RandomGraphs

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1.1

Let G_0 be a specific graph with n vertices and m_0 edges, then $\mathbb{P}[G \sim G(n,p) = G_0] = p^{m_0}(1-p)^{\binom{n}{2}-m_0} \Rightarrow \mathbb{P}[G \sim G(n,p) \in A] = \mathbb{P}[\bigcup_{G_0 \in A} G \sim G(n,p) = G_0] = \sum_{G_0 \in A} p^{m_0}(1-p)^{\binom{n}{2}-m_0}$. But this is a polynomial in p, hence continous, and for every $0 \le p \le q \le 1$. we have $\mathbb{P}[G \sim G(n,p) = G_0] = p^{m_0}(1-p)^{\binom{n}{2}-m_0} \le q^{m_0}(1-q)^{\binom{n}{2}-m_0} = \mathbb{P}[G \sim G(n,q) = G_0]$, which applies also for the sum of probabilities over $\bigcup_{G_0 \in A} G_0$. Thus, $f(p) := \mathbb{P}[G \sim G(n,p) \in A]$ is both continous and monotone increasing. We know that $\mathbb{P}[G \sim G(n,0) = \phi] = 1 \Rightarrow \mathbb{P}[G \sim G(n,0) \in A] = 0$, and that $\mathbb{P}[G \sim G(n,1) \in A] = 1$, hence $f(0) = 0$ and $f(1) = 1$, and by the intermediate value theorem, for each $0 \le v \le 1$, there must exist an argument $0 \le p \le 1$ s.t. $v = f(p)$. We choose $v = \frac{1}{2}$, hence there must exist some $0 \le p^* \le 1$ s.t. $v = f(p^*) = \frac{1}{2}$.

1.2

We check that $1 - kp \le (1-p)^k$, for all $k \in \mathbb{N}$, by induction.

For $k = 1$ it is trivial, for $k+1$, we have $(1-p)^{k+1} = (1-p)^k(1-p)$, and by the assumption, $(1-p)^k(1-p) \ge (1 - kp)(1 - p) = 1 - kp - p + kp^2 = 1 - (k+1)p + kp^2$, but $kp^2 > 0$, so $(1-p)^{k+1} = (1-p)^k(1-p) \ge 1 - (k+1)p + kp^2 > 1 - (k+1)p$, which proves the assumption.

But it means that for each potential edge e,

$\mathbb{P}[e \notin G \sim G(n,kp)] = 1 - kp \le (1-p)^k = (\mathbb{P}[e \notin G \sim G(n,p)])^k = \mathbb{P}[e \notin G \sim \bigcup_{i=1}^{k} G(n,p)] \Rightarrow \mathbb{P}[e \in G \sim G(n,kp)] \ge \mathbb{P}[e \in G \sim \bigcup_{i=1}^{k} G(n,p)]$,

but by a theorem coming from staged exposure, it means that if A is an increasing monotone property, then $\mathbb{P}[G \sim G(n,kp) \in A] \ge \mathbb{P}[G \sim \bigcup_{i=1}^{k} G(n,p) \in A] \Rightarrow \mathbb{P}[G \sim G(n,kp) \notin A] \le \mathbb{P}[G \sim \bigcup_{i=1}^{k} G(n,p) \notin A] = (\mathbb{P}[G \sim G(n,p) \notin A])^k$

1.3

$\omega := g(n)$, s.t. $\lim_{n \to \infty} g(n) = \infty$ and $\lim_{n \to \infty} \frac{g(n)}{n} = 0$.

Let $n_k := \min\{n \in \mathbb{N} : \lfloor g(n_k) \rfloor \geq k\}$, we are guaranteed to have such n_k for every $k \geq 1$, otherwise there exists some k_0 s.t. $\lfloor g(n) \rfloor \leq k_0$ for every $n \in \mathbb{N}$, in contradiction to $g(n) \rightarrow \infty$. Hence, for every $k \geq 1$ we have $n_k \in \mathbb{N}$, s.t. $\mathbb{P}[G(n_k, g(n_k)p^*) \notin A] \leq \mathbb{P}[G(n_k, kp^*) \notin A]$, this is true because $g(n) \geq \lfloor g(n) \rfloor \geq k$ and A is a monotone increasing property. But from 1.2 we know that $\mathbb{P}[G(n_k, kp^*) \notin A] \leq (\mathbb{P}[G(n_k, p^*) \notin A])^k$, but $\lim_{k \rightarrow \infty}(\mathbb{P}[G(n_k, p^*) \notin A])^k = \lim_{k \rightarrow \infty} \frac{1}{2^k} = 0 \Rightarrow \lim_{n \rightarrow \infty} \mathbb{P}[G(n, g(n)p^*) \in A] = \lim \mathbb{P}[G(n, \omega p^*) \in A] = 1$.

