Piecewise constant strain model for soft manipulator dynamics

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0.1 Continuous kinematics

A Cosserat beam is a one-dimensional continuum body in which the generic material element is considered as a infinitesimally small rigid element called cross-section which can rotate independently of the neighbouring fellows. According to the Cosserat beam theory, the configuration of a deformable body with respect to the inertial frame at a certain time characterized by a position vector $\boldsymbol{p}(s,t) \in \mathbb{R}^3$ and a rotational matrix $\boldsymbol{R}(s,t) \in SO(3)$, parameterized by the materail abscissa $s \in [0, L] \subset \mathbb{R}$ along the manipulator. Thus, the manipulator configuration space can be defined as a curve $\boldsymbol{g}(s,t) : s \mapsto g(s) \in SE(3)$ and time $t \in [0, \infty) \subset \mathbb{R}$ with[1]

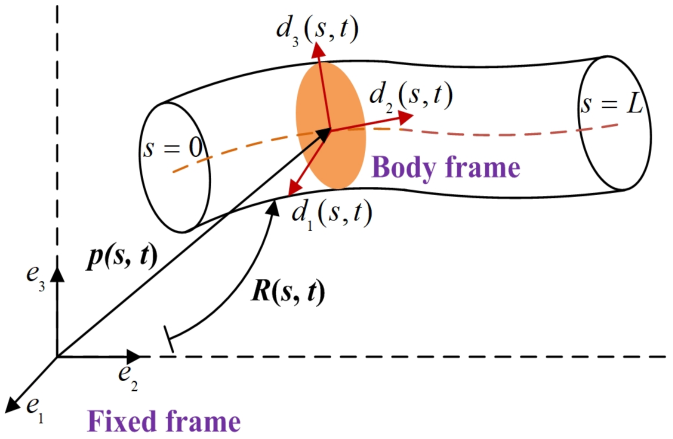

Figure 1: Configuration of a Cosserat beam

$$\boldsymbol{g}(s,t) = \begin{pmatrix} \boldsymbol{R} & \boldsymbol{p} \\ 0^T & 1 \end{pmatrix}$$

The strain state of the soft manipulator is defined by the strain twist $\hat{\boldsymbol{\xi}}(s,t) = \boldsymbol{g}^{-1}\frac{\partial \boldsymbol{g}}{\partial s} = \boldsymbol{g}^{-1}\boldsymbol{g}'$, and the hat represents the isomorphism between the twist space $\mathbb{R}^6$ and $se(3)$.

$$\hat{\boldsymbol{\xi}}(s,t) = \begin{pmatrix} \tilde{\boldsymbol{k}} & \boldsymbol{q} \\ 0^T & 0 \end{pmatrix} \in se(3), \qquad \boldsymbol{\xi}(s,t) = \left(\boldsymbol{k}^T, \boldsymbol{q}^T\right)^T \in \mathbb{R}^6$$

where $\hat{\boldsymbol{\xi}}$ is the strain twist measuring the bending and the torsion state of the manipulator, and defines the angular and linear strain with $\tilde{\boldsymbol{k}} \in so(3)$ which is a skew symmetric matrix, $\boldsymbol{k} \in \mathbb{R}^3$ and $\boldsymbol{q} \in \mathbb{R}^3$, respectively.
The change of configuration curve with time is represented by velocity twist $\hat{\boldsymbol{\eta}}(s,t) = \boldsymbol{g}^{-1}\frac{\partial \boldsymbol{g}}{\partial t} = \boldsymbol{g}^{-1}\dot{\boldsymbol{g}}$, as in the above case, $\hat{\boldsymbol{\eta}}$ can be detailed as follows

$$\hat{\boldsymbol{\eta}}(s,t) = \begin{pmatrix} \tilde{\boldsymbol{\omega}} & \boldsymbol{v} \\ 0^T & 0 \end{pmatrix} \in se(3), \qquad \boldsymbol{\eta}(s,t) = \left(\boldsymbol{\omega}^T, \boldsymbol{v}^T\right)^T \in \mathbb{R}^6$$

where $\hat{\boldsymbol{\eta}}$ is the velocity twist of the soft manipulator, and defines the angular and linear velocity with $\tilde{\boldsymbol{\omega}} \in so(3), \boldsymbol{\omega} \in \mathbb{R}^3$ and $\boldsymbol{v} \in \mathbb{R}^3$, respectively.
The continuous models of the position $\boldsymbol{g}(s)$, velocity $\boldsymbol{\eta}(s)$ and acceleration $\dot{\boldsymbol{\eta}}(s)$ of any microsolid along the soft manipulator can be obtained from the Cosserat theory, which gives [2][3]:

$$\boldsymbol{g}' = \boldsymbol{g}\hat{\boldsymbol{\xi}}, \qquad \dot{\boldsymbol{g}} = \boldsymbol{g}\hat{\boldsymbol{\eta}} \tag{1}$$

$$\boldsymbol{\eta}' = \dot{\boldsymbol{\xi}} - \mathrm{ad}_{\boldsymbol{\xi}}\boldsymbol{\eta} \tag{2}$$

Taking the derivative of the equation (2) with respect to time, we obtain the continuous model of acceleration

$$\dot{\boldsymbol{\eta}}' = \ddot{\boldsymbol{\xi}} - \mathrm{ad}_{\dot{\boldsymbol{\xi}}}\boldsymbol{\eta} - \mathrm{ad}_{\boldsymbol{\xi}}\dot{\boldsymbol{\eta}} \tag{3}$$

where $\mathrm{ad}_{(\cdot)} \in \mathbb{R}^{6\times 6}$ represents the adjoint operator of the Lie algebra, and $\mathrm{ad}_{\boldsymbol{\xi}} = \begin{pmatrix} \tilde{\boldsymbol{k}} & 0_{3\times 3} \\ \tilde{\boldsymbol{q}} & \tilde{\boldsymbol{k}} \end{pmatrix} \in \mathbb{R}^{6\times 6}$.
Solve the differential equation (1)-(3), respectively, then the geometric and kinematic models of the manipulator can be obtained

$$\boldsymbol{g}(s) = \boldsymbol{g}(s_0)e^{(s-s_0)\hat{\boldsymbol{\xi}}}, \qquad \boldsymbol{g}(t) = \boldsymbol{g}(t_0)e^{(t-t_0)\hat{\boldsymbol{\eta}}}$$

$$\boldsymbol{\eta}(s) = e^{-(s-s_0)\mathrm{ad}_{\boldsymbol{\xi}}}\boldsymbol{\eta}(s_0) + \int_{s_0}^{s} e^{(\tau-s)\mathrm{ad}_{\boldsymbol{\xi}}}\dot{\boldsymbol{\xi}}d\tau$$

$$\dot{\boldsymbol{\eta}}(s) = e^{-(s-s_0)\mathrm{ad}_{\boldsymbol{\xi}}}\dot{\boldsymbol{\eta}}(s_0) + \int_{s_0}^{s} e^{(\tau-s)\mathrm{ad}_{\boldsymbol{\xi}}}(\ddot{\boldsymbol{\xi}} - \mathrm{ad}_{\dot{\boldsymbol{\xi}}}\boldsymbol{\eta})d\tau$$

