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Beam model refinement and reduction

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ABSTRACT

In this paper we present a method for systematic construction of the stiffness matrix of an arbitrary spatial frame element by performing a series of elementary transformations. The procedure of this kind is capable of including a number of element refinements (addition of shear deformation, variable cross-section, etc.) that are not easily accessible to standard displacement-based method. We also discuss the necessary modifications of the element stiffness matrix in order to accommodate different constraints, such as point constraints in terms of joint releases (or hinges) for moments or shear forces. This is obtained by means of model reduction providing a more effective approach than the alternative one in which the global number of degrees of freedom has to be increased by one for each new release. Finally, we elaborate upon the global constraints imposing the length-invariant deformation of frame elements with an arbitrary position in space. Several numerical examples are used to illustrate the performance of the proposed procedures. The computations are carried out by a modified version of computer code CAL.

1. Introduction

The finite approximations, which is carried out by using the finite element methods, translates the continuous system analysis into discrete system analysis. In a mathematical sense, we switch from a set of differential equations to a set of algebraic equations within the framework of static analysis [1]. The basic advantage of such an approximation procedure is the possibility of creating a complete discrete model as a set of its constituent (finite) elements no matter what the complexity of the structural system is [2]. In this paper we briefly discuss the structural frame analysis methodology which pertains to stiffness assembly. The synthesis of stiffness matrices is illustrated using the computer program CAL [3].

In this paper, the special attention is dedicated to 3D frame analysis and enhancements of the basic beam element used for modelling. The construction of the stiffness matrix of an arbitrary (straight or curved) beam is derived by performing a series of elementary transformations. Also, we present possible modifications of the element stiffness matrix in the presence of joint releases (zero fields). The technique is described using simple 3D frame as an example. At the end, we discuss the implementation issues relevant for the technical displacement method which assumes axially rigid rods.

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2. Method of direct stiffness assembly

Static analysis of a discrete model which employs the assembly procedure inevitably reduces to solution of a system of linear algebraic equations classically written as

$$\mathbf{K}\mathbf{u} = \mathbf{f} \qquad (1)$$

where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{u}$ the displacement vector and $\mathbf{f}$ the external force vector. In the method of stiffness assembly the global stiffness matrix $\mathbf{K}$ is obtained as the assembly of the element stiffness matrices $\mathbf{K}^e$ as follows

$$\mathbf{K} = \sum_e \mathbf{L}^{eT}\mathbf{K}^e\mathbf{L}^e \qquad (2)$$

$\sum_e$ represents a symbolic summation of elementary stiffness matrices $\mathbf{K}^e$ which are placed in the corresponding slots of the global stiffness matrix $\mathbf{K}$, as determined by the transformation of displacements between local and global coordinate systems specified by the each element connectivity matrix $\mathbf{L}^e$ (e.g. see [10,11]).

If local and global displacements are defined in the same coordinate system (usually the perpendicular Cartesian coordinate system), then the summation in (2) is carried out using the Boolean matrices (with either unit or zero entries), i.e. only identifying the corresponding local and global displacements. In that case the element stiffness matrix $\mathbf{K}^e$ must be defined in the global coordinate system, which is shown in the next section.

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3. Element stiffness matrix

The element stiffness matrix of a beam element can be derived by employing the so-called Hermite polynomials for the real and virtual displacement fields (e.g. see [10,11]), to obtain the weak form interpretation of equilibrium equations for each element that can formally be written as:

$$\mathbf{f}_p = \mathbf{K}_p\mathbf{p} \tag{3}$$

where $\mathbf{f}_p$ are the generalized forces, p_1 to p_{12} are the generalized nodal displacements shown in Fig. 1, and $\mathbf{K}_p$ is the corresponding stiffness matrix.

The results are recalled here for 3D case:

$$\mathbf{K}_p = \begin{pmatrix}
\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 \\
 & \frac{12EI_z}{l^3} & 0 & 0 & 0 & \frac{6EI_z}{l^2} & 0 & -\frac{12EI_z}{l^3} & 0 & 0 & 0 & \frac{6EI}{l^2} \\
 & & \frac{12EI_y}{l^3} & 0 & -\frac{6EI_y}{l^2} & 0 & 0 & 0 & -\frac{12EI_y}{l^3} & 0 & -\frac{6EI_y}{l^2} & 0 \\
 & & & \frac{GJ}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{l} & 0 & 0 \\
 & & & & \frac{4EI_y}{l} & 0 & 0 & 0 & \frac{6EI_y}{l^2} & 0 & \frac{2EI_y}{l} & 0 \\
 & & & & & \frac{4EI_z}{l} & 0 & -\frac{6EI_z}{l^2} & 0 & 0 & 0 & \frac{2EI_z}{l} \\
 & & & & & & \frac{EA}{l} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & \frac{12EI_z}{l^3} & 0 & 0 & 0 & -\frac{6EI_z}{l^2} \\
 & & \text{sym.} & & & & & & \frac{12EI_y}{l^3} & 0 & \frac{6EI_y}{l^2} & 0 \\
 & & & & & & & & & \frac{GJ}{l} & 0 & 0 \\
 & & & & & & & & & & \frac{4EI_y}{l} & 0 \\
 & & & & & & & & & & & \frac{4EI_z}{l}
\end{pmatrix} \tag{4}$$

The polynomials of this kind are the exact solution to each elementary load case where only one of the generalized displacements is of the unit value, whereas all the other ones remain equal to zero. Moreover, due to property of superconvergence (e.g. see [10,11]) the generalized nodal displacements will also be exact for more general load case, as long as the equivalent nodal loads are also computed with these Hermite polynomials.

Furthermore, Eq. (3) can be transformed into global coordinate system with axes X, Y and Z (see Fig. 2). This is a usual element transformation from local to global coordinate system [7].

With that purpose, we first establish the relation between the local generalized displacements p_1 to p_{12} defined in the local coordinate system x, y, z and the local generalized displacements q_1 to q_{12} defined in the global coordinate system X, Y, Z as follows:

$$\mathbf{p} = \mathbf{T}_{pq}\mathbf{q}, \quad \mathbf{q}^T = \langle q_1, \ldots, q_{12}\rangle \tag{5}$$

where $\mathbf{T}_{pq}$ is a 12×12 block diagonal matrix whose 3×3 blocks represent the inclinations of the local coordinate system with respect to the global one [7]. Thereafter, the principle of virtual displacements (shape-wise equivalent to the real displacement field) leads to

$$\mathbf{f}_q = \mathbf{K}_q\mathbf{q}, \quad \mathbf{K}_q = \mathbf{T}_{pq}^T\mathbf{K}_p\mathbf{T}_{pq} \tag{6}$$

Since the element stiffness matrix $\mathbf{K}_q$ was derived for local degrees of freedom $\mathbf{q}$ in the global coordinate system X, Y, Z, it can directly be assembled into the structural stiffness matrix $\mathbf{K}$ in Eq. (2),

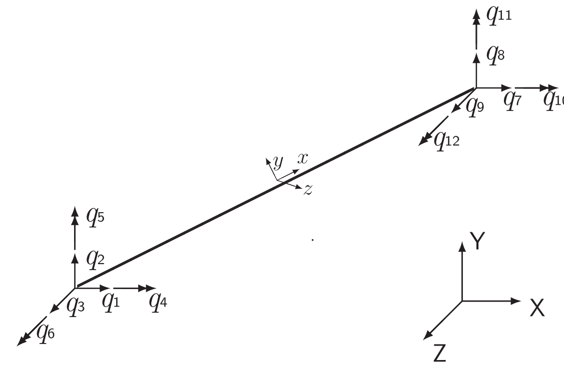

Fig. 2. Local degrees of freedom in the global coordinate system.

$$\mathbf{K}_q \equiv \mathbf{K}^e \tag{7}$$

However, the result for element stiffness matrix in the previous section remains valid only for a beam with uniform geometrical/mechanical properties. Moreover, we have shown in [12] that even accounting for shear deformation would require non-conventional interpolations with the increase of computational cost due to additional degrees of freedom.

4. Flexibility approach for accounting for variability of beam geometric/mechanical properties along the beam and shear deformation

We show here that the same element stiffness matrix can be obtained at neither extra cost nor additional degrees of freedom. The main idea is to start with the reduced model pertaining to the deformation space, which allows us to define the inverse of the corresponding stiffness matrix accounting (exactly) for any variation of mechanical/geometric beam properties and the shear deformation. The reduced deformation space matrix is then mapped into the full 3D form by employing the corresponding transformation between different sets of generalized displacements.

First, we define the natural local coordinate system for a given 3D beam (see Fig. 3) where x axis is directed along the beam and y and z axes are perpendicular to it as well as mutually perpendicular.

