

homework

haiml76

November 2024

Proposition 0.1. *Let $\varphi_r \in N(\mathbb{Q}_p)$ be a unipotent automorphism of $\mathcal{L}_{n,p}$ such that the matrix upper blocks M_{1s} , for all $s < r$, are zero matrices. Consider the $(n-1) \times (n-r)$ matrix $M_{1r} = (a_{ij})$. Then,*

1. *Let $2 \leq r < n-2$. If $a_{ij} \neq 0$, then either $i = j$ or $i = j + r - 1$, and we have the relation $a_{i+r,i+1} = -a_{ii}$.*
2. *Let $r = n-2$. If $a_{ij} \neq 0$, then either $i = j$ or $i = j + r - 1$ or $(i, j) \in \{(1, 2), (n-1, 1)\}$, with the same relation as above.*

Proof. From the relation $[\varphi_r(e_{k,k+1}), \varphi_r(e_{l,l+1})] = 0$ where $l > k+1$, we deduce that $a_{ij} \neq 0$ only if either $i = j$ or $i = j + r - 1$ or $(i, j) \in \{(r+1, 1), (r+2, 1), (n-r-2, n-r), (n-r-1, n-r)\}$. If $r < n-2$ then it follows from the conditions

$$\begin{aligned} [\varphi_r(e_{n-r-2, n-r-1}), \varphi_r(e_{n-r-2, n-r})] &= 0 \\ [\varphi_r(e_{n-r-1, n-r}), \varphi_r(e_{n-r-2, n-r})] &= 0 \\ [\varphi_r(e_{r+1, r+2}), \varphi_r(e_{r+1, r+3})] &= 0 \\ [\varphi_r(e_{r+2, r+3}), \varphi_r(e_{r+1, r+3})] &= 0 \end{aligned}$$

that the four exceptional cases cannot occur. When $r = n-2$, we have that $(n-r-2, n-r) = (0, 2)$ and $(r+2, 1) = (n, 1)$ so these cases do not exist, but so are the four conditions above, which means that the two remaining cases, $(r+1, 1) = (n-1, 1)$ and $(n-r-1, n-r) = (1, 2)$, do not necessarily vanish. \square

Proposition 0.2. *Denote by $N_r := \{\varphi_r : 2 \leq r \leq n-2\} \subset N(\mathbb{Q}_p)$ the set of all automorphisms of the form described in 0.1, then $N_r \leq N(\mathbb{Q}_p)$. Note that $N_2 = N(\mathbb{Q}_p)$.*

Proposition 0.3. *Let $2 \leq r \leq n-2$, and let $0 \leq k \leq n-r$, and let $a \in \mathbb{Q}_p$. We extend our notation of basis elements to include $e_{01} = e_{n, n+1} = 0$.*

1. *There is an automorphism $\varphi_{r,k}(a) \in N(\mathbb{Q}_p)$ determined by*

$$\varphi_{r,k}(a)(e_{i,i+1}) := \begin{cases} e_{k,k+1} + ae_{k,k+r} & : i = k \\ e_{k+r,k+r+1} - ae_{k+1,k+r+1} & : i = k+r \\ e_{i,i+1} & : i \notin \{k, k+r\} \end{cases}$$

2. Suppose that $r = n - 2$, let $(k, l) \in \{(1, 2), (n - 1, 1)\}$, and let $a \in \mathbb{Q}_p$. There is an automorphism $\varphi_{n-2,k,l}(a) \in G_n^0(\mathbb{Q}_p)$ determined by

$$\varphi_{n-2,k,l}(a)(e_{i,i+1}) := \begin{cases} e_{k,k+1} + ae_{l,l+r} & : i = k \\ e_{i,i+1} & : i \neq k \end{cases}$$

We denote $\varphi'_{n-2}(a) := \varphi_{n-2,1,2}(a)$ and $\varphi''_{n-2}(a) := \varphi_{n-2,n-1,1}(a)$.

