Abstract

Let G be any group. For any natural number $n \in \mathbb{N}$, let a_n be the number of subgroups $H \leq G$, such that $[G : H] = n$. Assume G is finitely-generated, then $a_n < \infty$, and we can define a Dirichlet series of the form $\zeta_G(s) := \sum_{i=1}^{\infty} a_n n^{-s}$, where $s \in \mathbb{C}$. Assume, in addition, that G is also nilpotent and torsion-free, then this function has some properties of the Riemann ζ-function, such as the Euler decomposition of ζ into a product of local factors indexed by primes. A version of this ζ-function counts pro-isomorphic subgroups, and an analogous function may be defined for appropriate Lie rings. We study here the pro-isomorphic ζ-functions for a family of nilpotent Lie rings of unbounded nilpotency class. We shall compute the automorphism groups of these Lie rings explicitly, prove uniformity of the local factors of the pro-isomorphic ζ-functions local factors, and aim to determine them explicitly.

1 Scientific Background

1.1 Introduction

We start our discussion with the following proposition, which stands at the very foundation of our subject.

Proposition 1.1.1. *Let G be any finitely generated group, and let $n \in \mathbb{N}$ any natural number. Then there is a finite number of subgroups $H \leq G$, such that $[G : H] = n$*

Proof. Let $H \leq G$ be a subgroup, such that $[G : H] = n$, and let ${}^{G}\!/\!{}_{H} := \{g_1 H, g_2 H, \ldots, g_n H\}$ be the set containing all left-cosets of H in G. We may consider the action of G by left multiplication on ${}^{G}\!/\!{}_{H}$: for all $g \in G$, and for all left-cosets $g_i H \in {}^{G}\!/\!{}_{H}$, we have that $g(g_i H) := (gg_i)H = g_j H$, where $g_j H \in {}^{G}\!/\!{}_{H}$ is some left-coset. This means that g maps every index $i \in [n]$ to some index $j \in [n]$, which means that g operates as a permutation on $[n]$. Therefore, there exists a homomorphism $f : G \to \mathcal{S}_n$, from G to the symmetric group of order n. Let $1 \leq k \leq n$ be the index for which $g_k H = H$. For all $g \in G$, $g \in H$ if and only if $gH = H$, which means that $gg_k H = g_k H$, which means that $f(g)(k) = k$, in words, k is fixed by the image $f(g)$. This means that $H = \{g \in G : f(g)(k) = k\}$, so H may be recovered from the homomorphism f, which clearly shows that the number of subgroups

$H \leq G$ of index n is less or equal to the number of maps $f : G \to \mathcal{S}_n$. G is finitely-generated, and $\mathcal{S}_n$ is finite. We know that group homomorphisms are uniquely determined by the maps of their generators. Therefore, there are finitely many maps from a finite set of generators to a finite group, which clearly shows that the number of subgroups $H \leq G$ of index n is necessarily finite. □

This proposition gives rise to an entire subject in group theory, called **subgroup growth**. We denote by $a_n(G)$ the number of subgroups of G of index n, and look at the sequence $\{a_n(G)\}$. The subject of subgroup growth aims to relate the properties of this sequence to the algebraic structure of G. For instance, Lubotzky, Mann and Segal showed that $a_n(G)$ grows polynomially if and only if G is virtually nilpotent of finite rank. That is, G has a finite-index nilpotent subgroup, and all finitely-generated subgroups may be generated by a finite number of generators. This research concentrates on the growth of **pro-isomorphic** subgroups, which we now define.

Definition 1.1.2. *Let G be any group, and let $\mathcal{N} := \{N_i \trianglelefteq G\}_{i \in I}$ the set of all normal subgroups of G. We define a partial order on $\mathcal{N}$, by inclusion, and assign G an infinite set of indices. The inverse limit $\widehat{G} = \underset{\leftarrow}{lim}\{{}^{G}/_{N_k}\}_{k \in I} := \{(h_k)_{k \in I} \in \prod_{k \in I} {}^{G}/_{N_k} : \pi_{ji}(h_j) = h_i, \forall i \leq j\}$, where $\pi_{ji} := {}^{G}/_{N_j} \to {}^{G}/_{N_i}$ is the natural projection, is called the **profinite closure** of G.*

Definition 1.1.3. *Let G be any group. a subgroup $H \leq G$ is called **pro-isomorphic**, if $\widehat{H} \cong \widehat{G}$.*

Definition 1.1.4. *Let G be any group, and let $\hat{a_n}(G) := \#\{H \leq G : \widehat{H} \cong \widehat{G}, [G : H] = n\}$, in words, the number of pro-isomorphic subgroups of G, of index n. Assume $\hat{a_n}(G) < \infty$, for all n. The **pro-isomorphic ζ-function** of G is defined by $\hat{\zeta_G}(s) := \sum_{n=1}^{\infty} \hat{a_n}(G) n^{-s}$, for $s \in \mathbb{C}$.*

If our ζ-function, which is a special case of the Dirichlet series, has some properties of convergence, on some subset of $\mathbb{C}$, one may reconstruct its coefficients, $\hat{a_n(G)}$, which, in our case, are the number of subgroups of our interest, using the **Perron's forumla**, which is an implementation of a **reverse Mellin transform**, as discussed, for example, in [2]. This method, including the specific properties of convergence required for the reconstruction, is out of the scope of our research, and therefore will not be further discussed, at this stage.

