

# The pro-isomorphic zeta-functions of some nilpotent Lie algebras over integer rings

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## Abstract

Let  $G$  be any group. For any natural number  $n \in \mathbb{N}$ , let  $a_n(G)$  be the number of subgroups  $H \leq G$ , such that  $[G : H] = n$ . Assume  $G$  is finitely-generated, then  $a_n(G) < \infty$ , and we can define a Dirichlet series of the form  $\zeta_G(s) := \sum_{n=1}^{\infty} a_n(G)n^{-s}$ , where  $s \in \mathbb{C}$ . Assume, in addition, that  $G$  is also nilpotent and torsion-free, then this function has some properties of the Riemann zeta-function  $\zeta$ , such as the Euler decomposition of  $\zeta$  into a product of local factors indexed by primes. A version of this zeta-function counts pro-isomorphic subgroups, and an analogous function may be defined for appropriate Lie rings. We study here the pro-isomorphic zeta-functions for a family of nilpotent Lie rings of unbounded nilpotency class. We shall compute the automorphism groups of these Lie rings explicitly, prove uniformity of the local factors of the pro-isomorphic zeta-functions, and aim to determine them explicitly.

# 1 Scientific Background

## 1.1 Introduction

Although we will work with Lie algebras, for motivation we first present analogous and more natural questions in the context of groups.

**Proposition 1.1.1.** *Let  $G$  be any finitely generated group, and let  $n \in \mathbb{N}$  be any natural number. Then there is a finite number of subgroups  $H \leq G$ , such that  $[G : H] = n$*

This proposition gives rise to an entire branch of group theory, called **subgroup growth**. We denote by  $a_n(G) = a_n^{\leq}(G) := |\{H \leq G : [G : H] = n\}|$  the number of subgroups of  $G$  of index  $n$ , and look at the sequence  $\{a_n(G)\}_{n=1}^{\infty}$ . The subject of subgroup growth aims to relate the properties of this sequence to the algebraic structure of  $G$ . We denote by  $a_n^{\trianglelefteq}(G) := |\{H \trianglelefteq G : [G : H] = n\}|$  the number of **normal** subgroups of  $G$ . Now we define another type of subgroups of  $G$ .

**Definition 1.1.2.** *Let  $G$  be any group, and let  $\mathcal{N}(G) := \{N_k \trianglelefteq G\}_{k \in I}$  be the set of all normal subgroups of  $G$ . We define a partial order on  $\mathcal{N}(G)$  by reverse inclusion, that is, for every two indices  $i, j$  we say that  $i \leq j$  if and*

only if  $N_j \subseteq N_i$ , hence for every  $i \leq j$  there exists a natural projection map  $\pi_{ji} : G/N_j \rightarrow G/N_i$ . The inverse limit

$$\widehat{G} = \varprojlim \{G/N_k\}_{k \in I} := \{(h_k)_{k \in I} \in \prod_{k \in I} G/N_k : \pi_{ji}(h_j) = h_i, \forall i \leq j\}$$

is called the **profinite closure** of  $G$ .

**Definition 1.1.3.** Let  $G$  be any group. A subgroup  $H \leq G$  is called **pro-isomorphic** if  $\widehat{H} \cong \widehat{G}$ .

We denote by  $\hat{a}_n(G) := |\{H \leq G : \widehat{H} \cong \widehat{G}, [G : H] = n\}|$  the number of **pro-isomorphic** subgroups of  $G$ .

**Definition 1.1.4.** Let  $G$  be a finitely-generated group, and let  $* \in \{\leq, \trianglelefteq, \wedge\}$ , then we define zeta-functions of the form  $\zeta_G^*(s) := \sum_{n=1}^{\infty} \hat{a}_n(G) n^{-s}$ .

**Proposition 1.1.5.** Let  $G$  be a  $\mathcal{T}$ -group, i.e. finitely-generated, nilpotent and torsion-free group, and let  $* \in \{\leq, \trianglelefteq, \wedge\}$ , then the zeta-functions for  $G$  have the following attributes:

**Polynomial growth and convergence.**  $\hat{a}_n(G) \leq Cn^b$ , for some  $b, C$  constant, thus, for all  $*$ , we get that  $\zeta_G^*(s)$  converges on some right half-plane  $\operatorname{Re}(s) > \alpha$ , for  $\alpha$  constant. The abscissa of convergence is  $\alpha^* := \inf\{\alpha : \zeta_G^*(s) < \infty, \operatorname{Re}(s) > \alpha\}$ , and we have that  $\alpha^* \in \mathbb{Q}$ .

**Euler decomposition.** For all  $*$ , we have that  $\zeta_G^*(s) = \prod_p \zeta_{G,p}^*(s)$ , where  $\zeta_{G,p}^*(s) = \sum_{k=0}^{\infty} \hat{a}_{p^k}^*(G) p^{-ks}$ .

