Academy for Teachers masterclass

The Joy of Abstraction

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Wednesday March 13, 2024

Plan

Part 1: What is abstract math?

Part 2: Examples

Part 3: Invertibility

Part 4: Sameness

Part 5: Universal properties

Part 1: What is abstract mathematics?

a) Introduction

b) Abstraction and analogies

c) Relations

d) Categories

Pure mathematics is a framework for agreeing on things.

1a. Introduction

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$$\begin{tikzcd}
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\text{applied mathematics} \arrow[d] \\
\text{science} \arrow[d] \\
\text{engineering, medicine,}\ldots \arrow[d] \\
\text{quantitative world}
\end{tikzcd}$$

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the study of how things work.

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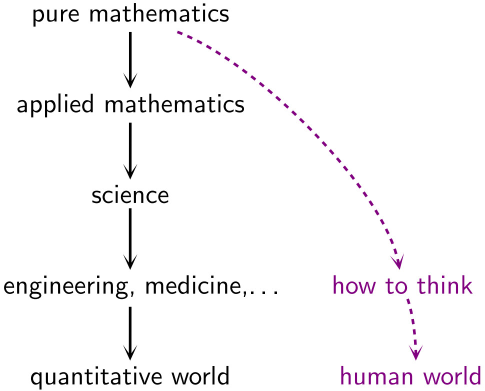

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What other things can you think of?

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pure mathematics

applied mathematics

science

engineering, medicine,. . .

quantitative world

how to think

human world

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logical study	math	philosophy
alogical study	numerology	astrology

What other things can you think of?

Previous teaching:

- math majors at research universities
- all students had done very well in high school math
- careers in math, science, engineering, finance, accountancy, law, teaching
- male dominated ($<30\%$ female).

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Current teaching:

- art students at art school (liberal arts)
- many students who did not do well in high school math
- careers in art, design, photography, teaching
- female dominated ($> 90\%$ female).

Art students: what put them off math?

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memorization

times tables

formulae

getting things wrong

art/math false dichotomy

lack of creativity

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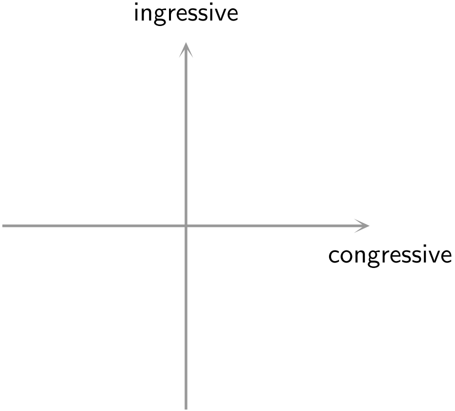

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This is not a classification of people,
but a descriptive characterization of behaviors and interactions.

Ingressive

focusing on oneself
imposing on people
independence and individualism
competitive and adversarial
selective or single-track

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Ingressive

focusing on oneself
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independence and individualism
competitive and adversarial
selective or single-track

focusing on society and community
taking others into account
interdependence and connectedness
collaborative and cooperative
circumspect

Congressive

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ingressive math $\longrightarrow$ **congressive math**

ingressive math → **congressive math**

outcome

facts and rules

solving problems

calculate answers

traditional lecturing

right/wrong

exams and competitions

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outcome	process
facts and rules	investigation, discovery, choices, reasons
solving problems	building structures
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> "Now I find that math is not just a tool, and the real reason for learning math is to practice our thinking".

$$\boxed{\text{Pure mathematics is a theory of analogies}}$$

Pure mathematics is a theory of analogies

2 apples

2 bananas

Pure mathematics is a theory of analogies

$$\begin{tikzcd} & \text{2 (things)} \arrow[dl] \arrow[dr] & \\ \text{2 apples} & & \text{2 bananas} \end{tikzcd}$$

1b. Abstraction and analogies

Pure mathematics is a theory of analogies

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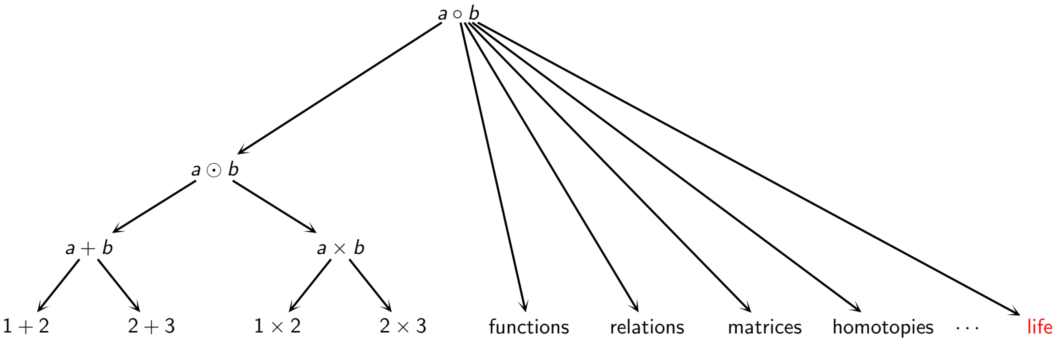

Pure mathematics is a theory of analogies

puff pastry

viruses

1b. Abstraction and analogies

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The idea of category theory

- relationships
- structure
- context
- nuanced ways to think about sameness

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   — transitive

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> **Definition:**
> If a relation is reflexive, symmetric and transitive it's called an equivalence relation.

Note that these properties only count as true if the condition is true for all objects.

If it sometimes holds and sometimes doesn't, the relation itself does not have the property.

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Explore

i. $\boxed{\text{is the same height as}}$ R, S, T
ii. is taller than ~~R~~, ~~S~~, T
iii. is mother of no
iv. is a friend of no
v. is east of
vi. is in an orchestra with ~~R~~, S, ~~T~~
vii. $\leq$ → R, ~~S~~, T
viii. is a factor of R, ~~S~~, T
ix. is roughly the same as (first create a definition)

Define numbers A and B to be "roughly the same" if they round to the same nearest integer

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Further: An equivalence relation is just a partition of the set into subsets with no overlap.

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Categories allow for more general kinds of relation so that we can include more examples like

- equivalence relations
- $\leq$
- a factor of
- functions
- matrices
- symmetry
- paths in a space
- $\vdots$

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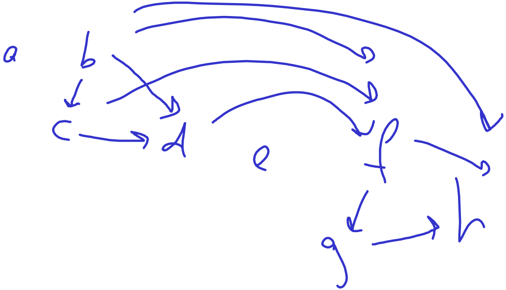

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- associativity: given $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$

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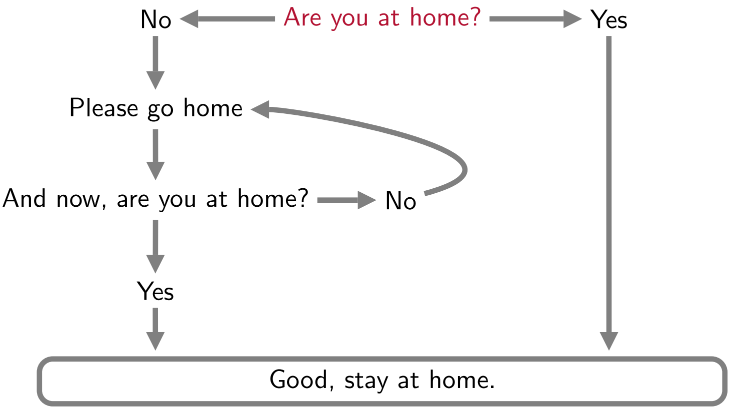
What happened to symmetry?

We don't demand it but we look for it afterwards. This is the notion of "sameness" in a category.

Note: arrows are also called morphisms or maps

In a way composition is the whole point of a category.
It is what makes it more than a flow chart.



Part 2: Examples

a) Examples: numbers, symmetries

b) Examples: symmetries

c) Examples: factors and privilege.

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We could take

- objects: natural numbers
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Thoughts.

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iii. Why do we have to use $\leq$ and not $<$ for this to be a category?

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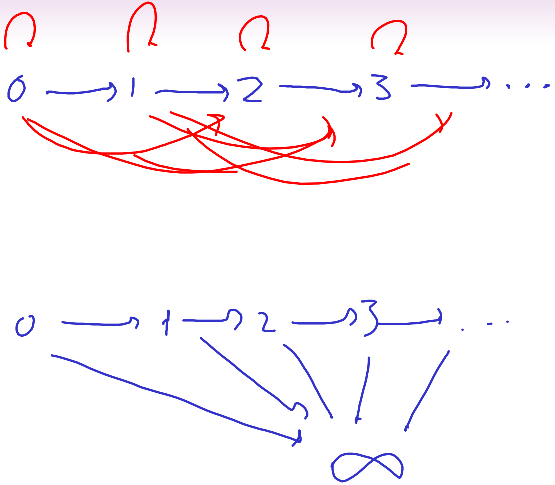

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2b) Examples: symmetries

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$$\square \xrightarrow{90^\circ} \square$$

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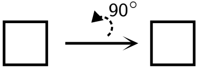

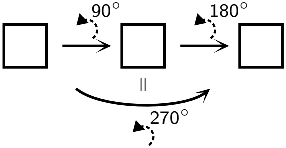

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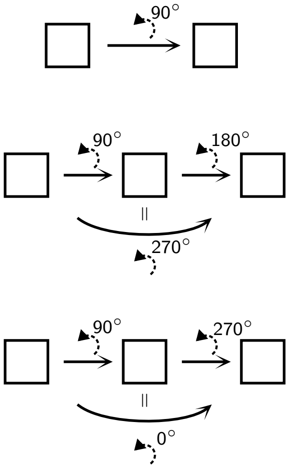

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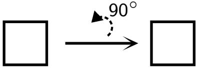
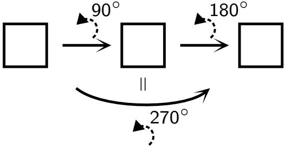
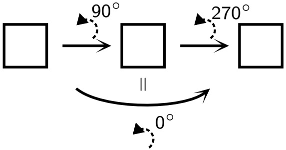
We can regard symmetry as a relation



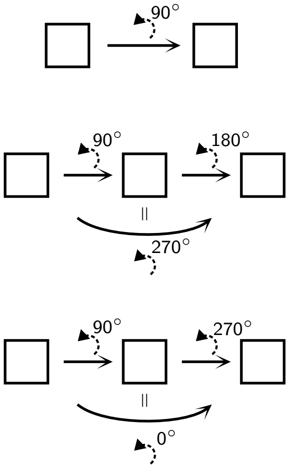




	0	90	180	270
0				
90			270	0
180				
270				

2b) Examples: symmetries

We can regard symmetry as a relation



	0	90	180	270
0	0	90	180	270
90	90	180	270	0
180	180	270	0	90
270	270	0	90	180

Fill in the rest of the table.

i. Do you see a pattern? Does it remind you of anything? If so, why?

ii. Theorize? Generalize?

2c) Examples: Factors and privilege

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

19.