It has been found that if $\hat{a_n}(G)$ grows polynomially, then $\hat{\zeta}_G(s)$ converges on some right half-plane of $\mathbb{C}$. For instance, we take the group $G = (\mathbb{Z}, +)$. $\mathbb{Z}$ is an abelian group, and every $H \leq \mathbb{Z}$ is of the form $H = n\mathbb{Z} = \langle n \rangle$, for some $n \in \mathbb{N}$, which means that $H \cong \mathbb{Z}$, as both are infinite cyclic groups, and so, $\widehat{H} \cong \widehat{\mathbb{Z}}$. Since we have only one $\mathbb{Z}$-subgroup of index n, for every $n \in \mathbb{N}$, then $a_n(\mathbb{Z}) = \hat{a_n}(\mathbb{Z}) = 1$, thus, its pro-isomorphic ζ-function is $\hat{\zeta_{\mathbb{Z}}} = \sum_{i=1}^{\infty} n^{-s} = \zeta(s)$, the Riemann ζ-function, which is known to converge for $Re(s) > 1$.

After establishing the basic definitions, we observe a fact that is a major motivation for this research, which says that the Riemann ζ-function decomposes to an infinite product of local ζ_p-functions, that is, $\zeta(s) = \prod_p \zeta_p(s) = \prod_p \sum_{k=0}^{\infty} p^{-ks} = \prod_p \frac{1}{1-p^{-s}}$, where the product runs over all the prime numbers. Following this fact, regarding the Riemann *zeta*-function, we observe that for any finitely-generated, nilpotent and torsion-free group, G, we have the same decomposition as above, for the pro-isomorphic ζ-function, $\hat{\zeta_G}(s) = \prod_p \hat{\zeta_{G,p}}(s)$, where $\hat{\zeta_{G,p}}(s) := \sum_{k=0}^{\infty} a_{p^{ks}}(G)p^{-ks}$ We hereby bring several basic definitions of group nilpotency, which are very important for this research.

The general construction of ζ-functions, as well as their Euler decomposition to local ζ_p-functions, for the study of subgroup growth, were well established by Grunewald, Segal and Smith, in [3].

Definition 1.1.5. *Let G be any group, then the* ***lower central series*** *of G is a sequence of subgroups of G, defined by the recursive rule, $G_n := [G, G_{n-1}]$, for every $n \in \mathbb{N}$, where $G_0 := G$. We recall that $[G, G_n] \leq G$ is the subgroup of commutators, $\{gg_ng^{-1}g_n^{-1} : g \in G, g_n \in G_n\}$*

Definition 1.1.6. *Let G be any group. the* ***nilpotency class*** *of G is $min\{n \in \mathbb{N} : G_n = [G, G_{n-1}] = \{e\}\}$, in words, the smallest natural number, such that the subgroup of commutators of the form $[G, G_n]$ is the trivial group. We can extend this definition, and say that the trivial group nilpotency class is 0.*

Definition 1.1.7. *Let G be a group. If G if of a finite nilpotency class, $n \in \mathbb{N}$, then G is said to be a* ***nilpotent*** *group.*

1.2 Linearization

For finitely-generated torsion-free nilpotent groups, G, we associate nilpotent Lie algebras over $\mathbb{Z}$. This, in general, is called the **Maltsev correspon-**

dance. If L is a $\mathbb{Z}$-Lie algebra, namely a free $\mathbb{Z}$-module of a finite rank, with a Lie bracket, then consider the (finite) number $\hat{a_n}(L)$ of subalgebras $M \leq L$, such that $M \otimes \mathbb{Z}_p \cong L \otimes \mathbb{Z}_p$, for all primes p. The Dirichlet series, $\hat{\zeta_L}(S) := \sum_{n=1}^{\infty} \hat{a_n}(L)n^{-1}$, is called the **pro-isomorphic ζ-function** of L. By the Maltsev correspondance, to every finitely-generated, nilpotent, torsion-free group, G, one may associate a Lie algebra, $L(G)$, such that $\hat{\zeta_{G,p}}(s) = \hat{\zeta_{L,p}}(s)$, for but finitely many primes, p. If G has nilpotency class 2, one may obtain the equality for all primes. Let L be a Lie algebra, and fix a $\mathbb{Z}$-basis, $\mathcal{B} = \{b_1, \ldots, b_d\}$. Let $\mathcal{L}_p = L \otimes_{\mathbb{Z}} \mathbb{Q}_p$, for any p. This is a $] = \mathbb{Q}_p$-Lie algebra, and our choice of basis allows us to identify the automorphism group $G(\mathbb{Q}_p) = Aut_{\mathbb{Q}_p}(\mathcal{L}_p)$ with a subgroup of $GL_d(\mathbb{Q}_p)$. Note that $\mathcal{L}_p$ contains a $\mathbb{Z}_p$-lattice, $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. If $\varphi \in G(\mathbb{Q}_p)$, then $\varphi(L_p) = L_p$ if and only if $\varphi \in G(\mathbb{Z}_p) = G(\mathbb{Q}_p) \cap GL_d(\mathbb{Z}_p)$. Here $GL_d(\mathbb{Z}_p)$ is the group of invertible $d \times d$ matrices over $\mathbb{Z}_p$. Similarly, $\varphi(L_p) \subseteq L_p$ if and only if $G^+(\mathbb{Q}_p) := G(\mathbb{Q}_p) \cap M_d(\mathbb{Z}_p)$, where $M_d(\mathbb{Z}_p)$ is the collection of $d \times d$ matrices with entries in $\mathbb{Z}_p$. Note that $G^+(\mathbb{Q}_p)$ is a monoid, not a group.

