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ARTICLE

Multivariate distribution for undrained shear strengths under various test procedures

Jianye Ching and Kok-Kwang Phoon

Abstract: The undrained shear strength (s_u) of a clay is not a constant. In particular, s_u values of a clay evaluated by different test procedures are different because these tests may have different stress states, stress histories, degrees of sampling disturbance, and strain rates. This study constructs the multivariate probability distribution of s_u from various test procedures based on a large clay database. This multivariate distribution provides an estimation of the normalized undrained shear strength based on four factors: test mode, overconsolidation ratio, strain rate, and plasticity. Once this multivariate distribution is constructed, interesting questions can be answered systematically using simple Bayesian analysis, e.g., given the s_u value for one test procedure, what is the updated mean and coefficient of variation of the s_u value for another test procedure?

Key words: undrained shear strength, uncertainty, correlation, multivariate distribution, site investigation.

Résumé : La résistance au cisaillement non drainé (s_u) d'une argile n'est pas une constante. Plus précisément, les s_u d'une argile évaluées par différentes procédures sont différentes puisque ces essais comportent différents états de contrainte, historique des contraintes, niveau de remaniement de l'échantillon, et taux de déformation. Cette étude présente la construction d'une distribution multi-variable des probabilités de su de différentes procédures d'essais à partir d'une grande base de données sur l'argile. Cette distribution multi-variable permet d'obtenir une estimation de la résistance au cisaillement non drainé normalisée basé sur quatre facteurs : le mode d'essai, le ratio de surconsolidation, le taux de déformation et la plasticité. Une fois que cette distribution multi-variable est construite, des questions intéressantes peuvent être répondues systématiquement à l'aide d'une analyse Bayésienne simple; par exemple, si l'on connait la valeur de s_u pour une procédure d'essai, quels sont la moyenne et le coefficient de variation de la valeur de s_u pour une autre procédure d'essai? [Traduit par la Rédaction]

Mots-clés : résistance au cisaillement non drainé, incertitude, corrélation, distribution multi-variable, investigation de site.

Introduction

It is well understood that the undrained shear strength (s_u) of a clay is not a fundamental material property, in the sense of being unique to a particular mineralogy. It depends on environmental factors such as stress state and stress history and test factors such as loading rate and loading mode, among others. It is useful to quantify the effects of these factors for two practical reasons. First, it allows s_u measured from different test types to be compared and cross-validated in a meaningful way and second, it allows s_u measured from a standard test, say a compression test, to be converted to a value more compatible with the failure mode observed in the field. Kulhawy and Mayne (1990) proposed the following systematic framework to estimate the normalized undrained shear strength as a function of three key factors, a_{test}, a_{OCR}, and a_{rate}:

$$(1) \qquad s_u/\sigma'_v = a_{test}a_{OCR}a_{rate}(\bar{s}_u/\sigma'_v)_{CIUC}$$

where σ'_v is the effective vertical stress; $(\bar{s}_u/\sigma'_v)_{CIUC}$ is the reference normalized undrained shear strength (dimensionless) under the isotropically consolidated undrained compression (CIUC) test with strain rate = 1%/h on a normally consolidated (NC) clay; a_{test}, a_{OCR}, and a_{rate} are dimensionless modifier factors that adjust the reference normalized undrained shear strength for the test mode, overconsolidation ratio, and strain rate, respectively. Equation (1) basically relates the undrained shear strength obtained from a particular test procedure to an undrained shear strength obtained from a reference test procedure, which is defined as test mode = CIUC, OCR = 1, and strain rate = 1%/h. Kulhawy and Mayne (1990) further proposed empirical–theoretical formulas for the a_{test}, a_{OCR}, and a_{rate} factors as reproduced in Table 1. The factors a_{test} and a_{OCR} depend on the test mode, but a_{rate} is thought to be independent of the test mode. It is clear that these factors reduce to unity when the test procedure coincides with the reference one, i.e., $a_{test\,=\,CIUC} = 1$, $a_{OCR\,=\,1} = 1$, and $a_{rate\,=\,1\%} = 1$. For instance, suppose the s_u/σ'_v values of a clay site are of interest, and the design goal is the shaft resistance of a pile (direct simple shear (DSS) test mode). A first-order estimate is that the clay's OCR = 5 and ϕ'_{TC} = 30°, and the time to failure is estimated to be 4 days (≈100 h). Assume that the strain at failure is 2%, then the target strain rate is therefore 2%/100 = 0.02%/h. With the above first-order information,

$$(2) \qquad \begin{aligned} a_{test} &= \frac{(s_u/\sigma'_v)_{DSS}}{(s_u/\sigma'_v)_{CIUC}} = 0.77 - 0.0064\phi'_{TC} = 0.578 \\ a_{OCR} &= \frac{(s_u/\sigma'_v)_{OCR=5}}{(s_u/\sigma'_v)_{NC}} = OCR^{0.776} = 3.487 \\ a_{rate} &= \frac{(s_u/\sigma'_v)_{strain\ rate=0.02\%/h}}{(s_u/\sigma'_v)_{strain\ rate=1\%/h}} = 1.0 + 0.1[\log_{10}(0.02\%/1\%)] = 0.83 \end{aligned}$$

The main purpose of this study is to develop a data-driven probabilistic version of eq. (1). Equation (1), to a certain degree, is not

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Table 1. Normalized undrained shear strength corrected for test mode (a_{test}), overconsolidation ratio (a_{OCR}), and strain rate (a_{rate}) (Kulhawy and Mayne 1990).

Factor	Test mode	Formula
a_{test}	CIUC	1
	CK_oUC	$1.13 - 0.0094\ \phi'_{TC}$
	DSS	$0.77 - 0.0064\ \phi'_{TC}$
	CK_oUE	$0.56 - 0.0046\ \phi'_{TC}$
a_{OCR}	CIUC	$OCR^{0.709}$
	CK_oUC	$OCR^{0.730}$
	DSS	$OCR^{0.776}$
	CK_oUE	$OCR^{0.818}$
a_{rate}	—	$1.0 + 0.1[\log_{10}(\text{strain rate}/1\%)]^a$

Note: CIU, isotropically consolidated undrained compression test; CK_0UC, K_0-consolidated undrained compression test; DSS, direct simple shear test; CK_0UE, K_0-consolidated undrained extension test; ϕ'_{TC}, effective friction angle in a triaxial compression test (TC stands for triaxial compression).

[a]Strain rate in % per hour.

purely data-driven, but relies on theories such as the modified Cam Clay model (Schofield and Wroth 1968). It is desirable to validate this equation more comprehensively with a larger database. More importantly, when practitioners try to obtain a first-order estimate of s_u based on this equation, it is essential to know the magnitude of uncertainty associated with such a first-order estimate. One would expect any first-order estimate in geotechnical engineering to be associated with significant uncertainty, rather than negligible uncertainty. This uncertainty will include inherent variability arising from a global database spanning diverse geographic locales, measurement errors, and transformation uncertainty associated with eq. (1). Estimating the magnitude of this "lumped" uncertainty is vital to reliability analysis and reliability-based design. It is useful to quantify this uncertainty to get a feel for the accuracy of the estimate from eq. (1), even if there is no requirement to perform a reliability-based design. In addition, it is fairly common to measure s_u using different test procedures. A multivariate probability distribution is more appropriate to characterize the uncertainties under this more general and more common condition. It is worth pointing out that uncertainty in one component of the multivariate distribution (s_u obtained from one test procedure) can be reduced if other components (s_u from other test procedures) are available. This reduction of uncertainty can be exploited to economize design in a reliability-based design framework. The rigorous construction of *complete* multivariate probability distributions for geotechnical parameters is rather limited at present (Ching and Phoon 2012*a*; Phoon et al. 2012).

This study constructs a complete multivariate distribution for s_u derived from seven different test procedures based on a large geographically diverse clay database covering 223 sites. The details for the database will be presented and the steps of constructing such a multivariate distribution will be explained below. The critical step is the estimation of the correlation matrix for an underlying multivariate normal distribution in which s_u from a variety test procedures can be extracted component-wise by a suitable lognormal transform. The estimation of this correlation matrix requires complete multivariate data, which is generally not available; for example, $s_u(CK_oUC)$ and s_u(UU) — where CK_oUC is the K_0-consolidated undrained compression test and UU is the unconsolidated undrained test — are not simultaneously available in our database. A reasonable approach to populate the entire correlation matrix in the presence of inevitable data gaps is discussed in this study. The main product of this paper is a seven-dimensional multivariate normal distribution that forms the basis for characterizing the mean values, coefficients of variation (COV), and correlations between the undrained shear strengths obtained under various test procedures.

Database

This study compiles a clay database from the literature consisting of a large number of s_u data points obtained from different test procedures. Many s_u data points are associated with a known test mode (6310 points), a known OCR (4584 points), and a known plasticity index (PI) (4541 points). There are some data points in which OCR is not known. The impact of using data points that cannot be standardized using eq. (1) for the estimation of correlation coefficients is discussed below. This database consists of data points from 164 studies. The number of data points associated with each study varies from 1 to 167 with an average 38.5 data points per study. The geographical regions cover Australia, Austria, Brazil, Canada, China, England, Finland, France, Germany, Hong Kong, Iraq, Italy, Japan, Korea, Malaysia, Mexico, New Zealand, Norway, Northern Ireland, Poland, Singapore, South African, Spain, Sweden, Thailand, Taiwan, United Kingdom, United States of America, and Venezuela. The clay properties cover a wide range of OCR values (mostly 1~10, with a few studies where OCR > 10, but nearly all studies are with OCR < 50) and a wide range of sensitivity, S_t (sites with S_t = 1~ tens or hundreds are fairly typical). The details for this database are shown in Table 2.

Standardization of undrained shear strength

Effect of plasticity index

In this study, eq. (1) is generalized to consider the effect of the plasticity index (PI) explicitly given that this additional piece of information is available in Table 2:

$$s_u/\sigma'_v = a_{test}a_{OCR}a_{rate}a_{PI}(\bar{s}_u/\sigma'_v)_{CIUC} \tag{3}$$

where a_{PI} is a modifier factor for the plasticity index and PI = 20% is considered an aspect of the reference test procedure, and $(\bar{s}_u/\sigma'_v)_{CIUC}$ is further restricted to the reference PI = 20%. This is a misnomer, because there is qualitative difference between eqs. (1) and (3). The former involves standardization of aspects relating to the test procedure, while the latter seeks to broaden eq. (1) to cover aspects relating to the material property. However, for brevity, the term "test procedure" will be used in this paper rather than the more accurate but lengthier term "test procedure and clay type". The reference plasticity index = 20% refers to a medium plastic clay.

Equation (3) is useful in itself in two ways. First, practitioners can obtain a first-order estimate of s_u under various stress states using this equation in the absence of data from a detailed site investigation or prior to a site investigation (note that $(\bar{s}_u/\sigma'_v)_{CIUC}$ can be estimated based on the modified Cam Clay model; see Appendix G in Kulhawy and Mayne (1990)). Second, this equation can convert s_u obtained from one test procedure to another. This conversion has a practical application — practitioners can estimate s_u under CK_oUC, K_0-consolidated undrained extension test (CK_oUE), and DSS based on the CIUC test result. The CIUC test is more commonly conducted than the other three tests.

Data standardization

An important step for the construction of the multivariate distribution is to convert all s_u data points in the database into the following standardized form:

$$\bar{s}_u/\sigma'_v = (s_u/\sigma'_v)/(a_{OCR}a_{rate}a_{PI}) \tag{4}$$

where $\bar{s}_u/\sigma'_v$ is the normalized undrained shear strength of a NC clay with PI = 20 subjected to a 1%/h strain rate — three factors,

Table 2. Basic information of the clay database.

