The pro-isomorphic zeta-functions of some nilpotent Lie algebras over integer rings

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Abstract

Let G be any group. For any natural number $n \in \mathbb{N}$, let $a_{\overline{n}}^{\leq}(G)$ be the number of subgroups $H \leq G$ such that $[G : H] = n$. If G is finitely-generated, then $a_{\overline{n}}^{\leq}(G) < \infty$, and we can define a Dirichlet series of the form $\zeta_G(s) := \sum_{n=1}^{\infty} a_{\overline{n}}^{\leq}(G)n^{-s}$, where $s \in \mathbb{C}$. Assume, in addition, that G is also nilpotent and torsion-free, then this function has some properties of the Riemann zeta-function ζ, such as the Euler decomposition of ζ into a product of local factors indexed by primes. A version of this zeta-function counts pro-isomorphic subgroups, and an analogous function may be defined for appropriate Lie rings. We study here the pro-isomorphic zeta-functions for a family of nilpotent Lie rings of unbounded nilpotency class. We shall compute the automorphism groups of these Lie rings explicitly and use them to prove uniformity of the local factors of the pro-isomorphic zeta-functions, and aim to determine them explicitly.

1 Scientific Background

1.1 Introduction

Although we will work with Lie algebras, for motivation we first present analogous and more natural questions in the context of groups.

Proposition 1.1.1. *Let G be any finitely generated group, and let $n \in \mathbb{N}$ be any natural number. Then there is a finite number of subgroups $H \leq G$ such that $[G : H] = n$.*

This proposition gives rise to an entire branch of group theory, called **subgroup growth**. We denote by

$$a_{\overline{n}}^{\leq}(G) := |\{H \leq G : [G : H] = n\}|$$

the number of subgroups of G of index n and consider the sequence $\{a_{\overline{n}}^{\leq}(G)\}_{n=1}^{\infty}$. The subject of subgroup growth aims to relate the properties of this sequence to the algebraic structure of G. We denote by

$$a_{\overline{n}}^{\trianglelefteq}(G) := |\{H \trianglelefteq G : [G : H] = n\}|$$

the number of **normal** subgroups of G. We now define another type of subgroups of G.

Definition 1.1.2. *Let G be a group. A subgroup $H \leq G$ is called* **pro-isomorphic** *if $\widehat{H} \cong \widehat{G}$, where $\widehat{H}$ and $\widehat{G}$ are the profinite closures of H and G, respectively.*

We denote by

$$a_n^\wedge(G) := |\{H \leq G : \widehat{H} \cong \widehat{G}, [G : H] = n\}|$$

the number of **pro-isomorphic** subgroups of G.

Now we define generating functions determined by these sequences.

Definition 1.1.3. *Let G be a finitely-generated group, and let $* \in \{\leq, \trianglelefteq, \wedge\}$, then we define the zeta-function $\zeta_G^*(s)$ to be the Dirichlet series $\sum_{n=1}^{\infty} a_n^*(G) n^{-s}$.*

Proposition 1.1.4. *Let G be a $\mathcal{T}$-group, i.e. finitely-generated, nilpotent and torsion-free group, and let $* \in \{\leq, \trianglelefteq, \wedge\}$. Then the zeta-functions for G have the following properties:*

*   **Polynomial growth and convergence.** *There exist constants b and C such that $a_n^{\leq}(G) \leq C n^b$ for all $n \in \mathbb{N}$. Hence $\zeta_G^*(s)$ converges on some right half plane $\operatorname{Re} s > \alpha$. The abscissae of convergence*

$$\alpha^* := \inf\{\alpha : \zeta_G^*(s) \text{ converges on } \operatorname{Re} s > \alpha\}$$

are known to be rational numbers.

*   **Euler decomposition.** *We have that $\zeta_G^*(s) = \prod_p \zeta_{G,p}^*(s)$, where the product runs over all primes p and $\zeta_{G,p}^*(s) = \sum_{k=0}^{\infty} a_{p^k}^*(G) p^{-ks}$.*

*   **Rationality.** *For all primes p, there is a rational function $W_p^* \in \mathbb{Q}(X)$ such that $\zeta_{G,p}^*(s) = W_p^*(p^{-s})$.*

Definition 1.1.5. *Let $\zeta_{L,p}^*$ be a* **finitely uniform** *zeta-function, i.e. there are rational functions $W_1^*(X,Y), \dots, W_r^*(X,Y) \in \mathbb{Q}(X,Y)$, such that for all p, there is some $1 \leq i \leq r$ such that $\zeta_{G,p}^*(s) = W_i^*(p, p^{-s})$. We say W_i^* satisfies a* **functional equation** *if $W_i^*(X^{-1}, Y^{-1}) = \pm X^a Y^b W_i^*(X,Y)$, where $a, b \in \mathbb{N} \cup \{0\}$. The factor $X^a Y^b$ is being called the* **symmetry factor**.