Denote by $G(\mathbb{Z}_p)g$, where $g \in G^+(\mathbb{Q}_p)$, a right-coset of $G(\mathbb{Z}_p)$, one checks that $G^+(\mathbb{Q}_p) = \bigsqcup_{i=1}^{m} G(\mathbb{Z}_p)g_i$, where $[G^+(\mathbb{Q}_p) : G(\mathbb{Z}_p)] = m$, in words, the monoid $G^+(\mathbb{Q}_p)$ is a disjoint union of right-cosets of $G(\mathbb{Z}_p)$.

The discussion above reveals the construction we base our research upon. We observe that there is a bijection between $G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)$ and $\{M \leq L_p : M \cong L_p\}$, in words, we have a bijective map between each right-coset of $G(\mathbb{Z}_p)$ in $G^+(\mathbb{Q}_p)$ and each L_p-subalgebra which is isomorphic to L_p itself. This bijection is canonical, by taking $M = \varphi(L_p)$, for some $\varphi \in G(\mathbb{Z}_p)g$. M is a L_p-subalgebra, because $M = \varphi(L_p) \subseteq L_p$. One checks that for every $\psi \in G(\mathbb{Z}_p)g$, we have that $\psi(L_p) = \varphi(L_p) = M$. We end this part, as a preparation for the final part of this technical background review, with the following result, which states that for each right-coset, $G(\mathbb{Z}_p)g$, if $M = \varphi(L_p)$, where $\varphi \in G(\mathbb{Z}_p)g$, then $[L_p : M] = |det\varphi|_p^{-1}$, and therefore,

$$\hat{\zeta_{L,p}}(s) = \sum_{\substack{M \leq L_p \\ M \cong L_p}} [L_p : M]^{-s} = \sum_{G(\mathbb{Z}_p)\varphi \in G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)} |det\varphi|_p^s.$$

1.3 p-adic Integration

In this final part of the technical background review, we finally get to the motivation for all the construction we have presented in the first parts. We now define a very central object for our research. We shall assume, without proof, the existence of such an object, under the prerequisites of the definition.

Definition 1.3.1. *Let Γ be a locally compact topological group, i.e., for all $\gamma \in \Gamma$, there is an open neighborhood of γ, U_γ, and a compact subset K_γ, such that $U_\gamma \subset K_\gamma$. Then there is a measure μ, with the following property: for any measurable subset, $U \subseteq \Gamma$, and any $\gamma \in \Gamma$, $\mu(U\gamma) = \mu(U)$, where $U\gamma := \{u\gamma : u \in U\}$. Such a measure μ is called a **right Haar measure**, and is unique up to multiplication by a non-zero constant.*

Equipped with the right Haar measure, we can finally make use of the construction from above. We start by claiming, without proof, that for every prime number p, the group $G(\mathbb{Q}_p)$ is a locally compact topological group. Therefore, it has a unique right Haar measure. We also claim that the right Haar measure has the property that $\mu(G(\mathbb{Z}_p)) = 1$, and that the measure on all the right-cosets of $G(\mathbb{Z}_p)$ equals to the measure on $G(\mathbb{Z}_p)$ itself, i.e., for every $g \in G^+(\mathbb{Q}_p)$, we have that $\mu(G(\mathbb{Z}_p)g) = \mu(G(\mathbb{Z}_p)) = 1$. With this observation, we go directly to the calculation of the p-adic norm of the determinant of every L_p-automorphism, as a p-adic integral over our measure space. First, we observe that given any L_p-automorphism in some right-coset, $\varphi \in G(\mathbb{Z}_p)\varphi$, we have that $|det\varphi|_p^s = \int_{G(\mathbb{Z}_p)\varphi} |det\varphi|_p^s d\mu$, because $\mu(G(\mathbb{Z}_p)\varphi) = 1$, and $|det\varphi|_p^{-1}$ is fixed on $G(\mathbb{Z}_p)\varphi$.

Going back to our desired function, we observe that

$$\hat{\zeta_{L,p}}(s) = \sum_{G(\mathbb{Z}_p)\varphi \in G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)} |det\varphi|_p^s = \sum_{G(\mathbb{Z}_p)\varphi \in G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)} \int_{G(\mathbb{Z}_p)\varphi} |det\varphi|_p^s d\mu = \int_{G^+(\mathbb{Z}_p)} |det\varphi|_p^s d\mu.$$

This calculation of the local ζ_p-function as a p-adic integral was established by the work of du Sautoy and Lubotzky, in [4]. This integral is the main object we shall study, in this research. We end this part, of the technical background review, by a definition and a theorem, which stand in the center of our research goals.

Definition 1.3.2. *Let L be a $\mathbb{Z}$-algebra. $\hat{\zeta_L}(s)$ is called **uniform**, if there exists a rational function, $W \in \mathbb{Q}(X, Y)$, such that for every prime number p, the local function $\zeta_{L,p}(s) = W(p, p^{-1})$.*

*Sometimes, we prefer to say that $\zeta_{L,p}(s)$ is **uniform**, if there exists a function $W_p \in \mathbb{Q}(X)$, such that $\zeta_{L,p}(s) = W_p(p^{-s})$.*

Both definitions are just two dif and only iferent ways of writing the rational function $W_p(p^{-s}) = W(p, p^{-s})$.

Here, $\mathbb{Q}(X)$ and $\mathbb{Q}(X, Y)$ are the fields of rational functions in one variable and two variables, respectively.