**Rationality.** For all  $*$ , all  $p$ , there is a rational function  $W_p^* \in \mathbb{Q}(X)$ , such that  $\zeta_{G,p}^*(s) = W_p^*(p^{-s})$ .

**Functional equation.** Suppose we have **finite uniformity**, i.e. we have  $r$  rational functions  $W_1^*(X, Y), \dots, W_r^*(X, Y) \in \mathbb{Q}(X, Y)$ , such that for all  $p$ , there is some  $1 \leq i \leq r$  such that  $\zeta_{G,p}^*(s) = W_i^*(p, p^{-s})$ . We say  $W_i^*$  satisfies a **functional equation** if  $W_i^*(X^{-1}, Y^{-1}) = X^a Y^b W_i^*(X, Y)$ , where  $a, b \in \mathbb{N} \cup \{0\}$ ,  $X^a Y^b$  being called the **symmetry factor**. Thus, if  $G$  is a  $\mathcal{T}$ -group, then  $\zeta_{G,p}^{\leq}(s)$  satisfies a functional equation, for all but finitely many  $p$ , with the same symmetry factor. If  $G$  is a  $\mathcal{T}$ -group of nilpotency class 2, same is true for  $\zeta_{G,p}^{\trianglelefteq}(s)$ .

If any zeta-function, which is a special case of the Dirichlet series, has some properties of convergence on some subset of  $\mathbb{C}$ , one may reconstruct its

coefficients  $a_n^*(G)$ , the number of subgroups of our interest, using the **Peron's formula**, which is an implementation of a **reverse Mellin transform**, as discussed, for example, in [7], but this discussion is out of the scope of our research.

This research concentrates on the growth of **pro-isomorphic** subgroups defined above, hence we shall restrict our further discussion to the pro-isomorphic case only. For example, we look at the additive group of integers  $G = (\mathbb{Z}, +)$ , for which, every subgroup  $H \leq \mathbb{Z}$  is of the form  $H = n\mathbb{Z} = \langle n \rangle$ , for some  $n \in \mathbb{N}$ , which means that  $H \cong \mathbb{Z}$ , as both are infinite cyclic groups, and so,  $\hat{H} \cong \hat{\mathbb{Z}}$ . Since we have only one subgroup of index  $n$ , for every  $n \in \mathbb{N}$ , then  $a_n(\mathbb{Z}) = \hat{a}_n(\mathbb{Z}) = 1$ . Thus, its pro-isomorphic zeta-function is  $\hat{\zeta}_{\mathbb{Z}}(s) = \sum_{i=1}^{\infty} n^{-s} = \zeta(s)$ , the Riemann zeta-function, which is known to converge for  $\text{Re}(s) > 1$ . We recall that the Riemann zeta-function decomposes into an infinite product of local zeta-functions, that is,  $\zeta(s) = \prod_p \zeta_p(s) = \prod_p \sum_{k=0}^{\infty} p^{-ks} = \prod_p \frac{1}{1-p^{-s}}$ , where the product runs over all the prime numbers.

## 1.2 Linearization

We want to transfer the ideas from the above discussion about groups to a linear context, where we can use tools from linear algebra. Hence, for finitely-generated torsion-free nilpotent groups  $G$ , we associate nilpotent Lie algebras over  $\mathbb{Z}$ . This, in general, is called the **Mal'cev correspondence**. If  $L$  is a  $\mathbb{Z}$ -Lie algebra, namely a free  $\mathbb{Z}$ -module of finite rank with a Lie bracket operation, then consider the number  $\hat{a}_n(L)$  of subalgebras  $M \leq L$ , where  $n = [L : M]$ , such that  $M \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ , where  $\mathbb{Z}_p$  is the ring of  $p$ -adic integers, for all primes  $p$ , and it is also known that  $\hat{a}_n(L) < \infty$  for all  $n \in \mathbb{N}$ . The Dirichlet series  $\hat{\zeta}_L(s) := \sum_{n=1}^{\infty} \hat{a}_n(L)n^{-s}$ , is called the **pro-isomorphic zeta-function** of  $L$ . By the Mal'cev correspondence, to every finitely-generated, nilpotent, torsion-free group  $G$ , one may associate a Lie algebra  $L = L(G)$ , such that  $\hat{\zeta}_{G,p}(s) = \hat{\zeta}_{L,p}(s)$ , for all but finitely many primes  $p$ . If  $G$  has nilpotency class 2, one may obtain the equality for all primes. For this  $L$ , choose a basis  $B = \{b_1, \dots, b_r\}$ , where  $r = \text{rank} L$ . Let  $\mathcal{L}_p = L \otimes_{\mathbb{Z}} \mathbb{Q}_p$ , for any  $p$ . This is a  $\mathbb{Q}_p$ -Lie algebra, and our choice of basis allows us to identify the automorphism group  $G(\mathbb{Q}_p) = \text{Aut}_{\mathbb{Q}_p}(\mathcal{L}_p)$  with a subgroup of  $GL_r(\mathbb{Q}_p)$ . Note that  $\mathcal{L}_p$  contains a  $\mathbb{Z}_p$ -lattice,  $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ . If  $\varphi \in G(\mathbb{Q}_p)$ , then  $\varphi(L_p) = L_p$  if and only if  $\varphi \in G(\mathbb{Z}_p) = G(\mathbb{Q}_p) \cap GL_r(\mathbb{Z}_p)$ . Here  $GL_r(\mathbb{Z}_p)$  is the group of  $r \times r$  matrices which are invertible over  $\mathbb{Z}_p$ .