For $\mathbb{P}[G(n, \frac{p^*}{g(n)}) \in A]$, we observe that $0 \leq p^* \leq 1 \Rightarrow \lim_{n \rightarrow \infty} \frac{p^*}{g(n)} \leq \lim_{n \rightarrow \infty} \frac{1}{g(n)} = 0$, as $g(n)$ tends to infinity. But then $\lim_{n \rightarrow \infty} \mathbb{P}[G(n, \frac{p^*}{g(n)}) \in A] = \lim_{p \rightarrow 0} \mathbb{P}[G(n, p) \in A] = \mathbb{P}[G(n, 0) \in A]$, as $\mathbb{P}[G(n, p) \in A]$ is continuous, but from 1.1 we know that $\mathbb{P}[G(n, 0) \in A] = 0$.

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Intuitively, we observe that for any finite graph on $n \geq 3$ vertices a triangle can appear starting from $m = 3$ which is a constant, but the graph cannot be connected before at least $n - 1$ edges appear, which depends on n, so when $n \rightarrow \infty$, the distance $(n-1)-3$ also tends to infinity. But we need to prove this formally. We shall partially repeat some well-known facts. Let $G \sim G(n, p)$, the probability of having at least one triangle in G has a threshold. Let T be the number of triangles in G, then $\mathbb{E}[T] = \binom{n}{3}p^3 \Rightarrow \lim_{n \rightarrow \infty} \mathbb{E}[T] = \lim_{n \rightarrow \infty} \binom{n}{3}p^3 = \lim_{n \rightarrow \infty} \frac{n(n-1)(n-2)}{3!}p^3$. For every $0 \leq p(n) \leq 1$, we have

$$\lim_{n \rightarrow \infty} \frac{n(n-1)(n-2)}{6}p(n)^3 = \begin{cases} 0, & p(n) << \frac{1}{n} \\ 1, & p(n) >> \frac{1}{n} \end{cases}$$

But by Markov's inequality, $\mathbb{P}[T \geq 1] \leq \frac{\mathbb{E}[T]}{1}$, so in the case of $p(n) << \frac{1}{n}$ we have that $\lim_{n \rightarrow \infty} \mathbb{P}[T \geq 1] \leq \lim_{n \rightarrow \infty} \mathbb{E}[T] = 0$. For $p >> \frac{1}{n}$, we use the second moment method, we have the inequality $\mathbb{P}[T \geq 1] \geq \frac{\mathbb{E}[T]^2}{\mathbb{E}[T^2]} = \frac{[\binom{n}{3}p^3]^2}{\mathbb{E}[T^2]}$. But $T^2 = \sum \mathbb{1}_{\{i,j,k\} \in \mathcal{T}} \sum \mathbb{1}_{\{f,g,h\} \in \mathcal{T}}$, where $\{i, j, k\}$ and $\{f, g, h\}$ are triplets of vertices and $\mathcal{T}$ is the set of all potential triangles in G. There are different ways to choose one triplet and then choose a second triplet, the one that yields the largest count of vertices is when $\{i, j, k\} \cap \{f, g, h\} = \phi$, we denote T_0 the potential number of triplets in the case of the intersection being ϕ, which means no joint vertex, T_1 the case of one joint vertex, T_2 the case of two joint vertices, and T_3 the case of choosing the same triangle twice.

Thus $T_0 = \binom{n}{3}\binom{n-3}{3}$.

For T_1 we have $\binom{n}{3}$ choices of the first triplet, and 3 options for choosing one of the vertices of the first triplet, and then $\binom{n-3}{2}$ choices for the other two vertices of the second triplet, so $T_1 = 3\binom{n}{3}\binom{n-3}{2}$, and $T_2 = 3\binom{n}{3}(n-3)$, and $T_3 = \binom{n}{3}$.

We observe that in the case of T_0 and the case of T_1 we have 6 distinct edges for every two triangles, so their probability is p^6, but in the case of T_2 we have one joint edges, so only 5 distinct edges for every two triangles, and

in the case of choosing the same triplet twice, we have the same three edges chosen twice, but the probability for that remains p^3. Thus the corresponding expectations of T_1, T_2 and T_3 are $O(n^5 p^6)$, $O(n^4 p^5)$ and $O(n^3 p^3)$, respectively. Thus $\frac{\mathbb{E}[T]^2}{\mathbb{E}[T^2]} = \frac{\binom{n}{3}^2 p^6}{\binom{n}{3}\binom{n-3}{3} p^6 + o(n^6 p^6)} = 1 - o(1)$, but $\mathbb{P}[T \geq 1] \geq \frac{\mathbb{E}[T]^2}{\mathbb{E}[T^2]} = 1 - o(1)$. Notice that this is true only if $p >> \frac{1}{p}$, for which $\mathbb{E}[T]$ tends to infinity.