The term uniform expresses that the rational function depends, for every p, only on p and its inverse.

Theorem 1.3.3. *Let p be a prime number, $s \in \mathbb{C}$, then $\zeta_{\hat{L},p}(s)$ is uniform.*

The uniformity is also established in the work of Grunewald, Segal and Smith, see [3].

2 Research Goals and Methodology

2.1 The unipotent group U_n

We start by this following definition.

Definition 2.1.1. *Let $\mathcal{R}$ be a commutative ring. Let $U_n(\mathcal{R}) \leq GL_n(\mathcal{R})$ be the subgroup of upper unitriangular matrices, i.e. $U_n(\mathcal{R}) = \left\{ \begin{pmatrix} 1 & a_{12} & \\ & \ddots & \ddots \\ & & 1 \end{pmatrix} \right\}$, then $U_n(\mathcal{R})$ is called a* ***unipotent group****.*

Looking into the structure of $U_n(\mathcal{R})$, we observe that we can find a set of generators, $\mathcal{U}_n(\mathcal{R})$. Denote by E_{ij}, where $i < j$, an elmentary matrix of the form $\begin{pmatrix} 1 & 0 & \dots & 0 \\ & \ddots & 1 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \in U_n(\mathcal{R})$, where, in addition to the main diagonal, only the element in row i and column j is 1, and all the other elements are 0. One checks that if $i < j$ and $k < l$, then the commutator $[E_{ij}, E_{kl}] = \begin{cases} E_{il}, & j = k \\ -E_{kj}, & i = l \\ I_n, & \text{otherwise} \end{cases}$

In addition, it is easy to observe that $E_{ij}^m = \begin{pmatrix} 1 & 0 & \dots & 0 \\ & \ddots & m & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$, for every $m \in \mathbb{Z}$, by simple induction. All this means that we can generate $U_n(\mathcal{R})$ entirely by the set $\mathcal{U}_n := \{E_{1,2}, \dots, E_{n-1,n}\}$. From this, it is also clear that $U_n(\mathcal{R})$ is nilpotent, of nilpotency class n, because the longest chain of commutators in $U_n(\mathcal{R})$, i.e., chaining all the elements in the set of generators, $\mathcal{U}_n$, by the order of their indices, yields $[E_{1,2}, [E_{2,3}, [\dots, [E_{n-1,n}]]]] = E_{1,n}$, which means that $\gamma_{n-1}U_n(\mathcal{R}) = \{E_{1,n}\}$, which means that $\gamma_n U_n(\mathcal{R}) = I_n$.

This also shows that $U_n(\mathcal{R})$ is torsion-free, if and only if $\mathcal{R}$ itself is torsion-free, as can be readily seen from the fact that E_{ij}^m has an m in row i and column j, and $m \neq 0$ if and only if $\mathcal{R}$ is torsion-free, for instance, $\mathcal{R} = \mathbb{Z}$.

These facts, regarding $U_n(\mathbb{Z})$ place it as a group of our interest, for this research, and bring us next to its associated Lie algebra.

2.2 The Lie algebra L_p

We start with $E_{i,i+1}$ from above and define matrices of the form $e_{i,i+1} = E_{i,i+1} - I_n$, in words, $e_{i,i+1}$ is obtained by replacing all the 1 on the main diagonal with 0. It is readily seen that the standard brackets operation between these matrices, $[e_{i,i+1}, e_{j,j+1}] = e_{i,i+1}e_{j,j+1} - e_{j,j+1}e_{i,i+1}$, obeys the commutator operation between the $U_n(\mathcal{R})$ matrices, i.e., $[e_{i,i+1}, e_{j,j+1}] = [E_{i,i+1} - I_n, E_{j,j+1} - I_n] = [E_{i,i+1}, E_{j,j+1}] - I_n$. From this, we construct a nilpotent $\mathbb{Z}$-Lie algebra, which we denote by L, with the standard brackets operation as Lie brackets. As discussed above, this $\mathbb{Z}$-Lie algebra can be extended to a $\mathbb{Z}_p$-algebra, which we denote by L_p, and then to a $\mathbb{Q}_p$-algebra, which we denote by $\mathcal{L}_p$. Since the Lie brackets obey the group commutator, we can obviously create any matrix of the form e_{ij}, where $i < j$, which has only 1 in row i and column j, and 0 anywhere else.

it is readily seen that all these matrices span the whole $\mathbb{Z}$-algebra and are $\mathbb{Z}$-linearly independent of each other, therefore, they form the standard basis for the $\mathbb{Z}$-Lie algebra, which we denote by

$\mathcal{B}_n := \{e_{1,2}, e_{1,3}, \dots, e_{1,n}, e_{2,3}, \dots, e_{2,n}, \dots, e_{n-1,n}\}$. One easily checks that $d = dimL = |\mathcal{B}_n| = \binom{n}{2}$, which is the number of elements above the main diagonal. Obviously, the same goes also for the extensions of L, namely L_p and $\mathcal{L}_p$. And so, we have reached the target of our research, which is studying the $\hat{\zeta_{L,p}}$-function on the $\mathbb{Z}_p$-Lie algebra associated with $U_n(\mathbb{Z}_p)$.

