ETH, MAVT

Flow past an airfoil

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1 D2Q9 scheme

This project is to simulate the flow past an airfoil using lattice Boltzmann method. Most of the contents are according to Dorschner et al.[1]. The D2Q9 scheme is used here, which is

$$f_i(\boldsymbol{x}+\boldsymbol{v_i}, t+1) - f_i(\boldsymbol{x}, t) = \alpha\beta(f_i^{eq} - f_i)$$

where f_i is the population and $\boldsymbol{v_i}$ is the corresponding velocity. The set of $\boldsymbol{v_i}$ is

$$\boldsymbol{v} = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & -1 & -1 \end{bmatrix}$$

and the corresponding weights are

$$\boldsymbol{W} = \begin{bmatrix} 4/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/36 & 1/36 & 1/36 & 1/36 \end{bmatrix}$$

The density and velocity of fluid are related by

$$\rho = \sum_i f_i, \rho\boldsymbol{u} = \sum_i \boldsymbol{v_i} f_i$$

and the equilibrium f_i^{eq} in the collision term is

$$f_i^{eq} = \rho W_i \prod_{j=1}^{D=2} \left(2 - \sqrt{1+3u_j^2}\right) \left(\frac{2u_j + \sqrt{1+3u_j^2}}{1-u_j}\right)^{v_{ij}}$$

[1] Grad's approximation for moving and stationary walls in entropic lattice Boltzmann simulations. JOURNAL OF COMPUTATIONAL PHYSICS

Other parameters in the collision term are related with Reynolds number Re, dynamical viscosity ν and the speed of sound $c_s = 1/\sqrt{3}$,

$$\alpha = 2.0, \beta = \frac{1}{2\frac{\nu}{c_s^2}+1}, c_s = 1/\sqrt{3}, \nu = u_0 c/Re$$

The airfoil is inside a channel i.e.(left side: inlet, right side: outlet, top/bottom side: free-slip). In detail, the channel boundary conditions can be implemented as

- inlet: $f_i^{t+1} = f_i^{eq}(\rho_0, u_0), i = 1,5,8$
- outlet: $f_i^{t+1} = f_i^t, i = 3,6,7$
- bottom: $f_i^{t+1} = f_j^t, (i,j) = (2,4),(5,8),(6,7)$
- top: $f_i^{t+1} = f_j^t, (i,j) = (4,2),(8,5),(7,6)$

where ρ_0 and u_0 are initial conditions ($\rho_0 = 1, u_0 = 0.05$ here).

2 Grad's approximation

For the airfoil boundary condition, the Grad's approximation is used at the boundary points which are generated by the intersection of the grid points around the airfoil and the outline of the airfoil. The missing populations (point from the grid points inside the airfoil to those outside the airfoil) can be expressed as

$$f_i^*(\rho_{tgt}, \rho_{tgt}\boldsymbol{u}_{tgt}, \boldsymbol{P}) = W_i\left[\rho_{tgt} + \frac{\rho_{tgt}\boldsymbol{u}_{tgt}\cdot\boldsymbol{v}_i}{c_s^2} + \frac{1}{2c_s^4}(\boldsymbol{P} - \rho_{tgt}c_s^2\boldsymbol{I}) : (\boldsymbol{v}_i\boldsymbol{v}_i - c_s^2\boldsymbol{I})\right]$$

where $\boldsymbol{I}$ is the identity matrix and the pressure tensor $\boldsymbol{P} = \boldsymbol{P}^{eq} + \boldsymbol{P}^{neq}$. The equilibrium and non-equilibrium parts are

$$\boldsymbol{P}^{eq} = \rho_{tgt}c_s^2\boldsymbol{I} + \rho_{tgt}\boldsymbol{u}_{tgt}\boldsymbol{u}_{tgt}$$

$$\boldsymbol{P}^{neq} = \frac{\rho_{tgt}c_s^2}{2\beta}(\nabla\boldsymbol{u}_{tgt} + \nabla\boldsymbol{u}_{tgt}^T)$$

