

# RandomGraphs

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## 1

### 1.1

Let  $G_0$  be a specific graph with  $n$  vertices and  $m_0$  edges, then  $\mathbb{P}[G \sim G(n, p) = G_0] = p^{m_0}(1-p)^{\binom{n}{2}-m_0} \Rightarrow \mathbb{P}[G \sim G(n, p) \in A] = \mathbb{P}[\bigcup_{G_0 \in A} G \sim G(n, p) = G_0] = \sum_{G_0 \in A} p^{m_0}(1-p)^{\binom{n}{2}-m_0}$ . But this is a polynomial in  $p$ , hence continuous, and for every  $0 \leq p \leq q \leq 1$ . we have  $\mathbb{P}[G \sim G(n, p) = G_0] = p^{m_0}(1-p)^{\binom{n}{2}-m_0} \leq q^{m_0}(1-q)^{\binom{n}{2}-m_0} = \mathbb{P}[G \sim G(n, q) = G_0]$ , which applies also for the sum of probabilities over  $\bigcup_{G_0 \in A} G_0$ . Thus,  $f(p) := \mathbb{P}[G \sim G(n, p) \in A]$  is both continuous and monotone increasing. We know that  $\mathbb{P}[G \sim G(n, 0) = \phi] = 1 \Rightarrow \mathbb{P}[G \sim G(n, 0) \in A] = 0$ , and that  $\mathbb{P}[G \sim G(n, 1) \in A] = 1$ , hence  $f(0) = 0$  and  $f(1) = 1$ , and by the intermediate value theorem, for each  $0 \leq v \leq 1$ , there must exist an argument  $0 \leq p \leq 1$  s.t.  $v = f(p)$ . We choose  $v = \frac{1}{2}$ , hence there must exist some  $0 \leq p^* \leq 1$  s.t.  $v = f(p^*) = \frac{1}{2}$ .

### 1.2

We check that  $1 - kp \leq (1-p)^k$ , for all  $k \in \mathbb{N}$ , by induction.

For  $k = 1$  it is trivial, for  $k+1$ , we have  $(1-p)^{k+1} = (1-p)^k(1-p)$ , and by the assumption,  $(1-p)^k(1-p) \geq (1-kp)(1-p) = 1-kp-p+kp^2 = 1-(k+1)p+kp^2$ , but  $kp^2 > 0$ , so  $(1-p)^{k+1} = (1-p)^k(1-p) \geq 1-(k+1)p+kp^2 > 1-(k+1)p$ , which proves the assumption.

But it means that for each potential edge  $e$ ,

$$\mathbb{P}[e \notin G \sim G(n, kp)] = 1 - kp \leq (1-p)^k = (\mathbb{P}[e \notin G \sim G(n, p)])^k = \mathbb{P}[e \notin G \sim \bigcup_{i=1}^k G(n, p)] \Rightarrow \mathbb{P}[e \in G \sim G(n, kp)] \geq \mathbb{P}[e \in G \sim \bigcup_{i=1}^k G(n, p)],$$

but by a theorem coming from staged exposure, it means that if  $A$  is an increasing monotone property, then  $\mathbb{P}[G \sim G(n, kp) \in A] \geq \mathbb{P}[G \sim \bigcup_{i=1}^k G(n, p) \in A] \Rightarrow \mathbb{P}[G \sim G(n, kp) \notin A] \leq \mathbb{P}[G \sim \bigcup_{i=1}^k G(n, p) \notin A] = (\mathbb{P}[G \sim G(n, p) \notin A])^k$

### 1.3

$\omega := g(n)$ , s.t.  $\lim_{n \rightarrow \infty} g(n) = \infty$  and  $\lim_{n \rightarrow \infty} \frac{g(n)}{n} = 0$ .