0.2 Continuous dynamics

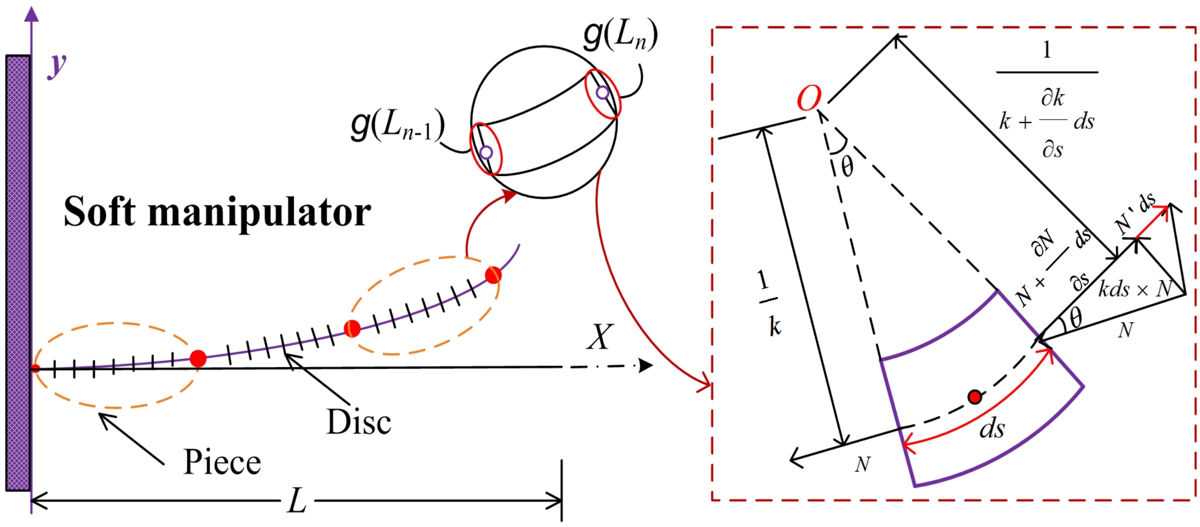

Figure 2: Schematic of the dynamics of soft manipulator via Cosserat

Considering the Cosserat beam with shearing and torsional deformations, as shown in the figure 2. Take infinitesimal material element ds from the beam for the force and torque analysis according to the laws of Newtonian mechanics. From the force and torque analysis of the element ds, we can obtain the dynamics equation of the infinitesimal material ds as follows:

$$\boldsymbol{N}'ds + \boldsymbol{k}ds \times \boldsymbol{N} + \boldsymbol{n}ds = \rho A\boldsymbol{I}ds\dot{\boldsymbol{v}} + \boldsymbol{\omega} \times \rho A\boldsymbol{I}ds\boldsymbol{v}$$

$$\boldsymbol{M}'ds + \boldsymbol{k}ds \times \boldsymbol{M} + \boldsymbol{q}ds \times \boldsymbol{N} + \boldsymbol{m}ds = \rho\boldsymbol{J}ds\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \rho\boldsymbol{J}ds\boldsymbol{\omega}$$

where $\boldsymbol{N}(s,t)$ and $\boldsymbol{M}(s,t) \in \mathbb{R}^3$ are the internal force and torque vectors, respectively, while $\boldsymbol{n}(s,t)$ and $\boldsymbol{m}(s,t) \in \mathbb{R}^3$ are the external force and torque for unit of s. ρ and $A(s) \in \mathbb{R}$ represent the density of the manipulator and the section area, respectively, $\boldsymbol{I}$ is the identity matrix, and $\boldsymbol{J}(s) \in \mathbb{R}^3 \otimes \mathbb{R}^3$ is the second moment of the area tensor. $\boldsymbol{N}'$ and $\boldsymbol{M}'$ represent changes in internal force and torque vectors on both sides of the unit s, respectively.
The internal and external components of wrenches for unit of s are specified as:

$$\boldsymbol{\mathcal{F}}_i = \begin{pmatrix} \boldsymbol{M} \\ \boldsymbol{N} \end{pmatrix} \in \mathbb{R}^6, \qquad \boldsymbol{\mathcal{F}}_e = \begin{pmatrix} \boldsymbol{m} \\ \boldsymbol{n} \end{pmatrix} \in \mathbb{R}^6$$

Define the screw inertia matrix and coadjoint map as follows:

$$\boldsymbol{\mathcal{M}} = \begin{pmatrix} \rho\boldsymbol{J} & 0 \\ 0 & \rho A\boldsymbol{I} \end{pmatrix} \in \mathbb{R}^{6\times 6}, \qquad \mathrm{ad}^*_{\boldsymbol{\xi}} = \begin{pmatrix} \tilde{\boldsymbol{k}} & \tilde{\boldsymbol{q}} \\ 0_{3\times 3} & \tilde{\boldsymbol{k}} \end{pmatrix} \in \mathbb{R}^{6\times 6}, \qquad \mathrm{ad}^*_{\boldsymbol{\eta}} = \begin{pmatrix} \tilde{\boldsymbol{\omega}} & \tilde{\boldsymbol{v}} \\ 0_{3\times 3} & \tilde{\boldsymbol{\omega}} \end{pmatrix} \in \mathbb{R}^{6\times 6}$$

The above dynamics equation can be expressed by twist and wrench form:

$$\boldsymbol{\mathcal{F}}_{\boldsymbol{i}}' + \mathrm{ad}^*_{\boldsymbol{\xi}}\boldsymbol{\mathcal{F}}_{\boldsymbol{i}} + \boldsymbol{\mathcal{F}}_{\boldsymbol{e}} = \boldsymbol{\mathcal{M}}\dot{\boldsymbol{\eta}} + \mathrm{ad}^*_{\boldsymbol{\eta}}\boldsymbol{\mathcal{M}}\boldsymbol{\eta} \tag{4}$$

Then, the cable-driven actuation mechanism is considered. And for cable-driven actuation, the distributed load $\overline{\boldsymbol{\mathcal{F}}}_a$ generated by cables at any arbitrary microsolid is obtained [3]

$$\overline{\boldsymbol{\mathcal{F}}}_a(s,t) = -(\boldsymbol{\mathcal{F}}_{\boldsymbol{a}}' + \mathrm{ad}^*_{\boldsymbol{\xi}}\boldsymbol{\mathcal{F}}_{\boldsymbol{a}})$$

where $\boldsymbol{\mathcal{F}}_a$ is the actuation load generated by a cable and expressed in the body coordinate.
Therefore, the dynamics equation (4) including the distributed load with respect to the body frames can be expressed as

$$\begin{gathered} \boldsymbol{\mathcal{M}}\dot{\boldsymbol{\eta}} + \mathrm{ad}^*_{\boldsymbol{\eta}}\boldsymbol{\mathcal{M}}\boldsymbol{\eta} = \boldsymbol{\mathcal{F}'}_{\boldsymbol{i}} - \boldsymbol{\mathcal{F}}'_{\mathrm{a}} + \mathrm{ad}^*_{\boldsymbol{\xi}}\boldsymbol{\mathcal{F}}_{\boldsymbol{i}} - \mathrm{ad}^*_{\boldsymbol{\xi}}\boldsymbol{\mathcal{F}}_{\boldsymbol{a}} + \boldsymbol{\mathcal{F}}_{\boldsymbol{e}} = \boldsymbol{\mathcal{F}}'_{\boldsymbol{i-a}} + \mathrm{ad}^*_{\boldsymbol{\xi}}\boldsymbol{\mathcal{F}}_{\boldsymbol{i-a}} + \boldsymbol{\mathcal{F}}_{\boldsymbol{e}} \\ \boldsymbol{\mathcal{F}}_{\boldsymbol{i-a}}(L_{n-1}) = -\boldsymbol{\mathcal{F}}_{\boldsymbol{J}_n}, \qquad \boldsymbol{\mathcal{F}}_{\boldsymbol{i-a}}(L_n) = -\mathrm{Ad}^*_{\boldsymbol{g}(L_n)}\boldsymbol{\mathcal{F}}_{\boldsymbol{J}_{n+1}} \end{gathered} \tag{5}$$

where, $\mathcal{F}_{J_n}$ is the wrench transmitted by the previous piece in the chain to the base of the piece n, while $\mathcal{F}_{J_{n+1}}$ is the wrench transmitted by the tip of the piece n to the next one in the chain.
Finally, the external force $\mathcal{F}_e$ and internal force $\mathcal{F}_i$ appearing in the equation (5) for the general case of the soft manipulator moving in the air are specified [3].

