$(u+v) \times (-v)]$

$= (u \times \ldots) \quad \because v \times v = 0$

$= (v \times u) - (u \times v)$

$= (v \times u) - (-(v \times u))$

$= 2(v \times u)$

(EX : 4.1)

Q7) $k(x, y, z) = (k^2 x, k^2 y, k^2 z)$

let $u = (x_1, y_1, z_1)$

$v = (x_2, y_2, z_2)$

$w = (x_3, y_3, z_3)$

Axiom 1 :

$u + v \in V$

$u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

Proved

Axiom 2 :

$u + v = v + u$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= u + v$

Also Proved

Axiom 3 :

$u + (v + w) = (u + v) + w$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2) + (x_3, y_3, z_3)$

$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3, z_1 + \ldots$

$=$ LHS (Proved)

Axiom 4:

$u+0 = u$

$u+0 = (x_1, y_1, z_1) + (0,0,0)$

$= (x_1, y_1, z_1) = u$

Proved

Axiom 5:

$-u \in V$

$-u = (-x_1, -y_1, -z_1)$

$u-u = (x_1-x_1, y_1-y_1, z_1-z_1)$

$= (0,0,0)$

Holds.

Axiom 6:

$ku \in V$

$ku = k(x_1, y_1, z_1)$

$= (k^2x_1, k^2y_1, k^2z_1) \in V$

Holds.

Axiom 7:

$k(u+v) = ku+kv$

$k(u+v) = k(x_1+x_2, y_1+y_2, z_1+z_2)$

$= (k^2x_1+k^2x_2, k^2y_1+k^2y_2, k^2z_1+k^2z_2)$

$= (k^2x_1, k^2y_1, k^2z_1) + (k^2x_2, k^2y_2, k^2z_2)$

$= ku+kv$

Holds.

Axiom 8:

$(k+m)u = km + ku$

$(k+m)u = (k+m)^2 (x_1, y_1, y_2)$

$km + mu = k^2(x_1, y_1, z_1)$

$+ m^2(x_1, y_1, z_1)$

$= (k^2+m^2)(x_1, y_1, z_1)$

$\neq (k+m)u$

Hence it doesn't holds.

V is **not vector space.**

Q11) $k(1,y) = (1, ky)$

$u = (1, y_1), \quad v = (1, y_2), \quad w = (1, y_3)$

Axiom 1:

$u+v \in V$

$= (1+y_1) + (1+y_2)$

Axiom: 2:

$u+v = v+u$

$u+v = (1, y_1+y_2)$

$= (1, y_1 + y_2)$

$= (1, y_2 + y_1)$

$= (1, y_2) + (1, y_1)$

$= v + u$

Axiom 3:

$u + (v + w) = (u + v) + w$

$= (1, y_1) + (1, y_2 + y_3)$

$= (1, y_1 + y_2 + y_3)$

$= (1, (y_1 + y_2)) + (1, y_3)$

$= (u + v) + w$

Axiom 4:

$\vec{0} = (1, 0)$

$u + 0 = (1, y_1) + (1, 0)$

$= (1, y_1)$

$= u$

Axiom 5:

$u + (-u) = 0$

$-u = (-1, -y_1)$

$= (1, y_1) + (-1, -y_1)$

$= (1, 0)$

$= 0$

V is a vector space.

Axiom 6:

$ku = k(1, y_1)$

$= (1, ky_1)$

Axiom 7:

$k(u + v) = ku + kv$

$= k(1, y_1 + y_2)$

$= (1, ky_1) + (1, ky_2)$

$= k(1, y_1) + k(1, y_2)$

$= ku + kv$

Axiom 8:

$= (1, (k+m)y_1)$

$= (1, ky_1) + (1, my_1)$

$= ku + mu$

Axiom 9:

$k(mu) = k(1, my_1)$

$= km(1, y_1)$

$= (km)u$

Axiom 10:

$1u = u$

$= (1, 1y_1) = u$

EX: 4.3

Q2) (a) $u = (2,1,4)$
$v = (1,-1,3)$
$w = (3,2,5)$

$(-9,-7,15) = k_1 u + k_2 v + k_3 w$

In Augmented Matrix

$$\begin{bmatrix} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & 15 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & -1 & 2 & -7 \\ 2 & 1 & 3 & -9 \\ 4 & 3 & 5 & 15 \end{bmatrix}$$