For connectedness, the threshold is $\frac{\log n}{n}$. We know that for a random graph process, each step $0 \leq m \leq \binom{n}{2}$ has the same distribution of $G(n, m)$, so we translate our thresholds from $G(n, p)$ to $G(n, m)$ by taking $m = \binom{n}{2} p + O(\sqrt{\binom{n}{2} p(1-p)})$. Thus, the $G(n, m)$ threshold for triangles becomes $m_t := \binom{n}{2} \frac{1}{n} + O(\sqrt{\binom{n}{2} \frac{1}{n}(1 - \frac{1}{n})}) \approx \binom{n}{2} \frac{1}{n}$, and the threshold for connectedness becomes $m_c := \binom{n}{2} \frac{\log n}{n} + O(\sqrt{\binom{n}{2} \frac{\log n}{n}(1 - \frac{\log n}{n})}) \approx \binom{n}{2} \frac{\log n}{n}$. We take the sequence $M_n := \{\frac{m_c}{m_t}\} \approx \log n$, but $\lim_{n \to \infty} \log n = \infty$, hence $\lim_{n \to \infty} \mathbb{P}[m_c \geq m_t] = \lim_{n \to \infty} \mathbb{P}[\frac{m_c}{m_t} \geq 1] = \infty$.

A more accurate solution is based on the book by Frieze and Karonski.

1. Theorem 4.1 (from Erdos and Reyni)

   Let $m = \frac{1}{2} n (\log n + c_n)$. Then

   $$\lim_{n \to \infty} \mathbb{P}[G \sim G(n, m) \in \mathcal{C}] = \begin{cases} 0, & c_n \to -\infty \\ e^{-e^{-c}}, & c_n \to c \\ 1, & c_n \to \infty \end{cases}$$

   where $\mathcal{C}$ is the property of being connected, and c is some constant.

2. Theorem 1.12

   If $\lim_{n \to \infty} \frac{m}{n} = \infty$ then $\lim_{n \to \infty} \mathbb{P}[G \sim G(n, m) \in \mathcal{T}] = 1$, where $\mathcal{T}$ is the property of having at least one triangle.

3. Using the two theorems above. Let c_0 be some constant, and $m_0 := \frac{1}{2} n (\log n + c_0)$, then $\lim_{n \to \infty} \frac{m_0}{n} = \lim_{n \to \infty} \frac{n(\log n + c_0)}{2n} = \lim_{n \to \infty} \frac{\log n + c_0}{2} = \infty \Rightarrow \lim_{n \to \infty} \mathbb{P}[G \sim G(n, m_0) \in \mathcal{T}] = 1$, by theorem 4.1, but $\lim_{n \to \infty} \mathbb{P}[G \sim G(n, m_0) \in \mathcal{C}] = e^{-e^{-c_0}}$, by theorem 1.12, which proves the claim.

3

We use the fact that a graph G is planar only if G is 5-degenerate. To illustrate a 5-degenerate graph on n vertices, we choose a permutation σ of the vertices, so each vertex $i \in [n]$ is assigned with the image $\sigma(i)$, and we arrange all the vertices in one row, from left to right, that is $\sigma(1), \sigma(2), \ldots, \sigma(n)$. Suppose we have such an arrangement, and we want to construct a maximal 5-degenerate graph. We do this by going over every $i \in \sigma([n])$, and drawing 5 edges between i and 5 of its predecessors. Easy to show that there are no other edges except the ones drawn by this procedure, because any edge from i to another vertex j is either included

