RandomGraphs

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1.1

Let G_0 be a specific graph with n vertices and m_0 edges, then $\mathbb{P}[G \sim G(n,p) = G_0] = p^{m_0}(1-p)^{\binom{n}{2}-m_0} \Rightarrow \mathbb{P}[G \sim G(n,p) \in A] = \mathbb{P}[\bigcup_{G_0 \in A} G \sim G(n,p) = G_0] = \sum_{G_0 \in A} p^{m_0}(1-p)^{\binom{n}{2}-m_0}$. But this is a polynomial in p, hence continous, and for every $0 \leq p \leq q \leq 1$. we have $\mathbb{P}[G \sim G(n,p) = G_0] = p^{m_0}(1-p)^{\binom{n}{2}-m_0} \leq q^{m_0}(1-q)^{\binom{n}{2}-m_0} = \mathbb{P}[G \sim G(n,q) = G_0]$, which applies also for the sum of probabilities over $\bigcup_{G_0 \in A} G_0$. Thus, $f(p) := \mathbb{P}[G \sim G(n,p) \in A]$ is both continous and monotone increasing. We know that $\mathbb{P}[G \sim G(n,0) = \phi] = 1 \Rightarrow \mathbb{P}[G \sim G(n,0) \in A] = 0$, and that $\mathbb{P}[G \sim G(n,1) \in A] = 1$, hence $f(0) = 0$ and $f(1) = 1$, and by the intermediate value theorem, for each $0 \leq v \leq 1$, there must exist an argument $0 \leq p \leq 1$ s.t. $v = f(p)$. We choose $v = \frac{1}{2}$, hence there must exist some $0 \leq p^* \leq 1$ s.t. $v = f(p^*) = \frac{1}{2}$.

1.2

We check that $1 - kp \leq (1-p)^k$, for all $k \in \mathbb{N}$, by induction.

For $k = 1$ it is trivial, for $k+1$, we have $(1-p)^{k+1} = (1-p)^k(1-p)$, and by the assumption, $(1-p)^k(1-p) \geq (1-kp)(1-p) = 1-kp-p+kp^2 = 1-(k+1)p+kp^2$, but $kp^2 > 0$, so $(1-p)^{k+1} = (1-p)^k(1-p) \geq 1-(k+1)p+kp^2 > 1-(k+1)p$, which proves the assumption.

But it means that for each potential edge e,

$\mathbb{P}[e \notin G \sim G(n,kp)] = 1 - kp \leq (1-p)^k = (\mathbb{P}[e \notin G \sim G(n,p)])^k = \mathbb{P}[e \notin G \sim \bigcup_{i=1}^k G(n,p)] \Rightarrow \mathbb{P}[e \in G \sim G(n,kp)] \geq \mathbb{P}[e \in G \sim \bigcup_{i=1}^k G(n,p)]$,

but by a theorem coming from staged exposure, it means that if A is an increasing monotone property, then $\mathbb{P}[G \sim G(n,kp) \in A] \geq \mathbb{P}[G \sim \bigcup_{i=1}^k G(n,p) \in A] \Rightarrow \mathbb{P}[G \sim G(n,kp) \notin A] \leq \mathbb{P}[G \sim \bigcup_{i=1}^k G(n,p) \notin A] = (\mathbb{P}[G \sim G(n,p) \notin A])^k$

1.3

$\omega := g(n)$, s.t. $\lim_{n\to\infty} g(n) = \infty$ and $\lim_{n\to\infty} \frac{g(n)}{n} = 0$.

Let $n_k := \min\{n \in \mathbb{N} : \lfloor g(n_k) \rfloor \geq k\}$, we are guaranteed to have such n_k for every $k \geq 1$, otherwise there exists some k_0 s.t. $\lfloor g(n) \rfloor \leq k_0$ for every $n \in \mathbb{N}$, in contradiction to $g(n) \to \infty$. Hence, for every $k \geq 1$ we have $n_k \in \mathbb{N}$, s.t. $\mathbb{P}[G(n_k, g(n_k)p^*) \notin A] \leq \mathbb{P}[G(n_k, kp^*) \notin A]$, this is true because $g(n) \geq \lfloor g(n) \rfloor \geq k$ and A is a monotone increasing property. But from 1.2 we know that $\mathbb{P}[G(n_k, kp^*) \notin A] \leq (\mathbb{P}[G(n_k, p^*) \notin A])^k$, but $\lim_{k\to\infty}(\mathbb{P}[G(n_k, p^*) \notin A])^k = \lim_{k\to\infty} \frac{1}{2^k} = 0 \Rightarrow \lim_{n\to\infty} \mathbb{P}[G(n, g(n)p^*) \in A] = \lim \mathbb{P}[G(n, \omega p^*) \in A] = 1$.