Let  $n_k := \min\{n \in \mathbb{N} : \lfloor g(n_k) \rfloor \geq k\}$ , we are guaranteed to have such  $n_k$  for every  $k \geq 1$ , otherwise there exists some  $k_0$  s.t.  $\lfloor g(n) \rfloor \leq k_0$  for every  $n \in \mathbb{N}$ , in contradiction to  $g(n) \rightarrow \infty$ . Hence, for every  $k \geq 1$  we have  $n_k \in \mathbb{N}$ , s.t.  $\mathbb{P}[G(n_k, g(n_k)p^*) \notin A] \leq \mathbb{P}[G(n_k, kp^*) \notin A]$ , this is true because  $g(n) \geq \lfloor g(n) \rfloor \geq k$  and  $A$  is a monotone increasing property. But from 1.2 we know that  $\mathbb{P}[G(n_k, kp^*) \notin A] \leq (\mathbb{P}[G(n_k, p^*) \notin A])^k$ , but  $\lim_{k \rightarrow \infty} (\mathbb{P}[G(n_k, p^*) \notin A])^k = \lim_{k \rightarrow \infty} \frac{1}{2^k} = 0 \Rightarrow \lim_{n \rightarrow \infty} \mathbb{P}[G(n, g(n)p^*) \in A] = \lim \mathbb{P}[G(n, \omega p^*) \in A] = 1$ .

For  $\mathbb{P}[G(n, \frac{p^*}{g(n)}) \in A]$ , we observe that  $0 \leq p^* \leq 1 \Rightarrow \lim_{n \rightarrow \infty} \frac{p^*}{g(n)} \leq \lim_{n \rightarrow \infty} \frac{1}{g(n)} = 0$ , as  $g(n)$  tends to infinity. But then  $\lim_{n \rightarrow \infty} \mathbb{P}[G(n, \frac{p^*}{g(n)}) \in A] = \lim_{p \rightarrow 0} \mathbb{P}[G(n, p) \in A] = \mathbb{P}[G(n, 0) \in A]$ , as  $\mathbb{P}[G(n, p) \in A]$  is continuous, but from 1.1 we know that  $\mathbb{P}[G(n, 0) \in A] = 0$ .

## 2

Intuitively, we observe that for any finite graph on  $n \geq 3$  vertices, a triangle may appear starting from  $m = 3$  which is a constant, but the graph cannot be connected before at least  $n - 1$  edges appear, which depends on  $n$ , so when  $n \rightarrow \infty$ , the distance  $(n - 1) - 3$  also tends to infinity. But we need to prove this formally.

We use two theorems which are common in the literature, and their results were studied in class.

1. Let  $m = \frac{1}{2}n(\log n + c_n)$ . Then

$$\lim_{n \rightarrow \infty} \mathbb{P}[G \sim G(n, m) \in \mathcal{C}] = \begin{cases} 0, & c_n \rightarrow -\infty \\ e^{-e^{-c}}, & c_n \rightarrow c \\ 1, & c_n \rightarrow \infty \end{cases}$$

where  $\mathcal{C}$  is the property of being connected, and  $c$  is some constant.

### Proof essence:

We go through the  $G(n, p)$  model. Let  $C_k$  be the number of connected components with  $k$  vertices. Clearly we count only up to  $k = \frac{n}{2}$ . Then the probability of  $G \sim G(n, p)$  having more than one connected component is the probability of the union of the events  $\{C_k > 0\}_{k=1}^{\frac{n}{2}}$ , that is  $\mathbb{P}[\bigcup_{k=1}^{\frac{n}{2}} C_k > 0]$ . Denote by  $\mathcal{C}$  the property of being connected, we notice that  $\mathbb{P}[G \sim G(n, p) \notin \mathcal{C}] \leq \mathbb{P}[\bigcup_{k=1}^{\frac{n}{2}} C_k > 0] \leq \sum_{k=1}^{\frac{n}{2}} \mathbb{P}[C_k > 0]$ , because if there is at least one connected component of up to  $k = \frac{n}{2}$  vertices in  $G$ , then  $G$  is not connected, so not being connected is a partial event to the union of having connected components. We separate the sum  $\sum_{k=1}^{\frac{n}{2}} \mathbb{P}[C_k > 0] = \mathbb{P}[C_1 > 0] + \sum_{k=2}^{\frac{n}{2}} \mathbb{P}[C_k > 0]$ , because we can bound each of two cases  $k = 1$  and  $2 \leq k \leq \frac{n}{2}$  separately. By Markov inequality,  $\sum_{k=2}^{\frac{n}{2}} \mathbb{P}[C_k \geq 1] \leq \sum_{k=2}^{\frac{n}{2}} \mathbb{E}C_k$ . For each component of size  $k$  we have  $\binom{n}{k}$  choices of  $k$  out of  $n$  vertices. We need to have at least  $k - 1$  edges connecting the  $k$  vertices in the component, that is  $p^{k-1}$ , and we must not have

