Comparison of Vector, Euler Angle, Lie Group, Quaternion, and Lie Algebra

Content

Rotation

*Vector

*Vector method is mathematic and easy to undertand, but fussy :-)

Upper letters (e.g. A) denote the corrdinate systems. Letters in lowercase with a arrow above (e.g. $\vec{p}$) denote vectors. Vectors with subscript $1, 2, 3$ (e.g. $\vec{a}_1, \vec{a}_2, \vec{a}_3$) show the basic vectors (x, y, z direction) in coordinate system A coordinate system. An additional underline (e.g. $\underline{\vec{a}}$) is the combination of basic vectors, where

$$\underline{\vec{a}} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix}.$$

Letters with underline ($\underline{p}$) denote a $n \times 1$ vector. Underlined letters with additional coordinate system symbol as subscript (e.g. $\vec{p}_A$) denote the projections of vector $\vec{p}$ in coordinate system A, where

$$\underline{p}_A = \begin{bmatrix} \vec{p} \cdot \vec{a}_1 \\ \vec{p} \cdot \vec{a}_2 \\ \vec{p} \cdot \vec{a}_3 \end{bmatrix}.$$

Obviously, it is valid that

$$\vec{p} = \underline{p}_A^{\mathrm{T}} \cdot \underline{\vec{a}},$$

which also means that $\underline{p}_A$ is the coordinate of $\vec{p}$ in coordinate system A.

Suppose we have 2 coordinate systems A and B. When B rotate θ degree around axis $\vec{b}_1$, we have the following relationship

$$\underline{\vec{a}} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix} =: R_{AB} \cdot \underline{\vec{b}}.$$

To describe vector $\vec{p}$, we have

$$\vec{p} = \underline{p}_A^{\mathrm{T}} \cdot \underline{\vec{a}} = \underline{p}_A^{\mathrm{T}} \cdot R_{AB} \cdot \underline{\vec{b}} = \underline{p}_B^{\mathrm{T}} \cdot \underline{\vec{b}},$$

which means the coordinate of $\vec{p}$ in B can be described as

$$\underline{p}_B^{\mathrm{T}} = \underline{p}_A^{\mathrm{T}} \cdot R_{AB}.$$

After transposition, it holds

$$\begin{bmatrix} p_B^1 \\ p_B^2 \\ p_B^3 \end{bmatrix} = \underline{p}_B = R_{AB}^{\mathrm{T}} \cdot \underline{p}_A = R_{AB}^{\mathrm{T}} \cdot \begin{bmatrix} p_A^1 \\ p_A^2 \\ p_A^3 \end{bmatrix}.$$

Explaination. We will explain the equation in two ways:

- Rotate of coordinate system: Vector $\vec{p}$ stay static, while coordinate system A rotate to B .

  For multiple coordinate systems:

  $$\underline{\vec{a}} = R_{AB} \cdot \underline{\vec{b}} = R_{AB} R_{BC} \cdot \underline{\vec{c}} = R_{AB} R_{BC} R_{CD} \cdot \underline{\vec{d}} =: R_{AD} \cdot \underline{\vec{d}}.$$

  We can see, the order of coordinate transformation is from left to right (R_{AB}, R_{BC}, R_{CD})

- Rotate of vector: the coordinate B didn't rotate, but the vector rotate $-\theta$ degree from $\vec{p}_A$ to $\vec{p}'_A$, where

  $$\underline{p}'_A = \underline{p}_B = R_{AB}^{\mathrm{T}} \cdot \underline{p}_A.$$

  For multiple coordintate systems:

  $$\underline{p}'_A = R_{CD}^{\mathrm{T}} \cdot R_{BC}^{\mathrm{T}} \cdot R_{AB}^{\mathrm{T}} \cdot \underline{p}_A = R_{AD}^{\mathrm{T}} \cdot \underline{p}_A$$

  We can see, the order of vector transformation is from right to left ($R_{AB}^{\mathrm{T}}, R_{BC}^{\mathrm{T}}, R_{CD}^{\mathrm{T}}$)

  Further, we define the rotate matrix $R(\theta)$ where

  $$\underline{p}'_A := R(\underline{\theta}) \cdot \underline{p}_A = R_{AD}^{\mathrm{T}}(-\underline{\theta}) \cdot \underline{p}_A,$$

  Then:

  $$R(\theta) = R_{AD}^{\mathrm{T}}(-\underline{\theta}) = R_{AD}.$$

> **Example**
> We choose $\theta = \pi/6, \underline{p}_A^{\mathrm{T}} = \begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \end{bmatrix}$ then:

$$R_{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix},$$

Explained as coordinate A rotating θ degree to B:

$$\begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{bmatrix}.$$

Explained as $\underline{p}_A$ rotating $-\theta$ degree to $\underline{p}'_A$:

$$\underline{p}'_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix} \approx \begin{bmatrix} 0.577 \\ 0.789 \\ 0.211 \end{bmatrix}.$$

Note: to simplify the expression, coordinates are usually calculated to the identical coordinate system (usually the world coordinate W), thus, we throw the vector $\underline{\vec{w}}$, e.g. we denote the following three vectors

$$\vec{p} = \underline{p}_A^{\mathrm{T}} \cdot \underline{\vec{a}}, \quad \vec{q} = \underline{q}_B^{\mathrm{T}} \cdot \underline{\vec{b}}, \quad \vec{k} = \underline{k}_C^{\mathrm{T}} \cdot \underline{\vec{c}}$$

as

$$\vec{p} = \underline{p}_A^{\mathrm{T}} \cdot R_{AW} \cdot \underline{\vec{w}}, \quad \vec{q} = \underline{q}_B^{\mathrm{T}} \cdot R_{BW} \cdot \underline{\vec{w}}, \quad \vec{k} = \underline{k}_C^{\mathrm{T}} \cdot R_{CW} \cdot \underline{\vec{w}},$$

then

$$\vec{p} = \underline{p}_W^{\mathrm{T}} \cdot \underline{\vec{w}}, \quad \vec{q} = \underline{q}_W^{\mathrm{T}} \cdot \underline{\vec{w}}, \quad \vec{k} = \underline{k}_W^{\mathrm{T}} \cdot \underline{\vec{w}},$$

then we denote vectors $\vec{p}, \vec{q}, \vec{k}$ easily with $\underline{p}_W, \underline{q}_W, \underline{k}_W$, and further to $\underline{p}, \underline{q}, \underline{k}$. In the following parts, we will drop the *arrow* symbol (e.g. $\vec{p}$) and the subscript for coordinate system (e.g. $\underline{p}_A$), instead, we use only $\underline{p}$ to denote a point.