in the 5 edges from i to its predecessors if $j < i$, or included in the 5 edges from j to its predecessors if $i < j$. Hence, the number of edges in a 5-degenerate graph is $m \leq 5n$ (minus a constant, because of the first 5 vertices). Fix some arrangement σ_0 and some $m_0 \leq 5n$, so the total number of possible 5-degenerated graphs with m_0 edges, for the arrangement σ_0, is $\binom{5n}{m_0}$, but then, the total number of graphs with m_0 edges is $n!\binom{5n}{m_0}$, because there are $n!$ permutations on $[n]$. Denote the property $\mathcal{P}$ that a graph G is planar, so $\mathbb{P}[G \sim G(n, m_0) \in \mathcal{P}] \leq \frac{n!\binom{5n}{m_0}}{\binom{N}{m_0}} \leq$ $n!(\frac{5n}{N})^{m_0} = n!\frac{5n}{\frac{n(n-1)}{2}} = n!(\frac{10}{n-1})^{m_0}$. By Stirling, we have $\mathbb{P}[G \sim G(n, m_0)] \approx$ $\sqrt{2\pi n}(\frac{n}{e})^n(\frac{10}{n-1})^{m_0} = \sqrt{2\pi n} \cdot n^n e^{-n} 10^{m_0}(n-1)^{-m_0}$. Take $m_0 = (1+\epsilon)n$, for an arbitrary $\epsilon > 0$, so $\mathbb{P}[G \sim G(n, m_0)] \approx \sqrt{2\pi} \cdot e^{-n} n^{\frac{1}{2}+n-n(1+\epsilon)} 10^{(1+\epsilon)n} = \sqrt{2\pi} \cdot$ $e^{-n} n^{\frac{1}{2}-\epsilon n} 10^{(1+\epsilon)n}$, but $n^{\frac{1}{2}-\epsilon n}$ goes to zero faster than $10^{(1+\epsilon)n}$ goes to infinity, thus $\lim_{n \to \infty} \mathbb{P}[G \sim G(n, n(1+\epsilon)) \in \mathcal{P}] = 0 \Rightarrow \lim_{n \to \infty} \mathbb{P}[G \sim G(n, n(1+\epsilon)) \notin \mathcal{P}] = 1$.

We prove this through the $G(n, p)$ model. Let $p = p(n)$ be a probability for which $\omega = \omega(n) = np \to \infty$ as $n \to \infty$, but $\omega(n) \leq \log n$. Denote by T the number of triangles in G, so $\mathbb{E}T = \binom{n}{3}p^3 = \frac{n(n-1)(n-2)}{3!}p^3 \approx \frac{n^3}{6}p^3 = \frac{(np)^3}{6} \to \infty$. Denote by m_t the number of triangles $m_t = \binom{m}{3}$, then $\mathbb{E}T^2 = \sum_{i,j=1}^{m_t} \mathbb{P}[T_i, T_j \in G] = \sum_{i=1}^{m_t} \mathbb{P}[T_i \in G] \sum_{j=1}^{m_t} \mathbb{P}[T_j \in G|T_i \in G]$. But the probability is equal for each of the triangles, hence $\mathbb{E}T^2 = m_t \mathbb{P}[T_i \in G] \sum_{j=1}^{m_t} \mathbb{P}[T_j \in G|T_i \in G] = \mathbb{E}T \sum_{j=1}^{m_t} \mathbb{P}[T_j \in G|T_i \in G]$

The conditional expectation depends on the number of joint edges,

3: this is the case where the two triangles are identical, so $\sum_{j=1}^{m_t} \mathbb{P}[T_j \in G|T_i \in G] = 1$.

2: there is no such case.

1: we have 3 ways to choose two joint vertices and one new vertex, so $\sum_{j=1}^{m_t} \mathbb{P}[T_j \in G|T_i \in G] = 3(n-3)p^2$

0: we have $\binom{n-3}{3}$ ways to choose three new vertices, and we have 3 ways to choose one joint vertex and two other vertices, so $\sum_{j=1}^{m_t} \mathbb{P}[T_j \in G|T_i \in G] = \binom{n-3}{3}p^3 + 3\binom{n-1}{2}p^2$.

$\mathbb{E}X = \binom{n}{2}p^9$

$p = \frac{1}{n^{\frac{6}{3}}} \Rightarrow \mathbb{E}X = \frac{n^6}{6}\frac{1}{(n^{\frac{2}{3}-\epsilon})^9} = \frac{n^6}{n^{5+a}}$

$\mathbb{P}[X > 0] \geq \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}$

$p > \frac{1}{n^{\frac{2}{3}}} \Rightarrow m = p\binom{n}{2} = \frac{n(n-1)}{2n^{\frac{2}{3}}} = \frac{n^2}{2n^{\frac{2}{3}}} = \frac{n^{2-\frac{2}{3}}}{2} = \frac{n^{\frac{4}{3}}}{2}$

$m = p\binom{n}{2} = n(1+\epsilon) \Rightarrow \frac{n(n-1)}{2}p = n(1+\epsilon) \Rightarrow \frac{n-1}{2}p = 1+\epsilon \to p = \frac{2(1+\epsilon)}{n-1}$