cases

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Denote $p^{\min} := p^{\min\{0,v_2+v_{11}\}}$
Denote $\rho := (1-p^{-1})$
Denote $f(p,t)$ the p and t product $p^{7v_1+10v_2+11v_3+7v_4+\min\{v_1,v_4\}}t^{4v_1+6v_2+6v_3+4v_4}$

1 Case 1

$v_3 > v_1 \geq v_4 > v_2 \geq 0$
$\Rightarrow v_{11} \geq -v_2 - v_1$
$\Rightarrow \alpha := \min\{v_2, v_2+v_{11}+v_1\} = \begin{cases} v_2, & v_{11} \geq -v_1 \\ v_2-1, v_2-2, \ldots, & v_{11} < -v_1 \end{cases}$
$\Rightarrow f(p,t) = p^{7v_1+10v_2+11v_3+7v_4+v_4}t^{4v_1+6v_2+6v_3+4v_4} = p^{7v_1+10v_2+11v_3+8v_4}t^{4v_1+6v_2+6v_3+4v_4}$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1+v_2\rho)$.
2. For $-v_1 \leq v_{11} \leq -v_2-1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{-v_1}^{-v_2-1} p^{v_2}\rho(1+v_2\rho) = p^{v_2}(v_1-v_2)\rho(1+v_2\rho) = p^{v_2}(v_1-v_2)\rho + p^{v_2}v_2(v_1-v_2)\rho^2$.
3. For $-v_2-v_1 \leq v_{11} \leq -v_1-1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_1}^{-v_1-1} p^{v_2}\rho(1+\alpha\rho) = \sum_{i=1}^{v_2} p^{v_2}\rho(1+(v_2-i)\rho) = p^{v_2}v_2\rho + p^{v_2}v_2^2\rho^2 - p^{v_2}\binom{v_2+1}{2}\rho^2$.

So we have $\sum_{v_{11}\geq -v_2-v_1} = p^{v_2}(1+v_2\rho+(v_1-v_2)\rho+v_2(v_1-v_2)\rho^2+v_2\rho+v_2^2\rho^2-\frac{(v_2+1)v_2}{2}\rho^2) = p^{v_2}(1+v_1\rho+v_2v_1\rho^2+v_2\rho-\frac{v_2^2}{2}\rho^2-\frac{v_2}{2}\rho^2)$. We compute the differences between v_1, v_2, v_3, v_4,

$v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_4 \overset{b}{>} v_2 \overset{a}{\geq} 0$

Thus, we substitute $v_1 \to a+b+c$, $v_2 \to a$, $v_3 \to a+b+c+d$, $v_4 \to a+b$, and compute series with the new variables as indices, where $a, c \geq 0$ and $b, d \geq 1$.

2 Case 2

$v_1 \geq v_3 \geq 0$
$v_1 \geq v_4 > v_2 \geq 0$
$\Rightarrow v_{11} \geq -v_2 - v_3$

$\Rightarrow \alpha := \min\{v_2, v_2+v_{11}+v_1\} = \begin{cases} v_2, & v_{11} \geq -v_1 \\ v_2-1, v_2-2, \ldots, & v_{11} < -v_1 \end{cases}$

$\Rightarrow f(p,t) = p^{7v_1+10v_2+11v_3+7v_4+v_4} t^{4v_1+6v_2+6v_3+4v_4} = p^{7v_1+10v_2+11v_3+8v_4} t^{4v_1+6v_2+6v_3+4v_4}$

We have two sub cases.

2.1 Sub case 2.1

$-v_2 - v_3 \geq -v_1 \Rightarrow v_1 \geq v_2 + v_3$

Here there is no phase transition, because the phase transition occurs at $-v_1 - 1 < -v_2 - v_3 \leq v_{11}$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1+v_2\rho)$.
2. For $-v_2-v_3 \leq v_{11} \leq -v_2-1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_3}^{-v_2-1} p^{v_2}\rho(1+v_2\rho) = p^{v_2}v_3\rho(1+v_2\rho) = p^{v_2}v_3\rho + p^{v_2}v_3v_2\rho^2$.

So we have $\sum_{v_{11}\geq -v_2-v_3} = p^{v_2}(1+v_2\rho+v_3\rho+v_2v_3\rho^2)$.

We compute the differences between v_1, v_2, v_3, v_4, we have several arrangements,

1. $v_1 \overset{d}{\geq} v_2 + v_3 \overset{a}{\geq} v_3 \overset{c}{\geq} v_4 \overset{b}{>} v_2 \overset{a}{\geq} 0$

   Thus, we substitute $v_1 \to 2a+b+c+d$, $v_2 \to a$, $v_3 \to a+b+c$, $v_4 \to a+b$, and compute series with the new variables as indices, where $a, c, d \geq 0$ and $b \geq 1$.

2. $v_1 \overset{d}{\geq} v_2 + v_3 \overset{c}{\geq} v_4 \overset{b}{>} v_3 \overset{a}{>} v_2 \overset{e}{\geq} 0$

   But $v_2 + v_3 - c - b = v_3 \Rightarrow v_2 = v_2 + v_3 - v_3 = b + c$

   Thus, we substitute $v_1 \to a+2b+2c+d$, $v_2 \to b+c$, $v_3 \to a+b+c$, $v_4 \to a+2b+c$, and compute series with the new variables as indices, where $c, d \geq 0$ and $a, b \geq 1$.

3. $v_1 \overset{d}{\geq} v_2 + v_3 \overset{c}{\geq} v_4 \overset{b}{>} v_2 \overset{a}{\geq} v_3 \overset{e}{\geq} 0$

   But $v_2 + v_3 - c - b = v_2 \Rightarrow v_3 = v_2 + v_3 - v_2 = b + c$

   Thus, we substitute $v_1 \to a+2b+2c+d$, $v_2 \to a+b+c$, $v_3 \to b+c$, $v_4 \to a+2b+c$, and compute series with the new variables as indices, where $a, c, d \geq 0$ and $b \geq 1$.

4. $v_1 \overset{d}{\geq} v_4 \overset{c}{>} v_2 + v_3 \overset{a}{\geq} v_2 \overset{b}{\geq} v_3 \overset{a}{\geq} 0$

   Thus, we substitute $v_1 \to 2a+b+c+d$, $v_2 \to a+b$, $v_3 \to a$, $v_4 \to 2a+b+c$, and compute series with the new variables as indices, where $a, b, d \geq 0$ and $c \geq 1$.

5. $v_1 \overset{d}{\geq} v_4 \overset{c}{>} v_2 + v_3 \overset{a}{\geq} v_3 \overset{b}{>} v_2 \overset{a}{\geq} 0$

   Thus, we substitute $v_1 \to 2a+b+c+d$, $v_2 \to a$, $v_3 \to a+b$, $v_4 \to 2a+b+c$, and compute series with the new variables as indices, where $a, d \geq 0$ and $b, c \geq 1$.

2.2 Sub case 2.2

$-v_2 - v_3 < -v_1 \Rightarrow v_1 < v_2 + v_3$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1+v_2\rho)$.

2. For $-v_1 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{-v_1}^{-v_2-1} p^{v_2}\rho(1+v_2\rho) = p^{v_2}(v_1 - v_2)\rho(1+v_2\rho) = p^{v_2}(v_1 - v_2)\rho + p^{v_2}(v_1 - v_2)v_2\rho^2$.