$$\mathcal{F}_i = \mathbf{\Gamma}(\boldsymbol{\xi} - \boldsymbol{\xi}_0) + \boldsymbol{\gamma}\dot{\boldsymbol{\xi}} \tag{6}$$

where $\mathbf{\Gamma} = \mathrm{diag}(GJ_x, GJ_y, GJ_z, EA, GA, GA)$, $\boldsymbol{\gamma} = \mathrm{diag}(\mu J_x, \mu 3J_y, \mu 3J_z, 3\mu A, \mu A, \mu A)$, and μ is the shear viscosity modulus.

$$\mathcal{F}_e = \boldsymbol{\mathcal{M}}\mathrm{Ad}^{-1}_{\boldsymbol{g}(s)\boldsymbol{g}_r}\boldsymbol{\mathcal{G}} + \delta(s - \overline{s})\mathcal{F}_p \tag{7}$$

where $\boldsymbol{\mathcal{G}} = [0, 0, 0, -9.81, 0, 0]^T$ is the gravity acceleration twist with respect to the inertial frame, $\delta(s)$ is the Dirac distribution. $\mathcal{F}_p$ is the wrench generated by external concentrated load. $\boldsymbol{g}_r$ is the transformation between the inertial frame and the microsolid frame.

0.3 PCS kinematics

The above continuous model is discretized by an analytic spatial integration according to the piecewise constant strain (PCS) assumption [1] which provides the condition to analytically integrate the continuous model. The materail abscissa $s \in [0, L]$ is divided into N pieces of the form $[L_{n-1}, L_n](n \in 1, 2, \cdots, N)$. To model constrained rod, it is useful to consider constrained strain state, which can be the result of geometric properties, mechanical properties or modeling assumptions, so the strain field of the soft manipulator j is specified as [3]

$$\boldsymbol{\xi}_j = \boldsymbol{B}_j\boldsymbol{q}_j + \overline{\boldsymbol{\xi}}_j$$

where $\boldsymbol{B}_j \in \mathbb{R}^{6\times 6}$ forms a basis for the allowed motion subspace, $\boldsymbol{q}_j \in \mathbb{R}^6$ contains the values of the allowed strains, and $\overline{\boldsymbol{\xi}}_j \in \mathbb{R}^6$ is the undeformed space twist in the straight reference configuration.

Under the PCS assumption, we can specify $s = L_{n-1}$ as the initial value for the differential equation of the piece n. The position $\boldsymbol{g}(s)$, velocity $\boldsymbol{\eta}(s)$ and acceleration $\dot{\boldsymbol{\eta}}(s)$ of any piece n at s along the soft manipulator at a certain instant t can be analytically solved using the matrix exponential method, which gives respectively:

$$\boldsymbol{g}_n(s) = \boldsymbol{g}(L_{n-1})e^{(s-L_{n-1})\hat{\boldsymbol{\xi}}_n} \tag{8}$$

For the straight configuration, the equation (8) becomes $\boldsymbol{g}_n(s) = \boldsymbol{g}(L_{n-1})\left(\boldsymbol{I}_4 + s\hat{\boldsymbol{\xi}}_n\right)$.

$$\boldsymbol{\eta}_n(s) = \mathrm{Ad}^{-1}_{\boldsymbol{g}_n(s)}\left(\boldsymbol{\eta}(L_{n-1}) + \mathrm{T}_{\boldsymbol{g}_n(s)}\boldsymbol{B}_n\dot{\boldsymbol{q}}_n\right) = \mathrm{Ad}^{-1}_{\boldsymbol{g}_n(s)}\boldsymbol{\eta}(L_{n-1}) + \mathrm{Ad}^{-1}_{\boldsymbol{g}_n(s)}\mathrm{T}_{\boldsymbol{g}_n(s)}\boldsymbol{B}_n\dot{\boldsymbol{q}}_n \tag{9}$$

where the tangent operator of the exponential map is defined as [4]

$$\begin{aligned}
\mathrm{T}_{\boldsymbol{g}_n(s)} &:= \int_{L_{n-1}}^{s}\mathrm{Ad}_{\boldsymbol{g}_n(l)}dl \\
&= (s - L_{n-1})\boldsymbol{I}_6 + \frac{1}{2\theta_n^2}\left(4 - 4\mathrm{cos}((s - L_{n-1})\theta_n) - (s - L_{n-1})\theta_n\mathrm{sin}((s - L_{n-1})\theta_n)\right)\mathrm{ad}_{\boldsymbol{\xi}_n} \\
&+\frac{1}{2\theta_n^3}\left(4(s - L_{n-1})\theta_n - 5\mathrm{sin}((s - L_{n-1})\theta_n) + (s - L_{n-1})\theta_n\mathrm{cos}((s - L_{n-1})\theta_n)\right)\mathrm{ad}^2_{\boldsymbol{\xi}_n} \\
&+\frac{1}{2\theta_n^4}\left(2 - 2\mathrm{cos}((s - L_{n-1})\theta_n) - (s - L_{n-1})\theta_n\mathrm{sin}((s - L_{n-1})\theta_n)\right)\mathrm{ad}^3_{\boldsymbol{\xi}_n} \\
&+\frac{1}{2\theta_n^5}\left(2(s - L_{n-1})\theta_n - 3\mathrm{sin}((s - L_{n-1})\theta_n) + (s - L_{n-1})\theta_n\mathrm{cos}((s - L_{n-1})\theta_n)\right)\mathrm{ad}^4_{\boldsymbol{\xi}_n}
\end{aligned}$$

$$\begin{aligned}
\dot{\boldsymbol{\eta}}_n(s) &= \mathrm{Ad}^{-1}_{\boldsymbol{g}_n(s)}\left(\dot{\boldsymbol{\eta}}(L_{n-1}) + \mathrm{T}_{\boldsymbol{g}_n(s)}\boldsymbol{B}_n\ddot{\boldsymbol{q}}_n - \int_{L_{n-1}}^{s}\mathrm{Ad}_{\boldsymbol{g}_n(l)}\mathrm{ad}_{\boldsymbol{B}_n\dot{\boldsymbol{q}}_n}\boldsymbol{\eta}(l)dl\right) \\
&= \mathrm{Ad}^{-1}_{\boldsymbol{g}_n(s)}\dot{\boldsymbol{\eta}}(L_{n-1}) + \mathrm{Ad}^{-1}_{\boldsymbol{g}_n(s)}\mathrm{T}_{\boldsymbol{g}_n(s)}\boldsymbol{B}_n\ddot{\boldsymbol{q}}_n + \mathrm{Ad}^{-1}_{\boldsymbol{g}_n(s)}\int_{L_{n-1}}^{s}\mathrm{Ad}_{\boldsymbol{g}_n(l)}\mathrm{ad}_{\boldsymbol{\eta}(l)}dl\boldsymbol{B}_n\dot{\boldsymbol{q}}_n
\end{aligned} \tag{10}$$