Reference	Country	Liquid index, LI	OCR	S_t	s_u[a] CIUC	$\mathrm{CK_0UC}$	$\mathrm{CK_0UE}$	DSS	FV	UU	UC	Availability of PI
Aas et al. (1986)	Norway	0.1~1.5	1.6~1.8	—	—	—	—	—	21	—	—	Yes
Agarwal (1967)	UK	—	—	—	—	—	—	—	—	—	16	No
Alberro and Santoyo (1973)	Mexico	—	—	—	14	—	3	—	—	—	—	No
Amerasinghe and Parry (1975)	UK	—	—	—	10	3	—	—	—	—	—	No
Anderson et al. (1982)	Brazil	1.1~1.6	1.6~7.4	1.7~5.0	—	—	—	—	27	—	—	Yes
Andersen et al. (1980)	Norway	—	1~50	—	5	—	1	5	—	—	—	No
Azzouz and Lutz (1986)	USA	0.1~0.5	1~1.4	1~1.1	—	—	—	—	7	29	24	Yes
Azzouz et al. (1983	Venezuela	0.5~0.8	1.1~3.3	—	—	—	—	30	27	18	—	Yes
Balasubramanian and Chaudhry (1976)	Thailand	—	—	—	8	—	—	—	—	—	—	Yes
Baligh et al. (1980)	USA	0.2~1.7	1.1~6.1	—	—	—	—	—	50	—	—	Yes
Baracos et al. (1980)	Canada	–0.0~0.6	2.7~9.2	—	32	—	—	—	—	—	6	Yes
Berre and Bjerrum (1973)	Norway	0.6~1.0	1~1.8	5~11	—	7	6	13	6	—	—	Yes
Bishop (1971)	UK	—	—	—	9	—	—	—	—	—	—	No
Bjerrum (1954)	Norway	2.4	—	—	—	1	—	—	—	—	—	Yes
Bjerrum and Lo (1963)	Norway	—	—	5	12	12	—	—	—	—	—	No
Bjerrum (1967)	Norway	2.1~3.9	—	6~200	—	—	—	—	38	—	—	Yes
Bjerrum (1972)	Thailand	0.9	—	—	—	—	—	1	1	—	—	Yes
	Canada	1.1	—	—	—	—	—	1	1	—	—	Yes
	Norway	0.6~0.7	—	—	—	—	—	3	3	—	—	Yes
	Scandinavia	0.3	—	—	—	—	—	1	1	—	—	Yes
Bjerrum (1973)	Norway	0.8~5.3	—	8~100	—	12	—	—	—	14	—	Yes
Bozozuk and Leonards (1972)	Canada	1.2~2.5	1.3~2.0	8~100	6	15	—	—	19	—	—	Yes
Broms and Ratnam (1963)	Norway	—	3.5~13	—	3	3	—	—	—	—	—	No
Burland et al. (1977)	UK	–0.2~0.1	—	—	11	—	—	—	—	—	—	Yes
Cadling and Odenstad (1950)	Sweden	0.9~1	—	5~167	4	—	—	—	84	—	18	Yes
Campanella et al. (1988)	Canada	—	1.0~4.5	—	—	—	—	—	20	—	—	No
Cancelli (1981)	Italy	0.0~0.3	—	—	—	—	—	—	—	—	13	Yes
Cancelli and Cividini (1984)	Italy	0.0~0.4	1.0~26.7	—	—	—	—	—	—	5	—	Yes
Chandler (1988)	Norway	0.6~4	1~3.2	3~42.5	—	9	—	—	9	—	—	Yes
	USA	0.6~0.9	4.3~6.3	4~5	—	2	—	—	2	—	—	Yes
	Canada	1.7~2.3	1.8~2.4	>15~80	—	2	—	—	2	—	—	Yes
	Sweden	0.9~1.4	1.3~1.4	17.5~20	—	2	—	—	2	—	—	Yes
	Italy	0.5~0.8	1.1	2~5	—	2	—	—	2	—	—	Yes
	Iraq	0.6	2.7	—	—	1	—	—	1	—	—	Yes
	Japan	0.7~0.9	2~3.8	—	—	2	—	—	2	—	—	Yes
	UK	0.8	1.1	5	—	1	—	—	1	—	—	Yes
	Australia	1.0	1.7	2.5	—	1	—	—	1	—	—	Yes
Chang (1991)	Singapore	0.1~1.0	0.8~2.2	1.4~10.7	—	—	—	—	55	18	13	Yes
	Malaysian	0.5~1.8	0.8~3.5	1.5~24.9	—	—	—	—	38	—	—	Yes
Chen and Kulhawy (1993)	Mexico	0.4~0.5	1.1~3.5	2.5	5	—	—	—	—	2	3	Yes
	Canada	0.4~2.4	1.1~7	26~128	20	—	—	—	—	9	11	Yes
	France	–0.1~0.4	1.2~17	—	3	—	—	—	—	1	2	Yes
	USA	–0.2~1.2	1~22	7.5~8	33	—	—	—	—	19	9	Yes
	Japan	—	1.2	—	3	—	—	—	—	3	—	Yes
	Brazil	1.2~1.4	1.7~2.1	2.6	4	—	—	—	—	4	—	Yes
	New Zealand	2~3.1	2.1~2.4	—	3	—	—	—	—	3	—	Yes
	Norway	0.9~1.2	1.1~1.3	9	2	—	—	—	—	—	2	Yes
	UK	0.1~0.3	4~20	—	4	—	—	—	—	4		Yes
	—	0.0~2.3	1.1~12	5~50	14	—	—	—	—	9	5	Yes
Chin and Liu (1997)	Taiwan	—	1~4.1	—	—	6	6	—	—	—	—	No
Chin et al. (1989)	Taiwan	—	1~8	—	32	—	—	—	—	—	—	Yes
Clough and Denby (1980)	China	0.5~2.0	—	—	—	—	—	—		21	—	Yes
Coutinho (2007)	Brazil	0.4~2.2	0.7~2.3	5.0~17.1	—	—	—	—	38	—	12	Yes
Crawford and Eden (1965)	Canada	0.9~2.5	1.5~7.9	10~500	—	—	—	—	9	—	—	Yes
Croce et al. (1969)	Italy	–0.1~0.6	1.1~4.3	—	—	—	—	—	—	—	46	No
Crooks (1981)	Northern Ireland	—	1.4~2.8	—	—	15	15	—	—	—	—	No
D'Appolonia et al. (1971)	USA	0.7	9.3	—	—	—	—	—	—	1	—	Yes
D'Appolonia and Saada (1972)	Norway	—	1.0~37.5	—	—	—	—	—	—	104	—	No
Dascal and Tournier (1975)	Canada	1.2~3.9	1.4~3.3	—	—	—	—	—	13	—	—	Yes
Degroot and Lutenegger(2003)	USA	—	1.4~9.5	—	11	—	—	12	30	—	—	No
De Lory and Salvas (1967)	Canada	0.8~1.3	—	—	25	15	—	—	—	9	—	Yes
De Lory and Salvas (1969)	Canada	—	—	—	2	2	—	—	11	—	11	No
Donaghe and Townsend (1978)	USA	—	—	—	24	4	—	—	—	—	—	No

Table 2 (*continued*).

Reference	Country	Liquid index, LI	OCR	S_t	s_u[a] CIUC	CK_0UC	CK_0UE	DSS	FV	UU	UC	Availability of PI
Eden and Bozozuk (1962)	Canada	1.3~2.1	1.2~1.9	8.3~18	—	—	—	—	9	—	—	Yes
Eden and Hamilton (1957)	Canada	0.6~3.2	1.0~3.5	3.5~1160	—	—	—	—	49	—	—	Yes
Eden and Crawford (1957)	Canada	0.8~3.8	1.6~4.4	39.2~1323	—	—	—	—	35	—	—	Yes
Eide and Holmberg (1972)	Thailand	0.7~0.9	1.6~2.2	5.5~12.4	—	—	—	—	28	3	—	Yes
Flaate and Preber (1974)	Norway	−0.5~4.0	—	—	—	—	—	—	49	—	89	Yes
Finno (1989)	USA	0.1~0.5	0.8~1.5	1.6~3.4	—	—	—	—	35	27	35	Yes
Hanzawa (1977*a*)	Iraq	0.6~0.7	—	—	—	—	—	—	11	—	—	Yes
Hanzawa (1977*b*)	Iraq	0.7~1.1	1	—	—	18	6	—	—	—	15	Yes
Hanzawa et al. (1979)	Iraqi	0.6~0.7	1.3~5.7	—	—	9	9		12		8	Yes
Hanzawa (1981)	Japan	—	—	—	—	3	—	—	—	—	—	No
Helenelund (1977)	Sweden	—	—	—	—	—	—	—	36	—	—	Yes
Hight et al. (1992)	U.K.	—	1.3~1.6	—	—	—	—	—	9	—	—	Yes
Holmberg (1977)	Thailand	—	—	—	—	12	—	18	—	—	—	No
Hong et al. (2010)	Korea	—	—	—	—	—	—	—	—	34	—	No
Horn and Lambe (1964)	USA	0.4~0.8	0.8~3.7	4.1~9.6	—	—	—	—	11	11	—	Yes
Jamiolkowski et al. (1982)	Italy	—	—	—	—	2	1	1	—	—	—	No
Jamiolkowski et al. (1988)	Italy	—	7.9~20.7	—	—	—	—	—		17	—	No
Karlsrud and Myrvoll (1976)	Norway	0.5~3.6	—	3~39.3	10	20	18	3	17	—	—	Yes
Karlsson and Pusch (1967)	Sweden	0.3~2.8	—	13.9~307	—	—	—	—	18		12	Yes
Kenney (1966)	Norway	0.5~2.9	—	3~60	—	67	—	21	—	—	—	Yes
Kinner and Ladd (1970)	USA	—	1~80.5	—	26	13	—	23	—	—	—	No
Kitago et al. (1976)	Japan	—	—	—	—	1	1	—	—	—	—	No
Kjekstad and Lunne (1981)	Norway	−0.1~0.3	—	—	1	21	14	25	—	—	30	Yes
Konrad and Law (1987)	Canada	0.5~4.5	1.0~9.7	7.4~500	—	—	—	—	55		3	Yes
Koutsoftas and Fischer (1976)	USA	0.3~1.1	1~29.5	108~413.8	—	8	—	4	23	62	—	Yes
Koutsoftas (1981)	USA	—	1~12	—	—	14	6	12	—	—	—	No
Koutsoftas and Ladd (1985)	USA	—	1~7.7	—	—	—	—	5	—	—	—	No
Koutsoftas et al. (1987)	Hong Kong	0.2~1.8	1~8.6	—	7	4	4	12	30	—	—	Yes
Kulhawy and Mayne (1990)	—	−0.4~2.4	1	—	74	91	25	85	—	—	—	Yes
Kulkarani et al. (1967)	India	0.6~0.9	0.7~1.0	3.4~4.4	—	—	—	—	5	—	5	Yes
Lacasse et al. (1977)	Canada	—	1~2.9	—	20	—	—	—	—	—	—	No
	USA	—	1~4.8	—		—	—	7	—	—	—	No
Lacasse et al. (1981)	Norway	—	1.1~2.7	—		11	11	11	11	—	—	No
Lacasse et al. (1985)	Norway	0.9~5.5	1.4~7.1	—	—	—	—	—	18	—	—	Yes
Ladd (1964)	USA	—	5.7	—	4	1	2	—	—	8	—	No
Ladd (1965)	USA	—	—	—	4	4	—	—	—	—	—	No
Ladd (1967)	USA	—	—	—	—	—	—	—	—	28	—	Yes
Ladd (1972)	USA	0.8~2.5	1.3~7.8	—	—	—	—	14	17	—	16	Yes
Ladd (1981)	USA	—	—	—	—	—	—	7	—	—	—	Yes
Ladd (1991)	USA	1.4~2.3	1~4.3	—	—	13	4	37	39	4	—	Yes
Ladd et al. (1971)	USA	1.0	1~8.1	8	3	10	2	—	—	—	—	Yes
Ladd et al. (1972)	USA	1.3~2.5	1.3~2.8	6.5~10.1	—	—	—	—	12	—	—	Yes
Ladd et al. (1977)	—	—	3.0~18.2	—	10	—	—	—	—	—	—	No
Ladd and Azzouz (1983)	Venezuela	−0.7~0.5	1.3	—	—	—	—	23	—	—	30	Yes
Ladd and Edgers (1971)	USA	—	1~12.1	5~100	7	21	13	99	7	—	—	Yes
Ladd and Foott (1974)	USA	0.6~2.7	1~4.9	—	—	9	9	—	18	13	16	Yes
Ladd and Lambe (1963)	—	—	1~3	5~10	26	6	—	—	—	27	—	Yes
Lafleur et al. (1988)	Canada	1.2~1.7	1.1~6.1	—	—	—	—	—	16	—	—	Yes
Lacasse and Lunne (1982)	Norway	0.3~2.0	1.0~15.5	—	—	—	—	—	151	—	—	Yes
Lambe (1963)	Venezuela	—	—	—	5	—	—	—	—	—	—	No
Lambe (1964)	Venezuela	—	—	—	10	—	—	—	—	—	—	No
La Rochelle et al. (1974)	Canada	1.1~3.3	2.0~5.2	—	5	—	—	—	23	11	—	Yes
Larsson (1980)	Norway	—	—	—	—	4	4	3	—	—	—	Yes
Larsson and Mulabdic (1991)	Sweden	0.5~1.5	1.1~4.2	—	—	—	—	—	167	—	—	Yes
Leathers and Ladd (1978)	USA	0.7~1.0	1.1~9.9	—	—	—	—	—	7	—	—	Yes
Lee and Morrison (1970)	USA	—	—	—	1	1	—	—	—	—	—	No
Lefebvre and LeBoeuf (1987)	Canada	—	—	—	17	9	—	—	—	—	—	No
Leroueil et al. (1983)	Canada	1.1~2.9	1.4~2.6	—	—	—	—	—	10	—	—	Yes
Lew (1981)	Canada	—	—	—	6	6	—	—	—	—	—	No
Liu (1999)	Taiwan	—	—	—	23	11	9	5	—	—	—	Yes
Lowe and Karafiath (1960)	Venezuela	—	—	—	7	6	—	—	—	—	—	No
Lumb and Holt (1968)	Hong Kong	0.7~1.1	—	—	—	—	—	—	41	—	—	Yes
Lunne et al. (1985)	Norway	−0.1~1.2	1.2~10	—	—	26	—	30	6	—	—	Yes