$R_2 - 2R_1$, $R_3 - 4R_1$

$$\begin{bmatrix} 1 & -1 & 2 & -7 \\ 0 & 3 & -1 & 5 \\ 0 & 7 & -3 & 13 \end{bmatrix}$$

$R_2/3$

$$\begin{bmatrix} 1 & -1 & 2 & -7 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 7 & -3 & 13 \end{bmatrix}$$

$R_3 - 7R_2$, $R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 5/3 & -16/3 \\ 0 & 1 & -1/3 & 5/3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_1 - 5/3 R_3$, $R_2 + 1/3 R_3$

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Hence,

$k_1 = -2$

$k_2 = 1$

$k_3 = -2$

$(-9,-7,15) = -2u + 1v + (-2w)$

Q4) (a) $p_1 = 2+x+x^2$

$p_2 = 1-x^2$

$p_3 = 1+2x$

$k_1p_1 + k_2p_2 + k_3p_3 = 1+x$

$k_1(2+x+x^2) + k_2(1-x^2) + k_3(1+2x) = 1+x$

In Matrix form

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$R_1/2$

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$R_2 - R_1,\ R_3 - R_1$

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 \\ 0 & -1/2 & 3/2 & 1/2 \\ 0 & -3/2 & -1/2 & -1/2 \end{bmatrix}$$

$R_2 * (-2)$

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & -3 & -1 \\ 0 & -3/2 & -1/2 & -1/2 \end{bmatrix}$$

$R_3 + 3/2\, R_2$

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -5 & -2 \end{bmatrix}$$

$R_3 * (-1/5)$

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 2/5 \end{bmatrix}$$

$R_2 + 3R_3,\ R_1 - 1/2\, R_3$

$$\begin{bmatrix} 1 & 1/2 & 0 & 3/10 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 2/5 \end{bmatrix}$$

$R_1 - 1/2\, R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 1/5 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 2/5 \end{bmatrix}$$

$k_1 = 1/5 \quad k_2 = 1/5 \quad k_3 = 2/5$.

Q5) (a) $A\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}$ $B\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ $C\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ $D\begin{bmatrix}2 & 0\\1 & -1\end{bmatrix}$

$$\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} = k_1\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix} + k_2\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} + k_3\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + k_4\begin{bmatrix}2 & 0\\1 & -1\end{bmatrix}$$

In Augmented Matrix

$$\begin{bmatrix}1 & 0 & 0 & 2 & 1\\-1 & 1 & 1 & 0 & 2\\0 & 0 & 0 & 1 & 2\\2 & 1 & 0 & -1 & 4\end{bmatrix}$$

R_2+R_1 , R_4-2R_1

$$\begin{bmatrix}1 & 0 & 0 & 2 & 1\\0 & 1 & 1 & 2 & 3\\0 & 0 & 0 & 1 & 2\\0 & 1 & 0 & -5 & 2\end{bmatrix}$$

R_4-R_2

$$\begin{bmatrix}1 & 0 & 0 & 2 & 1\\0 & 1 & 1 & 2 & 3\\0 & 0 & 0 & 1 & 2\\0 & 0 & -1 & -7 & -1\end{bmatrix}$$

$R_4*(-1)$

$R_4 \leftrightarrow R_3$

$$\begin{bmatrix}1 & 0 & 0 & 2 & 1\\0 & 1 & 1 & 2 & 3\\0 & 0 & 1 & 7 & 1\\0 & 0 & 0 & 1 & 2\end{bmatrix}$$

R_3-7R_4 , R_2-2R_4 , R_1-2R_4

R_2-R_3

$$\begin{bmatrix}1 & 0 & 0 & 0 & -3\\0 & 1 & 0 & 0 & 12\\0 & 0 & 1 & 0 & -13\\0 & 0 & 0 & 1 & 2\end{bmatrix}$$

$k_1 = -3$

$k_2 = 12$

$k_3 = -13$

$k_4 = 2$

$$\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} = -3A + 12B - 13C + 2D$$

Q7) (a) $v_1 = (2,2,2)$
$v_2 = (0,0,3)$
$v_3 = (0,1,1)$

Vectors span R^3 if they are independent

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\det(A) = 2 \cdot \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix}$$

$$= 2(-3) = -6 \neq 0 \quad \text{(independent)}$$

Hence they span R^3.