The generalized displacements (see Fig. 3) are defined according to the following: v_1 and v_2 are rotations around y axis at the element start and end, v_3 and v_4 are the corresponding rotations around z axis, v_5 is the rotation around x of the element end and finally v_6 represents the extension of the element end. The displacements v_1 to v_6 are six generalized (independent) beam displacements induced by deformation, which are separated from the six degrees of freedom corresponding to the rigid body modes. The rigid body displacements are restrained with the element supports (see Fig. 3). The idea of distinguishing between generalized deformations and rigid body modes is similar to the one presented in [4], which is given for large displacement and small deformation frame analysis, but the choice of deformation degrees of freedom in our case is different from the one in [4]. The relation between

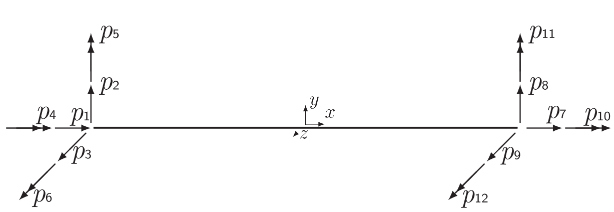

Fig. 1. Set of degrees of freedom in local coordinate system of 2-node beam element.

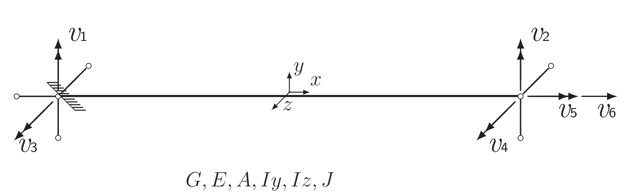

Fig. 3. Local coordinate system of a beam with deformation degrees of freedom.

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generalized displacements and the work-conjugate forces can be written as

$$\mathbf{f}_v = \mathbf{K}_v \mathbf{v} \tag{8}$$

where

$$\mathbf{K}_v = \begin{pmatrix} \frac{4EI_y}{l} & \frac{2EI_y}{l} & 0 & 0 & 0 & 0 \\ \frac{2EI_y}{l} & \frac{4EI_y}{l} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4EI_z}{l} & \frac{2EI_z}{l} & 0 & 0 \\ 0 & 0 & \frac{2EI_z}{l} & \frac{4EI_z}{l} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{GJ}{l} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{EA}{l} \end{pmatrix} \tag{9}$$

First four equations, for our choice of generalized displacements (deformations), directly stem from the equations of Takabeya (see [5]), whereas the 5th and 6th equation represent the well known torsional and axial beam stiffness.

The stiffness matrix $\mathbf{K}_v$ in Eq. (9) is given for a straight beam with constant mechanical and geometrical properties. For a curved beam with variable properties $\mathbf{K}_v$ will generally be fully populated matrix. However, Eq. (8) can be derived for an arbitrarily curved beam in space as well, with the unknown displacements described in Fig. 3. Geometrically exact relation of that sort (exact within the framework of infinitesimal displacements) can be constructed first by finding the 6×6 flexibility matrix $\mathbf{F}_v$ and then inverting in order to get

$$\mathbf{K}_v = \mathbf{F}_v^{-1} \tag{10}$$

The computation of the entries of the flexibility matrix $\mathbf{F}_v$ for an arbitrary (straight or curved) spatial beam can be performed by employing some established method, for example Mohr's analogy (the method of conjugate structure, see [6]).

From Figs. 1 and 3 one can develop mapping between displacements v_1 to v_6 and displacements p_1 to p_{12} in the following format

$$\mathbf{v} = \mathbf{T}_{vp}\mathbf{p}, \quad \mathbf{v}^T = \langle v_1, \ldots, v_6 \rangle, \quad \mathbf{p}^T = \langle p_1, \ldots, p_{12} \rangle \tag{11}$$

where the transformation matrix $\mathbf{T}_{vp}$ is given with

$$\mathbf{T}_{vp} = \begin{pmatrix} 0 & 0 & -\frac{1}{l} & 0 & 1 & 0 & 0 & 0 & \frac{1}{l} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{l} & 0 & 0 & 0 & 0 & 0 & \frac{1}{l} & 0 & 1 & 0 \\ 0 & \frac{1}{l} & 0 & 0 & 0 & 1 & 0 & -\frac{1}{l} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{l} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{12}$$

If we apply the principle of virtual displacements (for example see [7]) on the previously shown beam, constructing virtual displacements (denoted with hat) in the same fashion as real displacements (Figs. 1 and 3), then we have

$$\hat{\mathbf{p}}^T \mathbf{f}_p = \hat{\mathbf{v}}^T \mathbf{f}_v \tag{13}$$

where $\mathbf{f}_p$ is a set of generalized forces which are work-conjugate to generalized displacements described in Fig. 1. Consecutively exploiting Eqs. (11) and (8) and again (11) we wind up with

$$\hat{\mathbf{p}}^T \left(\mathbf{f}_p - \mathbf{T}_{vp}^T \mathbf{K}_v \mathbf{T}_{vp} \mathbf{p} \right) = 0 \tag{14}$$

Since virtual displacements are mutually independent (which is equivalent to the fundamental lemma of variational calculus in discrete formulation [8]), Eq. (14) results with

$$\mathbf{f}_p = \mathbf{K}_p \mathbf{p}, \quad \mathbf{K}_p = \mathbf{T}_{vp}^T \mathbf{K}_v \mathbf{T}_{vp} \tag{15}$$

In order to include shear deformation for Euler–Bernoulli beam, one can exploit the possibilities of flexibility approach, namely Eq. (10). For a 3D beam element with deformation degrees of freedom shown in Fig. 3 we can evaluate the flexibility matrix coefficients including displacements due to shear and obtain:

$$\mathbf{F}_v = \begin{pmatrix} \frac{l}{3EI_y} + \frac{1}{k_z GAl} & -\frac{l}{6EI_y} + \frac{1}{k_z GAl} & 0 & 0 & 0 & 0 \\ -\frac{l}{6EI_y} + \frac{1}{k_z GAl} & \frac{l}{3EI_y} + \frac{1}{k_z GAl} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{l}{3EI_z} + \frac{1}{k_y GAl} & -\frac{l}{6EI_z} + \frac{1}{k_y GAl} & 0 & 0 \\ 0 & 0 & -\frac{l}{6EI_z} + \frac{1}{k_y GAl} & \frac{l}{3EI_z} + \frac{1}{k_y GAl} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{l}{GJ} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{l}{EA} \end{pmatrix} \tag{16}$$

After inverting we get $\mathbf{K}_v$ that includes shear deformability. Finally, exploiting Eqs. (12) and (15) we finally determine the element stiffness matrix $\mathbf{K}_p$ which is in case of horizontal beam equal to $\mathbf{K}^e$:

$$\mathbf{K}^e = \begin{pmatrix} K_x & 0 & 0 & 0 & 0 & 0 & -K_x & 0 & 0 & 0 & 0 & 0 \\ & K_{y1} & 0 & 0 & 0 & K_{y2} & 0 & -K_{y1} & 0 & 0 & 0 & K_{y2} \\ & & K_{z1} & 0 & -K_{z2} & 0 & 0 & 0 & -K_{z1} & 0 & -K_{z2} & 0 \\ & & & T & 0 & 0 & 0 & 0 & 0 & -T & 0 & 0 \\ & & & & K_{z3} & 0 & 0 & 0 & K_{z2} & 0 & K_{z4} & 0 \\ & & & & & K_{y3} & 0 & -K_{y2} & 0 & 0 & 0 & K_{y4} \\ & & & & & & K_x & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & K_{y1} & 0 & 0 & 0 & -K_{y2} \\ & & sym. & & & & & & K_{z1} & 0 & K_{z2} & 0 \\ & & & & & & & & & T & 0 & 0 \\ & & & & & & & & & & K_{z3} & 0 \\ & & & & & & & & & & & K_{y3} \end{pmatrix} \tag{17}$$

where

$$\begin{aligned} &K_x = \frac{EA}{l} && K_{y1} = \frac{12EI_z}{(1+\phi_y)l^3} && K_{y2} = \frac{6EI_z}{(1+\phi_y)l^2} \\ &K_{y3} = \frac{(1+0.25\phi_y)4EI_z}{(1+\phi_y)l} && K_{y4} = \frac{(1-0.5\phi_y)4EI_z}{(1+\phi_y)l} && \phi_y = \frac{12EI_z}{k_y GAl^2} \\ &T = \frac{GJ}{l} && K_{z1} = \frac{12EI_y}{(1+\phi_z)l^3} && \cdots \end{aligned} \tag{18}$$

The remaining coefficients are easily obtained by subscript interchange. We mention that k_yA is the effective shear for transverse shear deformation in y direction (analogous for direction z).

With this we finalize the development of the element stiffness matrix $\mathbf{K}^e$ through a series of elementary transformations which enhance the stiffness matrix of the basic element (the one with all rigid body modes restrained). This facilitates the construction of $\mathbf{K}^e$ matrix as well as the understanding of the principles of that construction. Additionally, this approach enables us to form a stiffness matrix for a special beam element that has one or more joint releases (zero fields). Before we elaborate on such a possibility, we discuss the inclusion of shear strain in different Timoshenko beam elements in the next section.

