The pro-isomorphic zeta-functions of some nilpotent Lie algebras over integer rings

Research proposal for a Ph.D. Thesis
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Contents

Abstract

Let G be any group. For any natural number $n \in \mathbb{N}$, let $a_n(G)$ be the number of subgroups $H \leq G$, such that $[G : H] = n$. Assume G is finitely-generated, then $a_n(G) < \infty$, and we can define a Dirichlet series of the form $\zeta_G(s) := \sum_{n=1}^{\infty} a_n(G)n^{-s}$, where $s \in \mathbb{C}$. Assume, in addition, that G is also nilpotent and torsion-free, then this function has some properties of the Riemann zeta-function ζ, such as the Euler decomposition of ζ into a product of local factors indexed by primes. A version of this zeta-function counts pro-isomorphic subgroups, and an analogous function may be defined for appropriate Lie rings. We study here the pro-isomorphic zeta-functions for a family of nilpotent Lie rings of unbounded nilpotency class. We shall compute the automorphism groups of these Lie rings explicitly, prove uniformity of the local factors of the pro-isomorphic zeta-functions, and aim to determine them explicitly.

1 Scientific Background

1.1 Introduction

Although we will work with Lie algebras, for motivation we first present analogous and more natural questions in the context of groups.

Proposition 1.1.1. *Let G be any finitely generated group, and let $n \in \mathbb{N}$ be any natural number. Then there is a finite number of subgroups $H \leq G$, such that $[G : H] = n$*

This proposition gives rise to an entire branch of group theory, called **subgroup growth**. We denote by $a_n(G) = a_n^{\leq}(G) := |\{H \leq G : [G : H] = n\}|$ the number of subgroups of G of index n, and look at the sequence $\{a_n(G)\}_{n=1}^{\infty}$. The subject of subgroup growth aims to relate the properties of this sequence to the algebraic structure of G. We denote by $a_n^{\trianglelefteq}(G) := |\{H \trianglelefteq G : [G : H] = n\}|$ the number of **normal** subgroups of G. Now we define another type of subgroups of G.

Definition 1.1.2. *Let G be any group, and let $\mathcal{N}(G) := \{N_k \trianglelefteq G\}_{k \in I}$ be the set of all normal subgroups of G. We define a partial order on $\mathcal{N}(G)$ by reverse inclusion, that is, for every two indices i, j we say that $i \leq j$ if and* *only if $N_j \subseteq N_i$, hence for every $i \leq j$ there exists a natural projection map $\pi_{ji} : {}^{G}/_{N_j} \to {}^{G}/_{N_i}$. The inverse limit*

$$\widehat{G} = \varprojlim \{{}^{G}/_{N_k}\}_{k \in I} := \{(h_k)_{k \in I} \in \prod_{k \in I} {}^{G}/_{N_k} : \pi_{ji}(h_j) = h_i, \forall i \leq j\}$$

*is called the **profinite closure** of G.*

Definition 1.1.3. *Let G be any group. A subgroup $H \leq G$ is called **pro-isomorphic** if $\widehat{H} \cong \widehat{G}$.*

We denote by $\hat{a_n}(G) := |\{H \leq G : \widehat{H} \cong \widehat{G}, [G : H] = n\}|$ the number of **pro-isomorphic** subgroups of G.

Definition 1.1.4. *Let G be a finitely-generated group, and let $* \in \{\leq, \trianglelefteq, \wedge\}$, then we define zeta-functions of the form $\zeta_G^*(s) := \sum_{n=1}^{\infty} a_n^*(G) n^{-s}$.*

Proposition 1.1.5. *Let G be a $\mathcal{T}$-group, i.e. finitely-generated, nilpotent and torsion-free group, and let $* \in \{\leq, \trianglelefteq, \wedge\}$, then the zeta-functions for G have the following attributes:*

***Polynomial growth and convergence**. $a_n(G) \leq Cn^b$, for some b, C constant, thus, for all $*$, we get that $\zeta_G^*(s)$ converges on some right half-plane $\mathrm{Re}(s) > \alpha$, for α constant. The abscissa of convergence is $\alpha^* := \inf\{\alpha : \zeta_G^*(s) < \infty, \mathrm{Re}(s) > \alpha\}$, and we have that $\alpha^* \in \mathbb{Q}$.*