Q8) (a) $v_1 = (2,1,0,3)$
$v_2 = (3,-1,5,2)$
$v_3 = (-1,0,2,1)$

$(2,3,-7,3) = k_1 v_1 + k_2 v_2 + k_3 v_3$

In Matrix form

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 3 & -1 & 2 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{bmatrix}$$

$R_2 - 2R_1,\ R_4 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 5 & -1 & -4 \\ 0 & 5 & 2 & -7 \\ 0 & 5 & 1 & -6 \end{bmatrix}$$

$R_3 - R_2, R_4 - R_2$

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 5 & -1 & -4 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

$R_2 * (1/5)$

$R_3 * (1/3)$

$R_4 * (1/2)$

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -1/5 & -4/5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_4 - R_3, R_2 + 1/5 R_3$

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$k_1 = 2$

$k_2 = -1$

$k_3 = -1$

Hence $(2,3,-7,3)$ is in span$\{v_1, v_2, v_3\}$

Q20) span$\{v_1, v_2, v_3\}$ = span$\{w_1, w_2\}$

We will show this by proving that each vector v_1, v_2, v_3 can be written as linear combinatio of w_1 and w_2.

$v_1 = k_1 w_1 + k_2 w_2$

$(1,6,4) = k_1(1,-2,-5) + k_2(0,8,9)$

$1 = k_1$

$6 = -2k_1 + 8k_2$

$6 = -2(1) + 8k_2$

$k_2 = 8/8 = 1$

$v_1 = w_1 + w_2$

Now,

$v_2 = k_1 w_1 + k_2 w_2$

$(2, 4, -1) = k_1(1, -2, -5) + k_2(0, 8, 9)$

$k_1 = 2$

$4 = -2k_1 + 8k_2$

$4 = -2(2) + 8k_2$

$k_2 = 8/8 = 1$

$v_2 = 2w_1 + w_2$

Now,

$v_3 = k_1 w_1 + k_2 w_2$

$(-1, 2, 5) = k_1(1, -2, -5) + k_2(0, 8, 9)$

$k_1 = -1$

$2 = -2k_1 + 8k_2$

$2 = -2(-1) + 8k_2$

$k_2 = 0/8 = 0$

$v_3 = -w_1$

Hence proved that span $\{v_1, v_2, v_3\}$ = span $\{w_1, w_2\}$

EX: 4.4

Q1) (b) $u_1 = (3, -1)$

$u_2 = (4, 5)$

$u_3 = (-4, 7)$ in R^2

$\{v_1, v_2, v_3 \dots v_r\}$ in R^n is linearly dependent when $r > n$. Hence u_1, u_2, u_3 are dependent as $3 > 2$.

Q2) (b) As number of vectors (4) is greater than 'n' (R^3) therefore these vectors are linearly dependent $(4 > 3)$.

Q3) (a) let
$$A = \begin{bmatrix} 3 & 1 & 2 & 4 \\ 8 & 5 & -1 & 2 \\ 7 & 3 & 2 & 6 \\ -3 & -1 & 6 & 4 \end{bmatrix}$$

If $|A| = 0$, then v_1, v_2, v_3, v_4 are dependent.

$$|A| = \begin{vmatrix} 3 & 1 & 2 & 4 \\ 8 & 5 & -1 & 2 \\ 7 & 3 & 2 & 6 \\ -3 & -1 & 6 & 4 \end{vmatrix}$$

$R_2 - 5R_1$

$R_3 - 3R_1$

$R_4 + R_1$

$$|A| = \begin{vmatrix} 3 & 1 & 2 & 4 \\ -7 & 0 & -11 & -18 \\ -2 & 0 & -4 & -6 \\ 0 & 0 & 8 & 8 \end{vmatrix}$$

Expand from C_2

$$|A| = -1 \begin{vmatrix} -7 & -11 & -18 \\ -2 & -4 & -6 \\ 0 & 8 & 8 \end{vmatrix}$$

$C_2 - C_3$

$$= -1 \begin{vmatrix} -7 & 7 & -18 \\ -2 & 2 & -6 \\ 0 & 0 & 8 \end{vmatrix}$$

$$= -1(8) \begin{vmatrix} -7 & 7 \\ -2 & 2 \end{vmatrix}$$

$$= -1(8)(-14+14) = 0$$

Hence, vectors are dependent.