Proof. We need to verify that for all $1 \leq i < j \leq n$ and $1 \leq l < m \leq n$ we have the following relations

$$[\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{lm})] = \begin{cases} \varphi_{r,k}(a)(e_{im}) & : j = l \\ -\varphi_{r,k}(a)(e_{lj}) & : i = m \\ 0 & : \text{otherwise} \end{cases}$$

We can verify explicitly for $n = 4$ that these relations are true. Alternatively, Berman did this in [?, §3.3.7]. For $n > 4$, let $m = n$, then

$$[\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{ln})] = [\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{ln})].$$

If $i > 1$, then we consider the inclusion $\iota : \mathcal{L}_{n-1,p} \hookrightarrow \mathcal{L}_{n,p}$, mapping each $e_{i,i+1} \in \mathcal{L}_{n-1,p}$ to $e_{i+1,i+2} \in \mathcal{L}_{n,p}$ for all $1 \leq i \leq n - 2$. By the assumption on $\mathcal{L}_{n-1,p}$, we have that

$$\begin{aligned} (\iota \circ \iota^{-1})([\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{ln})]) &= \iota([\iota^{-1}(\varphi_{r,k}(a)(e_{ij})), \iota^{-1}(\varphi_{r,k}(a)(e_{ln}))]) = \\ &= \iota([\varphi_{r,k}(a)(e_{i-1,j-1}), \varphi_{r,k}(a)(e_{l-1,n-1})]) = \\ &= \begin{cases} \iota(\varphi_{r,k}(a)(e_{i-1,n-1})) = \varphi_{r,k}(a)(e_{in}) & : j = l \\ 0 & : j \neq l \end{cases} \end{aligned}$$

If $i = 1$, then

$$\begin{aligned} [\varphi_{r,k}(a)(e_{1j}), \varphi_{r,k}(a)(e_{ln})] &= [\varphi_{r,k}(a)(e_{1j}), \varphi_{r,k}(a)(e_{ln})] = \\ &= [\varphi_{r,k}(a)(e_{1j}), [\varphi_{r,k}(a)(e_{l,n-1}), \varphi_{r,k}(a)(e_{n-1,n})]]. \end{aligned}$$

By the Jacobi identity, we have that

$$\begin{aligned} &[\varphi_{r,k}(a)(e_{1j}), [\varphi_{r,k}(a)(e_{l,n-1}), \varphi_{r,k}(a)(e_{n-1,n})]] = \\ &= -[\varphi_{r,k}(a)(e_{n-1,n}), [\varphi_{r,k}(a)(e_{1j}), \varphi_{r,k}(a)(e_{l,n-1})]]. \end{aligned}$$

Now we use the inclusion $\iota' : \mathcal{L}_{n-1,p} \hookrightarrow \mathcal{L}_{n,p}$, where $\iota'(e_{i,i+1}) = e_{i,i+1}$ for all $1 \leq i \leq n - 1$, to obtain, same as above, that

$$[\varphi_{r,k}(a)(e_{n-1,n}), [\varphi_{r,k}(a)(e_{1j}), \varphi_{r,k}(a)(e_{l,n-1})]] = [\varphi_{r,k}(a)(e_{n-1,n}), [\varphi_{r,k}(a)(e_{1,n-1})]].$$

To continue, we need to prove the following fact: For any $i < j$ such that $j - i > 1$, we have $\varphi_{r,k}(a)(e_{ij}) = e_{ij}$. It is easy to see that this is true for

$h = j - i = 2$. For $h > 2$, $\varphi_{r,k}(a)(e_{ij}) = [\varphi_{r,k}(a)(e_{i,j-1}), \varphi_{r,k}(a)(e_{j-1,j})]$. By the assumption, $[e_{i,j-1}, \varphi_{r,k}(a)(e_{j-1,j})]$. If $j - 1 = k$ then $\varphi_{r,k}(a)(e_{j-1,j}) = e_{k,k+1} + ae_{k,k+r} \Rightarrow [e_{i,j-1}, \varphi_{r,k}(a)(e_{j-1,j})] = [e_{i,j-1}, e_{k,k+1} + ae_{k,k+r}] = e_{ij}$, so our induction step for $j - 1 = k$ holds, and it is easy to check that it holds also for $j - 1 = k + r$ and $j \notin \{k + 1, k + r + 1\}$. \square

Fix the two parameters $2 \leq r \leq n - 2$ and $0 \leq k \leq n - r$, and denote by $N_{r,k} := \{\varphi_{r,k}(a) : a \in \mathbb{Q}_p\} \subset G_n(\mathbb{Q}_p)$ the set of all automorphisms of this form. Also denote $N'_{n-2} := \{\varphi'_{n-2}(a) : a \in \mathbb{Q}_p\}$ and $N''_{n-2} := \{\varphi''_{n-2}(a) : a \in \mathbb{Q}_p\}$.