Similarly,  $\varphi(L_p) \subseteq L_p$  if and only if  $\varphi \in G^+(\mathbb{Q}_p) := G(\mathbb{Q}_p) \cap \mathcal{M}_r(\mathbb{Z}_p)$ , where  $\mathcal{M}_r(\mathbb{Z}_p)$  is the collection of  $r \times r$  matrices with entries in  $\mathbb{Z}_p$ . Note that  $G^+(\mathbb{Q}_p)$  is a monoid, not a group.

Denote by  $G(\mathbb{Z}_p)g$ , where  $g \in G^+(\mathbb{Q}_p)$ , a right-coset of  $G(\mathbb{Z}_p)$ . One checks that the monoid  $G^+(\mathbb{Q}_p)$  is a disjoint union of right-cosets of  $G(\mathbb{Z}_p)$ .

The discussion above reveals the construction we base our research upon. We observe that there is a bijection between the set  $G(\mathbb{Z}_p) \backslash G^+(\mathbb{Q}_p)$  of right-cosets of  $G(\mathbb{Z}_p)$  and the set  $\{M \leq L_p : M \cong L_p\}$  of  $L_p$ -subalgebras which are isomorphic to  $L_p$  itself. This bijection takes  $G(\mathbb{Z}_p)g$  to  $M = \varphi(L_p)$ . For any  $\varphi \in G(\mathbb{Z}_p)g$ , this is well-defined. One checks that for every  $\psi \in G(\mathbb{Z}_p)g$ , we have that  $\psi(L_p) = \varphi(L_p) = M$ . We end this part, as a preparation for the final part of this technical background review, with the following result, which states that for each right-coset  $G(\mathbb{Z}_p)g$ , if  $M = \varphi(L_p)$ , where  $\varphi \in G(\mathbb{Z}_p)g$ , then  $[L_p : M] = |\det \varphi|_p^{-1}$ , where  $|\det \varphi|_p$  is the  $p$ -adic norm of  $\det \varphi$ , and therefore,

$$\hat{\zeta}_{L,p}(s) = \sum_{\substack{M \leq L_p \\ M \cong L_p}} [L_p : M]^{-s} = \sum_{G(\mathbb{Z}_p)\varphi \in G(\mathbb{Z}_p) \backslash G^+(\mathbb{Q}_p)} |\det \varphi|_p^s.$$

**Theorem 1.2.1.** *Let  $*$   $\in \{\leq, \preceq, \wedge\}$ , then  $\zeta_{L,p}^*(s)$  is rational, i.e. there is a rational function in one variable  $W_p \in \mathbb{Q}(X)$  such that  $\zeta_{L,p}^*(s) = W_p(p^{-s})$ , for all  $p$  prime.*

After establishing rationality for the local zeta-functions in the Euler decomposition of  $\zeta_{L,p}^*(s)$ , one may study the uniformity or finite-uniformity of  $\zeta_L^*(s)$  itself, where  $\zeta_L^*$  is said to be **finitely-uniform** if the local zeta-functions in its Euler decomposition are represented by a finite set of  $r$  rational functions, for all but finitely many  $p$ , and **uniform** if  $r = 1$ . The uniformity of the zeta-functions of some algebras is established in the work of Grunewald, Segal and Smith, see [5]. We aim to show that our target  $\mathbb{Z}$ -Lie algebra is uniform, or at least finitely-uniform.

### 1.3 $p$ -adic Integration

**Definition 1.3.1.** *Let  $\Gamma$  be a locally compact topological group, i.e. for all  $\gamma \in \Gamma$ , there is an open neighborhood of  $\gamma \in U_\gamma$  and a compact subset  $K_\gamma$ , such that  $U_\gamma \subset K_\gamma$ . Then there is a measure  $\mu$ , with the following property: for any measurable subset  $U \subseteq \Gamma$  and any  $\gamma \in \Gamma$ ,  $\mu(U\gamma) = \mu(U)$ , where  $U\gamma := \{u\gamma : u \in U\}$ . Such a measure  $\mu$  is called a **right Haar measure**, and is unique up to multiplication by a non-zero constant.*