***Euler decomposition**. For all $*$, we have that $\zeta_G^*(s) = \prod_p \zeta_{G,p}^*(s)$, where $\zeta_{G,p}^*(s) = \sum_{k=0}^{\infty} a_{p^k}^*(G) p^{-ks}$.*

***Rationality**. For all $*$, all p, there is a rational function $W_p^* \in \mathbb{Q}(X)$, such that $\zeta_{G,p}^*(s) = W_p^*(p^{-s})$.*

***Functional equation**. Suppose we have **finite uniformity**, i.e. we have r rational functions $W_1^*(X,Y), \ldots, W_r^*(X,Y) \in \mathbb{Q}(X,Y)$, such that for all p, there is some $1 \leq i \leq r$ such that $\zeta_{G,p}^*(s) = W_i^*(p, p^{-s})$. We say W_i^* satisfies a **functional equation** if $W_i^*(X^{-1}, Y^{-1}) = X^a Y^b W_i^*(X,Y)$, where $a, b \in \mathbb{N} \cup \{0\}$, $X^a Y^b$ being called the **symmetry factor**. Thus, if G is a $\mathcal{T}$-group, then $\zeta_{G,p}^{\leq}(s)$ satisfies a functional equation, for all but finitely many p, with the same symmetry factor. If G is a $\mathcal{T}$-group of nilpotency class 2, same is true for $\zeta_{G,p}^{\trianglelefteq}(s)$.*

If any zeta-function, which is a special case of the Dirichlet series, has some properties of convergence on some subset of $\mathbb{C}$, one may reconstruct its coefficients $a_n^*(G)$, the number of subgroups of our interest, using the **Perron's formula**, which is an implementation of a **reverse Mellin transform**, as discussed, for example, in [7], but this discussion is out of the scope of our research.

This research concentrates on the growth of **pro-isomorphic** subgroups defined above, hence we shall restrict our further discussion to the pro-isomorphic case only. For example, we look at the additive group of integers $G = (\mathbb{Z}, +)$, for which, every subgroup $H \leq \mathbb{Z}$ is of the form $H = n\mathbb{Z} = \langle n \rangle$, for some $n \in \mathbb{N}$, which means that $H \cong \mathbb{Z}$, as both are infinite cyclic groups, and so, $\widehat{H} \cong \widehat{\mathbb{Z}}$. Since we have only one subgroup of index n, for every $n \in \mathbb{N}$, then $a_n(\mathbb{Z}) = \hat{a_n}(\mathbb{Z}) = 1$. Thus, its pro-isomorphic zeta-function is $\hat{\zeta_{\mathbb{Z}}}(s) = \sum_{i=1}^{\infty} n^{-s} = \zeta(s)$, the Riemann zeta-function, which is known to converge for $\mathrm{Re}(s) > 1$. We recall that the Riemann zeta-function decomposes into an infinite product of local zeta-functions, that is, $\zeta(s) = \prod_p \zeta_p(s) = \prod_p \sum_{k=0}^{\infty} p^{-ks} = \prod_p \frac{1}{1-p^{-s}}$, where the product runs over all the prime numbers.

1.2 Linearization

We want to transfer the ideas from the above discussion about groups to a linear context, where we can use tools from linear algebra. Hence, for finitely-generated torsion-free nilpotent groups G, we associate nilpotent Lie algebras over $\mathbb{Z}$. This, in general, is called the **Mal'cev correspondence**. If L is a $\mathbb{Z}$-Lie algebra, namely a free $\mathbb{Z}$-module of finite rank with a Lie bracket operation, then consider the number $\hat{a_n}(L)$ of subalgebras $M \leq L$, where $n = [L : M]$, such that $M \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers, for all primes p, and it is also known that $\hat{a_n}(L) < \infty$ for all $n \in \mathbb{N}$. The Dirichlet series $\hat{\zeta_L}(s) := \sum_{n=1}^{\infty} \hat{a_n}(L) n^{-s}$, is called the **pro-isomorphic zeta-function** of L. By the Mal'cev correspondence, to every finitely-generated, nilpotent, torsion-free group G, one may associate a Lie algebra $L = L(G)$, such that $\hat{\zeta_{G,p}}(s) = \hat{\zeta_{L,p}}(s)$, for all but finitely many primes p. If G has nilpotency class 2, one may obtain the equality for all primes. For this L, choose a basis $B = \{b_1, \ldots, b_r\}$, where $r = \mathrm{rank} L$. Let $\mathcal{L}_p = L \otimes_{\mathbb{Z}} \mathbb{Q}_p$, for any p. This is a $\mathbb{Q}_p$-Lie algebra, and our choice of basis allows us to identify the automorphism group $G(\mathbb{Q}_p) = Aut_{\mathbb{Q}_p}(\mathcal{L}_p)$ with a subgroup of $GL_r(\mathbb{Q}_p)$. Note that $\mathcal{L}_p$ contains a $\mathbb{Z}_p$-lattice, $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. If $\varphi \in G(\mathbb{Q}_p)$, then $\varphi(L_p) = L_p$ if and only if $\varphi \in G(\mathbb{Z}_p) = G(\mathbb{Q}_p) \cap GL_r(\mathbb{Z}_p)$. Here $GL_r(\mathbb{Z}_p)$ is the group of $r \times r$ matrices which are invertible over $\mathbb{Z}_p$.