2.3 Research goals

As stated above, this research will focus on studying $U_n(\mathbb{Z}_p)$ and its associated $\mathbb{Z}_p$-Lie algebra, namely L_p. We have three major steps in mind, and we shall strive to achieve the most of them:

1. **Calculating the automorphism group of the $\mathbb{Z}_p$-Lie algebra associated with $U_n(\mathbb{Z}_p)$, for all $n \in \mathbb{N}$.**
2. **Showing that the pro-isomorphic $\hat{\zeta_{L,p}}$-function, for L_p, is unified for all $n \in \mathbb{N}$.**
3. **Giving an explicit formula for the $\hat{\zeta_{L,p}}$-function, for specific values of n, if not for all $n \in \mathbb{N}$.**

We will have to give a full description of $G(\mathbb{Z}_p)$, for dimension of L_p. This approach is based on the natural assumption that all L_p, regardless of their dimensions, obey the same rules, and therefore, their automorphism groups behave in the same way, which we need to display and prove, and that will achieve the first step. These automorphism groups have been studied for decades from a dif and only iferent point of view. There are classical results showing that any automorphism may be expressed as a product of automorphisms of a specific type; see, for instance, the main result of Gibbs [1]. These results are not explicit enough for our purposes; indeed, the submonoid $G^+(\mathbb{Q}_p)$ arises, for us, as the domain of integration of a p-adic integral. In order to compute this integral, we need to decompose the automorphism group $G(\mathbb{Q}_p)$ into a repeated semi-direct product of groups with a simple structure.

After we have analyzed the structure of $G(\mathbb{Z}_p)$, we will need to construct the monoid $G^+(\mathbb{Q}_p)$ and its $G(\mathbb{Z}_p)$ right-cosets, as we have seen above. This will give us both the function to integrate, which is $det\varphi$, for every $G(\mathbb{Z}_p)$ right-coset, $G(\mathbb{Z}_p)\varphi$, and the domain of integration, which is the right-coset itself. We will use this information to analyze the behavior the p-adic integral we have described above, and prove that its calculation depends only on p, thus showing that the $\hat{\zeta_{L,p}}$-function is unified, regardless of $dimL_p$

2.4 The Heisenberg group

We show the very basic approach to this problem, by the simplest example, which is the **Heisenberg group**, which is simply $U_3(\mathbb{Z})$, and its associated $\mathbb{Z}$-Lie algebra. Since $U_3(\mathbb{Z})$ is the group of unipotent 3×3 matrices, we observe that the basis for the associated $\mathbb{Z}$-Lie algebra is $\mathcal{B} = \{e_{12}, e_{13}, e_{23}\}$. We observe that we can apply a linear order to the basis, where $e_{ij} < e_{kl}$ if the dif and only iference $j-i$ is less then $l-k$, or if both dif and only iferences are equal, and $i < k$. Therefore, we change the basis according to this order, $\mathcal{B} = \{e_{12}, e_{23}, e_{13}\}$. Every $\varphi \in G(\mathbb{Z}_p)$ must obey the Lie brackets. This means that if we analyze φ by its operation on the basis elements, then φ must be any 3×3 matrix over $\mathbb{Z}_p$, such that multiplying, from the right, with any vector $v = xe_{12} + ye_{23} + ze_{13} \in L_p$, yields a vector $u = x\varphi(e_{12}) + y\varphi(e_{23}) + z\varphi(e_{13}) \in L_p$, such that $\varphi(u) = \varphi^2(v) = v$, i.e., $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \varphi(x) \\ \varphi(y) \\ \varphi(z) \end{pmatrix}$ then we observe that $a_{31}e_{12} + a_{32}e_{23} + a_{33}e_{13} = \varphi(e_{13}) = \varphi[(e_{12}), (e_{23})] = [\varphi(e_{12}), \varphi(e_{23})] = [a_{11}e_{12} + a_{12}e_{23} + a_{13}e_{13}, a_{21}e_{12} + a_{22}e_{23} + a_{23}e_{13}]$, which gives that . gives the following relations,

$$a_{31} = 0$$

$$a_{32} = 0$$

$$a_{33} = (a_{11}a_{22} - a_{12}a_{21}) \neq 0$$

which means that φ is the following matrix, $\varphi = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & detA \end{pmatrix}$ where A is the minor $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

2.5 $U_n(\mathbb{Z}_p)$ groups of higher cardinality

Mark N. Berman, in his doctoral thesis[5], has displayed an explicit formula for $n = 4$, and proved that for $n = 5$, the ζ_p-function is, indeed, uniform. Following his work, we can observe that for every $v \in \text{Ł}_p$, where L_p is the $\mathbb{Z}$-Lie algebra associated with $U_n(\mathbb{Z})$, where $n > 3$, writing $M = \varphi$ as a $d \times d$ matrix, where $d = dimL_p\binom{n}{2}$, whose lines are set, by the order we have defined above, i.e., in the following way,

$$M = \begin{pmatrix} \varphi(e_{12}) \\ \varphi(e_{23}) \\ \varphi(e_{n-1n}) \\ \varphi(e_{13}) \\ \vdots \\ \varphi(e_{n-1n}) \\ \vdots \\ \varphi(e_{1n}) \end{pmatrix}$$

divides M to a block matrix,

$$M = \left(\begin{array}{c|c|c|c|c} M_{11} & M_{12} & \dots & M_{1n-1} & M_{1n} \\ \hline M_{21} & M_{22} & \dots & M_{2n-1} & M_{2n} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline M_{n1} & M_{n2} & \dots & M_{nn-1} & M_{nn} \end{array}\right)$$

where $M_{ij} \in \mathcal{M}_{k\times l}(\mathbb{Z}_p)$, $k = \dim(\gamma_i L_p), l = \dim(\gamma_j L_p)$. From this, we can understand that the blocks on the main diagonal of M are squared matrices, $A_{ii} \in \mathcal{M}_{n-i}$. From the calculation on the Heisenberg group, we understand also that any element $e_{ii+k} \in \gamma_k L_p$ must vanish in the images of elements of higher nilpotency classes, i.e. $\varphi(e_{i,i+l})$, where $l > k$, which means that all the elements under every squared block on the main diagonal must be zero, so M has the form,

$$M = \left(\begin{array}{c|c|cc|c|c} M_{11} & M_{12} & M_{13} & \dots & M_{1m-1} & M_{1m} \\ \hline 0 & M_{22} & M_{23} & \dots & M_{2m-1} & M_{2m} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \dots & M_{2m-1} & M_{2m} \\ \hline 0 & 0 & 0 & \dots & 0 & M_{mm} \end{array}\right)$$