Let G be a $\mathcal{T}$-group, then $\zeta_{G,p}^{\leq}(s)$ satisfies a functional equation, for all but finitely many p, with the same symmetry factor. If G is a $\mathcal{T}$-group of nilpotency class 2, the same is true for $\zeta_{G,p}^{\trianglelefteq}(s)$.

There are specific examples of $\mathcal{T}$-groups of nilpotency class 3, such that $\zeta_{G,p}^{\triangleleft}(s)$ is **uniform**, i.e. $\zeta_{G,p}^{\triangleleft}(s) = W(p, p^{-s})$, but $W(p, p^{-s})$ does not satisfy a functional equation.

In the case of pro-isomorphic zeta-functions, there are results showing that under restrictive conditions, $\zeta_{G,p}^{\wedge}(s)$ is uniform and satisfies a functional equation, see, for instance, [3]. However, there are examples of G of nilpotency class 4 such that $\zeta_{G,p}^{\wedge}(s)$ is uniform but does not have a functional equation.

If any Dirichlet series, in particular $\zeta_G^*(s)$, convergences on some subset of $\mathbb{C}$, one may reconstruct its coefficients $a_n^*(G)$, the sequence of our interest, using Perron's formula, which is an implementation of a reverse Mellin transform; see, for instance, [8].

As an example, consider the group $G = \mathbb{Z}$, for which every finite-index subgroup $H \leq \mathbb{Z}$ is of the form $H = n\mathbb{Z} = \langle n \rangle$ for some $n \in \mathbb{N}$. Thus $H \cong \mathbb{Z}$, as both are infinite cyclic groups, and so $\widehat{H} \cong \widehat{\mathbb{Z}}$. Since we have only one subgroup of index n for every $n \in \mathbb{N}$, then $a_n^*(\mathbb{Z}) = 1$, for $* \in \{\leq, \triangleleft, \wedge\}$. Thus, its pro-isomorphic zeta-function is $\zeta_{\mathbb{Z}}^{\wedge}(s) = \sum_{i=1}^{\infty} n^{-s} = \zeta(s)$, the Riemann zeta-function, which is known to converge for $\operatorname{Re} s > 1$. It is known that the Riemann zeta-function decomposes into an infinite product of local zeta-functions, that is, $\zeta(s) = \prod_p \zeta_p(s) = \prod_p \sum_{k=0}^{\infty} p^{-ks} = \prod_p \frac{1}{1-p^{-s}}$, where the product runs over all the prime numbers.

This research concentrates on the growth of **pro-isomorphic** subgroups defined above, hence we shall restrict our further discussion to the pro-isomorphic case only.

1.2   Linearization

We may transfer the ideas from the above discussion about groups to a linear context, where we can use tools from linear algebra. If L is a $\mathbb{Z}$-Lie algebra, namely a free $\mathbb{Z}$-module of finite rank with a Lie bracket operation, then consider the number $a_n^{\wedge}(L)$ of subalgebras $M \leq L$ of index n such that $M \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for all primes p. It is known that $a_n^{\wedge}(L) < \infty$ for all $n \in \mathbb{N}$. The Dirichlet series $\zeta_L^{\wedge}(s) := \sum_{n=1}^{\infty} a_n^{\wedge}(L) n^{-s}$, is called the **pro-isomorphic zeta-function** of L. By the Mal'cev correspondence, to every $\mathcal{T}$-group G one may associate a Lie algebra $L = L(G)$ such that $\zeta_{G,p}^{\wedge}(s) = \zeta_{L,p}^{\wedge}(s)$ for all but finitely many primes p. If G has nilpotency class 2, one may obtain the equality for all primes.

Let L be a $\mathbb{Z}$-Lie algebra. Choose a basis $(b_1, \ldots, b_r)$, where $r = \operatorname{rank} L$.