5. Timoshenko beam element for including shear strain

As opposed to adding flexibility on conventional Euler–Bernoulli beam, one can introduce independent interpolation for nodal rotations which produces Timoshenko beam formulation. The basic assumptions are that the sections still remain plane, but no longer normal to the neutral axis after bending. In this way it is possible to take into account the shear deformation (and stress) necessary for modelling thick beams. Following the kinematic hypotheses $u(x) = -y\theta(x)$ and $\theta(x) = \mathrm{d}v(x)/\mathrm{d}x - \gamma(x)$, where $\gamma(x)$ represents constant transverse shear strain, we interpolate deflections and rotations with linear shape functions (so called equal-order interpolation) expressed as:

$$\begin{aligned} v^h(\xi) &= N_1(\xi)v_1 + N_2(\xi)v_2 \\ \theta^h(\xi) &= N_1(\xi)\theta_1 + N_2(\xi)\theta_2 \end{aligned} \tag{19}$$

The element stiffness matrix $\mathbf{K}^e$ and the load vector $\mathbf{f}^e$ corresponding to uniform loading obtained by standard displacement based finite element procedure using two Gauss points (in this case equivalent to analytically integrated results) have the following form:

$$\begin{aligned} \mathbf{K}^e &= \begin{pmatrix} \frac{kGA}{l} & \frac{kGA}{2} & -\frac{kGA}{l} & \frac{kGA}{2} \\ \frac{kGA}{2} & \frac{EI}{l}+\frac{kGAl}{3} & -\frac{kGA}{2} & \frac{kGAl}{6}-\frac{EI}{l} \\ -\frac{kGA}{l} & -\frac{kGA}{2} & \frac{kGA}{l} & -\frac{kGA}{2} \\ \frac{kGA}{2} & \frac{kGAl}{6}-\frac{EI}{l} & -\frac{kGA}{2} & \frac{EI}{l}+\frac{kGAl}{3} \end{pmatrix} \\ \mathbf{f}^e &= \begin{pmatrix} \frac{ql}{2} & 0 & \frac{ql}{2} & 0 \end{pmatrix}^T \end{aligned} \tag{20}$$

It can be shown that this element is very stiff and that it yields poor results (shear locking) when the beam becomes slender. Furthermore, there are no forces on the nodal rotations which additionally disturbs the solution quality. Thus, the formulation is not generally applicable. This problem can be overcome in several ways and the first remedy is selective reduced integration. If we integrate the bending stiffness terms using appropriate order Gaussian quadrature, whereas the shear stiffness term is under-integrated by evaluating the shear strain at the element midpoint, we wind up with the following stiffness matrix which does not exhibit locking:

$$\mathbf{K}^e = \begin{pmatrix} \frac{kGA}{l} & \frac{kGA}{2} & -\frac{kGA}{l} & \frac{kGA}{2} \\ \frac{kGA}{2} & \frac{EI}{l}+\frac{kGAl}{4} & -\frac{kGA}{2} & \frac{kGAl}{4}-\frac{EI}{l} \\ -\frac{kGA}{l} & -\frac{kGA}{2} & \frac{kGA}{l} & -\frac{kGA}{2} \\ \frac{kGA}{2} & \frac{kGAl}{4}-\frac{EI}{l} & -\frac{kGA}{2} & \frac{EI}{l}+\frac{kGAl}{4} \end{pmatrix} \tag{21}$$

The other possibilities pertain to enhancing the displacement field with additional terms for example:

$$\begin{aligned} v^h(\xi) &= N_1(\xi)v_1 + N_2(\xi)v_2 + N_3(\xi)\Delta v \\ \theta^h(\xi) &= N_1(\xi)\theta_1 + N_2(\xi)\theta_2 \end{aligned} \tag{22}$$

The interpolation functions are Lagrangian polynomials superimposed hierarchically, hence the quadratic $N_3(\xi) = 1 - \xi^2$ is added to linear $N_1(\xi) = \frac{1}{2}(1-\xi)$ and $N_2(\xi) = \frac{1}{2}(1+\xi)$. N_3 corresponds to transverse displacement of the middle node, invisible to neighbouring elements which enables us to perform static condensation and again obtain 4×4 element stiffness matrix $\mathbf{K}^e$. First we determine the coefficient of the additional term, which is chosen such that it eliminates the linear term in shear strain responsible for shear locking. Thus, we have

$$v^h(\xi) = N_1(\xi)v_1 + N_2(\xi)v_2 + N_3(\xi)\frac{l}{8}(\theta_1 - \theta_2) \tag{23}$$

Plugging the assumed solution in the weak form of equilibrium equations, we get the following element equations:

$$\begin{pmatrix} \frac{kGA}{l} & \frac{kGA}{2} & -\frac{kGA}{l} & \frac{kGA}{2} \\ \frac{kGA}{2} & \frac{EI}{l}+\frac{kGAl}{4} & -\frac{kGA}{2} & \frac{kGAl}{4}-\frac{EI}{l} \\ -\frac{kGA}{l} & -\frac{kGA}{2} & \frac{kGA}{l} & -\frac{kGA}{2} \\ \frac{kGA}{2} & \frac{kGAl}{4}-\frac{EI}{l} & -\frac{kGA}{2} & \frac{EI}{l}+\frac{kGAl}{4} \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \frac{ql}{2} \\ \frac{ql^2}{12} \\ \frac{ql}{2} \\ -\frac{ql^2}{12} \end{pmatrix} \tag{24}$$

The stiffness matrix is identical to the one obtained with reduced integration, however here we have employed exact integration. In addition, the equivalent load vector gives forces on the nodal rotations, as was the case with conventional beam element.

Finally, we present the element that gives exact solutions of the governing differential equations for uniform Timoshenko beam with no distributed loads (so called linked interpolation [10]). It is developed using the standard displacement based formulation by enhancing the displacements in the following manner:

$$\begin{aligned} v^h(\xi) &= N_1(\xi)v_1 + N_2(\xi)v_2 + N_3(\xi)\frac{l}{8}(\theta_1 - \theta_2) + N_4(\xi)\alpha l \Delta\theta \\ \theta^h(\xi) &= N_1(\xi)\theta_1 + N_2(\xi)\theta_2 + N_3(\xi)\Delta\theta \end{aligned} \tag{25}$$

where $N_4(\xi) = \xi(1-\xi^2)$ and $\Delta\theta$ represents the rotation of the element middle node. The coefficient $\alpha = \frac{1}{6}$ is determined from the condition that the shear strain is constant within the element. Now we can easily write the relationship between strains and displacements connected through $\mathbf{B}$ matrix as

$$\begin{pmatrix} \kappa \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{l} & 0 & \frac{1}{l} & -4\frac{\xi}{l} \\ -\frac{1}{l} & -\frac{1}{2} & \frac{1}{l} & -\frac{1}{2} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ \Delta\theta \end{pmatrix} \tag{26}$$

Using the material elastic property array $\mathbf{D}$

$$\mathbf{D} = \begin{pmatrix} EI & 0 \\ 0 & kGA \end{pmatrix} \tag{27}$$

and the jacobian $j = \frac{l}{2}$ derived from $x(\xi) = N_1(\xi)x_1 + N_2(\xi)x_2$, we can evaluate the element stiffness matrix $\mathbf{K}^e$ and the equivalent load vector $\mathbf{f}^e$ for constant q

$$\begin{aligned} \mathbf{K}^e &= \int_{-1}^{+1} \mathbf{B}^T\mathbf{D}\mathbf{B} j \, d\xi \\ \mathbf{f}^e &= \int_{-1}^{+1} \mathbf{N}^T q j \, d\xi \end{aligned} \tag{28}$$

where

$$\mathbf{N} = \begin{pmatrix} \frac{1}{2}(1-\xi) & \frac{l}{8}(1-\xi^2) & \frac{1}{2}(1+\xi) & -\frac{l}{8}(1-\xi^2) & \frac{l}{6}\xi(1-\xi^2) \end{pmatrix} \tag{29}$$

and obtain the following results

$$\begin{aligned} \mathbf{K}^e &= \begin{pmatrix} \frac{kGA}{l} & \frac{kGA}{2} & -\frac{kGA}{l} & \frac{kGA}{2} & \frac{2kGA}{3} \\ \frac{kGA}{2} & \frac{kGAl}{4}+\frac{EI}{l} & -\frac{kGA}{2} & -\frac{kGAl}{4}-\frac{EI}{l} & \frac{kGAl}{3} \\ -\frac{kGA}{l} & -\frac{kGA}{2} & \frac{kGA}{l} & -\frac{kGA}{2} & -\frac{2kGA}{3} \\ \frac{kGA}{2} & \frac{kGAl}{4}-\frac{EI}{l} & -\frac{kGA}{2} & \frac{kGAl}{4}+\frac{EI}{l} & \frac{kGAl}{3} \\ \frac{2kGA}{3} & \frac{kGAl}{3} & -\frac{2kGA}{3} & \frac{kGAl}{3} & \frac{4kGAl}{9}+\frac{16EI}{3l} \end{pmatrix} \\ \mathbf{f}^e &= \begin{pmatrix} \frac{ql}{2} & \frac{ql^2}{12} & \frac{ql}{2} & -\frac{ql^2}{12} & 0 \end{pmatrix}^T \end{aligned} \tag{30}$$

