Problem Set 01

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1 Gradients and Hessians

a. The gradient of $f(x) = \frac{1}{2}x^T Ax + b^T x$ is:

$$\nabla f(x) = Ax + b$$

b. $f(x) = g(h(x))$

- We have:

$$\frac{\partial g(h(x))}{\partial x_i} = \frac{\partial g(h(x))}{\partial h(x)} \frac{\partial h(x)}{\partial x_i} = g'(h(x))\nabla h(x)$$

- Therefore:

$$\nabla f(x) = \nabla g(h(x)) = g'(h(x))\nabla h(x)$$

c. We have:

$$\Delta^2 f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2(\frac{1}{2}x_1^T A_{11} x_1)}{\partial x_1^2} & \frac{\partial^2(\frac{1}{2}x_1^T A_{12} x_2)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2(\frac{1}{2}x_1^T A_{1n} x_n)}{\partial x_1 \partial x_n} \\ \frac{\partial^2(\frac{1}{2}x_2^T A_{21} x_1)}{\partial x_2 \partial x_1} & \frac{\partial^2(\frac{1}{2}x_2^T A_{22} x_2)}{\partial x_2^2} & \cdots & \frac{\partial^2(\frac{1}{2}x_2^T A_{2n} x_n)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2(\frac{1}{2}x_n^T A_{n1} x_1)}{\partial x_n \partial x_1} & \frac{\partial^2(\frac{1}{2}x_2^T A_{2n} x_n)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2(\frac{1}{2}x_n^T A_{nn} x_n)}{\partial x_n^2} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} = A$$

d. The gradient of $f(x) = g(a^T x)$ is:

$$\nabla f(x) = \nabla g(a^T x) = g'(a^T x)a$$

- So now, we will compute $\nabla^2 f(x)$:

$$\frac{\partial^2 g(h(x))}{\partial x_i \partial x_j} = \frac{\partial g^2(h(x))}{\partial h^2(x)} \frac{\partial h(x)}{\partial x_i} \frac{\partial h(x)}{\partial x_j} = g''(h(x))\nabla^2 h(x)$$

$$\Rightarrow \frac{\partial^2 g(a^T x)}{\partial x_i \partial x_j} = g''(a^T x)\frac{\partial h(x)}{\partial x_i} \frac{\partial h(x)}{\partial x_j} = g''(a^T x)a_i a_j$$

- Therefore, we come to the result that:

$$\nabla^2 f(x) = g''(a^T x)aa^T$$

2 Positive definite matrices

a. We have

$$A^T = (zz^T)^T = zz^T = A$$
$$x^T Ax = x^T zz^T x = (z^T x)^T I(z^T x) \geq 0$$

b. N(A) $= x \in R^n$, $z^T x = 0$, R(A) $=$ R(zz^T) $= 1$

c. We have:

$$(BAB^T)^T = BA^T B^T = BAB^T$$
$$x^T BAB^T x = (B^T x)^T A(B^T x) >= 0$$

3 Eigenvectors, eigenvalues, and the spectral theorem

a. We have:

$$A = T\Lambda T^{-}1 \Leftrightarrow AT = T\Lambda$$

Therefore:

$$At^{(i)} = t^{(i)}\Lambda_{(i)} = \lambda_i t^{(i)}$$

b Consider:

$$A = U\Lambda U^T \Leftrightarrow AU = U\Lambda$$

From (a), we can prove that $u^{(i)}$ is an eigenvector of A, and $Au^{(i)} = \lambda_i u^{(i)}$

c. If A is PSD, we have $A = A^T$ and $x^T Ax \geq 0$, $\forall x \in R^n$.

- We have:

$$At^{(i)} = \lambda_i t^{(i)} \Leftrightarrow (t^{(i)})^T At^{(i)} = \lambda_i \|t^{(i)}\|^2 \geq 0 \Rightarrow \lambda_i \geq 0$$