3. For $-v_2 - v_3 \leq v_{11} \leq -v_1 - 1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_3}^{-v_1-1} p^{v_2}\rho(1+\alpha\rho) = \sum_{i=1}^{v_2+v_3-v_1} p^{v_2}\rho(1+(v_2-i)\rho) = p^{v_2}(v_2+v_3-v_1)\rho + p^{v_2}(v_2+v_3-v_1)v_2\rho^2 - p^{v_2}\binom{v_2+v_3-v_1+1}{2}\rho^2$.

So we have $\sum_{v_{11}\geq -v_2-v_3} = p^{v_2}(1+v_2\rho+(v_1-v_2)\rho+(v_1-v_2)v_2\rho^2+(v_2+v_3-v_1)\rho+(v_2+v_3-v_1)v_2\rho^2 - \frac{(v_2+v_3-v_1+1)(v_2+v_3-v_1)}{2}\rho^2) = p^{v_2}(1+v_2\rho+v_3\rho - \frac{v_2^2}{2}\rho^2 + \frac{v_2v_1}{2}\rho^2 - \frac{v_3^2}{2}\rho^2 + \frac{v_3v_1}{2}\rho^2 + \frac{v_1v_2}{2}\rho^2 + \frac{v_1v_3}{2}\rho^2 - \frac{v_1^2}{2}\rho^2 - \frac{v_2}{2}\rho^2 - \frac{v_3}{2}\rho^2 + \frac{v_1}{2}\rho^2) = p^{v_2}(1+v_2\rho+v_3\rho - \frac{v_2^2}{2}\rho^2 + v_2v_1\rho^2 - \frac{v_3^2}{2}\rho^2 + v_3v_1\rho^2 - \frac{v_1^2}{2}\rho^2 - \frac{v_2}{2}\rho^2 - \frac{v_3}{2}\rho^2 + \frac{v_1}{2}\rho^2)$.
We compute the differences between v_1, v_2, v_3, v_4, we have several arrangements,

1. $v_2 + v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_3 \overset{b}{\geq} v_4 \overset{a}{>} v_2 \overset{e}{\geq} 0$

   But $v_2 + v_3 - d - c = v_3 \Rightarrow v_2 = c + d$

   Thus, we substitute $v_1 \to a+b+2c+d$, $v_2 \to c+d$, $v_3 \to a+b+c+d$, $v_4 \to a+c+d$, and compute series with the new variables as indices, where $b, c \geq 0$ and $a, d \geq 1$.

2. $v_2 + v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_4 \overset{b}{>} v_3 \overset{a}{>} v_2 \overset{e}{\geq} 0$

   But $v_2 + v_3 - d - c - b = v_3 \Rightarrow v_2 = v_2 + v_3 - v_3 = b + c + d$

   Thus, we substitute $v_1 \to a+2b+2c+d$, $v_2 \to b+c+d$, $v_3 \to a+b+c+d$, $v_4 \to a+2b+c+d$, and compute series with the new variables as indices, where $c \geq 0$ and $a, b, d \geq 1$.

3. $v_2 + v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_4 \overset{b}{>} v_2 \overset{a}{\geq} v_3 \overset{e}{\geq} 0$

   But $v_2 + v_3 - d - c - b = v_2 \Rightarrow v_3 = v_2 + v_3 - v_2 = b + c + d$

   Thus, we substitute $v_1 \to a+2b+2c+d$, $v_2 \to a+b+c+d$, $v_3 \to b+c+d$, $v_4 \to a+2b+c+d$, and compute series with the new variables as indices, where $a, c \geq 0$ and $b, d \geq 1$.

3 Case 3

$v_3 > v_1 \geq 0$
$v_4 > v_1 \geq 0$
$v_4 > v_2 \geq 0$
$\Rightarrow v_{11} \geq -v_2 - v_1$

$$\Rightarrow \alpha := \min\{v_2, v_2+v_{11}+v_4\} = \begin{cases} v_2, & v_{11} \geq -v_4 \\ v_2-1, v_2-2, \ldots, & v_{11} < -v_4 \end{cases}$$
$\Rightarrow f(p,t) = p^{7v_1+10v_2+11v_3+7v_4+v_1} t^{4v_1+6v_2+6v_3+4v_4} = p^{8v_1+10v_2+11v_3+7v_4} t^{4v_1+6v_2+6v_3+4v_4}$
We have two sub cases

3.1 Sub case 3.1

$-v_2 - v_1 > -v_4 \Rightarrow v_4 > v_2 + v_1$
There is no phase transition here, since $v_{11} \geq -v_2 - v_1 > -v_4$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1+v_2\rho)$.
2. For $-v_2-v_1 \leq v_{11} \leq -v_2-1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_1}^{-v_2-1} p^{v_2}\rho(1+v_2\rho) = p^{v_2}v_1\rho + p^{v_2}v_1v_2\rho^2$.

So we have $\sum_{v_{11}\geq -v_2-v_1} p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1+v_2\rho+v_1\rho+v_1v_2\rho^2)$.
We have several arrangements,

1. $v_4 \overset{d}{>} v_1+v_2 \overset{c}{\geq} v_3 \overset{b}{>} v_2 \overset{a}{>} v_1 \overset{e}{\geq} 0$

   But $v_1+v_2-c-b = v_2 \Rightarrow v_1 = v_1+v_2-v_2 = b+c$

   Thus, we substitute $v_1 \to b+c$, $v_2 \to a+b+c$, $v_3 \to a+2b+c$, $v_4 \to a+2b+2c+d$, and compute series with the new variables as indices, where $c \geq 0$ and $a,b,d \geq 1$.

2. $v_4 \overset{d}{>} v_1+v_2 \overset{a}{\geq} v_2 \overset{c}{\geq} v_3 \overset{b}{>} v_1 \overset{a}{\geq} 0$

   We substitute $v_1 \to a$, $v_2 \to a+b+c$, $v_3 \to a+b$, $v_4 \to 2a+b+c+d$, and compute series with the new variables as indices, where $a,c \geq 0$ and $b,d \geq 1$.

3. $v_4 \overset{d}{>} v_1+v_2 \overset{c}{\geq} v_3 \overset{b}{>} v_1 \overset{a}{\geq} v_2 \overset{e}{\geq} 0$

   But $v_1+v_2-c-b = v_1 \Rightarrow v_2 = v_1+v_2-v_1 = b+c$

   We substitute $v_1 \to a+b+c$, $v_2 \to b+c$, $v_3 \to a+2b+c$, $v_4 \to a+2b+2c+d$, and compute series with the new variables as indices, where $a,c \geq 0$ and $b,d \geq 1$.

4. $v_4 \overset{d}{>} v_3 \overset{c}{>} v_1+v_2 \overset{a}{\geq} v_1 \overset{b}{\geq} v_2 \overset{a}{\geq} 0$

   We substitute $v_1 \to a+b$, $v_2 \to a$, $v_3 \to 2a+b+c$, $v_4 \to 2a+b+c+d$, and compute series with the new variables as indices, where $a,b \geq 0$ and $c,d \geq 1$.

