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1.1

Proposition 1.1.1. *Let* $\mathfrak{m} \in G_n(\mathbb{Q}_p)$ *be a coefficient matrix representing an* $\mathcal{L}_{n,p}$*-automorphism, and let* $\mathfrak{n} \in \mathcal{N}_n(\mathbf{Q}_p)$ *and* $\mathfrak{h} \in \mathcal{H}_n(\mathbf{Q}_p)$ *be the two matrices in the unique decomposition* $\mathfrak{m} = \mathfrak{n}\mathfrak{h}$*, then* $m \in G(\mathbb{Z}_p)$ *if and only if* $n \in \mathcal{N}(\mathbb{Z}_p)$ *and* $h \in \mathcal{H}(\mathbb{Z}_p)$.

Proof. We prove only the non-trivial direction. Assume $\mathfrak{m} \in G(\mathbf{Z}_p)$. By the construction of the automorphism coefficient matrices, the diagonal of $\mathfrak{m}$ is $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_1\lambda_2, \lambda_2\lambda_3, \ldots, \lambda_{n-2}\lambda_{n-1}, \ldots, \lambda_1\lambda_2 \cdots \lambda_{n-1}$, where $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{Q}_p$, but we assumed $\mathfrak{m} \in G_n(\mathbb{Z}_p)$, hence we must have all the λ coefficients and their products in $\mathbb{Z}_p$ itself. Moreover, if $\mathfrak{m} \in G_n(\mathbb{Z}_p)$, it means that $\det \mathfrak{m} \in \mathbb{Z}_p^*$, in words, the determinant of $\mathfrak{m}$ is invertible in $\mathbb{Z}_p$. The matrix $\mathfrak{h}$ has precisely the same diagonal, and determinant, as $\mathfrak{m}$, and zero in all the other cells. As seen earlier, the determinant of $\mathfrak{h}$, and of $\mathfrak{m}$, for every $n \geq 2$, is $\prod_{k=1}^{n-1} \lambda_k^{k(n-k)}$, so if $\det \mathfrak{h}$ is invertible, hence the valuation $v(\det \mathfrak{h}) = v(\prod_{k=1}^{n-1} \lambda_k^{k(n-k)}) = \sum_{k=1}^{n-1} v(\lambda_k^{k(n-k)}) = \sum_{k=1}^{n-1} \sum_{l=1}^{k(n-k)} v(\lambda_k) = 0$, but $\lambda_k \in \mathbb{Z}_p$, for every $1 \leq k \leq n-1$, hence $v(\lambda_k) \geq 0$, but if the total sum is zero, then we must have $v(\lambda_k) = 0$, for every k, which means that $\lambda_k \in \mathbb{Z}_p^*$, hence $\mathfrak{h} \in G_n(\mathbb{Z}_p)$. But if $\mathfrak{m}, \mathfrak{h} \in G_n(\mathbb{Z}_p)$, then $\mathfrak{n} = \mathfrak{m}\mathfrak{h}^{-1} \in G_n(\mathbb{Z}_p)$. $\square$