Assignment NO 1:

MTH 105

Multivariable Calculus

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Submitted To:

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(Question No. 01):-

(a) find equation of plane ABC, giving answer in the form of $ax + by + cz = d$

Solution!

$A(4, -4, 1)$

$B(-4, 3, -4)$

$C(4, -1, -2)$

Now,

$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$= \begin{bmatrix} -4 \\ 3 \\ -4 \end{bmatrix} - \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix}$$

$$\vec{AB} = \begin{bmatrix} -8 \\ 7 \\ -5 \end{bmatrix} \Rightarrow \boxed{-8\hat{i} + 7\hat{j} - 5\hat{k}}$$

$$\vec{AC} = \vec{OC} - \vec{OA}$$

$$= \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} \Rightarrow \boxed{0i + 3\hat{j} - 3k}$$

$$n = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -8 & 7 & -5 \\ 0 & 3 & -3 \end{vmatrix}$$

$$= \hat{i}(-21+15) - \hat{j}(24-0) + \hat{k}(-24-0)$$

$$= \hat{i} + 4\hat{j} + 4\hat{k}$$

$$d = a \cdot n = (4\hat{i} - 4\hat{j} + \hat{k}) \cdot (\hat{i} + 4\hat{j} + 4\hat{k}) \Rightarrow 4 - 16 + 4 = -8$$

Equation of plane is

$$r \cdot n = d$$

$$r \cdot (\hat{i} + 4\hat{j} + 4\hat{k}) = -8$$

$$\therefore (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + 4\hat{j} + 4\hat{k}) = -8$$

$$x + 4y + 4z = -8$$

$$\boxed{x + 4y + 4z + 8 = 0}$$

b) Find the perpendicular distance

perpendicular distance: $\Rightarrow \frac{d}{|n|} = \frac{+8}{\sqrt{1^2 + 4^2 + 4^2}}$

$$= \frac{8}{33} = 1.39$$

(c) Point D has position vector $2\hat{i}+3\hat{j}-3\hat{k}$ find the coordinates of the point of intersection of line OD with the plane ABC.

line OD: $\Rightarrow$ $r = a + \lambda b$

$$r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$$

Same as values of λ

$$\begin{pmatrix} 2\lambda \\ 3\lambda \\ -3\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} = -8$$

$$2\lambda + 12\lambda - 12\lambda = -8$$

$$2\lambda = -8$$

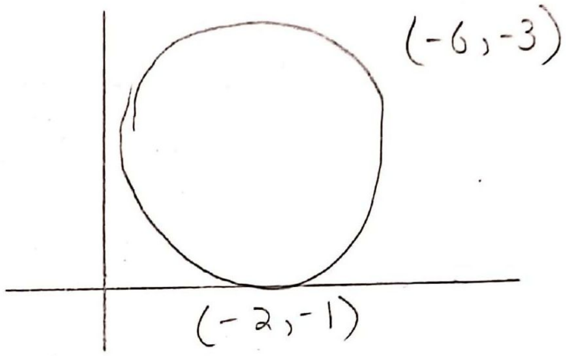
$$\boxed{\lambda = -4}$$

$$r = \begin{pmatrix} 2(-4) \\ 3(-4) \\ -3(-4) \end{pmatrix} = (-8, -12, 12)$$

(Question No. 05):-

a) If $P = (-2, -1)$ and $Q = (-6, -3)$ are the two end point of diameter of a circle, Find the equation of the circle

Solution!



Mid point:

$$\left(\frac{-2-6}{2}\right)\left(\frac{-1-3}{2}\right)$$

$$\frac{-8}{2}, \frac{-4}{2}$$

$$(-4, -2)$$

equation of the circle $= (x - h)^2 + (y - k)^2 = r^2$

$$(x+4)^2 + (y+2)^2 = r^2$$

$$x = -2 \ ; \ y = -1$$

$$(-2+4)^2 + (-1+2)^2 = r^2$$

$$(2)^2 + (1)^2 = r^2$$

$$4 + 1 = r^2$$

$$5 = r^2$$

So.

$$(x+4)^2 + (y+2)^2 = 5$$

b) If the circle pass through $(4, 0)$ and $(0, 2)$ and the center at y-axis then find the radius of the circle.