Let $\mathcal{L}_p = L \otimes_{\mathbb{Z}} \mathbb{Q}_p$, for any prime p. This is a $\mathbb{Q}_p$-Lie algebra, and our choice of basis allows us to identify the automorphism group $G(\mathbb{Q}_p) = Aut_{\mathbb{Q}_p}(\mathcal{L}_p)$ with a subgroup of $GL_r(\mathbb{Q}_p)$. Note that $\mathcal{L}_p$ contains a $\mathbb{Z}_p$-lattice $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. If $\varphi \in G(\mathbb{Q}_p)$, then $\varphi(L_p) = L_p$ if and only if $\varphi \in G(\mathbb{Z}_p) = G(\mathbb{Q}_p) \cap GL_r(\mathbb{Z}_p)$. Similarly, $\varphi(L_p) \subseteq L_p$ if and only if $\varphi \in G^+(\mathbb{Q}_p) := G(\mathbb{Q}_p) \cap M_r(\mathbb{Z}_p)$, where $M_r(\mathbb{Z}_p)$ is the collection of $r \times r$ matrices with entries in $\mathbb{Z}_p$. Note that $G^+(\mathbb{Q}_p)$ is a monoid, not a group.

We consider automorphisms as acting from the right, hence, considering right cosets of $G(\mathbb{Z}_p)$, we observe that there is a bijection between the set $G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)$ of right cosets of $G(\mathbb{Z}_p)$ and the set $\{M \leq L_p : M \cong L_p\}$ of L_p-subalgebras which are isomorphic to L_p itself. This bijection takes $G(\mathbb{Z}_p)g$ to $M = \varphi(L_p)$, for $\varphi \in G(\mathbb{Z}_p)g$. Observe that $[L_p : M] = |\det \varphi|_p^{-1}$, where $|\cdot|_p$ is the p-adic norm on $\mathbb{Q}_p$, and therefore,

$$\zeta_{L,p}^{\wedge}(s) = \sum_{\substack{M \leq L_p \\ M \cong L_p}} [L_p : M]^{-s} = \sum_{G(\mathbb{Z}_p)\varphi \in G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)} |\det \varphi|_p^s \qquad (1)$$

.

1.3 p-adic Integration

We fix a specific right Haar measure μ on the group $G(\mathbb{Q}_p)$ by normalizing $\mu(G(\mathbb{Z}_p)) = 1$. Now we are able to express $\zeta_{L,p}^{\wedge}(s)$ as a p-adic integral, as first done by du Sautoy and Lubotzky [4]. Since $|\det g|_p^s$ is constant on right cosets of $G(\mathbb{Z}_p)$, we have $|\det \varphi|_p^s = \int_{G(\mathbb{Z}_p)\varphi} |\det g|_p^s d\mu(g)$. By (1), we have

$$\zeta_{L,p}^{\wedge}(s) = \sum_{G(\mathbb{Z}_p)\varphi \in G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)} |\det \varphi|_p^s = \sum_{G(\mathbb{Z}_p)\varphi \in G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)} \int_{G(\mathbb{Z}_p)\varphi} |\det g|_p^s d\mu(g) =$$

$$= \int_{G^+(\mathbb{Z}_p)} |\det g|_p^s d\mu(g)$$

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2   Research Goals and Methodology

2.1   The Lie algebras L_n

Let R be a commutative ring, and let $L_n(R)$ be the set of strictly upper triangular $n \times n$ matrices with elements in R. The commutator Lie bracket

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$[A, B] = AB - BA$ gives $L_n(R)$ the structure of a nilpotent R-Lie algebra. Set $L_n = L_n(\mathbb{Z})$ and $\mathcal{L}_n = L_n(\mathbb{Q})$, while $L_{n,p} = L_n(\mathbb{Z}_p)$ and $\mathcal{L}_{n,p} = L_n(\mathbb{Q}_p)$.

Let $e_{ij} \in L_n$ be the matrix with 1 in the (i,j) entry and 0 elsewhere. Then $B := \{e_{ij} : 1 \leq i < j \leq n\}$ is a basis of L_n, which we order as follows: $(e_{12}, \ldots, e_{n-1,n}, e_{13}, \ldots, e_{n-2,n}, e_{14}, \ldots, e_{n-3,n}, \ldots, e_{1n})$. In other words we order the elements of B so that $e_{ij} < e_{kl}$ if either $j - i < l - k$ or if $j - i = l - k$ and $i < k$. Note that $\gamma_1 L_n, \ldots, \gamma_{n-1} L_n$ are spanned by final segments of B. The aim of our research is to study the pro-isomorphic zeta function of L_n. Berman has computed $\zeta^\wedge_{L_{4,p}}(s)$ explicitly for all p and shown that $\zeta^\wedge_{L_{5,p}}(s)$ are uniform.