Table 2 (*concluded*).

Reference	Country	Liquid index, LI	OCR	S_t	s_u[a] CIUC	CK_0UC	CK_0UE	DSS	FV	UU	UC	Availability of PI
Mahar and O'Neill (1983)	USA	0.2~0.7	3.7~5.0	—	—	—	—	—	4	—	4	Yes
Massarch et al. (1975)	Sweden	0.8~2.8	0.9~2.7	7~36	—	—	—	—	39	—	6	Yes
Massarch and Broms (1976)	Sweden	0.8~1.5	1.1~2.7	11~36	—	—	—	—	15	—	—	Yes
Mayne (1980)	USA	—	1~21.1	—	7	—	—	—	—	—	—	No
Mayne (1985*a*)	USA	—	1~75	—	56	56	8	—	—	—	—	No
Mayne (1985*b*)	—	—	1	—	15	38	34	98	—	—	—	Yes
Mayne (1988)	—	—	0.9~60.2	—	125	107	72	78	—	—	—	No
Mayne and Holtz (1985)	USA	—	—	—	13	53	53	—	—	—	—	No
Mitachi and Kitago (1979)	Japan	—	—	—	1	1	1	—	—	—	—	No
Moh et al. (1969)	Thailand	0.1~0.8	1.2~24.0	1.5~5.0	—	—	—	—	4	—	8	Yes
Morin et al. (1983)	Canada	0.7~2.9	1.1~28.9	—	—	—	—	—	51	—	—	Yes
Moum and Rosenqvist (1961)	USA	0.2~1.0	—	3.4~9.4	—	28	—	—	—	—	—	Yes
Nakase and Kamei (1983)	Japan	—	—	—	12	8	8	—	—	—	—	Yes
Nakase and Kamei (1986)	Japan	—	—	—	—	9	9	—	—	—	—	No
Nakase and Kamei (1988)	Japan	—	—	—	12	12	12	—	—	—	—	Yes
Ng and Lo (1985)	Canada		1.2~2.0	5.2~8.5	—	—	—	—	21		10	Yes
Niazi et al. (2010)	UK	—	2.9~11.5	—	—	—	—	9	—	—	—	No
Ohtsubo et al. (1995)	Japan	0.9~1.6	0.8~2.5	10.7~33.6	—	—	—	—	—	—	9	Yes
Ohtsubo et al. (2007)	Japan	0.7~2.7	0.7~5.1	3.9~1000	—	—	—	—	72	—	—	Yes
O'Riordan et al. (1982)	UK	0.7~2.2	—	—	—	—	—	—	35	—	—	Yes
Ou and Hsiao (1994)	Taiwan	—	1~5.9	—	7	7	—	—	—	—	—	No
Parry (1960)	England	—	1.0~24.5	—	40	23	16	—	—	—	—	No
Parry (1968)	Australia	0.7~1.2	—	2.3	—	—	—	—	17	—	—	Yes
Parry and Nadarajah (1974)	UK	—	—	—	4	4	4	—	—	—	—	No
Parry and Wroth (1981)	Canada	0.9~2.8	1.1~3.9	10~115	—	—	—	—	12	—	12	Yes
Phoon (2013)	Singapore	0.1~1.2	1.2~26.2	—	—	—	—	—	—	—	42	Yes
Powell and Quarterman (1988)	UK	0.6~1.0	1.4~3	—	—	—	—	—	—	—	11	Yes
Prevost (1979)	—	—	—	—	—	—	—	7	—	—	—	Yes
Quiros and Young (1988)	USA	—	1.0~6.2	—	4	5	—	24	—	—	—	No
Rad and Lunne (1988)	—	—	1.2~15	—	5	29	—	—	—	—	—	No
Raymond (1973)	Canada	—	1~2.5	1.6~11.4	—	—	—	—	28		13	No
Rocha-Filho and Alencar (1985)	Brazil	1.4~1.7	1.6~2.1	1.6~8.2	—	—	—	—	3	—	—	Yes
La Rochelle et al. (1988)	Canada	0.0~2.1	1.1~3.5	—	—	—	—	—	24	—	—	Yes
Roy et al. (1982)	Canada	0.4~3.5	2.3~6.1	—	5	—	—	—	26	6	—	Yes
Sanchez et al. (1979)	UK	—	1~2	—	—	10	—	—	—	—	—	No
Schofield and Wroth (1968)	UK	—	—	—	—	28	8	—	—	—	—	Yes
Senneset and Janbu (1985)	Norway	0.4~2.7	—	—	—	—	—	—	—	13	—	Yes
Silvestri and Aubertin (1988)	Canada	0.6~1.4	2.9~7.2	6.1~24	—	—	—	—	9	—	—	Yes
Simons (1960)	Norway	—	—	2.5~19.9	—	—	—	—	23	—	—	Yes
Simons (1976)	UK	—	—	—	—	—	—	7	7	—	—	Yes
Skempton (1948*a*)	UK	0.4~0.6	0.8~1.4	1.3~2.2	—	—	—	—	—	—	5	Yes
Skempton (1948*b*)	UK	0.4~0.9	—	2.4~3.3	—	—	—	—	17	—	—	Yes
Skempton (1957)	Hong Kong	−0.1~0.8	—	—	—	14	—	—	14	—	—	Yes
Skempton (1961)	UK	−0.1~0.7	1.4~44	—	23	—	—	—	—	—	—	Yes
Skempton and Bishop (1954)	—	—	1	—	—	5	—	—	—	—	—	Yes
Skempton and Henkel (1953)	UK	0.2~0.8	0.9~2.8	1.5~7	—	—	—	—	19	—	—	Yes
Skempton and Sowa (1963)	UK	—	—	—	5	46	—	—	—	—	—	No
Stille and Fredriksson (1979)	Sweden	1.2~1.5	—	—	—	—	—	—	6	—	—	Yes
Tan et al. (2003)	Singapore	—	—	—	—	—	—	—	16	—	—	Yes
Tavenas et al. (1975)	Canada	1.4~4.1	1.8~2.2	—	5	—	—	—	14	11	—	Yes
Tavenas and Leroueil (1977)	Canada	—	—	—	7	—	—	—	—	—	—	No
Wei and Pant (2010)	USA	0.2~0.7	—	—	—	—	—	—	—	28	—	Yes
Wroth and Houlsby (1985)	USA	0.6~1.3	1.0~5.6	—	—	—	—	—	15	8	—	Yes
Wu (1958)	USA	0.1~1.7	0.8~3.0	1.8~50.1	—	—	—	—	—	—	55	Yes
Wu et al. (1962)	USA	—	—	8	—	—	—	—	—	—	2	No
Wu et al. (1963)	USA	—	—	—	9	—	—	—	—	—	—	No
Wu et al. (1975)	USA	—	1.3~15.4	—	—	—	—	14	—	—	—	No
Wu et al. (1978)	USA	—	1.0~3.3	—	—	—	—	12	—	—	8	No
Yasuhara et al. (1982)	Japan	—	—	—	6	3	—	—	—	—	—	No

NOTE: LI, liquidity index; OCR, overconsolidation ratio; S_t, sensitivity; CIUC, isotropically consolidated undrained compression test; CK_0UC, K_0-consolidated undrained compression test; CK_0UE, K_0-consolidated undrained extension test; DSS, direct simple shear test; FV, field vane; UU, unconsolidated undrained compression test; UC, unconfined compression test; PI, plasticity index.

[a]Numbers in the "s_u" columns indicate the numbers of the s_u data points.

Table 3. Revised set of a_{OCR}, a_{rate}, and a_{PI} factors developed in Ching et al. (2013).

Factor	Test type	Formula
$a_{OCR} = OCR^{\Lambda}$	CIUC	$OCR^{0.602}$
	CK_0UC	$OCR^{0.681}$
	CK_0UE	$OCR^{0.898}$
	DSS	$OCR^{0.749}$
	FV	$OCR^{0.902}$
	UU	$OCR^{0.800}$
	UC	$OCR^{0.932}$
a_{rate}	—	$1.0 + 0.1[\log_{10}(\text{strain rate}/1\%)]$
$a_{PI} = (PI/20)^{\beta}$	CIUC	$(PI/20)^0 = 1$
	CK_0UC	$(PI/20)^0 = 1$
	CK_0UE	$(PI/20)^{0.178}$
	DSS	$(PI/20)^{0.0655}$
	FV	$(PI/20)^{0.124}$
	UU	$(PI/20)^0 = 1$
	UC	$(PI/20)^0 = 1$

Note: FV, field vane test; UC, unconfined compression test.

namely PI, OCR, and strain rate, are standardized, but the test mode is not standardized although the effects of test mode on a_{OCR}, a_{rate}, and a_{PI} are considered. The standardization process was presented elsewhere in Ching et al. (2013). The resulting a_{OCR}, a_{rate}, and a_{PI} factors are given in Table 3. Note that $a_{OCR} = OCR^{\Lambda}$ and $a_{PI} = (PI/20)^{\beta}$ depend on the test mode. The comparison made in Ching et al. (2013) between the new set of (a_{OCR}, a_{rate}, a_{PI}) factors in Table 3 and those in Table 1 generally shows good agreement.

Statistics of random variables

From hereon, the standardized $\bar{s}_u/\sigma'_v$ obtained in eq. (4) will be denoted by Y. An index i is used to denote the test mode: i = 1, 2, 3, 4, 5, 6, and 7 for CIUC, CK_0UC, CK_0UE, DSS, field vane test (FV), UU, and unconfined compression test (UC), respectively. For instance, Y_1 is the standardized $(\bar{s}_u/\sigma'_v)_{CIUC}$, Y_6 is $(\bar{s}_u/\sigma'_v)_{UU}$. Hence, there are seven random variables: Y_1, Y_2, …, Y_7. Table 4 shows the statistics of Y_i, including mean = E(Y_i); coefficient of variation, $COV_i = \sigma(Y_i)/E(Y_i)$ (where standard deviation = $\sigma(Y_i)$); minimum value = Min_i; and maximum value = Max_i. For Y_3 and Y_4 ($(\bar{s}_u/\sigma'_v)_{CK_0UE}$ and $(\bar{s}_u/\sigma'_v)_{DSS}$), the statistics are based on $(s_u/\sigma'_v)/(a_{OCR}a_{rate})$ data in the database, as there are insufficient data points with known PI (Ching et al. 2013).