The load vector is identical to the load vector of the Euler–Bernoulli theory, except for the last 0. Since $\Delta\theta$ is inner degree of freedom associated with each element individually, we can reduce the stiffness matrix to 4×4 by exploiting the static condensation algorithm presented in the next chapter. In this case, $\mathbf{r}_z$ is equal to $\Delta\theta$. With ϕ previously defined, the element stiffness matrix can finally be simplified to the same result already shown in Eq. (17), assuming 2D case and deliberately omitting axial terms.

$$\mathbf{K}^e = \begin{pmatrix} \frac{12EI}{(1+\phi)l^3} & \frac{6EI}{(1+\phi)l^2} & -\frac{12EI}{(1+\phi)l^3} & \frac{6EI}{(1+\phi)l^2} \\ \frac{6EI}{(1+\phi)l^2} & \frac{(1+0.25\phi)4EI}{(1+\phi)l} & -\frac{6EI}{(1+\phi)l^2} & \frac{(1-0.5\phi)2EI}{(1+\phi)l} \\ -\frac{12EI}{(1+\phi)l^3} & -\frac{6EI}{(1+\phi)l^2} & \frac{12EI}{(1+\phi)l^3} & -\frac{6EI}{(1+\phi)l^2} \\ \frac{6EI}{(1+\phi)l^2} & \frac{(1-0.5\phi)2EI}{(1+\phi)l} & -\frac{6EI}{(1+\phi)l^2} & \frac{(1+0.25\phi)4EI}{(1+\phi)l} \end{pmatrix} \tag{31}$$

As the beam becomes 'thin' (the shear stiffness increases), the ratio $\phi \to 0$ yielding the element stiffness matrix exactly the same as that for conventional beam element. The element will not lock, thus it is the most effective Timoshenko beam element among the formulations considered in this paper.

6. Beam element stiffness with joint releases

If we construct the beam element shown in Fig. 2 in a way that it has a spatial hinge (which precludes the existence of all three bending moment components), then Eq. (6) can be written as

$$\begin{pmatrix} \mathbf{f}_{q,n} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{K}^e_{nn} & \mathbf{K}^e_{nz} \\ \mathbf{K}^e_{zn} & \mathbf{K}^e_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{q}_n \\ \mathbf{q}_z \end{pmatrix} \tag{32}$$

where the vector $\mathbf{q}_n^T = \langle q_1, \ldots, q_9 \rangle$ represents the local independent beam displacements (which also defines $\mathbf{f}_{q,n}$ as conjugate forces) whereas $\mathbf{q}_z^T = \langle q_{10}, q_{11}, q_{12} \rangle$ is the vector of dependent generalized displacements that are to be eliminated. The procedure for elimination of $\mathbf{q}_z$ is identical to static condensation [7,9].

$$\mathbf{f}_{q,n} = \widehat{\mathbf{K}}^e_{nn}\mathbf{q}_n, \quad \widehat{\mathbf{K}}^e_{nn} = \mathbf{K}^e_{nn} - \mathbf{K}^e_{nz}\mathbf{K}^{e-1}_{zz}\mathbf{K}^e_{zn} \tag{33}$$

and can be implemented as a partial decomposition of $\mathbf{K}^e$ matrix employing Gaussian elimination, which is described in [9] (see Section 8).

The proposed method to modify the element stiffness matrix in the presence of joint releases can be applied to any of the three versions of the stiffness matrix discussed in the previous section, i.e. $\mathbf{K}_v$, $\mathbf{K}_p$ or $\mathbf{K}_q$. For example, the joint release described by a spatial hinge has the completely identical effect on the modification of $\mathbf{K}_p$ matrix, as the one given in Eq. (19) for $\mathbf{K}_q$ (since the total rotation vector is decomposed into another component set). For a joint release of the local hinge type at the element end, one needs to modify $\mathbf{K}_v$ matrix by eliminating degrees of freedom v_2 (hinge around the local y axis) and v_4 (hinge around the local z axis) from Eq. (8).

For a joint release pertaining to shear force at an element end, which modifies the matrix $\mathbf{K}_p$, it is necessary to eliminate the generalized displacements p_8 and p_9 from Eq. (15). Following the same principle, in the matrix $\mathbf{K}_p$ we will obtain the joint release for axial force by eliminating the displacement p_7, whereas we will impose the joint release for torsional moment by eliminating the displacement p_{10}.

The structural analysis of frames with joint releases can again be performed using the previously described assembly procedure, however we have to employ the element stiffness matrices modified for the presence of joint releases. Symbolically it can be written

$$\widehat{\mathbf{K}} = \sum_e \widehat{\mathbf{K}}^e \tag{34}$$

With this procedure, we have eliminated the dependent displacements at the local level, i.e. at the element (beam) level, and we consider this to be more effective analysis approach for larger structural systems.

The alternative to the previous method for analysis of frames with joint releases implies the elimination of dependent generalized displacements at the global level, i.e. the modification of the structural matrix $\mathbf{K}$. Namely, the presence of release imposes different generalized displacements on one and the other side of the considered node. It is understood that this pertains only to the generalized displacements influenced by the release, for example two sets of rotations for the spatial hinge connection (see example). This approach considers every beam element as standard (without the presence of joint release) and the assembly is conducted according to Eq. (2). If we denote the basic generalized displacements of the joints with $\mathbf{r}_n$, and the additional set of generalized displacements stemming from presence of joint releases with $\mathbf{r}_z$, then the system equilibrium equation can be written

$$\begin{pmatrix} \mathbf{f}_r \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{K}_{nn} & \mathbf{K}_{nz} \\ \mathbf{K}_{zn} & \mathbf{K}_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{r}_n \\ \mathbf{r}_z \end{pmatrix} \tag{35}$$

where $\mathbf{f}_r$ represents the vector of generalized nodal forces. The procedure of static condensation can now be applied on the entire system

$$\mathbf{f}_r = \widehat{\mathbf{K}}_{nn}\mathbf{r}_n, \quad \widehat{\mathbf{K}}_{nn} = \mathbf{K}_{nn} - \mathbf{K}_{nz}\mathbf{K}^{-1}_{zz}\mathbf{K}_{zn} \tag{36}$$

The global stiffness matrix $\widehat{\mathbf{K}}_{nn}$ is identical to the global stiffness matrix in Eq. (34) obtained by the assembly procedure of the modified stiffness matrices, hence the resulting displacements are the same. The condensation of the stiffness matrix at the global level is achieved in the triangular decomposition phase of the Gauss elimination method. Since this is the most costly part, the gain is negligible compared to solving system of algebraic equations without any reduction (see Section 8 for illustration of these ideas).

7. Reduction method for invariant length constraint

The displacement method (DM) for frame analysis has pushed out numerous iterative methods from engineering practise. With the increase of computational power the solution of the algebraic system (with the typical size for frame structures) was not a major problem, which led to the displacement based frame models implemented in a number of computer programs. In planar DM analysis the degrees of freedom (DOF) for each node are two translations and one rotation.

However, this is no longer the case in technical displacement method (TDM) which assumes axially rigid rods, and thus reduces the number of independent displacements. The TDM remains popular in the structural mechanics courses, because of its simplicity for analysis by hand. Moreover, this kind of method can also improve the system conditioning if the axial deformation is significantly smaller from the flexural one, where one should rather exclude it completely. This is the main motivation for further elaborating upon the appropriate implementation of such reduced model within the standard framework of the finite elements method.

The assembly of the global stiffness matrix $\mathbf{K}$ in TDM for frames with orthogonal beams can easily be implemented reducing the transformation matrix $\mathbf{T}$ (6×6) to identity (4×4) since the unknown translations are either vertical or horizontal (see Eq. (24)).