Similarly, $\varphi(L_p) \subseteq L_p$ if and only if $\varphi \in G^+(\mathbb{Q}_p) := G(\mathbb{Q}_p) \cap \mathcal{M}_r(\mathbb{Z}_p)$, where $\mathcal{M}_r(\mathbb{Z}_p)$ is the collection of $r \times r$ matrices with entries in $\mathbb{Z}_p$. Note that $G^+(\mathbb{Q}_p)$ is a monoid, not a group.

Denote by $G(\mathbb{Z}_p)g$, where $g \in G^+(\mathbb{Q}_p)$, a right-coset of $G(\mathbb{Z}_p)$. One checks that the monoid $G^+(\mathbb{Q}_p)$ is a disjoint union of right-cosets of $G(\mathbb{Z}_p)$.

The discussion above reveals the construction we base our research upon. We observe that there is a bijection between the set $G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)$ of right-cosets of $G(\mathbb{Z}_p)$ and the set $\{M \leq L_p : M \cong L_p\}$ of L_p-subalgebras which are isomorphic to L_p itself. This bijection takes $G(\mathbb{Z}_p)g$ to $M = \varphi(L_p)$. For any $\varphi \in G(\mathbb{Z}_p)g$, this is well-defined. One checks that for every $\psi \in G(\mathbb{Z}_p)g$, we have that $\psi(L_p) = \varphi(L_p) = M$. We end this part, as a preparation for the final part of this technical background review, with the following result, which states that for each right-coset $G(\mathbb{Z}_p)g$, if $M = \varphi(L_p)$, where $\varphi \in G(\mathbb{Z}_p)g$, then $[L_p : M] = |\det\varphi|_p^{-1}$, where $|\det\varphi|_p$ is the p-adic norm of $\det\varphi$, and therefore,

$$\hat{\zeta_{L,p}}(s) = \sum_{\substack{M \leq L_p \\ M \cong L_p}} [L_p : M]^{-s} = \sum_{G(\mathbb{Z}_p)\varphi \in G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)} |\det\varphi|_p^s.$$

Theorem 1.2.1. *Let $* \in \{\leq, \trianglelefteq, \wedge\}$, then $\zeta^*_{L,p}(s)$ is rational, i.e. there is a rational function in one variable $W_p \in \mathbb{Q}(X)$ such that $\zeta^*_{L,p}(s) = W_p(p^{-s})$, for all p prime.*

After establishing rationality for the local zeta-functions in the Euler decomposition of $\zeta^*_{L,p}(s)$, one may study the uniformity or finite-uniformity of $\zeta^*_L(s)$ itself, where ζ^*_L is said to be **finitely-uniform** if the local zeta-functions in its Euler decomposition are represented by a finite set of r rational functions, for all but finitely many p, and **uniform** if $r = 1$. The uniformity of the zeta-functions of some algebras is established in the work of Grunewald, Segal and Smith, see [5]. We aim to show that our target $\mathbb{Z}$-Lie algebra is uniform, or at least finitely-uniform.