Q4) (b) $p_1 = 1+3x+3x^2$ $\quad p_3 = 5+6x+3x^2$

$p_2 = x+4x^2$ $\quad p_4 = 7+2x-x^2$

$k_1p_1 + k_2p_2 + k_3p_3 + k_4p_4 = 0$

Corresponding matrix becomes

$$\begin{bmatrix} 1 & 0 & 5 & 7 & 0 \\ 3 & 1 & 6 & 2 & 0 \\ 3 & 4 & 3 & -1 & 0 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2-3R_1} \begin{bmatrix} 1 & 0 & 5 & 7 & 0 \\ 0 & 1 & -9 & -19 & 0 \\ 0 & 4 & -12 & -22 & 0 \end{bmatrix}$$

$R_3 - 4R_2$

$$\left[\begin{array}{ccccc} 1 & 0 & 5 & 7 & 0 \\ 0 & 1 & -9 & -19 & 0 \\ 0 & 0 & 24 & 54 & 0 \end{array}\right] \xrightarrow{R_3/24} \left[\begin{array}{ccccc} 1 & 0 & 5 & 7 & 0 \\ 0 & 1 & -9 & -19 & 0 \\ 0 & 0 & 1 & 9/4 & 0 \end{array}\right]$$

$R_2 + 9R_3$, $R_1 - 5R_3$

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & -17/4 & 0 \\ 0 & 1 & 0 & 5/4 & 0 \\ 0 & 0 & 1 & 9/4 & 0 \end{array}\right]$$

$k_1 = 17/4\, k_4$
$k_2 = -5/4\, k_4$
$k_3 = -9/4\, k_4$

Hence given vectors are dependent.

Q9) (a) $v_1 = (0,3,1,-1)$
$v_2 = (6,0,5,1)$
$v_3 = (4,-7,1,3)$

Vectors are linearly independent only if there is trivial solution for

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$$

In Matrix form

$$\left[\begin{array}{cccc} 0 & 6 & 4 & 0 \\ 3 & 0 & -7 & 0 \\ 1 & 5 & 1 & 0 \\ -1 & 1 & 3 & 0 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{cccc} 1 & 5 & 1 & 0 \\ 3 & 0 & -7 & 0 \\ 0 & 6 & 4 & 0 \\ -1 & 1 & 3 & 0 \end{array}\right]$$

$R_2 - 3R_1,\ R_1 + R_4$

$$\begin{bmatrix} 1 & 5 & 1 & 0 \\ 0 & -15 & -10 & 0 \\ 0 & 6 & 4 & 0 \\ 0 & 6 & 4 & 0 \end{bmatrix} \xrightarrow[R_4 - R_3]{R_2 * (-1/5)} \begin{bmatrix} 1 & 5 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 6 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 - R_2(2)$

$$\begin{bmatrix} 1 & 5 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$k_1 + 5k_2 + k_3 = 0$$
$$3k_2 + 2k_3 = 0$$
$$k_2 = -2/3\, k_3$$
$$k_1 = 7/3\, k_3$$

As solution is non-trivial so vectors are dependent.

(b) $k_1 v_1 = -k_2 v_2 - k_3 v_3$

$$v_1 = -\frac{k_2}{k_1} v_2 - \frac{k_3}{k_1} v_3$$

$$v_1 = \frac{2}{7} v_2 - \frac{3}{7} v_3$$

$$v_2 = -\frac{k_1}{k_2} v_1 - \frac{k_3}{k_2} v_3$$

$$v_2 = \frac{7}{2} v_1 + \frac{3}{2} v_3$$

$$v_3 = -\frac{k_1}{k_3} v_1 - \frac{k_2}{k_3} v_2$$

$$v_3 = -\frac{7}{3} v_1 + \frac{2}{3} v_2$$

Q19) (a) $1, x, e^x$

$$W = \begin{vmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix}$$

$= e^x \neq 0$

It is linearly independent.

(b) $1, x, x^2$

$$W = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

$= 2 \neq 0$

It is linearly independent.

EX : 4.5

Q7) (a) $(2,-3,1)$
$(4,1,1)$
$(0,-7,1)$

$k_1 v_1 + k_1 v_2 + k_3 v_3 = 0$

The corresponding matrix is

$$A = \begin{bmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{bmatrix}$$

$|A| = 2(1-(-7)) - 4(-3-(-7))$

$= 16 - 16 = 0$

Hence it doesn't form basis for R^3 as it's linearly dependent.