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} \tag{37}$$

However, the solution of frames with inclined members is more complicated. We want to generalize the stiffness matrix computation based on the assembly procedure as follows. Let us consider a system with n nodes, e elements, r supports and c unknown rotations. The starting point is the well-known equation for assembly of the global stiffness matrix:

$$\mathbf{K} = \sum_e \mathbf{L}^{eT}\mathbf{T}^{eT}\mathbf{K}^e\mathbf{T}^e\mathbf{L}^e \tag{38}$$

The system to be solved in displacement method $\mathbf{Ku} = \mathbf{f}$ has therefore the following unknowns:

$$\mathbf{u} = (\boldsymbol{\theta} \quad \bar{\mathbf{u}})^T \tag{39}$$

where $\mathbf{u}(1 \times m)$ comprises unknown rotations θ and translations of free nodes $\bar{\mathbf{u}}(1 \times 2n - r)$. For technical displacement method $\mathbf{K}^e$ are (4×4) matrices, $\mathbf{T}^e$ are (4×6) transformation matrices and $\mathbf{L}^e$ are $(6 \times m)$ location matrices, where

$$m = c + 2n - r \tag{40}$$

Due to the assumption of axial rigidity, the total number of unknown translations n_{transl} is reduced to:

$$n_{transl} = 2n - e - r \tag{41}$$

Now we can write the global equilibrium condition and unknown displacements:

$$\mathbf{K}_{TDM}\mathbf{u}_{TDM} = \mathbf{f}_{TDM} \quad \mathbf{u}_{TDM} = (\theta \quad \Delta)^T \tag{42}$$

where Δ comprises unknown translations. Exploiting the basic theorem of rigid body kinematics that the projections of nodal displacements on the bar axis have to be the same, we obtain e homogeneous linear equations with $2n - r$ unknown projections on the global axes:

$$\mathbf{W}\bar{\mathbf{u}} = \mathbf{0} \tag{43}$$

The previous equation can be written as:

$$(\mathbf{W_1} \quad \mathbf{W_2})(\mathbf{u_1} \quad \Delta)^T = \mathbf{0} \tag{44}$$

where $\mathbf{u_1}$ is the vector of displacements to be eliminated through Δ. Elements of Δ are chosen such that the matrix $\mathbf{W_1}$ is regular. The first element of vector Δ will be set to 1 and all others to zero. The solution of these equations represents the first column of matrix $\overline{\mathbf{G}}$ which relates $\bar{\mathbf{u}}$ and Δ:

$$\overline{\mathbf{G}}\Delta = \bar{\mathbf{u}} \tag{45}$$

Other columns of $\overline{\mathbf{G}}$ will be obtained by successively setting next elements of Δ to 1. Assembly procedure for the global stiffness matrix for the technical displacement method is now defined by:

$$\mathbf{K}_{TDM} = \mathbf{G}^T\left(\sum_e \mathbf{L}^{eT}\mathbf{T}^{eT}\mathbf{K}^e\mathbf{T}^e\mathbf{L}^e\right)\mathbf{G} = \mathbf{G}^T\mathbf{K}\mathbf{G} \tag{46}$$

where

$$\mathbf{G} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{G}} \end{pmatrix} \quad \mathbf{u}_{DM} = \mathbf{G}\mathbf{u}_{TDM} \tag{47}$$

Nodal force vector is defined by:

$$\mathbf{f}_{TDM} = \mathbf{G}^T\mathbf{f} \tag{48}$$

The internal nodal forces for each element can be determined after the corresponding reduction is applied:

$$\mathbf{f}^e = \mathbf{K}^e\mathbf{T}^e\mathbf{L}^e\mathbf{G}\mathbf{u}_{TDM} \tag{49}$$

8. Numerical examples

8.1. Shear locking in Timoshenko beam element

As an example to illustrate shear locking phenomenon pertinent to Timoshenko beam element, we consider a 2D beam of variable length clamped at one end and free at the other (see Fig. 4). We will compare the calculated vertical displacements for different elements with the exact deflection of the cantilever tip equal to

$$v_{exact} = v_{EB} + v_{shear} = \frac{Fl^3}{3EI} + \frac{Fl}{kGA} \tag{50}$$

where v_{EB} is the displacement portion corresponding to the bending of Euler–Bernoulli beam. The analyzed beam is discretized using one element for each case.

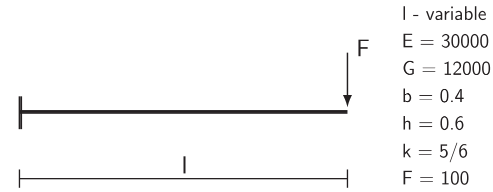

Fig. 4. Timoshenko beam – geometrical and mechanical properties with loading.

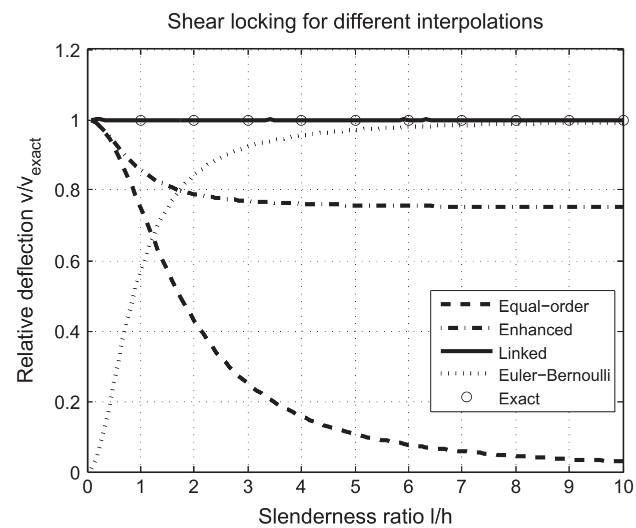

Fig. 5. Locking problem for different Timoshenko beam formulations.

The evaluated results are displayed in Fig. 5. From the diagram one can see that the element with linear shape functions (equal-order interpolation) yields acceptable results only for very deep structural elements. Otherwise, the element exhibits shear locking, being unable to adequately represent the deformation pattern in bending dominated problems. The displacement field enhanced with additional terms only partially solves the problem, resulting with cca. 20% stiffer response for slender elements. For comparison, Euler–Bernoulli beam element captures the relevant displacement in problems dominated by bending strain, however predicts poor results in shear dominated tasks. Finally, the beam element with linked interpolation successfully combines Euler–Bernoulli model for slender elements with the advantages of Timoshenko model for deep elements yielding accurate displacements regardless of l/h ratio.

8.2. Joint releases

In this section we present the results of several numerical simulations, which were carried out by the elements described herein and implemented in the computer program CAL [3]. In order to better understand the procedure, the list of used CAL commands is briefly explained in Appendix A. We analyze a simple 3D frame (see Fig. 6 for its properties), first for the case of moment release (hinge) placed at the elements' connection, second for the case of shear force release at the same location. Then we consider a 3D frame with larger number of degrees of freedom. Before advancing with the examples, we recall the number of operations needed for solving a system of algebraic equations by Gauss elimination (see [11]). Triangular decomposition is computationally the most expensive phase, dictating the total cost of the method to be approximately $\frac{2}{3}n^3$ for large n (exactly $\frac{2}{3}(n^3 - n) + \frac{5}{2}n(n-1) + n$). Also, the condensation of the stiffness matrix of order n to order $(n - k)$ is carried out by k steps of LU decomposition. If one element

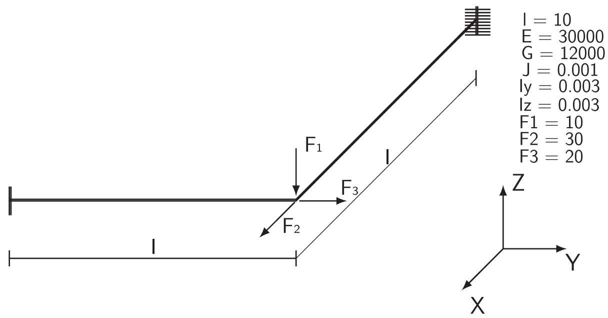

Fig. 6. Spatial frame: geometrical and mechanical properties with loading.

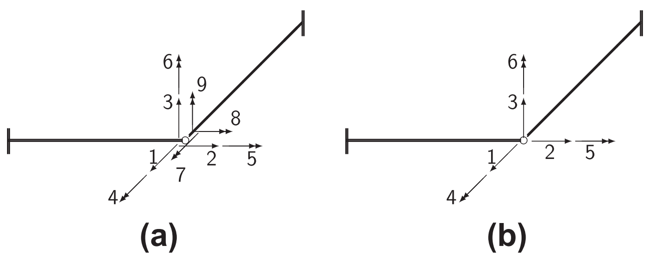

Fig. 7. Discrete frame models with moment releases: global degrees of freedom.

has s releases, we have taken it into account as s elements having 1 release. We remark that forward reduction and back substitution phases are quadratic functions of number of unknowns n.

8.2.1. Joint release for moments

In the case of joint release for moments, two analyses were conducted: first in which the components of the rotation vector left and right from the hinge in the given node were treated as independent degrees of freedom, and second in which we have modified the element stiffness matrix for the presence of joint release in one of the beams. In the first analysis, the global displacements 7–9 were eliminated at the structural level. For both models, the global displacements are marked in Fig. 7. In accordance with the previous explanations, the total number of operations required to perform the first analysis (global reduction) equals 540. On the other hand, the element reduction costs 980 operations which is much larger than the global reduction as well as the solution of the system without any reduction (equal to 669 exactly). We can conclude that both the element and the global reduction do not play significant role for small structural systems.

