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Spectral method in realistic modelling of bridges under moving vehicles

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ABSTRACT

This paper develops techniques for application of spectral method in dynamics of structures, particularly bridges under the moving loading. Spectral method is formulated in terms of matrix operators capable of discretizing and combining Kirchhoff–Love plates and Bernoulli beams. The boundary conditions have been imposed using Lagrange multipliers so combinations of Dirichlet and Neumann conditions could be dealt with. The proposed method allows a bridge to be represented as a 3D structure with realistic cross-section (ribs, box girders, etc.).

Moving loading can be any realistic model of a vehicle such as standard European three-axle road vehicle with 9 dynamic degrees of freedom. Vehicle behaviour is described with an additional system of differential equations that has to be solved together with a partial differential equation of the bridge model. Two different formulations and the corresponding methods of solution are presented: (i) formulation suitable for direct solution and (ii) formulation suitable for semi-direct (staggered) solution.

The proposed approach based upon the spectral matrix operators is sufficiently general to be suitable for dealing with strong form of any structure analysis problem. Example of dynamic analysis of a real bridge in Croatia illustrates the proposed method.

1. Introduction

Dealing with moving vehicles is very demanding tasks in structural analysis. One is facing a non-conservative problem in dynamic analysis of structures described as a system of differential equations in the case of discrete representation of structural masses, or a partial differential equation in the case of continuous mass representation.

When using the discrete model for computing the solution, the problem has to be discretized in space and time. Time discretization is usually based on finite difference technique leading to time integration scheme of the Newmark type (e.g. [1]). For space discretization two choices are mostly used, depending of the formulation chosen. The most popular choice is the finite element discretization that requires weak (integral) formulation of the problem [2]. The strong form (partial differential equation) is usually discretized using finite differences. Accuracy of finite differences can be improved if spectral analysis is applied to the strong form (like Fourier transforms, e.g. see [3]). In spite of their good properties spectral methods are just beginning to emerge in engineering applications. There is a similar situation regarding analysis of plates under the moving load; most authors prefer to model their structures as beams (e.g. [4,5,1]). However, there are formulations capable of dealing with 2 and 3 dimensional structures; quite a general formulation is presented in Andersen et al. [6]. They work in convected coordinates and with weak formulation of the problem.

The approach proposed in this paper is based upon the Chebyshev spectral method [7]. The main advantage of this method against the Fourier method is that there is no need for special representation (transformation) of loading. Moreover, we show that the Chebyshev spectral method can be somewhat modified, so that it can be completely formulated using matrix operators; thus in terms of solution steps it resembles the classical stiffness matrix approach used in structural analysis.

Boundary conditions are expressed through Lagrange multipliers and can be of Dirichlet type (representing restrictions on displacements) or Neumann type (representing restrictions on internal forces), or any combination of these two (as it is common in bridge analysis). This kind of approach also allows for nonholonomic conditions (e.g. boundary conditions involving velocity). Although they have not been treated in this paper, they could be important for moving load analysis [8]. One of the drawbacks of the Lagrange multipliers method when used for imposing the boundary conditions is the increase in size of the problem to be solved. However, spectral method requires quite modest number of equations. More important drawback is possible creation of a stiff

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system [9]. For certain boundary conditions, like partially supported and partially free plate, it is so pronounced that the resulting system can be so stiff that the penalty approach could not be used. However, some simple numerical tricks are enough to allow the use of standard equation solvers for the system of equations expanded with the Lagrange multipliers.

Loading moving over a bridge can be treated in several ways: (i) moving forces, (ii) moving masses, and (iii) moving vehicles (moving system with masses, springs and dampers). The later approach is the most realistic one and has advantages regarding accuracy [10]. The disadvantage is complexity of the model; bridge and vehicle are in interaction through contact points. When applying contact procedures the system matrix is usually time dependant since the position of each contact point changes over time (see e.g. [11]). As a result the system matrix must be updated and factorized at each time step. Usually, some assumptions are introduced to simplify the solution procedure, like vehicle and structure are permanently in contact and there is no sliding. This paper is based on realistic model of a vehicle, the standard European three-axle vehicle with 9 dynamic degrees of freedom and its minor simplifications. Two different formulations and the corresponding solution methods are presented: (i) formulation suitable for direct solution and (ii) formulation suitable for semi-direct (staggered) solution. The first one is more accurate but the later one offers better insight into vehicle's behaviour.

The paper has five chapters and is organized as follows. The second chapter presents Chebyshev spectral method using matrix operators and the Kronecker matrix product. Special attention is given to the thin plate equation and its combination with stiffeners (beams). There is description of boundary conditions that are entirely expressed in matrix form using Lagrange multipliers. In the end of the second chapter matrix properties are assessed using condition numbers. The third chapter describes the moving loading as a vehicle model and proposes solution procedures. The fourth chapter applies the presented procedure in analysis of a real bridge and compares the results with our previous analysis using different methods [12]. Bridge is described as a stiffened plate simply supported on two opposite sides and free on the other two sides. Such a plate is considered relevant since boundary conditions (arithmetic combination of Dirichlet and Neumann boundary conditions) are general enough to cover most practical cases.

2. Spectral method formulation

2.1. Spectral matrix operators

Spectral method has been chosen to replace the series expansion used in Fourier analysis solution of differential equations. Spectral methods offer high precision with minimal number of points used in spatial discretization.

In this work Chebyshev polynomials are chosen for spatial interpolation of the domain of the differential equation although there are other possibilities as well. Chebyshev polynomial is polynomial of degree N defined in points x_j according to the equation:

$$p(x) = \prod_{j=0}^{N}(x - x_j) \quad x_j = \cos\left(j \cdot \frac{\pi}{N}\right), \quad j = 0, 1, \ldots, N \tag{1}$$

Detailed description can be found in specialized literature (e.g. [13]). Here we will address some details specific to analysis of plates under the moving load.

