cases

haiml76

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Denote $p^{\min} := p^{\min\{0,v_2+v_{11}\}}$
Denote $\rho := (1-p^{-1})$

1 Case 1

$v_3 > v_1 \geq v_4 > v_2 \geq 0$
$\Rightarrow v_{11} \geq -v_2 - v_1$
$\Rightarrow \alpha := \min\{v_2, v_2+v_{11}+v_1\} = \begin{cases} v_2, & v_{11} \geq -v_1 \\ v_2-1, v_2-2, \ldots, & v_{11} < -v_1 \end{cases}$

For $v_{11} \geq -v_2$, we have $p^{\min}\mu(a_{11})(1+\alpha\rho) = p^{v_2}(1+v_2\rho)$.

For $-v_1 \leq v_{11} \leq -v_2-1$, we have $p^{\min}\mu(a_{11})(1+\alpha\rho) = \sum_{-v_1}^{-v_2-1} p^{v_2}\rho(1+v_2\rho) = p^{v_2}\rho(v_1-v_2)(1+v_2\rho) = p^{v_2}\rho(v_1-v_2) + p^{v_2}\rho^2 v_2(v_1-v_2)$.

For $-v_2-v_1 \leq v_{11} \leq -v_1-1$, we have $p^{\min}\mu(a_{11})(1+\alpha\rho) = \sum_{-v_2-v_1}^{-v_1-1} p^{v_2}\rho(1+\alpha\rho) = \sum_{i=1}^{v_2} p^{v_2}\rho(1+(v_2-i)\rho) = p^{v_2}v_2\rho + p^{v_2}v_2\rho^2 - p^{-v_2}\rho\binom{v_2+1}{2} = p^{v_2}v_2\rho + p^{v_2}\rho^2 - p^{v_2}\rho^2\binom{v_2+1}{2}$

So we have $\sum_{v_{11}\geq -v_2-v_1} = p^{v_2}(1+v_2\rho+(v_1-v_2)\rho+v_2(v_1-v_2)\rho^2+v_2\rho+v_2\rho^2-\frac{(v_2+1)v_2}{2}\rho^2)$. We compute the differences between v_1, v_2, v_3, v_4,

$v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_4 \overset{b}{>} v_2 \overset{a}{\geq} 0$

Thus, we substitute $v_1 \to a+b+c$, $v_2 \to a$, $v_3 \to a+b+c+d$, $v_4 \to a+b$, and compute series with the new variables as indices, where $a, c \geq 0$ and $b, d \geq 1$.

2 Case 2

$v_1 \geq v_3$
$v_1 \geq v_4 > v_2 \geq 0$
$\Rightarrow v_{11} \geq -v_2 - v_3$
$\Rightarrow \alpha := \min\{v_2, v_2+v_{11}+v_1\} = \begin{cases} v_2, & v_{11} \geq -v_1 \\ v_2-1, v_2-2, \ldots, & v_{11} < -v_1 \end{cases}$

We have two sub cases.

2.1 Sub case 2.1

$-v_2 - v_3 \geq -v_1 \Rightarrow v_1 \geq v_2 + v_3$

Here there is no phase transition, because the phase transition occurs at $-v_1 - 1 < -v_2 - v_3 \leq v_{11}$

For $v_{11} \geq -v_2$, we have $p^{\min}\mu(a_{11})(1+\alpha\rho) = p^{v_2}(1+v_2\rho)$.

For $-v_2 - v_3 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}\mu(a_{11})(1+\alpha\rho) = \sum_{-v_2-v_3}^{-v_2-1} p^{v_2}\rho(1+v_2\rho) = p^{v_2}v_3\rho(1+v_2\rho) = p^{v_2}v_3\rho + p^{v_2}v_3v_2\rho^2$.

So we have $\sum_{v_{11}\geq -v_2-v_3} = p^{v_2}(1+v_2\rho+v_3\rho+v_2v_3\rho^2)$. We compute the differences between v_1, v_2, v_3, v_4, we have several arrangements,

1. $v_1 \overset{d}{\geq} v_2 + v_3 \overset{e}{\geq} v_3 \overset{c}{\geq} v_4 \overset{b}{>} v_2 \overset{a}{\geq} 0$

   But $e = v_2 + v_3 - v_3 = v_2 = a$

   Thus, we substitute $v_1 \to 2a+b+c+d$, $v_2 \to a$, $v_3 \to a+b+c$, $v_4 \to a+b$, and compute series with the new variables as indices, where $a, c, d \geq 0$ and $b \geq 1$.