5. $v_4 \overset{d}{>} v_3 \overset{c}{>} v_1+v_2 \overset{a}{\geq} v_2 \overset{b}{>} v_1 \overset{a}{\geq} 0$

   We substitute $v_1 \to a$, $v_2 \to a+b$, $v_3 \to 2a+b+c$, $v_4 \to 2a+b+c+d$, and compute series with the new variables as indices, where $a \geq 0$ and $b,c,d \geq 1$.

6. $v_3 \overset{d}{\geq} v_4 \overset{c}{>} v_1 + v_2 \overset{a}{\geq} v_2 \overset{b}{>} v_1 \overset{a}{\geq} 0$

   We substitute $v_1 \to a$, $v_2 \to a+b$, $v_3 \to 2a+b+c+d$, $v_4 \to 2a+b+c$, and compute series with the new variables as indices, where $a, d \geq 0$ and $b, c \geq 1$.

7. $v_3 \overset{d}{\geq} v_4 \overset{c}{>} v_1 + v_2 \overset{a}{\geq} v_1 \overset{b}{\geq} v_2 \overset{a}{\geq} 0$

   We substitute $v_1 \to a+b$, $v_2 \to a$, $v_3 \to 2a+b+c+d$, $v_4 \to 2a+b+c$, and compute series with the new variables as indices, where $a, b, d \geq 0$ and $c \geq 1$.

3.2 Sub case 3.2

$-v_4 \geq -v_2 - v_1 \Rightarrow v_2 + v_1 \geq v_4$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1+v_2\rho)$.
2. For $-v_4 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{-v_4}^{-v_2-1} p^{v_2}\rho(1+v_2\rho) = p^{v_2}(v_4 - v_2)\rho + p^{v_2}(v_4 - v_2)v_2\rho^2$.
3. For $-v_2 - v_1 \leq v_{11} \leq -v_4 - 1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_1}^{-v_4-1} p^{v_2}\rho(1+v_2\rho) = \sum_{i=1}^{v_1+v_2-v_4} p^{v_2}\rho(1+(v_2-i)\rho) = p^{v_2}(v_1+v_2-v_4)\rho + p^{v_2}(v_1+v_2-v_4)v_2\rho^2 - p^{v_2}\binom{v_1+v_2-v_4+1}{2}\rho^2$.

So we have $\sum_{v_{11} \geq -v_2 - v_1} p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho + (v_4 - v_2)\rho + (v_4 - v_2)v_2\rho^2 + (v_1+v_2-v_4)\rho + (v_1+v_2-v_4)v_2\rho^2 - \frac{(v_1+v_2-v_4+1)(v_1+v_2-v_4)}{2}\rho^2) = p^{v_2}(1 + v_1\rho + v_2\rho - \frac{v_1^2}{2}\rho^2 + v_4 v_1\rho^2 - \frac{v_2^2}{2}\rho^2 + v_4 v_2\rho^2 - \frac{v_4^2}{2}\rho^2 - \frac{v_1}{2}\rho^2 - \frac{v_2}{2}\rho^2 + \frac{v_4}{2}\rho^2)$

We have several arrangements,

1. $v_1 + v_2 \overset{d}{\geq} v_4 \overset{c}{>} v_3 \overset{b}{>} v_2 \overset{a}{>} v_1 \overset{e}{\geq} 0$

   But $v_1 + v_2 - d - c - b = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = b + c + d$

   Thus, we substitute $v_1 \to b+c+d$, $v_2 \to a+b+c+d$, $v_3 \to a+2b+c+d$, $v_4 \to a+2b+2c+d$, and compute series with the new variables as indices, where $d \geq 0$ and $a, b, c \geq 1$

2. $v_1 + v_2 \overset{d}{\geq} v_4 \overset{c}{>} v_2 \overset{b}{\geq} v_3 \overset{a}{>} v_1 \overset{e}{\geq} 0$

   But $v_1 + v_2 - d - c = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = c + d$

   Thus, we substitute $v_1 \to c+d$, $v_2 \to a+b+c+d$, $v_3 \to a+c+d$, $v_4 \to a+b+2c+d$, and compute series with the new variables as indices, where $b, d \geq 0$ and $a, c \geq 1$

3. $v_1 + v_2 \overset{d}{\geq} v_4 \overset{c}{>} v_3 \overset{b}{>} v_1 \overset{a}{\geq} v_2 \overset{e}{\geq} 0$

   But $v_1 + v_2 - d - c - b = v_1 \Rightarrow v_2 = v_1 + v_2 - v_1 = b + c + d$

Thus, we substitute $v_1 \to a+b+c+d$, $v_2 \to b+c+d$, $v_3 \to a+2b+c+d$, $v_4 \to a+2b+2c+d$, and compute series with the new variables as indices, where $a, d \geq 0$ and $b, c \geq 1$

4. $v_1 + v_2 \overset{d}{\geq} v_3 \overset{c}{\geq} v_4 \overset{b}{>} v_2 \overset{a}{>} v_1 \overset{e}{\geq} 0$

   But $v_1 + v_2 - d - c - b = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = b + c + d$

   Thus, we substitute $v_1 \to b+c+d$, $v_2 \to a+b+c+d$, $v_3 \to a+2b+2c+d$, $v_4 \to a+2b+c+d$, and compute series with the new variables as indices, where $c, d \geq 0$ and $a, b \geq 1$

5. $v_1 + v_2 \overset{d}{>} v_3 \overset{c}{\geq} v_4 \overset{b}{>} v_1 \overset{a}{\geq} v_2 \overset{e}{\geq} 0$

   But $v_1 + v_2 - d - c - b = v_1 \Rightarrow v_2 = v_1 + v_2 - v_1 = b + c + d$

   Thus, we substitute $v_1 \to a+b+c+d$, $v_2 \to b+c+d$, $v_3 \to a+2b+2c+d$, $v_4 \to a+2b+c+d$, and compute series with the new variables as indices, where $a, c \geq 0$ and $b, d \geq 1$

6. $v_3 \overset{d}{\geq} v_1 + v_2 \overset{c}{\geq} v_4 \overset{b}{>} v_2 \overset{a}{>} v_1 \overset{e}{\geq} 0$

   But $v_1 + v_2 - c - b = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = b + c$

   Thus, we substitute $v_1 \to b+c$, $v_2 \to a+b+c$, $v_3 \to a+2b+2c+d$, $v_4 \to a+2b+c$, and compute series with the new variables as indices, where $c, d \geq 0$ and $a, b \geq 1$

7. $v_3 \overset{d}{\geq} v_1 + v_2 \overset{c}{\geq} v_4 \overset{b}{>} v_1 \overset{a}{\geq} v_2 \overset{e}{\geq} 0$

   But $v_1 + v_2 - c - b = v_1 \Rightarrow v_2 = v_1 + v_2 - v_1 = b + c$

   Thus, we substitute $v_1 \to a+b+c$, $v_2 \to b+c$, $v_3 \to a+2b+2c+d$, $v_4 \to a+2b+c$, and compute series with the new variables as indices, where $a, c, d \geq 0$ and $b \geq 1$

4 Case 4

$v_4 > v_1 \geq v_3 \geq 0$

$v_4 > v_2 \geq 0$

$\Rightarrow v_{11} \geq -v_2 - v_3$

$$\Rightarrow \alpha := \min\{v_2, v_2 + v_{11} + v_4\} = \begin{cases} v_2, & v_{11} \geq -v_4 \\ v_2 - 1, v_2 - 2, \ldots, & v_{11} < -v_4 \end{cases}$$

$\Rightarrow f(p,t) = p^{7v_1+10v_2+11v_3+7v_4+v_1} t^{4v_1+6v_2+6v_3+4v_4} = p^{8v_1+10v_2+11v_3+7v_4} t^{4v_1+6v_2+6v_3+4v_4}$

We have two sub cases

4.1 Sub case 4.1

$-v_4 < -v_2 - v_3 \Rightarrow v_4 > v_2 + v_3$

There is no phase transition here, since $v_{11} \geq -v_2 - v_3 > -v_4$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1+v_2\rho)$.
2. For $-v_2-v_3 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_3}^{-v_2-1} p^{v_2}\rho(1+v_2\rho) = p^{v_2}v_3\rho(1+v_2\rho) = p^{v_2}v_3\rho + p^{v_2}v_2v_3\rho^2$.