Remark 2.1.1. *Let $U_n(R)$ be the group of $n \times n$ upper unitriangular matrices over R. By the Mal'cev correspondence, $\zeta^\wedge_{U_n(\mathbb{Z}_p)}(s) = \zeta^\wedge_{L_{n,p}}(s)$, for all but finitely many primes p. This relates the subject of our research to the group-theoretic motivation presented at the beginning of this paper.*

2.2 Research goals

The project consists of three major steps:

1. Computing the automorphism group $G_n(\mathbb{Q}_p)$ of the $\mathbb{Q}_p$-Lie algebras $\mathcal{L}_{n,p}$, for all $n \in \mathbb{N}$ and all primes p, and understanding the submonoid $G_n^+(\mathbb{Q}_p)$.

2. Showing that the pro-isomorphic zeta-functions $\zeta^\wedge_{L_{n,p}}(s)$ are uniform, or finitely uniform, for all $n \in \mathbb{N}$. We expect the method used by Berman in the case $n = 5$ to generalize.

3. Giving an explicit uniform formula for the zeta-functions $\zeta^\wedge_{L_{n,p}}(s)$ for specific values of n, if not for all $n \in \mathbb{N}$. We aim to compute an explicit formula for $\zeta^\wedge_{L_{n,p}}(s)$ for all $n \geq 4$ and all primes p in terms of a suitable combinatorical framework that is compatible with the complicated structure of $G_n^+(\mathbb{Q}_p)$. We will study whether $\zeta^\wedge_{L_{n,p}}(s)$ have functional equations and in particular whether they satisfy the conjecture of [3], which says that if $\zeta^\wedge_{L,p}(s)$ has functional equations and L is graded, then the exponent b in the symmetry factor must be the weight of a minimal grading of L. Note that L_n is naturally graded by its lower central series, with minimal grading of weight $\sum_{i=1}^{n-1} i(n-i)$. In the event that we do not succeed in this task in general, we still expect to complete

Berman's unfinished calculation in the case $n = 5$. In this case, the combinatorics is just simple enough to handle directly, possibly with some assistance from a computer.

Automorphism groups of Lie algebras, and in particular $G_n(\mathbb{Q}_p)$, have been studied before from a different point of view, and there are classical results showing that any automorphism may be expressed as a product of automorphisms of a specific type; see, for instance, the main result of Gibbs [5]. These statements are not explicit enough for our purposes; indeed, the submonoid $G_n^+(\mathbb{Q}_p)$ arises for us as the domain of integration of a p-adic integral. In order to calculate this integral, it will be useful to decompose the automorphism group $G_n(\mathbb{Q}_p)$ into a repeated semi-direct product of groups with a simple structure.

We will need to analyze the monoid $G_n^+(\mathbb{Q}_p)$ and its $G_n(\mathbb{Z}_p)$ right-cosets, as we have seen above. This will give us both the integrand, which is $|\det \varphi|_p^s$, and the domain of integration, which is the monoid $G_n^+(\mathbb{Q}_p)$. It is expected that $G_n^+(\mathbb{Q}_p)$ will have a complicated structure, and expressing $\zeta_{L_{n,p}}^\wedge(s)$ explicitly in a way that allows us to verify functional equations will depend on finding a suitable combinatorical structure. However, we expect that the method of Berman's proof of uniformity in the case $n = 5$, which relies on generalizations of results by Stanley on generating functions of polyhedral cell complexes, will generalize to all $n \geq 4$.