Euler Angle

Euler Angle frames rotation into three rotation (e.g. α, β, γ) with associated axis order (e.g. $x-y-z$). **FORGET about the terms 'pitch', 'roll', and 'yaw', as the defination various from people to people, such as from Apple [1], from Niryo [2], and from some answers on Google [3].**

[1] https://developer.apple.com/documentation/coremotion/getting_processed_device-motion_data/understanding_reference_frames_and_device_attitude
[2] https://docs.niryo.com/dev/pyniryo/source/examples/examples_movement.html
[3] https://m.newsmth.net/article/Graphics/52100

$SO(3)$

SO(3) stands for Special Orthogonal Group. It is a 3×3 matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

SO(3) is not closure under *Addition*, only under *Multiplication*. The inverse of SO(3) is

$$R^{-1} = R^T$$

As the number of element in R is nine, but the Degree of Freedom (DoF) for a rotation is four (1 for rotation degree, 3 for rotation axis), there must be some constraints while doing interpolation oder optimization:

$$\det(R) = 1, \tag{1}$$
$$R \cdot R^{\mathrm{T}} = I. \tag{2}$$

Rodrigues's Formula describes a rotation with the axis $\vec{a}$ and the rotation degree θ:

$$R\left(\theta, \underline{a}\right) = \cos(\theta) \cdot I + (1 - \cos(\theta)) \cdot \underline{a} \cdot \underline{a}^{\mathrm{T}} + \sin(\theta) \cdot \underline{a}^{\wedge}, \tag{1}$$

where $\underline{a}^{\wedge}$ denote the skew matrix of the vector $\underline{a}$ with

$$\underline{a}^{\wedge} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

Rodrigues's Formula describes the rotation matrix using axis and degree.

As inversion, we want to calculate the axis $\vec{a}$ and degree θ from the rotation matrix R. After $\underline{a}$ rotating with matrix R, the vector stay unchanged, as $\underline{a}$ is the rotation axis itself, therefore:

$$R \cdot \underline{a} = \underline{a},$$

which means that $\underline{a}$ is a eigen vector of R associated with the eigen value $\lambda = 1$. From (1) we know that

$$\begin{aligned} \mathrm{Trace}(R) &= \mathrm{Trace}\left(\cos(\theta) \cdot I + (1 - \cos(\theta)) \cdot \underline{a} \cdot \underline{a}^{\mathrm{T}} + \sin(\theta) \cdot \underline{a}^{\wedge}\right) & (3) \\ &= \cos(\theta) \cdot \mathrm{Trace}(I) + (1 - \cos(\theta)) \cdot \mathrm{Trace}\left(\underline{a} \cdot \underline{a}^{\mathrm{T}}\right) + \sin(\theta) \cdot \mathrm{Trace}\left(\underline{a}^{\wedge}\right) & (4) \\ &= 3\cos(\theta) + (1 - \cos(\theta)) \cdot \|\underline{a}\|_2^2 & (5) \\ &= 1 + 2\cos(\theta) & (6) \end{aligned}$$

Then we have

$$\theta = \arccos\left(\frac{\mathrm{Tr}(R) - 1}{2}\right)$$

> **Relationship between SO(3) and Euler angle**: We utilize R with the rotate axis as subscript to denote the rotate matrix, e.g.,
>
> $$R_y(\alpha) = \begin{bmatrix} cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}.$$
>
> The total rotation can be then calculated by multipling three rotate matrix:
>
> $$r = R_z(\gamma) \cdot R_y(\beta) \cdot R_x(\alpha).$$

Quaternion

Analogous as imaginary number i, we introduce two more imaginary numbers j, k, where

$$i^2 = j^2 = k^2 = -1 \tag{7}$$
$$i \cdot j = k, \quad j \cdot k = i, \quad k \cdot i = j \tag{8}$$
$$i \cdot j = -j \cdot i, \quad j \cdot k = -k \cdot j, \quad i \cdot k = -k \cdot i \tag{9}$$

A quaternion has the following form:

$$\mathbf{q} = (a, \underline{u}^{\mathrm{T}}) = a + u_1 i + u_2 j + u_3 k$$

Caution: Sometimes (e.g. in SciPy) the quaternion is formed as $\mathbf{q} = (\underline{u}^{\mathrm{T}}, a)$ or $\mathbf{q} = (x, y, z, w)$.

Describe Vector $\underline{p}_A$ using Quaternion $\mathbf{p}$:

$$\mathbf{p} = (0, \underline{p}_A^{\mathrm{T}}).$$

Quaternion for rotation of θ degree around axis $\underline{a}$:

$$\mathbf{q} = \left(\cos\left(\frac{\theta}{2}\right), \underline{a}^{\mathrm{T}} \sin\left(\frac{\theta}{2}\right)\right).$$

After the rotation:

$$\mathbf{p}'_A = \mathbf{q} \cdot \mathbf{p} \cdot \mathbf{q}^{-1} = \mathbf{q} \cdot \mathbf{p} \cdot \mathbf{q}^*,$$

where $\mathbf{q}^* = \left(\cos\left(\frac{\theta}{2}\right), -\underline{a}^{\mathrm{T}} \sin\left(\frac{\theta}{2}\right)\right)$. Then the vector after rotation the $\underline{p}'_A$ is the imaginary part of $\mathbf{p}'_A$.