The PCS kinematics eqaution (9)-(10) are the recurrence relation for body frame suitable for computing velocity $\boldsymbol{\eta}$ and acceleration $\dot{\boldsymbol{\eta}}$ of any single piece in the chain. Applying these equations recursively from base to tip for all the pieces of the soft manipulator, we can obtain the expressions of the geometric Jacobian $\boldsymbol{J}_n(s) = [\,\boldsymbol{S}_1(s) \quad \boldsymbol{S}_2(s) \quad \cdots\ \boldsymbol{S}_n(s)] \in \mathbb{R}^{6\times 6n}$ whose components $\boldsymbol{S}_i \in \mathbb{R}^{6\times 6}$ and its derivative $\dot{\boldsymbol{S}}_i \in \mathbb{R}^{6\times 6}$. The geometric Jacobian is related to the generalized joint coordinate vector $\boldsymbol{q} = [\boldsymbol{q}_1^T, \boldsymbol{q}_2^T, \cdots, \boldsymbol{q}_n^T]^T \in \mathbb{R}^{6n}$. The non-recursive formula of velocity and acceleration twist of the n^{th} piece at s along the soft manipulator at a certain instant t in the body frame can be written as

$$\begin{aligned}
\boldsymbol{\eta}_n(s) &= \sum_{i=1}^{n}\mathrm{Ad}^{-1}_{\boldsymbol{g}_i}\cdots\mathrm{Ad}^{-1}_{\boldsymbol{g}_n}\mathrm{T}_{\boldsymbol{g}_i}\boldsymbol{B}_i\dot{\boldsymbol{q}}_i = \sum_{i=1}^{n}\boldsymbol{S}_i\,\dot{\boldsymbol{q}}_i = \boldsymbol{J}_n(s)\dot{\boldsymbol{q}} \\
\dot{\boldsymbol{\eta}}_n(s) &= \sum_{i=1}^{n}{}^{n}\boldsymbol{S}_i\,\ddot{\boldsymbol{q}}_i + \sum_{i=1}^{n}\mathrm{Ad}^{-1}_{\boldsymbol{g}_i\cdots\boldsymbol{g}_n}\int_{L_{i-1}}^{s}\mathrm{Ad}_{\boldsymbol{g}_i(l)}\mathrm{ad}_{\boldsymbol{\eta}_i(l)}dl\boldsymbol{B}_i\dot{\boldsymbol{q}}_i = \sum_{i=1}^{n}\boldsymbol{S}_i\,\ddot{\boldsymbol{q}}_i + \sum_{i=1}^{n}\dot{\boldsymbol{S}}_i\,\dot{\boldsymbol{q}}_i = \boldsymbol{J}_n(s)\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_n(s)\dot{\boldsymbol{q}}
\end{aligned} \tag{11}$$

which gives $\boldsymbol{S}_n(s) = \mathrm{Ad}^{-1}_{\boldsymbol{g}_n(s)} \mathrm{T}_{\boldsymbol{g}_n(s)} \boldsymbol{B}_n$ and $\dot{\boldsymbol{S}}_n(s) = \mathrm{Ad}^{-1}_{\boldsymbol{g}_n} \int_{L_{n-1}}^{s} \mathrm{Ad}_{\boldsymbol{g}_n(l)} \mathrm{ad}_{\boldsymbol{\eta}_n(l)} dl \boldsymbol{B}_n$, and where

$$
\boldsymbol{J}(s) = \left\{ \begin{array}{llllll}
\xrightarrow{n} & & & & & \\
\boldsymbol{S}_1(s) & \mathbf{0}_{6\times 6} & \mathbf{0}_{6\times 6} & \cdots & \mathbf{0}_{6\times 6} & 0 < s \leq L_1 \\
\boldsymbol{S}_1(s) & \boldsymbol{S}_2(s) & \mathbf{0}_{6\times 6} & \cdots & \mathbf{0}_{6\times 6} & L_1 < s \leq L_2 \\
\boldsymbol{S}_1(s) & \boldsymbol{S}_2(s) & \mathbf{0}_{6\times 6} & \cdots & \mathbf{0}_{6\times 6} & L_2 < s \leq L_3 \\
 & & \vdots & & & \vdots \\
\boldsymbol{S}_1(s) & \boldsymbol{S}_2(s) & \boldsymbol{S}_3(s) & \cdots & \boldsymbol{S}_N(s) & L_{N-1} < s \leq L_N
\end{array} \right.
$$

$$
\boldsymbol{J}^T(s) = \left\{ \begin{array}{llllll}
\xrightarrow{s} & & & & & \\
\boldsymbol{S}_1^T(s) & \boldsymbol{S}_1^T(s) & \boldsymbol{S}_1^T(s) & \cdots & \boldsymbol{S}_1^T(s) & n = 1 \\
\mathbf{0}_{6\times 6} & \boldsymbol{S}_2^T(s) & \boldsymbol{S}_2^T(s) & \cdots & \boldsymbol{S}_2^T(s) & n = 2 \\
\mathbf{0}_{6\times 6} & \mathbf{0}_{6\times 6} & \boldsymbol{S}_3^T(s) & \cdots & \boldsymbol{S}_3^T(s) & n = 3 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
\mathbf{0}_{6\times 6} & \mathbf{0}_{6\times 6} & \mathbf{0}_{6\times 6} & \cdots & \boldsymbol{S}_N^T(s) & n = N
\end{array} \right.
$$

And the Jacobian matrix $\boldsymbol{J}(s)$ and the derivative of the Jacobian matrix $\dot{\boldsymbol{J}}(s)$ are calculated directly as follows:

$$
\boldsymbol{J}(s) = \left\{ \begin{array}{llll}
\xrightarrow{n} & & & \\
\mathrm{Ad}^{-1}_{\boldsymbol{g}_1(s)} \mathrm{T}_{\boldsymbol{g}_1(s)} \boldsymbol{B}_1 & \mathbf{0}_{6\times 6} & \mathbf{0}_{6\times 6} \quad \cdots \quad \mathbf{0}_{6\times 6} & 0 < s \leq L_1 \\
\mathrm{Ad}^{-1}_{\boldsymbol{g}_2(s)} \mathrm{Ad}^{-1}_{\boldsymbol{g}_1(L_1)} \mathrm{T}_{\boldsymbol{g}_1(L_1)} \boldsymbol{B}_1 & \mathrm{Ad}^{-1}_{\boldsymbol{g}_2(s)} \mathrm{T}_{\boldsymbol{g}_2(s)} \boldsymbol{B}_2 & \mathbf{0}_{6\times 6} \quad \cdots \quad \mathbf{0}_{6\times 6} & L_1 < s \leq L_2 \\
\mathrm{Ad}^{-1}_{\boldsymbol{g}_3(s)} \mathrm{Ad}^{-1}_{\boldsymbol{g}_2(L_2)} \mathrm{Ad}^{-1}_{\boldsymbol{g}_1(L_1)} \mathrm{T}_{\boldsymbol{g}_1(L_1)} \boldsymbol{B}_1 & \mathrm{Ad}^{-1}_{\boldsymbol{g}_3(s)} \mathrm{Ad}^{-1}_{\boldsymbol{g}_2(L_2)} \mathrm{T}_{\boldsymbol{g}_2(L_2)} \boldsymbol{B}_2 & \cdots \quad \mathbf{0}_{6\times 6} & L_2 < s \leq L_3 \\
\vdots & & \vdots & \vdots \\
\prod_{i=1}^{N} \mathrm{Ad}^{-1}_{\boldsymbol{g}_i \min(L_i, s)} \mathrm{T}_{\boldsymbol{g}_1(L_1)} \boldsymbol{B}_1 & \cdots \quad \cdots \quad \cdots & \mathrm{Ad}^{-1}_{\boldsymbol{g}_N(s)} \mathrm{T}_{\boldsymbol{g}_N(s)} \boldsymbol{B}_N & L_{N-1} < s \leq L_N
\end{array} \right.
$$