Q13) (b) $v = (5, -12, 3)$

let

$$k_1(1,2,3) + k_2(-4,5,6) + k_3(7,-8,9) = (5,-12,3)$$

The corresponding system is

$$\left[\begin{array}{ccc|c} 1 & -4 & 7 & 5 \\ 2 & 5 & -8 & -12 \\ 3 & 6 & 9 & 3 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -4 & 7 & 5 \\ 0 & 13 & -22 & -22 \\ 0 & 18 & -12 & -12 \end{array}\right]$$

$R_2 * (1/13)$

$$\left[\begin{array}{ccc|c} 1 & -4 & 7 & 5 \\ 0 & 1 & \frac{-22}{13} & \frac{-22}{13} \\ 0 & 18 & -12 & -12 \end{array}\right] \xrightarrow{R_3 - 18R_2} \left[\begin{array}{ccc|c} 1 & -4 & 7 & 5 \\ 0 & 1 & -22/13 & -22/13 \\ 0 & 0 & \frac{240}{13} & \frac{240}{13} \end{array}\right]$$

$R_3 * (13/240)$

$$\left[\begin{array}{ccc|c} 1 & -4 & 7 & 5 \\ 0 & 1 & -22/13 & -22/13 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_2 + \frac{22}{13}R_3} \left[\begin{array}{ccc|c} 1 & -4 & 7 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_1 - 7R_3$

$$\left[\begin{array}{ccc|c} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 + 4R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$(v)_S = (-2, 0, 1)$.

(a) $v = (2, -1, 3)$

let

$$k_1(1,0,0) + k_2(2,2,0) + k_3(3,3,3) = (2,-1,3)$$

The corresponding system becomes

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 2 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & 3 & 3 \end{array}\right] \xrightarrow{R_2 * (1/2)} \left[\begin{array}{cccc} 1 & 2 & 3 & 2 \\ 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$R_3 * (1/3)$

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 2 \\ 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_1 - 3R_3]{R_2 - 3/2 R_3} \left[\begin{array}{cccc} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_1 - 2R_2$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$(v)_S = (3, -2, 1)$$

Q19) (a) In a vector space, the number of vectors which form basis must be equal to dimension but here.

$$n = 3$$

$$\dim(R^2) = 2$$

Thus, $\dim(R^n) \neq n$.

So u_1, u_2, u_3 don't form basis for R^2.

Ex : 4.6

Q5) $x_1 - 3x_2 + x_3 = 0$
$2x_1 - 6x_2 + 2x_3 = 0$
$3x_1 - 9x_2 + 3x_3 = 0$

In Matrix form

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 3x_2 + x_3 = 0$

let $x_2 = u$ $x_3 = v$

$x_1 = 3u - v$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3u - v \\ u \\ v \end{bmatrix} = u \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So bases are $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Dimension : 2

Q13) $v_1 = (1,-4,2,-3)$
$v_2 = (-3,8,-4,6)$

Since $\dim(R^4) = 4$ so we have to find 4 linearly independent vectors for basis.

let

$e_3 = (0,0,1,0)$ $e_4 = (0,0,0,1)$

Now, we prove that $\{v_1, v_2, e_3, e_4\}$ are linearly independent. In Matrix form:

$$\left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 0 \\ -4 & 8 & 0 & 0 & 0 \\ 2 & -4 & 1 & 0 & 0 \\ -3 & 6 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{R_2+4R_1 \\ R_3-2R_1 \\ R_4+3R_1}} \left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 \end{array}\right]$$

$R_2 * (-1/4)$

$$\left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{R_1+3R_2 \\ R_4+3R_2 \\ R_3-2R_2}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

As solution is trivial so, the set of vectors $\{v_1, v_2, e_3, e_4\}$ is standard basis for R^4.

Q17) $v_1 = (1,0,0)$
$v_2 = (1,0,1)$
$v_3 = (2,0,1)$
$v_4 = (0,0,-1)$

Here,

$$v_2 = v_1 - v_4$$

$$v_3 = 2v_1 - v_4$$

Hence, v_2 and v_3 can be removed as they are linearly dependent.

Now,

$$k_1 v_1 + k_2 v_4 = 0$$
$$k_1(1,0,0) + k_2(0,0,-1) = 0$$

$$k_1 = 0$$
$$k_2 = 0$$

Here, v_1 and v_4 are linearly independent.
Therefore the set $\{v_1, v_4\}$ forms standard basis for R^3.