Let K be a number field of degree $d = [K : \mathbb{Q}]$, and let $\mathcal{O}_K$ be its ring of integers. Let L be a $\mathbb{Z}$-Lie algebra of rank r. We can naturally consider the base extension $L_K = L \otimes_\mathbb{Z} \mathcal{O}_K$ as a $\mathbb{Z}$-Lie algebra of rank rd, and by extension of scalars we can consider also $\mathcal{L}_{K,p} = (L \otimes_\mathbb{Z} \mathcal{O}_K) \otimes_\mathbb{Z} \mathbb{Q}_p$ as a $\mathbb{Q}_p$-Lie algebra of the same rank. If $\mathcal{L}_p$ has a certain property originally introduced by Segal, which we call **rigidity**, then a simple modification of the calculation of $\zeta_{L,p}^\wedge(s)$ gives $\zeta_{L_K,p}^\wedge(s)$ for all K without any substantial extra effort. Berman, Glazer and Schein [2] give a criterion for rigidity in terms of the centralizers of elements of L_p. Note that the criterion does not necessarily apply for all p. We shall study whether the criterion applies to the $\mathbb{Q}_p$-Lie algebras $\mathcal{L}_{n,p}$ of our work. If so, then uniformity of $\zeta_{L,p}^\wedge(s)$ will imply finite uniformity of $\zeta_{L_K,p}^\wedge(s)$. Indeed, for each of the finitely many decomposition types of a prime in $\mathcal{O}_K$, there is a rational function in two variables $W \in \mathbb{Q}(X, Y)$ such that $\zeta_{\mathcal{L} \otimes \mathcal{O}_K,p}^\wedge(s) = W(p, p^{-s})$ for all p of that decomposition type.

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2.3   Representing matrices of automorphisms of $\mathcal{L}_{n,p}$

To sum up what is already known about automorphisms of $\mathcal{L}_{n,p}$, let $\varphi \in G_n(\mathbb{Q}_p)$. Let M be the matrix of φ with respect to the ordered basis B, where M acts from the right on row vectors. Thus:

$$M = \begin{pmatrix} \varphi(e_{12}) \\ \varphi(e_{23}) \\ \varphi(e_{n-1,n}) \\ \hdashline \varphi(e_{13}) \\ \vdots \\ \varphi(e_{n-2,n}) \\ \hdashline \vdots \\ \hdashline \varphi(e_{1n}) \end{pmatrix}$$

Since the elements $\{\gamma_k \mathcal{L}_{n,p}\}_{k=1}^{n-1}$ of the lower central series are characteristic subalgebras, M has the block-diagonal form

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} & \dots & M_{1,n-2} & M_{1,n-1} \\ 0 & M_{22} & M_{23} & \dots & M_{2,n-2} & M_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M_{2,n-2} & M_{2,n-1} \\ 0 & 0 & 0 & \dots & 0 & M_{n-1,n-1} \end{pmatrix}$$

where each block M_{ij} is an $(n-i) \times (n-j)$ matrix.

Note that $\begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}$ gives an element of $Aut_{\mathbb{Q}_p}(\mathcal{L}_{n,p}/\gamma_3\mathcal{L}_{n,p})$. Berman determined this automorphism group for all n and p [1, Prop. 3.6]. It follows that the block M_{11} must be either diagonal or anti-diagonal.

2.4   Preliminary results

We have made progress towards step 1 of our research program, namely determining the automorphism groups $G_n(\mathbb{Q}_p)$, based on Berman's results stated above. Define $\eta \in G_n(\mathbb{Q}_p)$ by $\eta(e_{ij}) := (-1)^{j-i-1}e_{n+1-j,n+1-i}$, for $1 \le i < j \le n$. Replacing φ by $\varphi \circ \eta$, if necessary we may assume without loss of generality that M_{11} is diagonal. Indeed, let $G_n^0(\mathbb{Q}_p) \le G_n(\mathbb{Q}_p)$ be the subgroup of automorphisms with diagonal block M_{11}, then $G_n(\mathbb{Q}_p) =$

$G_n^0(\mathbb{Q}_p) \coprod G_n(\mathbb{Q}_p)\eta$, and as in [4] we may replace the domain of integration in (1) by $G_n^0(\mathbb{Q}_p) \cap G_n^+(\mathbb{Q}_p)$ after a suitable renormalization of the Haar measure. We proceed to determine $G_n^0(\mathbb{Q}_p)$. We check, for any $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{Q}_p^*$, that the diagonal matrix

$$h = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_1\lambda_2, \lambda_2\lambda_3 \ldots, \lambda_{n-2}\lambda_{n-1}, \ldots, \lambda_1\lambda_2\cdots\lambda_{n-2}\lambda_{n-1})$$

represents an automorphism of $\mathcal{L}_{n,p}$. Thus every $g \in G_n^0(\mathbb{Q}_p)$ has a unique decomposition $g = uh$, where h is of the above form and u has 1's on the diagonal. The collection $H(\mathbb{Q}_p)$ of h's as above is the reductive part of $G_n^0(\mathbb{Q}_p)$. The collection $N(\mathbb{Q}_p)$ of u's as above is the unipotent radical of $G_n^0(\mathbb{Q}_p)$. We aim to determine the structure of the unipotent radical $N(\mathbb{Q}_p)$ by decomposing it into a semidirect product of abelian subgroups. For all $2 \leq r \leq n-1$ we denote by $N_r \leq N(\mathbb{Q}_p)$ the subgroup of all automorphisms of $\mathcal{L}_{n,p}$, such that $M_{11} = I_n$ and $M_{12} = M_{13} = \cdots = M_{1,r-1} = 0$. It is easy to see that