We will apply the spectral method for solution of the strong formulation (differential equation) of structural (static or dynamic) problem. The method is formulated using matrix differentiation operators.

$$P_x = D_N \cdot p \tag{2}$$

where p is a vector of discrete data of size N, p_x is its derivative and D_N is matrix differential operator, a square matrix of size $[N \times N]$. Such matrix differential operator can be constructed for various methods (e.g. finite differences). For spectral method based on Chebyshev polynomials, entries of matrix D_N [7] are

$$\begin{aligned}
&D_{N_{00}} = \frac{2N^2+1}{6} \quad D_{N_{NN}} = -\frac{2N^2+1}{6} \\
&D_{N_{ij}} = \frac{-x_j}{2\left[1-(x_j)^2\right]}, \quad j = 1, \ldots, N-1 \\
&D_{N_{ij}} = \frac{c_i}{c_j} \cdot \frac{-1^{i+j}}{x_i - x_j} \quad i \neq j \quad j = 0, \ldots, N \\
&c_i = \begin{vmatrix} 2 & \text{if } i = 0 \vee i = N \\ 1 & \text{otherwise} \end{vmatrix}
\end{aligned} \tag{3}$$

Spectral method produces full differentiation matrix, which means that all points are involved in getting the result. One may argue that is the reason for high accuracy of spectral methods.

With this matrix at hand, the solution of an ordinary differential equation:

$$\frac{d}{dx}u = f(x) \tag{4}$$

reduces to a solution of the linear system of equations. In this paper solution of the system of linear equations will be formally represented by matrix inverse

$$u = D_N^{-1} \cdot f \tag{5}$$

where f is vector of $f(x)$ evaluated at points $x_j (j = 0, N)$. Certainly, Eq. (5) is more efficiently solved using some other procedure instead of the matrix inversion.

Boundary conditions also have to be incorporated in D_N. We can note the analogy where the matrix differential operator plays the role of the stiffness matrix, but it is produced using the entirely different procedure.

Normally, structural equations require higher derivatives and two dimensional modelling requires partial derivatives and modifications of matrix operators. In modelling of plates there is a need for second, third and fourth order derivatives in x and y directions. Higher order operators are simply produced using matrix multiplications while expansion in more than one direction can be obtained using Kronecker product of two matrices. In the case when there are N points in the x direction and M points in the y direction and numeration is consecutive in the x direction, differential operators are, respectively

$$D_x = I_M \oplus D_N, \quad D_y = D_M \oplus I_N \tag{6}$$

which are Kronecker products of unit matrices and differential operators. It is to be mentioned that the Kronecker product is not commutative. Also, if one matrix is a unit matrix, then the product simply rearranges and expands the original matrix. In our example $[N \times N]$ matrix times $[M \times M]$ matrix produces a matrix of the size $[NM \times NM]$.

2.2. Application to thin plate equation

The thin plate equation (where w stands for plate deflection) is

$$\Delta\Delta \mathrm{w} = \frac{\partial^4}{\partial x^4}\mathrm{w} + 2 \cdot \frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}\mathrm{w} + \frac{\partial^4}{\partial y^4}\mathrm{w} = q(x, y) \tag{7}$$

Eq. (7) is discretized using Chebyshev polynomials with N points in the x direction and M points in the y direction. Transferring into differential operator form and using matrices and products defined above, we get

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$$\Delta\Delta w = \left(I_M \oplus D_N^4\right) + 2\cdot\left(D_M^2 \oplus I_N\right)\left(I_M \oplus D_N^2\right) + \left(D_M^4 \oplus I_N\right) = Q_{xy} \quad (8)$$

Powers of differential operators (matrices) gives us higher derivatives.

Chebyshev polynomials (and differential operators) have evaluation points that are not evenly spaced; they are denser near the boundaries. The arrangement of points appeared not to be a problem. Solution of a simply supported plate using spectral method with discretization $N = M = 4$ produces deflections that are accurate to 0.1%. This has been achieved solving only nine equations. Since results are so accurate, interpolations could be used for evaluation of extra points within the domain.

2.2.1. Addition of stiffeners

It is straightforward to add beams as plate stiffeners when spectral method is formulated using matrix operator formalism since it allows the same treatment of equations as in the stiffness method. In spectral method beam stiffness reads $EI * D_N^4$ where D_N^4 is the same matrix operator as in the plate equation. Members of that $[N \times N]$ matrix are added to the appropriate members of the plate stiffness matrix of size $[NM \times NM]$ described with Eq. (8). Stiffness correction can be performed on the basis of Steiner theorem. Number of added beams and their position depend on the structure. In the same manner a shallow box can be formed while for high boxes additional type of (membrane) structure would have to be taken into account.

2.3. Boundary conditions using Lagrange multipliers

Only the simple boundary conditions could be incorporated into the matrix D_N (or D_M). For the simple support condition (homogeneous Dirichlet boundary condition) it consists of removing the first and the last rows and columns from the matrix. The resulting problem is reduced in size and homogenous condition is simply restored after the solution. Non-homogeneous boundary conditions could be obtained by introducing 1s on the diagonal and replacing the corresponding row and column with zeros. However, the Neumann type of boundary condition could scarcely be enforced directly into the matrix differential operator. Plate with two free ends requires more elaborated boundary conditions [14]. We require that the reaction and the moment along the free boundary y vanish

$$\frac{\partial^3}{\partial y^3} w + (2-\nu)\cdot\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial y} w = 0, \quad \frac{\partial^2}{\partial y^2} w + \nu\cdot\frac{\partial^2}{\partial x^2} w = 0 \quad (9)$$

