incompressible Navier Stokes equation using fractional step method

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1 Incompressible Navier-Stokes equation

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \frac{1}{Re}\nabla^2 u$$

$$\nabla \cdot u = 0$$

discretization with a staggered mesh finite volume formulation or with lumped finite element formulation, using implicit Crank-Nicolson(CN) integration for the viscous terms and the explicit second-order Adams-Bashforth(AB2) scheme for the convective terms, namely:

$$\begin{aligned} &\frac{v^{n+1} - v^n}{\Delta t} + [\frac{3}{2}H(v^n) - \frac{1}{2}H(v^{n-1}] \\ &= -Gp^{n+1} + \frac{1}{2Re}L(v^{n+1} + v^n) + bc' \end{aligned}$$

$$Dv^{n+1} = 0 + bc''$$

in algebraic system :

$$\begin{bmatrix} A & G \\ D & 0 \end{bmatrix} \begin{pmatrix} v^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix} + \begin{pmatrix} bc' \\ bc'' \end{pmatrix}$$

where

$$A = \frac{1}{\Delta t}[I - \frac{\Delta t}{2Re}L]$$

$$r = \frac{1}{\Delta t}[I + \frac{\Delta t}{2Re}L]v^n - [\frac{3}{2}H(v^n) - \frac{1}{2}H(v^{n-1}]$$

For a algebraic system of equation above, usually we can approximate the divergence equation(the second block equation), or we can approximate the momentum equation by approximation the pressure term, as

$$\begin{bmatrix} A & (AB)G \\ D & 0 \end{bmatrix} \begin{pmatrix} v^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix} + \begin{pmatrix} bc' \\ bc'' \end{pmatrix}$$

Take block LU decomposition will obtain:

$$\begin{bmatrix} A & 0 \\ D & -DBG \end{bmatrix} \begin{pmatrix} v^{*} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix} + \begin{pmatrix} bc' \\ bc'' \end{pmatrix}$$

$$\begin{bmatrix} I & BG \\ 0 & I \end{bmatrix} \begin{pmatrix} v^{n+1} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} v^{*} \\ p^{n+1} \end{pmatrix}$$

or in the following sequence :

$$Av^* = r + bc'; DBGp^{n+1} = Dv^* - bc''; v^{n+1} = v * -BGp^{n+1}; \tag{1}$$

if B is chosen equal to Δt times the identity matrix, then it give the first-order block LU decomposition; if B is cosen to be an approximate inverse of A, then higher order accuray can be achieved.

2 Reference

1 An analysis of the fractional step method, J. B. Perot 1993 2 Analysis of an exact fractional step method, W. Chang 2002