So we have $\sum_{v_{11}\geq -v_2-v_3} p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1+v_2\rho+v_3\rho+v_2v_3\rho^2)$.

We have several arrangements,

1. $v_4 \overset{d}{>} v_2 + v_3 \overset{c}{\geq} v_1 \overset{b}{\geq} v_3 \overset{a}{>} v_2 \overset{e}{\geq} 0$

   But $v_2 + v_3 - c - b = v_3 \Rightarrow v_2 = v_2 + v_3 - v_3 = b + c$

   Thus, we substitute $v_1 \to a + 2b + c$, $v_2 \to b + c$, $v_3 \to a + b + c$, $v_4 \to a + 2b + 2c + d$, and compute series with the new variables as indices, where $b, c \geq 0$ and $a, d \geq 1$

2. $v_4 \overset{d}{>} v_2 + v_3 \overset{c}{\geq} v_1 \overset{b}{\geq} v_2 \overset{a}{\geq} v_3 \overset{e}{\geq} 0$

   But $v_2 + v_3 - c - b = v_2 \Rightarrow v_3 = v_2 + v_3 - v_2 = b + c$

   Thus, we substitute $v_1 \to a + 2b + c$, $v_2 \to a + b + c$, $v_3 \to b + c$, $v_4 \to a + 2b + 2c + d$, and compute series with the new variables as indices, where $a, b, c \geq 0$ and $d \geq 1$.

3. $v_4 \overset{d}{>} v_2 + v_3 \overset{a}{\geq} v_2 \overset{c}{>} v_1 \overset{b}{\geq} v_3 \overset{a}{\geq} 0$

   Thus, we substitute $v_1 \to a + b$, $v_2 \to a + b + c$, $v_3 \to a$, $v_4 \to 2a + b + c + d$, and compute series with the new variables as indices, where $a, b \geq 0$ and $c, d \geq 1$.

4. $v_4 \overset{d}{>} v_1 \overset{c}{\geq} v_2 + v_3 \overset{a}{\geq} v_2 \overset{b}{\geq} v_3 \overset{a}{\geq} 0$

   Thus, we substitute $v_1 \to 2a + b + c$, $v_2 \to a + b$, $v_3 \to a$, $v_4 \to 2a + b + c + d$, and compute series with the new variables as indices, where $a, b, c \geq 0$ and $d \geq 1$.

5. $v_4 \overset{d}{>} v_1 \overset{c}{\geq} v_2 + v_3 \overset{a}{\geq} v_3 \overset{b}{>} v_2 \overset{a}{\geq} 0$

   Thus, we substitute $v_1 \to 2a + b + c$, $v_2 \to a$, $v_3 \to a + b$, $v_4 \to 2a + b + c + d$, and compute series with the new variables as indices, where $a, c \geq 0$ and $b, d \geq 1$.

4.2 Sub case 4.2

$-v_2 - v_3 \leq -v_4 \Rightarrow v_2 + v_3 \geq v_4$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1+v_2\rho)$.

2. For $-v_4 \le v_{11} \le -v_2-1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{-v_4}^{-v_2-1} p^{v_2}\rho(1+v_2\rho) = p^{v_2}(v_4-v_2)\rho(1+v_2\rho) = p^{v_2}(v_4-v_2)\rho + p^{v_2}v_2(v_4-v_2)\rho^2$.

3. For $-v_2-v_3 \le v_{11} \le -v_4-1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_3}^{-v_4-1} p^{v_2}\rho(1+\alpha\rho) = \sum_{i=1}^{v_2+v_3-v_4} p^{v_2}\rho(1+(v_2-i)\rho) = p^{v_2}(v_2+v_3-v_4)\rho + (v_2+v_3-v_4)v_2\rho^2 - \binom{v_2+v_3-v_4+1}{2}\rho^2$.

So we have $\sum_{v_{11}\ge -v_2-v_3} p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1+v_2\rho+(v_4-v_2))\rho + v_2(v_4-v_2)\rho^2 + (v_2+v_3-v_4)\rho + (v_2+v_3-v_4)v_2\rho^2 - \frac{(v_2+v_3-v_4+1)(v_2+v_3-v_4)}{2}\rho^2) = p^{v_2}(1+v_2v_4\rho^2+v_2\rho+v_3\rho+v_3v_2\rho^2-v_4v_2\rho^2-\frac{v_2^2}{2}\rho^2-v_2v_3\rho^2+v_2v_4\rho^2-\frac{v_3^2}{2}\rho^2+v_3v_4\rho^2-\frac{v_4^2}{2}\rho^2-\frac{v_2}{2}\rho^2-\frac{v_3}{2}\rho^2+\frac{v_4}{2}\rho^2)$.

We have several arrangements,

1. $v_2+v_3 \overset{d}{\ge} v_4 \overset{c}{>} v_1 \overset{b}{\ge} v_2 \overset{a}{\ge} v_3 \overset{e}{\ge} 0$

   But $v_2+v_3-d-c-b = v_2 \Rightarrow v_3 = v_2+v_3-v_2 = b+c+d$

   Thus, we substitute $v_1 \to a+2b+c+d$, $v_2 \to a+b+c+d$, $v_3 \to b+c+d$, $v_4 \to a+2b+2c+d$, and compute series with the new variables as indices, where $a,b,d \ge 0$ and $c \ge 1$

2. $v_2+v_3 \overset{d}{\ge} v_4 \overset{c}{>} v_1 \overset{b}{\ge} v_3 \overset{a}{>} v_2 \overset{e}{\ge} 0$

   But $v_2+v_3-d-c-b = v_3 \Rightarrow v_2 = v_2+v_3-v_3 = b+c+d$

   Thus, we substitute $v_1 \to a+2b+c+d$, $v_2 \to b+c+d$, $v_3 \to a+b+c+d$, $v_4 \to a+2b+2c+d$, and compute series with the new variables as indices, where $b,d \ge 0$ and $a,c \ge 1$

3. $v_2+v_3 \overset{d}{\ge} v_4 \overset{c}{>} v_2 \overset{b}{>} v_1 \overset{a}{\ge} v_3 \overset{e}{\ge} 0$

   But $v_2+v_3-d-c = v_2 \Rightarrow v_3 = v_2+v_3-v_2 = c+d$

   Thus, we substitute $v_1 \to a+c+d$, $v_2 \to a+b+c+d$, $v_3 \to c+d$, $v_4 \to a+b+2c+d$, and compute series with the new variables as indices, where $a,d \ge 0$ and $b,c \ge 1$