It is evident from Table 4 that Y_1 (CIUC) has the largest mean, and Y_3 (= $(\bar{s}_u/\sigma'_v)_{CK_0UE}$) has the smallest. The ratio $\alpha_i = E(Y_i)/E(Y_1)$ shown in Table 4 has a physical significance similar to a_{test} in Table 1 — both involve the ratio $(\bar{s}_u/\sigma'_v)/(\bar{s}_u/\sigma'_v)_{CIUC}$. The numbers in the parenthesis in the α_i column in Table 4 are the a_{test} values in Table 1 when ϕ'_{TC} is assumed to be 30°. Note that the a_{test} factors in Table 1 are derived from the modified Cam Clay model (Schofield and Wroth 1968), but the α_i ratios shown in Table 4 are purely data driven. The agreement is rather remarkable.

Given a test mode i, the scatter in the Y_i data points, quantified by the COV_i in Table 4, may be due to measurement errors in s_u and (global) inherent variability in s_u (s_u from different geographic locales) as well as the transformation uncertainties associated with the standardization steps for PI, strain rate, and OCR. Y_5, Y_6, and Y_7 (FV, UU, UC) are associated with significantly larger COVs than the other Y values, probably because FV is a field test and UU and UC are susceptible to sample disturbance. The variables Y_1, Y_2, Y_3, and Y_4 ($\bar{s}_u/\sigma'_v$ for consolidated tests) are associated with smaller COVs. It is hypothesized that these four test modes are usually conducted for research rather than in practice and more care may have been exercised. In general, the COVs are low to medium: Y_1, Y_2, Y_3, and Y_4 are associated with 0.3 < COV < 0.4 and can be classified as low to medium variability based on the guidelines suggested by Kulhawy et al. (2012). This suggests that if the test mode (CIUC, CK_0UC, CK_0UE or DSS), OCR, PI, and strain rate are all known, the variability of s_u/σ'_v is low to medium. On the other hand, Y_5, Y_6, and Y_7 are associated with 0.4 < COV < 0.7 and can be classified as medium to high variability.

Application of statistics

Table 4 summarizes the statistics of a global database for s_u derived from different test procedures. The term "global" is used here in the sense that the sites are geographically diverse and not specific to a particular country, region or city. These statistics can be used to estimate the first two moments (mean and COV) of the s_u/σ'_v value when the test mode, OCR, PI, and strain rate are known. The COV already captures the global inherent variability, measurement errors in s_u, and transformation errors during the above standardization steps. For instance, suppose the mean and COV of s_u/σ'_v of a clay site are of interest. The design target is the shaft resistance of a pile (DSS test mode). A first-order estimate is that the clay is roughly characterized by OCR = 5 and PI = 30, and the possible time to failure is estimated to be 4 days (≈100 h). Assume that the strain at failure is 2%, the target strain rate is therefore 2%/100 = 0.02%/h. With the above first-order information, s_u/σ'_v can be roughly estimated prior to site investigation. This normalized undrained shear strength, s_u/σ'_v, is related to $Y_4 = (\bar{s}_u/\sigma'_v)_{DSS}$ through the following equation:

$$
\begin{aligned}
(5)\qquad s_u/\sigma'_v &= a_{OCR}a_{rate}a_{PI}\left(\bar{s}_u/\sigma'_v\right)_{DSS} \\
&= 5^{0.749}[1.0 + 0.1\log_{10}(0.02\%/1\%)](30/20)^{0.0655}\,Y_4 \\
&= 2.85Y_4
\end{aligned}
$$

As a result, the mean of s_u/σ'_v is 2.85 × 0.241 = 0.69 and the COV is 0.399 (0.241 and 0.399 are the mean and COV, respectively, of Y_4, as seen in Table 4). The mean 0.69 reflects the best estimate for s_u/σ'_v based on the global database, and the COV of 0.399 reflects the measurement errors and (global) inherent variability in the s_u values, as well as the transformation uncertainties during the standardization steps for PI, strain rate, and OCR. These statistics can be considered as "prior" statistics and they can be updated using standard Bayesian methods if additional site-specific data are made available.

Multivariate distribution of (Y_1, Y_2, …, Y_7)

Correlations between two variables are widely used in the geotechnical engineering literature, in part because of their simplicity and ready availability of bivariate information. However, they suffer from one important practical limitation. The uncertainty in estimating one variable cannot be further reduced when two or more variables are measured. This implies there is no incentive to conduct more than two tests, which is against prevailing practice where multiple tests are commonly conducted during site investigation. A multivariate probability model can exploit costly site data more effectively in terms of reducing the uncertainty associated with the estimation of a desired variable. In the opinion of the authors, there are compelling practical benefits to adopting a multivariate model, although it is admittedly more complicated and less familiar to engineers.

This section briefly reviews the multivariate normal distribution. The theory is elegant (Phoon 2006). The key challenge is to construct this distribution within the data constraints commonly encountered in geotechnical practice. The conventional bivariate model (correlation between two variables) is a special case of this multivariate normal model. It is widely used in part because it does not require more data than what are produced in most site investigation programs. The following section addresses the central issue pertaining to how a multivariate normal model can be constructed from real data. The last section presents one useful application of this multivariate model.

Table 4. Statistics of the Y data points.

	Variable	n	Mean $E(Y_i)$	COV_i	$\alpha_i = E(Y_i)/E(Y_1)^a$	Min_i	Max_i	$E[\ln(Y_i)](\lambda_i)$	$\sigma[\ln(Y_i)](\xi_i)$
Y_1	$(\bar{s}_u/\sigma'_v)_{CIUC}$	637	0.404	0.316	1.00 (1.00)	0.12	0.82	−0.955	0.315
Y_2	$(\bar{s}_u/\sigma'_v)_{CK_0UC}$	555	0.350	0.318	0.87 (0.85)	0.063	1.72	−1.090	0.280
Y_3	$(\bar{s}_u/\sigma'_v)_{CK_0UE}$	224	0.184	0.324	0.46 (0.42)	0.055	0.45	−1.748	0.355
Y_4	$(\bar{s}_u/\sigma'_v)_{DSS}$	573	0.241	0.399	0.60 (0.58)	0.081	1.83	−1.468	0.277
Y_5	$(\bar{s}_u/\sigma'_v)_{FV}$	1057	0.275	0.416	0.68	0.068	1.25	−1.363	0.372
Y_6	$(\bar{s}_u/\sigma'_v)_{UU}$	435	0.243	0.504	0.60	0.067	1.44	−1.523	0.463
Y_7	$(\bar{s}_u/\sigma'_v)_{UC}$	387	0.223	0.611	0.55	0.039	1.01	−1.640	0.523

[a]Numbers in parentheses in the α_i column are the a_{test} values in Table 1 when $\phi'_{TC} = 30°$.

Availability of multivariate and pairwise data

In principle, the construction of a multivariate distribution for ($Y_1, Y_2, \ldots, Y_7$) will require multivariate data points. In this study, this means a single set of values of ($Y_1, Y_2, \ldots, Y_7$) is obtained from the same soil sample or more practically, from soil samples extracted from adjacent boreholes at comparable depths. The criterion is that a single set of values must be measured from samples that are strongly correlated in a spatial sense. Table 2 will be a multivariate database if all seven tests were conducted in each cited reference. It is evident that such multivariate data are not available in Table 2. Nonetheless, it is not uncommon to have *pairwise* Y data in close spatial proximity, e.g., both Y_1 and Y_6 ($(\bar{s}_u/\sigma'_v)_{CIUC}$ and $(\bar{s}_u/\sigma'_v)_{UU}$) are measured in two adjacent boreholes at comparable depths. One would expect these Y_1 and Y_6 profiles to be correlated. Table 5 lists the number of pairwise data where (Y_i, Y_j) values under two test modes are known simultaneously. The diagonals are the numbers of available data points for individual s_u test modes (same as the third column "n" in Table 4). It is evident that there are many data points for each individual s_u test mode, but there are far fewer pairwise data points where two s_u tests are conducted in close proximity. It is natural to expect pairwise data points to be more limited in the literature as it is cheaper to measure one set than two sets of data. For a given limited budget, validation using a separate test is considered less important than increasing the spatial coverage of the tests.

The Y data points for each test mode are roughly lognormally distributed, i.e., $X_i = [\ln(Y_i) - \lambda_i]/\xi_i$ is roughly standard normal (λ_i is the sample mean of $\ln(Y_i)$, and ξ_i is the sample standard deviation), except for Y_3 (= $(\bar{s}_u/\sigma'_v)_{CK_0UE}$), as noted in Ching et al. (2013). It is analytically simpler to characterize a correlation structure for multiple variables if each variable is normally distributed, because only pairwise correlations (e.g., correlation between X_1 and X_6) are needed. Figure 1 shows the available pairwise (X_i, X_j) data points for various combinations of (i, j), i.e., various combinations of natural logarithm transformed Y_i from seven test modes.

Multivariate normality

As mentioned earlier, the random variable X_i for each test mode roughly follows a standard normal distribution. The multivariate normal probability density function is available analytically and can be defined uniquely by a mean vector and a covariance matrix (Ang and Tang 2007)

$$f(\boldsymbol{X}) = |\mathbf{C}|^{(-1/2)}(2\pi)^{(-n/2)}\exp[(-1/2)(\mathbf{X}-\boldsymbol{\mu})'\mathbf{C}^{-1}(\mathbf{X}-\boldsymbol{\mu})] \tag{6}$$

in which $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)'$ is a normal random vector with n components ($n = 7$ for this study), $\boldsymbol{\mu}$ is the mean vector, and $\mathbf{C}$ is the covariance matrix. For example, in the case of n = 3, the mean vector and covariance matrix are given by

$$\boldsymbol{\mu} = \begin{Bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{Bmatrix} \qquad \mathbf{C} = \begin{bmatrix} \sigma_1^2 & \delta_{12}\sigma_1\sigma_2 & \delta_{13}\sigma_1\sigma_3 \\ \delta_{12}\sigma_1\sigma_2 & \sigma_2^2 & \delta_{23}\sigma_2\sigma_3 \\ \delta_{13}\sigma_1\sigma_3 & \delta_{23}\sigma_2\sigma_3 & \sigma_3^2 \end{bmatrix} \tag{7}$$

in which μ_i and σ_i are the mean and standard deviation of X_i, respectively, and δ_{ij} is the product-moment (Pearson) correlation between X_i and X_j. If X_i is a standard normal random variable with zero mean and unit standard deviation (i.e., $\mu_i = 0$ and $\sigma_i = 1$), eq. (7) reduces to

$$\boldsymbol{\mu} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & \delta_{12} & \delta_{13} \\ \delta_{12} & 1 & \delta_{23} \\ \delta_{13} & \delta_{23} & 1 \end{bmatrix} \tag{8}$$

It is clear that the full multivariate dependency structure of a normal random vector only depends on a covariance matrix ($\mathbf{C}$) containing bivariate information (correlations) between all possible pairs of components, namely X_1 and X_2, X_1 and X_3, and X_2 and X_3. The practical advantage of capturing multivariate dependencies in any dimension (i.e., any number of random variables) using only bivariate dependency information is obvious. In practice, even bivariate data are limited as shown in Table 5. More details are discussed in Phoon (2006).

It is crucial to note here that *collectively* ($X_1, X_2, \ldots, X_7$) does not necessarily follow a multivariate normal distribution even if each component is normally distributed. For example, if the scatter plot of X_i versus X_j shows a distinct nonlinear trend, then the multivariate normal distribution assumption is incorrect. The scatter plots between X_i and X_j (Fig. 1) show that the trends are mostly linear. There are two exceptions, (X_2, X_6) and (X_2, X_7), which appear to exhibit nonlinear trends. However, the number of data points (around 15) in these scatter plots is small and it is not possible to establish that these nonlinear trends are statistically significant. Hence, for this particular database, there is no strong evidence to reject the hypothesis of multivariate normality.