$$
\dot{\boldsymbol{J}}(s) = \left\{ \begin{array}{llll}
\xrightarrow{n} & & & \\
\mathrm{Ad}^{-1}_{\boldsymbol{g}_1(s)} \int_{L_0}^{s} \mathrm{Ad}_{\boldsymbol{g}_1(l)} \mathrm{ad}_{\boldsymbol{\eta}_1(l)} dl \boldsymbol{B}_1 \quad \mathbf{0}_{6\times 6} \quad \mathbf{0}_{6\times 6} \quad \cdots \quad \mathbf{0}_{6\times 6} & & & 0 < s \leq L_1 \\
\mathrm{Ad}^{-1}_{\boldsymbol{g}_2(s)} \mathrm{Ad}^{-1}_{\boldsymbol{g}_1(L_1)} \int_{L_0}^{L_1} \mathrm{Ad}_{\boldsymbol{g}_1(l)} \mathrm{ad}_{\boldsymbol{\eta}_1(l)} dl \boldsymbol{B}_1 & \mathrm{Ad}^{-1}_{\boldsymbol{g}_2(s)} \int_{L_1}^{s} \mathrm{Ad}_{\boldsymbol{g}_2(l)} \mathrm{ad}_{\boldsymbol{\eta}_2(l)} dl \boldsymbol{B}_2 & \mathbf{0}_{6\times 6} \quad \cdots \quad \mathbf{0}_{6\times 6} & L_1 < s \leq L_2 \\
\mathrm{Ad}^{-1}_{\boldsymbol{g}_3(s)} \mathrm{Ad}^{-1}_{\boldsymbol{g}_2(L_2)} \mathrm{Ad}^{-1}_{\boldsymbol{g}_1(L_1)} \int_{L_0}^{L_1} \mathrm{Ad}_{\boldsymbol{g}_1(l)} \mathrm{ad}_{\boldsymbol{\eta}_1(l)} dl \boldsymbol{B}_1 & \mathrm{Ad}^{-1}_{\boldsymbol{g}_3(s)} \mathrm{Ad}^{-1}_{\boldsymbol{g}_2(L_2)} \int_{L_1}^{L_2} \mathrm{Ad}_{\boldsymbol{g}_2(l)} \mathrm{ad}_{\boldsymbol{\eta}_2(l)} dl \boldsymbol{B}_2 & \cdots \quad \mathbf{0}_{6\times 6} & L_2 < s \leq L_3 \\
\vdots & \vdots & & \vdots \\
\prod_{i=1}^{N} \mathrm{Ad}^{-1}_{\boldsymbol{g}_i \min(L_i, s)} \int_{L_0}^{L_1} \mathrm{Ad}_{\boldsymbol{g}_1(l)} \mathrm{ad}_{\boldsymbol{\eta}_1(l)} dl \boldsymbol{B}_1 \quad \cdots & \mathrm{Ad}^{-1}_{\boldsymbol{g}_N(s)} \int_{L_{N-1}}^{s} \mathrm{Ad}_{\boldsymbol{g}_N(l)} \mathrm{ad}_{\boldsymbol{\eta}_N(l)} dl \boldsymbol{B}_N & & L_{N-1} < s \leq L_N
\end{array} \right.
$$

0.4 PCS dynamics

In order to derive the discrete dynamics equation corresponding to the discrete kinematics equation (8), define the virtual displacement $\delta\boldsymbol{\zeta}(s)$ and introduce the relation $\delta\boldsymbol{\zeta}(s) = \boldsymbol{J}(s)\delta\boldsymbol{q} \in \mathbb{R}^6$. In addition to considering the differential kinematics equation (11), we can obtain the generalized dynamics for the soft manipulator [5]:

$$
\begin{aligned}
&\left(\int_0^{L_N} \boldsymbol{J}^T \boldsymbol{\mathcal{M}} \boldsymbol{J} ds \right) \ddot{\boldsymbol{q}} + \left[\int_0^{L_N} \boldsymbol{J}^T \left(\mathrm{ad}^*_{\boldsymbol{J}\dot{\boldsymbol{q}}} \boldsymbol{\mathcal{M}} \boldsymbol{J} + \boldsymbol{\mathcal{M}} \dot{\boldsymbol{J}} \right) ds \right] \dot{\boldsymbol{q}} \\
&= \int_0^{L_N} \boldsymbol{J}^T \left(\boldsymbol{\mathcal{F}'}_{\boldsymbol{i}} - \boldsymbol{\mathcal{F}}'_{\boldsymbol{a}} + \mathrm{ad}^*_{\boldsymbol{\xi}} \boldsymbol{\mathcal{F}}_{\boldsymbol{i}} - \mathrm{ad}^*_{\boldsymbol{\xi}} \boldsymbol{\mathcal{F}}_{\boldsymbol{a}} \right) ds + \left(\int_0^{L_N} \boldsymbol{J}^T \boldsymbol{\mathcal{M}} \mathrm{Ad}^{-1}_{\boldsymbol{g}} ds \right) \mathrm{Ad}^{-1}_{\boldsymbol{g}_r} \boldsymbol{\mathcal{G}} + \boldsymbol{J}^T(\bar{s}) \boldsymbol{\mathcal{F}}_p
\end{aligned} \tag{12}
$$

Finally, define the coefficient matrix of each term, and yield to the PCS dynamics equation in the classical form:

$$
\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{\tau}(\boldsymbol{q}) + \boldsymbol{G}(\boldsymbol{q}) \mathrm{Ad}^{-1}_{\boldsymbol{g}_r} \boldsymbol{\mathcal{G}} + \boldsymbol{F}(\boldsymbol{q}) \tag{13}
$$

where $\boldsymbol{M} \in \mathbb{R}^{6N\times 6N}$ is the generalized inertia matrix which is symmetric and positive definite, $\boldsymbol{C} \in \mathbb{R}^{6N\times 6N}$ is generalized Coriolis matrix, $\boldsymbol{G} \in \mathbb{R}^{6N\times 6}$ is the gravitational matrix, $\boldsymbol{F} \in \mathbb{R}^{6N}$ is the vector of generalized external concentrated forces, and $\boldsymbol{\tau} = [\boldsymbol{\tau}_1^T, \boldsymbol{\tau}_2^T, \cdots, \boldsymbol{\tau}_N^T]^T \in \mathbb{R}^{6N}$.

The following part will discribe inertia matrix, gravitational matrix, Coriolis matrix, the internal elastic and actuation force, and external force in the equation (13).

$$\boldsymbol{M}_{(n,m)} = \sum_{j=\max(n,m)}^{N} \int_{L_{j-1}}^{L_j} \boldsymbol{S}_n^T \boldsymbol{\mathcal{M}} \boldsymbol{S}_m ds$$

$$\boldsymbol{C}_{1(n,m)} = \sum_{j=\max(n,m)}^{N} \int_{L_{j-1}}^{L_j} \boldsymbol{S}_n^T \mathrm{ad}^*_{\boldsymbol{J}\dot{\boldsymbol{q}}} \boldsymbol{\mathcal{M}} \boldsymbol{S}_m ds$$

$$\boldsymbol{C}_{2(n,m)} = \sum_{j=\max(n,m)}^{N} \int_{L_{j-1}}^{L_j} \boldsymbol{S}_n^T \boldsymbol{\mathcal{M}} \dot{\boldsymbol{S}}_m ds$$

$$\boldsymbol{G}_{(n)} = \sum_{j=\max(n,m)}^{N} \int_{L_{j-1}}^{L_j} \boldsymbol{S}_n^T \boldsymbol{\mathcal{M}} \mathrm{Ad}^{-1}_{\boldsymbol{g}} ds$$

