

# Your Paper

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Denote  $G_5 := G_5(\mathbb{Z}_p)$ , and  $G_5^+ := G_5^+(\mathbb{Q}_p)$ .

$\zeta_{L_{5,p}}^\wedge(s) = \int_{G_5^+} |\det g|_p^s d\mu(G_5) = \int_{G_5^+} |\det uh|_p^s d\mu(G_5)$ , where  $h \in H$  and  $u \in N_h$ .

Each  $u$  is unipotent, hence  $\zeta_{L_{5,p}}^\wedge(s) = \int_{G_5^+} |\det h|_p^s d\mu(G_5) = \int_{G_5^+} |\lambda_1^4 \lambda_2^6 \lambda_3^6 \lambda_4^4|_p^s d\mu(G_5) = \int_{G_5^+} \left[ |\lambda_1^4|_p |\lambda_2^6|_p |\lambda_3^6|_p |\lambda_4^4|_p \right]^s d\mu(G_5)$ , by the inductive formula we have found for every  $|h|$ .

We denote  $v_i := v_p(\lambda_i)$ ,

and so  $\zeta_{L_{5,p}}^\wedge(s) = \int_{G_5^+} \left[ p^{-4v_1} p^{-6v_2} p^{-6v_3} p^{-4v_4} \right]^s d\mu(G_5) = \int_{G_5^+} p^{-(4v_1+6v_2+6v_3+4v_4)s} d\mu(G_5)$ .

We denote  $I_1 := p^{-(4v_1+6v_2+6v_3+4v_4)s}$ . Now we use the natural matrix decomposition of the  $N_h$  matrix of Berman's, which means that

$\zeta_{L_{5,p}}^\wedge(s) = \int_{G_5^+} I_1 d\mu(G_5) = \int_{\underline{\lambda}} \int_{\underline{a}} \int_{\underline{b}} \int_{\underline{c}} I_1 d\mu(\underline{c}) d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda})$ . Since  $I_1$  depends only on  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ , which appear only in the computation of the outermost integral, we consider them as constants for all the inner integrals, which means that we have  $\zeta_{L_{5,p}}^\wedge(s) = \int_{\underline{\lambda}} I_1 \int_{\underline{a}} \int_{\underline{b}} \int_{\underline{c}} 1 d\mu(\underline{c}) d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda})$ , hence all

the inner integrals evaluate to the measure of their domains of integration. now we compute the innermost integral by considering  $\underline{a}$ ,  $\underline{b}$  and  $\underline{\lambda}$  as constants, and integrating only over  $\underline{c}$ . Considering the multiplication  $uh$ , we observe that for each element  $c_j$ , we must have that  $\rho_j = c_j \lambda_1 \lambda_2 \lambda_3 \lambda_4 \in \mathbb{Z}_p$ , which means that  $v(\rho_i) = v(c_i \lambda_1 \lambda_2 \lambda_3 \lambda_4) \geq 0 \Rightarrow v(c_i) + v_1 + v_2 + v_3 + v_4 \geq 0 \Rightarrow v(c_i) \geq -(v_1 + v_2 + v_3 + v_4)$ . But this means that  $c_i \in p^{-(v_1+v_2+v_3+v_4)} \mathbb{Z}_p$ , and since the domain of integration for this integral is  $\underline{c} = \{c_1, c_2, c_3, c_4\}$ , then  $\mu(\underline{c}) = |c_j|_p^4 = p^{4(v_1+v_2+v_3+v_4)}$ . Denote  $I_2 := I_1 p^{4(v_1+v_2+v_3+v_4)}$ , we now have

that  $\zeta_{L_{5,p}}^\wedge(s) = \int_{\underline{\lambda}} I_2 \int_{\underline{a}} \int_{\underline{b}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda})$ .

Denote  $\lambda_{13} := \lambda_1 \lambda_2 \lambda_3$ ,  $\lambda_{24} := \lambda_2 \lambda_3 \lambda_4$ , and  $\lambda_{14} := \lambda_1 \lambda_2 \lambda_3 \lambda_4$ . We now consider the constraints on  $\underline{b}$ .

