1 Introduction

We have the following facts about expressions with v_{11}:

1. $v_{11} \geq -v_2 - \min\{v_1, v_3\}$
2. $p^{\min\{0,v_2+v_{11}\}} = \begin{cases} p^0 = 1 & v_{11} \geq -v_2 \\ p^{-j} & v_{11} = -v_2 - j, \text{where}\, j \geq 1 \end{cases}$
3. Following from the previous fact, we obtain that
$p^{\min\{0,v_2+v_{11}\}}\mu(a_{11}) = \begin{cases} p^0 p^{v_2} = p^{v_2} & v_{11} \geq -v_2 \\ p^{-j}(p^{v_2+j} - p^{v_2+j-1)} = p^{v_2}(1-p^{-1}) & v_{11} = -v_2 - j, \text{where}\, j \geq 1 \end{cases}$
4. $\alpha = \min\{\beta, v_{11} + v_2 + \gamma\}$, where $\beta = \min\{v_2, v_4\}$ and $\gamma = \max\{v_1, v_4\}$. Hence, we have the following settings:
   (a) $\beta = v_2$ and $\gamma = v_1$
   $\alpha = \begin{cases} v_2 & v_{11} \geq -v_1 \\ v_2 - j & v_{11} = -v_1 - j, \text{where}\, j \geq 1 \end{cases}$
   (b) $\beta = v_2$ and $\gamma = v_4$
   $\alpha = \begin{cases} v_2 & v_{11} \geq -v_4 \\ v_2 - j & v_{11} = -v_4 - j, \text{where}\, j \geq 1 \end{cases}$
   (c) $\beta = v_42$ and $\gamma = v_1$
   $\alpha = \begin{cases} v_4 & v_{11} \geq v_4 - v_1 - v_2 \\ v_4 - -j & v_{11} = v_4 - v_2 - v_1 - j, \text{where}\, j \geq 1 \end{cases}$
   (d) $\beta = v_42$ and $\gamma = v_41$
   $\alpha = \begin{cases} v_42 & v_{11} \geq -v_21 \\ v_42 - j & v_{11} = -v_21 - j, \text{where}\, j \geq 1 \end{cases}$

1. $\min\{v_1, v_3\} = v_3$, $\min\{v_2, v_4\} = v_2$, $\max\{v_1, v_4\} = v_1$
   Constraints
   (a) $v_{11} \geq -v_2 - v_3$
   (b) $\alpha = \min\{v_2, v_2 + v_{11} + v_1\}$

   Sub cases
   (a) $v_1 \geq v_2 + v_3 \Rightarrow -v_2 - v_3 \geq -v_1$
      i. $p^{\min\{0,v_2+v_{11}\}}\mu(a_{11}) = p^0 p^{v_2} + \sum_i^{v_3} p^{-i}(p^{v_2+i} - p^{v_2+i-1}) = p^{v_2} + \sum_i^{v_3} p^{v_2}(1-p^{-1}) =$
      $$= p^{v_2}(1 + v_3(1-p^{-1}))$$
      ii. $\alpha = \min\{v_2, v_2 + v_{11} + v_1\}$, and $v_{11} > -v_2 - v_3 \geq -v_1 \Rightarrow v_{11} + v_1 \geq 0 \Rightarrow v_2 \leq v_2 + v_{11} + v_1 \Rightarrow \alpha = v_2$.
      iii. $\Rightarrow S = p^{v_2}(1 + v_3(1-p^{-1}))(1 + v_2(1-p^{-1}))$.
   (b) $v_1 < v_2 + v_3 \Rightarrow -v_2 - v_3 < -v_1$
      i. $p^{\min\{0,v_2+v_{11}\}}\mu(a_{11}) = \begin{cases} p^{v_2}(1 + (v_1 - v_2)(1-p^{-1})) & v_{11} \geq -v_1 \\ p^{v_2}(v_2 + v_3 - v_1)(1-p^{-1})) & -v_2 - v_3 \leq v_{11} \leq -v_1 - 1 \end{cases}$
      ii. $\alpha = \begin{cases} v_2 & v_{11} \geq -v_1 \\ v_2 + v_{11} + v_1 & -v_2 - v_3 \leq v_{11} \leq -v_1 - 1 \end{cases}$
      iii. $\Rightarrow S = \begin{cases} p^{v_2}(1 + (v_1 - v_2)(1-p^{-1}))(1 + v_2(1-p^{-1})) & v_{11} \geq -v_1 \\ p^{v_2}[(v_2 - (v_1 - v_3))(1-p^{-1}) + [\binom{v_2}{2} - \binom{v_1-v_3}{2}](1-p^{-1})^2] & -v_2 - v_3 \leq v_{11} \leq -v_1 - 1 \end{cases}$.