Solution

Equation of circle:

Tim of direction

$$(x-h)^2 + (y-k)^2 = r^2$$

Let

$x=0$; $y = b$

at point $(4, 0)$

$$(4)^2 + (-b)^2 = r^2$$

$$16 + b^2 = r^2 \quad \text{——— (i)}$$

at point $(0, 2)$

$$(0)^2 + (2-b)^2 = k^2$$

$$(2-b)^2 = r^2 \quad \text{——— (ii)}$$

Compare eq (i) and eq (ii).

$$16 + b^2 = (2-b)^2$$

$$16 + b^2 = 4 - 4b + b^2$$

$$16 + \cancel{b^2} - \cancel{b^2} + 4b - 4 = 0.$$

$$4b = -12$$

$$b = -3$$

put in eq (i).

$$16 + (-3)^2 = r^2$$

$$16 + 9 = r^2$$

$$r^2 = 25$$

$$r = \pm 5$$

5

c) Find the equation of direction of parabole $y^2 = 100x$.

$y^2 = 100x$ compare with

$$y^2 = 4ax$$

$$4a\cancel{x} = 100\cancel{x}$$

$$a = 100/4 = 25$$

$$\boxed{a = 25}$$

equation of direction

$x = -a$

$x = -25$

d) Find the equation of the axis of the parabolla $x^2 = 24y$.

$x^2 = 24y$

compare with.

$x^2 = 4ay$

$4a = 24$

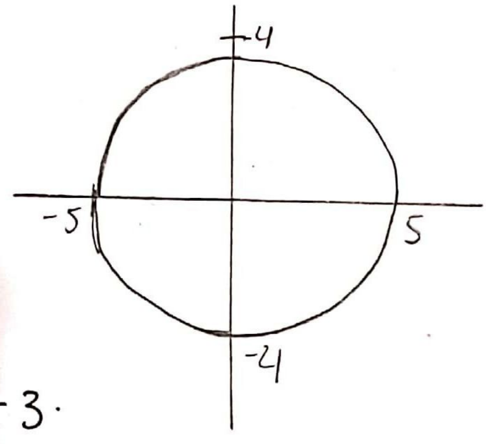
$a = 8.$

So focus is $f(a,0) = F(8,0)$ and equation of direction is.

$x = -a$

$x = -8.$

e) what is the major axis length for ellispe $\left(\frac{x}{25}\right)^2 + \left(\frac{y}{16}\right)^2 = 1.$



Compare with.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x^2}{5^2} + \frac{y^2}{4^2}$$

$a = 5$

$b = 4$

$$c = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = \pm 3.$$

$F_1 = (3,0)$

$F_2 = (-3,0)$

length of major axis $= 2a.$

$= 2(5)$

$= 10.$

f) If length of major axis is 10 and minor is 8 and major axis is along x-axis then find the equation of ellipse.

major axis = 10
minor axis = 8.

$2a = 10$
$a = 5$
$2b = 8$
$b = 4$

Equation of ellipse along x-axis.

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{4}\right)^2 = 1.$$

(Question No. 3):-

Let t be a positive constant. The line L_1 passes through position vector.......

(a) Find the value of t.

$L_1 = t\hat{i} + \hat{j}$ $\quad -2\hat{i} - \hat{j}$

$L_2 = \hat{j} + t\hat{k}$

$= 2\hat{j} + \hat{k}$

=> Shortest distance between L_1 and L_2 is $\sqrt{21}$

$r_1 = OA + \lambda AB$

$r_2 = OA + \lambda AB$

$r_1 = t\hat{i} + \hat{j} + 2\lambda\hat{i} - \hat{j}$

$$L_1 = \vec{r_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ -10 \\ 0 \end{bmatrix}$$

$$L_2 = \vec{r_2} = \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix} + u \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$D = \frac{(b_1 \times b_2)(a_2 - a_1)}{|b_1 \times b_2|}$$