Following this definition of a right Haar measure, we claim that for every prime number  $p$ , the group  $G(\mathbb{Q}_p)$  is a locally compact topological group. We also claim that a right Haar measure  $\mu$  on  $G$  can be normalized such that  $\mu(G(\mathbb{Z}_p)) = 1$ , and the normalized measure of any right-coset of  $G(\mathbb{Z}_p)$  equals to the measure of  $G(\mathbb{Z}_p)$  itself, i.e. for every  $g \in G^+(\mathbb{Q}_p)$ , we have that  $\mu(G(\mathbb{Z}_p)g) = \mu(G(\mathbb{Z}_p)) = 1$ . Following this, we calculate the  $p$ -adic norm of the determinant of every  $L_p$ -automorphism, as a  $p$ -adic integral over our measure space. Given any  $L_p$ -automorphism in some right-coset  $\varphi \in G(\mathbb{Z}_p)\varphi$ , we have that  $|\det \varphi|_p^s = \int_{G(\mathbb{Z}_p)\varphi} |\det \varphi|_p^s d\mu$ , because  $\mu(G(\mathbb{Z}_p)\varphi) = 1$ , and  $|\det \varphi|_p^{-1}$  is fixed on  $G(\mathbb{Z}_p)\varphi$ .

To calculate our target function, we observe that

$$\begin{aligned} \hat{\zeta}_{L,p}(s) &= \sum_{G(\mathbb{Z}_p)\varphi \in G(\mathbb{Z}_p) \backslash G^+(\mathbb{Q}_p)} |\det \varphi|_p^s = \sum_{G(\mathbb{Z}_p)\varphi \in G(\mathbb{Z}_p) \backslash G^+(\mathbb{Q}_p)} \int_{G(\mathbb{Z}_p)\varphi} |\det \varphi|_p^s d\mu = \\ &= \int_{G^+(\mathbb{Q}_p)} |\det \varphi|_p^s d\mu. \end{aligned}$$

This calculation of the local  $\zeta_p$ -function as a  $p$ -adic integral was established by the work of du Sautoy and Lubotzky, in [3], and we aim to study this integral and its attributes, where the integrand and domain of integration come from our target  $\mathbb{Z}_p$ -Lie algebra.

## 2 Research Goals and Methodology

### 2.1 The Lie algebras $L_{n,p}$

Let  $e_{ij}$  be an  $n \times n$  matrix, in which all the elements are zero, except for the element in row  $i$  and column  $j$  which has 1. On the set  $\mathcal{E} = \{e_{ij} : 1 \leq i \leq n-1 \wedge i+1 \leq j \leq n\}$  we define a bracket operation: for every  $1 \leq k, l \leq n-1$ , define  $[e_{k,k+1}, e_{l,l+1}] := e_{k,k+1}e_{l,l+1} - e_{l,l+1}e_{k,k+1}$ . Let  $\mathcal{R}$  be some commutative ring, then the standard operation of  $\mathcal{R}$  on  $\mathcal{E}$ , along with the defined bracket operation, form a nilpotent  $\mathcal{R}$ -Lie algebra. Considering  $\mathcal{R} = \mathbb{Z}$ , we obtain a nilpotent  $\mathbb{Z}$ -Lie algebra of strictly upper triangular matrices over  $\mathbb{Z}$ , which we denote by  $L_n$ , with the standard bracket operation as its Lie brackets. As discussed above, this  $\mathbb{Z}$ -Lie algebra can be extended to a  $\mathbb{Z}_p$ -algebra, which we denote by  $L_{n,p}$ , and then to a  $\mathbb{Q}_p$ -algebra, which we denote by  $\mathcal{L}_{n,p}$ . In all the following, we may set  $n$  and  $p$ , and denote  $L := L_{n,p}$  and  $\mathcal{L} := \mathcal{L}_{n,p}$  for abbreviation. It is readily seen that the set of

matrices of the form  $e_{ij}$  where  $i < j$ , spans the whole  $\mathbb{Z}$ -Lie algebra  $L$  and is  $\mathbb{Z}$ -linearly independent. Therefore, it forms a basis for  $L$  as a free module over  $\mathbb{Z}$ ,  $\mathcal{B}(\mathbb{Z}) := \{e_{12}, e_{13}, \dots, e_{1n}, e_{23}, \dots, e_{2n}, \dots, e_{n-1,n}\}$ , which we call **the standard basis of  $L$** . One easily checks that  $r = \text{rank} L = |\mathcal{B}(\mathbb{Z})| = \binom{n}{2}$ , which is the number of elements above the main diagonal for every  $n \in \mathbb{N}$ . To this standard basis we apply a linear order by defining  $e_{ij} < e_{kl}$  if  $j - i < l - k$  or if  $j - i = l - k$  and  $i < k$ . In other words, we apply an order that divides  $\mathcal{B}(\mathbb{Z})$  to basis elements of the quotients  $L/\gamma_2, \gamma_2/\gamma_3, \dots, \gamma_{n-2}/\gamma_{n-1}, \gamma_{n-1}$

Obviously, the same goes also for the extensions of  $L$ , namely  $L_{n,p}$  and  $\mathcal{L}$ . The target of our research is studying the  $\hat{\zeta}_{L,p}$ -function on these  $\mathbb{Z}_p$ -Lie algebras, and the related constructions.