$$\begin{aligned}\boldsymbol{\tau}_n &= \sum_{j=n}^{N} \int_{L_{j-1}}^{L_j} \boldsymbol{S}_n^T \left(\boldsymbol{\mathcal{F}'}_{\boldsymbol{i-a}} + \mathrm{ad}^*_{\boldsymbol{\xi}_n} \boldsymbol{\mathcal{F}}_{\boldsymbol{i-a}}\right) ds = \sum_{j=n}^{N} \int_{L_{j-1}}^{L_j} \mathrm{T}^T_{\boldsymbol{g}_n(s)} \mathrm{Ad}^*_{\boldsymbol{g}_n(s)} \left(\boldsymbol{\mathcal{F}'}_{\boldsymbol{i-a}} + \mathrm{ad}^*_{\boldsymbol{\xi}_n} \boldsymbol{\mathcal{F}}_{\boldsymbol{i-a}}\right) ds \\ &= \sum_{j=n}^{N} \int_{L_{j-1}}^{L_j} \mathrm{T}^T_{\boldsymbol{g}_n(s)} d\mathrm{Ad}^*_{\boldsymbol{g}_n(s)} \left(\boldsymbol{\mathcal{F}}_{\boldsymbol{i-a}}\right) = \sum_{j=n}^{N} \boldsymbol{S}_n^T \boldsymbol{\mathcal{F}}_{\boldsymbol{i-a}} \Big|_{L_{j-1}}^{L_j} - \sum_{j=n}^{N} \int_{L_{j-1}}^{L_j} \boldsymbol{\mathcal{F}}_{\boldsymbol{i-a}} ds\end{aligned}$$

$$\boldsymbol{F}_n = \boldsymbol{S}_n^T(\bar{s}) \boldsymbol{\mathcal{F}}_p$$

0.5 Recursive Newton Euler algorithm

0.5.1 Forward dynamics

In the forward dynamics problem, the applied force $\boldsymbol{\tau}$ is always given, and we need to know the joint vector $\boldsymbol{q}$. The forward dynamics algorithm can be divided into three steps:
(1) Outward iterations: Using (8) and (9) to recursively compute the position $\boldsymbol{g}(s)$ and velocity $\boldsymbol{\eta}(s)$ of each cross-section s of the piece n.
(2) Inward iterations: The objective of the second step is to calculate inertia and force for the piece n of the manipulator. First, project the dynamics equation (5) onto the frame $s = L_{n-1}$ with $\mathrm{Ad}^*_{\boldsymbol{g}_n(s)}$ and integrates along the interval $[L_{n-1}, L_n]$, we have:

$$\begin{aligned}&\left(\int_{L_{n-1}}^{L_n} \mathrm{Ad}^*_{\boldsymbol{g}_n(s)} \boldsymbol{\mathcal{M}} \mathrm{Ad}^{-1}_{\boldsymbol{g}_n(s)} ds\right) \dot{\boldsymbol{\eta}}(L_{n-1}) + \int_{L_{n-1}}^{L_n} \mathrm{Ad}^*_{\boldsymbol{g}_n(s)} \boldsymbol{\mathcal{M}} \left(\boldsymbol{S}_n \ddot{\boldsymbol{q}}_n + \mathrm{ad}_{\mathrm{Ad}^{-1}_{\boldsymbol{g}_n(s)} \boldsymbol{\eta}(L_{n-1})} \boldsymbol{S}_n \dot{\boldsymbol{q}}_n\right) ds \\ &\quad + \int_{L_{n-1}}^{L_n} \mathrm{Ad}^*_{\boldsymbol{g}_n(s)} \left(\mathrm{ad}^*_{\boldsymbol{\eta}} \boldsymbol{\mathcal{M}} \boldsymbol{\eta} - \boldsymbol{\mathcal{F}}_{\boldsymbol{e}}\right) ds = \int_{L_{n-1}}^{L_n} \mathrm{Ad}^*_{\boldsymbol{g}_n(s)} \left(\boldsymbol{\mathcal{F}'}_{\boldsymbol{i-a}} + \mathrm{ad}^*_{\boldsymbol{\xi}} \boldsymbol{\mathcal{F}}_{\boldsymbol{i-a}}\right) ds\end{aligned} \tag{14}$$

where

$$\begin{aligned}&\int_{L_{n-1}}^{L_n} \mathrm{Ad}^*_{\boldsymbol{g}_n(s)} \left(\boldsymbol{\mathcal{F}'}_{\boldsymbol{i-a}} + \mathrm{ad}^*_{\boldsymbol{\xi}} \boldsymbol{\mathcal{F}}_{\boldsymbol{i-a}}\right) ds = \int_{L_{n-1}}^{L_n} \left(\mathrm{Ad}^*_{\boldsymbol{g}_n(s)} \boldsymbol{\mathcal{F}}_{\boldsymbol{i-a}}\right)' ds \\ &= \mathrm{Ad}^*_{\boldsymbol{g}_n} \left(\boldsymbol{\mathcal{F}}_{\boldsymbol{i-a}}\right)(L_n) - \mathrm{Ad}^*_{\boldsymbol{g}_n} \left(\boldsymbol{\mathcal{F}}_{\boldsymbol{i-a}}\right)(L_{n-1}) = -\mathrm{Ad}^*_{\boldsymbol{g}_n(L_n)} \boldsymbol{\mathcal{F}}_{J_{n+1}} + \boldsymbol{\mathcal{F}}_{J_n}\end{aligned}$$

$\boldsymbol{\mathcal{M}}^t_n \in \mathbb{R}^{6\times 6}$ and $\boldsymbol{\mathcal{P}}^t_n \in \mathbb{R}^6$ are the inertia and total force of the piece n, respectively.

$$\int_{L_{n-1}}^{L_n} \mathrm{Ad}^*_{\boldsymbol{g}_n(s)} \boldsymbol{\mathcal{M}} \mathrm{Ad}^{-1}_{\boldsymbol{g}_n(s)} ds = \boldsymbol{\mathcal{M}}^t_n, \qquad \int_{L_{n-1}}^{L_n} \mathrm{Ad}^*_{\boldsymbol{g}_n(s)} \left(\mathrm{ad}^*_{\boldsymbol{\eta}} \boldsymbol{\mathcal{M}} \boldsymbol{\eta} - \boldsymbol{\mathcal{F}}_{\boldsymbol{e}}\right) ds = \boldsymbol{\mathcal{P}}^t_n$$

Then, rewritting equation (14) with respect to the frame at $s = L_{n-1}$, we obtain:

$$\boldsymbol{\mathcal{M}}^t_n \dot{\boldsymbol{\eta}}(L_{n-1}) + \boldsymbol{\mathcal{P}}^t_n + \int_{L_{n-1}}^{L_n} \mathrm{Ad}^*_{\boldsymbol{g}_n(s)} \boldsymbol{\mathcal{M}} \left(\boldsymbol{S}_n \ddot{\boldsymbol{q}}_n + \mathrm{ad}_{\mathrm{Ad}^{-1}_{\boldsymbol{g}_n(s)} \boldsymbol{\eta}(L_{n-1})} \boldsymbol{S}_n \dot{\boldsymbol{q}}_n\right) ds + \mathrm{Ad}^*_{\boldsymbol{g}_n(L_n)} \boldsymbol{\mathcal{F}}_{J_{n+1}} = \boldsymbol{\mathcal{F}}_{J_n} \tag{15}$$