$b_{11} \lambda_{13}, b_{31} \lambda_{13}, b_{41} \lambda_{13} \in \mathbb{Z}_p$ , and  $b_{12} \lambda_{24}, b_{22} \lambda_{24} \in \mathbb{Z}_p$ . These constraints are obtained by multiplying elements in block  $M_{13}$  with elements in  $h$ , but one

observes that we have  $b_{22}$  also in location  $(5, 10)$  of the matrix, and  $b_{31}$  in location  $(7, 10)$ , which means that  $b_{22}\lambda_{14}, b_{31}\lambda_{14} \in \mathbb{Z}_p$ . But since we already have  $b_{22}\lambda_{24}, b_{31}\lambda_{13} \in \mathbb{Z}_p$ , the constraints  $b_{22}\lambda_{14}$  and  $b_{31}\lambda_{14}$  do not contribute any new information. In addition, we have one of the elements of  $\underline{b}$  that forms a constraint together with elements from  $\underline{a}$ , namely  $(a_{11}a_{22} - b_{11})\lambda_{24} \in \mathbb{Z}_p$ . The constraints  $b_{31}\lambda_{13}, b_{41}\lambda_{13}, b_{12}\lambda_{24}, b_{22}\lambda_{24} \in \mathbb{Z}_p$  from above translate to  $p^{-2(v_1+v_2+v_3)}p^{-2(v_2+v_3+v_4)} = p^{-2(v_1+2v_2+2v_3+v_4)}$ . On the other hand,  $b_{11}$  is a part of two constraints, hence we must have both  $b_{11} \in p^{-(v_1+v_2+v_3)}\mathbb{Z}_p$  and  $a_{11}a_{22} - b_{11} \in p^{-(v_2+v_3+v_4)}\mathbb{Z}_p \Rightarrow b_{11} \in a_{11}a_{22} + p^{-(v_2+v_3+v_4)}\mathbb{Z}_p$ , which means that we need to compute the measure  $\mu(A)$ , where  $A = p^{-(v_1+v_2+v_3)}\mathbb{Z}_p \cap a_{11}a_{22} + p^{-(v_2+v_3+v_4)}\mathbb{Z}_p$ . Denote  $\alpha := v_1 + v_2 + v_3$ ,  $\beta := v_2 + v_3 + v_4$  and  $x := a_{11}a_{22}$ , and we need to find a formula for a generic intersection of the form  $A = p^{-\alpha}\mathbb{Z}_p \cap x + p^{-\beta}\mathbb{Z}_p$ . We need to find a formula for this generic form. Since  $b_{11}$  is in the intersection, we have that  $b_{11} = z = x + y$  where  $y \in p^{-\beta}$  and  $z \in p^{-\alpha}\mathbb{Z}_p \Rightarrow z - x \in p^{-\beta}\mathbb{Z}_p$ . Assume  $\beta \geq \alpha \Rightarrow -\beta \leq -\alpha$ , and since  $v_p(b_{11}) = v_p(z - x) \geq \min\{v_p(z), v_p(x)\}$ , and  $v_p(z) \geq -\alpha \geq -\beta$ , then we have two cases. If  $v_p(x) \geq -\beta$ , then  $v_p(z - x) \geq \beta \Rightarrow z - x \in p^{-\beta}\mathbb{Z}_p$ . But  $-\alpha \geq -\beta \Rightarrow p^{-\alpha}\mathbb{Z}_p \subseteq p^{-\beta}\mathbb{Z}_p \Rightarrow A = p^{-\alpha}\mathbb{Z}_p$ . If  $v_p(x) < -\beta$ , then  $v_p(z - x) = v_p(x) < -\beta \Rightarrow z - x \notin p^{-\beta}\mathbb{Z}_p$ , which means that  $A = \emptyset$ . One checks that if we assume  $\alpha \geq \beta$ , then we obtain that  $A = p^{-\beta}\mathbb{Z}_p$  if  $v_p(x) \geq -\alpha$ , and  $A = \emptyset$  if  $v_p(x) < -\alpha$ . Therefore,  $\mu(A) = p^{\min\{\alpha, \beta\}}$  for every  $x$  such that  $v_p(x) \geq \min\{-\alpha, -\beta\} = -\max\{\alpha, \beta\}$ , which means, in our case, that  $v_p(x) = v_p(a_{11}a_{22}) \geq -\max\{v_1 + v_2 + v_3, v_2 + v_3 + v_4\} = -v_2 - v_3 - \max\{v_1, v_4\}$ . Thus, denoting  $I_3 := I_2 p^{-(v_2+v_3) - \max\{v_1, v_4\}}$ , we have that  $\zeta_{L_{5,p}}^\wedge(s) = \int_{\underline{\lambda}} I_3 \int_{\underline{a}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda})$ . Denote  $v_{ij} := v_p(a_{ij})$ . For the constraints on  $\underline{a}$ , we have