Next, CAL commands for solving the problem in Fig. 7a are given.

```
C FIGURE 7A-FRAME WITH
  MOMENT RELEASES
C NODAL COORDINATES
LOAD XYZ R = 3 C = 3
  10 0 0
  10 10 0
  0 10 0
C FORM ELEMENT
  STIFFNESS MATRICES
C (Ki(12 * 12))
FRAME3 K1 T1 E = 30000
  A = 0.16 J = 0.001
I = 0.003,0.003 G = 12000
  N = 1,2 P = 1,0
FRAME3 K2 T2 E = 30000 A
  = 0.16 J = 0.001
I = 0.003, 0.003 G = 12000
  N = 2,3 P = 1,0
C LOCAL - GLOBAL
  DISPLACEMENTS
LOADI IDT R = 2 C = 12
  0 0 0 0 0 0 1 2 3 4 5 6
  1 2 3 7 8 9 0 0 0 0 0 0
TRAN IDT ID
C ASSEMBLE GLOBAL
  STIFFNESS MATRIX
ZERO KK R = 9 C = 9
ADDK KK K1 ID N = 1
ADDK KK K2 ID N = 2
C CONDENSATION OF GLOBAL
STIFFNESS MATRIX
LOADI IDKT R = 1 C = 9
4 5 6 7 8 9 1 2 3
TRAN IDKT IDK
ZERO KKP R = 9 C = 9
ADDK KKP KK IDK N = 1
ZERO A R = 9 C = 1
SOLVE KKP A EQ = 3
DUPSM KKP KKC R = 6 C = 6 L =
4,4
C LOAD VECTOR
LOAD RT R = 1 C = 6
30 20 -10 0 0 0
TRAN RT R
C SOLVE SYSTEM OF
EQUILIBRIUM EQUATIONS
SOLVE KKC R
C DISPLACEMENTS
P R
RETURN
```

Now we display CAL code for solving the problem in Fig. 7b.

```
C FIGURE 7B-FRAME WITH
  MOMENT RELEASES
C NODAL COORDINATES
LOAD XYZ R = 3 C = 3
  10 0 0
  10 10 0
  0 10 0
C FORM ELEMENT
  STIFFNESS MATRICES
C (Ki(12 * 12))
FRAME3 K1 T1 E = 30000 A
  = 0.16 J = 0.001
I = 0.003,0.003 G = 12000
  N = 1,2 P = 1,0
FRAME3 K2 T2 E = 30000 A =
  0.16 J = 0.001
I = 0.003,0.003 G = 12000
  N = 2,3 P = 1,0
C STATIC CONDENSATION
  FOR
C BEAM ELEMENT 2
LOADI IDKT R = 1 C = 12
  4 5 6 1 2 3 7 8 9 10 11 12
TRAN IDKT IDK
ZERO A R = 12 C = 1
ZERO K2P R = 12 C = 12
ADDK K2P K2 IDK N = 1
SOLVE K2P A EQ = 3
C LOCAL–GLOBAL
DISPLACEMENTS
LOADI IDT R = 2 C = 12
0 0 0 0 0 0 1 2 3 4 5 6
0 0 0 1 2 3 0 0 0 0 0 0
TRAN IDT ID
C ASSEMBLE GLOBAL
STIFFNESS MATRIX
ZERO KK R = 6 C = 6
ADDK KK K1 ID N = 1
ADDK KK K2P ID N = 2
C LOAD VECTOR
LOAD RT R = 1 C = 6
30 20 -10 0 0 0
TRAN RT R
C SOLVE SYSTEM OF
EQUILIBRIUM EQUATIONS
SOLVE KK R
C DISPLACEMENTS
P R
RETURN
```

8.2.2. Joint release for shear force

Very similar to the previous procedure is the one that we implement in the presence of joint release (zero field) for shear force at the elements' connection (see Fig. 8).

In this case, one model is created such that the transverse displacement components are independent considering sections left and right from the shear force joint release. Displacement components 7 and 8 are eliminated using static condensation algorithm at the global stiffness matrix level. The second model, however, employs the special beam element which is enhanced for the presence of the shear force release. The listing of CAL commands implemented in the solution process of the problem described in Fig. 8a is given next:

```
C FIGURE 8A - FRAME
  WITH SHEAR FORCE
C RELEASE
C NODAL
  COORDINATES
LOAD XYZ R = 3 C = 3
  10 0 0
  10 10 0
  0 10 0
C FORM ELEMENT
  STIFFNESS
  MATRICES
C (Ki(12*12))
FRAME3 K1 T1 E =
  30000 A = 0.16 J =
  0.001
I = 0.003,0.003
  G = 12000 N = 1,2
  P = 1,0
FRAME3 K2 T2
  E = 30000 A = 0.16
  J = 0.001
C LOCAL - GLOBAL
  DISPLACEMENTS
LOADI IDT R = 2 C = 12
  0 0 0 0 0 0 1 2 3 4 5
  6
  7 1 8 4 5 6 0 0 0 0 0
  0
TRAN IDT ID
C ASSEMBLE GLOBAL
  STIFFNESS
  MATRIX
ZERO KK R = 8 C = 8
ADDK KK K1 ID N = 1
ADDK KK K2 ID N = 2
C CONDENSATION OF
GLOBAL STIFFNESS
MATRIX
LOADI IDKT R = 1 C = 8
3 4 5 6 7 8 1 2
TRAN IDKT IDK
ZERO KKP R = 8 C = 8
ADDK KKP KK IDK N = 1
ZERO A R = 8 C = 1
SOLVE KKP A EQ = 2
DUPSM KKP KKC R = 6 C =
6 L = 3,3
C LOAD VECTOR
LOAD RT
R = 1
C = 6
30 20 -10 0 0 0
TRAN RT R
C SOLVE SYSTEM OF
EQUILIBRIUM
EQUATIONS
SOLVE KKC R
C DISPLACEMENTS
P R
RETURN
```

As opposed to the previous, the listing of CAL commands used to solve the problem in Fig. 8b is

```
C FIGURE 8B - FRAME WITH
  SHEAR FORCE
C RELEASE
C NODAL COORDINATES
LOAD XYZ R = 3 C = 3
  10 0 0
  10 10 0
  0 10 0
C FORM ELEMENT
  STIFFNESS MATRICES
C (Ki(12*12))
FRAME3 K1 T1 E = 30000
  A = 0.16 J = 0.001
SOLVE K2P A EQ = 2
C LOCAL - GLOBAL
  DISPLACEMENTS
LOADI IDT R = 2 C = 12
0 0 1 4 5 6 0 0 0 0 0 0
0 0 0 0 0 0 1 2 3 4 5 6
TRAN IDT ID
C ASSEMBLE GLOBAL
  STIFFNESS MATRIX
ZERO KK R = 6 C = 6
ADDK KK K1 ID N = 1
ADDK KK K2P ID N = 2
I = 0.003,0.003 G = 12000
N = 1,2 P = 1,0
FRAME3 K2 T2 E = 30000
A = 0.16 J = 0.001
I = 0.003,0.003 G = 12000
N = 2,3 P = 1,0
C STATIC CONDENSATION
LOADI IDKT R = 1 C = 12
  2 3 1 4 5 6 7 8 9 10 11 12
TRAN IDKT IDK
ZERO A R = 12 C = 1
ZERO K2P R = 12 C = 12
ADDK K2P K2 IDK N = 1
LOAD RT R = 1   C = 6
C LOAD VECTOR
30 20 −10 0 0 0
TRAN RT R
C SOLVE SYSTEM OF
EQUILIBRIUM EQUATIONS
SOLVE KK R
C DISPLACEMENTS
P R
RETURN
```

The analysis output is given in Appendix B. One can see that for both cases of joint releases (moment and shear force) the corresponding displacement components calculated from model a and model b are identical.

8.2.3. 3D frame

In the next example we analyze a 3D frame from the aspect of number of necessary operations to solve the system, hence the geometrical and mechanical properties are of no interest. The frame consists of continuous beams interconnected with beams

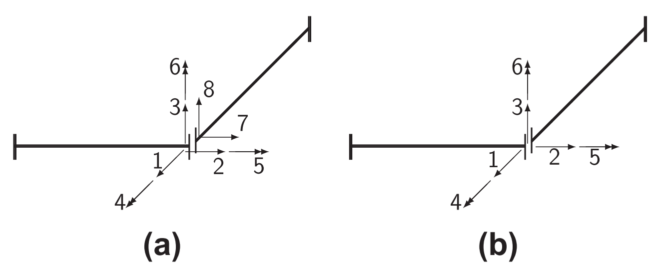

Fig. 8. Frame model with shear force releases: global degrees of freedom.