5 Case 5

$v_2 \ge v_4 > v_1 > v_3 \ge 0$

$\Rightarrow v_{11} \ge -v_2-v_3$

$$\Rightarrow \alpha := \min\{v_4, v_2+v_{11}+v_4\} = \begin{cases} v_4, & v_{11} \ge -v_2 \\ v_4-1, v_4-2, \dots, & v_{11} < -v_2 \end{cases}$$

$\Rightarrow f(p,t) = p^{7v_1+10v_2+11v_3+7v_4+v_1}t^{4v_1+6v_2+6v_3+4v_4} = p^{8v_1+10v_2+11v_3+7v_4}t^{4v_1+6v_2+6v_3+4v_4}$

1. For $v_{11} \ge -v_2$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1+v_4\rho)$.

2. For $-v_2-v_3 \le v_{11} \le -v_2-1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_3}^{-v_2-1} p^{v_2}\rho(1+\alpha\rho) = \sum_{i=1}^{v_3} p^{v_2}\rho(1+(v_4-i)\rho) = p^{v_2}(v_3\rho + v_3v_4\rho^2 - \binom{v_3+1}{2}\rho^2)$.

So we have $\sum_{v_{11}\geq -v_2-v_3} p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1+v_4\rho+v_3\rho+v_3v_4\rho^2 - \frac{(v_3+1)v_3}{2}\rho^2) = p^{v_2}(1+v_4\rho+v_3\rho+v_3v_4\rho^2 - \frac{v_3^2}{2}\rho^2 - \frac{v_3}{2}\rho^2)$.

$v_2 \overset{d}{\geq} v_4 \overset{c}{>} v_1 \overset{b}{\geq} v_3 \overset{a}{\geq} 0$

Thus, we substitute $v_1 \to a+b$, $v_2 \to a+b+c+d$, $v_3 \to a$, $v_4 \to a+b+c$, and compute series with the new variables as indices, where $a,b,d \geq 0$ and $c \geq 1$

6 Case 6

$v_2 \geq v_4 > v_1 \geq 0$

$v_3 > v_1 \geq 0$

$\Rightarrow v_{11} \geq -v_2 - v_1$

$\Rightarrow \alpha := \min\{v_4, v_2+v_{11}+v_4\} = \begin{cases} v_4, & v_{11} \geq -v_2 \\ v_4-1, v_4-2, \ldots, & v_{11} < -v_2 \end{cases}$

$\Rightarrow f(p,t) = p^{7v_1+10v_2+11v_3+7v_4+v_1}t^{4v_1+6v_2+6v_3+4v_4} = p^{8v_1+10v_2+11v_3+7v_4}t^{4v_1+6v_2+6v_3+4v_4}$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1+v_4\rho)$.

2. For $-v_2-v_1 \leq v_{11} \leq -v_2-1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_1}^{-v_2-1} p^{v_2}\rho(1+\alpha\rho) = \sum_{i=1}^{v_1} p^{v_2}\rho(1+(v_4-i)\rho) = p^{v_2}(v_1\rho+v_1v_4\rho^2 - \binom{v_1+1}{2}\rho^2)$.

So we have $\sum_{v_{11}\geq -v_2-v_1} p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1+v_4\rho+v_1\rho+v_1v_4\rho^2 - \frac{(v_1+1)v_1}{2}\rho^2) = p^{v_2}(1+v_4\rho+v_1\rho+v_1v_4\rho^2 - \frac{v_1^2}{2}\rho^2 - \frac{v_1}{2}\rho^2)$.

We have several arrangements,

1. $v_2 \overset{d}{\geq} v_4 \overset{c}{>} v_3 \overset{b}{>} v_1 \overset{a}{\geq} 0$

   Thus, we substitute $v_1 \to a$, $v_2 \to a+b+c+d$, $v_3 \to a+b$, $v_4 \to a+b+c$, and compute series with the new variables as indices, where $a,d \geq 0$ and $b,c \geq 1$

2. $v_2 \overset{d}{\geq} v_3 \overset{c}{\geq} v_4 \overset{b}{>} v_1 \overset{a}{\geq} 0$

   Thus, we substitute $v_1 \to a$, $v_2 \to a+b+c+d$, $v_3 \to a+b+c$, $v_4 \to a+b$, and compute series with the new variables as indices, where $a,c,d \geq 0$ and $b \geq 1$

3. $v_3 \overset{d}{>} v_2 \overset{c}{\geq} v_4 \overset{b}{>} v_1 \overset{a}{\geq} 0$

   Thus, we substitute $v_1 \to a$, $v_2 \to a+b+c$, $v_3 \to a+b+c+d$, $v_4 \to a+b$, and compute series with the new variables as indices, where $a,c \geq 0$ and $b,d \geq 1$

7 Case 7

$v_2 \geq v_4 \geq 0$

$v_3 > v_1 \geq v_4 \geq 0$
$\Rightarrow v_{11} \geq -v_2 - v_1$
$\Rightarrow f(p,t) = p^{7v_1+10v_2+11v_3+7v_4+v_4}t^{4v_1+6v_2+6v_3+4v_4} = p^{7v_1+10v_2+11v_3+8v_4}t^{4v_1+6v_2+6v_3+4v_4}$
$\Rightarrow \alpha := \min\{v_4, v_2+v_{11}+v_1\} = \begin{cases} v_4, & v_{11} \geq v_4 - v_1 - v_2 \\ v_4-1, v_4-2, \ldots, & v_{11} < v_4 - v_1 - v_2 \end{cases}$

There are no two sub cases here, because the phase transition occurs at $v_{11} = v_4 - v_1 - v_2 - 1$, thus

if $v_4 = 0$ then the transition would be at $v_{11} = -v_2 - v_1 - 1 < -v_2 - v_1$, which is a contradiction.

For $v_4 \geq 1$ the transition would be at $a_{11} \geq -v_2 - v_1$, which is always true for this case.

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1+v_4\rho)$.

2. For $v_4 - v_1 - v_2 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{v_4-v_1-v_2}^{-v_2-1} p^{v_2}\rho(1+v_4\rho) = \sum_{i=1}^{v_1-v_4} p^{v_2}\rho(1+v_4\rho) = p^{v_2}(v_1-v_4)\rho(1+v_4\rho) = p^{v_2}(v_1-v_4)\rho + p^{v_2}(v_1-v_4)v_4\rho^2$.

3. For $-v_1 - v_2 \leq v_{11} \leq v_4 - v_1 - v_2 - 1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{-v_1-v_2}^{v_4-v_1-v_2-1} p^{v_2}\rho(1+\alpha\rho) = \sum_{i=1}^{v_4} p^{v_2}\rho(1+(v_4-i)\rho) = p^{v_2}v_4\rho + p^{v_2}v_4^2\rho^2 - p^{v_2}\binom{v_4+1}{2}\rho^2$.