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

$1,$

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

$$1, \; 2,$$

2c) Examples: Factors and privilege

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b 19.

$$1, \ 2, \ 3,$$

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

$$1,\ 2,\ 3,\ 5,$$

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

$$1,\ 2,\ 3,\ 5,\ 6,$$

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

$$1,\ 2,\ 3,\ 5,\ 6,\ 10,$$

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b 19.

$$1, \ 2, \ 3, \ 5, \ 6, \ 10, \ 15,$$

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

$$1,\ 2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30$$

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

$$1,\ 2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30$$

$$30$$

2c) Examples: Factors and privilege

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

19.

$$1,\ 2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30$$

$$\begin{tikzcd}
 & 30 \arrow[dl] \arrow[d] \arrow[dr] & \\
6 & 10 & 15
\end{tikzcd}$$

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

$$1,\ 2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30$$

$$\begin{tikzcd}
 & 30 \arrow[dl] \arrow[d] \arrow[dr] & \\
6 & 10 \arrow[dr] & 15 \arrow[d] \\
 & & 5
\end{tikzcd}$$

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

19.

$$1,\ 2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30$$

$$\begin{tikzcd}
& 30 \arrow[dl] \arrow[d] \arrow[dr] & \\
6 \arrow[d] & 10 \arrow[dl] \arrow[dr] & 15 \arrow[d] \\
2 & & 5
\end{tikzcd}$$

2c) Examples: Factors and privilege

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

19.

$1,\ 2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30$

$$\begin{tikzcd}
 & 30 \arrow[dl] \arrow[d] \arrow[dr] & \\
6 \arrow[d] \arrow[dr] & 10 \arrow[dl] \arrow[dr] & 15 \arrow[dl] \arrow[d] \\
2 & 3 & 5
\end{tikzcd}$$

2c) Examples: Factors and privilege

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

$$1, \ 2, \ 3, \ 5, \ 6, \ 10, \ 15, \ 30$$

$$\begin{tikzcd}
 & 30 \arrow[dl] \arrow[d] \arrow[dr] & \\
6 \arrow[d] \arrow[dr] & 10 \arrow[dl] \arrow[dr] & 15 \arrow[dl] \arrow[d] \\
2 \arrow[dr] & 3 \arrow[d] & 5 \arrow[dl] \\
 & 1 &
\end{tikzcd}$$

2c) Examples: Factors and privilege

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever
  a is a multiple of b

$1,\ 2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30$

$$
\begin{tikzcd}
 & 30 \arrow[dl] \arrow[d] \arrow[dr] & \\
6 \arrow[d] \arrow[dr] & 10 \arrow[dl] \arrow[dr] & 15 \arrow[dl] \arrow[d] \\
2 \arrow[dr] & 3 \arrow[d] & 5 \arrow[dl] \\
 & 1 &
\end{tikzcd}
$$

Try some yourself

a) 6

b) 10

c) 8

d) 70

e) 42

f) 12

g) 24

h) hard: 60, 210

i) any number you like

2c) Examples: Factors and privilege

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

$1, \; 2, \; 3, \; 5, \; 6, \; 10, \; 15, \; 30$

$$\begin{tikzcd}
 & 30 \arrow[dl] \arrow[d] \arrow[dr] & \\
6 \arrow[d] \arrow[dr] & 10 \arrow[dl] \arrow[dr] & 15 \arrow[dl] \arrow[d] \\
2 \arrow[dr] & 3 \arrow[d] & 5 \arrow[dl] \\
 & 1 &
\end{tikzcd}$$

Try some yourself

a) 6
b) 10
c) 8
d) 70
e) 42
f) 12
g) 24
h) hard: 60, 210
i) any number you like

Some things to remember

- Each factor is drawn only once.
- We don't draw arrows that are redundant.

2c) Examples: Factors and privilege

- objects: factors of 30
- morphisms: $a \longrightarrow b$ whenever a is a multiple of b

1, 2, 3, 5, 6, 10, 15, 30

$$\begin{tikzcd}
 & 30 \arrow[dl] \arrow[d] \arrow[dr] & \\
6 \arrow[d] \arrow[dr] & 10 \arrow[dl] \arrow[dr] & 15 \arrow[dl] \arrow[d] \\
2 \arrow[dr] & 3 \arrow[d] & 5 \arrow[dl] \\
 & 1 &
\end{tikzcd}$$

Try some yourself

a) 6
b) 10
c) 8
d) 70
e) 42
f) 12
g) 24
h) hard: 60, 210
i) any number you like

Some things to remember

- Each factor is drawn only once.
- We don't draw arrows that are redundant.

Questions

What shapes are produced?
Why do some numbers produce the same shapes?
What is going on and why?

Examples

$6: \ 1, 2, 3, 6$

$$\begin{tikzcd} & 6 \arrow[dl] \arrow[dr] & \\ 2 \arrow[dr] & & 3 \arrow[dl] \\ & 1 & \end{tikzcd}$$

2c) Examples: Factors and privilege

Examples

$6: \; 1, 2, 3, 6$ $\qquad$ $10: \; 1, 2, 5, 10$

$$\begin{tikzcd}
& 6 \arrow[dl] \arrow[dr] & \\
2 \arrow[dr] & & 3 \arrow[dl] \\
& 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& 10 \arrow[dl] \arrow[dr] & \\
2 \arrow[dr] & & 5 \arrow[dl] \\
& 1 &
\end{tikzcd}$$

2c) Examples: Factors and privilege

Examples

$6: \; 1, 2, 3, 6$

$6 = 2 \times 3$

$$\begin{tikzcd} & 6 \arrow[dl] \arrow[dr] & \\ 2 \arrow[dr] & & 3 \arrow[dl] \\ & 1 & \end{tikzcd}$$

$10: \; 1, 2, 5, 10$

$10 = 2 \times 5$

$$\begin{tikzcd} & 10 \arrow[dl] \arrow[dr] & \\ 2 \arrow[dr] & & 5 \arrow[dl] \\ & 1 & \end{tikzcd}$$

Examples

$6: \; 1, 2, 3, 6$

$10: \; 1, 2, 5, 10$

$6 = 2 \times 3$

$10 = 2 \times 5$

$30 = 2 \times 3 \times 5$

$$\begin{tikzcd}
& 6 \arrow[dl] \arrow[dr] & \\
2 \arrow[dr] & & 3 \arrow[dl] \\
& 1 &
\end{tikzcd}$$

$$\begin{tikzcd}
& 10 \arrow[dl] \arrow[dr] & \\
2 \arrow[dr] & & 5 \arrow[dl] \\
& 1 &
\end{tikzcd}$$

$$\begin{tikzcd}
& 30 \arrow[dl] \arrow[d] \arrow[dr] & & & \\
6 \arrow[d] \arrow[dr] & 10 \arrow[dl] \arrow[dr] & 15 \arrow[dl] \arrow[d] & & \\
2 \arrow[dr] & 3 \arrow[d] & 5 \arrow[dl] & & \text{primes} \arrow[ll] \\
& 1 & & &
\end{tikzcd}$$

Examples

$6: \ 1, 2, 3, 6$

$10: \ 1, 2, 5, 10$

$6 = 2 \times 3$

$10 = 2 \times 5$

$30 = 2 \times 3 \times 5$

$$
\begin{tikzcd}
& 6 \arrow[dl] \arrow[dr] & \\
2 \arrow[dr] & & 3 \arrow[dl] \\
& 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& 10 \arrow[dl] \arrow[dr] & \\
2 \arrow[dr] & & 5 \arrow[dl] \\
& 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& 30 \arrow[dl] \arrow[d] \arrow[dr] & & {} & \text{product of 3 primes} \arrow[l] \\
6 \arrow[d] \arrow[dr] & 10 \arrow[dl] \arrow[dr] & 15 \arrow[dl] \arrow[d] & {} & \text{products of 2 primes} \arrow[l] \\
2 \arrow[dr] & 3 \arrow[d] & 5 \arrow[dl] & {} & \text{primes} \arrow[l] \\
& 1 & & &
\end{tikzcd}
$$

Examples

$6: \ 1, 2, 3, 6$ $\qquad$ $10: \ 1, 2, 5, 10$

$6 = 2 \times 3$ $\qquad$ $10 = 2 \times 5$ $\qquad$ $30 = 2 \times 3 \times 5$

$$
\begin{tikzcd}
& 6 \arrow[dl] \arrow[dr] & \\
2 \arrow[dr] & & 3 \arrow[dl] \\
& 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& 10 \arrow[dl] \arrow[dr] & \\
2 \arrow[dr] & & 5 \arrow[dl] \\
& 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& 30 \arrow[dl] \arrow[d] \arrow[dr] & & {} & \text{product of 3 primes} \arrow[l] \\
6 \arrow[d] \arrow[dr] & 10 \arrow[dl] \arrow[dr] & 15 \arrow[dl] \arrow[d] & {} & \text{products of 2 primes} \arrow[l] \\
2 \arrow[dr] & 3 \arrow[d] & 5 \arrow[dl] & {} & \text{primes} \arrow[l] \\
& 1 & & {} & \text{product of 0 primes} \arrow[l]
\end{tikzcd}
$$

Examples

$12 = 2 \times 2 \times 3$

2c) Examples: Factors and privilege

Examples

$12 = 2 \times 2 \times 3$

$$\begin{tikzcd}
2 \arrow[dr] & & 3 \arrow[dl] \\
& 1 &
\end{tikzcd}$$

Examples

$12 = 2 \times 2 \times 3$

$$
\begin{tikzcd}
4 \arrow[dr] & & 6 \arrow[dl] \arrow[dr] & \\
& 2 \arrow[dr] & & 3 \arrow[dl] \\
& & 1 &
\end{tikzcd}
$$

Examples

$12 = 2 \times 2 \times 3$

$$\begin{tikzcd}[column sep=small]
 & 12 \arrow[dl] \arrow[dr] & & \\
4 \arrow[dr] & & 6 \arrow[dl] \arrow[dr] & \\
 & 2 \arrow[dr] & & 3 \arrow[dl] \\
 & & 1 &
\end{tikzcd}$$

Examples

$12 = 2 \times 2 \times 3$

$24 = 2 \times 2 \times 2 \times 3$

$$
\begin{tikzcd}
 & 12 \arrow[dl] \arrow[dr] & & \\
4 \arrow[dr] & & 6 \arrow[dl] \arrow[dr] & \\
 & 2 \arrow[dr] & & 3 \arrow[dl] \\
 & & 1 &
\end{tikzcd}
$$

Examples

$12 = 2 \times 2 \times 3$

$24 = 2 \times 2 \times 2 \times 3$

$$\begin{tikzcd}
 & 24 \arrow[dl] \arrow[dr] & & & \\
8 \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & & \\
 & 4 \arrow[dr] & & 6 \arrow[dl] \arrow[dr] & \\
 & & 2 \arrow[dr] & & 3 \arrow[dl] \\
 & & & 1 &
\end{tikzcd}$$

Examples

$12 = 2 \times 2 \times 3$

$24 = 2 \times 2 \times 2 \times 3$

$$
\begin{tikzcd}
 & 24 \arrow[dl] \arrow[dr] & & & \\
8 \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & & \\
 & 4 \arrow[dr] & & 6 \arrow[dl] \arrow[dr] & \\
 & & 2 \arrow[dr] & & 3 \arrow[dl] \\
 & & & 1 &
\end{tikzcd}
$$

$8 = 2 \times 2 \times 2$

2c) Examples: Factors and privilege

Examples

$12 = 2 \times 2 \times 3$

$36 = 2 \times 2 \times 3 \times 3$

$24 = 2 \times 2 \times 2 \times 3$

$$\begin{tikzcd}
 & 24 \arrow[dl] \arrow[dr] & & & \\
8 \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & & \\
 & 4 \arrow[dr] & & 6 \arrow[dl] \arrow[dr] & \\
 & & 2 \arrow[dr] & & 3 \arrow[dl] \\
 & & & 1 &
\end{tikzcd}$$