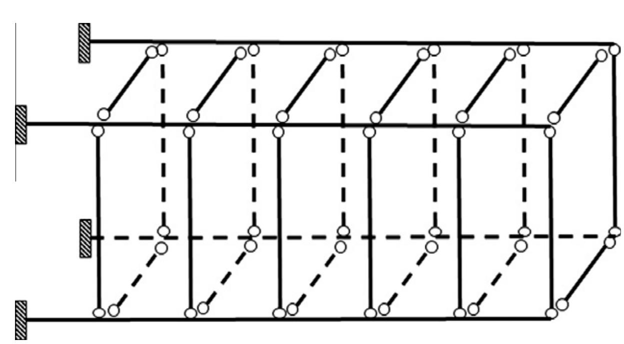

Fig. 9. 3 D frame with 288 DOFs without reduction.

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that have spatial hinges at both ends (see Fig. 9) and has a much higher number of degrees of freedom compared to the previous example. Here we have the total number of unknown displacements equal to 288, which, however, can be reduced to 144. The cost of solving the system without any reduction equals approximately 16×10^6, with the cost of global reduction slightly below. However, the element reduction costs only $\sim 2 \times 10^6$ operations and the advantage is obvious.

8.3. Global reduction to length-invariant beam model

As an example we look at the frame shown in Fig. 10 with the geometry and loads on the left and restrained independent translations on the right. Young's modulus is $E = 3 \times 10^7$ kN/m^2. Consequently, the displacement schemes are displayed in Fig. 11.

In contrast to the unknown displacements vector for displacement method (DM)

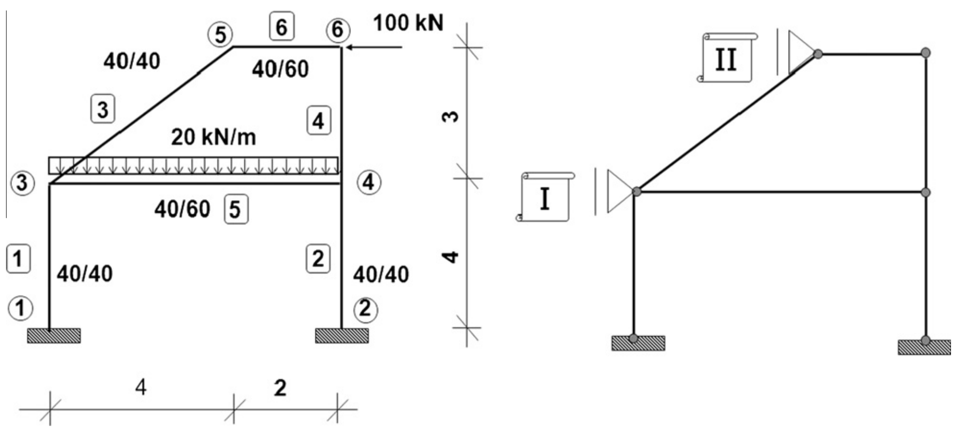

Fig. 10. Geometry and loads (left) and restrained translations (right).

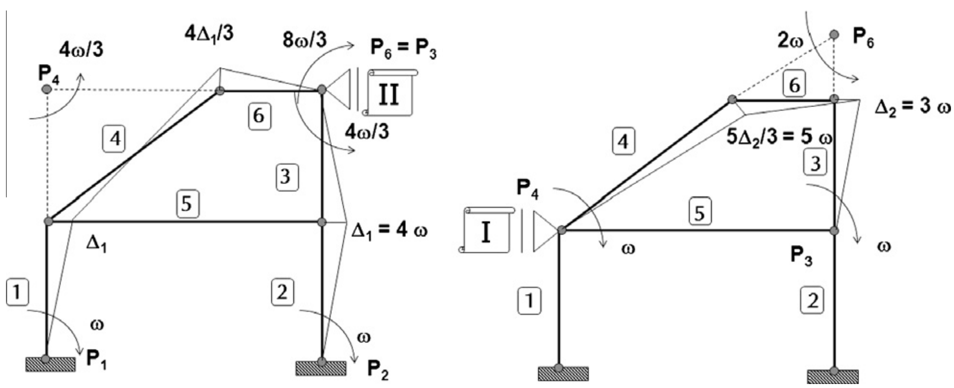

Fig. 11. Displacement scheme 1 (left) and scheme 2 (right).

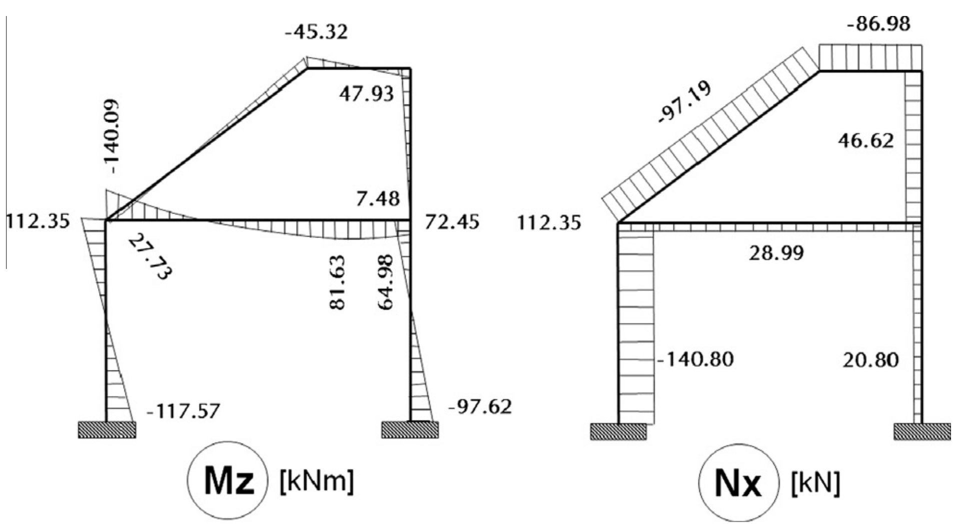

Fig. 12. Diagrams of bending moment (left) and normal force (right).

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$$\mathbf{u}_{DM} = (\,\theta_3 \quad \theta_4 \quad \theta_5 \quad \theta_6 \quad u_3 \quad v_3 \quad u_4 \quad v_4 \quad u_5 \quad v_5 \quad u_6 \quad v_6\,)^T \tag{51}$$

exploiting the axial rigidity constraint in technical displacement method (TDM) we wind up with the algebraic system with 6 unknowns that correspond to vector

$$\mathbf{u}_{TDM} = (\,\theta_3 \quad \theta_4 \quad \theta_5 \quad \theta_6 \quad u_4 \quad v_6\,)^T \tag{52}$$

Following the previously described procedure, some characteristic arrays for the analyzed example are given below:

$$\mathbf{u_1} = (\,u_3 \quad v_3 \quad v_4 \quad u_5 \quad v_5 \quad v_6\,)^T$$

$$\mathbf{W} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0.8 & 0.6 & 0 & 0 & -0.8 & -0.6 & 0 & 0 \end{pmatrix} \tag{53}$$

$$\mathbf{\Delta}_1 = (\,1 \quad 0\,)^T \quad \mathbf{\Delta}_2 = (\,0 \quad 1\,)^T$$

$$\overline{\mathbf{G}}^T = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 4/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4/3 & 1 & 0 \end{pmatrix}$$

In Fig. 12 we show the calculated bending moment and normal force diagrams.

9. Concluding remarks

In this work we have addressed several issues pertinent to constructing model refinements in materially and geometrically linear structural analysis of spatial frames. First we derived the element stiffness matrix starting from the reduced model pertaining to the deformation space, which allows to define the corresponding flexibility matrix, the inverse of the stiffness matrix. The proposed approach enables us to easily account for shear deformation in a more direct manner than the standard alternative based upon the displacement-based Timoshenko beam element, which requires non-conventional interpolations and the increase of computational cost (with additional degrees of freedom) to deliver the same quality of results. The merit of the proposed flexibility approach can further become clear when deriving the stiffness matrices of beam elements with varying cross section or Young's modulus, where standard cubic interpolation functions would produce rather poor results. However, the flexibility approach is based on the principle of virtual forces which assumes that there exists equilibrium between external and internal set of infinitesimal forces before the application of real loads/displacements. In the case of nonlinear analysis this principle cannot be used because the linear relationship between external and internal forces does not hold after the application of real loads. On the other hand, Timoshenko elements can be extended for material nonlinearity, but the additive decomposition of total rotations is not in accord with the multiplicative nature of large three-dimensional rotations (for possible extensions to geometrical nonlinearity see [13,14]). We have also addressed the corresponding model reduction issues with the element stiffness matrix in the presence of joint releases (zero fields) for moments and shear force. This is an alternative approach to doubling the corresponding nodal degrees of freedom for every release and it is applicable for nonlinear problems. The number of independent displacements is reduced compared to the standard beam element, thus yielding a system with lesser number of equations. In the case of structural system with many members, the element reduction is much cheaper than solving the system without condensation or with condensation at the global level. Hence we consider it to be computationally more effective. Additional model reduction pertaining strictly to linear analysis is discussed to provide the systematic computer implementation of technical displacement method with no change of element length. A detailed illustration is provided for implementation of the technical displacement method for frames with inclined members by exploiting the fundamental theorem from rigid body kinematics. Global reduction of the beam model which introduces length-invariance results in a system with fewer unknown displacements, and it could also be interesting for improving the system conditioning when the axial deformation remains very small.