edges between each of the vertices in the component and the other  $n - k$  vertices in the graph, that is  $(1 - p)^{k(n-k)}$ . Considering all the possible arrangements of  $k$  components for all the possible counts of  $C_k$ , we have  $\sum_{k=2}^{\frac{n}{2}} \mathbb{E}C_k \leq \sum_{k=2}^{\frac{n}{2}} \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)}$ . We can bound this sum by  $o(1)$ , so the probability  $\sum_{k=2}^{\frac{n}{2}} \mathbb{P}[C_k > 0]$  asymptotically contributes nothing to  $G$  not being connected. now we observe the probability of having no isolated vertices  $\mathbb{P}[C_1 = 0]$  behaves according to the claim that if we take  $p = \frac{\log n + c}{n}$ , then  $\lim_{n \rightarrow \infty} \mathbb{P}[C_1 = 0] = e^{-e^{-c}} \Rightarrow \lim_{n \rightarrow \infty} \mathbb{P}[G \sim G(n, p) \in \mathcal{C}] \approx \lim_{n \rightarrow \infty} \mathbb{P}[C_1 = 0] + o(1) \approx e^{-e^{-c}} < 1$ , for any constant  $c$ .

2. Let  $m$  be s.t.  $\lim_{n \rightarrow \infty} \frac{m}{n} = \infty$  then  $\lim_{n \rightarrow \infty} \mathbb{P}[G \sim G(n, m) \in \mathcal{T}] = 1$ , where  $\mathcal{T}$  is the property of having at least one triangle.

**Proof essence:**

We prove this through the  $G(n, p)$  model. Let  $p = p(n)$  be a probability for which  $\omega = \omega(n) = np \rightarrow \infty$  as  $n \rightarrow \infty$ , but  $\omega(n) = np \leq \log n \Rightarrow p \leq \frac{\log n}{n}$ . Denote by  $T$  the number of triangles in  $G$ , so  $\mathbb{E}T = \binom{n}{3} p^3 = \frac{n(n-1)(n-2)}{3!} p^3 \approx \frac{n^3}{6} p^3 = \frac{(np)^3}{6} \rightarrow \infty$ . Denote by  $m_t$  the number of all possible triangles  $m_t = \binom{n}{3}$ , then  $\mathbb{E}T^2 = \sum_{i,j=1}^{m_t} \mathbb{P}[T_i, T_j \in G] = \sum_{i=1}^{m_t} \mathbb{P}[T_i \in G] \sum_{j=1}^{m_t} \mathbb{P}[T_j \in G | T_i \in G]$ . But the probability is equal for each of the triangles, so fix some  $1 \leq i_0 \leq m_t$ ,  $\mathbb{E}T^2 = m_t \mathbb{P}[T_{i_0} \in G] \sum_{j=1}^{m_t} \mathbb{P}[T_j \in G | T_{i_0} \in G] = \mathbb{E}T \sum_{j=1}^{m_t} \mathbb{P}[T_j \in G | T_{i_0} \in G]$