Note: in some literatures the product of quaternion are denoted by "⊗," instead of "·".

Combination of quaternion: If we rotate $\underline{p}$ first using $\mathbf{q}$, then $\mathbf{r}$, we have

$$\mathbf{p}' = \mathbf{q} \cdot \mathbf{p} \cdot \mathbf{q}^* \tag{10}$$
$$\mathbf{p}'' = \mathbf{r} \cdot \mathbf{p}' \cdot \mathbf{r}^* = \mathbf{r} \cdot \mathbf{q} \cdot \mathbf{p} \cdot \mathbf{q}^* \cdot \mathbf{r}^* \tag{11}$$
$$:= \mathbf{R} \cdot \mathbf{p} \cdot \mathbf{R}^* \tag{12}$$
$$\mathbf{R} = \mathbf{r} \cdot \mathbf{q} \tag{13}$$

> **Example**
>
> $\underline{p} = [1 \quad 1 \quad 1]^{\mathrm{T}}$ rotate $\theta = \pi$ around $\underline{a} = [0 \quad 0 \quad 1]^{\mathrm{T}}$, then we have
>
> $$\mathbf{p} = (0, 1, 1, 1) \tag{14}$$
> $$\mathbf{q} = \left(\cos\left(\frac{\pi}{2}\right), [0 \quad 0 \quad 1] \sin\left(\frac{\pi}{2}\right)\right) = (0, 0, 0, 1), \tag{15}$$
>
> so that
>
> $$\mathbf{p}' = k \cdot (i + j + k) \cdot (-k) \tag{16}$$
> $$= (j - i - 1) \cdot (-k) \tag{17}$$
> $$= -i - j + k \tag{18}$$
> $$= [0 \quad -1 \quad -1 \quad 1] \tag{19}$$
> $$\vec{p}' = [-1 \quad -1 \quad 1]^{\mathrm{T}} \tag{20}$$

$\mathfrak{so}(3)$

Problems of $SO(3)$ is that, it is not closure under *Addition*, which means the rotations must be combined with *Multiplications*. This does harm to the derivation. Moreover, the $SO(3)$ is redudant.
Our intuition tells us to use Exponent operator to convert Addition from Multiplication. And to use a vector $\underline{\phi} = l \cdot \underline{k}$ (length l for rotation degree and direction $\underline{k}$ for rotate axis) to determinate the rotation:

$$\begin{aligned}
\exp\left\{\underline{\phi}^{\wedge}\right\} =& \exp\left\{\left(l \cdot \underline{k}\right)^{\wedge}\right\} = \exp\left\{l \cdot \underline{k}^{\wedge}\right\} && (21)\\
=& \sum_{n=0}^{\infty}\left\{\frac{1}{n!}\left(l \cdot \underline{k}^{\wedge}\right)^{n}\right\} && (22)\\
&\vdots && (23)\\
=& \cos\left(l\right) \cdot I + \left(1 - \cos\left(l\right)\right) \cdot \underline{k} \cdot \underline{k}^{\mathrm{T}} + \sin\left(l\right) \cdot \underline{k}^{\wedge}\ . && (24)
\end{aligned} \tag{2}$$

Reminding of the **Rodrigues's Formula**:

$$R\left(\theta, \underline{a}\right) = \cos\left(\theta\right) \cdot I + \left(1 - \cos\left(\theta\right)\right) \cdot \underline{a} \cdot \underline{a}^{\mathrm{T}} + \sin\left(\theta\right) \cdot \underline{a}^{\wedge}, \tag{3}$$

As we expected l and $\underline{k}$ are exactly the rotation degree and the rotation axis. By combining (2) and (3) we get

$$\exp\left\{\underline{\phi}^{\wedge}\right\} \overset{(2)}{=} \cos\left(\theta\right) \cdot I + \left(1 - \cos\left(\theta\right)\right) \cdot \underline{a} \cdot \underline{a}^{\mathrm{T}} + \sin\left(\theta\right) \cdot \underline{a}^{\wedge} \overset{(4)}{=} R(\theta, \underline{a}). \tag{25}$$

Obviously we know the relationship between R and $\underline{\phi}$. Moreover, $R \in \mathbb{R}^{3\times 3}$ is a $SO(3)$ (Lie Group) and $\vec{\phi} \in \mathbb{R}^{3}$ is a $\mathfrak{so}(3)$ (Lie Algebra).

> **Study on properitis of R and $\underline{\phi}$:**
>
> Assuming that R is a dependent variable of time t, we have
>
> $$R(t)R(t)^{\mathrm{T}} = I. \tag{26}$$
>
> Derivate both side w.r.t. time, we get
>
> $$\dot{R}(t)R(t)^{\mathrm{T}} + R(t)\dot{R}(t)^{\mathrm{T}} = 0.$$
>
> and thus
>
> $$\dot{R}(t)R(t)^{\mathrm{T}} = -R(t)\dot{R}(t)^{\mathrm{T}} = -\left(\dot{R}(t)R(t)^{\mathrm{T}}\right)^{\mathrm{T}}.$$
>
> That means $\dot{R}(t)R(t)^{\mathrm{T}}$ is a skew matrix, which can be written as
>
> $$\dot{R}(t)R(t)^{\mathrm{T}} = \begin{bmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{bmatrix} =: \Gamma =: \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}^{\wedge} = \underline{\gamma}(t)^{\wedge},$$
>
> where the *wedge* superscript "$^{\wedge}$" denote the operator converting a vector to its associated skew matrix, while a *vee* superscript describe the operation from skew matrix to vector, i. e.
>
> $$\Gamma^{\vee} = \begin{bmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \underline{\gamma}(t).$$
>
> **In the next section we will see that $\dot{R}(t)R(t)^{\mathrm{T}}$ is exactly the rotation speed, i.e. $\gamma \equiv \omega$**
>
> $$\dot{R} \cdot R^{\mathrm{T}} = \underline{\omega}^{\wedge}.$$