Appendix A

A.1. Partial list of program CAL commands

LOAD M1 R = ? C = ?
load matrix M1 of real numbers with ‘R=’ number of rows and ‘C=’ number of columns
LOADI M1 R = ? C = ?
load array M1 of integer numbers
ZERO M1 R = ? C = ?
load array M1 with all entries equal to zero
P M1
list array named M1
TRAN M1 M2
transpose matrix M1 to form matrix M2
DUPSM M1 M2 R = ? C = ? L = L1,L2
duplicates submatrix M2 from location M1(L1,L2) M2 is NR x NC
SOLVE M1 M2 EQ=?
solve symmetric system of equations M1 x = M2; EQ = number of equations to be decomposed
FRAME3 Ki Ti E = ? A = ? I = I3,I2 J = ? G = ? N = Ni,Nj P = P1,P2
forms a (12×12) spatial beam element stiffness matrix; E is Young's modulus, A is cross-sectional area, I = I2,I3 are second moments of inertia around axes 2 and 3, J is torsional moment of inertia, N=Ni,Nj are node numbers of element start and end used to extract coordinates from the previously defined coordinate matrix XYZ, P = P1,P2 are also defined by XYZ and are used to specify the direction of z axis in the local coordinate system; the degrees of freedom are 3 translations and 3 rotations at the element start and 3 translations and 3 rotations at the element end
ADDK K Ki ID N = ?
assembly od element stiffness matrices Ki into the global stiffness matrix K; ID is the connectivity matrix, N specifies the column number in ID corresponding to the considered element

Appendix B. analysis results

Fig. 5a – frame with moment release

```
LOAD XYZ R = 3 C = 3
ARRAY NAME = XYZ NUMBER OF
  ROWS = 3
NUMBER OF COLUMNS =  3
FRAME3 K1 T1 E = 30000
  A = 0.16 J = 0.001
I = 0.003,0.003 G = 12000
  N = 1,2 P = 1,0
FRAME3 K2 T2 E = 30000
  A = 0.16 J = 0.001
I = 0.003,0.003 G = 12000
  N = 2,3 P = 1,0
LOADI IDT R = 2 C = 12
```

```
NUMBER OF COLUMNS = 9
TRAN IDKT IDK

ZERO KKP R = 9 C = 9
ADDK KKP KK IDK N = 1

ZERO A R = 9 C = 1

SOLVE KKP A EQ = 3

TOTAL SOLUTION OF Ax = B

DUPSM KKP KKC R = 6 C = 6
```

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```
ARRAY NAME = IDT NUMBER OF
  ROWS = 2
NUMBER OF COLUMNS = 12
TRAN IDT ID
ZERO KK R = 9 C = 9
ADDK KK K1 ID N = 1
ADDK KK K2 ID N = 2
LOADI IDKT R = 1 C = 9
ARRAY NAME = IDKT NUMBER OF
  ROWS = 1
COL# = 1
ROW 1 .62465E−01
ROW 2 .41643E−01
ROW 3 −18.519
L = 4,4
LOAD RT R = 1 C = 6
ARRAY NAME = RT NUMBER
OF ROWS = 1
NUMBER OF COLUMNS = 6
TRAN RT R
SOLVE KKC R
TOTAL SOLUTION OF Ax = B
P R
ROW 4 −2.7778
ROW 5 .00000
ROW 6 −.93697E−02
RETURN
```

Fig. 5b – frame with moment release

```
LOAD XYZ R = 3 C = 3
ARRAY NAME = XYZ NUMBER OF
  ROWS = 3
NUMBER OF COLUMNS =  3
FRAME3 K1 T1 E = 30000
  A = 0.16 J = 0.001
I = 0.003,0.003 G = 12000
  N = 1,2 P = 1,0
FRAME3 K2 T2 E = 30000
  A = 0.16 J = 0.001
I = 0.003,0.003 G = 12000
  N = 2,3 P = 1,0
LOADI IDKT R = 1 C = 12
ARRAY NAME = IDKT NUMBER OF
  ROWS = 1
NUMBER OF COLUMNS =  12
TRAN IDKT IDK
ZERO A R = 12 C = 1
ZERO K2P R = 12 C = 12
ADDK K2P K2 IDK N = 1
SOLVE K2P A EQ = 3
TOTAL SOLUTION OF Ax = B
LOADI IDT R = 2 C = 12
ARRAY NAME = IDT NUMBER OF ROWS = 2
NUMBER OF COLUMNS =  12
TRAN IDT ID
ZERO KK R = 6 C = 6
ADDK KK K1 ID N = 1
ADDK KK K2P ID N = 2
LOAD RT R = 1 C = 6
ARRAY NAME = RT NUMBER
OF ROWS = 1
NUMBER OF COLUMNS =  6
TRAN RT R
SOLVE KK R
TOTAL SOLUTION OF Ax = B
P R
COL# =  1
ROW 1 .62465E−01
ROW 2 .41643E−01
ROW 3 −18.519
ROW 4 −2.7778
ROW 5 .00000
ROW 6 −.93697E−02
RETURN
```

Fig. 6a – frame with shear force release

```
LOAD XYZ R = 3 C = 3
ARRAY NAME = XYZ NUMBER OF
  ROWS = 3
NUMBER OF COLUMNS =  3
FRAME3 K1 T1 E = 30000 A
  = 0.16 J = 0.001
I = 0.003,0.003 G = 12000 N =
  1,2 P = 1,0
FRAME3 K2 T2 E = 30000 A
ZERO A R = 8 C = 1
SOLVE KKP A EQ = 2
TOTAL SOLUTION OF Ax = B
DUPSM KKP KKC R = 6 C = 6
L = 3,3
LOAD RT R = 1 C = 6
ARRAY NAME = RT NUMBER
  = 0.16 J = 0.001
I = 0.003,0.003 G = 12000 N =
  2,3 P = 1,0
LOADI IDT R = 2 C = 12
ARRAY NAME = IDT NUMBER OF
  ROWS = 2
NUMBER OF COLUMNS =  12
TRAN IDT ID
ZERO KK R = 8 C = 8
ADDK KK K1 ID N = 1
ADDK KK K2 ID N = 2
LOADI IDKT R = 1 C = 8
ARRAY NAME = IDKT NUMBER OF
  ROWS = 1
NUMBER OF COLUMNS = 8
TRAN IDKT IDK
ZERO KKP R = 8 C = 8
ADDK KKP KK IDK N = 1
OF ROWS = 1
NUMBER OF COLUMNS = 6
TRAN RT R
SOLVE KKC R
TOTAL SOLUTION OF Ax = B
P R
COL# = 1
ROW 1 13.889
ROW 2 .41667E−01
ROW 3 −33.769
ROW 4 −4.9020
ROW 5 .00000
ROW 6 .34266E−15
RETURN
```

Fig. 6b – frame with shear force release

```
LOAD XYZ R = 3 C = 3
ARRAY NAME = XYZ NUMBER
  OF ROWS = 3
NUMBER OF COLUMNS = 3
FRAME3 K1 T1 E = 30000 A =
  0.16 J = 0.001
I = 0.003,0.003 G = 12000
  N = 1,2 P = 1,0
TRAN IDKT IDK
ZERO A R = 12 C = 1
ZERO K2P R = 12 C = 12
ADDK K2P K2 IDK N = 1
SOLVE K2P A EQ = 2
TOTAL SOLUTION OF Ax = B
LOADI IDT R = 2 C = 12
ARRAY NAME = IDT NUMBER
  OF ROWS = 2
NUMBER OF COLUMNS =  12
TRAN IDT ID
ZERO KK R = 6 C = 6
ADDK KK K1 ID N = 1
FRAME3 K2 T2 E = 30000 A
= 0.16 J = 0.001
I = 0.003,0.003 G = 12000 N
= 2,3 P = 1,0
LOADI IDKT R = 1 C = 12
ARRAY NAME = IDKT NUMBER
OF ROWS  = 1
NUMBER OF COLUMNS =  12
NUMBER OF COLUMNS =  6
TRAN RT R
SOLVE KK R
TOTAL SOLUTION OF Ax = B
P R
COL# =  1
ROW 1 13.889
ROW 2 .41667E−01
ROW 3 −33.769
ROW 4 −4.9020
ROW 5 .00000
```

(continued on next page)

```
ADDK KK K2P ID N = 2
LOAD RT R = 1 C = 6
ARRAY NAME = RT NUMBER
  OF ROWS = 1
ROW 6 .34266E−15
RETURN
```

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