1.3 p-adic Integration

Definition 1.3.1. *Let Γ be a locally compact topological group, i.e. for all $\gamma \in \Gamma$, there is an open neighborhood of $\gamma \in U_\gamma$ and a compact subset K_γ, such that $U_\gamma \subset K_\gamma$. Then there is a measure μ, with the following property: for any measurable subset $U \subseteq \Gamma$ and any $\gamma \in \Gamma$, $\mu(U\gamma) = \mu(U)$, where $U\gamma := \{u\gamma : u \in U\}$. Such a measure μ is called a **right Haar measure**, and is unique up to multiplication by a non-zero constant.*

Following this definition of a right Haar measure, we claim that for every prime number p, the group $G(\mathbb{Q}_p)$ is a locally compact topological group. We also claim that a right Haar measure μ on G can be normalized such that $\mu(G(\mathbb{Z}_p)) = 1$, and the normalized measure of any right-coset of $G(\mathbb{Z}_p)$ equals to the measure of $G(\mathbb{Z}_p)$ itself, i.e. for every $g \in G^+(\mathbb{Q}_p)$, we have that $\mu(G(\mathbb{Z}_p)g) = \mu(G(\mathbb{Z}_p)) = 1$. Following this, we calculate the p-adic norm of the determinant of every L_p-automorphism, as a p-adic integral over our measure space. Given any L_p-automorphism in some right-coset $\varphi \in G(\mathbb{Z}_p)\varphi$, we have that $|\det\varphi|_p^s = \int_{G(\mathbb{Z}_p)\varphi} |\det\varphi|_p^s d\mu$, because $\mu(G(\mathbb{Z}_p)\varphi) = 1$, and $|\det\varphi|_p^{-1}$ is fixed on $G(\mathbb{Z}_p)\varphi$.

To calculate our target function, we observe that $\hat{\zeta_{L,p}}(s) = \sum_{G(\mathbb{Z}_p)\varphi \in G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)} |\det\varphi|_p^s = \sum_{G(\mathbb{Z}_p)\varphi \in G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)} \int_{G(\mathbb{Z}_p)\varphi} |\det\varphi|_p^s d\mu = \int_{G^+(\mathbb{Q}_p)} |\det\varphi|_p^s d\mu$.

This calculation of the local ζ_p-function as a p-adic integral was established by the work of du Sautoy and Lubotzky, in [3], and we aim to study this integral and its attributes, where the integrand and domain of integration come from our target $\mathbb{Z}_p$-Lie algebra.

2 Research Goals and Methodology

2.1 The Lie algebras $L_{n,p}$

Let e_{ij} be an $n \times n$ matrix, in which all the elements are zero, except for the element in row i and column j which has 1. On the set $E = \{e_{ij} : 1 \leq i \leq n-1 \wedge i+1 \leq j \leq n\}$ we define a bracket operation: for every $1 \leq k, l \leq n-1$, define $[e_{k,k+1}, e_{l,l+1}] := e_{k,k+1}e_{l,l+1} - e_{l,l+1}e_{k,k+1}$. Let $\mathcal{R}$ be some commutative ring, then the standard operation of $\mathcal{R}$ on E through scalar multiplication, along with the defined bracket operation, form a nilpotent $\mathcal{R}$-Lie algebra. Considering $\mathcal{R} = \mathbb{Z}$, we obtain a nilpotent $\mathbb{Z}$-Lie algebra of strictly upper triangular matrices over $\mathbb{Z}$, which we denote by L_n, with the bracket operation defined above as its Lie bracket. As discussed above, this $\mathbb{Z}$-Lie algebra can be extended to a $\mathbb{Z}_p$-algebra, which we denote by $L_{n,p}$, and then to a $\mathbb{Q}_p$-algebra, which we denote by $\mathcal{L}_{n,p}$. In all the following, we may set n and p, and denote $L := L_{n,p}$ and $\mathcal{L} := \mathcal{L}_{n,p}$, for abbreviation. One checks that the set of matrices of the form e_{ij}, where $i < j$, spans the whole $\mathbb{Z}$-Lie algebra L and is $\mathbb{Z}$-linearly independent, hence it forms a basis for L as a free module over $\mathbb{Z}$, namely $B := \{e_{12}, e_{13}, \ldots, e_{1n}, e_{23}, \ldots, e_{2,n}, \ldots, e_{n-1,n}\}$, which we call the **standard basis** of L. One also checks easily that $r = \text{rank} L = |B| = \binom{n}{2}$, which is the number of elements above the main diagonal for every $n \in \mathbb{N}$. To this standard basis we apply a linear order by defining $e_{ij} < e_{kl}$ if $j - i < l - k$ or if $j - i = l - k$ and $i < k$. In other words, we apply an order that divides B to elements of the quotients ${}^{L}/_{\gamma_2}, {}^{\gamma_2}/_{\gamma_3}, \ldots, {}^{\gamma_{n-2}}/_{\gamma_{n-1}}, \gamma_{n-1}$. The same goes also for the extensions of L, namely $L_{n,p}$ and $\mathcal{L}_{n,p}$. The target of our research is studying the zeta-function $\zeta^{\wedge}_{L,p}$ on these $\mathbb{Z}_p$-Lie algebras, and the related constructions.