$8 = 2 \times 2 \times 2$

2c) Examples: Factors and privilege

Examples

$12 = 2 \times 2 \times 3$

$24 = 2 \times 2 \times 2 \times 3$

$36 = 2 \times 2 \times 3 \times 3$

$$
\begin{tikzcd}
 & 24 \arrow[dl] \arrow[dr] & & & \\
8 \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & & \\
 & 4 \arrow[dr] & & 6 \arrow[dl] \arrow[dr] & \\
 & & 2 \arrow[dr] & & 3 \arrow[dl] \\
 & & & 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 12 \arrow[dl] \arrow[dr] & & \\
4 \arrow[dr] & & 6 \arrow[dl] \arrow[dr] & \\
 & 2 \arrow[dr] & & 3 \arrow[dl] \\
 & & 1 &
\end{tikzcd}
$$

$8 = 2 \times 2 \times 2$

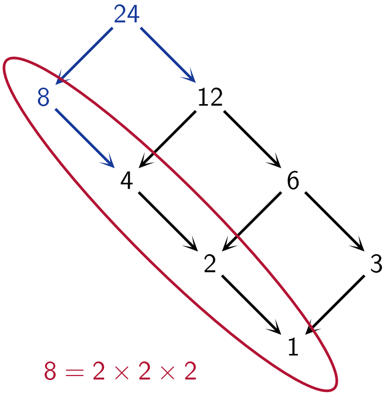
2c) Examples: Factors and privilege

Examples

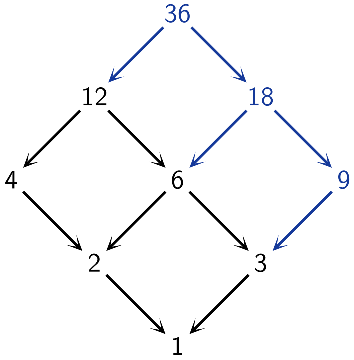
$12 = 2 \times 2 \times 3$

$24 = 2 \times 2 \times 2 \times 3$

$36 = 2 \times 2 \times 3 \times 3$



$8 = 2 \times 2 \times 2$

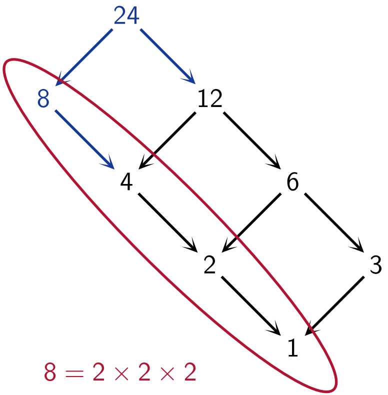


Examples

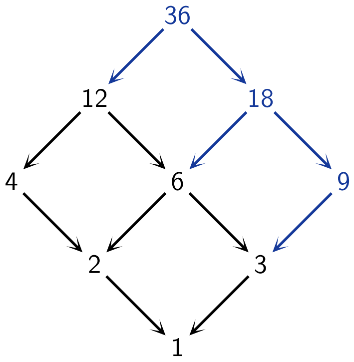
$12 = 2 \times 2 \times 3$

$24 = 2 \times 2 \times 2 \times 3$

$36 = 2 \times 2 \times 3 \times 3$



$8 = 2 \times 2 \times 2$



- Number of dimensions = number of distinct prime factors

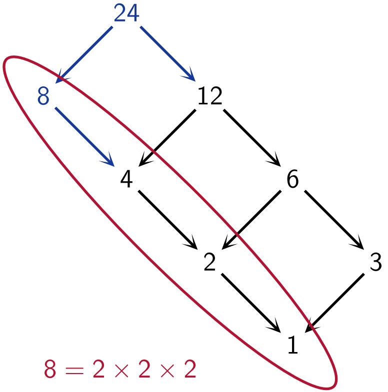
Examples

$12 = 2 \times 2 \times 3$

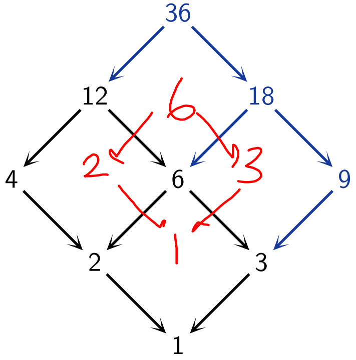
$24 = 2 \times 2 \times 2 \times 3$

$36 = 2 \times 2 \times 3 \times 3$

$= 6 \times 6$



$8 = 2 \times 2 \times 2$



- Number of dimensions $=$ number of distinct prime factors

- Lengths of paths $=$ number of repetitions of prime factors

Factors of 42

Factors of 42

$$1,\ 2,\ 3,\ 6,\ 7,\ 14,\ 21,\ 42$$

2c) Examples: Factors and privilege

Factors of 42

$$1,\ 2,\ 3,\ 6,\ 7,\ 14,\ 21,\ 42$$

$$\begin{tikzcd}
2 \arrow[dr] & 3 \arrow[d] & 7 \arrow[dl] \\
& 1 &
\end{tikzcd}$$

Factors of 42

1, 2, 3, 6, 7, 14, 21, 42

$$\begin{tikzcd}
6 \arrow[d] \arrow[dr] & 14 \arrow[dl] \arrow[dr] & 21 \arrow[dl] \arrow[d] \\
2 \arrow[dr] & 3 \arrow[d] & 7 \arrow[dl] \\
& 1 &
\end{tikzcd}$$

Factors of 42

1, 2, 3, 6, 7, 14, 21, 42

$$
\begin{tikzcd}
 & 42 \arrow[dl] \arrow[d] \arrow[dr] & \\
6 \arrow[d] \arrow[dr] & 14 \arrow[dl] \arrow[dr] & 21 \arrow[dl] \arrow[d] \\
2 \arrow[dr] & 3 \arrow[d] & 7 \arrow[dl] \\
 & 1 &
\end{tikzcd}
$$

Factors of 42

1, 2, 3, 6, 7, 14, 21, 42

$$
\begin{tikzcd}
 & 42 \arrow[dl] \arrow[d] \arrow[dr] & \\
6 \arrow[d] \arrow[dr] & 14 \arrow[dl] \arrow[dr] & 21 \arrow[dl] \arrow[d] \\
2 \arrow[dr] & 3 \arrow[d] & 7 \arrow[dl] \\
 & 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \{2,3,7\} \arrow[dl] \arrow[d] \arrow[dr] & \\
\{2,3\} \arrow[d] \arrow[dr] & \{2,7\} \arrow[dl] \arrow[dr] & \{3,7\} \arrow[dl] \arrow[d] \\
\{2\} \arrow[dr] & \{3\} \arrow[d] & \{7\} \arrow[dl] \\
 & \emptyset &
\end{tikzcd}
$$

Factors of 42

$1,\ 2,\ 3,\ 6,\ 7,\ 14,\ 21,\ 42$

$$
\begin{tikzcd}
 & 42 \arrow[dl] \arrow[d] \arrow[dr] & \\
6 \arrow[d] \arrow[dr] & 14 \arrow[dl] \arrow[dr] & 21 \arrow[dl] \arrow[d] \\
2 \arrow[dr] & 3 \arrow[d] & 7 \arrow[dl] \\
 & 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \{2,3,7\} \arrow[dl] \arrow[d] \arrow[dr] & \\
\{2,3\} \arrow[d] \arrow[dr] & \{2,7\} \arrow[dl] \arrow[dr] & \{3,7\} \arrow[dl] \arrow[d] \\
\{2\} \arrow[dr] & \{3\} \arrow[d] & \{7\} \arrow[dl] \\
 & \emptyset &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \{a,b,c\} \arrow[dl] \arrow[d] \arrow[dr] & \\
\{a,b\} \arrow[d] \arrow[dr] & \{a,c\} \arrow[dl] \arrow[dr] & \{b,c\} \arrow[dl] \arrow[d] \\
\{a\} \arrow[dr] & \{b\} \arrow[d] & \{c\} \arrow[dl] \\
 & \emptyset &
\end{tikzcd}
$$

Factors of 42

$$1,\ 2,\ 3,\ 6,\ 7,\ 14,\ 21,\ 42$$

$$\begin{tikzcd}
& 42 \arrow[dl] \arrow[d] \arrow[dr] & \\
6 \arrow[d] \arrow[dr] & 14 \arrow[dl] \arrow[dr] & 21 \arrow[dl] \arrow[d] \\
2 \arrow[dr] & 3 \arrow[d] & 7 \arrow[dl] \\
& 1 &
\end{tikzcd}$$

$$\begin{tikzcd}
& \{2,3,7\} \arrow[dl] \arrow[d] \arrow[dr] & \\
\{2,3\} \arrow[d] \arrow[dr] & \{2,7\} \arrow[dl] \arrow[dr] & \{3,7\} \arrow[dl] \arrow[d] \\
\{2\} \arrow[dr] & \{3\} \arrow[d] & \{7\} \arrow[dl] \\
& \emptyset &
\end{tikzcd}$$

$$\begin{tikzcd}
& \{a,b,c\} \arrow[dl] \arrow[d] \arrow[dr] & \\
\{a,b\} \arrow[d] \arrow[dr] & \{a,c\} \arrow[dl] \arrow[dr] & \{b,c\} \arrow[dl] \arrow[d] \\
\{a\} \arrow[dr] & \{b\} \arrow[d] & \{c\} \arrow[dl] \\
& \emptyset &
\end{tikzcd}$$

$6 < 7$

2c) Examples: Factors and privilege

rich white male

2c) Examples: Factors and privilege

$$\begin{tikzcd}
 & \text{rich white male} \arrow[dl] \\
\text{rich white} &
\end{tikzcd}$$

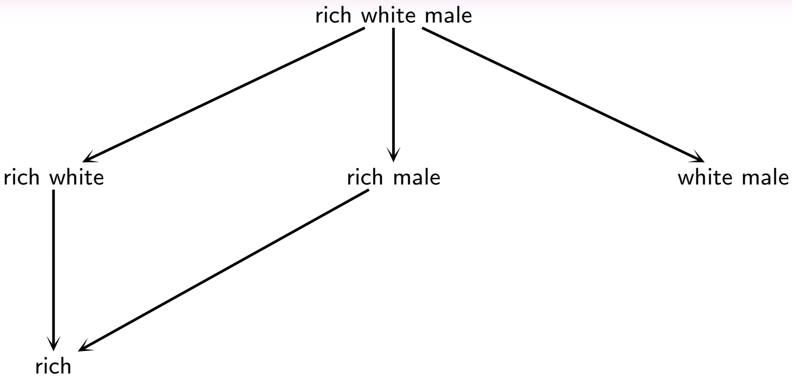
2c) Examples: Factors and privilege

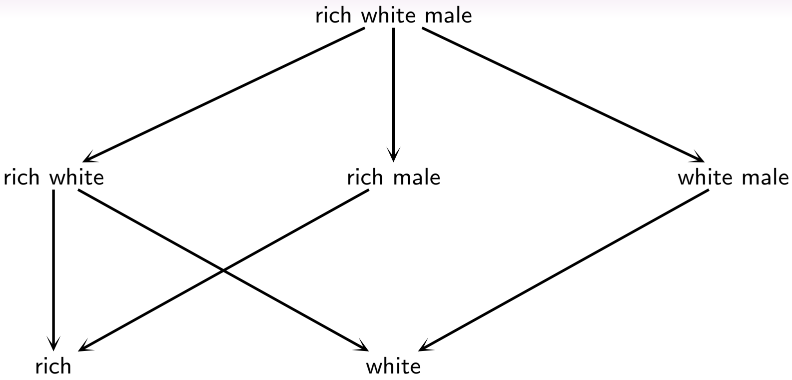
$$\begin{tikzcd}
& \text{rich white male} \arrow[dl] \arrow[d] \\
\text{rich white} & \text{rich male}
\end{tikzcd}$$