So we have $\sum_{v_{11}\geq -v_2-v_1} p^{\min}\mu(a_{11}) = p^{v_2}(1+v_4\rho+(v_4-v_1)\rho+(v_4-v_1)v_4\rho^2 + v_4\rho + v_4^2\rho^2 - \frac{v_4(v_4-1)}{2}\rho^2) = p^{v_2}(1+3v_4\rho - v_1\rho + 2v_4^2\rho^2 - v_1v_4\rho^2 - \frac{v_4^2}{2}\rho^2 + \frac{v_4}{2}\rho^2)$

We have several arrangements,

1. $v_2 \overset{d}{\geq} v_3 \overset{c}{>} v_1 \overset{b}{\geq} v_4 \overset{a}{\geq} 0$

   Thus, we substitute $v_1 \to a+b$, $v_2 \to a+b+c+d$, $v_3 \to a+b+c$, $v_4 \to a$, and compute series with the new variables as indices, where $a,b,d \geq 0$ and $c \geq 1$

2. $v_3 \overset{d}{>} v_2 \overset{c}{>} v_1 \overset{b}{\geq} v_4 \overset{a}{\geq} 0$

   Thus, we substitute $v_1 \to a+b$, $v_2 \to a+b+c$, $v_3 \to a+b+c+d$, $v_4 \to a$, and compute series with the new variables as indices, where $a,b \geq 0$ and $c,d \geq 1$

3. $v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_2 \overset{b}{\geq} v_4 \overset{a}{\geq} 0$

   Thus, we substitute $v_1 \to a+b+c$, $v_2 \to a+b$, $v_3 \to a+b+c+d$, $v_4 \to a$, and compute series with the new variables as indices, where $a,b,c \geq 0$ and $d \geq 1$

8 Case 8

$v_2 \geq v_4 \geq 0$

$v_1 \geq v_4 \geq 0$
$v_1 \geq v_3 \geq 0$
$\Rightarrow v_{11} \geq -v_2 - v_3$
$\Rightarrow \alpha := \min\{v_4, v_2 + v_{11} + v_1\} = \begin{cases} v_4, & v_{11} \geq v_4 - v_1 - v_2 \\ v_4 - 1, v_4 - 2, \ldots, & v_{11} < v_4 - v_1 - v_2 \end{cases}$
$\Rightarrow f(p,t) = p^{7v_1+10v_2+11v_3+7v_4+v_4} t^{4v_1+6v_2+6v_3+4v_4} = p^{7v_1+10v_2+11v_3+8v_4} t^{4v_1+6v_2+6v_3+4v_4}$
There are two sub cases here

8.1 Sub case 8.1

$-v_2 - v_3 \geq v_4 - v_1 - v_2 \Rightarrow -v_3 \geq v_4 - v_1 \Rightarrow v_1 \geq v_3 + v_4$

For $v_{11} \geq -v_2$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho)$.

For $-v_2 - v_3 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_3}^{-v_2-1} p^{v_2}\rho(1+v_4\rho) = \sum_{i=1}^{v_3} p^{v_2}\rho(1 + v_4\rho) = p^{v_2}(v_3\rho + v_3v_4\rho^2)$.

So we have $\sum_{v_{11} \geq -v_2-v_3} p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho + v_3\rho + v_3v_4\rho^2)$.

We have several arrangements,

1. $v_1 \overset{d}{\geq} v_3 + v_4 \overset{a}{\geq} v_3 \overset{c}{>} v_2 \overset{b}{\geq} v_4 \overset{a}{\geq} 0$

   But $e = v_2 + v_3 - v_3 = v_2 = a + b$.

   Thus, we substitute $v_1 \to 2a + b + c + d$, $v_2 \to a + b$, $v_3 \to a + b + c$, $v_4 \to a$, and compute series with the new variables as indices, where $a, b, d \geq 0$ and $c \geq 1$

2. $v_1 \overset{d}{\geq} v_3 + v_4 \overset{c}{\geq} v_2 \overset{b}{\geq} v_3 \overset{a}{\geq} v_4 \overset{e}{\geq} 0$

   But $v_3 = v_3 + v_4 - c - b \Rightarrow v_4 = b + c$

   Thus, we substitute $v_1 \to a + 2b + 2c + d$, $v_2 \to a + 2b + c$, $v_3 \to a + b + c$, $v_4 \to b + c$, and compute series with the new variables as indices, where $a, b, c, d \geq 0$.

3. $v_1 \overset{d}{\geq} v_3 + v_4 \overset{c}{\geq} v_2 \overset{b}{\geq} v_4 \overset{a}{>} v_3 \overset{e}{\geq} 0$

   Thus, we substitute $v_1 \to a + 2b + 2c + d$, $v_2 \to a + 2b + c$, $v_3 \to a + b + c$, $v_4 \to b + c$, and compute series with the new variables as indices, where $b, c, d \geq 0$ and $a \geq 1$

4. $v_1 \overset{d}{\geq} v_2 \overset{c}{>} v_3 + v_4 \overset{a}{\geq} v_4 \overset{b}{>} v_3 \overset{a}{\geq} 0$

   Thus, we substitute $v_1 \to 2a + b + c + d$, $v_2 \to 2a + b + c$, $v_3 \to a$, $v_4 \to a + b$, and compute series with the new variables as indices, where $a, d \geq 0$ and $b, c \geq 1$

5. $v_1 \overset{d}{\geq} v_2 \overset{c}{>} v_3 + v_4 \overset{a}{\geq} v_3 \overset{b}{\geq} v_4 \overset{a}{\geq} 0$

   Thus, we substitute $v_1 \to 2a + b + c + d$, $v_2 \to 2a + b + c$, $v_3 \to a + b$, $v_4 \to a$, and compute series with the new variables as indices, where $a, b, d \geq 0$ and $c \geq 1$

6. $v_2 \overset{d}{>} v_1 \overset{c}{\geq} v_3 + v_4 \overset{a}{\geq} v_3 \overset{b}{\geq} v_4 \overset{a}{\geq} 0$

   Thus, we substitute $v_1 \to 2a+b+c$, $v_2 \to 2a+b+c+d$, $v_3 \to a+b$, $v_4 \to a$, and compute series with the new variables as indices, where $a,b,c \geq 0$ and $d \geq 1$

7. $v_2 \overset{d}{>} v_1 \overset{c}{\geq} v_3 + v_4 \overset{a}{\geq} v_4 \overset{b}{>} v_3 \overset{a}{\geq} 0$

   Thus, we substitute $v_1 \to 2a+b+c$, $v_2 \to 2a+b+c+d$, $v_3 \to a$, $v_4 \to a+b$, and compute series with the new variables as indices, where $a,c \geq 0$ and $b,d \geq 1$

@@@@ CHECKED @@@@

8.2 Sub case 8.2

$-v_2 - v_3 < v_4 - v_1 - v_2 \Rightarrow -v_3 < v_4 - v_1 \Rightarrow v_1 < v_3 + v_4$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = p^{v_2}(1+v_4\rho)$.

2. For $v_4 - v_1 - v_2 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{v_4-v_1-v_2}^{-v_2-1} p^{v_2}\rho(1+v_4\rho) = p^{v_2}((v_1-v_4)\rho + (v_1-v_4)v_4\rho^2)$.

3. For $-v_2 - v_3 \leq v_{11} \leq v_4 - v_1 - v_2 - 1$, we have $p^{\min}(1+\alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_3}^{v_4-v_1-v_2-1} p^{v_2}\rho(1+\alpha\rho) = \sum_{i=1}^{v_3+v_4-v_1} p^{v_2}\rho(1+(v_4-i))\rho) = p^{v_2}((v_3+v_4-v_1)\rho + (v_3+v_4-v_1)v_4\rho^2 - \binom{v_3+v_4-v_1+1}{2}\rho^2)$.