These boundary conditions have to be translated into differential operators

$$\left(D_M^3 \oplus I_N\right) + (2-\nu)\cdot\left(D_M^2 \oplus I_N\right)(I_M \oplus D_N) = 0 \quad (10.a)$$

$$\left(D_M^2 \oplus I_N\right) + \nu\cdot\left(I_M \oplus D_N^2\right) = 0 \quad (10.b)$$

Differential operators from Eq. (10) act on the same points on the boundary (in each point on the boundary two conditions have to be fulfilled). That is the additional reason for application of the Lagrange multipliers in enforcing the boundary conditions. Operators in Eq. (10) have full size of the problem but they do not act on all the points of the plate. The extra points are removed through extraction of only those rows that belong to the degrees of freedom where the desired boundary condition is present. In our example that leaves us with two matrix operators of the size $[2N \times NM]$. Together with matrix operators resulting from the Dirichlet boundary conditions on the two supported sides of the plate they are assembled into constraint matrix C used in the Lagrange multiplier method

$$C = \begin{bmatrix} I_{PM} \\ I_{PN}\cdot\left[\left(D_M^3 \oplus +I_N\right) + (2-\nu)\cdot\left(D_M^2 \oplus I_N\right)(I_M \oplus D_N)\right] \\ I_{PN}\cdot\left[\left(D_M^2 \oplus I_N\right) + \nu\cdot\left(I_M \oplus D_N^2\right)\right] \end{bmatrix} \quad (11)$$

I_{PN} and I_{PM} are purging matrices for extraction of N and M points respectively and C is of size $[(2M + 2N + 2N) \times NM]$.

2.4. Assessment of the numerical characteristics of matrix operators

In order to asses quality of the solution condition number for spectral operators have been calculated. Among many possibilities we have chosen L1 norm based on singular values of a matrix. In contrast to condition numbers based on other norms it can be evaluated for singular matrices as well. Actually, we are dealing with nearly singular matrices but due to limited calculation capabilities the practical performance can be similar to singular matrices. In Table 1 there are condition numbers for some examples. Beam is simply supported beam, plate 4 is plate simply supported on four sides (with different aspect ratios) and plate 2 is plate simply supported on two sides (which is numerically a much more difficult example due to boundary conditions on free ends).

In the first column of Table 1 there is an explanation of boundary conditions: "Dirichlet" – homogenous boundary conditions imposed through removal of points where displacement values is known to be zero, "singular" – full sized matrix without any boundary conditions, it is singular, "r.b.m." – above matrix with removal of rigid body modes, not singular anymore, "Lagrange" – r.b.m. matrix expanded with boundary conditions and Lagrange multipliers, "Penalty" – r.b.m. matrix with boundary conditions imposed through penalty number (α from 10^8 to 10^{10}).

Note: condition number alone cannot give the whole picture about matrix usability (e.g. [15]). It is evident from Table 1 that

Table 1
Sizes and condition numbers of spectral stiffness matrices for various structures.

Beam	Plate 4 1:1	Plate 4 1:5	Plate 2 1:1	Plate 2 1:5
Dirichlet				
$N = 49$ 1.493×10^{10}	$N = 961$ 4.263×10^{8}	$N = 713$ 5.025×10^{7}	n.a.	n.a.
Singular				
$N = 51$ 7.761×10^{20}	$N = 1089$ 7.411×10^{19}	$N = 825$ 1.362×10^{20}	$N = 1089$ 3.797×10^{20}	$N = 825$ 1.053×10^{20}
r.b.m.				
$N = 51$ 1.750×10^{20}	$N = 1089$ 1.633×10^{21}	$N = 825$ 4.480×10^{19}	$N = 1089$ 9.66×10^{19}	$N = 825$ 8.853×10^{19}
Lagrange				
$N = 55$ 8.804×10^{15}	$N = 1345$ 1.226×10^{27}	$N = 1049$ 1.813×10^{27}	$N = 1353$ 7.234×10^{33}	$N = 1057$ 2.926×10^{32}
Penalty				
$N = 51$ 1.235×10^{17}	$N = 1089$ 2.889×10^{16}	$N = 825$ 2.630×10^{16}	$N = 1089$ 5.194×10^{20}	$N = 825$ 4.393×10^{20}

the removal of rigid body modes can worsen the condition number but at the same time matrix becomes non-singular and can be used in calculations. Also, introduction of Lagrange multipliers seem to deteriorate the performance but it is only the multiplier part of the matrix that is badly conditioned, the rest behaves well and the results are quite acceptable. Introduction of techniques that would separate Lagrange multipliers from the rest of the matrix would further confirm this statement. For example, the condition number for plate 2 is smaller for penalty method then for Lagrange multipliers method, but penalty procedure gives completely wrong results in eigenvalue analysis.

3. Moving loading description

3.1. Moving loading equation

By applying D'Alembert's principle differential equation describing two dimensional structural behaviour under the moving force is obtained.

$$\rho \cdot \frac{\partial^2}{\partial t^2} u + \frac{1}{K} \cdot \left(\frac{\partial^4}{\partial x^4} u + 2 \cdot \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} u + \frac{\partial^4}{\partial y^4} u \right) = p(t) \cdot \delta(x - v_x t) \cdot \delta(y - v_y t) \tag{12}$$

$u = u(x, y, t)$ is displacement in space and time, ρ is surface density, K is plate stiffness, δ is Dirac function. Load description can be simplified if we assume that it moves along one axis only. Using spectral operator (Eq. (8)) for space discretization and assuming load is moving along one coordinate only, we can write using sub matrices

$$\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \cdot \frac{\partial^2}{\partial t^2} \begin{pmatrix} u \\ \lambda \end{pmatrix} + \begin{pmatrix} \Delta\Delta & C^T \\ C & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} F(x,t) \\ 0 \end{pmatrix} \tag{13}$$
$$F(x,t) = p(t) \cdot \delta(x - v_x \cdot t)$$

M is mass matrix, 0s are zero matrices of the appropriate size, $\Delta\Delta$ is spectral operator playing the role of the stiffness matrix, λ is vector of Lagrange multipliers, u is the displacement and $F(x, t)$ is a loading function (known in the case of moving forces or unknown in the case of moving vehicle). The main characteristic of the moving load problem described with Eq. (13) is the right hand side. It is convenient to perform loading discretization prior to the time integration procedure (after load path and time integration parameters, like Δt are set). This has been recognized by many authors, like Ju and Lin [16] who are identifying loaded nodes prior to analysis.