We observe that the matrix M_{ij} blocks represent quotients of the form ${}^{\gamma_i \mathcal{L}_{p,n}}\!/\!{}_{\gamma_{i+1}\mathcal{L}_{p,n}}, {}^{\gamma_j \mathcal{L}_{p,n}}\!/\!{}_{\gamma_{j+1}\mathcal{L}_{p,n}}$. We shall state, as a fact, that the block M_{11} is either diagonal or anti-diagonal, i.e.,

$$M_{11} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

or

$$M_{11} = \begin{pmatrix} & & & \lambda_1 \\ & & \lambda_2 & \\ & \iddots & & \\ \lambda_n & & & \end{pmatrix}$$

In the case of an anti-diagonal block, we have the following proposition,

Proposition 2.5.1. *Let p be a prime number, and let $n \in \mathbb{N}$. $B_n = \{e_1, \ldots, e_{m-1}\}$, where $m = \binom{n}{2}$. Then, the map $\eta_n : B_n \to B_n$, defined by $\eta_n(e_i) := e_{m-i}$ is a $\mathcal{L}_{p,n}$-automorphism, which is also an involution.*

Proof. Clearly, η_n is the anti-diagonal $m \times m$ matrix,

$$\eta_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$$

η_n is an invertible matrix, which operates on any vector

$$v = (a_1, a_2, \ldots, a_{m-1}) = \sum_{i=1}^{m-1} a_i e_i$$

in the following way,

$$\eta_n(v) = \eta_n\left(\sum_{i=1}^{m-1} a_i e_i\right) = \begin{pmatrix} a_1 & a_2 & \ldots & a_{m-1} \end{pmatrix} \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} = \begin{pmatrix} a_{m-1} & a_{m-2} & \ldots & a_1 \end{pmatrix} =$$

$$\begin{pmatrix} \eta_n(a_1) & \eta_n(a_2) & \ldots & \eta_n(a_{m-1}) \end{pmatrix} = \sum_{i=1}^{m-1} \eta_n(a_i e_i) = \sum_{i=1}^{m-1} a_i \eta_n(e_i)$$

And, $\eta_n^2(v) = \eta_n(\eta_n(v)) = \eta_n\left(\eta_n\left(\sum_{i=1}^{m-1} a_i e_i\right)\right) = \eta_n\left(\sum_{i=1}^{m-1} a_i \eta_n(e_i)\right) = \sum_{i=1}^{m-1} a_i \eta_n^2(e_i) = \sum_{i=1}^{m-1} a_i \eta_n(e_{n-i}) = \sum_{i=1}^{m-1} a_i e_i$ $\square$

From this proposition, we realize that if M_{11} is anti-diagonal, then $\eta_n \varphi$ is the automorphism which has that M_{11} is diagonal.

Proposition 2.5.2. *Let p be a prime number, and let $n \in \mathbb{N}$, and let $M = \varphi \in \mathcal{L}_{p,n}$. Then, all the blocks on the main diagonal, $M_{ii}, \ldots, M_{n-1n-1}$, are diagonal, of the form,*

$$M = \varphi = \begin{pmatrix} \lambda_1 &&&&&&&&& \\ & \lambda_2 &&&&&&&& \\ && \ddots &&&&&&& \\ &&& \lambda_n &&&&&& \\ &&&& \lambda_1\lambda_2 &&&&& \\ &&&&& \lambda_2\lambda_3 &&&& \\ &&&&&& \ddots &&& \\ &&&&&&& \lambda_{n-1}\lambda_n && \\ &&&&&&&& \ddots & \\ &&&&&&&&& \lambda_1\lambda_2\cdots\lambda_n \end{pmatrix}$$

Proof. By simple induction. We have already assumed that M_{11} is diagonal. Every sequential block M_{ii} contains the coefficients of elements of ${}^{\gamma_i\mathcal{L}_{p,n}}\!/\!_{\gamma_{i+1}\mathcal{L}_{p,n}}$ as summands in images of elements of the same quotient algebra. So, $\varphi(e_{ii+2}) = \sum_{i=1}^{n-2} a_{ii+2}e_{ii+2}$, but $e_{ii+2} = [e_{ii+1}, e_{i+1i+2}]$, so $\varphi(e_{ii+2}) = [\varphi(e_{ii+1}), \varphi(e_{i+1i+2})]$, hence, $\lambda_{i+2} = a_{ii+2} = a_{ii+1}a_{i+1i+2} = \lambda_i\lambda_{i+1}$, which proves the proposition. □

Proposition 2.5.3. *Let $n \in \mathbb{N}$, and let*

$$A_n = \begin{pmatrix} \lambda_1 &&&&&&&&& \\ & \lambda_2 &&&&&&&& \\ && \ddots &&&&&&& \\ &&& \lambda_n &&&&&& \\ &&&& \lambda_1\lambda_2 &&&&& \\ &&&&& \lambda_2\lambda_3 &&&& \\ &&&&&& \ddots &&& \\ &&&&&&& \lambda_{n-1}\lambda_n && \\ &&&&&&&& \ddots & \\ &&&&&&&&& \lambda_1\lambda_2\cdots\lambda_n \end{pmatrix}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{Q}_p$, then, $det(A_n) = \prod_{i=1}^{n} \lambda_i^{i(n+1-i)}$.