Besides, the target velocity u_{tgt} and the target density ρ_{tgt} are

$$u_{tgt} = \frac{1}{n_{\bar{D}}}\sum_{i\in\bar{D}}\frac{q_i u_{f,i} + u_{w,i}}{1+q_i}$$

where $\bar{D}$ is the set of missing populations and $n_{\bar{D}}$ is its number. $\boldsymbol{u}_{tgt}$ is the average of the interpolation of surrounding fluid velocity $\boldsymbol{u}_f$ and wall velocity $\boldsymbol{u}_w$. q_i is the distance between the fluid node $\boldsymbol{x}_{f,i}$ and the wall node $\boldsymbol{x}_{w,i}$. Namely, $\boldsymbol{x}_{w,i} = \boldsymbol{x}_{f,i} + q_i\boldsymbol{v}_i$. The target density ρ_{tgt} is contributed by the bounce back effect ρ_{bb} and the local density change ρ_s. So

$$\rho_{tgt} = \rho_{bb} + \rho_s$$

$$\rho_{bb} = \sum_{i\in\bar{D}} f_i^{bb} + \sum_{i\notin\bar{D}} f_i$$

$$\rho_s = \sum_{i\in\bar{D}} 6W_i\rho_0\boldsymbol{v}_i\cdot\boldsymbol{u}_{w,i}$$

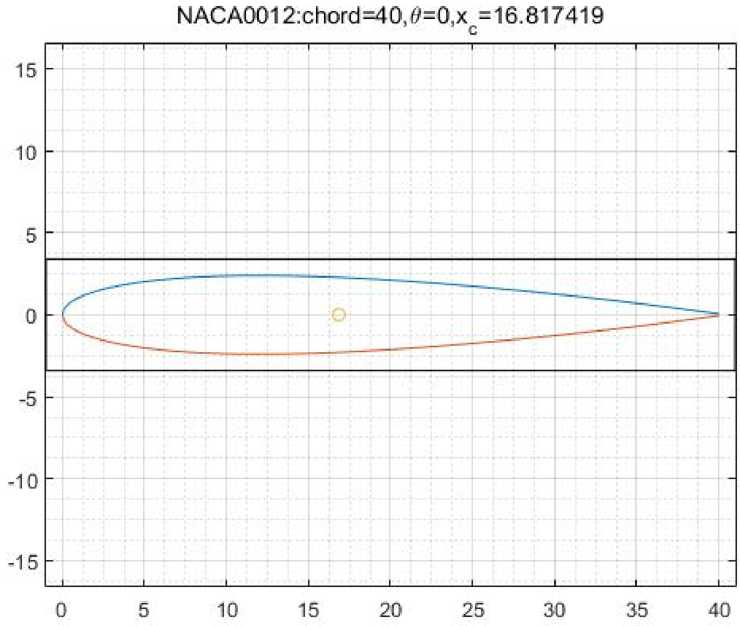


Figure 3.1: NACA0012: $c = 40$ confined in a box $[-1, c+1] \times [t/2-1, t/2+1]$

For the force evaluation, momentum exchange method is used. The total force $\boldsymbol{F}$ is formulated by the summation of momentum exchange in every boundary points x_b,

$$\boldsymbol{F} = \sum_{all\,x_b} \sum_{i \in \bar{D}} \tilde{\boldsymbol{v}}_i \left[f_i(x_b, t) + \tilde{f}_i(\boldsymbol{x}_b + \tilde{\boldsymbol{v}}_i, t) \right]$$

where $\tilde{\boldsymbol{v}}_i = -\boldsymbol{v}_i$ and $\tilde{f}_i$ the associated population.

3 Boundary points searching

To use the Grad's approximation, the position of boundary points $\boldsymbol{x}_b$ or the distance q_i need to be found. In this case, NACA00xx series airfoil are used. The shape of the airfoil can be described by an analytical function,

$$y_t(x) = 5tc \left[0.2969\sqrt{\frac{x}{c}} + (-0.1260)\left(\frac{x}{c}\right) + (-0.3516)\left(\frac{x}{c}\right)^2 + 0.2843\left(\frac{x}{c}\right)^3 + (-0.1015)\left(\frac{x}{c}\right)^4 \right]$$

where c is chord length, t is thickness (e.g. t=xx%c for NACA00xx). $y_t(x)$ is the upper curve of the airfoil. Due to symmetry, the whole shape of the airfoil is $|y_t(x)|$. It is noted that the curve is not closed at the trailing edge (i.e. $y_t(c) \neq 0$). So the boundary points are searched within a box around the airfoil as shown in Fig.(3.1). For an 2D airfoil, it has 3 degree of freedoms: mass center (x_c, y_c) and the angle θ with respect to the global coordinates. Their derivatives u_x, u_y, ω_z are also included so that the airfoil is fixed in phase space. The shape function