Estimation of correlation matrix in the absence of full multivariate data

The multivariate normal distribution is essentially characterized by the correlation matrix $\mathbf{C}$. It is easy to construct the correlation matrix *theoretically*. However, it is difficult to construct this matrix from *real data*. This paper is a first attempt to attack this challenge in a reasonably practical way. There are two practical challenges in the construction of $\mathbf{C}$:

1. *Estimation of correlation coefficients* — Each correlation coefficient constitutes a single entry in the correlation matrix $\mathbf{C}$. This correlation coefficient can be estimated if there are sufficient bivariate data. The database is divided into three data groups to address this estimation issue. Direct estimation is only possible for group 1 (good quality data). Methods to estimate

Table 5. Numbers of available (Y_i, Y_j) data pairs (or (X_i, X_j) data pairs). Indices in parentheses are the group indices.

	Y_1 (CIUC)	Y_2 (CK$_0$UC)	Y_3 (CK$_0$UE)	Y_4 (DSS)	Y_5 (FV)	Y_6 (UU)	Y_7 (UC)
Y_1 (CIUC)	637	129 (1)	30 (1)	24 (1)	20 (2)	84 (1)	38 (1)
Y_2 (CK$_0$UC)		555	69 (1)	135 (1)	79 (2)	13 (3)	14 (3)
Y_3 (CK$_0$UE)			224	66 (1)	43 (2)	7 (3)	14 (3)
Y_4 (DSS)				573	58 (2)	18 (3)	14 (3)
Y_5 (FV)					1057	123 (1)	140 (1)
Y_6 (UU)		Symmetry				435	53 (1)
Y_7 (UC)							387

the correlation coefficient in the presence of more data constraints (not uncommon in geotechnical engineering) in groups 2 and 3 are discussed below.

2. *Rectification of the correlation matrix* — The entire collection of correlation coefficients must satisfy a critical matrix property called positive definiteness. When the correlation matrix is formed from correlation coefficients estimated using separate bivariate databases out of practical necessity as is done in this study, it may not be positive definite. This matrix property is rather abstract, but it suffices to state that no simulation can be conducted in the absence of this property. The eigenspectrum approach discussed below to make the correlation matrix positive definite is thus critical to the objective of this paper.

Categories of data quality

The main challenge in the construction of a multivariate normal distribution is the estimation of the coefficient of correlation δ_{ij} in eq. (8), in the presence of limited bivariate (pairwise) data as shown in Table 5. Table 5 lists the numbers of available (X_i, X_j) data points. Figure 1 shows the scatter of the (X_i, X_j) data points. Based on the quantity and quality of the (X_i, X_j) data, the (X_i, X_j) pairs are categorized into three data groups as discussed below. The indices in the parentheses in Table 5 summarize the data group designation for each (X_i, X_j) pair.

Group 1 (good-quality data)

There is a sufficient number of sites in the pairwise data to mitigate bias towards one particular geographical location. Bias may be present if the majority of the data are contributed by one or two sites (Ching and Phoon 2012*b*). There are a sufficient number of pairwise data points to ensure that the statistical error introduced in the estimation of the correlation coefficient is reasonable. Group 1 includes all pairs extracted from the sets (X_1, X_2, X_3, X_4), (X_1, X_6, X_7), and (X_5, X_6, X_7). Group 1 refers to all pairs lying outside the solid and dashed boxes in Fig. 1 and Table 5.

Group 2 (medium-quality data)

There is insufficient pairwise data and (or) an insufficient number of sites. However, if one forgoes the standardization step given in eq. (4), particularly a_{OCR}, the quality and quantity of pairwise data are sufficient. The reason is that additional pairwise data associated with incomplete geotechnical information (e.g., missing OCR) can be used for the estimation of the correlation coefficients if standardization can be partially relaxed. Group 2 includes all pairs involving X_5 (FV). These pairs are highlighted by a solid box in Fig. 1 and Table 5.

Group 3 (poor-quality data)

There are insufficient pairwise data, even if one considers all data, including those associated with incomplete geotechnical information. Group 3 includes the pairs: (X_2, X_6), (X_2, X_7), (X_3, X_6), (X_3, X_7), (X_4, X_6), and (X_4, X_7). The correlation coefficients δ_{ij} for group 3 will be estimated based on judgment as discussed later. These pairs are highlighted by a dashed box in Fig. 1 and Table 5.

Estimation of correlation coefficients

Different approaches are applied for estimation of the correlation coefficients, depending on the data group category.

Group 1 (good-quality data)

For the pairs in group 1, the (X_i, X_j) data points in Fig. 1 will be directly used to estimate the correlation coefficient δ_{ij} based on the principle of maximum likelihood (Ang and Tang 2007). Details are given in Appendix A. The bootstrapping technique (Efron and Tibshirani 1993) is applied to obtain 1000 samples of the δ_{ij} estimates as follows: n random samples of (X_i, X_j) are drawn with replacement from the original (X_i, X_j) dataset, and the δ_{ij} estimate is obtained by solving eq. (A1). This is repeated 1000 times to obtain 1000 samples of the δ_{ij} estimates. Figure 2 shows the histogram of the 1000 δ_{ij} estimates for the (X_1, X_2) pair. Based on the δ_{ij} samples, the 90% confidence intervals (5% and 95% percentiles) and the median of δ_{ij} can be identified.

Group 2 (medium-quality data)

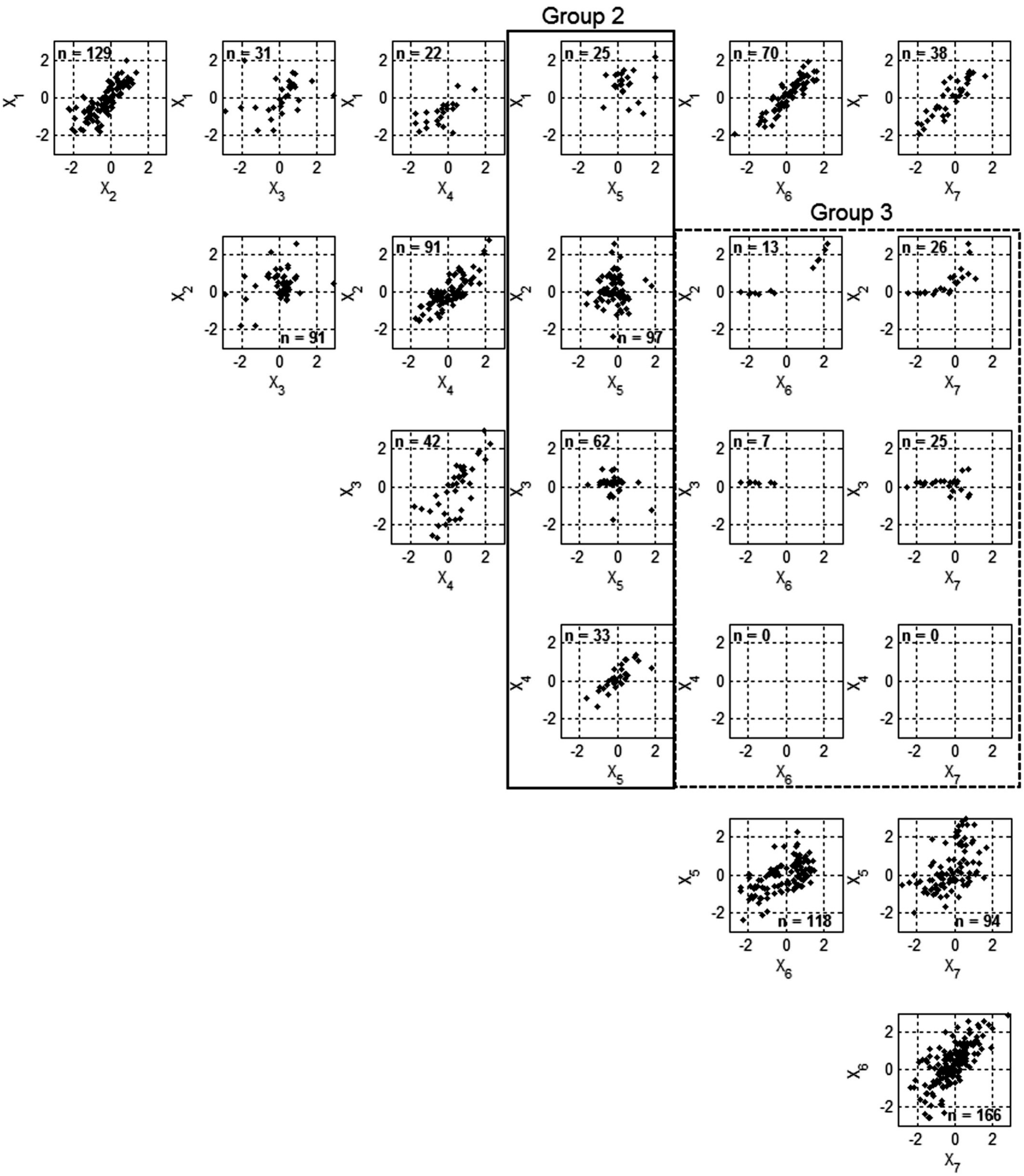
For pairwise data in group 2, direct calculation of δ_{ij} based on the insufficient (X_i, X_j) data (standardized data) can be misleading, as distinct trends cannot be visually discerned even in Fig. 1. The sample sizes of nonstandardized s_u/σ'_v data points are more abundant because the OCR information is not required. For instance, for the (CIUC, FV) pairs, although there are only 20 (X_1, X_5) (or (Y_1, Y_5)) data points, there are 37 $(s_u/\sigma'_v)_{\mathrm{CIUC}}$ versus $(s_u/\sigma'_v)_{\mathrm{FV}}$ data points (note: no "overbar" in s_u). In other words, there are 17 data points with missing OCR information in the source literature and standardization cannot be performed. For these data points, the standardization with respect to a_{rate} and a_{PI} is still possible

$$W = \left(s_u/\sigma'_v\right)/(a_{\mathrm{rate}}a_{\mathrm{PI}}) \tag{9}$$

where W denotes the normalized undrained shear strength with PI = 20 subjected to a 1%/h strain rate — only two factors, namely PI and strain rate, are standardized. It is critical to observe that the variable W is different from the variable Y, because a_{OCR} is not available as a deterministic number given missing OCR information. Figure 3 shows the (W_i, W_j) data points for the group 2 pairs.

For group 2, replacing the (X_i, X_j) dataset by the (W_i, W_j) dataset means that OCR is not known for some data points. In this circumstance, OCR is treated as a random variable in the group 2 analysis as explained in Appendix A. The bootstrapping technique is applied to obtain 1000 samples of the δ_{ij} estimate (computed using eq. (A2)). Based on the δ_{ij} samples, the 90% confidence intervals (5% and 95% percentiles) and the median of δ_{ij} can be identified. Figure 4 compares the δ_{ij} estimators produced by eqs. (A1) and (A2) for group 1 pairwise data. Note that eq. (A1) is based on X_i where OCR is treated as a deterministic number, while eq. (A2) is based on W_i where OCR is treated as a random variable. In other words, the OCR information in group 1 is deliberately ignored in eq. (A2) to assess the practicality of this approach. In Fig. 4,

Fig. 1. Pairwise (X_i, X_j) data points (i or j = 1 for CIUC, 2 for CK_0UC, 3 for CK_0UE, 4 for DSS, 5 for FV, 6 for UU, and 7 for UC).