$$\begin{aligned}
& a_{11}\lambda_1\lambda_2, -a_{11}\lambda_2\lambda_3, -a_{11}\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_{11} \geq -(v_1 + v_2), v_{11} \geq -(v_2 + v_3) \Rightarrow \\
& v_{11} \geq -v_2 - \min\{v_1, v_3\}. \\
& a_{21}\lambda_1\lambda_2, a_{21}\lambda_1\lambda_2\lambda_3, a_{21}\lambda_1\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_{21} \geq -(v_1 + v_2). \\
& a_{22}\lambda_2\lambda_3, -a_{22}\lambda_3\lambda_4, a_{22}\lambda_1\lambda_2\lambda_3 \in \mathbb{Z}_p \Rightarrow \\
& \Rightarrow v_{22} \geq -(v_2 + v_3), v_{22} \geq -(v_3 + v_4), v_{22} \geq -(v_1 + v_2 + v_3) \Rightarrow v_{22} \geq -v_3 - \min\{v_2, v_4\}. \\
& a_{33}\lambda_3\lambda_4, a_{33}\lambda_2\lambda_3\lambda_4, a_{33}\lambda_1\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_{33} \geq -(v_3 + v_4). \\
& a_{21}a_{22}\lambda_1\lambda_2\lambda_3 \in \mathbb{Z}_p \Rightarrow v_{21} + v_{22} \geq -(v_1 + v_2 + v_3). \\
& -a_{11}a_{33}\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_{11} + v_{33} \geq -(v_2 + v_3 + v_4). \\
& a_{21}a_{33}\lambda_1\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_{21} + v_{33} \geq -(v_1 + v_2 + v_3 + v_4). \\
& \text{And we also have the constraint found earlier,} \\
& v_{11} + v_{22} \geq -(v_2 + v_3 + \max\{v_1, v_4\}). \\
& \text{We have three constraints on } a_{21}
\end{aligned}$$

1.  $v_{21} \geq -(v_1 + v_2)$
2.  $v_{21} \geq -(v_1 + v_2 + v_3 + v_{22})$
3.  $v_{21} \geq -(v_1 + v_2 + v_3 + v_4 + v_{33})$

But the third constraint does not add new information, because we already have the two separate constraints  $v_{21}, v_{33} \geq -(v_1 + v_2 + v_3 + v_4)$ .

The two valid constraints translate to

$$v_{21} \geq \min\{-(v_1 + v_2), -(v_1 + v_2 + v_3 + v_{22})\} = -(v_1 + v_2) - \{0, v_3 + v_{22}\}.$$

In the same way, we obtain the constraint  $v_{33} \geq -(v_3 + v_4) - \min\{0, v_2 + v_{11}\}$ .

Thus, we decompose the inner integral for  $\underline{a}$  into separate integrals, to obtain

$$\begin{aligned} \zeta_{L_{5,p}}^\wedge(s) &= \int_{\underline{\lambda}} I_3 \int_{\underline{a}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda}) = \\ &= \int_{\underline{\lambda}} I_3 \int_{a_{11}} \int_{a_{22}} \int_{a_{33}} \int_{a_{21}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda}). \end{aligned}$$