Remark 2.1.1. *Let $\mathcal{R}$ be a commutative ring, and let $\mathcal{U}_n(\mathcal{R})$ be the group of $n \times n$ upper unitriangular matrices over $\mathcal{R}$, with the standard matrix multiplication as the group operation. Denote by $\mathcal{U}_{n,p} = \mathcal{U}_n(\mathbb{Z}_p)$ the unitriangular matrix group over $\mathbb{Z}_p$, then by Mal'cev correspondence, $\zeta^{\wedge}_{\mathcal{U}_{n,p}}(s) = \zeta^{\wedge}_{L_{n,p}}(s)$, for all but finitely many primes p. This relates the subject of research to the motivation presented at the beginning of this paper.*

2.2 Research goals

The project consists of three major steps:

1. **Computing the automorphism group of the $\mathbb{Q}_p$-Lie algebras $\mathcal{L}$, for all $n \in \mathbb{N}$ and all primes p.**
2. **Showing that the pro-isomorphic zeta-functions $\zeta^{\wedge}_{L_{n,p}}(s)$ are uniform for all $n \in \mathbb{N}$.**
3. **Giving an explicit uniform formula for the zeta-functions $\zeta^{\wedge}_{L_{n,p}}(s)$ for specific values of n, if not for all $n \in \mathbb{N}$.** Specifically for $n = 5$ we aim to continue the work of Mark N. Berman, who proved that $\zeta^{\wedge}_{L_{5,p}}(s)$ is uniform.

As we elaborate further, steps 1 and 2 are already known entirely for $n \leq 5$, and step 3 is known for $n \leq 4$. We start with the first step of calculating $Aut_{\mathbb{Q}_p}(\mathcal{L})$. These automorphism groups have been studied before from a different point of view, and there are classical results showing that any automorphism may be expressed as a product of automorphisms of a specific type; see, for instance, the main result of Gibbs [4]. These results are not explicit enough for our purposes; indeed, the submonoid $G^+(\mathbb{Q}_p)$ arises for us as the domain of integration of a p-adic integral. In order to calculate this integral, we need to decompose the automorphism group $G(\mathbb{Q}_p)$ into a repeated semi-direct product of groups with a simple structure.

After we have analyzed the structure of $G(\mathbb{Z}_p)$, we will need to construct the monoid $G^+(\mathbb{Q}_p)$ and its $G(\mathbb{Z}_p)$ right-cosets, as we have seen above. This will give us both the function to integrate, which is $\det \varphi$ for every $G(\mathbb{Z}_p)$ right-coset $G(\mathbb{Z}_p)\varphi$, and the domain of integration, which is the monoid $G^+(\mathbb{Q}_p)$. We will use this information to analyze the behavior of the p-adic integral we have described above and prove that its calculation depends only on p, thus showing that the $\hat{\zeta_{L,p}}$-function is uniform.

2.3 $L_{n,p}$-Lie algebras for $n \geq 5$

Mark N. Berman, in his doctoral thesis [1], has displayed an explicit formula for $\hat{\zeta_{L_{4,p}}}$, and proved that $\hat{\zeta_{L_{5,p}}}$ is indeed uniform. We aim to generalize his work to prove that $\hat{\zeta_{L_{n,p}}}$ is uniform for all n. We also aim to compute $\hat{\zeta_{\mathcal{L}_{n,p}}}(s)$ explicitly for all n, or at least to obtain explicit formulas for some $n \geq 5$, and specifically for $n = 5$. By analyzing carefully Berman's work on $L_{4,p}$ and $L_{5,p}$, we gain the basic understanding of the expected structure of the local zeta-functions in the general case. We begin our discussion of the first goal, which is computing $G(\mathbb{Z}_p)$, by first recalling that for every $v \in L_{n,p}$, where $n \geq 3$, we present $\varphi(v)$ as the multiplication of v by a matrix from the right $\varphi(v) = vM$. As stated earlier, M is an $r \times r$ matrix, where $r = \text{rank} L_{n,p} = \binom{n}{2}$, whose lines are set by the order we have defined above, i.e. considering the standard ordered basis