2c) Examples: Factors and privilege

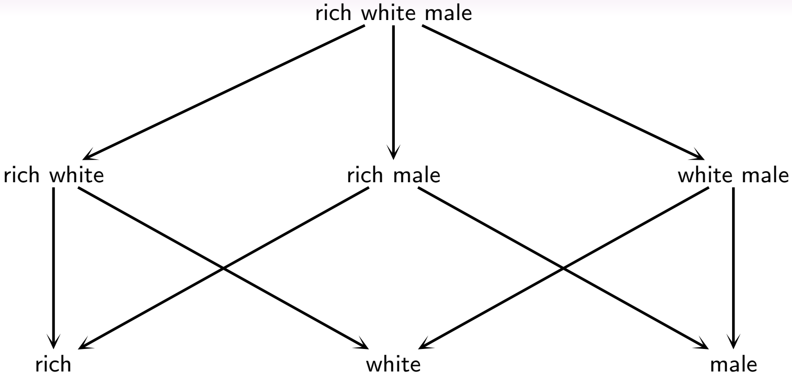
$$
\begin{tikzcd}
 & \text{rich white male} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{rich white} & \text{rich male} & \text{white male}
\end{tikzcd}
$$

2c) Examples: Factors and privilege

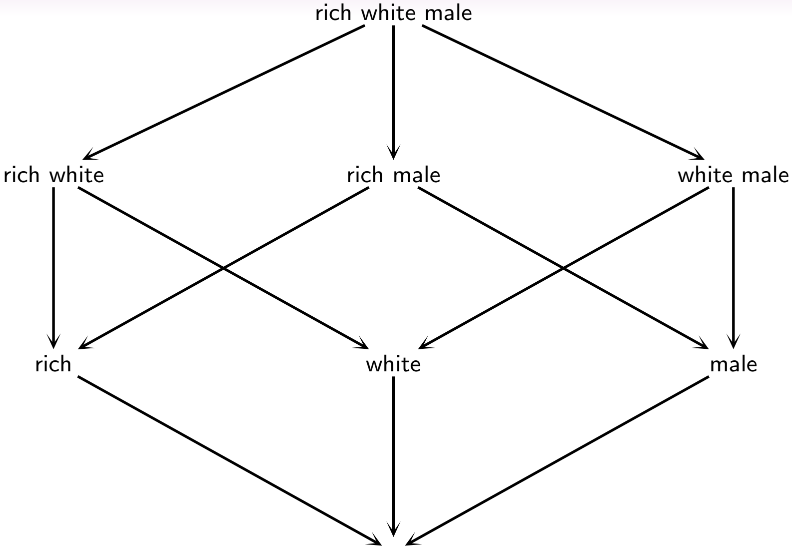



rich white male
rich white
rich male
white male
rich
white

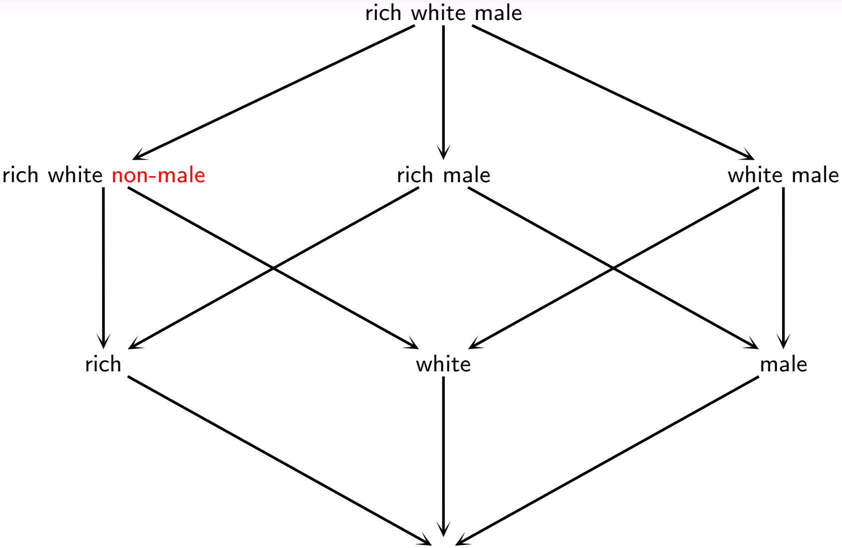
2c) Examples: Factors and privilege



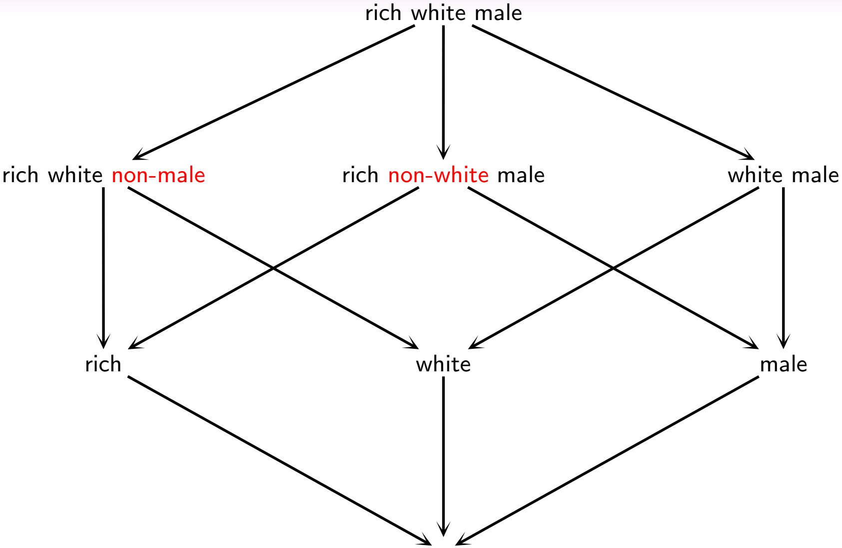
2c) Examples: Factors and privilege



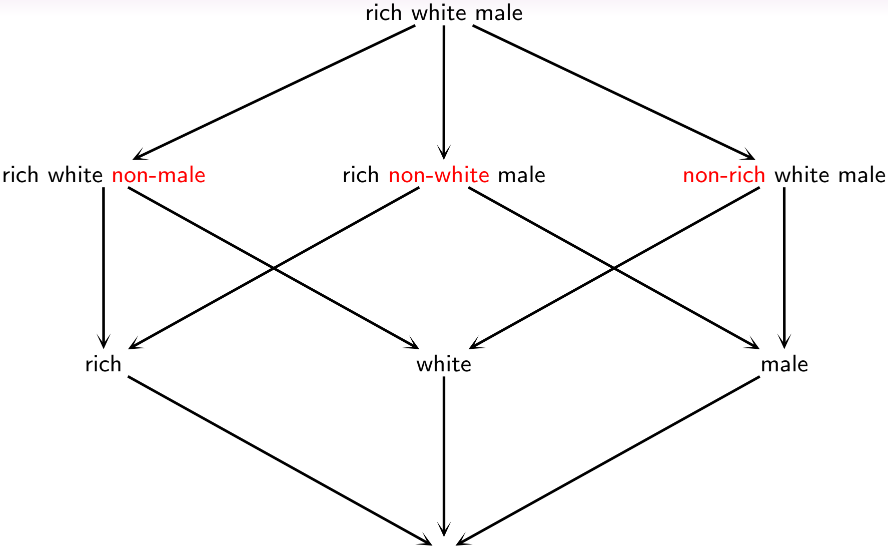
2c) Examples: Factors and privilege



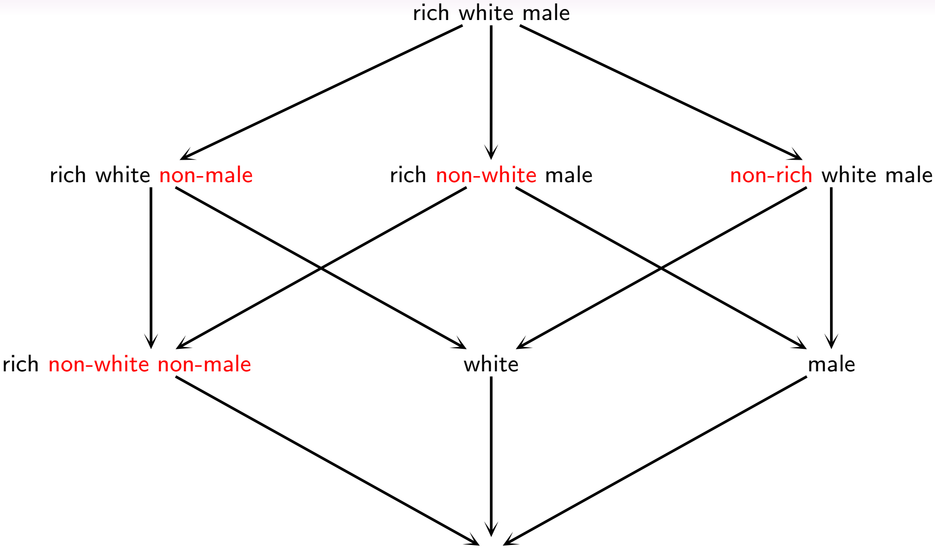
2c) Examples: Factors and privilege

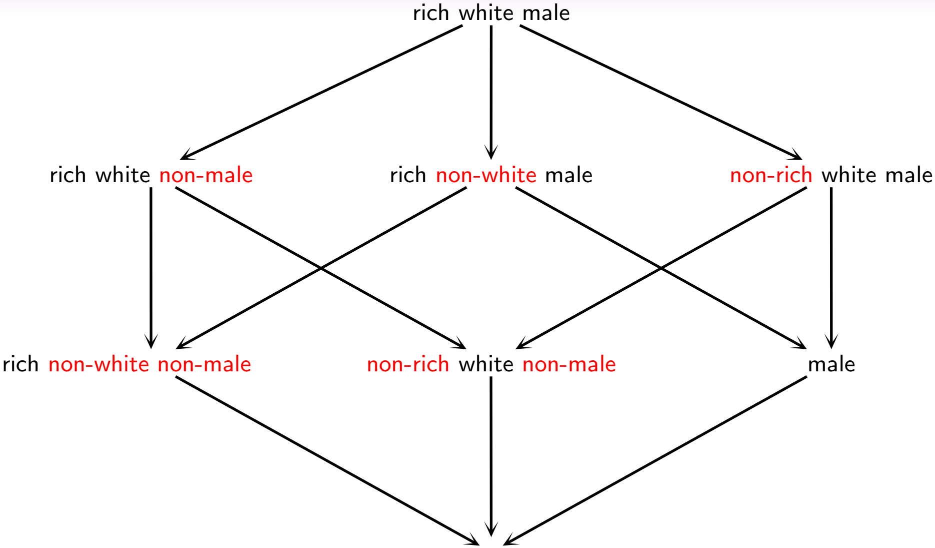


2c) Examples: Factors and privilege



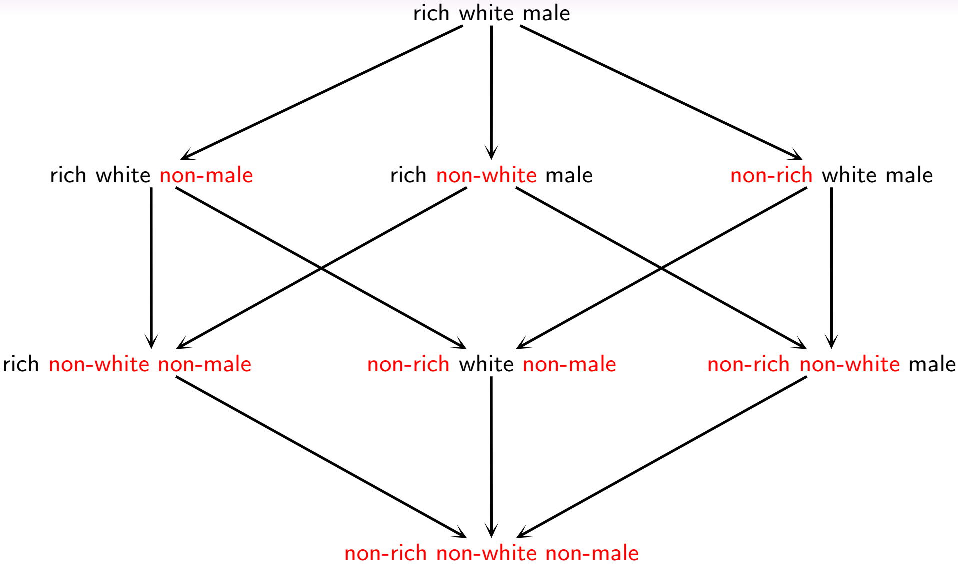
2c) Examples: Factors and privilege




rich white male
rich white non-male
rich non-white male
non-rich white male
rich non-white non-male
non-rich white non-male
male

$$
\begin{tikzcd}
 & & \text{rich white male} \arrow[dll] \arrow[d] \arrow[drr] & & \\