Proof. We observe that the determinants, for $n = 1, 2, 3, \ldots$, form a recursive sequence,

$$\begin{gathered} det(A_1) = \lambda_1 \\ det(A_2) = det(A_1)\lambda_1\lambda_2^2 \\ det(A_3) = det(A_2)\lambda_1\lambda_2^2\lambda_3^3 \\ \vdots \\ det(A_n) = det(A_{n-1})\lambda_1\lambda_2^2\lambda_3^3\cdots\lambda_n^n \end{gathered}$$

Calculating the general element, $a_n = det(A_n)$, we see that we have n times λ_1, $n-1$ times λ_2^2, $n-2$ times λ_3^3, and so forth. In general, we have $n-i+1$ times λ_i^i, which means that we have $i(n-i+1)$ times λ_i, and in total, $a_n = det(A_n) = \prod_{i=1}^{n} \lambda_i^{i(n+1-i)}$. □

This means that every $M = \varphi \in \mathcal{L}_{p,n}$ is of the form,

$$M = \varphi = \left(\begin{array}{c|c|cc|c|c} \begin{matrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{matrix} & M_{12} & M_{13} & \ldots & M_{1m-1} & M_{1m} \\ \hline 0 & \begin{matrix} \lambda_1\lambda_2 & & & \\ & \lambda_2\lambda_3 & & \\ & & \ddots & \\ & & & \lambda_{n-1}\lambda_n \end{matrix} & M_{23} & \ldots & M_{2m-1} & M_{2m} \\ \hline \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \ldots & M_{2m-1} & M_{2m} \\ \hline 0 & 0 & 0 & \ldots & 0 & \lambda 1\lambda 2\cdots\lambda_n \end{array}\right)$$

The above discussion gives rise to the decomposition of each $\varphi \in \mathcal{L}_{p,n}$ to two matrices, one is the diagonal matrix

$$h = \begin{pmatrix} \lambda_1 & & & & & & & & & \\ & \lambda_2 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & \lambda_n & & & & & & \\ & & & & \lambda_1\lambda_2 & & & & & \\ & & & & & \lambda_2\lambda_3 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \lambda_{n-1}\lambda_n & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \lambda_1\lambda_2\cdots\lambda_n \end{pmatrix}$$

and the other matrix is

$$n = \begin{pmatrix} 1 & * & * & * & * & * & * & * \\ & 1 & * & * & * & * & * & * \\ & & \ddots & * & * & * & * & * \\ & & & 1 & * & * & * & * \\ & & & & 1 & * & * & * \\ & & & & & 1 & * & * \\ & & & & & & \ddots & * \\ & & & & & & & 1 \end{pmatrix}$$

So, we have the following proposition,

Proposition 2.5.4. *Let p be a prime number, and let $n \in \mathbb{N}$, and let $M = \varphi \in \mathcal{L}_{p,n}$. Then, $M = \varphi = nh$, where n and h are of the above form.*

Proof. Trivially, h is an invertible matrix, and its inverse is the matrix

$$h^{-1} = \begin{pmatrix} \lambda_1^{-1} & & & & & & & & & & \\ & \lambda_2^{-1} & & & & & & & & & \\ & & \ddots & & & & & & & & \\ & & & \lambda_n^{-1} & & & & & & & \\ & & & & (\lambda_1\lambda_2)^{-1} & & & & & & \\ & & & & & (\lambda_2\lambda_3)^{-1} & & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & & (\lambda_{n-1}\lambda_n)^{-1} & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & (\lambda_1\lambda_2\cdots\lambda_n)^{-1} \end{pmatrix}$$

Easy to check that $n = Mh^{-1}$ is also an invertible matrix, with 1 on the main diagonal, and 0 below it. $\square$

We observe that all the matrices with non-zero elements on the main diagonal, and 0 everywhere else form an abelian subgroup of $G_n(\mathbb{Q}_p)$, since multiplying such matrices yields a matrix of the same specification. Let

$$h_\alpha = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_m \end{pmatrix}, h_\beta = \begin{pmatrix} \beta_1 & & & \\ & \beta_2 & & \\ & & \ddots & \\ & & & \beta_m \end{pmatrix}$$

Then,

$$h_\alpha h_\beta = \begin{pmatrix} \alpha_1\beta_1 & & & \\ & \alpha_2\beta_2 & & \\ & & \ddots & \\ & & & \alpha_m\beta_m \end{pmatrix} = \begin{pmatrix} \beta_1\alpha_1 & & & \\ & \beta_2\alpha_2 & & \\ & & \ddots & \\ & & & \beta_m\alpha_m \end{pmatrix} = h_\beta h_\alpha$$