$$\mathcal{B} = \{e_{12}, e_{23}, \ldots, e_{n-1,n}, e_{13}, \ldots, e_{n-2,n}, \ldots, e_{1n}\}$$

then M is the following matrix,

$$M = \begin{pmatrix} \varphi(e_{12}) \\ \varphi(e_{23}) \\ \varphi(e_{n-1,n}) \\ \hdashline \varphi(e_{13}) \\ \vdots \\ \varphi(e_{n-2,n}) \\ \hdashline \vdots \\ \hdashline \varphi(e_{1n}) \end{pmatrix}$$

Given an $\mathcal{L}$-automorphism φ, we denote by $\varphi_k : \gamma_k\mathcal{L} \to \gamma_k\mathcal{L}$ the operation of φ on all the $n-k$ elements of the lower central series starting from k, that is, we consider only the images

$$\varphi(e_{1,1+k}), \varphi(e_{2,2+k}), \ldots, \varphi(e_{n-k,n}), \varphi(e_{1,2+k}), \ldots, \varphi(e_{n-k-1,n}), \ldots, \varphi(e_{1n})$$

For every φ_k, we have the induced map denoted by φ_{kk}, from the quotient algebra ${}^{\gamma_k}\!/_{\gamma_{k+1}}$ to itself, defined by $\varphi_{kk}(e_{l,l+k} + \gamma_{k+1}\mathcal{L}) := a_{1,1+k}e_{1,1+k} + a_{2,2+k}e_{2,2+k} + \cdots + a_{n-k,n}e_{n-k,n} + z_{k+1}$, where $z_{k+1} \in \gamma_{k+1}\mathcal{L}$, for every $1 \leq l \leq n-k$. Clearly, φ_{kk} is well-defined, since $\varphi_k(\gamma_k\mathcal{L}) = \gamma_k\mathcal{L}$, for every $1 \leq k \leq n-1$. Following this division of $\mathcal{L}$ by the lower central series and its quotients, we view M as a block matrix,

$$M = \left(\begin{array}{c|c|c|c|c} M_{11} & M_{12} & \ldots & M_{1,n-2} & M_{1,n-1} \\ \hline M_{21} & M_{22} & \ldots & M_{2,n-2} & M_{2,n-1} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline M_{n-1,1} & M_{n-1,2} & \ldots & M_{n-1,n-2} & M_{n-1,n-1} \end{array}\right)$$

each block is denoted by $M_{kl} \in \mathcal{M}_{m\times r}(\mathbb{Q}_p)$, where $m = \dim {}^{\gamma_k}\!/_{\gamma_{k+1}}$ and $r = \dim {}^{\gamma_l}\!/_{\gamma_{l+1}}$. From this, we can understand that the blocks on the main diagonal of M, which are the induced quotient maps defined above, are square matrices $\varphi_{kk} = M_{kk} \in \mathcal{M}_{n-k}(\mathbb{Q}_p)$. A trivial observation is that since all the elements of the lower central series of $\mathcal{L}$ are characteristic subalgebras, then ${}^{\varphi(\gamma_k\mathcal{L})}\!/_{\gamma_k\mathcal{L}} = 0$, which means that all the matrix blocks M_{kl}, where $k < l$, must be zero, therefore M has the form,

$$M = \left(\begin{array}{c|c|cc|c|c} M_{11} & M_{12} & M_{13} & \ldots & M_{1,n-2} & M_{1,n-1} \\ \hline 0 & M_{22} & M_{23} & \ldots & M_{2,n-2} & M_{2,n-1} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \ldots & M_{2,n-2} & M_{2,n-1} \\ \hline 0 & 0 & 0 & \ldots & 0 & M_{n-1,n-1} \end{array}\right)$$

2.4 Preliminary results

We have made progress towards step 1 of our research program, namely determining the automorphism groups $G(\mathbb{Q}_p)$. The first result in this direction appears already in the thesis of M. N. Berman[1, Prop. 3.6].