\text{rich white non-male} \arrow[d] \arrow[drr] & & \text{rich non-white male} \arrow[dll] \arrow[drr] & & \text{non-rich white male} \arrow[dll] \arrow[d] \\
\text{rich non-white non-male} \arrow[drr] & & \text{non-rich white non-male} \arrow[d] & & \text{non-rich non-white male} \arrow[dll] \\
 & & {} & &
\end{tikzcd}
$$

2c) Examples: Factors and privilege



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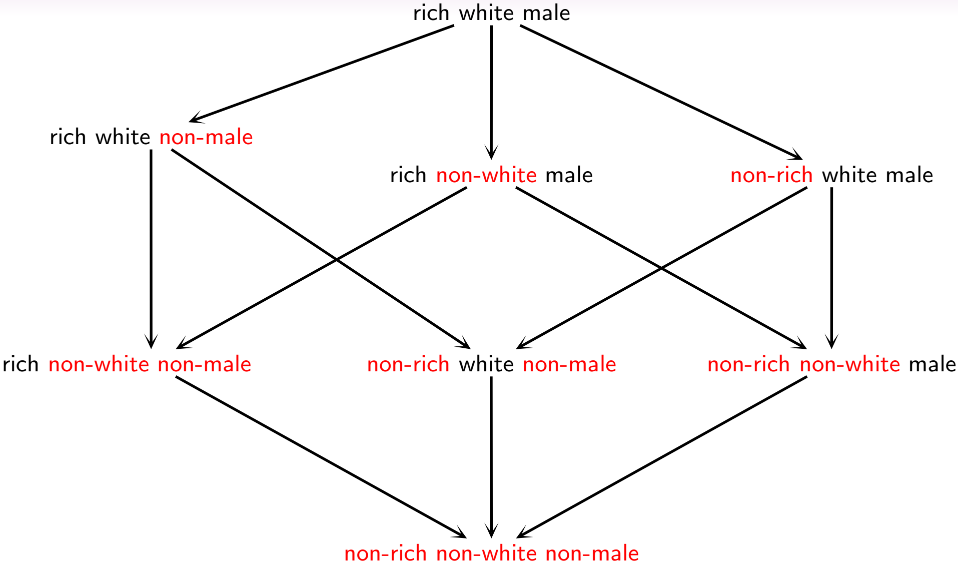

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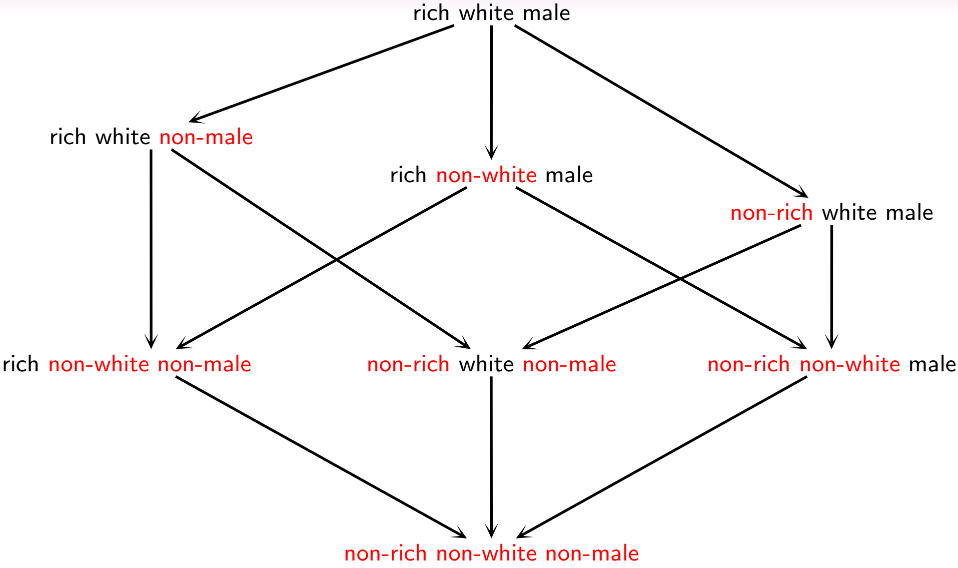

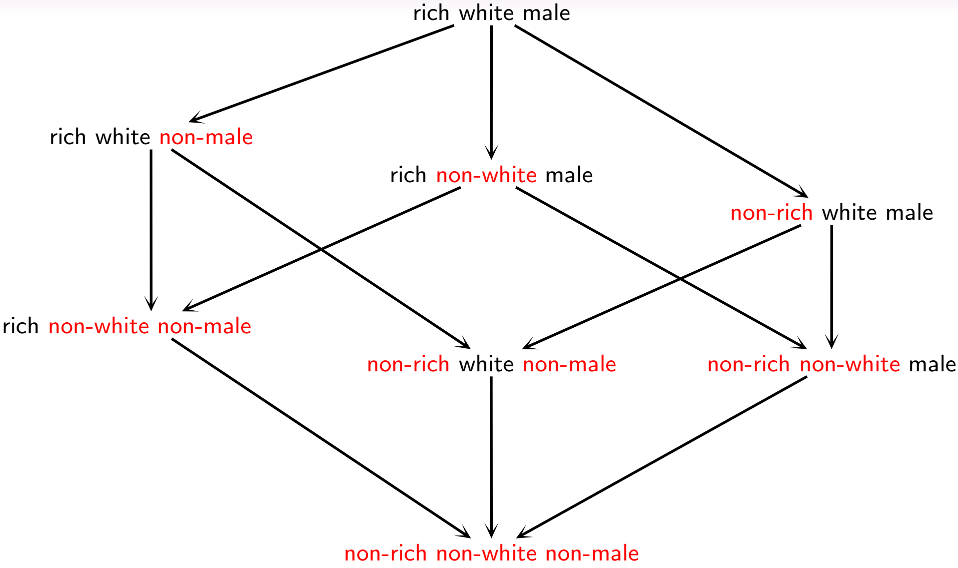
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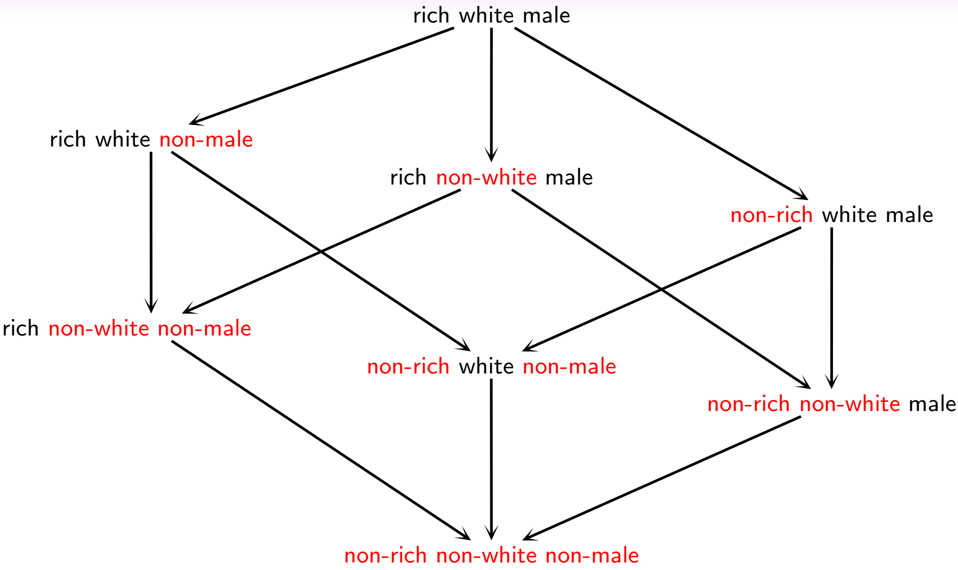

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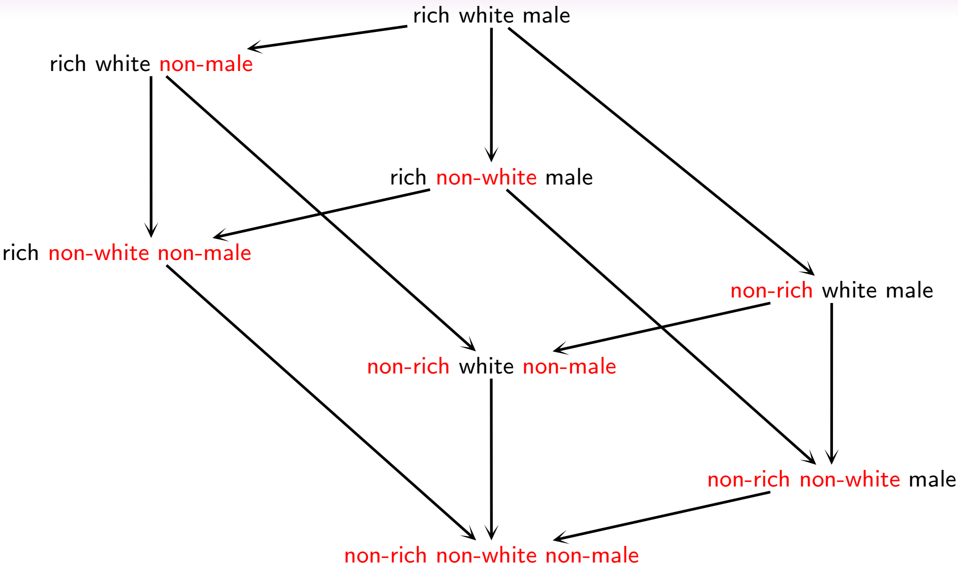

Part 3: Invertibility

1. More examples of categories: sets, ordered sets, groups
2. Zooming in and out
3. Invertibility

There is a large category with

- objects: all possible sets
- arrows: all possible functions

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Functions have a fixed domain and codomain (range).