Relationship between $SO(3)$ and $\mathfrak{so}(3)$:

- $SO(3)$ to $\mathfrak{so}(3)$:

$$\underline{\phi} = \theta \cdot \underline{a} = \arccos\left(\frac{\mathrm{Tr}(R) - 1}{2}\right) \cdot \underline{a}, \quad \theta \in (-\pi, \pi],$$

  where $\underline{a}$ is the eigen vector of R associated with the eigen value $\lambda = 1$. Another approach to calculate $\underline{\phi}$ is to use Logarithm Transformation:

$$\underline{\phi} = (\ln(R))^{\vee} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (R - I)^{n+1}\right)^{\vee},$$

- $\mathfrak{so}(3)$ to $SO(3)$:

$$R = \cos(\|\underline{\phi}\|) \cdot I + (1 - \cos(\|\underline{\phi}\|)) \cdot \frac{\underline{a}}{\|\underline{a}\|} \cdot \left(\frac{\underline{a}}{\|\underline{a}\|}\right)^{\mathrm{T}} + \sin(\|\underline{\phi}\|) \cdot \left(\frac{\underline{a}}{\|\underline{a}\|}\right)^{\wedge}.$$

Combination of Rotation

As the rotations are combined as product in $SO(3)$, i. e. $R = R_2 \cdot R_1$, and $\mathfrak{so}(3)$ describes the exponential map of $SO(3)$, we intuitively guess that

$$\underline{\phi} = \underline{\phi}_2 + \underline{\phi}_1.$$

Unfortunately, for matrix exponential this rule is not true [1], i.e.

$$\exp\left\{\left(\underline{\phi}_2 + \underline{\phi}_1\right)^{\wedge}\right\} \neq \exp\left\{\underline{\phi}_2^{\wedge}\right\} \exp\left\{\underline{\phi}_1^{\wedge}\right\}$$

and in fact

$$\ln\left\{\exp\left\{\underline{\phi}_2^{\wedge}\right\} \exp\left\{\underline{\phi}_1^{\wedge}\right\}\right\} \approx \underline{\phi}_2^{\wedge} + \underline{\phi}_1^{\wedge} + \frac{1}{2}\left[\underline{\phi}_2^{\wedge} + \underline{\phi}_1^{\wedge}\right],$$

when $\underline{\phi}_1^{\wedge}$ or $\underline{\phi}_2^{\wedge}$ is small. The **BCH Approximation** is

$$\left(\ln\left\{\exp\left\{\underline{\delta}^{\wedge}\right\} \exp\left\{\underline{\phi}^{\wedge}\right\}\right\}\right)^{\vee} \approx \left(J_l(\underline{\phi})\right)^{-1} \underline{\delta} + \underline{\phi}, \tag{27}$$

$$\left(\ln\left\{\exp\left\{\underline{\phi}^{\wedge}\right\} \exp\left\{\underline{\delta}^{\wedge}\right\}\right\}\right)^{\vee} \approx \left(J_r(\underline{\phi})\right)^{-1} \underline{\delta} + \underline{\phi}, \tag{28}$$

where the subscript l and r denote the relative position of $\underline{\delta}$ to $\underline{\phi}$, and after denoting $\underline{\phi} = \theta \underline{a}$,

$$J_l\left(\underline{\phi}\right) = \frac{\sin(\theta)}{\theta} I + \left(1 - \frac{\sin(\theta)}{\theta}\right) a a^{\mathrm{T}} + \frac{1 - \cos(\theta)}{\theta} a^{\wedge}, \tag{29}$$

$$J_r\left(\underline{\phi}\right) = J_l\left(-\underline{\phi}\right), \tag{30}$$

$$\left(J_l(\underline{\phi})\right)^{-1} =: J_l^{-1}\left(\underline{\phi}\right) = \frac{\theta}{2}\cot\left(\frac{\theta}{2}\right) I + \left(1 - \frac{\theta}{2}\cot\left(\frac{\theta}{2}\right)\right) a a^{\mathrm{T}} - \frac{\theta}{2} \underline{a}^{\wedge}, \tag{31}$$

$$\left(J_r(\underline{\phi})\right)^{-1} =: J_r^{-1}\left(\underline{\phi}\right) = \frac{\theta}{2}\cot\left(\frac{\theta}{2}\right) I + \left(1 - \frac{\theta}{2}\cot\left(\frac{\theta}{2}\right)\right) a a^{\mathrm{T}} + \frac{\theta}{2} \underline{a}^{\wedge}. \tag{32}$$

Inversly, if we add a $\underline{\delta}$ on $\underline{\phi}$:

$$\exp\left\{\underline{\phi}^{\wedge} + \underline{\delta}^{\wedge}\right\} = \exp\left\{\left(\underline{\phi} + \underline{\delta}\right)^{\wedge}\right\} \overset{!}{=} \exp\left\{\left(J_l^{-1}\left(\underline{\phi}\right) J_l\left(\underline{\phi}\right) \cdot \underline{\delta} + \underline{\phi}\right)^{\wedge}\right\},$$

or

$$\exp\left\{\underline{\phi}^\wedge + \underline{\delta}^\wedge\right\} = \exp\left\{\left(\underline{\phi} + \underline{\delta}\right)^\wedge\right\} \stackrel{!}{=} \exp\left\{\left(J_r^{-1}\left(\underline{\phi}\right) J_r\left(\underline{\phi}\right) \cdot \underline{\delta} + \underline{\phi}\right)^\wedge\right\},$$

i.e.

$$R' = \Delta R_l \cdot R = R \cdot \Delta R_r,$$

where ΔR_l and ΔR_r are associated with $J_l\left(\underline{\phi}\right) \cdot \underline{\delta}$ and $J_r\left(\underline{\phi}\right) \cdot \underline{\delta}$ respectively, I. e. $\Delta R_l = \Delta R_l\left(J_l\left(\underline{\phi}\right) \cdot \underline{\delta}\right)$ and $\Delta R_r = \Delta R_r\left(J_r\left(\underline{\phi}\right) \cdot \underline{\delta}\right)$.