$$N_{n-1} = \left\{ \begin{pmatrix} I_{n-1} & 0 & \cdots & 0 & M_{1,n-1} \\ 0 & I_{n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_2 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\}$$

where $M_{1,n-1} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{pmatrix} \in \mathbb{Q}_p^{n-1}$ is the collection of all $u \in N(\mathbb{Q}_p)$ that

induce the trivial automorphism on $\mathcal{L}_{n,p}/Z(\mathcal{L}_{n,p})$.

Proposition 2.4.1. *Consider the $(n-1) \times (n-r)$ matrix $M_{1r} = (a_{ij})$. Then,*

1. *Let $2 \leq r < n-2$. If $a_{ij} \neq 0$, then either $i = j$ or $i = j + r - 1$, and we have the relation $a_{i+r,i+1} = -a_{ii}$.*

2. *Let $r = n-2$. If $a_{ij} \neq 0$, then either $i = j$ or $i = j + r - 1$ or $(i,j) \in \{(1,2),(n-1,1)\}$, with the same relation as above.*

Proof. From the relation $[\varphi(e_{k,k+1}), \varphi(e_{l,l+1})] = 0$ where $l > k+1$, we deduce that $a_{ij} \neq 0$ only if either $i = j$ or $i = j + r - 1$ or $(i,j) \in \{(r+1,1),(r+$

$2,1), (n-r-2, n-r), (n-r-1, n-r)\}$. If $r < n-2$ then it follows from the conditions,

$$[\varphi(e_{n-r-2,n-r-1}), \varphi(e_{n-r-2,n-r})] = 0$$
$$[\varphi(e_{n-r-1,n-r}), \varphi(e_{n-r-2,n-r})] = 0$$
$$[\varphi(e_{r+1,r+2}), \varphi(e_{r+1,r+3})] = 0$$
$$[\varphi(e_{r+2,r+3}), \varphi(e_{r+1,r+3})] = 0$$

that the four exceptional cases cannot occur. Similar considerations eliminate two of the exceptional cases when $r = n-2$. $\square$

Proposition 2.4.2. *Let $2 \le r \le n-2$, and let $0 \le k \le n-r$, and let $a \in \mathbb{Q}_p$.*

1. *There is an automorphism $\varphi_{r,k}(a) \in G_n^0(\mathbb{Q}_p)$ determined by*

$$\varphi_{r,k}(a)(e_{i,i+1}) = \begin{cases} e_{k,k+1} + ae_{k,k+r} & : i = k \\ e_{k+r,k+r+1} - ae_{k+1,k+r+1} & : i = k+r \\ e_{i,i+1} & : i \notin \{k, k+r\} \end{cases}$$

2. *Suppose that $r = n-2$, let $(k,l) \in \{(1,2), (n-1,1)\}$, and let $a \in \mathbb{Q}_p$. There is an automorphism $\varphi_{n-2,k,l}(a) \in G_n^0(\mathbb{Q}_p)$ determined by*

$$\varphi_{n-2,k,l}(a)(e_{i,i+1}) = \begin{cases} e_{k,k+1} + ae_{l,l+r} & : i = k \\ e_{i,i+1} & : i \ne k \end{cases}$$

We denote $\varphi'_{n-2}(a) := \varphi_{n-2,1,2}(a)$ and $\varphi''_{n-2}(a) := \varphi_{n-2,n-1,1}(a)$.