Hence, we have the measures  $\mu(a_{21}) = p^{v_1+v_2+\min\{0, v_3+v_{22}\}}$  and  $\mu(a_{33}) = p^{v_3+v_4+\min\{0, v_2+v_{11}\}}$ . Denote  $I_4 := I_3 p^{v_1+v_2} p^{v_3+v_4}$ . We have

$$\begin{aligned} \zeta_{L_{5,p}}^\wedge(s) &= \int_{\underline{\lambda}} I_3 \int_{\underline{a}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda}) = \\ &= \int_{\underline{\lambda}} I_3 \int_{a_{11}} \int_{a_{22}} \int_{a_{33}} \int_{a_{21}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda}) = \\ &= \int_{\underline{\lambda}} I_4 \int_{a_{11}} p^{\min\{0, v_2+v_{11}\}} \int_{a_{22}} p^{\min\{0, v_3+v_{22}\}} d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda}). \end{aligned}$$

By the constraints we found earlier on  $a_{22}$ , we have the following.

1.  $v_{22} \geq -v_3 - \min\{v_2, v_4\}$
2.  $v_{22} \geq -(v_2 + v_3) - \max\{v_1, v_4\} - v_{11}$

which translates into  $v_{22} \geq -v_3 - \min\{\min\{v_2, v_4\}, v_2 + \max\{v_1, v_4\} + v_{11}\} = -v_3 - \min\{v_2, v_4, v_2 + \max\{v_1, v_4\} + v_{11}\}$ .

Denote  $\alpha := v_2 + \max\{v_1, v_4\} + v_{11}$  and  $\beta := \min\{v_2, v_4, \alpha\}$ . We already have the constraint  $v_{11} \geq -(v_2 + \min\{v_1, v_3\})$ , which means that, in either case,

$$\begin{aligned} v_{11} &\geq -(v_1 + v_2) \geq -\min\{v_1, v_4\} - v_2 \\ &\Rightarrow \alpha = v_2 + \max\{v_1, v_4\} + v_{11} \geq \max\{v_1, v_4\} - \min\{v_1, v_4\} \geq 0 \\ &\Rightarrow \beta = \min\{v_2, v_4, \alpha\} \geq 0 \Rightarrow v_3 + \alpha > 0 \Rightarrow v_{22} \geq -(v_3 + \beta). \end{aligned}$$

For the inner integral  $\int_{a_{22}} p^{\min\{0, v_3+v_{22}\}} d\mu(a_{22})$ , we have two cases. If  $v_3 +$

$$v_{22} \geq 0, \text{ then } \min\{0, v_3+v_{22}\} = 0 \Rightarrow \int_{a_{22}} p^{\min\{0, v_3+v_{22}\}} d\mu(a_{22}) = \int_{v_{22} \geq -v_3} 1 d\mu(a_{22}) = p^{v_3}.$$

$$\text{If } v_3 + v_{22} < 0, \text{ then } \int_{a_{22}} p^{\min\{0, v_3+v_{22}\}} d\mu(a_{22}) = \int_{v_{22} < -v_3} p^{v_3+v_{22}} d\mu(a_{22}).$$

But we saw earlier that  $v_{22} \geq -(v_3 + \beta)$ , hence  $-v_3 - \beta \leq v_{22} \leq -v_3 - 1 \Rightarrow -\beta \leq v_3 + v_{22} \leq -1$ , which means that we can compute the integral over  $a_{22}$  as

$$\text{a sum of } \beta \text{ integrals, } \int_{-v_3-\beta \leq v_{22} \leq -v_3-1} p^{v_3+v_{22}} d\mu(a_{22}) = \sum_{\tau=1}^{\beta} \int_{v_{22}=-v_3-\tau} p^{-\tau} d\mu(a_{22}).$$

To evaluate each integral in the sum, we need to calculate the measure of its domain, namely  $\mu(\{v_{22} = -(v_3 + \tau)\}) = \mu(\{v_{22} \geq -(v_3 + \tau + 1)\} \setminus \{v_{22} \geq -(v_3 + \tau)\}) = \mu(p^{-(v_3+\tau+1)} \mathbb{Z}_p \setminus p^{-(v_3+\tau)} \mathbb{Z}_p) = p^{v_3+\tau+1} - p^{v_3+\tau} = p^{v_3+\tau}(p-1)$ , which means that each integral evaluates as  $p^{v_3+\tau}(p-1)p^\tau = p^{v_3}(p-1)$ , and the sum is over

$$\beta \text{ such integrals, so we have that } \int_{a_{22}} p^{\min\{0, v_3+v_{22}\}} d\mu(a_{22}) = p^{v_3} + \beta p^{v_3}(p-1),$$

where  $\beta$  depends also on  $v_{11}$ .