Summary

- Multiplication on $SO(3)$:

$$\Delta R \cdot R = \exp\left\{\underline{\delta}^\wedge\right\} \exp\left\{\underline{\phi}^\wedge\right\} = \exp\left\{\left(\underline{\phi} + J_l^{-1}\left(\underline{\phi}\right) \cdot \underline{\delta}\right)^\wedge\right\} \tag{33}$$

$$R \cdot \Delta R = \exp\left\{\underline{\phi}^\wedge\right\} \exp\left\{\underline{\delta}^\wedge\right\} = \exp\left\{\left(\underline{\phi} + J_r^{-1}\left(\underline{\phi}\right) \cdot \underline{\delta}\right)^\wedge\right\} \tag{34}$$

  with ΔR associated with $\underline{\delta}$, which means

$$\Delta R \cdot R \mapsto \underline{\phi} + J_l^{-1}\left(\underline{\phi}\right) \cdot \underline{\delta} \tag{35}$$

$$R \cdot \Delta R \mapsto \underline{\phi} + J_r^{-1}\left(\underline{\phi}\right) \cdot \underline{\delta} \tag{36}$$

- Addition on $\mathfrak{so}(3)$:

$$\underline{\phi} + \underline{\delta} \mapsto \begin{cases} \Delta R_l \cdot R, & J_l(\underline{\phi}) \cdot \underline{\delta} \mapsto \Delta R_l \\ R \cdot \Delta R_r, & J_r(\underline{\phi}) \cdot \underline{\delta} \mapsto \Delta R_r \end{cases} \tag{37}$$

[1] https://en.wikipedia.org/wiki/Baker%E2%80%93Campbell%E2%80%93Hausdorff_formula

*Summary

	Description of vector	Description of Rotation	Rotation
Euler	$\underline{p} = [p_1 \quad p_2 \quad p_3]^\mathrm{T}$	$\underline{r} = [\alpha \quad \beta \quad \gamma \quad a_1 \quad a_2 \quad a_3], a_i \in \{x, y, z\}$	$\underline{r} \mapsto R \in \mathbb{R}^{3\times3}, \underline{p}' = R \cdot \underline{p}$
$SO(3)$	$\underline{p} = [p_1 \quad p_2 \quad p_3]^\mathrm{T}$	$R \in \mathbb{R}^{3\times3}$ with $R^\mathrm{T} = R^{-1}, \det(R) = 1$	$\underline{p}' = R \cdot \underline{p}$
$\mathfrak{so}(3)$	$\underline{p} = [p_1 \quad p_2 \quad p_3]^\mathrm{T}$	$\underline{\phi} = \theta \cdot \underline{a}$	$\underline{\phi} \mapsto R \in \mathbb{R}^{3\times3}, \underline{p}' = R \cdot \underline{p}$
Quaternion	$\mathbf{p} = [0 \quad p_1 \quad p_2 \quad p_3]$	$\mathbf{q} = \left[\cos\left(\frac{\theta}{2}\right), \underline{a}\sin\left(\frac{\theta}{2}\right)\right]$	$\underline{p}' \stackrel{\wedge}{=} \mathbf{p}' = \mathbf{q} \cdot \mathbf{p} \cdot \mathbf{q}^*$

Angular Velocity

*Vector

Suppose coordinate system B rotate while A stay static. We assume the rotate speed of B in A is

$${}^A\vec{\omega}^B = \omega_1 \cdot \vec{b}_1 + \omega_2 \cdot \vec{b}_2 + \omega_3 \cdot \vec{b}_3.$$

Let's take a static point $\vec{p}$ in B, i. e.

$$\vec{p} = \underline{p}_A^{\mathrm{T}} \cdot \underline{\vec{a}} = \underline{p}_B^{\mathrm{T}} \cdot \underline{\vec{b}},$$

where $\underline{p}_B^{\mathrm{T}}$ is constant. We have also the following relationship between rotating speed and velocity:

$$\frac{{}^A\mathrm{d}\vec{p}}{\mathrm{d}t} = {}^A\vec{\omega}^B \times \vec{p} \tag{38}$$

$$p_1\dot{\vec{b}}_1 + p_2\dot{\vec{b}}_2 + p_3\dot{\vec{b}}_3 = \left(\omega_1\vec{b}_1 + \omega_2\vec{b}_2 + \omega_3\vec{b}_3\right) \times \left(p_1\vec{b}_1 + p_2\vec{b}_2 + p_3\vec{b}_3\right). \tag{39}$$