Then project the equation (5) onto the joint space with $\boldsymbol{S}_n^T(s)$ and integrate along the interval $[L_{n-1}, L_n]$, we have

$$\begin{aligned}&\int_{L_{n-1}}^{L_n} \boldsymbol{S}_n^T \boldsymbol{\mathcal{M}} \mathrm{Ad}^{-1}_{\boldsymbol{g}_n(s)} ds \dot{\boldsymbol{\eta}}(L_{n-1}) + \int_{L_{n-1}}^{L_n} \boldsymbol{S}_n^T \boldsymbol{\mathcal{M}} \left(\boldsymbol{S}_n \ddot{\boldsymbol{q}}_n + \mathrm{ad}_{\mathrm{Ad}^{-1}_{\boldsymbol{g}_n(s)} \boldsymbol{\eta}(L_{n-1})} \boldsymbol{S}_n \dot{\boldsymbol{q}}_n\right) ds \\ &+ \int_{L_{n-1}}^{L_n} \boldsymbol{S}_n^T \left(\mathrm{ad}^*_{\boldsymbol{\eta}} \boldsymbol{\mathcal{M}} \boldsymbol{\eta} - \boldsymbol{\mathcal{F}}_{\boldsymbol{e}}\right) ds = \int_{L_{n-1}}^{L_n} \boldsymbol{S}_n^T(s) \left(\boldsymbol{\mathcal{F}'}_{\boldsymbol{i-a}} + \mathrm{ad}^*_{\boldsymbol{\xi}} \boldsymbol{\mathcal{F}}_{\boldsymbol{i-a}}\right) ds = \int_{L_{n-1}}^{L_n} \mathrm{T}^T_{\boldsymbol{g}_n(s)} d(\mathrm{Ad}^*_{\boldsymbol{g}_n(s)} \boldsymbol{\mathcal{F}}_{\boldsymbol{i-a}}) \\ &= \boldsymbol{S}_n^T \boldsymbol{\mathcal{F}}_{\boldsymbol{J}_n} - \boldsymbol{S}_n^T(L_n) \boldsymbol{\mathcal{F}}_{\boldsymbol{J}_{n+1}} = \boldsymbol{\tau}_n - \boldsymbol{S}_n^T(L_n) \boldsymbol{\mathcal{F}}_{\boldsymbol{J}_{n+1}}\end{aligned} \tag{16}$$

Sort out the equation (16), we can obtain

$$
\begin{aligned}
\left(\int_{L_{n-1}}^{L_n} \boldsymbol{S}_n^T \boldsymbol{\mathcal{M}} \boldsymbol{S}_n ds\right) \ddot{\boldsymbol{q}}_n = \boldsymbol{\tau}_n - \boldsymbol{S}_n^T(L_n) \boldsymbol{\mathcal{F}}_{\boldsymbol{J}_{n+1}} \\
- \int_{L_{n-1}}^{L_n} \boldsymbol{S}_n^T \boldsymbol{\mathcal{M}} \left(\mathrm{Ad}_{\boldsymbol{g}_n(s)}^{-1} \dot{\boldsymbol{\eta}}(L_{n-1}) + \mathrm{ad}_{\mathrm{Ad}_{\boldsymbol{g}_n(s)}^{-1}\boldsymbol{\eta}(L_{n-1})} \boldsymbol{S}_n \dot{\boldsymbol{q}}_n\right) ds - \int_{L_{n-1}}^{L_n} \boldsymbol{S}_n^T \left(\mathrm{ad}_{\boldsymbol{\eta}}^* \boldsymbol{\mathcal{M}} \boldsymbol{\eta} - \boldsymbol{\mathcal{F}}_{\boldsymbol{e}}\right) ds
\end{aligned} \tag{17}
$$

Suppose inertia $\boldsymbol{\mathcal{M}}_{n+1}^A$ and force $\boldsymbol{\mathcal{P}}_{n+1}^A$ of the piece $n+1$ is given, and we have

$$
\boldsymbol{\mathcal{F}}_{\boldsymbol{J}_{n+1}} = \boldsymbol{\mathcal{M}}_{n+1}^A \dot{\boldsymbol{\eta}}(L_n) + \boldsymbol{\mathcal{P}}_{n+1}^A \tag{18}
$$

Substitute the equation (18) into (17) and yields

$$
\begin{aligned}
\left[\int_{L_{n-1}}^{L_n} \boldsymbol{S}_n^T \boldsymbol{\mathcal{M}} \boldsymbol{S}_n ds + \left(\boldsymbol{S}_n^T \boldsymbol{\mathcal{M}}_{n+1}^A \boldsymbol{S}_n(L_n)\right)\right] \ddot{\boldsymbol{q}}_n = \boldsymbol{\tau}_n - \left[\int_{L_{n-1}}^{L_n} \boldsymbol{S}_n^T \boldsymbol{\mathcal{M}} \mathrm{Ad}_{\boldsymbol{g}_n(s)}^{-1} ds + \boldsymbol{S}_n^T \boldsymbol{\mathcal{M}}_{n+1}^A \mathrm{Ad}_{\boldsymbol{g}_n(s)}^{-1}(L_n)\right] \dot{\boldsymbol{\eta}}(L_{n-1}) \\
- \left[\int_{L_{n-1}}^{L_n} \boldsymbol{S}_n^T \boldsymbol{\mathcal{M}} \mathrm{ad}_{\mathrm{Ad}_{\boldsymbol{g}_n(s)}^{-1}\boldsymbol{\eta}(L_{n-1})} \boldsymbol{S}_n ds + \left(\boldsymbol{S}_n^T \boldsymbol{\mathcal{M}}_{n+1}^A \mathrm{ad}_{\mathrm{Ad}_{\boldsymbol{g}_n(s)}^{-1}\boldsymbol{\eta}(L_{n-1})} \boldsymbol{S}_n\right)(L_n)\right] \dot{\boldsymbol{q}}_n \\
- \left[\int_{L_{n-1}}^{L_n} \boldsymbol{S}_n^T \left(\mathrm{ad}_{\boldsymbol{\eta}}^* \boldsymbol{\mathcal{M}} \boldsymbol{\eta} - \boldsymbol{\mathcal{F}}_{\boldsymbol{e}}\right) ds + \boldsymbol{S}_n^T(L_n) \boldsymbol{\mathcal{P}}_{n+1}^A\right]
\end{aligned} \tag{19}
$$

In order to obtain inertia $\boldsymbol{\mathcal{M}}_n^A$ and force $\boldsymbol{\mathcal{P}}_n^A$ of the piece n, substitute the equation (18) and $\ddot{\boldsymbol{q}}_n$ obtained from the equation (19) into the equation (15), and yields

$$
\boldsymbol{\mathcal{F}}_{\boldsymbol{J}_n} = \boldsymbol{\mathcal{M}}_n^A \dot{\boldsymbol{\eta}}(L_{n-1}) + \boldsymbol{\mathcal{P}}_n^A \tag{20}
$$

(3) Outward iterations: Finally, the equation (19) and (9) can be used to recursively compute $\ddot{\boldsymbol{q}}$ and $\dot{\boldsymbol{\eta}}$ from base to tip.