For $\mathbb{P}[G(n, \frac{p^*}{g(n)}) \in A]$, we observe that $0 \leq p^* \leq 1 \Rightarrow \lim_{n\to\infty} \frac{p^*}{g(n)} \leq \lim_{n\to\infty} \frac{1}{g(n)} = 0$, as $g(n)$ tends to infinity. But then $\lim_{n\to\infty} \mathbb{P}[G(n, \frac{p^*}{g(n)}) \in A] = \lim_{p\to 0} \mathbb{P}[G(n,p) \in A] = \mathbb{P}[G(n,0) \in A]$, as $\mathbb{P}[G(n,p) \in A]$ is continuous, but from 1.1 we know that $\mathbb{P}[G(n,0) \in A] = 0$.

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Intuitively, we observe that for any finite graph on $n \geq 3$ vertices a triangle can appear starting from $m = 3$ which is a constant, but the graph cannot be connected before at least $n - 1$ edges appear, which depends on n, so when $n \to \infty$, the distance $(n-1) - 3$ also tends to infinity. But we need to prove this formally.

We use two theorems which are common in the literature, and were studied in class.

1. Let $m = \frac{1}{2}n(\log n + c_n)$. Then

   $\lim_{n\to\infty} \mathbb{P}[G \sim G(n,m) \in \mathcal{C}] = \begin{cases} 0, & c_n \to -\infty \\ e^{-e^{-c}}, & c_n \to c \\ 1, & c_n \to \infty \end{cases}$

   where $\mathcal{C}$ is the property of being connected, and c is some constant.

   Proof essence:

   We go through the $G(n,p)$ model. Let C_k be the number of connected components with k vertices. Clearly we count only up to $k = \frac{n}{2}$. Then the probability of $G \sim G(n,p)$ having more than one connected component is the probability of the union of the events $\{C_k > 0\}_{k=1}^{\frac{n}{2}}$, that is $\mathbb{P}[\bigcup_{k=1}^{\frac{n}{2}} C_k > 0]$. Denote by $\mathcal{C}$ the property of being connected, we notice that $\mathbb{P}[G \sim G(n,p) \notin \mathcal{C}] \leq \mathbb{P}[\bigcup_{k=1}^{\frac{n}{2}} C_k > 0] \leq \sum_{k=1}^{\frac{n}{2}} \mathbb{P}[C_k > 0]$, because if there is at least one connected component of up to $k = \frac{n}{2}$ vertices in G, then G is not connected, so not being connected is a partial event to the union of having connected components. We separate the sum $\sum_{k=1}^{\frac{n}{2}} \mathbb{P}[C_k > 0] = \mathbb{P}[C_1 > 0] + \sum_{k=2}^{\frac{n}{2}} \mathbb{P}[C_k > 0]$, because we can bound each of two cases $k = 1$ and $2 \leq k \leq \frac{n}{2}$ separately. By Markov inequality, $\sum_{k=2}^{\frac{n}{2}} \mathbb{P}[C_k \geq 1] \leq \sum_{k=2}^{\frac{n}{2}} \mathbb{E}C_k$. For each component of size k we have $\binom{n}{k}$ choices of k out of n vertices. We need to have at least $k-1$ edges connecting the k vertices in the component, that is p^{k-1}, and we must not have edges between each of the vertices in the component and the other $n-k$ vertices in the graph, that is $(1-p)^{k(n-k)}$. Considering all the possible arrangements of k components for all the possible counts of C_k, we have $\sum_{k=2}^{\frac{n}{2}} \mathbb{E}C_k \leq \sum_{k=2}^{\frac{n}{2}} \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)}$. We can bound this sum by $o(1)$, so the probability $\sum_{k=2}^{\frac{n}{2}} \mathbb{P}[C_k > 0]$ asymptotically contributes nothing to G not being connected. now we observe the probability of having no isolated vertices $\mathbb{P}[C_1 = 0]$ behaves according to the claim that if we take $p = \frac{logn+c}{n}$, then $\lim_{n\to\infty} \mathbb{P}[C_1 = 0] = e^{-e^{-c}} \Rightarrow \lim_{n\to\infty} \mathbb{P}[G \sim G(n,p) \in \mathcal{C}] \approx \lim_{n\to\infty} \mathbb{P}[C_1 = 0] + o(1) \approx e^{-e^{-c}} < 1$, for any constant c.