EX: 4.7

Q1) (a) $P_{B'} \to P_B$

let $u_1' = k_1 u_1 + k_2 u_2$

$$\begin{bmatrix}1\\3\end{bmatrix} = k_1\begin{bmatrix}2\\2\end{bmatrix} + k_2\begin{bmatrix}4\\-1\end{bmatrix}$$

$2k_1 + 4k_2 = 1$

$2k_1 - k_2 = 3$

$k_2 = 2k_1 - 3$

$2k_1 + 4(2k_1 - 3) = 1$

$2k_1 + 8k_1 - 12 = 1$

$k_1 = 13/10$

$k_2 = -2/5$

$$[u_1'] = \begin{bmatrix}13/10\\-2/5\end{bmatrix}$$

let $u_2' = k_1 u_1 + k_2 u_2$

$$\begin{bmatrix}-1\\-1\end{bmatrix} = k_1\begin{bmatrix}2\\2\end{bmatrix} + k_2\begin{bmatrix}4\\-1\end{bmatrix}$$

$2k_1 + 4k_2 = -1$

$2k_1 - k_2 = -1$

$5k_2 = 0$

$k_2 = 0$

$k_1 = -1/2$

$$[u_2'] = \begin{bmatrix}-1/2\\0\end{bmatrix}$$

$$P_{B'} \to P_B = \begin{bmatrix}13/10 & -1/2\\-2/5 & 0\end{bmatrix}$$

(b) $P_B \to P_{B'}$

let $u_1 = k_1 u_1' + k_2 u_2'$

$$\begin{bmatrix}2\\2\end{bmatrix} = k_1\begin{bmatrix}1\\3\end{bmatrix} + k_2\begin{bmatrix}-1\\-1\end{bmatrix}$$

$k_1 - k_2 = 2$

$3k_1 - k_2 = 2$

let $u_2 = k_1 u_1' + k_2 u_2'$

$$\begin{bmatrix}4\\-1\end{bmatrix} = k_1\begin{bmatrix}1\\3\end{bmatrix} + k_2\begin{bmatrix}-1\\-1\end{bmatrix}$$

$k_1 - k_2 = 4$

$3k_1 - k_2 = -1$

$$-2k_1 = 0$$
$$k_1 = 0$$
$$k_2 = -2$$

$$[u_1] = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$-2k_1 = 5$$
$$k_1 = -5/2$$
$$k_2 = -13/2$$

$$[u_2] = \begin{bmatrix} -5/2 \\ -13/2 \end{bmatrix}$$

$$P_B \rightarrow P_{B'} = \begin{bmatrix} 0 & -5/2 \\ -2 & -13/2 \end{bmatrix}$$

(c) $[w]_B$

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = k_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$2k_1 + 4k_2 = 3$$
$$2k_1 - k_2 = -5$$
$$5k_2 = 8$$
$$k_2 = 8/5$$
$$k_1 = -17/10$$

$$[w]_B = \begin{bmatrix} -17/10 \\ 8/5 \end{bmatrix}$$

$$[w]_{B'} = P_B \rightarrow P_{B'} [w]_B$$
$$= \begin{bmatrix} 0 & -5/2 \\ -2 & -13/2 \end{bmatrix} \begin{bmatrix} -17/10 \\ 8/5 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -7 \end{bmatrix}$$

(d) $[w]_{B'}$

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$k_1 - k_2 = 3$$
$$3k_1 - k_2 = -5$$
$$-2k_1 = 8$$
$$k_1 = -4$$
$$k_2 = -7$$

$$[w]_{B'} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$$

Q7) (a) $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 2 & 3 & 3 & 4 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 2 \end{array}\right]$

$R_1 + 2R_2$

$\left[\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & -1 & 1 & 2 \end{array}\right] \xrightarrow{R_2 * (-1)} \left[\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & -1 & -2 \end{array}\right]$

$$P_{B2 \to B1} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$$

(b) $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 2 \\ 3 & 4 & 2 & 3 \end{array}\right] \xrightarrow{R_2-3R_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -3 \end{array}\right]$

$R_1 - R_2$

$\left[\begin{array}{cc|cc} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 \end{array}\right]$ $\quad P_{B1\to B2} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$

(c) $P_{B1\to B2} \cdot P_{B2\to B1} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$P_{B2\to B1} \cdot P_{B1\to B2} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hence they are inverse to eachother!