After the dynamic equation has been discretized in space and time, its solution can proceed using any time integration scheme. (Note: space discretization has to obey properties of the Dirac δ function.) That results with a requirement of a constant force within one time increment! Also, it is straightforward to implement a non-constant force (like in [4]) since there are no restrictions on the forcing function $p(t)$.

Note that due to the Chebyshev polynomials used in space discretization even the constant loading does not have equal amplitude in every time increment (constants in Chebishev coordinates are not constant when transformed back into structure space–time coordinates). Also, other forms of moving load such as force changing in time are allowed by Eq. (13).

After the appropriate form for the right hand side (the loading) has been found, solution procedure can be applied. In order to apply Newmark class of integration schemes the governing Eq. (13) has to be rewritten in the incremental form:

$$\begin{pmatrix} M & 0 \\ 0 & 0^{-20} \end{pmatrix} \cdot \frac{\partial^2}{\partial t^2} \begin{pmatrix} \delta u \\ \delta\lambda \end{pmatrix} + \begin{pmatrix} \Delta\Delta & C^T \\ C & 0^{-20} \end{pmatrix} \cdot \begin{pmatrix} \delta u \\ \delta\lambda \end{pmatrix} = \begin{pmatrix} \delta F \\ 0 \end{pmatrix} \tag{14}$$

where δu is the displacement increment and δF is loading increment calculated from the discretized loading function. In cases when δF is not known (moving vehicle) or when accurate value of accelerations is required (moving masses) a modified Newmark method that obeys force impulse balance could be advantageous [17].

Mass and "stiffness" matrices could be rather stiff for some boundary conditions and introduction of 0^{-20} (very small number in place of 0s) often can improve the behaviour of numerical procedures (especially eigenvalue analysis). Additionally $\Delta\Delta$ matrix can be stabilized by introduction of rigid body modes [18]. In our case numerical procedures in MathCAD 13 performed well with 0s but those in MathCAD 11 required such a trick (but only for eigenvalue analysis).

3.2. Moving vehicle

Realistic vehicle model plays an important role in obtaining accurate results [12] of dynamic bridge analysis. Usually a vehicle model consists of a rigid body with a mass M_V representing the vehicle itself, supported on springs k_i with or without mass m_{ik} representing suspension and axles of the vehicle and additional springs k_{ik} representing tyres that are in contact with the structure.

One such vehicle model is standard European 3 axle road vehicle with 9 d.o.f. presented in Fig. 1. It should be mentioned that this type of model is valid for road vehicles since railway vehicles have some other characteristics that have to be taken into account (e.g. non-sprung masses in contact with the bridge, etc.).

Choice of freedoms and an arrangement of vehicle equations are very important for proper formulation of the solution procedure. Basic degrees of freedom of the vehicle from Fig. 1 are three rigid body modes: vertical movement Z (bouncing) and rotations θ (pitch) and α (roll). Additional degrees of freedom are vertical displacements on each line of springs (corresponding to wheels): z_i (at the connection of spring and the vehicle), z_{ik} (corresponding to mass representing the axle) and z_{ip} (corresponding to contact between the tyre and the structure). This is 18 d.o.f. in total but that can be reduced considering kinematic relations and assuming permanent contact between wheels and the structure which leaves only 6 z_{ik} unknowns.

Equations describing the vehicle behaviour are obtained by applying D'Alambert's principle of equilibrium. Very clear insight is obtained if contact forces are used in assembly of the equations. The first three equations describe equilibrium of rigid body movements $\Sigma Z = 0$, $\Sigma M_\theta = 0$, $\Sigma M_\alpha = 0$. Additional equations describe dynamic equilibrium along each line of springs

$$M_V \frac{d^2 z_{ik}}{dt^2} - k_{ik}(y_i - z_{ik}) - k_i(z_{ik} - Z - z_i) = 0 \tag{15}$$

In the above equation it has been assumed that vehicle displacement z_{ip} correspond to the structural displacements y_i. It is straightforward to incorporate surface roughness into the model simply by adding it to the structure displacement y. Solution procedures can be formed in several ways, maybe the simplest one consisting in separation of variables in Eq. (15)

$$M_V \frac{d^2 z_{ik}}{dt^2} - K_V Z = K_{VS} y + F \tag{16}$$

where K_V is stiffness corresponding to the vehicle and K_{VS} stiffness corresponding to the contact vehicle – structure and F are eventually some additional forces. If K_{VS} is kept on the right hand side (as in Eq. (16)) one has iterative procedure where y are assumed to be known (calculated before). If K_{VS} is kept on the left hand side then one has direct procedure, all equations have to be combined into one system and solved simultaneously.