The $\dot{\vec{b}}_i$ is defined as $\frac{{}^A\mathrm{d}\vec{b}_i}{\mathrm{d}t}$. As

$$\left(\omega_1\vec{b}_1 + \omega_2\vec{b}_2 + \omega_3\vec{b}_3\right) \times p_1\vec{b}_1 = p_1\left(-\omega_2\vec{b}_3 + \omega_3\vec{b}_2\right) \overset{!}{=} p_1\dot{\vec{b}}_1 \tag{40}$$

$$-\omega_2\vec{b}_3 + \omega_3\vec{b}_2 = \left(\dot{\vec{b}}_1\vec{b}_1\right)\cdot\vec{b}_1 + \left(\dot{\vec{b}}_1\vec{b}_2\right)\cdot\vec{b}_2 + \left(\dot{\vec{b}}_1\vec{b}_3\right)\cdot\vec{b}_3 \tag{41}$$

holds, and

$$\vec{b}_1 \cdot \vec{b}_1 = 1 \tag{42}$$

$$\frac{\mathrm{d}\left(\vec{b}_1\vec{b}_1\right)}{\mathrm{d}t} = 2\dot{\vec{b}}_1\vec{b}_1 = 0 \tag{43}$$

$$\dot{\vec{b}}_1\vec{b}_1 = 0 \tag{44}$$

$$\vec{b}_3\vec{b}_1 = 0 \tag{45}$$

$$\frac{\mathrm{d}\left(\vec{b}_3\vec{b}_1\right)}{\mathrm{d}t} = \dot{\vec{b}}_3\vec{b}_1 + \dot{\vec{b}}_1\vec{b}_3 = 0 \tag{46}$$

$$\dot{\vec{b}}_3\vec{b}_1 = -\dot{\vec{b}}_1\vec{b}_3 \tag{47}$$

is valid, then we have

$$\omega_3\vec{b}_2 - \omega_2\vec{b}_3 + = \left(\dot{\vec{b}}_1\vec{b}_2\right)\cdot\vec{b}_2 - \left(\dot{\vec{b}}_3\vec{b}_1\right)\cdot\vec{b}_3\ ,$$

as well as

$$\omega_2 = \dot{\vec{b}}_3\vec{b}_1\ , \tag{48}$$

$$\omega_3 = \dot{\vec{b}}_1\vec{b}_2\ , \tag{49}$$

analogous:

$$\omega_1 = \dot{\vec{b}}_2\vec{b}_3.$$

> **Example:**
>
> B rotate θ degree around axis $\vec{b}_1$, then
>
> $$\begin{bmatrix}\vec{b}_1\\ \vec{b}_2\\ \vec{b}_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta) & \sin(\theta)\\ 0 & -\sin(\theta) & \cos(\theta)\end{bmatrix}\cdot\begin{bmatrix}\vec{a}_1\\ \vec{a}_2\\ \vec{a}_3\end{bmatrix},$$
>
> i.e.
>
> $$\vec{b}_1 = \vec{a}_1, \quad \vec{b}_2 = \cos(\theta)\vec{a}_2 + \sin(\theta)\vec{a}_3, \quad \vec{b}_3 = -\sin(\theta)\vec{a}_2 + \cos(\theta)\vec{a}_3,$$
>
> the derivates are:

> $$\dot{\vec{b}}_1 = \frac{{}^A\mathrm{d}\vec{b}_1}{\mathrm{d}t} = 0 \tag{50}$$
>
> $$\dot{\vec{b}}_2 = \frac{{}^A\mathrm{d}\vec{b}_2}{\mathrm{d}t} = \frac{\mathrm{d}\cos(\theta)}{\mathrm{d}t}\vec{a}_2 + \frac{\mathrm{d}\sin(\theta)}{\mathrm{d}t}\vec{a}_3 = -\dot{\theta}\sin(\theta)\vec{a}_2 + \dot{\theta}\cos(\theta)\vec{a}_3 \tag{51}$$
>
> $$\dot{\vec{b}}_3 = -\dot{\theta}\cos(\theta)\vec{a}_2 - \dot{\theta}\sin(\theta)\vec{a}_3 \tag{52}$$
>
> Then
>
> $$
\begin{aligned}
{}^A\vec{\omega}^B =& \left(\dot{\vec{b}}_2\vec{b}_3\right)\vec{b}_1 + \left(\dot{\vec{b}}_3\vec{b}_1\right)\vec{b}_2 + \left(\dot{\vec{b}}_1\vec{b}_2\right)\vec{b}_3 && (53)\\
=& \left(\left(-\dot{\theta}\sin(\theta)\vec{a}_2 + \dot{\theta}\cos(\theta)\vec{a}_3\right)\left(-\sin(\theta)\vec{a}_2 + \cos(\theta)\vec{a}_3\right)\right)\vec{b}_1 && (54)\\
&+\left(\left(-\dot{\theta}\cos(\theta)\vec{a}_2 - \dot{\theta}\cos(\theta)\vec{a}_3\right)\vec{a}_1\right)\vec{b}_2 && (55)\\
=& \left(\dot{\theta}\sin^2(\theta) + \dot{\theta}\cos^2(\theta)\right)\vec{b}_1 && (56)\\
=& \dot{\theta}\cdot\vec{b}_1 && (57)\\
=& \dot{\theta}\cdot\vec{a}_1 && (58)
\end{aligned}
$$
>
> Then we have the rotating speed in A coordinate $\underline{\omega} = [\dot{\theta} \quad 0 \quad 0]^{\mathrm{T}}$.

Euler Angle

Take $\underline{r} = [\alpha \quad \beta \quad \gamma \quad x \quad y \quad z]$ as an example, and we utilize the coordinate systems A, B, C, D to describe the coordinate systems as

$$
\begin{aligned}
{}^A\vec{\omega}^B &= \dot{\alpha}\cdot\vec{a}_1 && (59)\\
{}^B\vec{\omega}^C &= \dot{\beta}\cdot\vec{b}_2 && (60)\\
{}^C\vec{\omega}^D &= \dot{\gamma}\cdot\vec{c}_3 && (61)
\end{aligned}
$$

Then we have

$$ {}^A\vec{\omega}^D = \dot{\alpha}\cdot\vec{a}_1 + \dot{\beta}\cdot\vec{b}_2 + \dot{\gamma}\cdot\vec{c}_3 $$