Hence, we need to compute the integral

$$\begin{aligned}
& \int_{a_{11}} p^{\min\{0, v_2 + v_{11}\}} p^{v_3} [1 + \beta(p-1)] d\mu(a_{11}). \text{ We denote } I_5 := I_4 p^{v_3}, \text{ so } \zeta_{L_{5,p}}^\wedge(s) = \\
& = \int_{\underline{\lambda}} I_5 \int_{a_{11}} p^{\min\{0, v_2 + v_{11}\}} [1 + \beta(p-1)] d\mu(a_{11}). \text{ But similar to what we} \\
& \text{saw earlier, } p^{\min\{0, v_2 + v_{11}\}} \text{ has two cases. If } v_{11} \geq -v_2, \text{ then } p^{\min\{0, v_2 + v_{11}\}} = \\
& p^0. \text{ If } v_{11} < -v_2, \text{ then } p^{\min\{0, v_2 + v_{11}\}} = p^{v_2 + v_{11}}. \text{ We saw earlier that } v_{11} \geq \\
& -(v_2 + v_3) \Rightarrow v_{11} + v_2 \geq -v_3, \text{ so for this case, we have that } -v_3 \leq v_{11} + v_2 \leq \\
& 0, \text{ which means that } \int_{a_{11}} p^{\min\{0, v_2 + v_{11}\}} [1 + \beta(p-1)] d\mu(a_{11}) = \int_{v_{11} \leq -v_2} 1 + \\
& \beta(p-1) d\mu(a_{11}) + \int_{v_{11} + v_2 \geq -v_3} p^{v_2 + v_{11}} [1 + \beta(p-1)] d\mu(a_{11}) = \int_{v_{11} \leq -v_2} 1 + \beta(p- \\
& 1) d\mu(a_{11}) + \sum_{\tau=1}^{v_3} \int_{v_{11} \geq -(v_3 + v_2)} p^{-\tau} [1 + \beta(p-1)] d\mu(a_{11}). \text{ Now we need to resolve} \\
& \beta = \min\{v_2, v_4, v_2 + \max\{v_1, v_4\} + v_{11}\}, \text{ hence we need to divide the inner} \\
& \text{integral to different orderings of } v_1, v_2, v_3, v_4. \\
& \text{Case 1: } v_1 \geq v_2 \geq v_3 \geq v_4. \text{ For this case, we have that } \beta = \min\{v_2, v_4, v_2 + \\
& \max\{v_1, v_4\} + v_{11}\} = \min\{v_4, v_1 + v_2 + v_{11}\}. \\
& \text{The two possible minimum values are equal when } v_4 = v_1 + v_2 + v_{11}, \text{ that} \\
& \text{is, when } v_2 + v_{11} = -(v_1 - v_4). \text{ But for this case we have two subcases. If} \\
& v_1 - v_4 \leq v_3, \text{ then, since } v_{11} \geq -(v_2 + v_3), \text{ we have that } v_{11} + v_2 \geq -v_3 \geq \\
& -(v_1 - v_4) \Rightarrow v_1 + v_2 + v_{11} \geq v_4 \Rightarrow \beta = v_4, \text{ hence} \\
& \int_{\underline{\lambda}} I_5 \int_{a_{11}} p^{\min\{0, v_2 + v_{11}\}} (1 + v_4(1 - p^{-1})) d\mu(a_{11}) = \\
& = \int_{\underline{\lambda}} I_5 (1 + v_4(1 - p^{-1})) \int_{v_{11} \geq -(v_2 + v_3)} p^{\min\{0, v_2 + v_{11}\}} d\mu(a_{11}), \text{ but same as ear-} \\
& \text{lier } \int_{v_{11} \geq -(v_2 + v_3)} p^{\min\{0, v_2 + v_{11}\}} d\mu(a_{11}) = p^0 \mu(\{v_{11} + v_2 \geq 0\}) + \int_{v_2 + v_{11} < 0} p^{v_2 + v_{11}} d\mu(a_{11}) = \\
& 1 \mu(\{v_{11} \geq -v_2\}) + \int_{v_2 + v_{11} < 0} p^{v_2 + v_{11}} d\mu(a_{11}) = p^{v_2} + \int_{-v_3 \leq v_{11} < -v_2} p^{v_2 + v_{11}} d\mu(a_{11}) = \\
& p^{v_2} + \sum_{\tau=-v_3}^{-(v_2+1)} \int_{v_{11}=\tau} p^{v_2+\tau} d\mu(a_{11}) = \\
& = p^{v_2} + \sum_{\tau=1}^{v_3} \int_{v_{11}=-(v_2+\tau)} p^{\tau} d\mu(a_{11}) = p^{v_2} + \sum_{\tau=1}^{v_3} p^{-\tau} (p^{v_2+\tau} - p^{v_2+\tau-1}) = \\
& p^{v_2} + \sum_{\tau=1}^{v_3} p^{v_2} (1 - p^{-1}) = p^2 + v_3 = p^{v_2} (1 + v_3(1 - p^{-1})). \text{ Denote } I_6 := \\
& I_5 p^{v_2} (1 + v_3(1 - p^{-1})), \text{ thus we have } \zeta_{L_{5,p}}^\wedge(s) = \int_{\underline{\lambda}} I_6 d\mu(\lambda). \text{ We compute the com-} \\
& \text{plete expression } I_6 = p^{4(v_1 + v_2 + v_3 + v_4)} p^{-(v_2 + v_3) - \max\{v_1, v_4\}} p^{v_1 + v_2} p^{v_3 + v_4} p^{v_3} p^{v_2} (1 + \\
& v_3(1 - p^{-1})) p^{-(4v_1 + 6v_2 + 6v_3 + 4v_4)s} = p^{7v_1 + 11v_2 + 11v_3 + 8v_4} (1 + v_3(1 - p^{-1})) (1 + v_4(1 + \\
& p^{-1})) p^{-(4v_1 + 6v_2 + 6v_3 + 4v_4)s} = p^{(7-4s)v_1} p^{(11-6s)v_2} p^{(11-6s)v_3} p^{(8-4s)v_4} (1 + v_3(1 - p^{-1})) (1 + \\
& v_4(1 - p^{-1})) \text{ and integrate it over } \underline{\lambda}, \text{ which translates to the infinite sum}
\end{aligned}$$