2. $\min\{v_1, v_3\} = v_3$, $\min\{v_2, v_4\} = v_2$, $\max\{v_1, v_4\} = v_4$
   Constraints
   (a) $v_{11} \geq -v_2 - v_3$
   (b) $\alpha = \min\{v_2, v_2 + v_{11} + v_4\}$

   Sub cases

(a) $v_4 \geq v_2 + v_3 \Rightarrow -v_2 - v_3 \geq -v_4$

i. $p^{\min\{0,v_2+v_{11}\}}\mu(a_{11}) = p^0 p^{v_2} + \sum_i^{v_3} p^{-i}(p^{v_2+i} - p^{v_2+i-1}) = p^{v_2} + \sum_i^{v_3} p^{v_2}(1-p^{-1}) =$

$$= p^{v_2}(1 + v_3(1-p^{-1}))$$

ii. $\alpha = \min\{v_2, v_2 + v_{11} + v_4\}$, and $v_{11} > -v_2 - v_3 \geq -v_4 \Rightarrow v_{11} + v_1 \geq 0 \Rightarrow v_2 \leq v_2 + v_{11} + v_4 \Rightarrow \alpha = v_2$.

iii. $\Rightarrow S = p^{v_2}(1 + v_3(1-p^{-1}))(1 + v_2(1-p^{-1}))$.

(b) $v_4 < v_2 + v_3 \Rightarrow -v_2 - v_3 < -v_4$

i. $p^{\min\{0,v_2+v_{11}\}}\mu(a_{11}) = \begin{cases} p^{v_2}(1 + (v_4 - v_2)(1-p^{-1})) & v_{11} \geq -v_4 \\ p^{v_2}(v_2 + v_3 - v_4)(1-p^{-1})) & -v_2 - v_3 \leq v_{11} \leq -v_4 - 1 \end{cases}$

ii. $\alpha = \begin{cases} v_2 & v_{11} \geq -v_4 \\ v_2 + v_{11} + v_4 & -v_2 - v_3 \leq v_{11} \leq -v_4 - 1 \end{cases}$

iii. $\Rightarrow S = \begin{cases} p^{v_2}(1 + (v_4 - v_2)(1-p^{-1}))(1 + v_2(1-p^{-1})) & v_{11} \geq -v_4 \\ p^{v_2}[(v_2 - (v_4 - v_3))(1-p^{-1}) + [\binom{v_2}{2} - \binom{v_4 - v_3}{2}](1-p^{-1})^2] & -v_2 - v_3 \leq v_{11} \leq -v_4 - 1 \end{cases}$.

3. $\min\{v_1, v_3\} = v_3$, $\min\{v_2, v_4\} = v_4$, $\max\{v_1, v_4\} = v_1$

Constraints

(a) $v_{11} \geq -v_2 - v_3$

(b) $\alpha = \min\{v_4, v_2 + v_{11} + v_1\}$

Sub cases

(a) $v_3 \leq v_1 - v_4 \Rightarrow -v_3 \geq v_4 - v_1 \Rightarrow -v_2 - v_3 \geq v_4 - v_1 - v_2$

i. $p^{\min\{0,v_2+v_{11}\}}\mu(a_{11}) = p^0 p^{v_2} + \sum_i^{v_3} p^{-i}(p^{v_2+i} - p^{v_2+i-1}) = p^{v_2} + \sum_i^{v_3} p^{v_2}(1-p^{-1}) =$

$$= p^{v_2}(1 + v_3(1-p^{-1}))$$

ii. $\alpha = \min\{v_4, v_2 + v_{11} + v_1\}$, and $v_{11} \geq -v_2 - v_3 \geq v_4 - v_1 - v_2 \Rightarrow v_{11} + v_1 + v_2 \geq v_4 \Rightarrow \alpha = v_4$.

iii. $\Rightarrow S = p^{v_2}(1 + v_3(1-p^{-1}))(1 + v_4(1-p^{-1}))$.