We calculate  $\sum_{j=1}^{m_t} \mathbb{P}[T_j \in G | T_{i_0} \in G]$  by counting the expected number of triangles in  $G$ , when there is already a triangle in  $G$ . For that, we need to break the expected count into different cases, by the number of joint edges between the existing triangle and the new triangles. Denote by  $\mathbb{E}_k$  the count of new triangles when the number of joint edges is  $k$ , then  $\sum_{j=1}^{m_t} \mathbb{P}[T_j \in G | T_{i_0} \in G] = \sum_{k=0}^3 \mathbb{E}_k$ . We check the different cases,

$k = 3$ : Only the existing triangle is a.s. (or surely) sharing 3 edges with itself, thus  $\mathbb{E}_3 = 1$ .

$k = 2$ : There is a.s. (surely) no triangle sharing only two edges with another, otherwise it is sharing also the third edge and this is the case  $\mathbb{E}_3$ , thus  $\mathbb{E}_2 = 0$ .

$k = 1$ : There are 3 choices of the 2 shared vertices, for each we choose a new vertex not in the triangle, and draw 2 edges from this vertex to the 2 existing vertices, thus  $\mathbb{E}_1 = 3(n-3)p^2$ .

$k = 0$ : There are two cases here, whether there is a joint vertex or not, but either way, the approximate number of choices for the new vertices is  $\binom{n-3}{3} \leq \binom{n}{3}$ , and we have 3 new edges, so the probability is  $p^3$ , thus  $\mathbb{E}_0 \leq \binom{n}{3} p^3 = \mathbb{E}T$ . Then we observe that  $\text{Var}T = \mathbb{E}T^2 - (\mathbb{E}T)^2 \leq \mathbb{E}T(1 + 3(n-3)p^2 + \mathbb{E}T) - (\mathbb{E}T)^2 = (1 + 3(n-3)p^2)\mathbb{E}T$ . But we chose  $p \leq \frac{\log n}{n} \Rightarrow (1 + \frac{(3n-9)(\log n)^2}{n^2})\mathbb{E}T \leq 2\mathbb{E}T$ . Hence, we can use the Chebyshev inequality on the following claim,

The probability that  $G$  has no triangles  $\{T = 0\}$  is a partial event of the event  $\{T : |T - \mathbb{E}T| \geq \mathbb{E}T\}$ , because  $T = 0 \Rightarrow |0 - \mathbb{E}T| = \mathbb{E}T$ . Hence,  $\mathbb{P}[T = 0] \leq \mathbb{P}[|T - \mathbb{E}T| \geq \mathbb{E}T] \leq \frac{\text{Var}T}{(\mathbb{E}T)^2}$ , by the Chebyshev inequality. But  $\text{Var}T \leq 2\mathbb{E}T \Rightarrow \mathbb{P}[T = 0] \leq \frac{\text{Var}T}{(\mathbb{E}T)^2} \leq \frac{2\mathbb{E}T}{(\mathbb{E}T)^2} = \frac{2}{\mathbb{E}T}$ , but  $\mathbb{E}T \rightarrow \infty$  when  $n \rightarrow \infty$ , thus  $\mathbb{P}[T = 0] = 0$ , when  $n \rightarrow \infty$ , so  $\lim_{n \rightarrow \infty} \mathbb{P}[G \sim G(n, p) \notin \mathcal{T}] = 0 \Rightarrow \lim_{n \rightarrow \infty} \mathbb{P}[G \sim G(n, p) \in \mathcal{T}] = 1$ , for  $p \leq \frac{\log n}{n}$ , and since  $\mathcal{T}$  is a monotone increasing property, this will be true for any greater  $p$ . But  $m \approx \binom{n}{2}p = \frac{n(n-1)}{2}p \approx \frac{n(n-1)\log n}{2n} \approx n \log n \Rightarrow \frac{n \log n}{n} = \log n \rightarrow \infty$ , when  $n \rightarrow \infty$ , which shows that it is enough to have  $m = O(\log n)$  edges in  $G \sim G(n, m)$  so that  $G \sim G(n, m) \in \mathcal{T}$ .