2. $v_1 \overset{d}{\geq} v_2 + v_3 \overset{c}{\geq} v_4 \overset{b}{>} v_3 \overset{a}{>} v_2 \overset{e}{\geq} 0$

   But $v_2 + v_3 - c - b = v_3 \Rightarrow v_2 = v_2 + v_3 - v_3 = b + c$

   Thus, we substitute $v_1 \to a+2b+c+d$, $v_2 \to b+c$, $v_3 \to a+b+c$, $v_4 \to a+2b+c$, and compute series with the new variables as indices, where $c, d \geq 0$ and $a, b \geq 1$.

3. $v_1 \overset{d}{\geq} v_2 + v_3 \overset{c}{\geq} v_4 \overset{b}{>} v_2 \overset{a}{\geq} v_3 \overset{e}{\geq} 0$

   But $v_2 + v_3 - c - b = v_2 \Rightarrow v_3 = v_2 + v_3 - v_2 = b + c$

   Thus, we substitute $v_1 \to a+2b+c+d$, $v_2 \to a+b+c$, $v_3 \to b+c$, $v_4 \to a+2b+c$, and compute series with the new variables as indices, where $a, c, d \geq 0$ and $b \geq 1$.

2.2 Sub case 2.2

$-v_2 - v_3 < -v_1 \Rightarrow v_1 < v_2 + v_3$

For $v_{11} \geq -v_2$, we have $p^{\min}\mu(a_{11})(1+\alpha\rho) = p^{v_2}(1+v_2\rho)$.

For $-v_1 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}\mu(a_{11})(1+\alpha\rho) = \sum_{-v_1}^{-v_2-1} p^{v_2}\rho(1+v_2\rho) = p^{v_2}(v_1 - v_2)\rho(1+v_2\rho) = p^{v_2}(v_1-v_2)\rho + p^{v_2}(v_1-v_2)v_2\rho^2$.

For $-v_2 - v_3 \leq v_{11} \leq -v_1 - 1$, we have $p^{\min}\mu(a_{11})(1+\alpha\rho) = \sum_{-v_2-v_3}^{-v_1-1} p^{v_2}\rho(1+\alpha\rho) = \sum_{i=1}^{v_2+v_3-v_1} p^{v_2}\rho(1+(v_2-i)\rho) = p^{v_2}(v_2-v_3+v_1)\rho + p^{v_2}(v_2-v_3+v_1)v_2\rho^2 - p^{v_2}\binom{v_2+v_3-v_1+1}{2}\rho^2$.

So we have $\sum_{v_{11}\geq -v_2-v_3} = p^{v_2}(1+v_2\rho+(v_1-v_2)\rho+(v_1-v_2)v_2\rho^2+(v_2-v_3+v_1)\rho+(v_2-v_3+v_1)v_2\rho^2 - \frac{(v_2-v_3+v_1+1)(v_2-v_3+v_1)}{2}\rho^2)$. We compute the differences between v_1, v_2, v_3, v_4, we have several arrangements,

1. $v_2 + v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_3 \overset{b}{\geq} v_4 \overset{a}{>} v_2 \overset{e}{\geq} 0$

   But $c = v_1 - v_3$ and $d = v_2 + v_3 - v_1$, so $e = v_2 = c + d$

   Thus, we substitute $v_1 \to a + b + 2c + d$, $v_2 \to c + d$, $v_3 \to a + b + c + d$, $v_4 \to a + c + d$, and compute series with the new variables as indices, where $b, c \geq 0$ and $a, d \geq 1$.

2. $v_2 + v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_4 \overset{b}{>} v_3 \overset{a}{>} v_2 \overset{e}{\geq} 0$

   But $v_2 + v_3 - d - c - b = v_3 \Rightarrow b + c + d = v_2 + v_3 - v_3 = v_2$

   Thus, we substitute $v_1 \to a + 2b + 2c + d$, $v_2 \to b + c + d$, $v_3 \to a + b + c + d$, $v_4 \to a + 2b + c + d$, and compute series with the new variables as indices, where $c \geq 0$ and $a, b, d \geq 1$.

3. $v_2 + v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_4 \overset{b}{>} v_2 \overset{a}{\geq} v_3 \overset{e}{\geq} 0$

   But $v_2 + v_3 - d - c - b = v_2 \Rightarrow v_3 = v_2 + v_3 - v_2 = b + c + d$

   Thus, we substitute $v_1 \to a + 2b + 2c + d$, $v_2 \to a + b + c + d$, $v_3 \to b + c + d$, $v_4 \to a + 2b + c + d$, and compute series with the new variables as indices, where $a, c \geq 0$ and $b, d \geq 1$.