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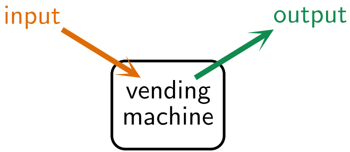
> Functions have a fixed domain and codomain (range).

A function is like a vending machine.



a - - - -> apple
b - - - -> banana
c - - - -> chocolate

3a. More examples: Sets and functions

There is a large category with

- objects: all possible sets
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Functions have a fixed domain and codomain (range).

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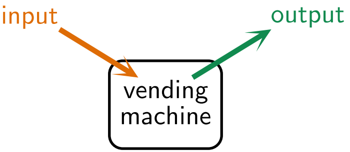
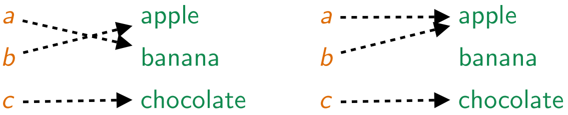


$a \dashrightarrow$ apple
$b \dashrightarrow$ banana
$c \dashrightarrow$ chocolate

But we could do less "sensible" things:



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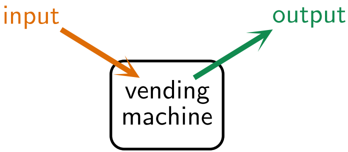

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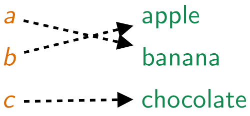
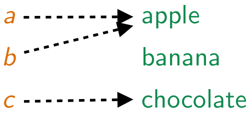
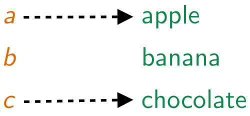
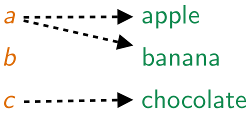




Every input has to produce exactly one thing so these don't count.





> In this category:
> a morphism $A \longrightarrow B$ is a function $A \longrightarrow B$.

3a. More examples: order-preserving functions

In some cases the lines don't have to cross

$$\begin{array}{lcl} a & \dashrightarrow & \text{apple} \\ b & \dashrightarrow & \text{banana} \\ c & \nearrow & \end{array}$$

What other possibilities are there?

a apple
b banana
c

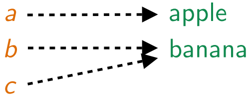
---

a apple
b banana
c

---

a apple
b banana
c

In some cases the lines don't have to cross



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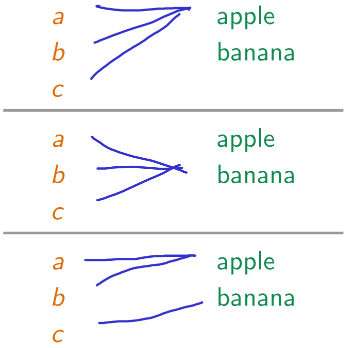
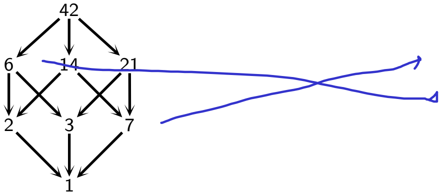


When lines don't cross this is called order-preserving.



Think about these:

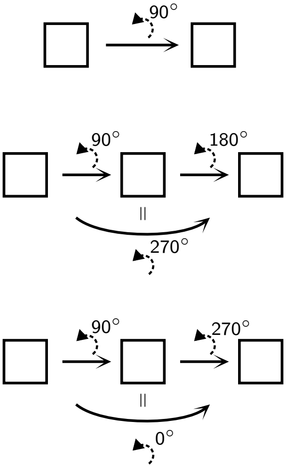
$\{\text{people, age}\} \longrightarrow \{\text{people, wealth}\}$

$\{\text{people, privilege}\} \longrightarrow \{\text{people, wealth}\}$

There is a category of ordered sets and order-preserving functions

3a. More examples: Sets with structure

We regarded symmetry as a relation and produced the table on the right



	0	90	180	270
0	0	90	180	270
90	90	180	270	0
180	180	270	0	90
270	270	0	90	180

The table gives a binary operation on the elements.

This is called a group. It has to satisfy some axioms:

- associativity
- there is an identity element which “does nothing”
- every element has an inverse which “undoes” it

This is a category with a single object.

3b. Zooming in and out

Categories work at different scales.

Zoom in: an ordered set is a category.

- objects: elements of the set
- morphisms: $\leq$

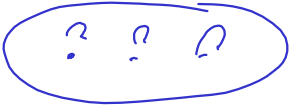
Zoom out: there is a large category of ordered sets.

- objects: ordered sets
- morphisms: order-preserving functions

We can do this on sets themselves.

Zoom in: a set is a category.

- objects: elements of the set
- morphisms: just identities



Zoom out: there is a large category of sets

- objects: sets
- morphisms: functions

Categories are a generalization of sets.

Zoom in: the group of symmetries of a square is a category

- objects: one single object, the square
- morphism: the symmetries

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This generalizes to other shapes.

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This generalizes to other shapes.

Zoom out: there is a large category of groups

- objects: groups
- morphisms: structure-preserving functions

Zoom in: the group of symmetries of a square is a category

- objects: one single object, the square
- morphism: the symmetries

This generalizes to other shapes.

Zoom out: there is a large category of groups

- objects: groups
- morphisms: structure-preserving functions

Technicalities:

- When we did ordered sets we also had structure-preserving morphisms.
- There the structure was the ordering, so the morphisms had to preserve the order.
- Here the structure is the binary operation, so the morphisms have to preserve that.
- This means

$$f(a \circ b) = f(a) \circ f(b)$$

We'll come back to this later.

3c. Invertibility

All equations are lies.

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$$8 + 1 \quad = \quad 1 + 8$$

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$$\begin{aligned} 8+1 &= 1+8 \\ 2\times 5 &= 5\times 2 \end{aligned}$$

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The only equation that is not a lie is

$x = x$.

All equations are lies. . . or useless.

$$
\begin{aligned}
8 + 1 &= 1 + 8 \\
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\end{aligned}
$$

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In category theory we try and express things just using relationships.

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For example, can we say these sets are "the same" without mentioning the elements?

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We use invertibility of morphisms.

This is about going back to where you started or undoing a process.

$$\begin{tikzcd} 1 \arrow[r, bend left, "+4"] & 5 \end{tikzcd}$$

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Are these processes undoable?

i. $\xrightarrow{+4}$

ii. $\xrightarrow{\times 4}$

iii. $\xrightarrow{+0}$

iv. $\xrightarrow{\times 0}$

v. $\xrightarrow{\text{squaring}}$ $\{1\} \longrightarrow \{1\}$

vi. All the functions $\{a, b\} \longrightarrow \{1, 2\}$. Which do you think should count as invertible?

vii. What about functions $\{a, b, c\} \longrightarrow \{1, 2\}$?

vi.

$$A \xrightarrow{f} B$$
$$\begin{tikzcd} a \arrow[r] & 1 \\ b \arrow[r] & 2 \end{tikzcd}$$

$$A \xrightarrow{f} B$$
$$\begin{tikzcd} a \arrow[dr] & 1 \\ b \arrow[ur] & 2 \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r] & 1 \\ b \arrow[ur] & 2 \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[dr] & 1 \\ b \arrow[r] & 2 \end{tikzcd}$$

vii.

$$A \xrightarrow{f} B$$
$$\begin{tikzcd} a \arrow[r] & 1 \\ b \arrow[r] & 2 \\ c \arrow[ur] & \end{tikzcd}$$

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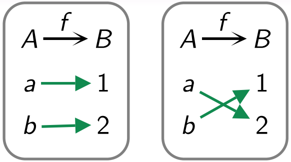
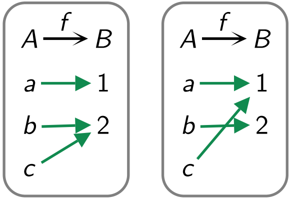
3c. Invertibility

vi.



vii.



- In what sense is/isn't divorce the inverse of marriage?
- In what sense is/isn't a pardon the inverse of a criminial conviction?

Definition of inverses in category theory

An inverse for f is g such that:

$$\begin{tikzcd} a \arrow[r, bend left, "f"] & b \arrow[l, bend left, "g"] \end{tikzcd}$$

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\end{tikzcd}
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\end{tikzcd}
$$

$$
\begin{tikzcd}
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\end{tikzcd}
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In that case

- f and g are inverses of each other
- they are called invertible, also isomorphisms
- a and b are called isomorphic

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$$\begin{tikzcd} & x \arrow[dl] & \\ a \arrow[rr, bend left, "f"] & & b \arrow[ll, bend left, "g"] \end{tikzcd}$$

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Isomorphic objects are treated as the same by the rest of the category.

$$\begin{tikzcd} & x \arrow[dl] \arrow[dr, dotted] & \\ a \arrow[rr, bend left, "f"] & & b \arrow[ll, bend left, "g"] \end{tikzcd}$$

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$$\begin{tikzcd} & x \arrow[dr, red] & \\ a \arrow[rr, bend left, "f"] & & b \arrow[ll, bend left, red, "g"] \end{tikzcd}$$

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$$\begin{tikzcd} & x \arrow[dl, dotted] \arrow[dr, red] & \\ a \arrow[rr, bend left, "f"] & & b \arrow[ll, bend left, red, "\color{red}g"] \end{tikzcd}$$

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Things to try proving:

i. Inverses are unique.

ii. Isomorphism is an equivalence relation.

Equality in category theory is not about when things are the same,
but when the category treats them as the same.

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but when the category treats them as the same.

Equality in society should be about
society treating people as the same.

Part 4: Sameness

a) Recap
b) Isomorphisms of sets
c) Isomorphisms of groups
d) Isomorphisms of ordered sets

Category theory is the "mathematics of mathematics"

Why do we even do math at all?

- To solve problems.
- It's fun and interesting.
- We can apply it to other parts of life.
- It helps us understand the world better.
- It helps us make connections between different things in the world.

Category theory does all these things for us in math and therefore also life.

Definition: a category $\mathcal{C}$ consists of:

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- a set of objects $\mathrm{ob}\,\mathcal{C}$
- for all $a, b \in \mathrm{ob}\,\mathcal{C}$ a set of arrows $\mathcal{C}(a, b)$

$$a \longrightarrow b$$

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$$a \longrightarrow b$$

Structure

- identities: for all objects a

  an identity arrow $a \xrightarrow{1_a} a$

- composition: given $a \xrightarrow{f} b \xrightarrow{g} c$

  a composite arrow $a \xrightarrow{g \circ f} c$

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Properties (axioms)

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$$a \xrightarrow{1_a} a \xrightarrow{f} b \quad = \quad a \xrightarrow{f} b$$

$$a \xrightarrow{f} b \xrightarrow{1_b} b \quad = \quad a \xrightarrow{f} b$$

- associativity: given $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$

$$(h \circ g) \circ f = h \circ (g \circ f)$$

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$$(h \circ g) \circ f = h \circ (g \circ f)$$

What happened to symmetry?