Proof. We need to verify that for all $1 \le i < j \le n$ and $1 \le l < m \le n$ we have the following relations

$$[\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{lm})] = \begin{cases} \varphi_{r,k}(a)(e_{im}) & : j = l \\ -\varphi_{r,k}(a)(e_{lj}) & : i = m \\ 0 & : \text{otherwise} \end{cases}$$

We can verify explicitly for $n = 4$ that these relations are true. For $n > 4$, one considers the injection map $\iota : \mathcal{L}_{n-1,p} \hookrightarrow \mathcal{L}_{n,p}$, mapping each $e_{i,i+1} \in \mathcal{L}_{n-1,p}$ to $e_{i,i+1} \in \mathcal{L}_{n,p}$ for all $1 \le i \le n-2$. Let $2 \le r \le n-3$ and $0 \le k \le$

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$n - 1 - r$ and $1 \leq i < j \leq n - 1$ and $1 \leq l < m \leq n - 1$, and consider $A = [\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{lm})]$. We have that $A = [\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{lm})] = [\varphi_{r,k}(a)(\iota(e_{ij})), \varphi_{r,k}(a)(\iota(e_{lm}))] = \iota([\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{lm})]) = \iota(A)$, but A satisfies the above relations by the assumption on $\mathcal{L}_{n-1,p}$, so $\iota(A)$ must satisfy these relations also as a product in $\mathcal{L}_{n,p}$. One verifies that considering $j = n$ or $m = n$ or $r = n - 2$ or $k = n - r$ also preserves these relations. □

Fix some r, k and denote by $N_{r,k} := \{\varphi_{r,k}(a) : a \in \mathbb{Q}_p\} \subset G_n(\mathbb{Q}_p)$ the set of all automorphisms of this form. It is easy to see that $N_{r,k} < G_n(\mathbb{Q}_p)$ and $N_{r,k} \cong \mathbb{Q}_p$. Also denote by $N'_{n-2} := \{\varphi'_{n-2}(a) : a \in \mathbb{Q}_p\}$ and by $N''_{n-2} := \{\varphi''_{n-2}(a) : a \in \mathbb{Q}_p\}$, and it is easy to see that the same applies for N'_{n-2} and N''_{n-2}.

Proposition 2.4.3. *For all $\varphi \in N_r$*

1. *when $r < n - 2$, denote by ψ the automorphism*

$$\psi := \varphi \circ \varphi_{r,n-r}(-a_{n-r,n-r}) \circ \cdots \circ \varphi_{r,1}(-a_{11}) \circ \varphi_{r,0}(-a_{r1})$$

   where $M_{1r} = (a_{ij})$ is the $(1, r)$ block of φ. Then $\psi \in N_r$ and the $(1, r)$ block of ψ is 0.

2. *when $r = n - 2$, denote by ψ_{n-2} the automorphism*

$$\psi_{n-2} := \varphi \circ \varphi'_{n-2}(-a_{12}) \circ \varphi''_{n-2}(-a_{n-1,1}) \circ$$

$$\circ \varphi_{n-2,2}(-a_{n-2,2}) \circ \varphi_{n-2,1}(-a_{n-2,1}) \circ \varphi_{n-2,0}(-a_{n-2,0})$$

   where $M_{1r} = (a_{ij})$ is the $(1, r)$ block of φ. Then $\psi_{n-2} \in N_{n-2}$ and the $(1, r)$ block of ψ_{n-2} is 0.

Proof. For ψ it is immediate to see that the matrix multiplication annihilates all the block entries except for $(r + 1, 1), (r + 2, 1), (n - r - 2, n - r)$ and $(n - r - 1, n - r)$. The vanishing of these particular four entries, when $r < n - 2$ and when $r = n - 2$ is already discussed in the proof of 2.4.1. □

Proposition 2.4.4. *We have the following decompositions*

1. *For all $2 \leq r < n - 2$, we have*

$$N_r = N_{r+1} \rtimes (N_{r,n-r} \rtimes (\cdots (N_{r,1} \rtimes N_{r,0}) \cdots))$$

2. For $r = n-2$, we have

$$N_{n-2} = N_{n-1} \rtimes (N''_{n-2} \rtimes (N'_{n-2} \rtimes (N_{n-2,2} \rtimes (N_{n-2,1} \rtimes N_{n-2,0}))))$$

Corollary 2.4.5. *For all $n \geq 4$, the unipotent radical of $G_n(\mathbb{Q}_p)$ has the decomposition $N(\mathbb{Q}_p) = N_2 = N_3 \rtimes (N_{2,n-2} \rtimes (\cdots \rtimes (N_{2,1} \rtimes N_{2,0}) \cdots))$, which decomposes recursively into a semidirect sum of abelian subgroups, recursion stops at $N_{n-1} \cong \mathbb{Q}_p^{n-1}$. This gives us a satisfactory description of $G_n(\mathbb{Q}_p) = N(\mathbb{Q}_p) \rtimes H(\mathbb{Q}_p)$.*