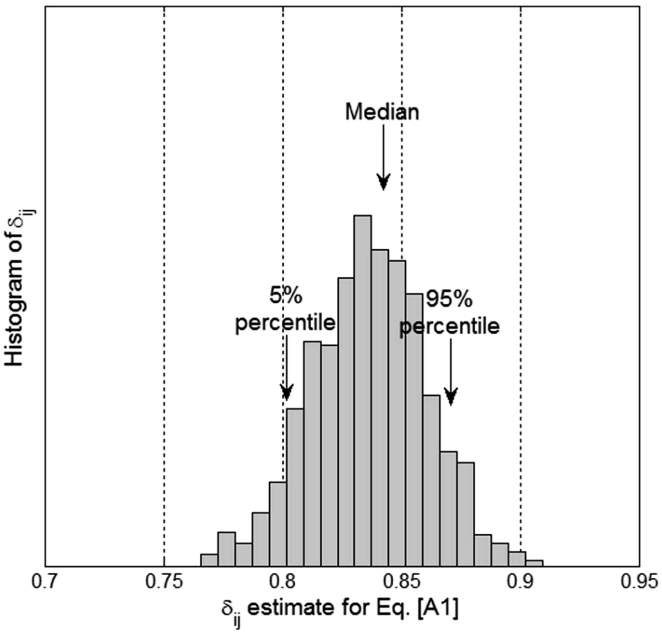


the marker and horizontal/vertical bar show the median value and the 90% confidence interval, respectively. It is reassuring to observe that these two δ_{ij} estimators produce comparable results. Based on this validation exercise using group 1 data, the δ_{ij} estimator produced by eq. (A2) is deemed satisfactory for group 2 data.

Group 3 (poor-quality data)

For the remaining pairs in group 3, judgment is applied to specify δ_{ij} based on some physical observations. It is postulated that the correlation coefficients should be ranked as follows: $\delta_{7j} < \delta_{6j} < \delta_{1j}$. This is because the stress state for the UU and UC test modes (i = 6 and

Fig. 2. Histogram of the bootstrap δ_{ij} samples for the (X_1, X_2) pair.



7, respectively) is roughly comparable to that for CIUC ($i = 1$), but UU is less reliable than CIUC, and UC is known to be even less reliable. Based on this observation, δ_{62}, δ_{63}, and δ_{64} could be comparable but no larger than δ_{12}, δ_{13}, and δ_{14}. Using the median values of (δ_{12}, δ_{13}, δ_{14}) = (0.84, 0.47, 0.72) as upper bounds, (δ_{62}, δ_{63}, δ_{64}) are chosen to be (0.7, 0.4, 0.6). The values for (δ_{72}, δ_{73}, δ_{74}) should be respectively smaller than (δ_{62}, δ_{63}, δ_{64}) and are thus chosen to be (0.6, 0.3, 0.5).

Is it reasonable to question how appropriate these choices are? It is important to observe that a similar question appears even for the good-quality group 1 data. For any finite sample size, δ_{ij} follows a sampling distribution as shown in Fig. 2. It is reasonable to select the median as the "best" point estimate, but this is also fundamentally a choice. The bottom line is that choices in statistical analysis are inevitable in the face of unavoidable practical limitations in data quality and (or) quantity. Nonetheless, these choices are not entirely arbitrary. The choices are also theoretically constrained by the need to maintain a positive definite correlation matrix. In short, despite data and (or) theoretical constraints, a range of choices remains; the range can usually be narrowed by more and better quality data. The crux is that data limitation can be manifested explicitly within a statistical analysis through indicators such as a sampling distribution or confidence intervals; an important practical aspect mostly absent from deterministic analysis.

Table 6 shows the 90% confidence intervals (range bounded by the 5% and 95% percentiles) and the median of the δ_{ij} estimates. The four triaxial compression (TC) test modes (X_1, X_2, X_6, X_7) seem mutually highly correlated (δ_{ij} median > 0.8), with the exception of (X_6, X_7) having a δ_{ij} median = 0.59. The CK$_0$UE test mode (X_3) has weak correlation with the four TC test modes (δ_{ij} median < 0.5), probably because it imposes a different stress state from the TC tests. The correlation coefficients between FV and TC are relatively weak as well (δ_{ij} median ≤ 0.63). Such relatively low correlation between FV and TC may be due to the fact that the FV test has several distinct aspects (stress state, drainage boundaries, strain rate, and failure mode). It is interesting that the correlation between FV and DSS is high (δ_{ij} median = 0.73).

Positive definiteness of the correlation matrix

The correlation matrix **C** presented in Table 6, which is formed from the median values of δ_{ij}, may not be positive definite. Positive definiteness can only be guaranteed if the correlation matrix were to be estimated from a full multivariate dataset (X_1, X_2, …, X_7). However, because only bivariate (pairwise) data are typically available, each δ_{ij} is estimated *independently* of other entries in Table 6. Moreover, the (X_i, X_j) dataset (or the (W_i, W_j) dataset) for one pair may come from sources (sites) that are different from the other pairs. The **C** matrix estimated in this entry-by-entry pairwise manner is not guaranteed to be positive definite. In fact, a correlation matrix with δ_{ij} randomly picked from the 90% confidence intervals can be nonpositive definite.

The eigenspectrum of **C** only contains positive values if and only if **C** is positive definite. There is one negative eigenvalue in the **C** matrix based on the median values of δ_{ij}. This negative eigenvalue is associated with the submatrix for (X_1, X_6, X_7)

$$\begin{bmatrix} 1 & \delta_{16} & \delta_{17} \\ & 1 & \delta_{67} \\ \text{sym} & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.90 & 0.89 \\ & 1 & 0.59 \\ \text{sym} & & 1 \end{bmatrix} \tag{10}$$

Physically, it is possible to observe that the three median values of δ_{ij} are somewhat incompatible. Large values of $\delta_{16} = 0.90$ and $\delta_{17} = 0.89$ imply that both X_6 and X_7 are highly correlated to X_1. One could loosely appreciate that X_6 and X_7 should be highly correlated as well. The actual median value $\delta_{67} = 0.59$ appears to be rather low from this perspective. As mentioned earlier, this incompatibility can arise because (δ_{16}, δ_{17}, δ_{67}) are estimated independently from one another and the pairwise data are from different sources. On the other hand, it has been observed previously that choosing the median value as a point estimate is a matter of convenience. In actuality, the possible values of (δ_{16}, δ_{17}, δ_{67}) lie within ranges, which can be quantified by 90% confidence intervals such as those shown in Table 6. If we pick (δ_{16}, δ_{17}) at their lower bounds (0.88, 0.85) and pick δ_{67} at its upper bound 0.68, all three eigenvalues become positive. These values of (δ_{16}, δ_{17}, δ_{67}) are still compatible with the available data in the statistical sense because they belong to the respective sampling distributions with a rather high confidence of 90%.

Table 7 shows a positive definite correlation matrix where most δ_{ij} entries are identical to the median values in Table 6 except that (δ_{16}, δ_{17}) are taken to be the 5% percentiles and δ_{67} is taken to be the 95% percentile. There are many other possible choices of such a positive definite matrix. Table 7 is one such example. One can randomly sample from the 90% confidence interval for δ_{ij} until a positive definite **C** is obtained. In short, in the presence of statistical uncertainties arising from finite sample sizes, it is not possible to identify a unique correlation matrix. The choices for the correlation matrix are however constrained by the positive definite requirement. The correlation coefficient δ_{ij} between (X_i, X_j) can be converted to the correlation coefficient ρ_{ij} between (Y_i, Y_j) through the following equation:

$$\rho_{ij} = \frac{\exp(\xi_i \xi_j \delta_{ij}) - 1}{\sqrt{[\exp(\xi_i^2) - 1][\exp(\xi_j^2) - 1]}} \tag{11}$$

Table 8 shows the correlation coefficient ρ_{ij} between (Y_i, Y_j) converted from the δ_{ij} estimates in Table 7.

The correlation coefficients in Tables 7 and 8 are based on the global database in Table 2. Ideally, it is better to resolve this global database into regional databases (North America, Europe, Asia, etc.), and the correlation coefficients for each individual region can be obtained. However, this resolution is not possible at present due to insufficient data. For example, if the database is restricted to Europe, only five (X_i, X_j) pairs will appear in group 1 or 2 (good- to medium-quality data). The remaining 16 (X_i, X_j) pairs will appear in group 3 (poor-quality data). It is well known in geotechnical engineering that site-specific information is better than global information. Nonetheless, it is also well appreciated

Fig. 3. (W_i, W_j) data points for the group 2 pairs.

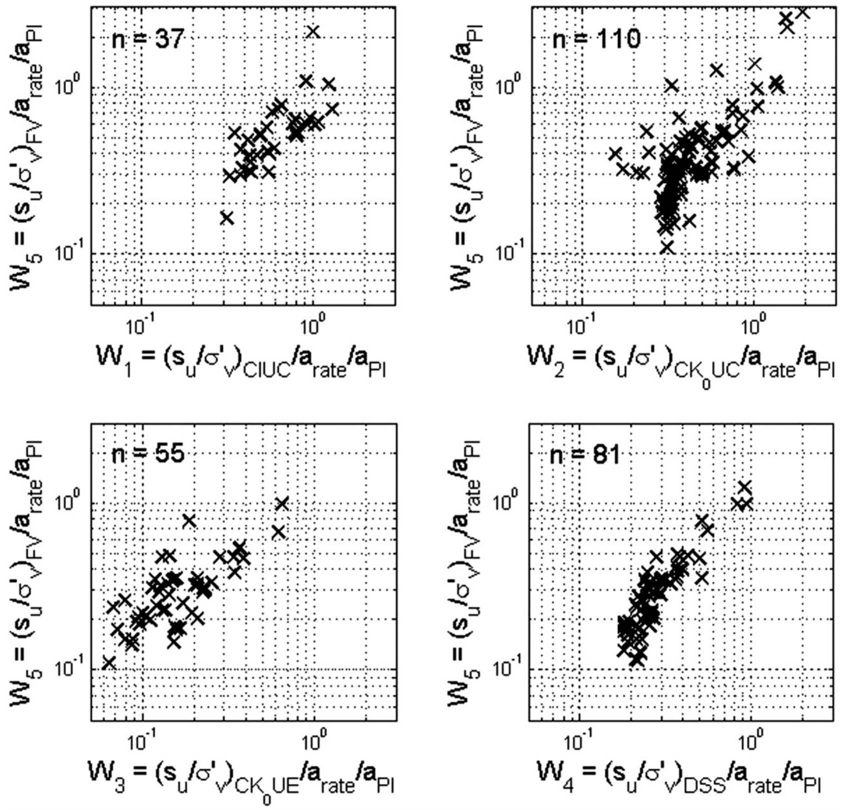
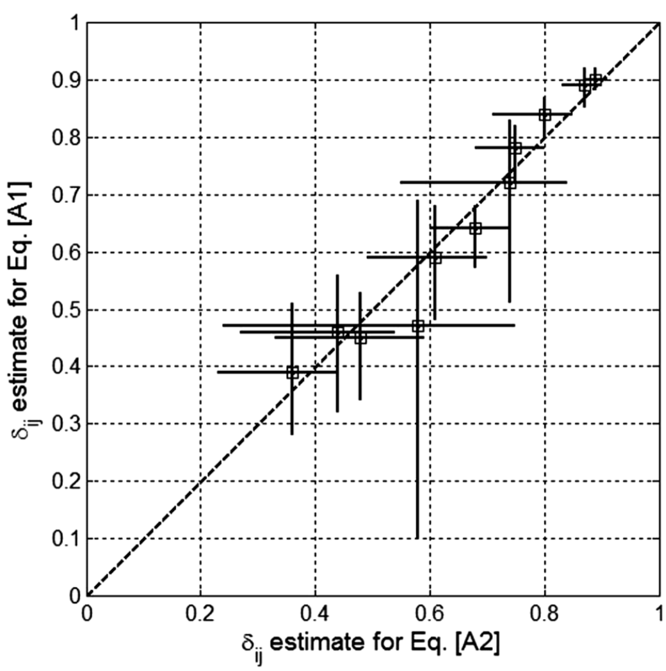


Fig. 4. Comparison between the δ_{ij} estimators for group 1: eq. (A1) is based on standardized pairwise data while eq. (A2) is based on treating OCR as a random variable.



that site-specific information is frequently limited and it is not uncommon in current practice to rely on global information published in the literature to provide preliminary first-order estimates of soil parameters in the absence of site-specific information. There is admittedly an element of a leap of faith in this approach, but one could quite safely say it is more sensible to do this, than to apply data from a single region to another region. Global data tend to provide middle-of-the-road estimates and the bias is smaller when applied to regions. The correlation coefficients presented in Tables 7 and 8 should thus be viewed in this context and with the associated limitations in mind.