$\because a_2 - a_1 = \hat{n}$ (normal vector)

$$b_1 \times b_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -1 & 0 \\ 0 & -2 & 1 \end{vmatrix} \Rightarrow \hat{i}(-1) - \hat{j}(-2) + \hat{k}(4)$$

$$\boxed{= -\hat{i} + 2\hat{j} + 4\hat{k}}$$

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$$|b_1 \times b_2| = \sqrt{(-1)^2 + (2)^2 + (4)^2}$$

$$= \sqrt{21}$$

$$(a_2 - a_1) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{= t\hat{i} + t\hat{k}}$$

$$D = \frac{(-\hat{i} + 2\hat{j} + 4\hat{k}) \cdot (-t\hat{i} + t\hat{k})}{\sqrt{21}}$$

$$\sqrt{21} = \frac{t + 4t}{\sqrt{21}}$$

$$21 = t + 4t$$

$$21 = 5$$

$$\boxed{t = \frac{21}{5}}$$

b)

$$r_1 = \frac{21}{5}\hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j})$$

$$r_2 = \hat{j} - \frac{21}{5}\hat{k} + \mu(-2\hat{j} + \hat{k})$$

$$\pi_1 = r = \vec{OA} + \lambda\vec{AB} + \mu\vec{AC}$$

$$\vec{r} = \frac{-21}{5}\hat{i} + \hat{j} + \lambda(-2\hat{i} - \hat{j}) + \mu(-2\hat{j} + \hat{k})$$

(c)

$\lambda_2 = 5x - 6y + 5z = 0$

$\lambda_1 = \frac{x-0}{0}$, $\lambda_2 = \frac{y-1}{-2}$

$\lambda_2 = \frac{2-4.2}{1}$

=> From L_2 direction vector is

$= \{0, -2, 1\}$

=> From π_2 normal vector is

$= (5, -6, 7)$

$$\cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{|a|\,|b|}$$

$$|a \cdot b| = \begin{bmatrix} 0 \\ 1 \\ 21/5 \end{bmatrix} \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix} \Rightarrow -6 + 29.4 \boxed{= 23.4}$$

$$|a| = \sqrt{(1)^2 + \left(\frac{21}{5}\right)^2} = \sqrt{1 + 17.64}$$

$$\boxed{|a| = 4.3}$$

$$\boxed{|b| = 10.49} \Rightarrow |b| = \sqrt{(5)^2 + (6)^2 + (7)^2}$$

$$\boxed{|b| = 10.49}$$

$$\theta = \cos^{-1}\left(\frac{23.4}{4.3 \times 10.49}\right)$$

$$\theta = \cos^{-1} \frac{23.4}{45.11} \Rightarrow \boxed{\theta = 59.34^\circ}$$

d)

$$\pi_1 = \begin{bmatrix} -21/5 \\ 1 \\ 0 \end{bmatrix} \quad , \quad \pi_2 \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$$

$$\pi_1 \cdot \pi_2 = \begin{bmatrix} -21/5 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$$

$$= -21 - 6$$

$$= -27$$

$$|a| = \sqrt{\left(\frac{-21}{5}\right)^2 + (1)^2} = \sqrt{17.64 + 1} = 4.3.$$

$$|b| = \sqrt{5^2 + (-6)^2 + 7^2} = 10.49.$$

$$\phi = \cos^{-1} \frac{-27}{4.3 \times 10.49} \quad , \quad \theta = 126.78.$$

Acute angle

$$\phi = 180 - 126.78$$

$$\phi = 53.23^\circ$$

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(Question No. 2):-

The position vector of points A, B, C, D are

$A = (7\hat{i} + 4\hat{j} - \hat{k})$ $D = (2\hat{i} + 7\hat{j} + \lambda k)$

$B = (11\hat{i} + 3\hat{j})$

$C = (2\hat{i} + 6\hat{j} + 3\hat{k})$

(a)

$\lambda^2 - 5\lambda + 4 = 0$ — (a)

$\vec{r_1} = \vec{OA} - \vec{AB}$

$\vec{AB} = \vec{OB} - \vec{OA} \Rightarrow \begin{bmatrix} 11 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$

