immersed boundary method: using fractional step method

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1 Incompressible Navier-Stokes equation with boundary force

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \frac{1}{Re}\nabla^2 u + \int f(\xi)\delta(\xi - x)ds \nabla \cdot u = 0 u(\xi) = \int u(x)\delta(x-\xi)dx \tag{1}$$

where $f(\xi), u(\xi)$ describe the boundary Lagrangian points
discritization as incompressible Navier-Stokes equation as before.

$$\begin{bmatrix} A & G & -H \\ D & 0 & 0 \\ E & 0 & 0 \end{bmatrix} \begin{pmatrix} v^{n+1} \\ p^{n+1} \\ f(\xi) \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ u(\xi)^{n+1} \end{pmatrix} + \begin{pmatrix} bc' \\ bc'' \\ 0 \end{pmatrix}$$

Note in this system, the variable v, p are fluid's paramters, while $f(\xi)$ is the boundary force on the Lagrangian points. So introducing a transformed forcing function $\tilde{f}$ that satisfies:

$$Hf = -E^T \tilde{f}$$

To discrete delta function,

$$d(r) = \begin{cases} \frac{1}{6\Delta r}[5 - 3\frac{|r|}{\Delta r} - \sqrt{-3(1 - \frac{|r|}{\Delta r})^2 + 1}] & \text{for} 0.5\Delta r << |r| << 1.5\Delta r \\ \frac{1}{3\Delta r}[1 + \sqrt{-3(\frac{r}{\Delta r})^2 + 1} & \text{for} |r| << 0.5\Delta r \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

By the discrete equation above,
$u(\xi) = \int u(x)\delta(x - \xi)dx$, yielding

$$u_k = \Delta x \Delta y \sum_i u_i d(x_i - \xi_k) d(y_i - \eta_k)$$

defining $E_{k,i} = \alpha d(x_i - \xi_k)d(y_i - \eta_k)$
namely, $u_k = E_{k,j} u_i$
similar, for $\int f(\xi)\delta(\xi - x)ds = f_i$, yielding
defining $H_{i,k} = \beta d(\xi_k - x_i)d(\eta_k - y_i) = \frac{\beta}{\alpha}E^T_{k,i}$
namely, $f_i = H_{i,k} f(\xi)_k = -E^T_{k,i}\tilde{f}_i$
now, the formualted system

$$\begin{bmatrix} A & G & E^T \\ G^T & 0 & 0 \\ E & 0 & 0 \end{bmatrix} \begin{pmatrix} v^{n+1} \\ p^{n+1} \\ \tilde{f} \end{pmatrix} = \begin{pmatrix} r^n \\ 0 \\ u(\xi)^{n+1} \end{pmatrix} + \begin{pmatrix} bc' \\ bc'' \\ 0 \end{pmatrix}$$

now all the variables are fluid's parameters, and consider both the discrete pressure and boundary forcing fucntions are Lagrange multipliers, so define $Q = [G, E^T]$, $\lambda = (p, \tilde{f})^T$, $r_1 = r^n + bc1$, $r_2 = (-bc2, u(\xi)^{n+1})$, then the system can be reduced as

$$\begin{bmatrix} A & Q \\ Q^T & 0 \end{bmatrix} \begin{pmatrix} v^{n+1} \\ \lambda \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

samilar do the LU decomposition, which give the fractional steps. in the following sequence :

$$Av^* = r_1; Q^T BQ\lambda = Q^T v^* - r_2; v^{n+1} = v * -BQ\lambda; \tag{3}$$

for moving immersed bodies, the location of Lagragian points must be updated at each time so

$$E_{k,i}^{n+1} = E(\xi_k(t^{n+1}), x_i)$$

2 Reference

the immersed boundary method: a projection approach K. Taira, T. Colonius