Obviously, this subgroup, which we shall denote as $H < G_n(\mathbb{Q}_p)$ is not normal, as we observe by taking the n matrix described above, and multiplying $A = nhn^{-1}$, clearly $A \notin H$. On the other hand, the set of all n matrices if a normal subgroup of $G_n(\mathbb{Q}_p)$, because if

$$n_\alpha = \begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1m} \\ & 1 & \alpha_{23} & \dots & \alpha_{2m} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & \alpha_{m-1m} \\ & & & & 1 \end{pmatrix}, n_\beta = \begin{pmatrix} 1 & \beta_{12} & \beta_{13} & \dots & \beta_{1m} \\ & 1 & \beta_{23} & \dots & \beta_{2m} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & \beta_{m-1m} \\ & & & & 1 \end{pmatrix}$$

Then

$$n_\alpha n_\beta = \begin{pmatrix} 1 & \alpha_{12}+\beta_{12} & * & \dots & * \\ & 1 & * & \dots & * \\ & & \ddots & \vdots & \vdots \\ & & & 1 & \alpha_{m-1m}+\beta_{m-1m} \\ & & & & 1 \end{pmatrix}$$

which proves that all the n matrices form a subgroup, which we shall denote by $N \in G_n(\mathbb{Q}_p)$. taking any matrix, $g \in G_n(\mathbb{Q}_p)$, and taking the product $A = gng^{-1}$, if we look at the main diagonals, we see that the product is of the general form

$$gng^{-1} = \begin{pmatrix} \lambda_1 & a_{12} & \dots & a_{1m} \\ 0 & \lambda_2 & \dots & a_{2m} \\ & & \ddots & * \\ & & & \lambda_m \end{pmatrix} \begin{pmatrix} 1 & b_{12} & \dots & b_{1m} \\ 0 & 1 & \dots & b_{2m} \\ & & \ddots & * \\ & & & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^{-1} & c_{12} & \dots & c_{1m} \\ 0 & \lambda_2^{-1} & \dots & c_{2m} \\ & & \ddots & * \\ & & & \lambda_m^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} 1 & d_{12} & \dots & d_{1m} \\ 0 & 1 & \dots & d_{2m} \\ & & \ddots & * \\ & & & 1 \end{pmatrix} \in N$$

So, $N \lhd G_n(\mathbb{Q}_p)$ is a normal subgroup. This discussion gives rise to the decomposition of $G_n(\mathbb{Q}_p)$. Since only N is a normal subgroup of $G_n(\mathbb{Q}_p)$, we decompose $G_n(\mathbb{Q}_p)$ to a semi-direct product, $G \cong N \rtimes H$, where the map $\phi : H \to Aut(N)$, given by $\phi(h)(n) := hnh^{-1}$, for every $h \in H$, and $n \in N$, is a homomorphism, as we can see by the fact that for every $h_1, h_2 \in H$, and for every $n \in N$, $\phi(h_1)\phi(h_2)(n) = h_1 n(h_2 n h_2^{-1})h_1^{-1} = h_1 h_2 n h_2^{-1} h_1^{-1} = (h_1 h_2) n (h_1 h_2)^{-1} = \phi(h1h2)(n)$ This means that calculating the integral, for $G_n(\mathbb{Q}_p)$, reduces to calculating a double integral, $\displaystyle\int_{N \rtimes H}$. We mean to show in the research that the normal subgroup N can itself be decomposed to a semi-direct product of several subgroups, thus simplyifing the integration.

By **??**, we have that any $L_p(\mathbb{Z}_p)$-automorphism must be in $G_n(\mathbb{Z}_p)$, in words, any $\varphi \in G(\mathbb{Z}_p)$ is an invertible matrix with elements in $\mathbb{Z}_p$. Our goal is to find a way to compute $G(\mathbb{Z}_p)$, the automorphism group of $L_n(\mathbb{Z}_p)$, for any $n \in \mathbb{N}$. After finding a general formula for this calculation, we shall be able to show a way to compute the n-multiple p-adic integral of the form $\int\int\cdots\int\int_{D_1 \times D_2 \cdots \times D_{n-1} \times D_n} f(h_1, h_2, \dots, h_{n-1}, h_n) d(\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n)$, where D_i is the set of $G(\mathbb{Z}_p)$-cosets, for $G(\mathbb{Z}_p)$, the group of $\mathbb{Z}_p$-automorphisms on the algebra $L_i(\mathbb{Z}_p)$, and h_i is any element of this group, and μ_i is the Haar measure on this group. By Fubini, this multiple integral can be calculated as the iterated integral

$$\int_{D_n} \left(\int_{D_{n-1}} \cdots \left(\int_{D_2} \left(\int_{D_1} f(h_1, h_2, \dots, h_{n-1}, h_n) d\mu_1 \right) d\mu_2 \right) \dots d\mu_{n-1} \right) d\mu_n$$

. Alternatively, if we do not find an explicit formula for this calculation, we will show the general approach for this calculation, and prove the necessary conditions for its validity.

2.6 Possible Extension

To conclude this part, of the research goals review, we shall display a possible extension, considering a general ring of integers, as our associated Lie algebra. Taking some number field, $K = {}^{K}\!/\mathbb{Q}$, of a finite degree, d, over $\mathbb{Q}$, and its ring of integers, $\mathcal{O}_K$, which is a d-dimensional $\mathbb{Z}$-algebra. We can try applying everything we have established, regarding algebras over $\mathbb{Z}_p$ and $\mathbb{Q}_p$, to, for example, algebras over $\mathbb{Z}[\sqrt{p}]$ and $\mathbb{Q}[\sqrt{p}]$, respectively. This approach is only one possible way to extend our research, and therefore, we shall not go into the details of it, at this stage.

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