**Remark 2.1.1.** *Let  $\mathcal{R}$  be a commutative ring, and let  $\mathcal{U}_n(\mathcal{R})$  be the group of  $n \times n$  upper unitriangular matrices over  $\mathcal{R}$ , with the standard matrix multiplication as the group operation. Denote by  $\mathcal{U}_{n,p} = \mathcal{U}_n(\mathbb{Z}_p)$  the unitriangular matrix group over  $\mathbb{Z}_p$ , then by Mal'cev correspondence,  $\hat{\zeta}_{\mathcal{U}_{n,p}}(s) = \hat{\zeta}_{L_{n,p}}(s)$ , for all but finitely many primes  $p$ . This relates the subject of research to the motivation presented at the beginning of this paper.*

## 2.2 Research goals

The project consists of three major steps:

1. **Computing the automorphism group of the  $\mathbb{Q}_p$ -Lie algebras  $\mathcal{L}$ , for all  $n \in \mathbb{N}$  and all primes  $p$ .**
2. **Showing that the pro-isomorphic zeta-functions  $\hat{\zeta}_{L_{n,p}}(s)$  are uniform for all  $n \in \mathbb{N}$ .**
3. **Giving an explicit uniform formula for the zeta-functions  $\hat{\zeta}_{L_{n,p}}(s)$  for specific values of  $n$ , if not for all  $n \in \mathbb{N}$ .** Specifically for  $n = 5$  we aim to continue the work of Mark N. Berman, who proved that  $\hat{\zeta}_{L_{5,p}}(s)$  is uniform.

As we elaborate further, steps 1 and 2 are already known entirely for  $n \leq 5$ , and step 3 is known for  $n \leq 4$ . We start with the first step of calculating  $\text{Aut}_{\mathbb{Q}_p}(\mathcal{L})$ . These automorphism groups have been studied for decades from a different point of view. There are classical results showing that any automorphism may be expressed as a product of automorphisms of a specific type; see, for instance, the main result of Gibbs [4]. These results are not explicit enough for our purposes; indeed, the submonoid  $G^+(\mathbb{Q}_p)$  arises for us as the domain of integration of a  $p$ -adic integral. In order to calculate



this integral, we need to decompose the automorphism group  $G(\mathbb{Q}_p)$  into a repeated semi-direct product of groups with a simple structure.

After we have analyzed the structure of  $G(\mathbb{Z}_p)$ , we will need to construct the monoid  $G^+(\mathbb{Q}_p)$  and its  $G(\mathbb{Z}_p)$  right-cosets, as we have seen above. This will give us both the function to integrate, which is  $\det \varphi$  for every  $G(\mathbb{Z}_p)$  right-coset  $G(\mathbb{Z}_p)\varphi$ , and the domain of integration, which is the monoid  $G^+(\mathbb{Q}_p)$ . We will use this information to analyze the behavior of the  $p$ -adic integral we have described above and prove that its calculation depends only on  $p$ , thus showing that the  $\hat{\zeta}_{L,p}$ -function is uniform.

### 2.3 $L_{n,p}$ -Lie algebras for $n > 3$

Mark N. Berman, in his doctoral thesis [1], has displayed an explicit formula for  $\hat{\zeta}_{L_{4,p}}$ , and proved that  $\hat{\zeta}_{L_{5,p}}$  is indeed uniform. We aim to generalize his work to prove that  $\hat{\zeta}_{L_{n,p}}$  is uniform for all  $n$ . We also aim to compute  $\hat{\zeta}_{L_{n,p}}(s)$  explicitly for all  $n$ , or at least to obtain explicit formulas for some  $n \geq 5$ , and specifically for  $n = 5$ . By analyzing carefully Berman's work on  $L_{4,p}$  and  $L_{5,p}$ , we gain the basic understanding of the expected structure of the local zeta-functions in the general case. We begin our discussion of the first goal, which is computing  $G(\mathbb{Z}_p)$ , by first recalling that for every  $v \in L_{n,p}$ , where  $n \geq 3$ , we present  $\varphi(v)$  as the multiplication of  $v$  by a matrix from the right  $\varphi(v) = vM$ . As stated earlier,  $M$  is an  $r \times r$  matrix, where  $r = \text{rank} L_{n,p} = \binom{n}{2}$ , whose lines are set by the order we have defined above, i.e. considering the standard ordered basis

$$\mathcal{B} = \{e_{12}, e_{23}, \dots, e_{n-1,n}, e_{13}, \dots, e_{n-2,n}, \dots, e_{1n}\}$$

then  $M$  is the following matrix,

$$M = \begin{pmatrix} \varphi(e_{12}) \\ \varphi(e_{23}) \\ \varphi(e_{n-1,n}) \\ \hline \varphi(e_{13}) \\ \vdots \\ \varphi(e_{n-2,n}) \\ \hline \vdots \\ \varphi(e_{1n}) \end{pmatrix}$$