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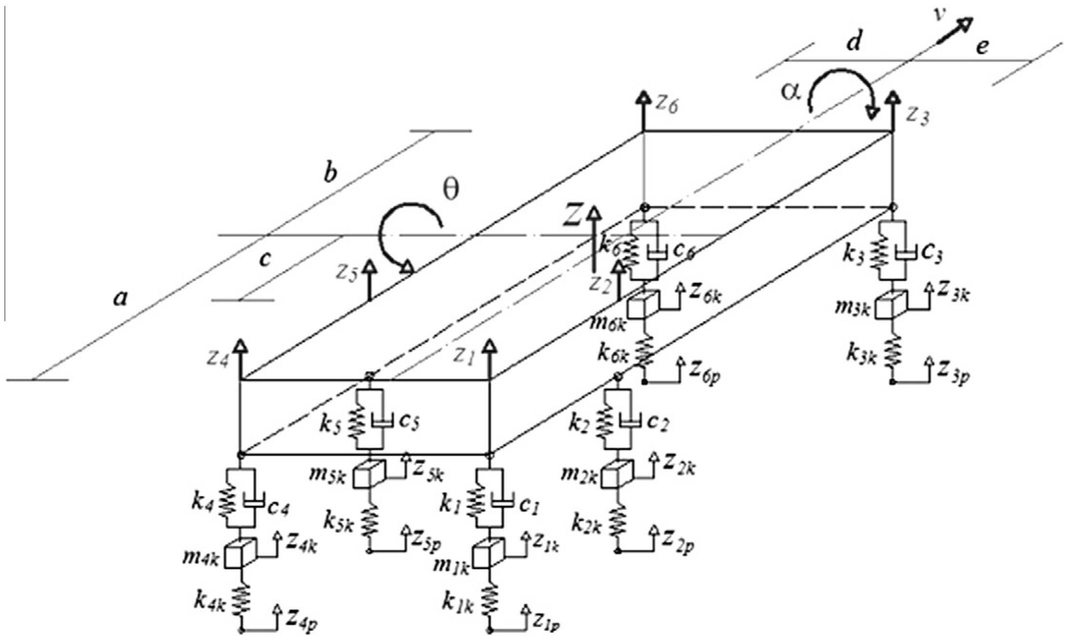

Fig. 1. European vehicle model with 9 d.o.f.

Equations pertaining to the vehicle used will be fully given in the examples.

3.3. Solution procedures

Above system of differential equations describing the vehicle has to be combined with a partial differential equation describing the structure. The resulting system can be solved either directly or in a staggered manner: first structure then vehicle iterating until contact forces are within the prescribed tolerance. Both approaches have advantages and disadvantages; direct solution is more accurate and faster in cases of more than one vehicle. In that case impulse acceleration method [17] is advantageous since it readily enables different integration parameters on different time steps for the structure and the vehicle. Iterative process is fast when there is not much interaction between structure and vehicle (only one vehicle, light vehicle on stiff and heavy structure, etc.). In cases of sufficiently small time steps, the iterative procedure might even not be needed [16]. Also, iterative process has an advantage of showing vehicle's behaviour and whenever is that information required, it should be used. Clear insight into vehicle's behaviour is obtained using Eq. (16) and with iterations performed in a staggered manner between the structure alone and the vehicle(s).

4. Numerical example

It is not so often that meshless methods are used for engineering analysis so a real bridge loaded with a real truck has been chosen as an example to illustrate the presented procedure. Details of the structure are given in Fig. 2.

The bridge in Fig. 2 is the same as analyzed in Torić Malić and Kožar [12] so that the results could be compared. However, the vehicle is not quite the same as it has been somewhat simplified in this example: (i) axle mass has been removed since it has been established that it's contribution is not decisive when combined with hard springs and (ii) two rear axles have been combined into one (added).

Equations describing the vehicle are

$$M_V \cdot \frac{d^2}{dt^2} z - M_V \cdot g - P_1 - P_2 - P_3 - P_4 = 0 \tag{17}$$

$$I_\theta \cdot \frac{d^2}{dt^2} \theta + (P_1 + P_3) \cdot a - (P_2 + P_4) \cdot b = 0 \tag{18}$$

$$I_\alpha \cdot \frac{d^2}{dt^2} \alpha + (P_3 + P_4) \cdot c - (P_1 + P_2) \cdot d = 0 \tag{19}$$

Eq. (17) represents force equilibrium, Eq. (18) moment equilibrium about transversal axis of the vehicle and Eq. (19) about the longitudinal axis. This system of ordinary differential equations has to be solved for vehicle displacements and new contact forces P_1, P_2, P_3 and P_4 can be determined. In the above equations contact forces are

$$\begin{aligned} P_1 &= k_1 \cdot (y_1 - z - z_1) \\ P_2 &= k_2 \cdot (y_2 - z - z_2) \\ P_3 &= k_3 \cdot (y_3 - z - z_3) \\ P_4 &= k_4 \cdot (y_4 - z - z_4) \end{aligned} \tag{20}$$

Applying kinematic conditions

$$\begin{aligned} z_1 &= b \cdot \theta - d \cdot \alpha \\ z_2 &= -a \cdot \theta - d \cdot \alpha \\ z_3 &= b \cdot \theta + c \cdot \alpha \\ z_4 &= -a \cdot \theta + c \cdot \alpha \end{aligned} \tag{21}$$

reduces the number of unknowns to only three rigid body movements (assuming support movements y_1, y_2, y_3 and y_4 are known). It suffices to solve the system of Eqs. (17)–(19). The result is 3D vehicle model with 3 d.o.f. Table 2 gives the parameters of the structure and Table 3 dimensions and parameters of the vehicle.

Other analysis parameters are: $\Delta t = 0.002$ s., number of time steps $m = 550$ so the total analysis time is 1.1 s. Vehicle velocity in the example was $v_0 = 30.0$ m/s so that whole bridge was passed over by both axles (distance between the axles $\Delta l = 3.0$ m).