$$S := \sum_{v_1 \geq v_2 \geq v_3 \geq v_4} p^{(7-4s)v_1} p^{(11-6s)v_2} p^{(11-6s)v_3} p^{(8-4s)v_4} (1+v_3(1-p^{-1}))(1+v_4(1-p^{-1})).$$

We notice that we have no constraint which dictates an order relation between  $v_1$  and  $v_2$ , thus we have the following constraints

$v_2 \geq v_3 \geq v_4$  and  $v_1 \geq v_3 + v_4 \geq v_3 \geq v_4$ . But this allows us to break the computed sum into separate sums, where the index of summation must preserve the constraints between  $v_1, v_2, v_3, v_4$ .

Denote  $u_4 := v_4$ ,  $u_3 := v_3 - v_4$ ,  $u_2 := v_2 - v_3$  and  $u_1 := v_1 - (v_3 + v_4)$ . With these notations, we have

$$S = \sum_{u_1=0}^{\infty} \sum_{u_2=0}^{\infty} \sum_{u_3=0}^{\infty} \sum_{u_4=0}^{\infty} p^{(7-4s)(u_1+u_3+2u_4)} p^{(11-6s)(u_2+u_3+u_4)} p^{(11-6s)(u_3+u_4)} p^{(8-4s)u_4} (1+(u_3+u_4)(1-p^{-1}))(1+u_4(1-p^{-1})) = \sum_{u_1=0}^{\infty} p^{(7-4s)u_1} \sum_{u_2=0}^{\infty} p^{(11-6s)u_2} \sum_{u_3=0}^{\infty} p^{(29-16s)u_3} \sum_{u_4=0}^{\infty} p^{(44-24s)u_4} (1+(u_3+2u_4)(1-p^{-1}) + (u_3u_4 + u_4^2(1-p^{-1}))).$$