$y_t(x)$ is described in the local coordinates with its origin at the leading ledge. Thus, to find the intersection between the grid points and the airfoil shape, all the vectors in the global coordinates should be converted to the local coordinates. For instance, the mass center (x_c, y_c) maps (x_m, y_m) in the local coordinates. For NACA00xx series, $x_m = \frac{\int_0^c x f_t(x)\mathrm{d}x}{\int_0^c f_t(x)\mathrm{d}x}$ and $y_m = 0$. The vectors $\boldsymbol{v}_i$ is also rotated by θ.

$$\begin{aligned} v_{ix}^{\theta} &= v_{ix}\cos\theta + v_{iy}\sin\theta \\ v_{iy}^{\theta} &= -v_{ix}\sin\theta + v_{iy}\cos\theta \end{aligned}$$

In addition, a grid point (i, j) in the local coordinates is

$$\begin{aligned} x &= x_m + (i - x_c)\cos\theta + (j - y_c)\sin\theta \\ y &= y_m - (j - x_c)\sin\theta + (j - y_c)\cos\theta \end{aligned}$$

In the local coordinates, Newton-Raphson method can be used to find the distance q_i, that is

$$\min_{q_i\in[0,1]} \left[\left| y_t(x + q_i v_{ix}^{\theta}) \right| - (y + q_i v_{iy}^{\theta}) \right]^2$$

It is noted that some grid points within the searching box $[-1, c+1] \times [t/2-1, t/2+1]$ have negative x component in the range of $[-1, 0]$. They may have intersections but Newton-Raphson method cannot find them because the Jacobian near the origin comes to infinity. Therefore, the grid points in the region $[-1, 0]$ should be search manually. A searching result of the airfoil NACA0012 at certain angle θ is shown in Fig.(3.2).

Another drawbacks of the analytical description occurs when the airfoil is moving. For example, the flapping wing follows a trajectory as

$$\begin{aligned} x(t) &= \frac{N_x}{2} + \frac{A_0}{2}\cos(2\pi f t), y(t) = \frac{N_y}{2} \\ \alpha(t) &= \alpha_0 + \beta\sin(2\pi f t + \phi) \end{aligned}$$

where $N_x = N_y = 25c$, $Re = 75$, $A_0 = 2.8c$, $u_{max} = \pi f A_0 = 0.01$, $c = 40$, $\alpha_0 = \frac{\pi}{2}$, $\beta = \frac{\pi}{4}$, $\phi = \frac{\pi}{4}$. At certain time t, some grid nodes inside the airfoil come to the fluid region. To reinitialize the populations f_i of those nodes, the equilibrium distribution $f_i^{eq}(\rho_0, \boldsymbol{u}_w)$ is used. To get the wall velocity $\boldsymbol{u}_w$ at certain time t, binary searching is adopted. In other words, the objective is to find $t \in [t_0, t_1]$ such that $\left| |y_t(x(t))| - y(t) \right| < \epsilon$ where $|y_t(x(t_0))| - y(t_0) > 0$ and $|y_t(x(t_1))| - y(t_1) < 0$. For some nodes, $x(t_0) > 0$ and $x(t_1) < 0$ will happen, however, $y_t(x(t_1))$ yields unreasonable results so the binary searching fails. In such situation, take $t = (t_0 + t_1)/2$ to keep the computation running.

