Interpolation of DCMs

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1 Logarithmic maps for DCMs

For any direction cosine matrix (DCM), Λ, the logarithmic map for the matrix is a skew-symmetric matrix, λ:

$$\lambda = \log(\Lambda) = \begin{bmatrix} 0 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & 0 & \lambda_1 \\ \lambda_2 & -\lambda_1 & 0 \end{bmatrix} \tag{1}$$

2 Matrix exponentials

The angle of rotation for the skew-symmetric matrix, λ is

$$\theta = \|\lambda\| = \sqrt{{\lambda_1}^2 + {\lambda_2}^2 + {\lambda_3}^2} \tag{2}$$

The matrix exponential is

$$\Lambda = \exp(\lambda) = \begin{cases} I & \theta = 0 \\ I + \frac{\sin\theta}{\theta}\lambda + \frac{1-\cos\theta}{\theta^2}\lambda^2 & \theta > 0 \end{cases} \tag{3}$$

3 Solving for λ

If the logarithmic map and matrix exponential are truly inverses, we need

$$\exp(\log(\Lambda)) = \Lambda. \tag{4}$$

Using the expression for λ from Equation 1, we get

$$\exp\left(\begin{bmatrix} 0 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & 0 & \lambda_1 \\ \lambda_2 & -\lambda_1 & 0 \end{bmatrix}\right) = \Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{bmatrix} \tag{5}$$

Doing a little algebra for $\theta > 0$, Equation 5 becomes

$$\Lambda = \begin{bmatrix} 1 - \frac{1-\cos\theta}{\theta^2}\left(\lambda_3^2 + \lambda_2^2\right) & \frac{\sin\theta}{\theta}\lambda_3 + \frac{1-\cos\theta}{\theta^2}\lambda_1\lambda_2 & -\frac{\sin\theta}{\theta}\lambda_2 + \frac{1-\cos\theta}{\theta^2}\lambda_1\lambda_3 \\ -\frac{\sin\theta}{\theta}\lambda_3 + \frac{1-\cos\theta}{\theta^2}\lambda_1\lambda_2 & 1 - \frac{1-\cos\theta}{\theta^2}\left(\lambda_3^2 + \lambda_1^2\right) & \frac{\sin\theta}{\theta}\lambda_1 + \frac{1-\cos\theta}{\theta^2}\lambda_2\lambda_3 \\ \frac{\sin\theta}{\theta}\lambda_2 + \frac{1-\cos\theta}{\theta^2}\lambda_1\lambda_3 & -\frac{\sin\theta}{\theta}\lambda_1 + \frac{1-\cos\theta}{\theta^2}\lambda_2\lambda_3 & 1 - \frac{1-\cos\theta}{\theta^2}\left(\lambda_2^2 + \lambda_1^2\right) \end{bmatrix} \tag{6}$$

It follows that the trace is

$$\begin{aligned}
\mathrm{Tr}(\Lambda) &= 3 - 2\frac{1-\cos\theta}{\theta^2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \\
&= 3 - 2\,(1-\cos\theta) \\
&= 1 + 2\cos\theta
\end{aligned}$$

or

$$\theta = \cos^{-1}\left(\tfrac{1}{2}\,(\mathrm{Tr}(\Lambda) - 1)\right) \quad \theta \in [0, \pi] \tag{7}$$

It also follows that

$$\Lambda - \Lambda^T = \frac{2\sin\theta}{\theta}\begin{bmatrix} 0 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & 0 & \lambda_1 \\ \lambda_2 & -\lambda_1 & 0 \end{bmatrix} \tag{8}$$

or, when $\sin\theta \neq 0$

$$\lambda = \frac{\theta}{2\sin\theta}\left(\Lambda - \Lambda^T\right) \tag{9}$$

We need an equation that works when $\sin\theta$ approaches 0, that is, when θ approaches 0 or θ approaches π. When θ approaches 0, Equation 9 actually behaves well:

$$\lim_{\theta\to 0}\frac{\theta}{2\sin\theta} = \frac{1}{2} \tag{10}$$

and since θ is the l_2 norm of the individual components of λ, it follows that they approach zero, and we get

$$\lambda = 0 \tag{11}$$

However, when θ approaches π, $\Lambda - \Lambda^T$ approaches 0, and

$$\lim_{\theta\to\pi}\frac{\theta}{2\sin\theta} = \infty \tag{12}$$

We need a different method to find λ. Going back to Equations 5 and 6, we can compute the following:

$$\Lambda_{11} + \Lambda_{22} - \Lambda_{33} = 1 - \frac{2\lambda_3^2(1-\cos\theta)}{\theta^2} \tag{13}$$

or

$$\lambda_3 = \pm\theta\sqrt{\frac{(1 + \Lambda_{33} - \Lambda_{11} - \Lambda_{22})}{2\,(1-\cos\theta)}} \tag{14}$$