Denote  $w_1 := p^{(7-4s)}$ ,  $w_2 := p^{(11-6s)}$ ,  $w_3 := p^{(29-16s)}$  and  $w_4 := p^{(44-24s)}$ . We shall compute each summand separately.

$$S_1 := \sum_{u_1}^{\infty} w_1^{u_1} \sum_{u_2}^{\infty} w_2^{u_2} \sum_{u_3}^{\infty} w_3^{u_3} \sum_{u_4}^{\infty} w_4^{u_4} = \frac{1}{1-w_1} \frac{1}{1-w_2} \frac{1}{1-w_3} \frac{1}{1-w_4}.$$

$$S_2 := \sum_{u_1}^{\infty} w_1^{u_1} \sum_{u_2}^{\infty} w_2^{u_2} \sum_{u_3}^{\infty} w_3^{u_3} \sum_{u_4}^{\infty} w_4^{u_4} (u_3 + 2u_4)(1-p^{-1}) = (1-p^{-1}) \sum_{u_1}^{\infty} w_1^{u_1} \sum_{u_2}^{\infty} w_2^{u_2} \sum_{u_3}^{\infty} w_3^{u_3} \sum_{u_4}^{\infty} w_4^{u_4} (u_3 + 2u_4) = (1-p^{-1}) \sum_{u_1}^{\infty} w_1^{u_1} \sum_{u_2}^{\infty} w_2^{u_2} \sum_{u_3}^{\infty} w_3^{u_3} \sum_{u_4}^{\infty} w_4^{u_4} (u_3 + 2u_4) = (1-p^{-1}) \left[ \sum_{u_1}^{\infty} w_1^{u_1} \sum_{u_2}^{\infty} w_2^{u_2} \sum_{u_3}^{\infty} u_3 w_3^{u_3} \sum_{u_4}^{\infty} w_4^{u_4} + \sum_{u_1}^{\infty} w_1^{u_1} \sum_{u_2}^{\infty} w_2^{u_2} \sum_{u_3}^{\infty} w_3^{u_3} \sum_{u_4}^{\infty} 2u_4 w_4^{u_4} \right] = (1-p^{-1}) \left[ \frac{1}{1-w_1} \frac{1}{1-w_2} \frac{w_3}{(1-w_3)^2} \frac{1}{1-w_4} + 2 \frac{1}{1-w_1} \frac{1}{1-w_2} \frac{1}{1-w_3} \frac{w_4}{(1-w_4)^2} \right].$$

$$S_3 := \sum_{u_1}^{\infty} w_1^{u_1} \sum_{u_2}^{\infty} w_2^{u_2} \sum_{u_3}^{\infty} w_3^{u_3} \sum_{u_4}^{\infty} w_4^{u_4} u_4 (1-p^{-1}) = (1-p^{-1}) \sum_{u_1}^{\infty} w_1^{u_1} \sum_{u_2}^{\infty} w_2^{u_2} \sum_{u_3}^{\infty} u_3 w_3^{u_3} \sum_{u_4}^{\infty} w_4^{u_4} u_4 = \sum_{u_1}^{\infty} w_1^{u_1} \sum_{u_2}^{\infty} w_2^{u_2} \sum_{u_3}^{\infty} w_3^{u_3} \sum_{u_4}^{\infty} \frac{1}{1-w_1} \frac{1}{1-w_2} \frac{1}{1-w_3} \frac{w_4}{(1-w_4)^2}.$$

The second sub case is where  $\beta = \min\{v_4, \alpha\} = \alpha < v_4$ , where  $\alpha = v_1 + v_2 + v_{11}$ . But  $v_{11} \geq -(v_2 + v_3) \Rightarrow v_1 + v_2 + v_{11} \geq v_1 - v_3$ , which means that  $v_4 > v_1 - v_3$  is a necessary condition for this sub case. We use a strong inequality here, because we have already counted the case where  $v_4 = v_1 + v_2 + v_{11}$ .