We don't demand it but we look for it afterwards.

This is the notion of "sameness" in a category.

Definition of inverses in category theory

An inverse for f is g such that:

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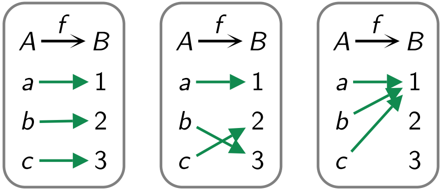
When is a function an isomorphism?



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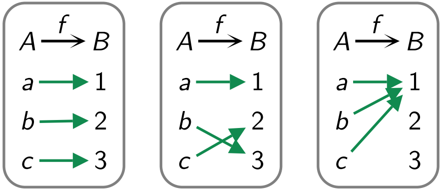

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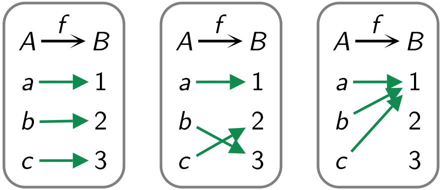

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Similar to permutations!

Isomorphisms $A \longrightarrow B$ are bijections.

- For finite sets: same number of elements.
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4b. Isomorphisms of sets

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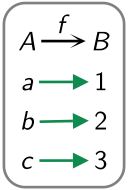

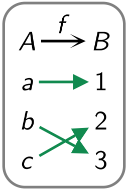

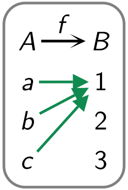

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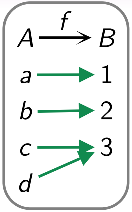

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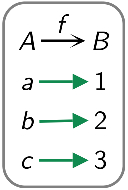

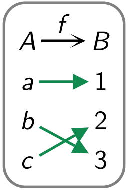

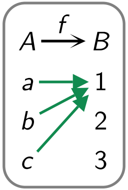

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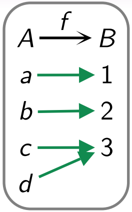

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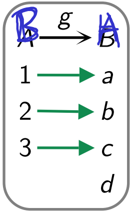
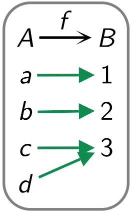
We can try to define an inverse g.





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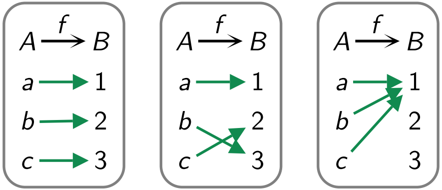

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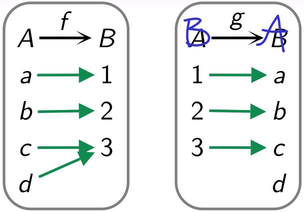

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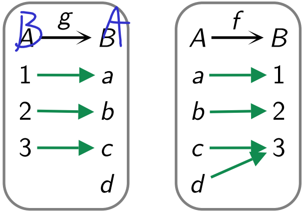

A group is a set with a binary operation satisfying associativity, identities, and inverses.

- It is a category with one object in which every morphism is an isomorphism.
- But we can zoom out: what is an isomorphism of groups?

Key: patterns in tables

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Key: patterns in tables

Rotations of square $\equiv$ addition on 4-hour clock

	0	90	180	270
0	0	90	180	270
90	90	180	270	0
180	180	270	0	90
270	270	0	90	180

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

These patterns are "the same".

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	0	1	2	3
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2	2	3	0	1
3	3	0	1	2

These patterns are "the same".

Try $\mathbb{Z}_{10}$ ("10-hour clock") $\times$

	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

	1	3	9	7
1	1	3	9	7
3	3	9	7	1
9	9	7	1	3
7	7	1	3	9

We have to re-order this to see the pattern.

Try $\mathbb{Z}_8$ (8-hour clock) $\times$.
What pattern do you see?

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Here are two functions from the rotations of a square to $\{1, 3, 9, 7\}$ in $\mathbb{Z}_{10}$

f:

$$0 \mapsto 1,\quad 90 \mapsto 3,\quad 180 \mapsto 7,\quad 270 \mapsto 9$$

g:

$$0 \mapsto 1,\quad 90 \mapsto 3,\quad 180 \mapsto 9,\quad 270 \mapsto 7$$

Which do you think should count as "pattern preserving" and which not?

	0	90	180	270
0	0	90	180	270
90	90	180	270	0
180	180	270	0	90
270	270	0	90	180

	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
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	1	3	9	7
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7	7	~~9~~ 1	3	9

4c. Isomorphisms of groups

Here are two functions from the rotations of a square to $\{1, 3, 9, 7\}$ in $\mathbb{Z}_{10}$

$$
f:\quad \begin{array}{l} 0 \longrightarrow 1 \\ 90 \longrightarrow 3 \\ 180 \longrightarrow 7 \\ 270 \longrightarrow 9 \end{array}
\qquad\qquad
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	0	90	180	270
0	0	90	180	270
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270	270	0	90	180

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9	9	7	3	1

	1	3	9	7
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3	3	9	7	1
9	9	7	1	3
7	7	9	3	9

Preserving pattern is about respecting $\circ$

A group homomorphism $G \xrightarrow{f} H$ is a function such that for all $a, b, \in G$, $f(a \circ b) = f(a) \circ f(b)$

$$
\begin{tikzcd}
\boxed{a \,\vdots\, b} \arrow[r, "\text{do } F"] \arrow[dd, "\text{do } \circ"'] & \boxed{F(a) \,\vdots\, F(b)} \arrow[d, "\text{do } \circ"] \\
 & \boxed{F(a) \circ F(b)} \arrow[d, phantom, "\parallel\ ?"] \\
\boxed{a \circ b} \arrow[r, "\text{do } F"'] & \boxed{F(a \circ b)}
\end{tikzcd}
$$

Groups and homomorphisms form a category.

- Isomorphisms in the category of groups and group homomorphisms are group homomorphisms with an inverse.
- This turns out to mean they have the same pattern.
- There is only one possible pattern for a group of 2 elements. We say there is only one group with 2 elements "up to isomorphism".
- There is also only one group with 3 elements "up to isomorphism".

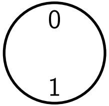
For example addition on a "3-hour clock" (integers modulo 3).

+	0	1	2
0	0	1	2
1	1	2	0
2	2	1	0

This would also be the same pattern as rotations of an equilateral triangle.

4c. Isomorphisms of groups

Here are some examples of the 2-element group.



$+$	0	1
0	0	1
1	1	0

$+$	even	odd
even	even	odd
odd	odd	even

Battenberg Cake

$\times$	0	1
0		
1		

4c. Isomorphisms of groups

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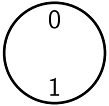


$+$	0	1
0	0	1
1	1	0

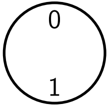
$+$	even	odd
even	even	odd
odd	odd	even

Battenberg Cake

$\times$	0	1
0	0	0
1	0	1

4c. Isomorphisms of groups

Here are some examples of the 2-element group.



$+$	0	1
0	0	1
1	1	0

$+$	even	odd
even	even	odd
odd	odd	even

Battenberg Cake

$\times$	0	1
0	0	0
1	0	1

$\times$	even	odd
even	even	even
odd	even	odd

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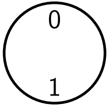


+	0	1
0	0	1
1	1	0

+	even	odd
even	even	odd
odd	odd	even

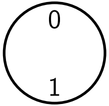
Battenberg Cake

×	0	1
0	0	0
1	0	1

×	even	odd
even	even	even
odd	even	odd

4c. Isomorphisms of groups

Here are some examples of the 2-element group.



$+$	0	1
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even	even	odd
odd	odd	even

Battenberg Cake

$\times$	0	1
0	0	0
1	0	1

$\times$	even	odd
even	even	even
odd	even	odd

$\times$	real	imaginary
real	real	imaginary
imaginary	imaginary	real

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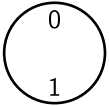


+	0	1
0	0	1
1	1	0

+	even	odd
even	even	odd
odd	odd	even

Battenberg Cake

×	0	1
0	0	0
1	0	1

×	even	odd
even	even	even
odd	even	odd

×	real	imaginary
real	real	imaginary
imaginary	imaginary	real

×	tolerant	intolerant
tolerant	tolerant	intolerant
intolerant	intolerant	tolerant

- There are only two possible patterns for a group of 4 elements.
- We say there are only two groups with 4 elements, "up to isomorphism".

We have seen the two possible patterns:

	0	90	180	270
0	0	90	180	270
90	90	180	270	0
180	180	270	0	90
270	270	0	90	180

$\mathbb{Z}_8$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Isomorphisms of ordered sets are functions that are both order-preserving and invertible.

Consider these functions $\mathbb{Z} \longrightarrow \mathbb{Z}$.
Are they order-preserving? Invertible?

i. $f(n) = 2n$

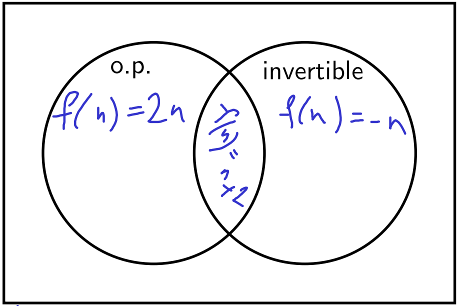
ii. $f(n) = -n$

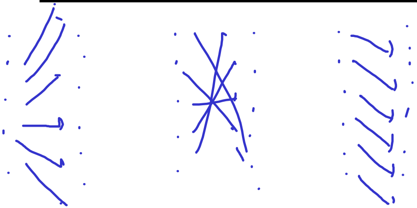
iii. $f(n) = n + 2$

iv. Can you figure out what *all* the order-preserving isomorphisms are? See if you can put things in areas of the Venn diagram.

if $x \leq y$ then $f(x) \leq f(y)$

functions $\mathbb{Z} \longrightarrow \mathbb{Z}$





When we draw an ordered set like a category
an isomorphism shows "the same pattern" of arrows.