(b) $v_3 > v_1 - v_4 \Rightarrow -v_3 < v_4 - v_1 \Rightarrow -v_2 - v_3 < v_4 - v_1 - v_2$

i. $p^{\min\{0,v_2+v_{11}\}}\mu(a_{11}) = \begin{cases} p^{v_2}(1 + (v_1 - v_4)(1-p^{-1})) & v_{11} \geq v_4 - v_1 - v_2 \\ p^{v_2}(v_2 + v_3 - v_4)(1-p^{-1})) & -v_2 - v_3 \leq v_{11} \leq v_4 - v_1 - v_2 - 1 \end{cases}$

ii. $\alpha = \begin{cases} v_4 & v_{11} \geq v_4 - v_1 - v_2 \\ v_4 + v_{11} + v_1 & -v_2 - v_3 \leq v_{11} \leq v_4 - v_1 - v_2 - 1 \end{cases}$

iii. $\Rightarrow S = \begin{cases} p^{v_2}(1 + (v_1 - v_4)(1-p^{-1}))(1 + v_2(1-p^{-1})) & v_{11} \geq v_4 - v_1 - v_2 \\ p^{v_2}[(v_2 - (v_4 - v_3))(1-p^{-1}) + [\binom{v_4}{2} - \binom{v_1 - v_3}{2}](1-p^{-1})^2] & -v_2 - v_3 \leq v_{11} \leq v_4 - v_1 - v_2 - 1 \end{cases}$.

4. $\min\{v_1, v_3\} = v_3$, $\min\{v_2, v_4\} = v_4$, $\max\{v_1, v_4\} = v_4$

Constraints

(a) $v_{11} \geq -v_2 - v_3$

(b) $\alpha = \min\{v_4, v_2 + v_{11} + v_4\}$

There are no sub cases here

(a) $p^{\min\{0,v_2+v_{11}\}}\mu(a_{11}) = \begin{cases} p^0 p^{v_2} & v_{11} \geq -v_2 \\ p^{v_2}(v_3(1-p^{-1})) & -v_2 - v_3 \leq v_{11} \leq -v_2 - 1 \end{cases}$

(b) $\alpha = \begin{cases} v_2 & v_{11} \geq -v_4 \\ v_2 + v_{11} + v_4 & -v_2 - v_3 \leq v_{11} \leq -v_4 - 1 \end{cases}$

(c) $\Rightarrow S = \begin{cases} p^{v_2}(1 + v_2(1-p^{-1})) & v_{11} \geq -v_2 \\ p^{v_2}[(v_3 - (v_1 - v_4))(1-p^{-1}) + [\binom{v_4}{2} - \binom{v_1 - v_3}{2}](1-p^{-1})^2] & -v_2 - v_3 \leq v_{11} \leq v_4 - v_1 - v_2 - 1 \end{cases}$.

5. $\min\{v_1, v_3\} = v_1$, $\min\{v_2, v_4\} = v_2$, $\max\{v_1, v_4\} = v_1$

Constraints

(a) $v_{11} \geq -v_2 - v_1$

(b) $\alpha = \min\{v_2, v_2 + v_{11} + v_1\}$

There are no sub cases here, because $-v_2 - v_1 \le -v_1$

(a) $p^{\min\{0,v_2+v_{11}\}}\mu(a_{11}) = \begin{cases} p^{v_2}(1+(v_1-v_2)(1-p^{-1})) & v_{11} \ge -v_1 \\ p^{v_2}v_2(1-p^{-1})) & -v_2-v_1 \le v_{11} \le -v_1-1 \end{cases}$

(b) $\alpha = \begin{cases} v_2 & v_{11} \ge -v_1 \\ v_2+v_{11}+v_1 & -v_2-v_1 \le v_{11} \le -v_1-1 \end{cases}$

(c) $\Rightarrow S = \begin{cases} p^{v_2}(1+(v_1-v_4)(1-p^{-1}))(1+v_2(1-p^{-1})) & v_{11} \ge -v_1 \\ p^{v_2}[v_2(1-p^{-1})+[\binom{v_2}{2}](1-p^{-1})^2] & -v_2-v_1 \le v_{11} \le -v_1-1 \end{cases}$.