4 Implementation and results

The program structure is basically as follows,

Algorithm 1 lattice Boltzmann method progamming procedures

for time=0;time<timesteps;time++ **do**

update the airfoil $x_c, y_c, \theta, u_x, u_y, \omega_z$ {for moving airfoil, stationary airfoil could not be updated}

find new boundary nodes x_b, solid nodes x_s and mark the solid to fluid nodes x_{sf}

if time>0 **then**

reinitialization: $f_i^{t+1}(\boldsymbol{x}_{sf}) = f_i^{eq}$

end if

advect: $f_i^{t+1} = f_i^t(\boldsymbol{x} + \boldsymbol{v_i})$ {periodic boundary condition can be added here for the flapping wing case}

boundary condition (walls): {for stationary airfoil inside flow channel}

- inlet: $f_i^{t+1} = f_i^{eq}(\rho_0, u_0), i = 1, 5, 8$
- outlet: $f_i^{t+1} = f_i^t, i = 3, 6, 7$
- bottom: $f_i^{t+1} = f_j^t, (i, j) = (2, 4), (5, 8), (6, 7)$
- top: $f_i^{t+1} = f_j^t, (i, j) = (4, 2), (8, 5), (7, 6)$

boundary condition (airfoil): Grad's approximation

collide: $f_i^{t+1} = f_i^{t+1} + \alpha * \beta(f_i^{eq} - f_i^{t+1})$

force valuation: momentum exchange method

update f: $f_i^t = f_i^{t+1}$

if time>0 **then**

delete the solid to fluid nodes x_{sf}

end if

end for

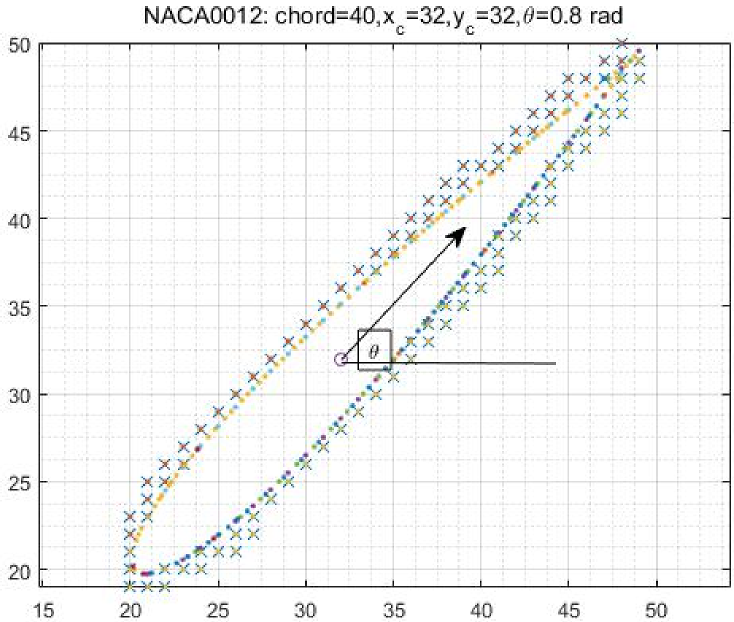


Figure 3.2: Boundary points searching by Newton-Raphson method, crosses are the boundary points and dots are intersection points

4.1 STATIONARY AIRFOIL IN THE FLOW CHANNEL

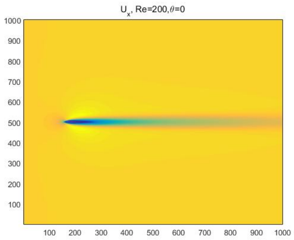
The airfoil is fixed inside the flow channel ($N_x = N_y = 1000$) with $(x_c, y_c) = (2c, N_y/2)$. Choose the Reynold number $Re = 200$, initial condition $\rho_0 = 1, \boldsymbol{u}_0 = 0.05$ and only change the angle θ, we can find at which attack angle $-\theta$ the vortex street will occur. From Fig.(4.1), the vortex street starts at $\theta = -0.2 \approx -23°$.

4.2 FLAPPING WING

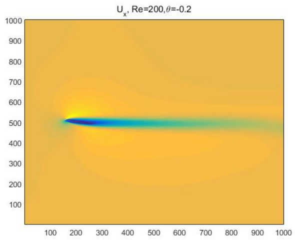
The airfoil follows the trajectory mentioned before,

$$x(t) = \frac{N_x}{2} + \frac{A_0}{2}\cos(2\pi f t), y(t) = \frac{N_y}{2}$$
$$\alpha(t) = \alpha_0 + \beta \sin(2\pi f t + \phi)$$

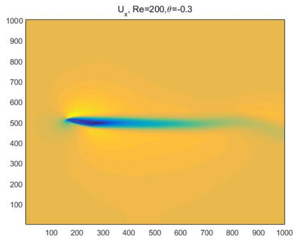
where $N_x = N_y = 500, Re = 75, A_0 = 2.8c, u_{max} = \pi f A_0 = 0.01, c = 40, \alpha_0 = \frac{\pi}{2}, \beta = \frac{\pi}{4}, \phi = \frac{\pi}{4}$. It is noted that the fluid is stagnant and periodic boundary condition is used here. The results after $2.75T$ time steps is shown in Fig.(4.2).



