RandomGraphs

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1.1

Let G_0 be a specific graph with n vertices and m_0 edges, then $\mathbb{P}[G \sim G(n,p) = G_0] = p^{m_0}(1-p)^{\binom{n}{2}-m_0} \Rightarrow \mathbb{P}[G \sim G(n,p) \in A] = \mathbb{P}[\bigcup_{G_0 \in A} G \sim G(n,p) = G_0] = \sum_{G_0 \in A} p^{m_0}(1-p)^{\binom{n}{2}-m_0}$. But this is a polynomial in p, hence continous, and for every $0 \leq p \leq q \leq 1$. we have $\mathbb{P}[G \sim G(n,p) = G_0] = p^{m_0}(1-p)^{\binom{n}{2}-m_0} \leq q^{m_0}(1-q)^{\binom{n}{2}-m_0} = \mathbb{P}[G \sim G(n,q) = G_0]$, which applies also for the sum of probabilities over $\bigcup_{G_0 \in A} G_0$. Thus, $f(p) := \mathbb{P}[G \sim G(n,p) \in A]$ is both continous and monotone increasing. We know that $\mathbb{P}[G \sim G(n,0) = \phi] = 1 \Rightarrow \mathbb{P}[G \sim G(n,0) \in A] = 0$, and that $\mathbb{P}[G \sim G(n,1) \in A] = 1$, hence $f(0) = 0$ and $f(1) = 1$, and by the intermediate value theorem, for each $0 \leq v \leq 1$, there must exist an argument $0 \leq p \leq 1$ s.t. $v = f(p)$. We choose $v = \frac{1}{2}$, hence there must exist some $0 \leq p^* \leq 1$ s.t. $v = f(p^*) = \frac{1}{2}$.

1.2

We check that $1 - kp \leq (1-p)^k$, for all $k \in \mathbb{N}$, by induction.

For $k = 1$ it is trivial, for $k+1$, we have $(1-p)^{k+1} = (1-p)^k(1-p)$, and by the assumption, $(1-p)^k(1-p) \geq (1-kp)(1-p) = 1-kp-p+kp^2 = 1-(k+1)p+kp^2$, but $kp^2 > 0$, so $(1-p)^{k+1} = (1-p)^k(1-p) \geq 1-(k+1)p+kp^2 > 1-(k+1)p$, which proves the assumption.

But it means that for each potential edge e,

$\mathbb{P}[e \notin G \sim G(n,kp)] = 1 - kp \leq (1-p)^k = (\mathbb{P}[e \notin G \sim G(n,p)])^k = \mathbb{P}[e \notin G \sim \bigcup_{i=1}^k G(n,p)] \Rightarrow \mathbb{P}[e \in G \sim G(n,kp)] \geq \mathbb{P}[e \in G \sim \bigcup_{i=1}^k G(n,p)]$,

but by a theorem coming from staged exposure, it means that if A is an increasing monotone property, then $\mathbb{P}[G \sim G(n,kp) \in A] \geq \mathbb{P}[G \sim \bigcup_{i=1}^k G(n,p) \in A] \Rightarrow \mathbb{P}[G \sim G(n,kp) \notin A] \leq \mathbb{P}[G \sim \bigcup_{i=1}^k G(n,p) \notin A] = (\mathbb{P}[G \sim G(n,p) \notin A])^k$

1.3

$\omega := g(n)$, s.t. $\lim_{n\to\infty} g(n) = \infty$ and $\lim_{n\to\infty} \frac{g(n)}{n} = 0$.

Let $n_k := \min\{n \in \mathbb{N} : \lfloor g(n_k) \rfloor \geq k\}$, we are guaranteed to have such n_k for every $k \geq 1$, otherwise there exists some k_0 s.t. $\lfloor g(n) \rfloor \leq k_0$ for every $n \in \mathbb{N}$, in contradiction to $g(n) \to \infty$. Hence, for every $k \geq 1$ we have $n_k \in \mathbb{N}$, s.t. $\mathbb{P}[G(n_k, g(n_k)p^*) \notin A] \leq \mathbb{P}[G(n_k, kp^*) \notin A]$, this is true because $g(n) \geq \lfloor g(n) \rfloor \geq k$ and A is a monotone increasing property. But from 1.2 we know that $\mathbb{P}[G(n_k, kp^*) \notin A] \leq (\mathbb{P}[G(n_k, p^*) \notin A])^k$, but $\lim_{k\to\infty}(\mathbb{P}[G(n_k, p^*) \notin A])^k = \lim_{k\to\infty} \frac{1}{2^k} = 0 \Rightarrow \lim_{n\to\infty} \mathbb{P}[G(n, g(n)p^*) \in A] = \lim \mathbb{P}[G(n, \omega p^*) \in A] = 1$.

For $\mathbb{P}[G(n, \frac{p^*}{g(n)}) \in A]$, we observe that $0 \leq p^* \leq 1 \Rightarrow \lim_{n\to\infty} \frac{p^*}{g(n)} \leq \lim_{n\to\infty} \frac{1}{g(n)} = 0$, as $g(n)$ tends to infinity. But then $\lim_{n\to\infty} \mathbb{P}[G(n, \frac{p^*}{g(n)}) \in A] = \lim_{p\to 0} \mathbb{P}[G(n,p) \in A] = \mathbb{P}[G(n,0) \in A]$, as $\mathbb{P}[G(n,p) \in A]$ is continuous, but from 1.1 we know that $\mathbb{P}[G(n,0) \in A] = 0$.