Given an  $\mathcal{L}$ -automorphism  $\varphi$ , we denote by  $\varphi_k : \gamma_k \mathcal{L} \rightarrow \gamma_k \mathcal{L}$  the operation of  $\varphi$  on all the  $n - k$  elements of the lower central series starting from  $k$ , that is, we consider only the images

$$\varphi(e_{1,1+k}), \varphi(e_{2,2+k}), \dots, \varphi(e_{n-k,n}), \varphi(e_{1,2+k}), \dots, \varphi(e_{n-k-1,n}), \dots, \varphi(e_{1n})$$

For every  $\varphi_k$ , we have the induced map denoted by  $\varphi_{kk}$ , from the quotient algebra  $\gamma_k / \gamma_{k+1}$  to itself, defined by  $\varphi_{kk}(e_{l,l+k} + \gamma_{k+1} \mathcal{L}) := a_{1,1+k} e_{1,1+k} + a_{2,2+k} e_{2,2+k} + \dots + a_{n-k,n} e_{n-k,n} + z_{k+1}$ , where  $z_{k+1} \in \gamma_{k+1} \mathcal{L}$ , for every  $1 \leq l \leq n - k$ . Clearly,  $\varphi_{kk}$  is well-defined, since  $\varphi_k(\gamma_k \mathcal{L}) = \gamma_k \mathcal{L}$ , for every  $1 \leq k \leq n - 1$ . Following this division of  $\mathcal{L}$  by the lower central series and its quotients, we view  $M$  as a block matrix,

$$M = \left( \begin{array}{c|c|c|c|c} M_{11} & M_{12} & \dots & M_{1,n-2} & M_{1,n-1} \\ \hline M_{21} & M_{22} & \dots & M_{2,n-2} & M_{2,n-1} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline M_{n-1,1} & M_{n-1,2} & \dots & M_{n-1,n-2} & M_{n-1,n-1} \end{array} \right)$$

each block is denoted by  $M_{kl} \in \mathcal{M}_{m \times r}(\mathbb{Q}_p)$ , where  $m = \dim \gamma_k / \gamma_{k+1}$  and  $r = \dim \gamma_l / \gamma_{l+1}$ . From this, we can understand that the blocks on the main diagonal of  $M$ , which are the induced quotient maps defined above, are square matrices  $\varphi_{kk} = M_{kk} \in \mathcal{M}_{n-k}(\mathbb{Q}_p)$ . A trivial observation is that since all the elements of the lower central series of  $\mathcal{L}$  are characteristic subalgebras, then  $\varphi(\gamma_k \mathcal{L}) / \gamma_k \mathcal{L} = 0$ , which means that all the matrix blocks  $M_{kl}$ , where  $k < l$ , must be zero, therefore  $M$  has the form,

$$M = \left( \begin{array}{c|c|c|c|c} M_{11} & M_{12} & M_{13} & \dots & M_{1,n-2} & M_{1,n-1} \\ \hline 0 & M_{22} & M_{23} & \dots & M_{2,n-2} & M_{2,n-1} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \dots & M_{2,n-2} & M_{2,n-1} \\ \hline 0 & 0 & 0 & \dots & 0 & M_{n-1,n-1} \end{array} \right)$$

## 2.4 Preliminary results

We have made progress towards step 1 of our research program, namely determining the automorphism groups  $G(\mathbb{Q}_p)$ . The first result in this direction appears already in the thesis of M. N. Berman[1, Prop. 3.6].

**Proposition 2.4.1.** *Let  $\varphi \in G(\mathbb{Q}_p)$  be a  $\mathcal{L}$ -automorphism, and  $M$  its representing matrix, divided into matrix blocks, as shown earlier. Then,  $M_{11} \in \mathcal{M}_{n-1}(\mathbb{Q}_p)$  is either diagonal or anti-diagonal.*

Define an involution  $\eta \in G(\mathbb{Q}_p)$  by  $\eta(e_{ij}) := (-1)^{j-i-1} e_{n+1-j, n+1-i}$ , for  $1 \leq i \leq n-1$  and  $i+1 \leq j \leq n$ . Replacing  $\varphi$  by  $\varphi \circ \eta$ , we assume without loss of generality that  $M_{11}$  is diagonal. We check, for any  $\lambda_1, \lambda_2, \dots, \lambda_{n-1} \in \mathbb{Q}_p^*$ , that the diagonal matrix

$$h = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_1 \lambda_2, \lambda_2 \lambda_3 \dots, \lambda_{n-2} \lambda_{n-1}, \dots, \lambda_1 \lambda_2 \dots \lambda_{n-2} \lambda_{n-1})$$

