RandomGraphs

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1.1

Let G_0 be a specific graph with n vertices and m_0 edges, then $\mathbb{P}[G \sim G(n,p) = G_0] = p^{m_0}(1-p)^{\binom{n}{2}-m_0} \Rightarrow \mathbb{P}[G \sim G(n,p) \in A] = \mathbb{P}[\bigcup_{G_0 \in A} G \sim G(n,p) = G_0] = \sum_{G_0 \in A} p^{m_0}(1-p)^{\binom{n}{2}-m_0}$. But this is a polynomial in p, hence continous, and for every $0 \le p \le q \le 1$. we have $\mathbb{P}[G \sim G(n,p) = G_0] = p^{m_0}(1-p)^{\binom{n}{2}-m_0} \le q^{m_0}(1-q)^{\binom{n}{2}-m_0} = \mathbb{P}[G \sim G(n,q) = G_0]$, which applies also for the sum of probabilities over $\bigcup_{G_0 \in A} G_0$. Thus, $f(p) := \mathbb{P}[G \sim G(n,p) \in A]$ is both continous and monotone increasing. We know that $\mathbb{P}[G \sim G(n,0) = \phi] = 1 \Rightarrow \mathbb{P}[G \sim G(n,0) \in A] = 0$, and that $\mathbb{P}[G \sim G(n,1) \in A] = 1$, hence $f(0) = 0$ and $f(1) = 1$, and by the intermediate value theorem, for each $0 \le v \le 1$, there must exist an argument $0 \le p \le 1$ s.t. $v = f(p)$. We choose $v = \frac{1}{2}$, hence there must exist some $0 \le p^* \le 1$ s.t. $v = f(p^*) = \frac{1}{2}$.

1.2

We check that $1 - kp \le (1-p)^k$, for all $k \in \mathbb{N}$, by induction.

For $k = 1$ it is trivial, for $k+1$, we have $(1-p)^{k+1} = (1-p)^k(1-p)$, and by the assumption, $(1-p)^k(1-p) \ge (1-kp)(1-p) = 1-kp-p+kp^2 = 1-(k+1)p+kp^2$, but $kp^2 > 0$, so $(1-p)^{k+1} = (1-p)^k(1-p) \ge 1-(k+1)p+kp^2 > 1-(k+1)p$, which proves the assumption.

But it means that for each potential edge e,

$\mathbb{P}[e \notin G \sim G(n,kp)] = 1 - kp \le (1-p)^k = (\mathbb{P}[e \notin G \sim G(n,p)])^k = \mathbb{P}[e \notin G \sim \bigcup_{i=1}^k G(n,p)] \Rightarrow \mathbb{P}[e \in G \sim G(n,kp)] \ge \mathbb{P}[e \in G \sim \bigcup_{i=1}^k G(n,p)]$,

but by a theorem coming from staged exposure, it means that if A is an increasing monotone property, then $\mathbb{P}[G \sim G(n,kp) \in A] \ge \mathbb{P}[G \sim \bigcup_{i=1}^k G(n,p) \in A] \Rightarrow \mathbb{P}[G \sim G(n,kp) \notin A] \le \mathbb{P}[G \sim \bigcup_{i=1}^k G(n,p) \notin A] = (\mathbb{P}[G \sim G(n,p) \notin A])^k$

1.3

$\omega(n) \to \infty$

Let $n_k := \min\{n \in \mathbb{N} : \lfloor \omega(n_k) \rfloor \geq k\}$, we are guaranteed to have such n_k for every $k \geq 1$, otherwise there exists some k_0 s.t. $\lfloor \omega(n) \rfloor \leq k_0$ for every $n \in \mathbb{N}$, in contradiction to $\omega(n) \to \infty$. Hence, for every $k \geq 1$ we have $n_k \in \mathbb{N}$, s.t. $\mathbb{P}[G(n_k, \omega(n_k)p^*) \notin A] \leq \mathbb{P}[G(n_k, kp^*) \notin A]$, this is true because $\omega(n) \geq \lfloor \omega(n) \rfloor \geq k$ and A is a monotone increasing property. But from 1.2 we know that $\mathbb{P}[G(n_k, kp^*) \notin A] \leq (\mathbb{P}[G(n_k, p^*) \notin A])^k$, but $\lim_{k\to\infty}(\mathbb{P}[G(n_k, p^*) \notin A])^k = \lim_{k\to\infty} \frac{1}{2^k} = 0 \Rightarrow \lim_{n\to\infty} \mathbb{P}[G(n, \omega p^*) \in A] = 1$.

For $\mathbb{P}[G(n, \frac{p^*}{\omega(n)}) \in A]$, we observe that $0 \leq p^* \leq 1 \Rightarrow \lim_{n\to\infty} \frac{p^*}{\omega(n)} \leq \lim_{n\to\infty} \frac{1}{\omega(n)} = 0$, as $\omega(n)$ tends to infinity. But then $\lim_{n\to\infty} \mathbb{P}[G(n, \frac{p^*}{\omega(n)}) \in A] = \lim_{p\to 0} \mathbb{P}[G(n, p) \in A] = \mathbb{P}[G(n, 0) \in A]$, as $\mathbb{P}[G(n, p) \in A]$ is continuous, but from 1.1 we know that $\mathbb{P}[G(n, 0) \in A] = 0$.