Proposition 0.0.1. *Let $N_{r,k}$, N'_{n-2} and N''_{n-2} be the subsets defined above, then*

1. $N_{r,k}, N'_{n-2}, N''_{n-2} \leq G_n^0(\mathbb{Q}_p)$.
2. $N_{r,k}, N'_{n-2}, N''_{n-2} \cong \mathbb{Q}_p$.

Proof. A simple check shows that these subsets are subgroups of $G_n^0(\mathbb{Q}_p)$. Define $\tau_{r,k} : \mathbb{Q}_p \rightarrow N_{r,k}$. For every $a, b \in \mathbb{Q}_p$, it is easy to see that the image of the sum, $\tau_{r,k}(a + b) = \tau_{r,k}(a) \cdot \tau_{r,k}(b)$, is the product of the images of a and b , and that $\tau_{r,k}^{-1}(I) = \{0\}$. \square

The following proposition follows from a simple computation.

Proposition 0.4. *Consider $\varphi_r \in N_r$.*

1. *If $r < n - 2$, denote by ψ_r the automorphism*

$$\psi_r := \varphi_r \circ \varphi_{r,n-r}(-a_{n-r,n-r}) \circ \cdots \circ \varphi_{r,1}(-a_{11}) \circ \varphi_{r,0}(-a_{r+1,1}).$$

Then $\psi_r \in N_{r+1}$.

2. *If $r = n - 2$, denote by ψ_{n-2} the automorphism*

$$\begin{aligned} \psi_{n-2} &:= \varphi_r \circ \varphi'_{n-2}(-a_{12}) \circ \varphi''_{n-2}(-a_{n-1,1}) \circ \\ &\quad \circ \varphi_{n-2,2}(-a_{n-2,2}) \circ \varphi_{n-2,1}(-a_{n-2,1}) \circ \varphi_{n-2,0}(-a_{n-2,0}). \end{aligned}$$

Then $\psi_{n-2} \in N_{n-1}$.

Corollary 0.5. *We have the following decompositions:*

1. *For all $2 \leq r < n - 2$, we have*

$$N_r = N_{r+1} \rtimes (N_{r,0} \rtimes (\cdots (N_{r,n-r-1} \rtimes N_{r,n-r}) \cdots)).$$

2. *For $r = n - 2$, we have*

$$N_{n-2} = N_{n-1} \rtimes (N_{n-2,0} \rtimes (N_{n-2,1} \rtimes (N_{n-2,2} \rtimes (N''_{n-2} \rtimes N'_{n-2}))))).$$

This is immediate from Proposition 0.4.

Corollary 0.5 provides a recursive decomposition of the unipotent radical $N(\mathbb{Q}_p)$ as an iterated semidirect product of N_{n-1} and subgroups isomorphic to \mathbb{Q}_p .

As we saw earlier, the calculation of $\zeta_{L_{n,p}}^\wedge(s)$ requires understanding $G_n(\mathbb{Z}_p)$ and $G_n^+(\mathbb{Q}_p)$ first. As $G_n(\mathbb{Z}_p)$ is a group, its structure is easily deduced from the above.

Proposition 0.6. *For all $n \geq 4$ the group $G_n^0(\mathbb{Z}_p)$ has the decomposition $G_n^0(\mathbb{Z}_p) = N(\mathbb{Z}_p) \rtimes H(\mathbb{Z}_p)$, where*

$$N(\mathbb{Z}_p) := M_{\binom{n}{2}}(\mathbb{Z}_p) \cap N(\mathbb{Q}_p),$$

$$H(\mathbb{Z}_p) := \{\text{diag}(\lambda_1, \lambda_2, \dots) : \lambda_1, \dots, \lambda_{n-1} \in \mathbb{Z}_p^*\}.$$