2. Let m be s.t. $\lim_{n\to\infty} \frac{m}{n} = \infty$ then $\lim_{n\to\infty} \mathbb{P}[G \sim G(n,m) \in \mathcal{T}] = 1$, where $\mathcal{T}$ is the property of having at least one triangle.

   Proof essence:

   We prove this through the $G(n,p)$ model. Let $p = p(n)$ be a probability for which $\omega = \omega(n) = np \to \infty$ as $n \to \infty$, but $\omega(n) = np \leq \log n \Rightarrow p \leq \frac{\log n}{n}$. Denote by T the number of triangles in G, so $\mathbb{E}T = \binom{n}{3} p^3 = \frac{n(n-1)(n-2)}{3!} p^3 \approx \frac{n^3}{6} p^3 = \frac{(np)^3}{6} \to \infty$. Denote by m_t the number of triangles $m_t = \binom{m}{3}$, then $\mathbb{E}T^2 = \sum_{i,j=1}^{m_t} \mathbb{P}[T_i, T_j \in G] = \sum_{i=1}^{m_t} \mathbb{P}[T_i \in G] \sum_{j=1}^{m_t} \mathbb{P}[T_j \in G | T_i \in G]$. But the probability is equal for each of the triangles, hence $\mathbb{E}T^2 = m_t \mathbb{P}[T_i \in G] \sum_{j=1}^{m_t} \mathbb{P}[T_j \in G | T_i \in G] = \mathbb{E}T \sum_{j=1}^{m_t} \mathbb{P}[T_j \in G | T_i \in G]$

   We calculate $\sum_{j=1}^{m_t} \mathbb{P}[T_j \in G | T_i \in G]$ by counting the expected number of triangles in G, when there is already a triangle in G. For that, we need to break the expected count into different cases, by the number of joint edges between the existing triangle and the new triangles. Denote by $\mathbb{E}_k$ the count of new triangles when the number of joint edges is k, then $\sum_{j=1}^{m_t} \mathbb{P}[T_j \in G | T_i \in G] = \sum_{k=0}^{3} \mathbb{E}_k$. We check the different cases,

   $k = 3$: Only the existing triangle is a.s. (or surely) sharing 3 edges with itself, thus $\mathbb{E}_3 = 1$.

   $k = 2$: There is a.s. (surely) no triangle sharing only two edges with another, otherwise it is sharing also the third edge and this is the case $\mathbb{E}_3$, thus $\mathbb{E}_2 = 0$.

   $k = 1$: There are 3 choices of the 2 shared vertices, for each we choose a new vertex not in the triangle, and draw 2 edges from this vertex to the 2 existing vertices, thus $\mathbb{E}_1 = 3(n-3)p^2$.

   $k = 0$: There are two cases here, whether there is a joint vertex a not, but either way, the approximated number of choices for the new vertices is $\binom{n-3}{3} \leq \binom{n}{3}$, and we have 3 new edges, so the probability is p^3, thus $\mathbb{E}_0 \leq \binom{n}{3} p^3 = \mathbb{E}T$. Then we observe that $VarT = \mathbb{E}T^2 - (\mathbb{E}T)^2 \leq \mathbb{E}T(1 + 3(n-3)p^2 + \mathbb{E}T) - (\mathbb{E}T)^2 = (1 + 3(n-3)p^2)\mathbb{E}T$. But we chose $p \leq \frac{\log n}{n} \Rightarrow (1 + \binom{n}{3} p^2)\mathbb{E}T \leq 2\mathbb{E}T$. Hence, we can use the Chebyshev inequality on the following claim, The probability that G has no triangles $\{T = 0\}$ is a partial event of the event $\{T : |T - \mathbb{E}T| \geq \mathbb{E}T\}$, because $T = 0 \Rightarrow |0 - \mathbb{E}T| = \mathbb{E}T$. Hence, $\mathbb{P}[T = 0] \leq \mathbb{P}[|T - \mathbb{E}T| \geq \mathbb{E}T] \leq \frac{VarT}{(\mathbb{E}T)^2}$, by the Chebyshev inequality. But $VarT \leq 2\mathbb{E}T \Rightarrow \mathbb{P}[T = 0] \leq \frac{VarT}{(\mathbb{E}T)^2} \leq \frac{2\mathbb{E}T}{(\mathbb{E}T)^2} = \frac{2}{\mathbb{E}T}$, but $\mathbb{E}T \to \infty$ when $n \to \infty$, thus $\mathbb{P}[T = 0] = 0$, when $n \to \infty$, so $\lim_{n\to\infty} \mathbb{P}[G \sim G(n,p) \notin \mathcal{T}] = 0 \Rightarrow \lim_{n\to\infty} \mathbb{P}[G \sim G(n,p) \in \mathcal{T}] = 1$, for $p \leq \frac{\log n}{n}$, and since $\mathcal{T}$ is a monotone increasing property, this will be true for any greater p. But $m \approx \binom{n}{2}p = \frac{n(n-1)}{2}p \approx \frac{n(n-1)\log n}{2n} \approx n \log n \Rightarrow \frac{n \log n}{n} = \log n \to \infty$, when $n \to \infty$, which shows that it is enough to have $m = O(\log n)$ edges in $G \sim G(n,m)$ so that $G \sim G(n,m) \in \mathcal{T}$.