represents an automorphism of  $\mathcal{L}$ . Thus, multiplying  $\varphi$  from the right by a unique such automorphism, we may now assume that  $M$  has 1's on the diagonal. By this point we know  $H(\mathbb{Q}_p) := \{\text{diag}(\lambda_1, \lambda_2, \dots) : \lambda_1, \lambda_2, \dots \in \mathbb{Q}_p^*\}$  is the **reductive part** of  $\mathbb{Q}_p$ , while  $N(\mathbb{Q}_p) := \{\varphi \in G(\mathbb{Q}_p) \text{ with 1's in diagonal}\}$  is the **unipotent radical** of  $G(\mathbb{Q}_p)$ . Every  $g \in G(\mathbb{Q}_p)$  has a unique decomposition  $g = \mathbf{n}\mathbf{h}$ , with  $\mathbf{n} \in N(\mathbb{Q}_p)$ ,  $\mathbf{h} \in H(\mathbb{Q}_p)$ . We aim to determine the structure of the unipotent radical  $N(\mathbb{Q}_p)$  by decomposing it into a semidirect product of abelian subgroups. We can simplify the domain of integration, for the  $p$ -adic integral that we aim to calculate, at the price of replacing a single integral by multiple integrals. As we saw earlier, the calculation of  $\hat{\zeta}_L(s)$  requires computing  $G(\mathbb{Z}_p)$  and  $G^+(\mathbb{Q}_p)$  first. Assuming we have already computed  $G(\mathbb{Q}_p)$ , based on the strategy that we have presented above, we need to identify  $G(\mathbb{Z}_p)$  as a subgroup of  $G(\mathbb{Q}_p)$ , which is expected not to be difficult, and continue from there to identify the monoid  $G^+(\mathbb{Q}_p)$ , which is expected to be a substantial challenge. By applying **Fubini's theorem** for semidirect products of topological groups, we have that

$$\hat{\zeta}_{\mathcal{L}}(s) = \int_{G^+(\mathbb{Q}_p)} |\det \varphi|_p^s d\mu_{G(\mathbb{Z}_p)\varphi} = \int_{H^+(\mathbb{Q}_p)} \left( \int_{N_{\mathbf{h}}^+} |\det \mathbf{n}\mathbf{h}|_p^s d\mu_{N(\mathbb{Q}_p)} \right) d\mu_{H(\mathbb{Q}_p)}$$

where  $H^+(\mathbb{Q}_p)$  consists of all  $\mathbf{h} \in H(\mathbb{Q}_p)$  that appear in the decomposition  $\varphi = \mathbf{n}\mathbf{h}$  for some  $\varphi \in G^+(\mathbb{Q}_p)$ , and, for a given  $\mathbf{h} \in H^+(\mathbb{Q}_p)$ , we set  $N_{\mathbf{h}}^+(\mathbb{Q}_p) := \{\mathbf{n} \in N(\mathbb{Q}_p) : \mathbf{n}\mathbf{h} \in G^+(\mathbb{Q}_p)\}$ . The integrand of the inner integral is constant, so the integral amounts to computing the measure of the set  $N_{\mathbf{h}}^+(\mathbb{Q}_p)$ . The advantage that we gain by this decomposition is that it simplifies the calculation of the integral. The integral function  $|\det \mathbf{n}\mathbf{h}|_p^s = |\det \mathbf{h}|_p^s$  depends only on  $\mathbf{h}$ , so computing the inner integral amounts to finding the measure of  $N_{\mathbf{h}}^+(\mathbb{Q}_p)$ . For the unipotent matrix  $\mathbf{n}$ , the determinant is 1, for the diagonal matrix  $\mathbf{h}$ , we have the following proposition,

**Proposition 2.4.2.** *Let  $n = 2, 3, 4, \dots$ , and let  $A_n$  be the diagonal  $m \times m$  matrix where  $m = \binom{n}{2}$ , of the same form of  $h$  from above for  $n - 1$  scalars, then  $\det A_n = \prod_{i=1}^{n-1} \lambda_i^{i(n-i)}$ .*

*Proof.* We observe that the determinants, for  $n = 2, 3, 4, \dots$ , form a recursive sequence,

$$\begin{aligned} a_2 &= \det A_2 = \lambda_1 \\ a_3 &= \det A_3 = \det A_2 \lambda_1 \lambda_2^2 = \lambda_1 \lambda_1 \lambda_2^2 = \lambda_1^2 \lambda_2^2 \\ a_4 &= \det A_4 = \det A_3 \lambda_1 \lambda_2^2 \lambda_3^3 = \lambda_1 \lambda_1 \lambda_1 \lambda_2^2 \lambda_2^2 \lambda_3^3 = \lambda_1^3 \lambda_2^4 \lambda_3^3 \\ &\vdots \\ a_n &= \det A_n = \det A_{n-1} \lambda_1 \lambda_2^2 \lambda_3^3 \cdots \lambda_{n-1}^{n-1} \end{aligned}$$