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Intuitively, we observe that for any graph on $n \geq 3$ vertices a triangle can appear starting from $m = 3$ which is a constant, but the graph cannot be connected before at least $n-1$ edges appear, which depends on n, so when $n \to \infty$, the distance $(n-1)-3$ also tends to infinity. But we need to prove this formally. We shall partially repeat some well-known facts. Let $G \sim G(n,p)$, the probability of having at least one triangle in G has a threshold. Let T be the number of triangles in G, then $\mathbb{E}[T] = \binom{n}{3}p^3 \Rightarrow \lim_{n\to\infty}\mathbb{E}[T] = \lim_{n\to\infty} \binom{n}{3}p^3 = \lim_{n\to\infty} \frac{n(n-1)(n-2)}{3!}p^3$. For every $0 \leq p(n) \leq 1$, we have

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)}{6}p(n)^3 = \begin{cases} 0, & p(n) << \frac{1}{n} \\ 1, & p(n) >> \frac{1}{n} \end{cases}$$

But by Markov's inequality, $\mathbb{P}[T \geq 1] \leq \frac{\mathbb{E}[T]}{1}$, so in the case of $p(n) << \frac{1}{n}$ we have that $\lim_{n\to\infty} \mathbb{P}[T \geq 1] \leq lim_{n\to\infty}\mathbb{E}[T] = 0$. For $p >> \frac{1}{n}$, we use the second moment method, we have the inequality $\mathbb{P}[T \geq 1] \geq \frac{\mathbb{E}[T]^2}{\mathbb{E}[T^2]} = \frac{[\binom{n}{3}p^3]^2}{\mathbb{E}[T^2]}$. But $T^2 = \sum \mathbb{1}_{\{i,j,k\}\in\mathcal{T}} \sum \mathbb{1}_{\{f,g,h\}\in\mathcal{T}}$, where $\{i,j,k\}$ and $\{f,g,h\}$ are triplets of vertices and $\mathcal{T}$ is the set of all potential triangles in G. There are different ways to choose one triplet and then choose a second triplet, the one that yields the largest count of vertices is when $\{i,j,k\} \cap \{f,g,h\} = \phi$, we denote T_0 the potential number of triplets in the case of the intersection being ϕ, which means no joint vertex, T_1 the case of one joint vertex, T_2 the case of two joint vertices, and T_3 the case of choosing the same triangle twice.

Thus $T_0 = \binom{n}{3}\binom{n-3}{3}$.

For T_1 we have $\binom{n}{3}$ choices of the first triplet, and 3 options for choosing one of the vertices of the first triplet, and then $\binom{n-3}{2}$ choices for the other two vertices of the second triplet, so $T_1 = 3\binom{n}{3}\binom{n-3}{2}$, and $T_2 = 3\binom{n}{3}(n-3)$, and $T_3 = \binom{n}{3}$.

We observe that in the case of T_0 and the case of T_1 we have 6 distinct edges for every two triangles, so their probability is p^6, but in the case of T_2 we have one joint edges, so only 5 distinct edges for every two triangles, and in the case of choosing the same triplet twice, we have the same three edges chosen twice, but the probability for that remains p^3. Thus the corresponding expectations of T_1, T_2 and T_3 are $O(n^5p^6)$, $O(n^4p^5)$ and $O(n^3p^3)$, respectively. Thus $\frac{\mathbb{E}[T]^2}{\mathbb{E}[T^2]} = \frac{\binom{n}{3}^2 p^6}{\binom{n}{3}\binom{n-3}{3}p^6+o(n^6p^6)} = 1-o(1)$, but $\mathbb{P}[T \geq 1] \geq \frac{\mathbb{E}[T]^2}{\mathbb{E}[T^2]} = 1-o(1)$. Notice that this is true only if $p >> \frac{1}{p}$, for which $\mathbb{E}[T]$ tends to infinity.

For connectedness, the threshold is $\frac{\log n}{n}$. We know that for a random graph process, each step $0 \leq m \leq \binom{n}{2}$ has the same distribution of $G(n,m)$, so we translate our thresholds from $G(n,p)$ to $G(n,m)$ by taking $m = \binom{n}{2}p + O(\sqrt{\binom{n}{2}p(1-p)})$. Thus, the $G(n,m)$ threshold for triangles becomes $m_t := \binom{n}{2}\frac{1}{n} + O(\sqrt{\binom{n}{2}\frac{1}{n}(1-\frac{1}{n})}) \approx \binom{n}{2}\frac{1}{n}$, and the threshold for connectedness becomes $m_c := \binom{n}{2}\frac{\log n}{n} + O(\sqrt{\binom{n}{2}\frac{\log n}{n}(1-\frac{\log n}{n})}) \approx \binom{n}{2}\frac{\log n}{n}$. We take the sequence $M_n := \{\frac{m_c}{m_t}\} \approx \log n$, but $\lim_{n\to\infty}\log n = \infty$, hence $\lim_{n\to\infty}\mathbb{P}[m_c \geq m_t] = \lim_{n\to\infty}\mathbb{P}[\frac{m_c}{m_t} \geq 1] = \infty$.