Using the two theorems above. Let c_0 be some constant, and $m_0 := \frac{1}{2}n(\log n+ c_0)$, then $\lim_{n\to\infty} \frac{m_0}{n} = \lim_{n\to\infty} \frac{n(\log n+c_0)}{2n} = \lim_{n\to\infty} \frac{\log n+c_0}{2} = \infty \Rightarrow \lim_{n\to\infty} \mathbb{P}[G \sim G(n,m_0) \in \mathcal{T}] = 1$, by theorem 4.1, but $\lim_{n\to\infty} \mathbb{P}[G \sim G(n,m_0) \in \mathcal{C}] = e^{-e^{-c_0}}$, by theorem 1.12, which proves the claim.

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We use the fact that a graph G is planar only if G is 5-degenerate. To illustrate a 5-degenerate graph on n vertices, we choose a permutation σ of the vertices, so each vertex $i \in [n]$ is assigned with the image $\sigma(i)$, and we arrange all the vertices in one row, from left to right, that is $\sigma(1), \sigma(2), \ldots, \sigma(n)$. Suppose we have such an arrangement, and we want to construct a maximal 5-degenerate graph. We do this by going over every $i \in \sigma([n])$, and drawing 5 edges between i and 5 of its predecessors. Easy to show that there are no other edges except the ones drawn by this procedure, because any edge from i to another vertex j is either included in the 5 edges from i to its predecessors if $j < i$, or included in the 5 edges from j to its predecessors if $i < j$. Hence, the number of edges in a 5-degenerate graph is $m \leq 5n$ (minus a constant, because of the first 5 vertices). Fix some arrangement σ_0 and some $m_0 \leq 5n$, so the total number of possible 5-degenerated graphs with m_0 edges, for the arrangement σ_0, is $\binom{5n}{m_0}$, but then, the total number of graphs with m_0 edges is $n!\binom{5n}{m_0}$, because there are $n!$ permutations on $[n]$. Denote the property $\mathcal{P}$ that a graph G is planar, so $\mathbb{P}[G \sim G(n,m_0) \in \mathcal{P}] \leq \frac{n!\binom{5n}{m_0}}{\binom{N}{m_0}} \leq n!(\frac{5n}{N})^{m_0} = n!\frac{5n}{\frac{n(n-1)}{2}} = n!(\frac{10}{n-1})^{m_0}$. By Stirling, we have $\mathbb{P}[G \sim G(n,m_0)] \approx \sqrt{2\pi n}(\frac{n}{e})^n(\frac{10}{n-1})^{m_0} = \sqrt{2\pi n} \cdot n^n e^{-n} 10^{m_0}(n-1)^{-m_0}$. Take $m_0 = (1+\epsilon)n$, for an arbitrary $\epsilon > 0$, so $\mathbb{P}[G \sim G(n,m_0)] \approx \sqrt{2\pi} \cdot e^{-n} n^{\frac{1}{2}+n-n(1+\epsilon)} 10^{(1+\epsilon)n} = \sqrt{2\pi} \cdot e^{-n} n^{\frac{1}{2}-\epsilon n} 10^{(1+\epsilon)n}$, but $n^{\frac{1}{2}-\epsilon n}$ goes to zero faster than $10^{(1+\epsilon)n}$ goes to infinity, thus $\lim_{n\to\infty} \mathbb{P}[G \sim G(n, n(1+\epsilon)) \in \mathcal{P}] = 0 \Rightarrow \lim_{n\to\infty} \mathbb{P}[G \sim G(n, n(1+\epsilon)) \notin \mathcal{P}] = 1$.