$$
\begin{aligned}
\boldsymbol{\mathcal{M}}_n^A = \boldsymbol{\mathcal{M}}_n^t + \left(\mathrm{Ad}_{\boldsymbol{g}_n}^* \boldsymbol{\mathcal{M}}_{n+1}^A \mathrm{Ad}_{\boldsymbol{g}_n}^{-1}(L_n) - \left[\int_{L_{n-1}}^{L_n} \mathrm{Ad}_{\boldsymbol{g}_n}^* \boldsymbol{\mathcal{M}} \boldsymbol{S}_n ds + \left(\mathrm{Ad}_{\boldsymbol{g}_n}^* \boldsymbol{\mathcal{M}}_{n+1}^A \boldsymbol{S}_n\right)(L_n)\right]\right) \\
\left[\int_{L_{n-1}}^{L_n} \boldsymbol{S}_n^T \boldsymbol{\mathcal{M}} \boldsymbol{S}_n ds + \left(\boldsymbol{S}_n^T \boldsymbol{\mathcal{M}}_{n+1}^A \boldsymbol{S}_n\right)(L_n)\right]^{-1} \left[\int_{L_{n-1}}^{L_n} \boldsymbol{S}_n^T \boldsymbol{\mathcal{M}} \mathrm{Ad}_{\boldsymbol{g}_n}^{-1} ds + \left(\boldsymbol{S}_n^T \boldsymbol{\mathcal{M}}_{n+1}^A \mathrm{Ad}_{\boldsymbol{g}_n}^{-1}\right)(L_n)\right]
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\boldsymbol{\mathcal{P}}_n^A = \boldsymbol{\mathcal{P}}_n^t + \mathrm{Ad}_{\boldsymbol{g}_n}^*(L_n) \boldsymbol{\mathcal{P}}_{n+1}^A + \left[\int_{L_{n-1}}^{L_n} \mathrm{Ad}_{\boldsymbol{g}_n}^* \boldsymbol{\mathcal{M}} \mathrm{ad}_{\mathrm{Ad}_{\boldsymbol{g}_n(s)}^{-1}\boldsymbol{\eta}(L_{n-1})} \boldsymbol{S}_n ds + \left(\mathrm{Ad}_{\boldsymbol{g}_n}^* \boldsymbol{\mathcal{M}}_{n+1}^A \mathrm{ad}_{\mathrm{Ad}_{\boldsymbol{g}_n(s)}^{-1}\boldsymbol{\eta}(L_{n-1})} \boldsymbol{S}_n\right)(L_n)\right] \dot{\boldsymbol{q}}_n \\
+ \left[\int_{L_{n-1}}^{L_n} \mathrm{Ad}_{\boldsymbol{g}_n}^* \boldsymbol{\mathcal{M}} \boldsymbol{S}_n ds + \left(\mathrm{Ad}_{\boldsymbol{g}_n}^* \boldsymbol{\mathcal{M}}_{n+1}^A \boldsymbol{S}_n\right)(L_n)\right] \left[\int_{L_{n-1}}^{L_n} \boldsymbol{S}_n^T \boldsymbol{\mathcal{M}} \boldsymbol{S}_n ds + \left(\boldsymbol{S}_n^T \boldsymbol{\mathcal{M}}_{n+1}^A \boldsymbol{S}_n\right)(L_n)\right]^{-1} \\
\left\{\boldsymbol{\tau}_n - \left[\int_{L_{n-1}}^{L_n} \boldsymbol{S}_n^T \boldsymbol{\mathcal{M}} \mathrm{ad}_{\mathrm{Ad}_{\boldsymbol{g}_n(s)}^{-1}\boldsymbol{\eta}(L_{n-1})} \boldsymbol{S}_n ds + \left(\boldsymbol{S}_n^T \boldsymbol{\mathcal{M}}_{n+1}^A \mathrm{ad}_{\mathrm{Ad}_{\boldsymbol{g}_n(s)}^{-1}\boldsymbol{\eta}(L_{n-1})} \boldsymbol{S}_n\right)(L_n)\right] \dot{\boldsymbol{q}}_n \right. \\
\left. - \left[\int_{L_{n-1}}^{L_n} \boldsymbol{S}_n^T \left(\mathrm{ad}_{\boldsymbol{\eta}}^* \boldsymbol{\mathcal{M}} \boldsymbol{\eta} - \boldsymbol{\mathcal{F}}\right) ds + \boldsymbol{S}_n^T(L_n) \boldsymbol{\mathcal{P}}_{n+1}^A\right]\right\}
\end{aligned} \tag{22}
$$

where exponential map in the Adjoint and coAdjoint representation are as follows

$$
\begin{aligned}
\mathrm{Ad}_{\boldsymbol{g}_n}(s) = e^{s\mathrm{ad}_{\boldsymbol{\xi}}} = \boldsymbol{I}_6 + \frac{1}{2\boldsymbol{\theta}}\left(3\sin(s\boldsymbol{\theta}) - s\boldsymbol{\theta}\cos(s\boldsymbol{\theta})\right)\mathrm{ad}_{\boldsymbol{\xi}} + \frac{1}{2\boldsymbol{\theta}^2}\left(4 - 4\cos(s\boldsymbol{\theta}) - s\boldsymbol{\theta}\sin(s\boldsymbol{\theta})\right)\mathrm{ad}_{\boldsymbol{\xi}}^2 \\
+ \frac{1}{2\boldsymbol{\theta}^3}\left(\sin(s\boldsymbol{\theta}) - s\boldsymbol{\theta}\cos(s\boldsymbol{\theta})\right)\mathrm{ad}_{\boldsymbol{\xi}}^3 + \frac{1}{2\boldsymbol{\theta}^4}\left(2 - 2\cos(s\boldsymbol{\theta}) - s\boldsymbol{\theta}\sin(s\boldsymbol{\theta})\right)\mathrm{ad}_{\boldsymbol{\xi}}^4
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ad}_{\boldsymbol{g}_n}^*(s) = \left(e^{s\mathrm{ad}_{\boldsymbol{\xi}}}\right)^{-T} = \boldsymbol{I}_6 + \frac{1}{2\boldsymbol{\theta}}\left(3\sin(s\boldsymbol{\theta}) - s\boldsymbol{\theta}\cos(s\boldsymbol{\theta})\right)\mathrm{ad}_{\boldsymbol{\xi}}^* + \frac{1}{2\boldsymbol{\theta}^2}\left(4 - 4\cos(s\boldsymbol{\theta}) - s\boldsymbol{\theta}\sin(s\boldsymbol{\theta})\right)\mathrm{ad}_{\boldsymbol{\xi}}^{*2} \\
+ \frac{1}{2\boldsymbol{\theta}^3}\left(\sin(s\boldsymbol{\theta}) - s\boldsymbol{\theta}\cos(s\boldsymbol{\theta})\right)\mathrm{ad}_{\boldsymbol{\xi}}^{*3} + \frac{1}{2\boldsymbol{\theta}^4}\left(2 - 2\cos(s\boldsymbol{\theta}) - s\boldsymbol{\theta}\sin(s\boldsymbol{\theta})\right)\mathrm{ad}_{\boldsymbol{\xi}}^{*4}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ad}_{\boldsymbol{g}_n}^{-1}(s) = e^{-s\mathrm{ad}_{\boldsymbol{\xi}}} = \boldsymbol{I}_6 - \frac{1}{2\boldsymbol{\theta}}\left(3\sin(s\boldsymbol{\theta}) - s\boldsymbol{\theta}\cos(s\boldsymbol{\theta})\right)\mathrm{ad}_{\boldsymbol{\xi}} + \frac{1}{2\boldsymbol{\theta}^2}\left(4 - 4\cos(s\boldsymbol{\theta}) - s\boldsymbol{\theta}\sin(s\boldsymbol{\theta})\right)\mathrm{ad}_{\boldsymbol{\xi}}^2 \\
- \frac{1}{2\boldsymbol{\theta}^3}\left(\sin(s\boldsymbol{\theta}) - s\boldsymbol{\theta}\cos(s\boldsymbol{\theta})\right)\mathrm{ad}_{\boldsymbol{\xi}}^3 + \frac{1}{2\boldsymbol{\theta}^4}\left(2 - 2\cos(s\boldsymbol{\theta}) - s\boldsymbol{\theta}\sin(s\boldsymbol{\theta})\right)\mathrm{ad}_{\boldsymbol{\xi}}^4
\end{aligned}
$$

0.5.2 Inverse dynamics

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