSBH solution is located at a merged instability if one can deduce the similar elastic instability as that of perturbative blackfold equations. Otherwise there is no interesting connection between KSBH and blackfold described KPV.

3. The Page charge is negative, and the Maxwell charge has a small region of negative value (metastability) under perturbation $\Rightarrow$ In addition to the conclusions of case one, there exists a geometric transition from metastability to instability, corresponding to the negative charge and positive charge, respectively. And this fact also implies that the topological term perspective can reveal us some non-extremal perturbative behaviors with respect to KSBH. This would be the most interesting case.

5.5.5 Restore the Instability Range

5.6 Non-Extremal Perturbation

The perturbation for the charge conservation equations are the same as that derived in extremal limit, the only difference is that we require

$$\partial_a \delta(Q_3^{Page} - Q_3^{Max}) = 0 \tag{5.115}$$

rather than $\partial_a \delta(Q_3^{Page}) = 0$. This then gives

$$\delta \tan\theta + \frac{2\cot\psi_0 \tan\theta - b_0^2(\cot^2\psi_0 - \frac{3}{b_0^4})}{\frac{b_0^2}{4\pi M}(\cot^2\psi_0 - \frac{1}{b_0^4}) + 1}\delta\psi = 0. \tag{5.116}$$

Taking the extremal limit for this equation with $\cot\psi_0 = \frac{1}{b_0^2}$, it nicely comes to

$$\delta \tan\theta + 2(\tan\theta\cot\psi + \frac{1}{b_0^2})\delta\psi = 0, \tag{5.117}$$

which perfectly match to the extremal perturbation equation (5.43)! This is a strong hint that the topological perspective can display some non-extremal perturbative behaviors of blackfold described KPV.

5.6.1 Instability Insights Geometry

[**?**] A. Buchel, *Klebanov–Strassler Black Holes*.

Chapter 6

Epilogue

6.1 Conclusions

6.2 Outlooks

[]

Appendix A

Complex Geometry

A.1 Kähler Geometry

A.2 Orbifolds and Orientifolds

A.3 Moduli Space

Appendix B

Bestiary of Superstring Theory, F-Theory and M-Theory

B.1 Tadpole Cancellation

Appendix C

Klebanov–Strassler Geometry

C.1 Warped Conifold

From AdS/CFT correspondence, Maldacena pointed out that a stack of N coinciding D3 branes can realize a $\mathcal{N} = 4$ four-dimensional supersymmetric SU(N) gauge theory. Meanwhile, it also creates a curved background of ten-dimensional theory of closed superstrings with metric:

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-\frac{1}{2}} [-(dx^0)^2 + (dx^i)^2] + \left(1 + \frac{L^4}{r^4}\right)^{\frac{1}{2}} (dr^2 + r^2 d\Omega_5^2), \tag{C.1}$$

where L is the curvature of the embedding geometry and Ω_5 denotes the angular basis of a unit five-sphere. In the full text part, we usually take the notion $h(r) = 1 + \frac{L^4}{r^4}$.

For this metric, if one requires $r \to 0$ with low energy approximation, the metric becomes

$$ds^2 = \frac{L^2}{z^2}[-(dx^0)^2 + (dx^i)^2 + dz^2] + L^2 d\Omega_5^2, \tag{C.2}$$

which describes the direct product of AdS_5 and S^5 with equal radius of curvature L, $z \equiv \frac{L^2}{r}$.

To study the generalization of the basic AdS/CFT correspondence, physicists found that the behaviors of branes at the conical singularities are remarkable. Left the definition of conical singularity to later discussion, let us first consider a stack of N

D3 branes placed at the tip of a six-dimensional Ricci-flat cone X_6 whose base space is a five-dimensional Einstein manifold Y_5 so that $ds^2_{X_6} = dr^2 + r^2 ds^2_{Y_5}$ (as depicted in Fig. **A.1**). This construction can significantly reduce the number of supersymmetries in AdS/CFT. Taking the near horizon limit of the background created by the N D3 branes, we can obtain the spacetime $AdS_5 \times Y_5$ whose radius can match to the brane tensions. And then a generalized version of AdS/CFT correspondence can be proposed as[Kachru, Silverstein]:

Type IIB string theory on $AdS_5 \times X_5$ is dual to the IR limit of the worldvolume theory on the parallel coincident D3 branes located at the conical singularity.

And the Klebanov–Strassler geometry is motivated from such conical singularity of type IIB supergravity solution. Let us explain these elements in more detail.

C.1.1 D3 Branes on the Conifold

Definition **A.1**: The conifold is a Calabi–Yau three-fold cone X described by the constraint $\sum_{a=1}^4 z_a^2 = 0$ which is invariant under an overall real rescaling of the local coordinates, where z_a are four complex variables.

As depicted in Fig. **A.1**, the base space of the conifold X is a coset space $T^{1,1}$ which has symmetry[1] SU(2) $\times$ SU(2)/U(1). According to the definition, the metric on the conifold can be written as

$$ds_6^2 = dr^2 + r^2 ds^2_{T^{1,1}}, \tag{C.3}$$

[1]To identify this, we should note that $z_a = 0$ denote the singularities and transform in SO(4), besides, the U(1) transformation $z_a \to e^{i\theta} z_a$ preserves the constraint equation for all $\theta \in \mathbb{C}$.

After omitting the singularity at the origin, dividing the conifold by $\mathbb{R}_+$ is equivalent to intersect it to the unit sphere $\sum_a |z_a|^2 = 1$. Since SO(4) acts transitively on this intersection, any given point on the intersection is invariant under only a single U(1) $\subset$ SO(4). For instance, define $P: (z_1, z_2, z_3, z_4) = (1, i, 0, 0)$, it is only invariant under the subgroup U(1) $\cong$ SO(2) that rotates z_3 and z_4. Similar construction can be taken for z_1 and z_2. Therefore, Sym(X_5) = SO(4)/U(1) = SO(2) $\times$ SO(2)/U(1) and the base space metric can be constructed by its symmetry as:

$$ds^2_{T^{1,1}} = \frac{1}{9}\left(d\psi + \sum_{i=1}^{2}\cos\theta_i d\phi_i\right)^2 + \frac{1}{6}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2), \tag{C.4}$$

where ψ is the angular coordinate runs over 0 to 4π and (θ_i, ϕ_i) parametrize two two-cycles, which has a S^2 structure, on the manifold. Thus the topology of $T^{1,1}$ is $S^2 \times S^3$.

C.2 Kähler Potential and Superpotential

Bibliography

[Buchel] A. Buchel, Klebanov–Strassler Black Holes.