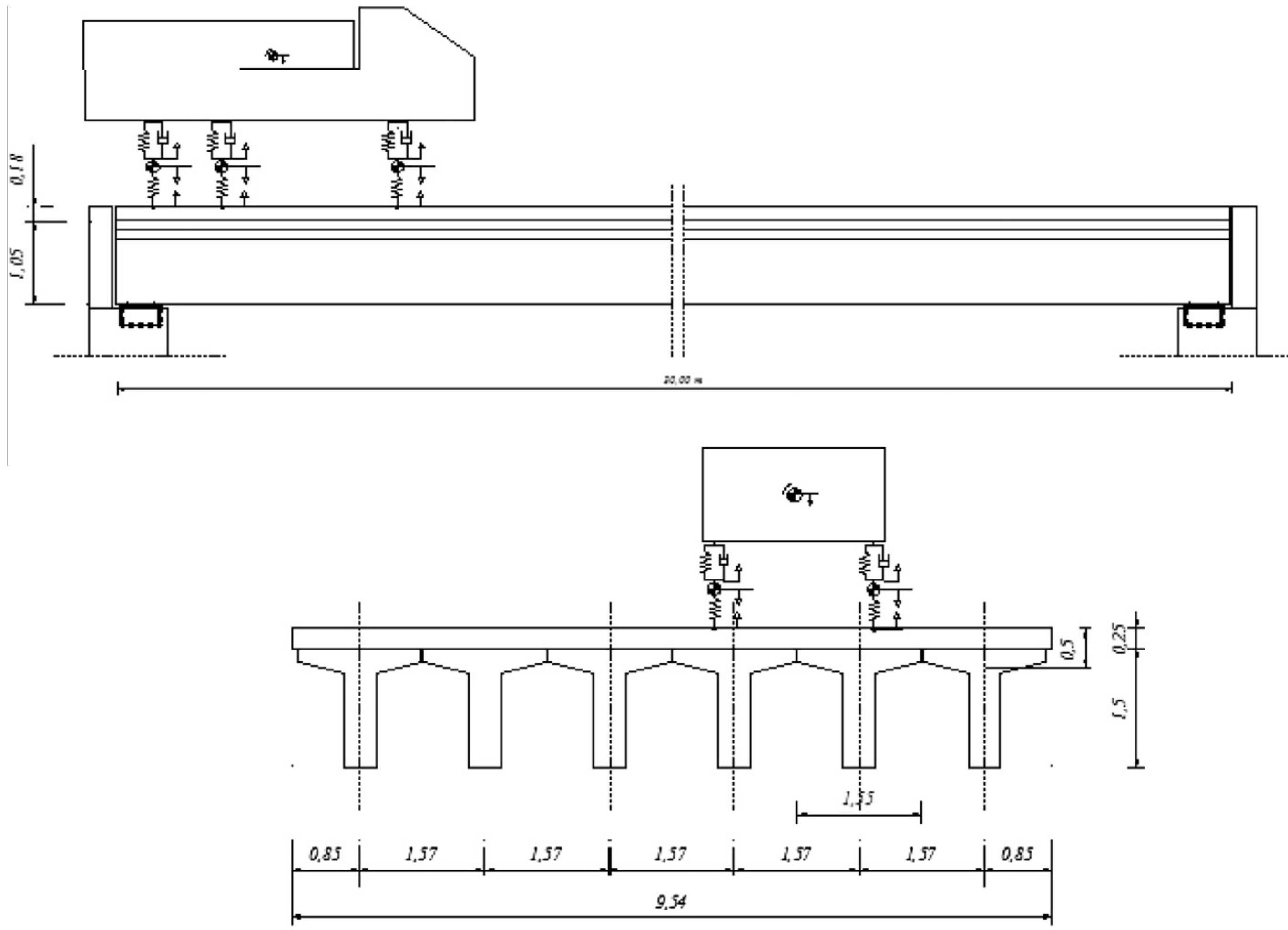

Fig. 2. Real bridge used for analysis.

Table 2
Parameters of the structure.

Model	L (m)	b (m)	h (m)	E (kN/m^2)	ρ (kg/m^3)	ν	I (m^4)
Stiffened plate	30.0	10.0	0.7	3.5×10^7	2500.0	0.2	2.66

Table 3
Dimensions and parameters of the road vehicle used in the analysis.

Dimensions	a (m)	b (m)	d (m)	e (m)
3D model	1.0	2.0	0.75	0.75
Parameters	Axle stiffness front (kN/m)	Axle stiffness back (kN/m)	I_θ (kg m^4)	I_α (kg m^4)
3 d.o.f.	15.0×10^5	20.0×10^5	22.50	5.625

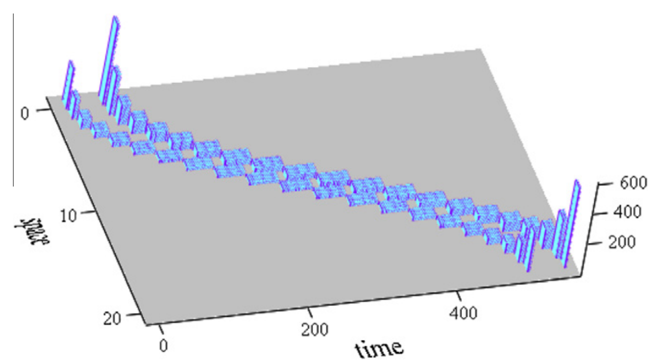

Fig. 3. Contact forces acting on the bridge in the first iteration.

Structure–vehicle interaction has been calculated using iterative method since that gives a better insight into the solution process.

Contact forces acting on the bridge in the first iteration are calculated according to Eq. (13) and are shown in Fig. 3 (for the sake of clarity only front and rear forces on the left side are shown). It is interesting to observe that in this iteration the two forces are constant in time, they only appear to be changing because they are shown in Chebyshev coordinates. In later iterations when contact forces are not constant in time the changes will be greater and completely irregular.