Proposition 2.4.1. *Let $\varphi \in G(\mathbb{Q}_p)$ be a $\mathcal{L}$-automorphism, and M its representing matrix, divided into matrix blocks, as shown earlier. Then, $M_{11} \in \mathcal{M}_{n-1}(\mathbb{Q}_p)$ is either diagonal or anti-diagonal.*

Define an involution $\eta \in G(\mathbb{Q}_p)$ by $\eta(e_{ij}) := (-1)^{j-i-1}e_{n+1-j,n+1-i}$, for $1 \leq i \leq n-1$ and $i+1 \leq j \leq n$. Replacing φ by $\varphi \circ \eta$, we assume without loss of generality that M_{11} is diagonal. We check, for any $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{Q}_p^*$, that the diagonal matrix

$$h = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_1\lambda_2, \lambda_2\lambda_3 \ldots, \lambda_{n-2}\lambda_{n-1}, \ldots, \lambda_1\lambda_2 \cdots \lambda_{n-2}\lambda_{n-1})$$

represents an automorphism of $\mathcal{L}$. Thus, multiplying φ from the right by a unique such automorphism, we may now assume that M has 1's on the diagonal. By this print we know $H(\mathbb{Q}_p) := \{\mathrm{diag}(\lambda_1, \lambda_2, \ldots) : \lambda_1, \lambda_2, \cdots \in \mathbb{Q}_p^*\}$ is the **reductive part** of $\mathbb{Q}_p$, while $N(\mathbb{Q}_p) := \{\varphi \in G(\mathbb{Q}_p)$ with 1's in diagonal $\}$ is the **unipotent radical** of $G(\mathbb{Q}_p)$. Every $g \in G(\mathbb{Q}_p)$ has a unique decomposition $g = \mathfrak{n}\mathfrak{h}$, with $\mathfrak{n} \in N(\mathbb{Q}_p)$, $\mathfrak{h} \in H(\mathbb{Q}_p)$. We aim to determine the structure of the unipotent radical $N(\mathbb{Q}_p)$ by decomposing it into a semidirect product of abelian subgroups. We can simplify the domain of integration, for the p-adic integral that we aim to calculate, at the price of replacing a single integral by multiple integrals. As we saw earlier, the calculation of $\hat{\zeta}_L(s)$ requires computing $G(\mathbb{Z}_p)$ and $G^+(\mathbb{Q}_p)$ first. Assuming we have already computed $G(\mathbb{Q}_p)$, based on the strategy that we have presented above, we need to identify $G(\mathbb{Z}_p)$ as a subgroup of $G(\mathbb{Q}_p)$, which is expected not to be difficult, and continue from there to identify the monoid $G^+(\mathbb{Q}_p)$, which is expected to be a substantial challenge. By applying **Fubini's theorem** for semidirect products of topological groups, we have that

$$\hat{\zeta}_{\mathcal{L}}(s) = \int_{G^+(\mathbb{Q}_p)} |\det \varphi|_p^s d\mu_{G(\mathbb{Z}_p)\varphi} = \int_{H^+(\mathbb{Q}_p)} \left(\int_{N_{\mathfrak{h}}^+} |\det \mathfrak{n}\mathfrak{h}|_p^s d\mu_{N(\mathbb{Q}_p)} \right) d\mu_{H(\mathbb{Q}_p)}$$

where $H^+(\mathbb{Q}_p)$ consists of all $\mathfrak{h} \in H(\mathbb{Q}_p)$ that appear in the decomposition $\varphi = \mathfrak{n}\mathfrak{h}$ for some $\varphi \in G^+(\mathbb{Q}_p)$, and, for a given $\mathfrak{h} \in H^+(\mathbb{Q}_p)$, we set $N_{\mathfrak{h}}^+(\mathbb{Q}_p) := \{\mathfrak{n} \in N(\mathbb{Q}_p) : \mathfrak{n}\mathfrak{h} \in G^+(\mathbb{Q}_p)\}$. The integrand of the inner integral is constant, so the integral amounts to computing the measure of the set $N_h^+(\mathbb{Q}_p)$. The advantage that we gain by this decomposition is that it simplifies the calculation of the integral. The integral function $|\det \mathfrak{n}\mathfrak{h}|_p^s = |\det \mathfrak{h}|_p^s$ depends only on $\mathfrak{h}$, so computing the inner integral amounts to finding the measure of $N_{\mathfrak{h}}^+(\mathbb{Q}_p)$. For the unipotent matrix $\mathfrak{n}$, the determinant is 1, for the digonal matrix $\mathfrak{h}$, we have the following proposition,

Proposition 2.4.2. *Let $n \geq 2$, and let*

$$h = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_1\lambda_2, \lambda_2\lambda_3 \ldots, \lambda_{n-2}\lambda_{n-1}, \ldots, \lambda_1\lambda_2 \cdots \lambda_{n-2}\lambda_{n-1})$$

as defined above, then $\det \mathfrak{h} = \prod_{i=1}^{n-1} \lambda_i^{i(n-i)}$.