$\boxed{\text{These whole categories are isomorphic.}}$

$$\begin{tikzcd}
\text{male people} \arrow[d] & & \text{white people} \arrow[d] & & \text{rich people} \arrow[d] \\
\text{female people} & \cong & \text{black people} & \cong & \text{poor people}
\end{tikzcd}$$

When we draw an ordered set like a category
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$$
\begin{tikzcd}
\text{male people} \arrow[d] \arrow[r, phantom, "\cong"] & \text{white people} \arrow[d] \arrow[r, phantom, "\cong"] & \text{rich people} \arrow[d] \\
\text{female people} & \text{black people} & \text{poor people}
\end{tikzcd}
$$

However:

$$
\begin{tikzcd}
\text{male people} \arrow[d] \arrow[r, phantom, "\ncong"] \arrow[dr] & \text{female people} \\
\text{female people} \arrow[ur] & \text{male people} \arrow[u]
\end{tikzcd}
$$

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\end{tikzcd}
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$$
\begin{tikzcd}
\text{male people} \arrow[d] & \ncong & \text{female people} \\
\text{female people} & & \text{male people} \arrow[u]
\end{tikzcd}
$$

These are also isomorphic.

$$
\begin{tikzcd}
 & 30 \arrow[dl] \arrow[d] \arrow[dr] & \\
6 \arrow[d] \arrow[dr] & 10 \arrow[dl] \arrow[dr] & 15 \arrow[dl] \arrow[d] \\
2 \arrow[dr] & 3 \arrow[d] & 5 \arrow[dl] \\
 & 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \text{rich white men} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{rw} \arrow[d] \arrow[dr] & \text{rm} \arrow[dl] \arrow[dr] & \text{wm} \arrow[dl] \arrow[d] \\
\text{r} \arrow[dr] & \text{w} \arrow[d] & \text{m} \arrow[dl] \\
 & \emptyset &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \text{rich white cis women} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{rich white trans women} \arrow[d] \arrow[dr] & \text{rich non-white cis women} \arrow[dl] \arrow[dr] & \text{non-rich white cis women} \arrow[dl] \arrow[d] \\
\text{rich non-white trans women} \arrow[dr] & \text{non-rich white trans women} \arrow[d] & \text{non-rich non-white cis women} \arrow[dl] \\
 & \text{non-rich non-white trans women} &
\end{tikzcd}
$$

Part 5: Universal properties

a) Role vs character

b) Extremes

c) Example: sets

d) Context

In normal language we mix up role and intrinsic character

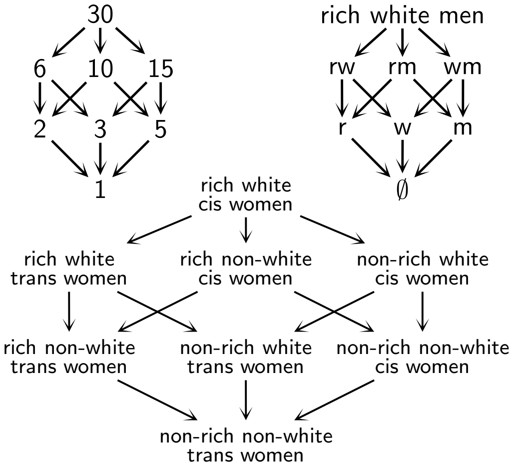
For example: film.

Think about role and intrinsic character for these examples.

i. baseball cap
ii. pumpkin spice
iii. whipped cream
iv. fat-free half-and-half
v. teaspoon
vi. pound cake
vii. cookie, biscuit

5b. Extremes

Last time we saw that these categories are isomorphic.



- 30
- rich white men
- rich white cis women

play analogous roles within those contexts.

Can you describe their role, just by talking about arrows in the category?

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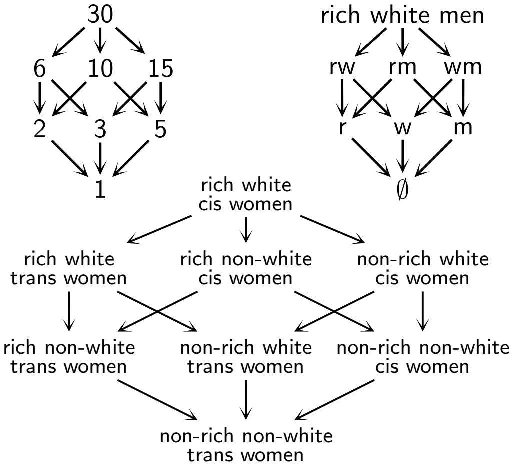

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play analogous roles within those contexts.

Can you describe their role, just by talking about arrows in the category?

Category theory seeks to characterize things by the role they play in a category, just by describing interactions between arrows.

Advantage: we can go round looking for analogous situations in any category.

Another example: natural numbers

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow \cdots$$

5b. Extremes: initial and terminal objects

We look for an object where all arrows "start".

Definition

An initial object in a category $\mathcal{C}$ is an object I such that $\forall X \in \mathcal{C}, \ \exists!\, I \longrightarrow X$

translation:

For all X in C, there exists a unique arrow I -> X

A terminal object in a category $\mathcal{C}$ is an object T such that $\forall X \in \mathcal{C}, \ \exists!\, X \longrightarrow T$

translation:

For all X in C, there exists a unique arrow X -> T

Terminal objects are dual to initial objects:
the definitions are the same
just with all the arrows reversed.

Questions one might ask

- When are there terminal and initial objects?
- How many can there be? Are there ever none?
- Can something be both terminal and initial?

Thinking mathematically is as much about asking good questions as about finding answers.

Example

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow \cdots$$

does not have a terminal object because:

It is "trying" to be: ∞

Explore: try and draw some categories which fail to have a terminal/initial object. Try and classify the types of failure into which features cause them to fail.

Something to try and prove: Can a category have more than one initial object? (Think about whether a room can have two "tallest people" and if so, what does that tell us about them?)

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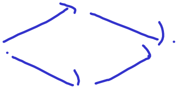

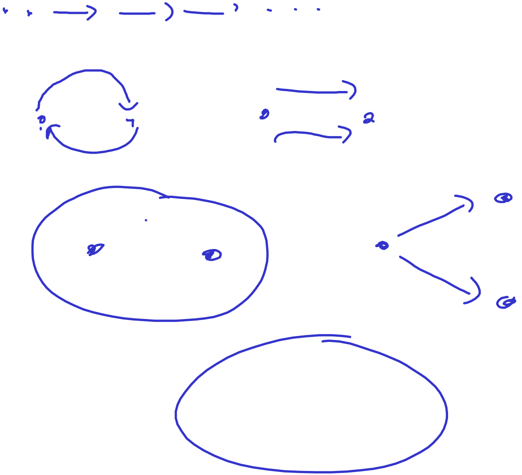

infinite, loops, parallel, branching, disconnected, no arrows, no objects

In the category of sets and functions.

How many functions $A \longrightarrow B$.

We write $|A|$ for the number of elements in A.

Number of functions $A \longrightarrow B$ is $|B|^{|A|}$.

What values of $|A|$ and $|B|$ would make this 1?

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We write $|A|$ for the number of elements in A.

Number of functions $A \longrightarrow B$ is $|B|^{|A|}$.

What values of $|A|$ and $|B|$ would make this 1?

- If $|B| = 1$ then raising it to any power gives 1.

This tells us that if B has 1 object there is always exactly 1 function to it.

That is: any 1-element set is terminal.

- If $|A| = 0$ then the formula always gives 1.

This tells us that if A has 0 objects then there is always exactly 1 function out of it.

That is: the empty set $\varnothing$ is initial.

How is this possible??

The definition of being a function is vacuously satisfied.

5d. Context

Something not universal in one place can be universal in another.

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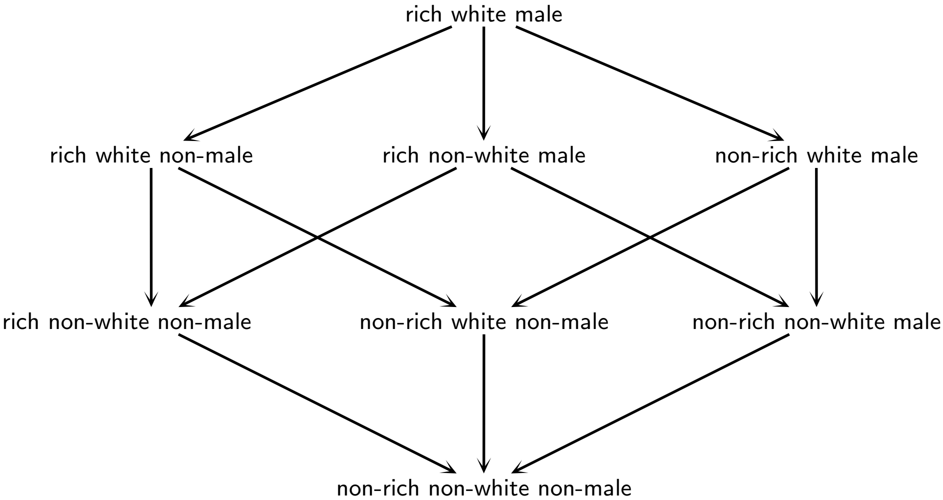

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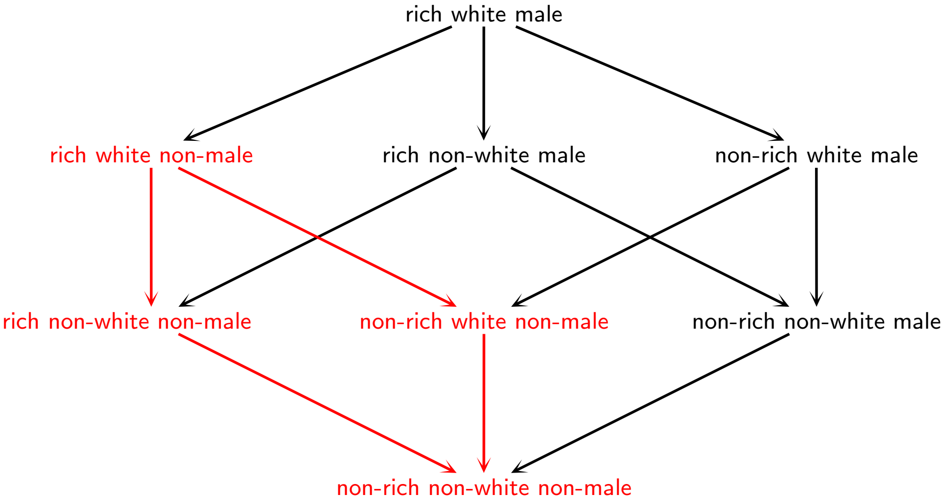

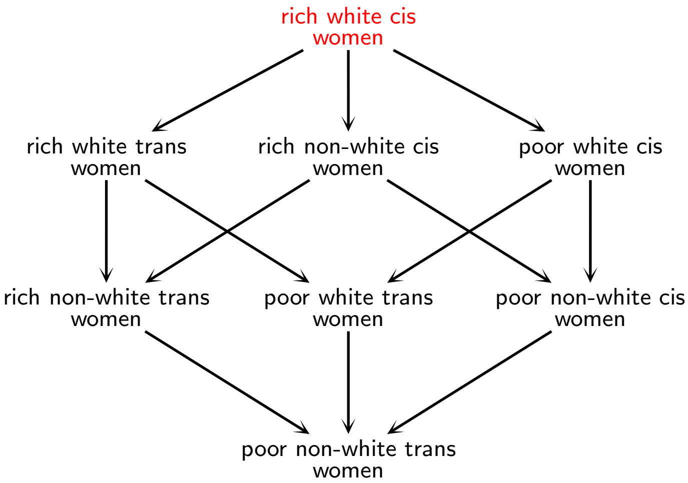

We can always restrict context to make something universal.

If you're the only person in the room then you're definitely the tallest. And the shortest.

Example: my life.

Further reading

- The ideas of category theory mostly covered in my first book *How to Bake* π.
- All the formality of what we covered, and more, is in my most recent book *The Joy of Abstraction*.
- Thoughts on social and political arguments are in *The Art of Logic in an Illogical World*.
- Thoughts on ingressive/congressive math and life are in $x + y$: A Mathematician's Manifesto For Re-thinking Gender.
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Thoughts to take away

- Abstraction has a point: it takes us further from "real life" but enables us to include a wider range of examples.
- So abstract math can be more relevant to our daily lives than applied math.
- Math is not just about solving problems; it is also about shedding light.
- I think the "shedding light" aspect of math is more congressive and therefore can be more broadly inclusive.

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- I think the "shedding light" aspect of math is more congressive and therefore can be more broadly inclusive.

Category theory is a congressive form of math that can be accessible and relevant to everyone.