As we saw earlier, the calculation of $\zeta^{\wedge}_{L_{n,p}}(s)$ requires computing $G_n(\mathbb{Z}_p)$ and $G_n^+(\mathbb{Q}_p)$ first. After computing $G_n(\mathbb{Q}_p)$ through on the decomposition presented in the corollary, we need to identify $G_n(\mathbb{Z}_p)$ as a subgroup of $G_n(\mathbb{Q}_p)$, which is expected not to be difficult, and continue from there to identify the monoid $G_n^+(\mathbb{Q}_p)$, which is expected to be a substantial challenge. By applying **Fubini's theorem** for semidirect products of topological groups, we have that

$$\zeta^{\wedge}_{L_{n,p}}(s) = \int_{G_n^+(\mathbb{Q}_p)} |\det \varphi|_p^s d\mu_{G_n(\mathbb{Z}_p)\varphi} = \int_{H^+(\mathbb{Q}_p)} \left(\int_{N_h^+(\mathbb{Q}_p)} |\det uh|_p^s d\mu_{N(\mathbb{Q}_p)} \right) d\mu_{H(\mathbb{Q}_p)}$$

where $H^+(\mathbb{Q}_p)$ consists of all $h \in H(\mathbb{Q}_p)$ that appear in the decomposition $\varphi = uh$ for some $\varphi \in G_n^+(\mathbb{Q}_p)$, and, for a given $h \in H^+(\mathbb{Q}_p)$, we set $N_h^+(\mathbb{Q}_p) := \{u \in N(\mathbb{Q}_p) : uh \in G_n^+(\mathbb{Q}_p)\}$.

The computation of $\det h$ of the form stated above is easy and given by $\det h = \prod_{i=1}^{n-1} \lambda_i^{i(n-i)}$, which can be shown by induction on n, and since $\det u = 1$ for all $u \in N(\mathbb{Q}_p)$, we have that $|\det uh|_p^s = |\det h|_p^s$ for all $h \in H^+(\mathbb{Q}_p)$, which means that the integrand is constant for $N_h^+(\mathbb{Q}_p)$, so computing the inner integral amounts to finding the measure of $N_h^+(\mathbb{Q}_p)$, which is complicated, as stated above, but using the decomposition from the corollary, we can simplify $N_h^+(\mathbb{Q}_p)$, at the price of replacing a single integral by multiple integrals. Define a sequence

$$\{\mathcal{N}_1 = N_{2,0}, \mathcal{N}_2 = N_{2,1}, \ldots, \mathcal{N}_n = N_{3,0}, \mathcal{N}_{n+1} = N_{3,1}, \ldots, \mathcal{N}_m = N_{n-1}\}$$

which enumerates all the components in the decomposition of $N(\mathbb{Q}_p)$ by the order of r and k. Then for all $h \in H^+(\mathbb{Q}_p)$ define a sequence $\{\mathcal{N}_i^+\}_{i=1}^m$ where $\mathcal{N}_i^+ := \{u \in N_i : \varphi = uh\}$. Thus, we can decompose the inner integral of

$N_h^+(\mathbb{Q}_p)$ into a multiple integral over multiple subdomains, each of them is simple and easy to compute, that is

$$\zeta_{L_{n,p}}^\wedge(s) = \int_{G_n^+(\mathbb{Q}_p)} |\det\varphi|_p^s d\mu_{G_n(\mathbb{Z}_p)\varphi} = \int_{H^+(\mathbb{Q}_p)} \left(\int_{N_h^+(\mathbb{Q}_p)} |\det uh|_p^s d\mu_{N(\mathbb{Q}_p)} \right) d\mu_{H(\mathbb{Q}_p)} =$$

$$= \int_{H^+(\mathbb{Q}_p)} \left(\int_{\mathcal{N}_m^+} \cdots \left(\int_{\mathcal{N}_2^+} \left(\int_{\mathcal{N}_1^+} |\det u_1 u_2 \cdots u_m h|_p^s d\mu_{\mathcal{N}_1} \right) d\mu_{\mathcal{N}_2} \right) \cdots d\mu_{\mathcal{N}_m} \right) d\mu_{H(\mathbb{Q}_p)}$$

Each factor u_i in the decomposition of u is itself a unipotent matrix, hence $\det u_i = 1$ for all $1 \le i \le m$. This means that computing the inner integrals amounts to determining the measure of the sets $\mathcal{N}_i^+$ in terms of $h, u_1, u_2, \ldots, u_m$.

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