# 1 The computation of $G_n(\mathbb{Q}_p)$

## 1.1 The computation of the first block $M_{11}$

**Proposition 1.1.1.** *Let $x = \sum_{i=1}^{n-1} \lambda_i e_{i,i+1}$, where $\lambda_i \in \mathbb{Q}_p$ not all zero. Then $\dim \mathcal{C}_{\gamma_3}(x) = l+m$, where **l** is the number of sequences of non-zero coefficients of the form $\lambda_j, \lambda_{j+1}, \ldots, \lambda_{j+k-1}, \lambda_{j+k}$ and $\lambda_{j-1} = \lambda_{j+k+1} = 0$[1], and **m** is the number of zero coefficients $\lambda_j = 0$, such that also $\lambda_{j-1} = \lambda_{j+1} = 0$.*

*Proof.* Let $y = \sum_{i=1}^{n-1} \mu_i e_{i,i+1} \in \mathcal{C}_{\gamma_3}(x)$, where $\lambda_i \in \mathbb{Q}_p$. For every $1 \leq i \leq n-1$, denote by $c_i$ the constraint equation $[\lambda_i e_{i,i+1}, \mu_{i+1} e_{i+1,i+2}] - [\lambda_{i+1} e_{i+1,i+2}, \mu_i e_{i,i+1}] = (\lambda_i \mu_{i+1} - \lambda_{i+1}\mu_i) e_{i,i+2} = 0$. Let $1 \leq j \leq n-1$ and $1 \leq k \leq n-1-j$ be two indices, such that $\lambda_{j-1} = \lambda_{j+k+1} = 0$, and $\lambda_j, \lambda_{j+1}, \ldots, \lambda_{j+k-1}, \lambda_{j+k}$ are all non-zero, then by constraints $c_j, c_{j+1}, \ldots, c_{m-1}$, we have that $\mu_m = \frac{\lambda_m}{\lambda_{m-1}} \mu_{m-1} = \frac{\lambda_m}{\lambda_{m-1}} \frac{\lambda_{m-1}}{\lambda_{m-2}} \mu_{m-2} = \frac{\lambda_m}{\lambda_{m-2}} \mu_{m-2} = \cdots = \frac{\lambda_m}{\lambda_j} \mu_j$, for every $j+1 \leq m \leq j+k-1$, which means that all $\mu$ coefficients of $y$, with indices from $j+1$ to $j+k$, depend on the first coefficient, namely $\mu_j$. We denote the free choice of $\mu_j$ by $\mu_j = *$. One easily checks that we can choose freely any coefficient $\mu_m$ from $j+1$ to $j+k$, instead of $\mu_j$, and all other coefficients in that range will depend on our choice of $\mu_m$. By constraint $c_{j-1}$, we have that $\lambda_{j-1}\mu_j - \lambda_j \mu_{j-1} = 0$, but $\lambda_{j-1} = 0$, hence $\lambda_j \mu_{j-1}$ must vanish, but $\lambda_j \neq 0$, which obviously means that $\mu_{j-1} = 0$. Similarly, we have that $\mu_{j+k+1} = 0$, due to constraint $c_{j+k}$. By constraint $c_{j+k+1}$, we have that $\lambda_{k+k+1}\mu_{j+k+2} - \lambda_{j+k+2}\mu_{j+k+1} = 0$, but $\lambda_{j+k+1} = \mu_{j+k+1} = 0$, hence, $\lambda_{j+k+1}\mu_{j+k+2}$ must vanish, but $\lambda_{j+k+1} = 0$, which means that we need to look at constraint $c_{j+k+2}$, that is, $\lambda_{j+k+2}\mu_{j+k+3} - \lambda_{j+k+3}\mu_{j+k+2} = 0$. We check the different options. If $\lambda_{j+k+2} = 0$, then $\lambda_{j+k+3}\mu_{j+k+2}$ must vanish. Therefore, if $\lambda_{j+k+3} \neq 0$, then $\mu_{j+k+2} = 0$, but if $\lambda_{j+k+3} = 0$, then $\mu_{j+k+2} = *$. If $\lambda_{j+k+2} \neq 0$, then again $\mu_{j+k+2} = *$. If $\lambda_{j+k+2} \neq 0$, then $\mu_{j+k+2} = *$, and we continue the same way as for $\lambda_j$ and its following coefficients. $\square$

[1] We extend our notation of indices, to include also the case where $j = 1$ or $j+k = n-1$, and define that $\lambda_{j-1} = \lambda_0 = 0$ or $\lambda_{j+k+1} = \lambda_n = 0$, respectively