(d) $[w]_{B1}$

$(0,1) = k_1(1,2) + k_2(2,3)$

$k_1 + 2k_2 = 0$

$2k_1 + 3k_2 = 1$

$2(Eq\,1) - (Eq\,2)$

$k_2 = -1$

$k_1 = 3+1/2 = 2$

$[w]_{B1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$[w]_{B2} = P_{B1\to B2}[w]_{B1} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

(e) $[w]_{B2}$

$(2,5) = k_1(1,3) + k_2(1,4)$

$k_1 + k_2 = 2$

$3k_1 + 4k_2 = 5$

$k_2 = 5-6 = -1$

$k_1 = 5+4/3 = 3$

$[w]_{B2} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$[w]_{B1} = P_{B2 \to B1} \cdot [w]_{B2} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

EX: 4.8

Q3) (a) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ $b = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Reducing A into echelon form

$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$

$R_2 * (-1)$
$R_3 + R_2$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Since $x_3 \cdot 0 \neq -2$, so b is not column space of A.

Q7) (a) $x_1 - 3x_2 = 1$

$2x_1 - 6x_2 = 2$

let $x_2 = r$

$x_1 = 1 + 3r$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 + 3r \\ r \end{bmatrix}$$

$Ax = b$:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$Ax = 0$:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = r \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(b) $x_1 + x_2 + 2x_3 = 5$

$x_1 + x_3 = -2$

$2x_1 + x_2 + 3x_3 = 3.$

let $x_3 = r$

$x_1 = -2 - r$

$x_2 = 3 + 4 + 2r - 3r = 7 - r$

$Ax = b$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 0 \end{bmatrix} + r \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$Ax = 0$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = r \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Q9) (a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

$R_2 - 5R_1$
$R_3 - 7R_1$

$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$

$R_1 + R_2$

$\begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$

$x_3 = t$
$x_1 = 16t$
$x_2 = 19t$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$

bases for null space for A is $\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$

The basis for the row space : $r_1 = [1 \quad 0 \quad -16]$
$r_2 = [0 \quad 1 \quad -19]$

Q12) (b)

basis for row spaces : {[1,2,-1,5], [0,1,4,3], [0,0,1,-7], [0,0,0,1]}

basis for column space : $\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\4\\1\\0\end{bmatrix}, \begin{bmatrix}5\\3\\-7\\1\end{bmatrix} \right\}$

(a)

basis for row space : {[1,2,4,5], [0,1,-3,0], [0,0,1,-3], [0,0,0,1]}

basis for column space : $\left\{ \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}4\\-3\\1\\0\\0\end{bmatrix}, \begin{bmatrix}5\\0\\-3\\1\\0\end{bmatrix} \right\}$

EX: 4.9

Q3) $A = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & -3 \\ 1 & 1 & 4 \end{bmatrix}$ $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a) Rank(A) = 3
Nullity(A) = 0

(c) leading variables = 3
parameters = 0

(b) Rank(A) + Nullity(A) = n
3 + 0 = 3
Hence, satisfies the formula.

Find Dimension and Basis

Q11) $A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \\ -9 & 0 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{cc|ccc} 1 & 4 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ -9 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2 * (1/3)]{R_3 + 9R_1} \left[\begin{array}{cc|ccc} 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/3 & 0 \\ 0 & 36 & 9 & 0 & 1 \end{array}\right]$$

$R_1 - 4R_2$
$R_3 - 36R_2$

$$\left[\begin{array}{cc|ccc} 1 & 0 & 1 & -4/3 & 0 \\ 0 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & 9 & -12 & 1 \end{array}\right]$$

R E

Using R and E

r=2

Basis of row space : $\{[1,0], [0,1]\}$ Dim $(row(A)) = 2$

Basis of column space : $\left\{ \begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \right\}$ Dim $(col(A)) = 2$

mxn 3x2 n-r =2-2=0

Basis for null (A) : { } ∴ No parameters Dim $(null(A)) = 0$

Basis for null (A^T) : $\left\{ \begin{bmatrix} 9 \\ -12 \\ 1 \end{bmatrix} \right\}$ Dim $(null(A^T)) = 1$

bottom 1 row of E

m-r=3-2=1

Q23) (a) $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

$R_2 - R_3$

$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

Rank (A) = 3.

(b) Rank (A) + Nullity (A) = 5

Nullity (A) = 5-3 = 2