Equations for the other two components of λ are similar:

$$\lambda_1 = \pm\theta\sqrt{\frac{(1 + \Lambda_{11} - \Lambda_{22} - \Lambda_{33})}{2\,(1-\cos\theta)}} \tag{15}$$

$$\lambda_2 = \pm\theta\sqrt{\frac{(1 + \Lambda_{22} - \Lambda_{11} - \Lambda_{33})}{2\,(1-\cos\theta)}} \tag{16}$$

Equations 14-16 give us the magnitude, not the sign of the vector we are looking for. Here is one possibility for choosing the sign: If $(\lambda_1) \neq 0$, choose $\mathrm{sign}(\lambda_1)$ positive.

$$\Lambda_{12} + \Lambda_{21} = \frac{2\,(1-\cos\theta)}{\theta^2}\lambda_1\lambda_2 \tag{17}$$

so

$$\mathrm{sign}(\lambda_2) = \mathrm{sign}(\Lambda_{12} + \Lambda_{21}) \tag{18}$$

and similarly,

$$\mathrm{sign}(\lambda_3) = \mathrm{sign}(\Lambda_{13} + \Lambda_{31}) \tag{19}$$

If $(\lambda_1) = 0$, similar arguments can be used to choose $\mathrm{sign}(\lambda_2)$ positive, and

$$\mathrm{sign}(\lambda_3) = \mathrm{sign}(\Lambda_{23} + \Lambda_{32}) \tag{20}$$

At this point, the relationships between the components of λ are set, so we have computed $\pm\lambda$. If $\theta = \pi$, both values are a solution, so this good enough.

If θ is close to π, we will need to determine if we have the negative of the solution. This is required for numerical stability of the solution. In this case, $\Lambda - \Lambda^T$ is not exactly zero, so we can look at the sign of the solution we would have computed if we had used Equation 9:

$$\Lambda_{23} - \Lambda_{32} = \left|\frac{\sin\theta}{\theta}\right|\lambda_1 \tag{21}$$

$$\Lambda_{31} - \Lambda_{13} = \left|\frac{\sin\theta}{\theta}\right|\lambda_2 \tag{22}$$

$$\Lambda_{12} - \Lambda_{21} = \left|\frac{\sin\theta}{\theta}\right|\lambda_3 \tag{23}$$

For numerical reasons, we don't want to use these equations to get the magnitude of the components of λ, but we can look at the sign of the element with largest magnitude and ensure our λ has the same sign.

4 Interpolation

4.1 Periodicity of solutions

Given $\lambda_k = \lambda\left(1 + \frac{2k\pi}{\|\lambda\|}\right)$ for any integer k, it follows that

$$\theta_k = \left|1 + \frac{2k\pi}{\|\lambda\|}\right|\theta \tag{24}$$

or

$$\theta_k = |\theta + 2k\pi| \tag{25}$$

$$
\begin{aligned}
\Lambda_k &= \exp(\lambda_k) \\
&= I + \frac{\sin\theta_k}{\theta_k}\lambda_k + \frac{1-\cos\theta_k}{\theta_k^2}\lambda_k^2 \\
&= I + \frac{\sin|\theta+2k\pi|}{|\theta+2k\pi|}\left(\frac{\theta+2k\pi}{\theta}\right)\lambda + \frac{1-\cos|\theta+2k\pi|}{|\theta+2k\pi|^2}\left(\frac{\theta+2k\pi}{\theta}\right)^2\lambda^2 \\
&= I + \frac{\sin|\theta+2k\pi|}{\theta}\frac{\theta+2k\pi}{|\theta+2k\pi|}\lambda + \frac{1-\cos|\theta+2k\pi|}{\theta^2}\lambda^2 \\
&= I + \frac{\sin\theta}{\theta}\lambda + \frac{1-\cos\theta}{\theta^2}\lambda^2 \\
&= \exp(\lambda) \\
&= \Lambda
\end{aligned}
$$

Thus, if λ is one solution to $\log(\Lambda)$, then so is $\lambda_k = \lambda\left(1+\frac{2k\pi}{\|\lambda\|}\right)$ for any integer k.

4.2 Finding values of λ for interpolation

Given a set of λ^j to be interpolated, find equivalent $\tilde{\lambda}^j$ for integers $j = 1, 2, ...n$: Set $\tilde{\lambda}^1 = \lambda^1$. For each $j \in [2, n]$, check to see if $\tilde{\lambda}^{j-1}$ is closer (in the l_2-norm sense) to λ^j or $\lambda^j\left(1+\frac{2\pi}{\|\lambda^j\|}\right)$. If the latter, set $\tilde{\lambda}^j = \lambda^j\left(1+\frac{2\pi}{\|\lambda^j\|}\right)$ and continue checking if we need to add more 2π periods. Otherwise, check to see if $\tilde{\lambda}^{j-1}$ is closer to λ^j or $\lambda^j\left(1-\frac{2\pi}{\|\lambda^j\|}\right)$. If the latter, set $\tilde{\lambda}^j = \lambda^j\left(1-\frac{2\pi}{\|\lambda^j\|}\right)$ and continue checking if we need to subtract more 2π periods. Otherwise set $\tilde{\lambda}^j = \lambda^j$.

Interpolation must occur on the $\tilde{\lambda}^j$ and not the λ^j.