Transformations between different s_u/σ'_v values

This section presents one practical application of the constructed multivariate normal model; namely, given the s_u/σ'_v value from a particular test procedure (test mode, strain rate, OCR, and PI) for a clay of interest, what is the mean value and "updated" COV of the s_u/σ'_v value for the same clay under another test procedure? This question can be answered once the transformations between (Y_i, Y_j) pair are established. Table 9 summarizes some useful (Y_i, Y_j) transformations derived from Bayesian analysis (Ang and Tang 2007). In Table 9, "source" denotes the given Y, whereas "target" denotes the Y of interest. For instance, given the source information $Y_5 = (\bar{s}_u/\sigma')_{FV}$, the mean value and updated COV of the target $Y_1 = (\bar{s}_u/\sigma')_{CIUC}$ can be easily checked in Table 9 (see the entry in grey):

$$\text{(12)} \qquad E(Y_1) = 0.820Y_5^{0.534} \qquad \mathrm{COV}(Y_1) = 0.249$$

The transformations are derived from Bayesian analysis based on the correlation coefficients δ_{ij} given in Table 7. Appendix B demonstrates the derivations for the transformation with source = Y_5 and target = Y_1. For other (Y_i, Y_j) pairs, the same derivation processes hold.

The updated COV in Table 9 reflects the amount of remaining uncertainty in the target Y given the information of the source Y. In the last row of the table, the mean and "prior" COV represent the case without any information. It is remarkable that the up-

Table 6. Statistics of the estimated δ_{ij} in the form of 90% confidence interval (median).

C =	Y_1 (CIUC)	Y_2 (CK_0UC)	Y_3 (CK_0UE)	Y_4 (DSS)	Y_5 (FV)	Y_6 (UU)	Y_7 (UC)
Y_1 (CIUC)	1.00	0.80–0.87 (0.84)	0.10–0.69 (0.47)	0.51–0.83 (0.72)	0.48–0.72[a] (0.63)	0.88-0.92 (0.90)	0.85–0.92 (0.89)
Y_2 (CK_0UC)	0.80–0.87 (0.84)	1.00	0.28–0.51 (0.39)	0.72–0.82 (0.78)	0.14–0.49[a] (0.35)	0.7[b]	0.6[b]
Y_3 (CK_0UE)	0.10–0.69 (0.47)	0.28–0.51 (0.39)	1.00	0.34–0.53 (0.45)	0.21–0.53[a] (0.41)	0.4[b]	0.3[b]
Y_4 (DSS)	0.51–0.83 (0.72)	0.72–0.82 (0.78)	0.34–0.53 (0.45)	1.00	0.64–0.79[a] (0.73)	0.6[b]	0.5[b]
Y_5 (FV)	0.48–0.72[a] (0.63)	0.14–0.49[a] (0.35)	0.21–0.53* (0.41)	0.64–0.79[a] (0.73)	1.00	0.57–0.68 (0.64)	0.32–0.56 (0.46)
Y_6 (UU)	0.88–0.92 (0.90)	0.7[b]	0.4[b]	0.6[b]	0.57–0.68 (0.64)	1.00	0.48–0.68 (0.59)
Y_7 (UC)	0.85–0.92 (0.89)	0.6[b]	0.3[b]	0.5[b]	0.32–0.56 (0.46)	0.48–0.68 (0.59)	1.00

[a]Group 2 (δ_{ij} estimated based on eq. (A2)).
[b]Group 3 (δ_{ij} estimated based on judgment).

Table 7. Estimated correlation matrix **C** between (X_i, X_j).

	X_1 (CIUC)	X_2 (CK_0UC)	X_3 (CK_0UE)	X_4 (DSS)	X_5 (FV)	X_6 (UU)	X_7 (UC)
X_1 (CIUC)	1.00	0.84	0.47	0.72	0.63	0.88[a]	0.85[a]
X_2 (CK_0UC)	0.84	1.00	0.39	0.78	0.35	0.7	0.6
X_3 (CK_0UE)	0.47	0.39	1.00	0.45	0.41	0.4	0.3
X_4 (DSS)	0.72	0.78	0.45	1.00	0.73	0.6	0.5
X_5 (FV)	0.63	0.35	0.41	0.73	1.00	0.64	0.46
X_6 (UU)	0.88[a]	0.7	0.4	0.6	0.64	1.00	0.68[b]
X_7 (UC)	0.85[a]	0.6	0.3	0.5	0.46	0.68[b]	1.00

[a]5% percentile of δ_{ij} is taken.
[b]95% percentile of δ_{ij} is taken.

Table 8. Estimated correlation matrix **C** between (Y_i, Y_j).

	Y_1 (CIUC)	Y_2 (CK_0UC)	Y_3 (CK_0UE)	Y_4 (DSS)	Y_5 (FV)	Y_6 (UU)	Y_7 (UC)
Y_1 (CIUC)	1.00	0.83	0.46	0.71	0.62	0.87	0.83
Y_2 (CK_0UC)	0.83	1.00	0.38	0.77	0.34	0.68	0.57
Y_3 (CK_0UE)	0.46	0.38	1.00	0.44	0.39	0.38	0.28
Y_4 (DSS)	0.71	0.77	0.44	1.00	0.72	0.58	0.47
Y_5 (FV)	0.62	0.34	0.39	0.72	1.00	0.62	0.43
Y_6 (UU)	0.87	0.68	0.38	0.58	0.62	1.00	0.65
Y_7 (UC)	0.83	0.57	0.28	0.47	0.43	0.65	1.00

dated COV is always less than the prior COV, and whenever correlation is strong between (Y_i, Y_j) (see Table 8 for the correlations), the updated COV is significantly smaller than the prior COV, and vice versa. It is also remarkable that all updated COVs are mostly less than 0.3, i.e., classified as low variability based on the guidelines suggested by Kulhawy et al. (2012). This suggests that the use of inexpensive tests such as FV, UU, and UC to infer other s_u values seems to be fairly effective in the sense of uncertainty reduction. The only exception is the inference of CK_0UE, where the updated COVs are all larger than 0.3, due to the poor correlation to CK_0UE. It is noteworthy that the multivariate probability can systematically update the s_u/σ'_v value from one test procedure in presence of s_u/σ'_v values from multiple test procedures as well. This more general application is outside the scope of present study.

The following simple example is used to demonstrate the transformation between the nonstandardized $(s_u/\sigma'_v)_{FV}$ and the nonstandardized $(s_u/\sigma'_v)_{CIUC}$. Consider a clay site. A FV test reveals that $(s_u/\sigma'_v)_{FV} = 0.488$ at a depth of 5 m. Suppose the OCR at 5 m depth is 2.0 and PI = 30. Based on the above FV information, the target is to estimate the mean and COV of $(s_u/\sigma'_v)_{CIUC}$ at a strain rate of 0.02%/h for a clay at 10 m deep, assuming that OCR = 1.5 at this depth and the PI of the clay remains the same (the same clay). That is to say, the source information is $(s_u/\sigma'_v)_{FV}$ at 5 m depth, while the target variable is $(s_u/\sigma'_v)_{CIUC}$ at 10 m depth. This can be achieved with the following steps:

1. Convert the source $(s_u/\sigma'_v)_{FV}$ to Y_5 using eq. (4)

$$Y_5 = (\bar{s}_u/\sigma'_v)_{FV} = \frac{(s_u/\sigma'_v)_{FV}}{a_{OCR}a_{rate}a_{PI}} = \frac{0.488}{2^{\Lambda_5}[1.0 + 0.1\log_{10}(60)](30/20)^{\beta_5}} = 0.211 \tag{13}$$

where Λ_5 and β_5 are the Λ and β values for FV, which are respectively 0.902 and 0.124 (obtained from Table 3). The strain rate for FV is taken to be 60%/h.

2. Transform Y_5 into Y_1 using eq. (12)

$$E(Y_1) = 0.820Y_5^{0.534} = 0.357 \qquad COV(Y_1) = 0.249 \tag{14}$$

3. Convert Y_1 back to $(s_u/\sigma'_v)_{CIUC}$. The mean value of $(s_u/\sigma'_v)_{CIUC}$ can be computed from the reverse equation of eq. (4), whereas the updated COV of $(s_u/\sigma'_v)_{CIUC}$ remains the same

$$\begin{aligned} E\left[(s_u/\sigma'_v)_{CIUC}\right] &= a_{OCR}a_{rate}a_{PI}E\left[(\bar{s}_u/\sigma'_v)_{CIUC}\right] \\ &= 1.5^{\Lambda_1}[1.0 + 0.1\log_{10}(0.02)](30/20)^{\beta_1}E(Y_1) \\ &= 0.378 \\ COV\left[(s_u/\sigma'_v)_{CIUC}\right] &= COV(Y_1) = 0.249 \end{aligned} \tag{15}$$

where Λ_1 and β_1 are the Λ and β values for CIUC, which are respectively 0.602 and 0 (obtained from Table 3). Hence, $(s_u/\sigma'_v)_{CIUC}$ is lognormally distributed with posterior mean = 0.378 and posterior COV = 0.249.

Conclusion

This study summarizes the development of a large clay database that contains thousands of data points of undrained shear strengths (s_u) from various test modes, including CIUC, CK_0UC, CK_0UE, DSS, FV, UU, and UC. Based on simple analysis on this database, this study summarizes useful statistics (such as mean and COV) for s_u/σ'_v values that are standardized with respect to OCR, strain rate, and PI. These statistics can be used to further obtain preliminary estimates for the mean and COV of the s_u/σ'_v value of a clay with known OCR, stress state, strain rate, and PI, to facilitate reliability analysis and reliability-based design. It is concluded that if the stress state, OCR, PI, and strain rate for a clay are all known, the COV of s_u/σ'_v can be fairly low — maybe less than 0.3.

The analysis approach taken in this study is purely data driven, and the resulting s_u estimation method basically reflects the behaviors in the s_u database. Detailed study of the s_u database shows that the data behaviors are consistent to the behaviors exhibited by equations (4–55) in Kulhawy and Mayne (1990) (i.e., eq. (1) in this paper). This is a significant finding, as eq. (1) is not purely data driven but, to a certain degree, relies on theories such as the modified Cam Clay model.

Another important contribution of this study is to construct the multivariate distribution of s_u from various test modes in the

Table 9. Transformations between various source–target pairs.

	Target = Y_1 (CIUC)		Target = Y_2 (CK$_0$UC)		Target = Y_3 (CK$_0$UE)		Target = Y_4 (DSS)	
Source	Mean	COV	Mean	COV	Mean	COV	Mean	COV
Y_1 (CIUC)	—	—	$0.694Y_1^{0.747}$	0.153	$0.303Y_1^{0.529}$	0.321	$0.429Y_1^{0.632}$	0.194
Y_2 (CK$_0$UC)	$1.093Y_2^{0.945}$	0.172	—	—	$0.315Y_2^{0.494}$	0.336	$0.541Y_2^{0.770}$	0.175
Y_3 (CK$_0$UE)	$0.830Y_3^{0.417}$	0.284	$0.596Y_3^{0.308}$	0.263	—	—	$0.439Y_3^{0.351}$	0.251
Y_4 (DSS)	$1.314Y_4^{0.820}$	0.222	$1.089Y_4^{0.790}$	0.177	$0.427Y_4^{0.577}$	0.325	—	—
Y_5 (FV)	**$0.820Y_5^{0.534}$**	**0.249**	$0.498Y_5^{0.264}$	0.267	$0.313Y_5^{0.391}$	0.333	$0.491Y_5^{0.543}$	0.191
Y_6 (UU)	$0.970Y_6^{0.600}$	0.151	$0.655Y_6^{0.424}$	0.202	$0.293Y_6^{0.307}$	0.334	$0.408Y_6^{0.359}$	0.224
Y_7 (UC)	$0.904Y_7^{0.513}$	0.167	$0.584Y_7^{0.323}$	0.227	$0.258Y_7^{0.204}$	0.349	$0.366Y_7^{0.265}$	0.243
No info	0.404	0.323	0.350	0.286	0.185	0.367	0.239	0.282

presence of two key data limitations. One data limitation is the lack of complete multivariate data taken from the same site. The multivariate distribution is characterized by a correlation matrix constructed from independent pairwise data, including pairwise data from different sites. The second data limitation is that even the quantity and (or) quality of pairwise data may not be sufficient for reliable estimation of the correlation coefficient. In the presence of this limitation, each entry in the correlation matrix should be reported as a confidence interval to reflect the underlying statistical uncertainties. The range of possible correlation matrices satisfying these confidence intervals can be narrowed by a positive definite requirement, which is not guaranteed unless a complete multivariate dataset from the same set is available. An important outcome of these data limitations is that a unique correlation matrix cannot be established nor is it realistic to expect uniqueness. A plausible positive definite matrix can however be established.