Thus,  $v_1 - v_3 \leq \alpha < v_4$ , which means that for this case, the value of  $\beta$  is not constant, but rather  $\beta \in \{v_1 - v_3, v_1 - v_3 + 1, v_1 - v_3 + 2, \dots, v_4 - 2, v_4 - 1\}$ .

Hence, the lower bound for  $v_{11}$  remains  $-(v_2 + v_3)$ , but our upper bound comes from either  $\min\{0, v_2 + v_{11}\}$  or  $\alpha$ . We have

$$\min\{0, v_2 + v_{11}\} \leq 0 \Rightarrow v_2 + v_{11} \leq 0 \Rightarrow v_{11} \leq -v_2$$

and

$$\alpha < v_4 \Rightarrow v_1 + v_2 + v_{11} \leq v_4 - 1 \Rightarrow v_{11} \leq v_4 - (v_1 + v_2 + 1)$$

But  $v_4 \leq v_1 \Rightarrow v_4 - v_1 \leq 0$ , and hence  $v_4 - v_1 - v_2 - 1 \leq -v_2 - 1$ , which means that  $-(v_2 + v_3) \leq v_{11} \leq v_4 - (v_1 + v_2 + 1)$ .

Therefore, we have

$$\begin{aligned}
S_{11} &= \int_{a_{11}} p^{\min\{0, v_2 + v_{11}\}} [1 + \beta(1 - p^{-1})] d\mu(a_{11}) = \int_{a_{11}} p^{\min\{0, v_2 + v_{11}\}} [1 + \alpha(1 - p^{-1})] d\mu(a_{11}) = \\
&= \sum_{v_{11} = -(v_2 + v_3)}^{v_4 - (v_1 + v_2 + 1)} p^{\min\{0, v_2 + v_{11}\}} [1 + (v_1 + v_2 + v_{11})(1 - p^{-1})] d\mu(a_{11})
\end{aligned}$$

We notice that for this case we do not have  $\min\{0, v_2 + v_{11}\} = 0$ , because  $\max v_{11} = v_4 - (v_1 + v_2 + 1)$ , hence if  $v_2 + v_{11} = 0$ , then  $v_{11} = -v_2 = v_4 - v_1 - v_2 - 1 \Rightarrow v_4 - v_1 - 1 = 0 \Rightarrow v_4 - v_1 = 1$ , which contradicts  $v_1 \geq v_4$ . Therefore,

$$\begin{aligned}
S_{11} &= \sum_{v_{11} = -(v_2 + v_3)}^{v_4 - (v_1 + v_2 + 1)} p^{v_2 + v_{11}} [1 + (v_1 + v_2 + v_{11})(1 - p^{-1})] d\mu(a_{11}) = \\
&= p^{-v_3} [1 + (v_1 - v_3)(1 - p^{-1})] \mu(p^{-(v_2 + v_3)} \mathbb{Z}_p \setminus p^{-(v_2 + v_3 - 1)} \mathbb{Z}_p) + \\
&+ p^{-v_3 + 1} [1 + (v_1 - v_3 + 1)(1 - p^{-1})] \mu(p^{-(v_2 + v_3 - 1)} \mathbb{Z}_p \setminus p^{-(v_2 + v_3 - 2)} \mathbb{Z}_p) + \\
&\quad \vdots \\
&+ p^{v_4 - v_1 - 1} [1 + (v_4 - 1)(1 - p^{-1})] \mu(p^{v_1 + v_2 + 1 - v_4} \mathbb{Z}_p \setminus p^{v_1 + v_2 + 2 - v_4} \mathbb{Z}_p) = \\
&= p^{-v_3} [1 + (v_1 - v_3)(1 - p^{-1})] (p^{v_2 + v_3} - p^{v_2 + v_3 - 1}) + \\
&+ p^{-v_3 + 1} [1 + (v_1 - v_3 + 1)(1 - p^{-1})] (p^{v_2 + v_3 - 1} - p^{v_2 + v_3 - 2}) + \\
&\quad \vdots \\
&+ p^{v_4 - v_1 - 1} [1 + (v_4 - 1)(1 - p^{-1})] (p^{v_4 - (v_1 + v_2 + 1)} - p^{v_4 - (v_1 + v_2 + 2)})
\end{aligned}$$