$\vec{r_1} = \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$

$\vec{r_2} = \vec{OC} - \vec{OD}$

$\vec{CD} = \vec{OD} - \vec{OC}$

$= \begin{bmatrix} 2 \\ 7 \\ \lambda \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 1 \\ \lambda - 3 \end{bmatrix}$

$$b_1 \times b_2 = \begin{bmatrix} i & j & k \\ 4 & -1 & 1 \\ 0 & 1 & \lambda-3 \end{bmatrix}$$

$$= \hat{i}\left(-(\lambda-3)-1\right) - \hat{j}\left(4(\lambda-3)\right) + 4\hat{k}$$

$$= (2-\lambda)\hat{i} - \hat{j}\left(4\lambda-12\right) + 4\hat{k}$$

$$a_2 - a_1 = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix} - \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 2 \\ 4 \end{bmatrix}$$

$$|b_1 \times b_2| = \sqrt{(2-\lambda)^2 + (4\lambda-12)^2 + (4)^2}$$

$$= \sqrt{17\lambda^2 - 100\lambda + 164}$$

$$|b_1 \times b_2| \cdot (a_1 - a_2) = \left[(2-\lambda)\hat{i} - \hat{j}\left(4\lambda-12\right) + 4\hat{k}\right]$$

$$= -5(2-\lambda) - 2(4\lambda-12) + 16$$

$$= -10 + 5\lambda - 8\lambda + 24 + 16$$

$$= 30 - 3\lambda$$

$$d = \frac{(b_1 \times b_2) \cdot (a_1 - a_2)}{|b_1 \times b_2|}$$

$$3 = \frac{30 - 3\lambda}{\sqrt{17\lambda^2 - 100\lambda + 164}} \Rightarrow a = \frac{(30 - 3\lambda)^2}{\sqrt{17\lambda^2 - 100\lambda + 164}}$$

$$a = \frac{900 + 9\lambda^2 - 180\lambda}{17\lambda^2 - 100\lambda + 164}$$

(1

$9(17\lambda^2 - 100\lambda + 164) = (900 + 9\lambda^2 - 180)$

$153\lambda^2 - 900\lambda + 1476 - 900 - 9\lambda^2 + 180 = 0$

$144\lambda^2 - 720\lambda + 576 = 0$

$4(36\lambda^2 - 180\lambda + 144) = 0$

$36\lambda^2 - 180\lambda + 144 = 0$

$4(9\lambda^2 - 45\lambda + 36) = 0$

$9(\lambda^2 - 5\lambda + 4) = 0$

$\lambda^2 - 5\lambda + 4 = 0$

$\lambda^2 - 4\lambda - \lambda + 4 = 0$

$\lambda(\lambda - 4) - 1(\lambda - 4) = 0$

$\lambda - 1 = 0$ | $\lambda - 4 = 0$

$\lambda = 1$ | $\lambda = 4$

put $\lambda = 1$ in equation (a)

$(1)^2 - 5(1) + 4 = 0$

$0 = 0$

put $\lambda = 4$ in equation (a)

$(4)^2 - 5(4) + 4 = 0$

$0 = 0$

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(b)

Let Π_1 be the plane ABD when $\lambda = 1$. Let Π_2 be the plane ABD when $\lambda = 4$.

i) write down an equation of Π_1 giving your answer in the form.

$$r = a + sb + tc$$

$$\Pi_1 = \vec{r_1} = \vec{OA} + s\vec{AB} + t\vec{AD}$$

when $\lambda = 1$

$$\vec{AD} = \vec{OD} - \vec{OA}$$

$$= \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}.$$

$$\Pi_1 = 7\hat{i} + 4\hat{j} - \hat{k} + s(4\hat{i} - \hat{j} + \hat{k}) + t(-5\hat{i} + 3\hat{j} + 2\hat{k})$$

ii) Find an equation of Π_2 giving your answer in the form $ax + by + cz = d$.