Converting all coordinate bases to initial coordinate system A:

$$
\begin{aligned}
{}^A\vec{\omega}^D &= \dot{\alpha}\cdot\vec{a}_1 + \dot{\beta}\cdot\left(\cos(\alpha)\vec{a}_2 + \sin(\alpha)\vec{a}_3\right) + \dot{\gamma}\cdot\left(\sin(\beta)\vec{a}_1 - \sin(\alpha)\cos(\beta)\vec{a}_2 + \cos(\alpha)\cos(\beta)\vec{a}_3\right) && (62)\\
&= \begin{bmatrix} \dot{\alpha} + \dot{\gamma}\sin(\beta) \\ \dot{\beta}\cos(\alpha) - \dot{\gamma}\sin(\alpha)\cos(\beta) \\ \dot{\beta}\sin(\alpha) + \dot{\gamma}\cos(\alpha)\cos(\beta) \end{bmatrix}^{\mathrm{T}} \cdot \underline{\vec{a}} && (63)\\
\underline{\omega} &= \begin{bmatrix} \dot{\alpha} + \dot{\gamma}\sin(\beta) \\ \dot{\beta}\cos(\alpha) - \dot{\gamma}\sin(\alpha)\cos(\beta) \\ \dot{\beta}\sin(\alpha) + \dot{\gamma}\cos(\alpha)\cos(\beta) \end{bmatrix} && (64)
\end{aligned}
$$

It is obvious that the angular velocity $\underline{\omega}$ depends not only on the rotate speed $\dot{\alpha}, \dot{\beta}, \dot{\gamma}$, but also **strongly on the order of the rotation**. It's not useful.

$SO(3)$

Considering the vector $\vec{p}$ in "vector" section as

$$\underline{p}_A = R_{AB} \cdot \underline{p}_B,$$

and derivate it w.r.t. time, we have

$$\dot{\underline{p}}_A = \dot{R}_{AB} \cdot \underline{p}_B + R_{AB} \cdot \dot{\underline{p}}_B = \dot{R}_{AB} \cdot \underline{p}_B \tag{4}$$

> Excursion:
>
> the *cross multiplication* $\vec{p} \times \vec{q}$ can be written as the *dot product* of a skew matrix with a vector if they are in the same coordinate system, i.e.
>
> $$\vec{p} = \underline{p}_A^{\mathrm{T}} \cdot \underline{\vec{a}}, \quad \vec{q} = \underline{q}_A^{\mathrm{T}} \cdot \underline{\vec{a}},$$
>
> then
>
> $$\vec{p} \times \vec{q} = \left((\underline{p}_A)^{\wedge} \cdot \underline{q}_A \right)^{\mathrm{T}} \cdot \underline{\vec{a}}$$

The formula $\frac{{}^{A}\mathrm{d}\vec{p}}{\mathrm{d}t} = {}^{A}\vec{\omega}^{B} \times \vec{p}$ can be then written in matrix form:

$$\dot{\underline{p}}_A^{\mathrm{T}} \cdot \underline{\vec{a}} = \left(\left({}^{A}\underline{\omega}^{B} \right)_A^{\wedge} \cdot \underline{p}_A \right)^{\mathrm{T}} \cdot \underline{\vec{a}},$$

i. e.

$$\dot{\underline{p}}_A = \left({}^{A}\underline{\omega}^{B} \right)_A^{\wedge} \cdot \underline{p}_A = \left({}^{A}\underline{\omega}^{B} \right)_A^{\wedge} \cdot R_{AB} \cdot \underline{p}_B. \tag{5}$$

Combining (4) and (5), we have

$$\left({}^{A}\underline{\omega}^{B} \right)_A^{\wedge} \cdot R_{AB} \cdot \underline{p}_B = \dot{R}_{AB} \cdot \underline{p}_B \tag{65}$$

$$\left({}^{A}\underline{\omega}^{B} \right)_A^{\wedge} \cdot R_{AB} = \dot{R}_{AB} \tag{66}$$

At the end we get

$$\underline{\omega}^{\wedge} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \dot{R}_{AB}(R_{AB})^{\mathrm{T}}.$$

Quaternion

According to the definition of rotating speed $\vec{\omega} = \omega_1 \cdot \vec{a}_1 + \omega_2 \cdot \vec{a}_2 + \omega_3 \cdot \vec{a}_3$. First, we drop the coordinate system $\underline{\vec{a}}$ as we handle them always in A. It can be then describes as the product of its direction $\frac{\omega}{\|\underline{\omega}\|}$ and the associated rotating speed (i.e. its length) $\|\underline{\omega}\|$. Describe the derivative of the rotation quaternion $\mathbf{q}(t)$ as :

$$\frac{\mathrm{d}\mathbf{q}(t)}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\mathbf{q}(t + \Delta t) - \mathbf{q}(t)}{\Delta t} \tag{67}$$

$$= \lim_{\Delta t \to 0} \frac{\delta \mathbf{q}(\Delta t, \underline{\omega}) \cdot \mathbf{q}(t) - \mathbf{q}(t)}{\Delta t}, \tag{68}$$

where $\delta \mathbf{q}(\Delta t, \underline{\omega}) = \delta \mathbf{q}(\|\underline{\omega}\| \Delta t, \frac{\omega}{\|\underline{\omega}\|})$