Moreover, $N(\mathbb{Z}_p)$ itself has the decomposition:

$$\begin{aligned} N(\mathbb{Z}_p) = \tilde{N}_2(\mathbb{Z}_p) = \tilde{N}_{n-1} \rtimes (\tilde{N}_{n-2,0} \rtimes (\tilde{N}_{n-2,1} \rtimes (\tilde{N}_{n-2,2} \rtimes (\tilde{N}_{n-2}'' \rtimes \tilde{N}_{n-2}')))) \rtimes \dots \\ \dots \rtimes (\tilde{N}_{2,0} \rtimes (\dots (\tilde{N}_{2,n-3} \rtimes \tilde{N}_{2,n-2}) \dots)), \end{aligned}$$

where $\tilde{N}_r = N_r \cap N(\mathbb{Z}_p)$ and $\tilde{N}_{r,k} = \{\varphi_{r,k}(a) : a \in \mathbb{Z}_p\}$.

By contrast, describing the structure of the monoid $G_n^+(\mathbb{Q}_p)$ is expected to be a substantial challenge.

By applying Fubini's theorem for semidirect products of topological groups [?, Proposition 28], we have that

$$\zeta_{L_{n,p}}^\wedge(s) = \int_{G_n^+(\mathbb{Q}_p)} |\det \varphi|_p^s d\mu_{G_n(\mathbb{Z}_p)} \varphi = \int_{H^+(\mathbb{Q}_p)} \left(\int_{N_h^+} |\det uh|_p^s d\mu_{N(\mathbb{Q}_p)} \right) d\mu_{H(\mathbb{Q}_p)},$$

where

$$H^+(\mathbb{Q}_p) := \{\text{diag}(\lambda_1, \dots, \lambda_{n-1}, \lambda_1 \lambda_2, \dots, \lambda_1 \lambda_2 \cdots \lambda_{n-2} \lambda_{n-1}) : \lambda_i \in \mathbb{Z}_p \setminus \{0\}\},$$

that is, $H^+(\mathbb{Q}_p)$ consists of all $h \in H(\mathbb{Q}_p)$ that appear in the decomposition $\varphi = uh$ for some $\varphi \in G_n^+(\mathbb{Q}_p)$, and, for a given $h \in H^+(\mathbb{Q}_p)$, we set $N_h^+ := \{u \in N(\mathbb{Q}_p) : uh \in G_n^+(\mathbb{Q}_p)\}$. The integrand is constant on N_h^+ , so computing the inner integral amounts to finding the measure of N_h^+ , which is complicated, but using the decomposition from Corollary 0.5, we can simplify N_h^+ at the price of replacing a single integral by multiple integrals.

Let $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_m$, where $m = \binom{n}{2}$, be an enumeration of the subgroups

$$\begin{aligned} N_{2,n-2}, N_{2,n-3}, \dots, N_{2,0}, N_{3,n-3}, \dots, N_{3,0}, \dots \\ \dots, N_{n-2,2}, N_{n-2,1}, N_{n-2,0}, N_{n-2}', N_{n-2}'', N_{n-1}. \end{aligned}$$

Every $\varphi \in G_n^0(\mathbb{Q}_p)$ can be written uniquely as $\varphi = u_m \cdots u_1 h$, where $u_i \in \mathcal{N}_i$. Thus, by Fubini

$$\zeta_{L_{n,p}}^\wedge(s) = \int_{H^+} \int_{\mathcal{N}_1^+(h)} \int_{\mathcal{N}_2^+(h, u_1)} \cdots \int_{\mathcal{N}_m^+(h, u_1, \dots, u_{m-1})} |\det h|_p^s d\mu_H d\mu_{\mathcal{N}_1} \cdots d\mu_{\mathcal{N}_m},$$

where each $\mu_{\mathcal{N}_i}$ is Haar measure on $\mathcal{N}_i(\mathbb{Q}_p)$ normalized so that $\mu_{\mathcal{N}_i}(\mathcal{N}_i(\mathbb{Z}_p)) = 1$, and

$$\mathcal{N}_i^+(h, u_1, \dots, u_{i-1}) := \{u_i \in \mathcal{N}_i : \exists u_{i+1}, \dots, u_m \text{ such that } u_m \cdots u_1 h \in G_n^+(\mathbb{Q}_p)\}.$$