$$\Pi_2 = \vec{r_2} = \vec{OA} + \lambda\vec{AB} + u\vec{AD}$$

when $\lambda = 4$

$$\vec{AD} = \vec{OD} - \vec{OA}$$

$$= \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} - \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}$$

$$\Pi_2 = \vec{r_2} = \vec{OA} + \lambda\vec{AB} + \mu\vec{AD}$$

$$= \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} + \lambda\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} + \mu\begin{bmatrix} -5 \\ 3 \\ 5 \end{bmatrix}$$

$$x = 7 + 4\lambda - 5\mu \quad \text{—— (i)}$$

$$y = 7 - \lambda + 3\mu \quad \text{—— (ii)}$$

$$z = -1 + \lambda + 5\mu \quad \text{— (iii)}$$

eq (ii) + eq (iii)

$$y = 4 - \cancel{\lambda} + 3\mu$$
$$z = -1 + \cancel{\lambda} + 5\mu$$
$$y + z = 3 + 8\mu \quad \text{—— (iv)}$$

Multiplying eq (ii) by 2.

$$y = 4 - \lambda + 3\mu$$
$$\times 4$$
$$4y = 16 - 4\lambda + 12\mu \quad \text{— (v).}$$

eq (i) + eq (v)

$$x = 7 + \cancel{4\lambda} - 5\mu$$
$$4y = 16 - \cancel{4\lambda} + 12\mu$$
$$x + 4y = 23 + 7\mu$$

$$\frac{x + 4y - 23}{7} = \mu$$

put the values in eq (iv)

$$y + z = 3 + 8\left(\frac{x + 4y - 23}{7}\right)$$

$$y + z = 3 + \frac{8x + 32y - 184}{7} \quad \text{Xing by (7)}$$

$$7y + 7z = 21 + 8x + 32y - 184$$

$$7y + 7z - 21 - 8x - 32y + 184 = 0$$

$$-(8x + 25y - 7z - 163) = 0$$

$$8x + 25y - 7z = 163.$$

(c)

$\pi_1 = \vec{r_1} = \overrightarrow{OA} + \lambda\,\overrightarrow{AB} + \mu\,\overrightarrow{AD}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}$$

$x = 7 + 4\lambda - 5\mu$ — (i)

$y = 4 - \lambda + 3\mu$ — (ii)

$z = -1 + \lambda + 2\mu$ — (iii).

eq (ii) + eq (iii).

$$\begin{aligned} y &= 4 - \cancel{\lambda} + 3\mu \\ z &= -1 + \cancel{\lambda} + 2\mu \\ \hline y + z &= 3 + 5\mu \end{aligned}$$ — (iv).

Multiply eq (ii) by 4.

$4y = 16 - 4\lambda + 12$ — (v).

ad eq (i) and eq (v).

$$\begin{aligned} x &= 7 + 4\cancel{\lambda} - 5\mu \\ 4y &= 16 - \cancel{4\lambda} + 12 \\ \hline x + 4y &= 23 + 7\mu \end{aligned}$$

$\dfrac{x + 4y - 23}{7} = \mu$

$y + z = 3 + 5\left(\dfrac{x + 4y - 23}{7}\right)$ xing by (7).

$7y + 7z = 21 + 5y + 20y - 115.$

$7y + 7z - 21 - 5x - 20y + 115 = 0.$

$-5x - 13y + 7z + 94 = 0.$

$-\S(5x + 13y - 7z - 94) = 0.$

$5x + 13y - 7z = 94.$

$$\theta = \cos^{-1} \frac{\pi_1 \cdot \pi_2}{|\pi_1| \, |\pi_2|}$$

$$\pi_1 \cdot \pi_2 = \begin{bmatrix} 5 \\ 13 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 25 \\ -7 \end{bmatrix}$$

$$= 40 + 385 + 49$$

$$= 414$$

$$|n_1| = \sqrt{5^2 + 13^2 + (-1)^2}$$

$$= 9\sqrt{3}$$

$$|n_2| = \sqrt{8^2 + 25^2 + (-7)^2}$$

$$= 3\sqrt{82}$$

$$\theta = \cos^{-1}\left(\frac{414}{9\sqrt{3} \times 3\sqrt{82}}\right)$$

$$\theta = \cos^{-1}\left(\frac{414}{27\sqrt{248}}\right)$$

$$\theta = 12.15^\circ$$