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{q}(t)}{\mathrm{d}t} &= \lim_{\Delta t \to 0} \frac{\left[\cos\left(\frac{\|\omega\|}{2}\Delta t\right) - 1 \quad \frac{\omega_1}{\|\underline{\omega}\|}\sin\left(\frac{\|\omega\|}{2}\Delta t\right) \quad \frac{\omega_2}{\|\underline{\omega}\|}\sin\left(\frac{\|\omega\|}{2}\Delta t\right) \quad \frac{\omega_3}{\|\underline{\omega}\|}\sin\left(\frac{\|\omega\|}{2}\Delta t\right)\right] \cdot \mathbf{q}(t)}{\Delta t} && (69)\\
&= \lim_{\Delta t \to 0} \left[\frac{\cos\left(\frac{\|\omega\|}{2}\Delta t\right) - 1}{\Delta t} \quad \frac{\frac{\omega_1}{\|\underline{\omega}\|}\sin\left(\frac{\|\omega\|}{2}\Delta t\right)}{\Delta t} \quad \frac{\frac{\omega_2}{\|\underline{\omega}\|}\sin\left(\frac{\|\omega\|}{2}\Delta t\right)}{\Delta t} \quad \frac{\frac{\omega_3}{\|\underline{\omega}\|}\sin\left(\frac{\|\omega\|}{2}\Delta t\right)}{\Delta t}\right] \cdot \mathbf{q}(t) && (70)\\
&= \left[0 \quad \frac{\omega_1}{\|\underline{\omega}\|}\frac{\|\omega\|}{2} \quad \frac{\omega_2}{\|\underline{\omega}\|}\frac{\|\omega\|}{2} \quad \frac{\omega_3}{\|\underline{\omega}\|}\frac{\|\omega\|}{2}\right] \cdot \mathbf{q}(t) && (71)\\
&= \frac{1}{2}[0 \quad \omega_1 \quad \omega_2 \quad \omega_3] \cdot \mathbf{q}(t) && (72)
\end{aligned}
$$

I. e.

$$
\begin{aligned}
2\dot{\mathbf{q}} \cdot \mathbf{q}^* &= [0 \quad \underline{\omega}] && (73)\\
\dot{\mathbf{q}} &= \frac{1}{2}[0 \quad \underline{\omega}] \cdot \mathbf{q} && (74)
\end{aligned}
$$

$\mathfrak{so}(3)$

As we have known:

$$
\begin{aligned}
\exp\{\phi^\wedge\} &= R && (75)\\
\frac{\mathrm{d}}{\mathrm{d}t}(\exp\{\phi^\wedge\}) &= \frac{\mathrm{d}R}{\mathrm{d}t} && (76)\\
\left(\dot{\phi}^\wedge\right) \cdot \exp\{\phi^\wedge\} &= \dot{R} && (77)\\
\left(\dot{\phi}^\wedge\right) \cdot R &= \dot{R} && (78)\\
\left(\dot{\phi}^\wedge\right) &= \dot{R}R^{\mathrm{T}} && (79)
\end{aligned}
$$

Besides, we have

$$\dot{R}R^{\mathrm{T}} = \omega^\wedge,$$

that means

$$\omega = \dot{\phi}$$

With Uncertainty

In this section we suppose that the noises v obey Gaussian Distribution with $v \sim \mathcal{N}\{0, \sigma\} \in \mathbb{R}^{3\times 3}$.

$SO(3)$

Suppose there is a noise in $SO(3)$, i. e.

$$\hat{R} = R + v.$$

The associated rotation aixs is

$$\hat{\underline{a}} = \underline{a} + \delta\underline{a}.$$

Moreover, we have

$$
\begin{aligned}
\underline{a} + \delta\underline{a} = \hat{\underline{a}} &= \hat{R} \cdot \hat{\underline{a}} && (80)\\
&= (R + v) \cdot \hat{\underline{a}} && (81)\\
&= R \cdot \hat{\underline{a}} + v \cdot \hat{\underline{a}} && (82)\\
&= R \cdot (\underline{a} + \delta\underline{a}) + v \cdot \hat{\underline{a}} && (83)\\
&= R \cdot \underline{a} + R \cdot \delta\underline{a} + v \cdot \hat{\underline{a}} && (84)\\
&= \underline{a} + R \cdot \delta\underline{a} + v \cdot \hat{\underline{a}} && (85)\\
(I - R) \cdot \delta\underline{a} &= v \cdot \hat{\underline{a}} && (86)\\
\delta\underline{a} &= (I - R)^{-1} \cdot v \cdot \hat{\underline{a}} && (87)
\end{aligned}
$$

and thus

$$
\begin{aligned}
\mathbb{E}\left\{\delta\underline{a}\right\} &= \mathbb{E}\left\{(I - R)^{-1} \cdot v \cdot \hat{\underline{a}}\right\} = 0 && (88)\\
\mathrm{Cov}\left\{\delta\underline{a}\right\} &= \mathbb{E}\left\{(v \cdot \hat{\underline{a}}) \cdot (v \cdot \hat{\underline{a}})^{\mathrm{T}}\right\} && (89)\\
&= \mathbb{E}\left\{v \cdot \hat{\underline{a}} \cdot \hat{\underline{a}}^{\mathrm{T}} \cdot v^{\mathrm{T}}\right\} && (90)
\end{aligned}
$$

$R_3 \cdot R_2 \cdot R_1$

From (1) we know that

$$
\begin{aligned}
\mathrm{Trace}(R) &= \mathrm{Trace}\left(\cos(\theta) \cdot I + (1 - \cos(\theta)) \cdot \underline{a} \cdot \underline{a}^{\mathrm{T}} + \sin(\theta) \cdot \underline{a}^{\wedge}\right) && (91)\\
&= \cos(\theta) \cdot \mathrm{Trace}(I) + (1 - \cos(\theta)) \cdot \mathrm{Trace}\left(\underline{a} \cdot \underline{a}^{\mathrm{T}}\right) + \sin(\theta) \cdot \mathrm{Trace}\left(\underline{a}^{\wedge}\right) && (92)\\
&= 3\cos(\theta) + (1 - \cos(\theta)) \cdot \|\underline{a}\|_2^2 && (93)\\
&= 1 + 2\cos(\theta) && (94)
\end{aligned}
$$

Then we have

$$
\theta = \arccos\left(\frac{\mathrm{Tr}(R) - 1}{2}\right)
$$

TODO

description of rotation through

- [ ] Euler Angle
- [ ] Quaternion
- [ ] $\mathfrak{so}(3)$

description of rotation + translation

- [ ] Vector
- [ ] $SE(3)$
- [ ] $\mathfrak{se}(3)$
- [ ] Dual Quaternion

Optimization on manifold