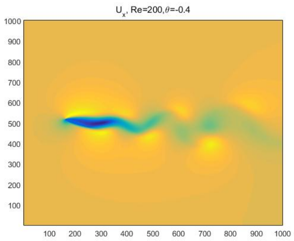
(a) $\theta = 0$



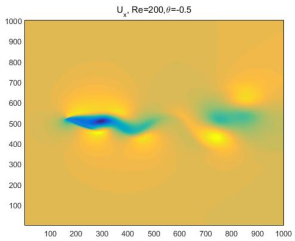
(b) $\theta = -0.2$



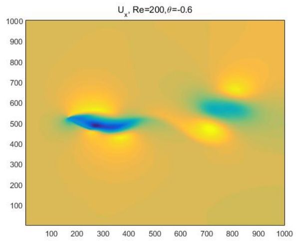
(c) $\theta = -0.3$



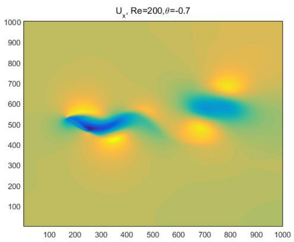
(d) $\theta = -0.4$



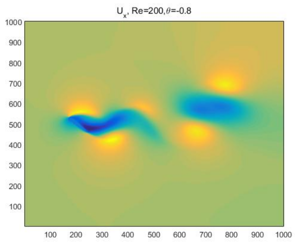
(e) $\theta = -0.5$



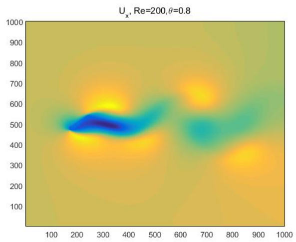
(f) $\theta = -0.6$



(g) $\theta = -0.7$



(h) $\theta = -0.8$



(i) $\theta = 0.8$

Figure 4.1: u_x plot of NACA0012:Re=200 at different angle θ

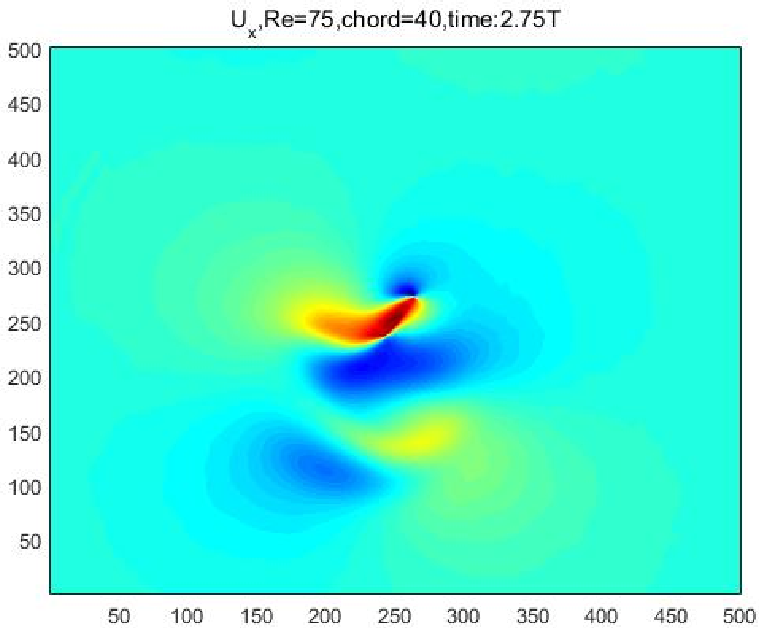


Figure 4.2: u_x plot of the flapping wing NACA0012 at 2.75T