Calculating the general element,  $a_n = \det A_n$ , we see that we have  $n - 1$  times  $\lambda_1$ ,  $n - 2$  times  $\lambda_2^2$ ,  $n - 3$  times  $\lambda_3^3$ , and so forth. In general, we have  $n - i$  times  $\lambda_i^i$ , for every  $1 \leq i \leq n - 1$ , which means that we have  $i(n - i)$  times  $\lambda_i$ , and in total,  $a_n = \det A_n = \prod_{i=1}^{n-1} \lambda_i^{i(n-i)}$ .  $\square$

We conclude this section with the observation that the sets  $N_h^+(\mathbf{Q}_p)$  arising in our computation are quite complicated. We will decompose  $N(\mathbf{Q}_p)$  into an iterated semidirect product of a large number of subgroups, each abelian with a simple structure. This will decompose the integral over  $N_h^+(\mathbf{Q}_p)$  into a multiple integral that can be computed explicitly, given a suitable combinatorial framework. Hence, we strive to decompose  $N_n(\mathbf{Q}_p)$  itself into a product of finitely many simpler subgroups,  $N_n(\mathbf{Q}_p) = N_n(\mathbf{Q}_p)_1 \rtimes_\phi N_n(\mathbf{Q}_p)_2 \rtimes_\phi \cdots \rtimes_\phi N_n(\mathbf{Q}_p)_{m_n}$ , where  $m_n$  is the number of subgroups in the decomposition of  $N_n(\mathbf{Q}_p)$ , for every  $n \in \mathbb{N}$ , which means that

$$\int_{G^+(\mathbf{Q}_p)} |\det \varphi|_p^s d\mu_{G(\mathbf{Q}_p)} = \int_{H_n^+(\mathbf{Q}_p)} \left( \int_{\mathcal{N}_{m_n}^+} \cdots \left( \int_{\mathcal{N}_3^+} \left( \int_{\mathcal{N}_2^+} \left( \int_{\mathcal{N}_1^+} |\det \varphi|_p^s d\mu_{\mathcal{N}_1} \right) d\mu_{\mathcal{N}_2} \right) d\mu_{\mathcal{N}_3} \right) \cdots d\mu_{\mathcal{N}_{m_n}} \right) d\mu_{H_n(\mathbf{Q}_p)},$$

where we denote  $\mathcal{N}_i := N_n(\mathbf{Q}_p)_i$  and  $\mathcal{N}_i^+ := N_n^+(\mathbf{Q}_p)_i$ , for every  $1 \leq i \leq m_n$ . One checks that every  $\mathcal{N}_i^+$  depends on  $h, n_1, n_2, \dots, n_{i-1}$ , if  $\varphi = n_{m_n} \cdots n_2 n_1 h$ , where  $h \in H_n^+(\mathbf{Q}_p)$  and  $n_k \in \mathcal{N}_k$ , for every  $1 \leq k \leq i - 1$ . All the subgroups in the decomposition of  $N_n(\mathbf{Q}_p)$  are obviously unipotent as well, which means that their determinants are also 1. This means that computing the inner integrals amounts to determining the measure of the sets  $\mathcal{N}_i^+$  in terms of  $h, n_1, n_2, \dots, n_{n-1}$ .

## 2.5 Base Extension

Let  $K$  be a number field of degree  $d = [K : \mathbb{Q}]$ , and let  $\mathcal{O}_K$  be its ring of integers. Let  $L$  be a  $\mathbb{Z}$ -Lie algebra of rank  $r$ . By base extension we can consider  $L \otimes_{\mathbb{Z}} \mathcal{O}_K$  as a  $\mathbb{Z}$ -Lie algebra of rank  $rd$ , and by extension of scalars we can consider also  $L_{K,p} = (L \otimes_{\mathbb{Z}} \mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Q}_p$  as a  $\mathbb{Q}_p$ -Lie algebra of the same rank. Berman-Glazer-Schein give a criterion in [2], under which the pro-isomorphic zeta-function of  $L_{K,p}$  can be calculated without a significant extra effort relative to that of  $L_p$  itself. Note that the criterion does not necessarily apply for all  $p$ . We shall research whether the criterion applies to the  $\mathbb{Q}_p$ -Lie algebras  $\mathcal{L}$  of our work. If so, then  $\hat{\zeta}_{\mathcal{L} \otimes \mathcal{O}_{K,p}}$  will be finitely uniform (see ??). In other words, for each of the finitely many decomposition types of a prime in  $\mathcal{O}_K$ , there is a rational function in two variables  $W \in \mathbb{Q}(X, Y)$  such that  $\hat{\zeta}_{\mathcal{L} \otimes \mathcal{O}_{K,p}}(s) = W(p, p^{-s})$  for all  $p$  of that decomposition type.

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