So we have $\sum_{v_{11} \geq -v_2-v_3} \rho(1+\alpha\rho) = p^{v_2}(1+v_4\rho+(v_1-v_4)\rho+(v_1-v_4)v_4\rho^2+(v_3+v_4-v_1)\rho+(v_3+v_4-v_1)v_4\rho^2 - \frac{(v_3+v_4-v_1+1)(v_3+v_4-v_1)}{2}\rho^2) = p^{v_2}(1+v_3\rho+v_4\rho - \frac{v_3^2}{2}\rho^2 + v_1v_3\rho^2 - \frac{v_4^2}{2}\rho^2 + v_1v_4\rho^2 - \frac{v_1^2}{2}\rho^2 - v_3\rho^2 - v_4\rho^2 + v_1\rho^2)$.

We have several arrangements,

1. $v_3 + v_4 \overset{d}{>} v_1 \overset{c}{\geq} v_3 \overset{b}{>} v_2 \overset{a}{\geq} v_4 \overset{e}{\geq} 0$

   But $v_3 + v_4 - d - c = v_3 \Rightarrow v_4 = c + d$.

   Thus, we substitute $v_1 \to a+b+2c+d$, $v_2 \to a+c+d$, $v_3 \to a+b+c+d$, $v_4 \to c+d$, and compute series with the new variables as indices, where $a,c \geq 0$ and $b,d \geq 1$

2. $v_3 + v_4 \overset{d}{>} v_1 \overset{c}{\geq} v_2 \overset{b}{\geq} v_3 \overset{a}{\geq} v_4 \overset{e}{\geq} 0$

   But $v_3 + v_4 - d - c - b = v_3 \Rightarrow v_4 = b + c + d$.

   Thus, we substitute $v_1 \to a+2b+2c+d$, $v_2 \to a+2b+c+d$, $v_3 \to a+b+c+d$, $v_4 \to b+c+d$, and compute series with the new variables as indices, where $a,b,c \geq 0$ and $d \geq 1$.

3. $v_3 + v_4 \overset{d}{>} v_1 \overset{c}{\geq} v_2 \overset{b}{\geq} v_4 \overset{a}{>} v_3 \overset{e}{\geq} 0$

   But $v_3 + v_4 - d - c - b = v_4 \Rightarrow v_3 = b + c + d$.

Thus, we substitute $v_1 \to a+2b+2c+d$, $v_2 \to a+2b+c+d$, $v_3 \to b+c+d$, $v_4 \to a+b+c+d$, and compute series with the new variables as indices, where $b,c \geq 0$ and $a,d \geq 1$

4. $v_3 + v_4 \overset{d}{\geq} v_2 \overset{c}{>} v_1 \overset{b}{\geq} v_4 \overset{a}{>} v_3 \overset{e}{\geq} 0$

   But $v_3 + v_4 - d - c - b = v_4 \Rightarrow v_3 = b+c+d$.

   Thus, we substitute $v_1 \to a+2b+c+d$, $v_2 \to a+2b+2c+d$, $v_3 \to b+c+d$, $v_4 \to a+b+c+d$, and compute series with the new variables as indices, where $b,d \geq 0$ and $a,c \geq 1$

5. $v_3 + v_4 \overset{d}{>} v_2 \overset{c}{>} v_1 \overset{b}{\geq} v_3 \overset{a}{\geq} v_4 \overset{e}{\geq} 0$

   But $v_3 + v_4 - d - c - b = v_3 \Rightarrow v_4 = b+c+d$.

   Thus, we substitute $v_1 \to a+2b+c+d$, $v_2 \to a+2b+2c+d$, $v_3 \to a+b+c+d$, $v_4 \to b+c+d$, and compute series with the new variables as indices, where $a,b \geq 0$ and $c,d \geq 1$

6. $v_2 \overset{d}{\geq} v_3 + v_4 \overset{c}{>} v_1 \overset{b}{\geq} v_4 \overset{a}{>} v_3 \overset{e}{\geq} 0$

   But $v_3 + v_4 - c - b = v_4 \Rightarrow v_3 = b+c$.

   Thus, we substitute $v_1 \to a+2b+c$, $v_2 \to a+2b+2c+d$, $v_3 \to b+c$, $v_4 \to a+b+c$, and compute series with the new variables as indices, where $b,d \geq 0$ and $a,c \geq 1$

7. $v_2 \overset{d}{\geq} v_3 + v_4 \overset{c}{>} v_1 \overset{b}{\geq} v_3 \overset{a}{\geq} v_4 \overset{e}{\geq} 0$

   But $v_3 + v_4 - c - b = v_3 \Rightarrow v_4 = b+c$.

   Thus, we substitute $v_1 \to a+2b+c$, $v_2 \to a+2b+2c+d$, $v_3 \to a+b+c$, $v_4 \to b+c$, and compute series with the new variables as indices, where $a,b,d \geq 0$ and $c \geq 1$

Checking all the cases by the 4! arrangements.

1. $v_1 \geq v_2 \geq v_3 \geq v_4 \geq 0$

   $v_1 \geq v_3 + v_4 \geq v_2 \geq v_3 \geq v_4 \geq 0$ (8.1.2)

   $v_1 \geq v_2 > v_3 + v_4 \geq v_3 \geq v_4 \geq 0$ (8.1.5)

   $v_3 + v_4 > v_1 \geq v_2 \geq v_3 \geq v_4 \geq 0$ (8.2.2)

2. $v_1 \geq v_2 \geq v_4 > v_3 \geq 0$

   $v_1 \geq v_3 + v_4 \geq v_2 \geq v_4 > v_3 \geq 0$ (8.1.3)

   $v_1 \geq v_2 > v_3 + v_4 \geq v_4 > v_3 \geq 0$ (8.1.4)

   $v_3 + v_4 > v_1 \geq v_2 \geq v_4 > v_3 \geq 0$ (8.2.3)

3. $v_1 \geq v_3 > v_2 \geq v_4 \geq 0$

   $v_1 \geq v_3 + v_4 \geq v_3 > v_2 \geq v_4 \geq 0$ (8.1.1)

   $v_3 + v_4 > v_1 \geq v_3 > v_2 \geq v_4 \geq 0$ (8.2.1)

4. $v_1 \geq v_3 \geq v_4 > v_2 \geq 0$

   $v_1 \geq v_2 + v_3 \geq v_3 \geq v_4 > v_2 \geq 0$ (2.1.1)

   $v_2 + v_3 > v_1 \geq v_3 \geq v_4 > v_2 \geq 0$ (2.2.1)

5. $v_1 \geq v_4 > v_2 \geq v_3 \geq 0$

   $v_1 \geq v_2 + v_3 \geq v_4 > v_2 \geq v_3 \geq 0$ (2.1.3)

   $v_1 \geq v_4 > v_2 + v_3 \geq v_2 \geq v_3 \geq 0$ (2.1.4)

   $v_2 + v_3 > v_1 \geq v_4 > v_2 \geq v_3 \geq 0$ (2.2.3)

6. $v_1 \geq v_4 > v_3 > v_2 \geq 0$

   $v_1 \geq v_2 + v_3 \geq v_4 > v_3 > v_2 \geq 0$ (2.1.2)

   $v_2 + v_3 > v_1 \geq v_4 > v_3 > v_2 \geq 0$ (2.2.2)