4.1. Results for the bridge

Bridge results consist of the deflections in time from which the stress resultants are calculated. Eq. (14) is solved and Fig. 4 shows bridge displacements at 75th time increment magnified 100,000

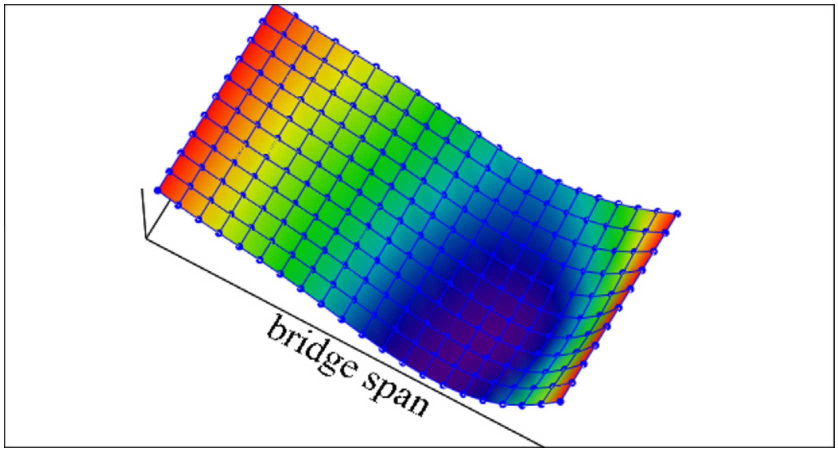

Fig. 4. Displacements of the bridge at 75th time increment.

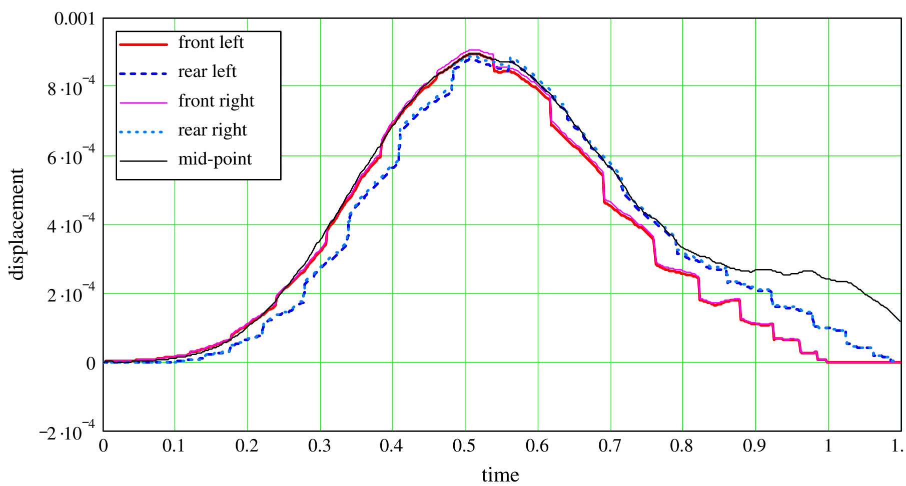

Fig. 5. Bridge displacements in time.

times. That is 0.15 s of the analysis and the first vehicle axle is at 4.5 m from the edge of the bridge.

Fig. 4 is not very practical for presentation of results, mid-point deflection in time is used instead. Also, in the case of iterative calculation of the vehicle–bridge interaction deflections at contact points are of importance as well. One should bear in mind that it is not so straightforward to extract contact displacements since they change in time and move in space. Vehicle speed determines the convected coordinate system that connects general time and space coordinates and results are shown in Fig. 5 as displacements in time.

It is visible form Fig. 5 that the left and right side of the vehicle do not differ much. Sudden changes in vehicle displacements are consequence of use of the Dirac function in forces description (the only correct description). Also, bridge mid-point has time shift due to large inertial forces.

4.2. Results for the vehicle

Vehicle results consist of rigid body movements Z, θ and α and contact forces between the vehicle and the structure. Usually one is interested in vehicle results in special cases, e.g. there is some measuring equipment on the vehicle or damage detection is performed [19].

Resulting structure displacements from Eq. (14) are substituted into Eqs. (17)–(19) that are solved for vehicle displacements. Fig. 6 shows vertical vehicle displacement (bouncing). It is evident that due to it is much smaller mass vehicle really bounces on the bridge. Fig. 7 show rotations of the vehicle (pitch and roll).

After vehicle displacements are known contact forces are recalculated using Eqs. (20) and (21). Fig. 8 shows front wheel contact forces and Fig. 9 rear wheel contact forces.

There is a number of results in Figs. 8 and 9. Thin line is resulting directly from Eqs. (17)–(19) and it is evident that forces oscillate very much. On the other hand if one seeks the median of the resulting data is very close to the starting value of contact forces: 46.24 kN left and 49.36 right versus initial 50 kN for the front wheels and 98.63 kN left and 98.95 kN right versus initial 100 kN for the rear wheels. The resulting forces change to much in time for calculation purposes so they are smoothed using median smoothing with window of 19 data points. Figs. 8 and 9 show smoothed contact forces from the first (previous) iteration and how they change from the next iteration. The smoothed contact forces are used as structure loading in the next iteration. The form of the resulting contact forces brings up the question of convergence criterion. Namely, it is evident that it will be difficult to obtain small differences in forces, especially for longer simulations since oscillations increase with time. As a consequence, the greatest difference in median is used as a convergence criterion.