We show this by induction on n. We conclude this section with the observation that the sets $N_h^+(\mathbf{Q_p})$ arising in our computation are quite complicated. We will decompose $N(\mathbb{Q}_p)$ into an iterated semidirect product of a large number of subgroups, each abelian with a simple structure. This will decompose the integral over $N_h^+(\mathbb{Q}_p)$ into a multiple integral that can be computed explicitly, given a suitable combinatorical framework. Hence, we strive to decompose $N_n(\mathbb{Q}_p)$ itself into a product of finitely many simpler subgroups, $N_n(\mathbb{Q}_p) = N_n(\mathbb{Q}_p)_1 \rtimes_\phi N_n(\mathbb{Q}_p)_2 \rtimes_\phi \cdots \rtimes_\phi N_n(\mathbb{Q}_p)_{m_n}$, where m_n is the number of subgroups in the decomposition of $N_n(\mathbb{Q}_p)$, for every $n \in \mathbb{N}$, which means that

$$\int_{G^+(\mathbb{Q}_p)} |\det \varphi|_p^s d\mu_{G(\mathbb{Q}_p)} = \int_{H_n^+(\mathbb{Q}_p)} \left(\int_{\mathcal{N}_{m_n}^+} \cdots \left(\int_{\mathcal{N}_3^+} \left(\int_{\mathcal{N}_2^+} \left(\int_{\mathcal{N}_1^+} |\det \varphi|_p^s d\mu_{\mathcal{N}_1} \right) d\mu_{\mathcal{N}_2} \right) d\mu_{\mathcal{N}_3} \right) \cdots d\mu$$

where we denote $\mathcal{N}_i := N_n(\mathbb{Q}_p)_i$ and $\mathcal{N}_i^+ := N_n^+(\mathbb{Q}_p)_i$, for every $1 \leq i \leq m_n$. One checks that every $\mathcal{N}_i^+$ depends on $h, n_1, n_2, \ldots, n_{i-1}$, if $\varphi = n_{m_n} \cdots n_2 n_1 h$, where $h \in H_n^+(\mathbb{Q}_p)$ and $n_k \in \mathcal{N}_k$, for every $1 \leq k \leq i-1$. All the subgroups in the decomposition of $N_n(\mathbb{Q}_p)$ are obviously unipotent as well, which means that their determinants are also 1. This means that computing the inner integrals amounts to determining the measure of the sets $\mathcal{N}_i^+$ in terms of $h, n_1, n_2, \ldots, n_{n-1}$.

2.5 Base Extension

Let K be a number field of degree $d = [K : \mathbb{Q}]$, and let $\mathcal{O}_K$ be its ring of integers. Let L be a $\mathbb{Z}$-Lie algebra of rank r. By base extension we can consider $L \otimes_{\mathbb{Z}} \mathcal{O}_K$ as a $\mathbb{Z}$-Lie algebra of rank rd, and by extension of scalars we can consider also $L_{K,p} = (L \otimes_{\mathbb{Z}} \mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ as a $\mathbb{Q}_p$-Lie algebra of the same rank. Berman-Glazer-Schein give a criterion in [2], under which the pro-isomorphic zeta-function of $L_{K,p}$ can be calculated without a significant extra effort relative to that of L_p itself. Note that the criterion does not necessarily apply for all p. We shall research whether the criterion applies to the $\mathbb{Q}_p$-Lie algebras $\mathcal{L}$ of our work. If so, then $\hat{\zeta}_{\mathcal{L}\otimes\mathcal{O}_K,p}$ will be finitely-uniform. In other words, for each of the finitely many decomposition types of a prime in $\mathcal{O}_K$, there is a rational function in two variables $W \in \mathbb{Q}(X, Y)$ such that $\hat{\zeta}_{\mathcal{L}\otimes\mathcal{O}_K,p}(s) = W(p, p^{-s})$ for all p of that decomposition type.

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