   $v_1 \geq v_4 > v_2 + v_3 \geq v_3 > v_2 \geq 0$ (2.1.5)

7. $v_2 > v_1 \geq v_3 \geq v_4 \geq 0$

   $v_2 > v_1 \geq v_3 + v_4 \geq v_3 \geq v_4 \geq 0$ (8.1.6)

   $v_2 \geq v_3 + v_4 > v_1 \geq v_3 \geq v_4 \geq 0$ (8.2.7)

   $v_3 + v_4 > v_2 > v_1 \geq v_3 \geq v_4 \geq 0$ (8.2.5)

8. $v_2 > v_1 \geq v_4 > v_3 \geq 0$

   $v_2 > v_1 \geq v_3 + v_4 \geq v_4 > v_3 \geq 0$ (8.1.7)

   $v_2 \geq v_3 + v_4 > v_1 \geq v_4 > v_3 \geq 0$ (8.2.6)

   $v_3 + v_4 \geq v_2 > v_1 \geq v_4 > v_3 \geq 0$ (8.2.4)

9. $v_2 \geq v_3 > v_1 \geq v_4 \geq 0$ (7.1)

10. $v_2 \geq v_3 \geq v_4 > v_1 \geq 0$ (6.2)

11. $v_2 \geq v_4 > v_1 \geq v_3 \geq 0$ (5)

12. $v_2 \geq v_4 > v_3 > v_1 \geq 0$ (6.1)

13. $v_3 > v_1 \geq v_2 \geq v_4 \geq 0$ (7.3)

14. $v_3 > v_1 \geq v_4 > v_2 \geq 0$ (1)

15. $v_3 > v_2 > v_1 \geq v_4 \geq 0$ (7.2)

16. $v_3 > v_2 \geq v_4 > v_1 \geq 0$ (6.3)

17. $v_3 \geq v_4 > v_1 \geq v_2 \geq 0$

    $v_3 \geq v_4 > v_1 + v_2 \geq v_1 \geq v_2 \geq 0$ (3.1.7)

    $v_3 \geq v_1 + v_2 \geq v_4 > v_1 \geq v_2 \geq 0$ (3.2.7)

    $v_1 + v_2 > v_3 \geq v_4 > v_1 \geq v_2 \geq 0$ (3.2.5)

18. $v_3 \geq v_4 > v_2 > v_1 \geq 0$

    $v_3 \geq v_4 > v_1 + v_2 \geq v_2 > v_1 \geq 0$ (3.1.6)

    $v_3 \geq v_1 + v_2 \geq v_4 > v_2 > v_1 \geq 0$ (3.2.6)

    $v_1 + v_2 \geq v_3 \geq v_4 > v_2 > v_2 \geq 0$ (3.2.4)

19. $v_4 > v_1 \geq v_2 \geq v_3 \geq 0$

    $v_4 > v_2 + v_3 \geq v_1 \geq v_2 \geq v_3 \geq 0$ (4.1.2)

    $v_2 + v_3 \geq v_4 > v_1 \geq v_2 \geq v_3 \geq 0$ (4.2.1)

    $v_4 > v_1 \geq v_2 + v_3 \geq v_2 \geq v_3 \geq 0$ (4.1.4)

20. $v_4 > v_1 \geq v_3 > v_2 \geq 0$

    $v_4 > v_2 + v_3 \geq v_1 \geq v_3 > v_2 \geq 0$ (4.1.1)

    $v_2 + v_3 \geq v_4 > v_1 \geq v_3 > v_2 \geq 0$ (4.2.2)

    $v_4 > v_1 \geq v_2 + v_3 \geq v_3 > v_2 \geq 0$ (4.1.5)

21. $v_4 > v_2 > v_1 \geq v_3 \geq 0$

    $v_4 > v_2 + v_3 \geq v_2 > v_1 \geq v_3 \geq 0$ (4.1.3)

    $v_2 + v_3 \geq v_4 > v_2 > v_1 \geq v_3 \geq 0$ (4.2.3)

22. $v_4 > v_2 \geq v_3 > v_1 \geq 0$

    $v_4 > v_1 + v_2 \geq v_2 \geq v_3 > v_1 \geq 0$ (3.1.2)

    $v_1 + v_2 \geq v_4 > v_2 \geq v_3 > v_1 \geq 0$ (3.2.2)

23. $v_4 > v_3 > v_1 \geq v_2 \geq 0$

    $v_4 > v_1 + v_2 \geq v_3 > v_1 \geq v_2 \geq 0$ (3.1.3)

    $v_1 + v_2 \geq v_4 > v_3 > v_1 \geq v_2 \geq 0$ (3.2.3)

    $v_4 > v_3 > v_1 + v_2 \geq v_1 \geq v_2 \geq 0$ (3.1.4)

24. $v_4 > v_3 > v_2 > v_1 \geq 0$

    $v_4 > v_1 + v_2 \geq v_3 > v_2 > v_1 \geq 0$ (3.1.1)

    $v_1 + v_2 \geq v_4 > v_3 > v_2 > v_1 \geq 0$ (3.2.1)

    $v_4 > v_3 > v_1 + v_2 \geq v_2 > v_1 \geq 0$ (3.1.5)

9 Sum of Series

$$\sum_{a\geq 0} x^a = \frac{1}{1-x}$$

$$\sum_{a\geq 1} x^a = \sum_{a\geq 0} x^a - x^0 = \frac{1}{1-x} - 1 = \frac{x}{1-x}$$

$$\sum_{a\geq 0} ax^a = x\sum_{a\geq 0} ax^{a-1} = x(\sum_{a\geq 0} x^a)' = x((1-x)^{-1})' = -1\cdot x(1-x)^{-2}\cdot -1 = \frac{x}{(1-x)^2}$$

$$\sum_{a\geq 1} ax^a = \sum_{a\geq 0} ax^a - 0x^0 = \sum_{a\geq 0} ax^a = \frac{x}{(1-x)^2}$$

$$(\sum_{a\geq 0} x^a)'' = (\sum_{a\geq 0} ax^{a-1})' = \sum_{a\geq 0} a(a-1)x^{a-2} = \frac{1}{x^2}\sum_{a\geq 0} a(a-1)x^a =$$

$$= \frac{1}{x^2}\sum_{a\geq 0} a^2x^a - \frac{1}{x^2}\sum_{a\geq 0} ax^a = \frac{1}{x^2}\sum_{a\geq 0} a^2x^a - \frac{1}{x^2}\frac{x}{(1-x)^2}$$

But

$$(\sum_{a\geq 0} x^a)'' = [\frac{1}{(1-x)^2}]' = ((1-x)^{-2})' = -2\cdot(1-x)^-3\cdot -1 = \frac{2}{(1-x)^3} \Rightarrow$$

$$\Rightarrow \frac{1}{x^2}\sum_{a\geq 0} a^2x^a = \frac{2}{(1-x)^3} + \frac{x}{x^2(1-x)^2} \Rightarrow \sum_{a\geq 0} a^2x^a = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} =$$

$$= \frac{2x^2+x-x^2}{(1-x)^3} = \frac{x^2+x}{(1-x)^3}$$

$$\sum_{a\geq 1} a^2x^a = \sum_{a\geq 0} a^2x^a - 0^2x^0 = \sum_{a\geq 0} a^2x^a = \frac{x^2+x}{(1-x)^3}$$