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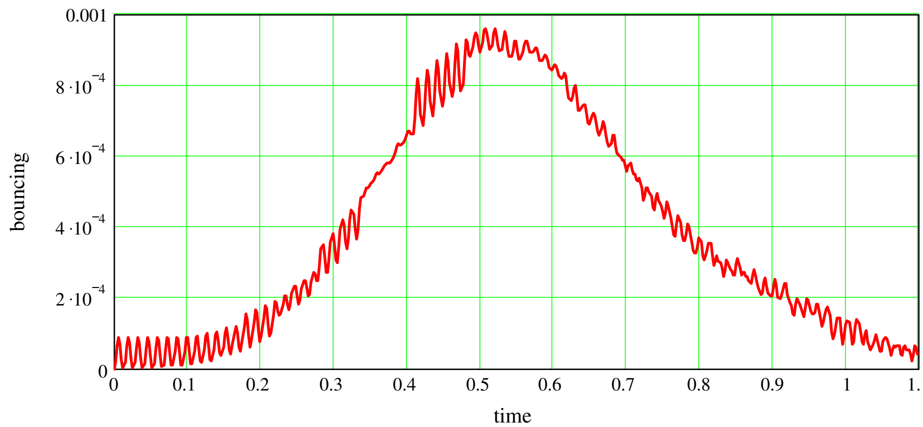

Fig. 6. Bouncing of the vehicle over the bridge.

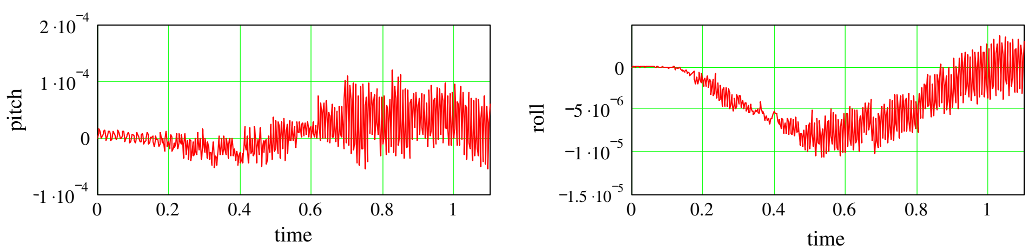

Fig. 7. Pitch and roll of the vehicle over the bridge.

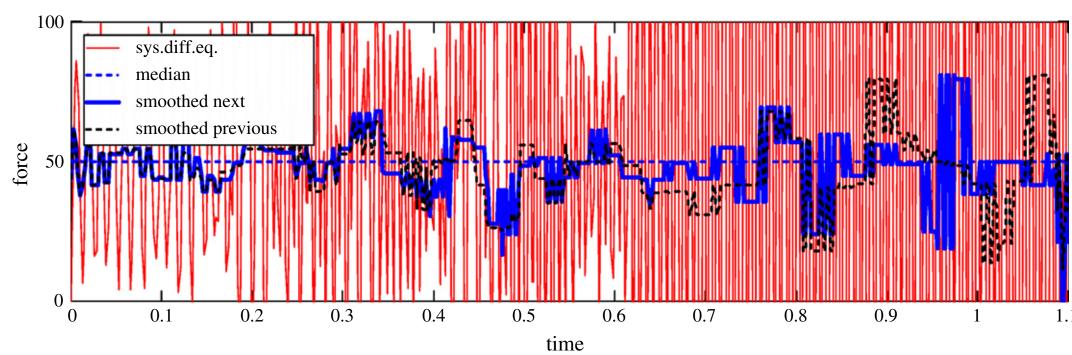

Fig. 8. Contact forces at front left wheel.

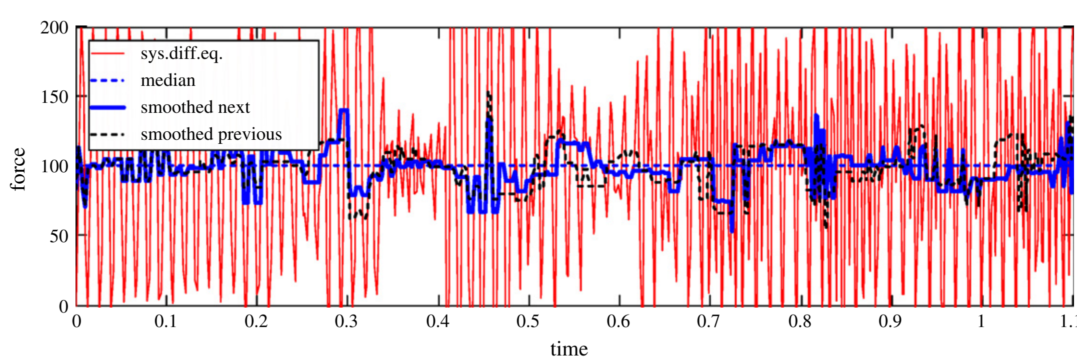

Fig. 9. Contact forces at rear left wheel.

5. Conclusions

Spectral methods are not often used for dynamic analysis of real engineering structures. This paper presents a procedure for realistic analysis of a class of road bridges loaded with moving vehicles. The space discretization is obtained by the Chebyshev spectral method that is implemented in form of a matrix differential operator. Resulting discretization in its form resembles the stiffness matrix and time discretization can be performed using any suitable method. The spectral differential matrix operator is fast and simple to construct; it is sufficient to construct one dimensional operator and expand it into two or more dimensions using the formalism of matrix Kronecker product. The resulting matrix is rather dense and small in size since spectral methods achieve high accuracy with a modest number of points. Owing to the use of matrix operators it is easy to combine various load carrying elements (like plates and beams) into one structure so that complex objects could be analyzed. Boundary conditions can be treated in several ways but Lagrange multipliers offer the most general approach suitable even for the most complicated boundary conditions. Vehicle loading is modelled in a general and realistic way with an arbitrary number of local degrees of freedom. Presented method of combination of local (vehicle) and global (structure) degrees of freedom allows for iterative procedure that is simple and efficient and gives good insight into both structure and vehicle behaviour.

The proposed procedure has been tried on a real example of a bridge loaded with a moving vehicle. Obtained results compare well with other methods of analysis and demonstrate realistic contact forces between the structure and the moving vehicle. The matrix operator formalism of the spectral method produces accurate results while retaining small size of the problem; it is very suitable for integration of all strong forms of engineering problems.

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