Using the two theorems above. Let  $c_0$  be some constant, and  $m_0 := \frac{1}{2}n(\log n + c_0)$ , then  $\lim_{n \rightarrow \infty} \frac{m_0}{n} = \lim_{n \rightarrow \infty} \frac{n(\log n + c_0)}{2n} = \lim_{n \rightarrow \infty} \frac{\log n + c_0}{2} = \infty \Rightarrow \lim_{n \rightarrow \infty} \mathbb{P}[G \sim G(n, m_0) \in \mathcal{T}] = 1$ , by theorem 4.1, but  $\lim_{n \rightarrow \infty} \mathbb{P}[G \sim G(n, m_0) \in \mathcal{C}] = e^{-e^{-c_0}}$ , by theorem 1.12, which proves the claim.

### 3

We use the fact that a graph  $G$  is planar only if  $G$  is 5-degenerate. To illustrate a 5-degenerate graph on  $n$  vertices, we choose a permutation  $\sigma$  of the vertices, so each vertex  $i \in [n]$  is assigned with the image  $\sigma(i)$ , and we arrange all the vertices in one row, from left to right, that is  $\sigma(1), \sigma(2), \dots, \sigma(n)$ . Suppose we have such an arrangement, and we want to construct a maximal 5-degenerate graph. We do this by going over every  $i \in \sigma([n])$ , and drawing 5 edges between  $i$  and 5 of its predecessors. Easy to show that there are no other edges except the ones drawn by this procedure, because any edge from  $i$  to another vertex  $j$  is either included in the 5 edges from  $i$  to its predecessors if  $j < i$ , or included in the 5 edges from  $j$  to its predecessors if  $i < j$ . Hence, the number of edges in a 5-degenerate graph is  $m \leq 5n$  (minus a constant, because of the first 5 vertices). Fix some arrangement  $\sigma_0$  and some  $m_0 \leq 5n$ , so the total number of possible 5-degenerated graphs with  $m_0$  edges, for the arrangement  $\sigma_0$ , is  $\binom{5n}{m_0}$ , but then, the total number of graphs with  $m_0$  edges is  $n! \binom{5n}{m_0}$ , because there are  $n!$  permutations on  $[n]$ . Denote the property  $\mathcal{P}$  that a graph  $G$  is planar, so  $\mathbb{P}[G \sim G(n, m_0) \in \mathcal{P}] \leq \frac{n! \binom{5n}{m_0}}{\binom{5n}{m_0}} \leq n! \left(\frac{5n}{N}\right)^{m_0} = n! \left(\frac{5n}{\frac{n(n-1)}{2}}\right)^{m_0} = n! \left(\frac{10}{n-1}\right)^{m_0}$ . By Stirling, we have  $\mathbb{P}[G \sim G(n, m_0)] \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(\frac{10}{n-1}\right)^{m_0} = \sqrt{2\pi n} \cdot n^n e^{-n} 10^{m_0} (n-1)^{-m_0}$ . Take  $m_0 = (1+\epsilon)n$ , for an arbitrary  $\epsilon > 0$ , so  $\mathbb{P}[G \sim G(n, m_0)] \approx \sqrt{2\pi} \cdot e^{-n} n^{\frac{1}{2}+n-n(1+\epsilon)} 10^{(1+\epsilon)n} = \sqrt{2\pi} \cdot e^{-n} n^{\frac{1}{2}-\epsilon n} 10^{(1+\epsilon)n}$ , but  $n^{\frac{1}{2}-\epsilon n}$  goes to zero faster than  $10^{(1+\epsilon)n}$  goes to infinity, thus  $\lim_{n \rightarrow \infty} \mathbb{P}[G \sim G(n, n(1+\epsilon)) \in \mathcal{P}] = 0 \Rightarrow \lim_{n \rightarrow \infty} \mathbb{P}[G \sim G(n, n(1+\epsilon)) \notin \mathcal{P}] = 1$ .