Using a multivariate distribution characterized by such a plausible correlation matrix, steps are proposed to infer the mean and COV of s_u for one scenario given the information of another s_u. This is intrinsically the probabilistic extension of eq. (1) — the major addition is the ability of updating the COV for the target (or desired) s_u value, which is not possible with eq. (1). This addition is essential for reliability analysis and reliability-based design. It is remarkable that the updated COVs are always less than the prior COV. Moreover, the updated COVs are significantly less than 0.3 for many cases, i.e., low variability. This suggests that the use of inexpensive tests such as FV, UU, and UC to infer of other s_u values seems to be fairly useful insofar as uncertainty reduction. The results appear to be insensitive to the choice of a correlation matrix among the range of plausible ones.

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List of symbols

a_{OCR} unitless modifier factor for overconsolidation ratio
a_{PI} unitless modifier factor for plasticity index
a_{rate} unitless modifier factor for strain rate
a_{test} unitless modifier factor for test mode
C covariance matrix
COV coefficient of variation
COV(.) coefficient of variation of the enclosed variable
CV(.,.) covariance between the enclosed variables
E(.) expected value of the enclosed variable
f(.) probability density function
LI liquidity index
m_i mean value of W_i
OCR overconsolidation ratio (OCR > 1 indicates overconsolidated (OC); OCR = 1 indicates normally consolidated)
PI plasticity index
S_t sensitivity
s_u undrained shear strength
s_u/σ'_v normalized undrained shear strength (with respect to the effective vertical stress)
$\bar{s}_u/\sigma'_v$ normalized undrained shear strength (unitless) for strain rate = 1% per hour on a normally consolidated (NC) clay with PI = 20%
Var(.) variance of the enclosed variable
W_i normalized undrained shear strength (unitless) for strain rate = 1%/h on a clay with PI = 20%
X_i $[\ln(Y_i) - \lambda_i]/\xi_i$
Y_i short notation for $\bar{s}_u/\sigma'_v$ (i = 1 for CIUC, i = 2 for CK_0UC, i = 3 for CK_0UE, i = 4 for DSS, i = 5 for FV, i = 6 for UU, i = 7 for UC)
α_i $E(Y_i)/E(Y_1)$
β exponent in a_{PI} ($a_{\mathrm{PI}} = (\mathrm{PI}/20)^{\beta}$)
δ_{ij} product-moment (Pearson) correlation coefficient between X_i and X_j
Λ SHANSEP (Ladd and Foott 1974) exponent ($a_{\mathrm{OCR}} = \mathrm{OCR}^{\Lambda}$)
λ_i mean of $\ln(Y_i)$
μ_i mean value of X_i
ξ_i standard deviation of $\ln(Y_i)$
ρ_{ij} product-moment (Pearson) correlation coefficient between Y_i and Y_j
σ(.) standard deviation of the enclosed variable
σ'_v effective vertical stress
σ_i standard deviation of X_i
ϕ'_{TC} effective friction angle from a triaxial compression test

Appendix A. Estimation of δ_{ij} for groups 1 and 2 data

For group 1, the (X_i, X_j) data points in Fig. 1 will be directly used to estimate the correlation coefficient δ_{ij} based on the principle of maximum likelihood (Ang and Tang 2007)

$$\text{(A1)}\qquad \delta_{ij} = \underset{\delta_{ij}}{\operatorname{argmax}} \prod_{k=1}^{n} f\left(X_i^{(k)}, X_j^{(k)} \middle| \delta_{ij}\right)$$
$$= \underset{\delta_{ij}}{\operatorname{argmax}} \prod_{k=1}^{n} \frac{1}{2\pi\sqrt{1-\delta_{ij}^2}} \times \exp\left\{-\frac{1}{2}\begin{bmatrix} X_i^{(k)} \\ X_j^{(k)} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & \delta_{ij} \\ \delta_{ij} & 1 \end{bmatrix}^{-1} \begin{bmatrix} X_i^{(k)} \\ X_j^{(k)} \end{bmatrix}\right\}$$

where $X^{(k)}$ is the kth sample in the (X_i, X_j) dataset, and $f(X_i, X_j|\delta_{ij})$ is the joint probability density function (PDF) of (X_i, X_j) given the correlation coefficient δ_{ij}. This estimator is found to be more robust than the traditional product moment estimators for the correlation coefficient δ_{ij}.

For group 2, replacing the (X_i, X_j) dataset by the (W_i, W_j) dataset means that OCR is treated as a random variable. In the mathematical derivation detailed below, the OCR is treated as a diffuse random variable to reflect the lack of information. The (W_i, W_j) data points can be used to estimate δ_{ij} based on the principle of maximum likelihood (Ang and Tang 2007)

$$\text{(A2)}\qquad \delta_{ij} = \underset{\delta_{ij}}{\operatorname{argmax}} \prod_{k=1}^{n} f\left(W_i^{(k)}, W_j^{(k)} \middle| \delta_{ij}\right) = \underset{\delta_{ij}}{\operatorname{argmax}} \prod_{k=1}^{n} \frac{\exp\left\{-\frac{1}{2}\begin{bmatrix} W_i^{(k)} - E(W_i) \\ W_j^{(k)} - E(W_j) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathrm{Var}(W_i) & \mathrm{CV}(W_i, W_j) \\ \text{symmetry} & \mathrm{Var}(W_j) \end{bmatrix}^{-1} \begin{bmatrix} W_i^{(k)} - E(W_i) \\ W_j^{(k)} - E(W_j) \end{bmatrix}\right\}}{2\pi\sqrt{\mathrm{Var}(W_i)\,\mathrm{Var}(W_j) - \mathrm{CV}(W_i, W_j)^2}}$$

where CV(.,.) denotes the covariance between W_i and W_j;

$$\text{(A3)}\quad \begin{aligned} \mathrm{E}(W_i) &= \mathrm{E}[\ln(Y_i) + \Lambda_i \ln(\mathrm{OCR})] = \lambda_i + \Lambda_i \mathrm{E}[\ln(\mathrm{OCR})] \\ \mathrm{Var}(W_i) &= \mathrm{Var}[\ln(Y_i) + \Lambda_i \ln(\mathrm{OCR})] = \xi_i^2 + \Lambda_i^2 \mathrm{Var}[\ln(\mathrm{OCR})] \\ \mathrm{CV}(W_i, W_j) &= \mathrm{CV}[\ln(Y_i), \ln(Y_j)] + \Lambda_i\Lambda_j \mathrm{Var}[\ln(\mathrm{OCR})] = \xi_i\xi_j \mathrm{CV}(X_i, X_j) + \Lambda_i\Lambda_j \mathrm{Var}[\ln(\mathrm{OCR})] = \xi_i\xi_j(\delta_{ij}) + \Lambda_i\Lambda_j \mathrm{Var}[\ln(\mathrm{OCR})] \end{aligned}$$

The parameters (λ, ξ) are given in Table 4, and Λ is given in Table 3. Note that $\ln(\mathrm{OCR}) = [W_i - \ln(Y_i)])/\Lambda_i = (W_j - \ln(Y_j))/\Lambda_j$, hence E[ln(OCR)] can be estimated as

$$\text{(A4)}\quad \mathrm{E}[\ln(\mathrm{OCR})] \approx \frac{1}{2}\left[\left(\frac{1}{n}\sum_{k=1}^{n} W_i^{(k)} - \lambda_i\right)/\Lambda_i + \left(\frac{1}{n}\sum_{k=1}^{n} W_j^{(k)} - \lambda_j\right)/\Lambda_j\right]$$

and Var[ln(OCR)] can be estimated as

$$\text{(A5)}\quad \mathrm{Var}[\ln(\mathrm{OCR})] \approx \frac{1}{2}\left(\left[\frac{1}{n-1}\sum_{k=1}^{n}\left(W_i^{(k)} - m_i\right)^2 - \xi_i^2\right]/\Lambda_i^2 + \left[\frac{1}{n-1}\sum_{k=1}^{n}\left(W_j^{(k)} - m_j\right)^2 - \xi_j^2\right]/\Lambda_j^2\right)$$

where m_i is the sample mean of ($W_i^{(1)}, \ldots, W_i^{(n)}$). Equation (A5) is based on the assumption that Y and ln(OCR) are uncorrelated.

Appendix B. Transformation from Y_5 to Y_1

Consider the scenario where $Y_5 = (\bar{s}_u/\sigma'_v)_{FV}$ is known. The purpose of this appendix is to derive the updated mean value and COV of $Y_1 = (\bar{s}_u/\sigma'_v)_{CIUC}$, given that Y_5 is known. This can be done in few steps. For other (Y_i, Y_j) pairs, the same derivation processes hold.

1. The source information Y_5 is converted to a standard normal variable $X_5 = [\ln(Y_5) - \lambda_5]/\xi_5$, where λ_5 and ξ_5 are given in Table 4. For FV, $\lambda_5 = -1.363$ and $\xi_5 = 0.372$, hence

$$\text{(B1)}\quad X_5 = \frac{\ln(Y_5) + 1.363}{0.372} = 2.688 \ln(Y_5) + 3.664$$

2. Recall that (X_1, X_5) are multivariate normal. Given the source information X_5, the updated mean and variance of the target X_1 are (Ang and Tang 2007)

$$\text{(B2)}\quad \begin{aligned} \mathrm{E}(X_1|\text{source}) &= E(X_1) + \delta_{15}\frac{\sqrt{\mathrm{Var}(X_1)}}{\sqrt{\mathrm{Var}(X_5)}}[X_5 - \mathrm{E}(X_5)] \\ &= \delta_{15}X_5 \\ \mathrm{Var}(X_1|\text{source}) &= \mathrm{Var}(X_1)\left(1 - \delta_{15}^2\right) = 1 - \delta_{15}^2 \end{aligned}$$

Here we have implemented the fact that $\mathrm{E}(X_1) = \mathrm{E}(X_5) = 0$ and $\mathrm{Var}(X_1) = \mathrm{Var}(X_5) = 1$ because X_1 and X_5 are both standard normal and that $\mathrm{CV}(X_1, X_5) = \delta_{15}$. The correlation $\delta_{15} = 0.63$ can be obtained from Table 7, hence

$$\text{(B3)}\quad \begin{aligned} \mathrm{E}(X_1|\text{source}) &= 0.63X_5 = 1.694 \ln(Y_5) + 2.308 \\ \mathrm{Var}(X_1|\text{source}) &= 0.60 \end{aligned}$$

3. Recall that $\ln(Y_1) = \lambda_1 + \xi_1 X_1 = -0.955 + 0.315X_1$ ($\lambda_1 = -0.955$ and $\xi_1 = 0.315$ can be obtained from Table 4). The updated mean and variance of $\ln(Y_1)$ is therefore

$$\text{(B4)}\quad \begin{aligned} \mathrm{E}[\ln(Y_1)|\text{source}] &= -0.955 + 0.315\mathrm{E}(X_1|\text{source}) \\ &= 0.534 \ln(Y_5) - 0.228 \\ \mathrm{Var}[\ln(Y_1)|\text{source}] &= 0.315^2 \mathrm{Var}(X_1|\text{source}) = 0.0595 \end{aligned}$$

The updated mean of Y_1 is simply $\exp[0.534 \ln(Y_5) - 0.228 + 0.5(0.0595)] = 0.820Y_5^{0.534}$, and the updated COV of Y_1 is simply $[\exp(0.0595) - 1]^{0.5} = 0.249$. These results are consistent to those in the entry in grey in Table 9.