3 Case 3

$v_4 > v_3 > v_1 \geq 0$

$v_4 > v_2 \geq 0$

$\Rightarrow v_{11} \geq -v_2 - v_1$

$\Rightarrow \alpha := \min\{v_2, v_2 + v_{11} + v_4\} = \begin{cases} v_2, & v_{11} \geq -v_4 \\ v_2 - 1, v_2 - 2, \ldots, & v_{11} < -v_4 \end{cases}$

We have two sub cases

3.1 Sub case 3.1

$-v_2 - v_1 \geq -v_4 \Rightarrow v_2 + v_1 \leq v_4$

There is no phase transition here, since $v_{11} \geq -v_2 - v_1 \geq -v_4$

For $v_{11} \geq -v_2$, we have $p^{\min}\mu(a_{11})(1 + \alpha\rho) = p^{v_2}(1 + v_2\rho)$.

For $-v_2 - v_1 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}\mu(a_{11})(1+\alpha\rho) = \sum_{-v_2-v_1}^{-v_2-1} p^{v_2}\rho(1+ v_2\rho) = p^{v_2}v_1\rho + p^{v_2}v_1v_2\rho^2)$.

So we have $\sum_{v_{11} \geq -v_2+v_1} p^{\min}\mu(a_{11})(1 + \alpha\rho) = p^2(1 + v_2\rho + v_1\rho + v_1v_2\rho^2$.

We have several arrangements,

1. $v_4 \overset{d}{\geq} v_1 + v_2 \overset{c}{\geq} v_3 \overset{b}{>} v_2 \overset{a}{>} v_1 \overset{e}{\geq} 0$

   But $v_1 + v_2 - c - b = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = b + c$

   Thus, we substitute $v_1 \to b + c$, $v_2 \to a + b + c$, $v_3 \to a + 2b + c$, $v_4 \to a + 2b + 2c + d$, and compute series with the new variables as indices, where $c, d \geq 0$ and $a, b \geq 1$.

2. $v_4 \overset{d}{\geq} v_1 + v_2 \overset{a}{\geq} v_2 \overset{c}{\geq} v_3 \overset{b}{>} v_1 \overset{a}{\geq} 0$

   We substitute $v_1 \to a$, $v_2 \to a+b+c$, $v_3 \to a+b$, $v_4 \to 2a+b+c+d$, and compute series with the new variables as indices, where $a, c, d \geq 0$ and $b \geq 1$.

3.2 Sub case 3.2

$-v_2 - v_1 < -v_4 \Rightarrow v_2 + v_1 > v_4$

For $v_{11} \geq -v_2$, we have $p^{\min}\mu(a_{11})(1+\alpha\rho) = p^{v_2}(1+v_2\rho)$.

For $-v_4 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}\mu(a_{11})(1+\alpha\rho) = \sum_{-v_4}^{-v_2-1} p^{v_2}\rho(1+v_2\rho) = p^{v_2}(v_4 - v_2)\rho + p^{v_2}(v_4 - v_2)v_2\rho^2$.

For $-v_2 - v_1 \leq v_{11} \leq -v_4 - 1$, we have $p^{\min}\mu(a_{11})(1+\alpha\rho) = \sum_{-v_2-v_1}^{-v_4-1} p^{v_2}\rho(1+v_2\rho) = p^{v_2}(v_1+v_2-v_4)\rho + p^{v_2}(v_1+v_2-v_4)v_2\rho^2 - p^{v_2}\binom{v_1+v_2-v_4+1}{2}\rho^2$.

So we have $\sum_{v_{11} \geq -v_2+v_1} p^{\min}\mu(a_{11})(1+\alpha\rho) = p^2(1+v_2\rho+(v_4-v_2)\rho+(v_4-v_2)v_2\rho^2 + (v_1+v_2-v_4)\rho + (v_1+v_2-v_4)v_2\rho^2 - \frac{(v_1+v_2-v_4+1)(v_1+v_2-v_4)}{2}\rho^2)$

We have several arrangements,

1. $v_1 + v_2 \overset{d}{>} v_4 \overset{c}{\geq} v_3 \overset{b}{>} v_2 \overset{a}{>} v_1 \overset{e}{\geq} 0$

   But $v_1 + v_2 - d - c - b = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = b + c + d$

   Thus, we substitute $v_1 \to b+c+d$, $v_2 \to a+b+c+d$, $v_3 \to a+2b+c+d$, $v_4 \to a+2b+2c+d$, and compute series with the new variables as indices, where $c \geq 0$ and $a, b, d \geq 1$

2. $v_1 + v_2 \overset{d}{>} v_4 \overset{c}{>} v_2 \overset{b}{\geq} v_3 \overset{a}{>} v_1 \overset{e}{\geq} 0$

   But $v_1 + v_2 - d - c = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = c + d$

   Thus, we substitute $v_1 \to c+d$, $v_2 \to a+b+c+d$, $v_3 \to a+c+d$, $v_4 \to a+b+2c+d$, and compute series with the new variables as indices, where $b \geq 0$ and $a, c, d \geq 1$