6. $\min\{v_1, v_3\} = v_3$, $\min\{v_2, v_4\} = v_2$, $\max\{v_1, v_4\} = v_4$

Constraints

(a) $v_{11} \ge -v_2 - v_1$

(b) $\alpha = \min\{v_2, v_2 + v_{11} + v_4\}$

Sub cases

(a) $v_2 + v_1 \le v_4 \Rightarrow -v_2 - v_1 \ge -v_4$

i. $p^{\min\{0,v_2+v_{11}\}}\mu(a_{11}) = p^0 p^{v_2} + \sum_i^{v_1} p^{-i}(p^{v_2+i} - p^{v_2+i-1}) = p^{v_2} + \sum_i^{v_1} p^{v_2}(1-p^{-1}) =$

$$= p^{v_2}(1+v_1(1-p^{-1}))$$

ii. $\alpha = \min\{v_2, v_2+v_{11}+v_4\}$, and $v_{11} \ge -v_2 - v_1 \ge -v_4 \Rightarrow v_{11} + v_4 + v_2 \ge v_2 \Rightarrow \alpha = v_2$.

iii. $\Rightarrow S = p^{v_2}(1+v_1(1-p^{-1}))(1+v_2(1-p^{-1}))$.

(b) $v_2 + v_1 > v_4 \Rightarrow -v_4 > -v_2 - v_1$

i. $p^{\min\{0,v_2+v_{11}\}}\mu(a_{11}) = \begin{cases} p^{v_2}(1+(v_4-v_2)(1-p^{-1})) & v_{11} \ge -v_4 \\ p^{v_2}(v_2+v_1-v_4)(1-p^{-1})) & -v_2-v_1 \le v_{11} \le -v_4-1 \end{cases}$

ii. $\alpha = \begin{cases} v_2 & v_{11} \ge -v_4 \\ v_2+v_{11}+v_4 & -v_2-v_1 \le v_{11} \le v_4-v_1 \end{cases}$

iii. $\Rightarrow S = \begin{cases} p^{v_2}(1+(v_4-v_2)(1-p^{-1}))(1+v_2(1-p^{-1})) & v_{11} \ge v_4-v_1-v_2 \\ p^{v_2}[(v_2-(v_4-v_3))(1-p^{-1})+[\binom{v_2}{2}-\binom{v_4-v_1}{2}](1-p^{-1})^2] & -v_2-v_1 \le v_{11} \le -v_4-1 \end{cases}$.

7. $\min\{v_1, v_3\} = v_1$, $\min\{v_2, v_4\} = v_4$, $\max\{v_1, v_4\} = v_1$

Constraints

(a) $v_{11} \ge -v_2 - v_1$

(b) $\alpha = \min\{v_4, v_2 + v_{11} + v_1\}$

There are no sub cases here, because $-v_2 - v_1 \le v_4 - v_2 - v_1$

(a) $p^{\min\{0,v_2+v_{11}\}}\mu(a_{11}) = \begin{cases} p^{v_2}(1+(v_1-v_4)(1-p^{-1})) & v_{11} \ge v_4-v_2-v_1 \\ p^{v_2}v_4(1-p^{-1})) & -v_2-v_1 \le v_{11} \le v_4-v_2-v_1-1 \end{cases}$

(b) $\alpha = \begin{cases} v_4 & v_{11} \ge v_4-v_2-v_1 \\ v_2+v_{11}+v_1 & -v_2-v_1 \le v_{11} \le v_4-v_2-v_1-1 \end{cases}$

(c) $\Rightarrow S = \begin{cases} p^{v_2}(1+(v_1-v_4)(1-p^{-1}))(1+v_4(1-p^{-1})) & v_{11} \ge v_4-v_2-v_1 \\ p^{v_2}[v_4(1-p^{-1})+[\binom{v_4}{2}](1-p^{-1})^2] & -v_2-v_1 \le v_{11} \le v_4-v_2-v_1-1 \end{cases}$.