Exercise Let $\{E_{i,j}\}_{i<j}$ be the set of all elementary matrices, of this form. Prove that $E_{i,j}^{-1} = (b_{l,k})$ is $E_{i,j} = (a_{l,k})$, when we substitute $a_{i,j} = 1$ with $b_{i,j} = -1$

Proof We can see that directly from the fact that if we multiply $E_{i,j}^{-1}$ by $E_{i,j}$ from the left then $E_{i,j}$ is operating on $E_{i,j}^{-1}$ by adding row j to row i
So, in the product matrix, $(c_{l,k})$, in order to have 1
on the main diagonal, we need them to exist on the main diagonal of $E_{i,j}^{-1}$, to begin with. Now, in order to have $c_{i,j} = 0$, we need to have the addition of j to i giving $c_{i,j} = a_{i,j} + b_{i,j} = 0 \Rightarrow b_{i,j} = -a_{i,j} = -1$

Exercise Prove that if $(a_{ij}) = E_{i,j}, i < j$ is an elementary matrix, then $\forall m \in (N), E_{i,j}^m$ is $E_{i,j}$, but with $a_{ij} = m$

$$E_{i,j} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$E_{i,j}^2 = E_{i,j} \cdot E_{i,j}$$

Since $E_{i,j}$ is en elemntary matrix, then it operates on the right matrix as an addition of row j to row i

So,

$$E_{i,j}^2 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 2 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

We assume this is true for all $E_{i,j}^m$, now we prove for $E_{i,j}^{m+1}$

$$E_{i,j}^{m+1} = E_{i,j} \cdot E_{i,j}^m = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}^m$$

(by the assumption)

$$= \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & m & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & m+1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

From these two exercises, we obtain an almost trivial corollary
Exercise Prove that if $(a_{ij}) = E_{i,j}, i < j$ is an elementary matrix, then $\forall m \in (N), (E_{i,j}^{-1})^m = (E_{i,j}^m)^{-1} = E_{i,j}^{-m}$ is $E_{i,j}$, but with $a_{ij} = -m$

Proof We immitate both proofs from above (we can either show how the power of m is operating on $E_{i,j}^{-1}$, or show how the inversion is operating on $E_{i,j}^m$).

Commutators of elementary matrices

Let $\{E_{i,j}\}_{i<j}$ be the set of all elementary matrices of this form.

Exercise $(a_{l,m}) = E_{i,j}^{-1}$ is the matrix with 1 on the main diagonal, and -1 in $a_{i,j}$

Proof We can see that directly from the fact that in order to have
$(c_{l,m}) = (a_{l,m}) \cdot (b_{l,m}) = E_{i,j} \cdot E_{i,j}^{-1} = I$,
we need to have $c_{i,j} = 0$, which means that adding row j to row i, in $E_{i,j}^{-1}$ (by the left multiplication of $E_{i,j}$)
must give $a_{i,j} + b_{i,j} = c_{i,j} = 0 \Rightarrow b_{i,j} = -a_{i,j} = -1$

Exercise $[E_{i,j}, E_{j,k}] = E_{i,k}$

Proof $E_{i,j}$ is operating from left on $E_{j,k}$ by addition of row j to row i, so, the product matrix, $(a_{l,m}) = E_{i,j} \cdot E_{j,k}$ has 1 on the main diagonal and in $a_{j,k}, a_{i,j}, a_{i,k}$
$E_{i,j}^{-1}$ is operating from left on $E_{j,k}^{-1}$ by subtraction of row j from row i, so, the product matrix, $(b_{l,m}) = E_{i,j}^{-1} \cdot E_{j,k}^{-1}$ has 1 on the main diagonal and in $b_{i,k}$, and -1 in $b_{j,k}, b_{i,j}$
Multiplying $(a_{l,m}) \cdot (b_{l,m})$ yields a product matrix, $(c_{l,m})$ with 1 on the main diagonal, and,
since $a_{i,i} = a_{i,j} = a_{i,k} = 1$, with all other cells in row j being 0, and since $b_{i,k} = b_{k,k} = 1$, and $b_{j,k} = -1$, multiplying row $(a_{l,m})_i$ by column $(b_{l,m})_k$ yields the value $c_{i,k} = b_{i,k} + b_{j,k} + b_{k,k} = 1 - 1 + 1 = 1$
We can see that multiplying $(a_{l,m})_i \cdot (b_{l,m})_j$ yields $c_{i,j} = a_{i,i} \cdot b_{i,j} + a_{i,j} \cdot b_{j,j} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$
And, we can see that multiplying $(a_{l,m})_j \cdot (b_{l,m})_k$ yields $c_{j,k} = a_{j,j} \cdot b_{j,k} + a_{j,k} \cdot b_{k,k} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

Conclusion

$[E_{j,k}, E_{i,j}] = E_{j,k} \cdot E_{i,j} \cdot E_{j,k}^{-1} \cdot E_{i,j}^{-1} = ((E_{i,j}^{-1})^{-1} \cdot (E_{j,k}^{-1})^{-1} \cdot E_{i,j}^{-1} \cdot E_{j,k}^{-1})^{-1} = (E_{i,j} \cdot E_{j,k} \cdot E_{i,j}^{-1} \cdot E_{j,k}^{-1})^{-1} = [E_{i,j}, E_{j,k}]^{-1}$

For example, $n = 4$,

$$E_{1,2} \cdot E_{2,3} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_{1,2}^{-1} \cdot E_{2,3}^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[E_{1,2} \cdot E_{2,3}] = E_{1,2} \cdot E_{2,3} \cdot E_{1,2}^{-1} \cdot E_{2,3}^{-1} =$$

$$= \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{1,3}$$

Exercise $[E_{i,j}, E_{l,k}] = I$, where $j \neq l$

Proof $E_{i,j}$ is operating from left on $E_{l,k}$ by addition of row j to row i, so, the product matrix, $(a_{n,m} = E_{i,j} \cdot E_{l,k}$ has 1 on the main diagonal and in $a_{l,k}, a_{i,j}$

$E_{i,j}^{-1}$ is operating from left on $E_{l,k}^{-1}$ by subtraction of row j from row i, so, the product matrix, $(b_{n,m} = E_{i,j}^{-1} \cdot E_{l,k}^{-1}$ has 1 on the main diagonal,

and -1 in $b_{l,k}, b_{i,j}$

We can see that multiplying $(a_{n,m})_i \cdot (b_{n,m})_j$ yields $c_{i,j} = a_{i,i} \cdot b_{i,j} + a_{i,j} \cdot b_{j,j} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

And, we can see that multiplying $(a_{n,m})_l \cdot (b_{n,m})_k$ yields $c_{l,k} = a_{l,l} \cdot b_{l,k} + a_{l,k} \cdot b_{k,k} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

For example, $n = 4$,

$$E_{1,2} \cdot E_{3,4} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_{1,2}^{-1} \cdot E_{3,4}^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[E_{1,2} \cdot E_{3,4}] = E_{1,2} \cdot E_{3,4} \cdot E_{1,2}^{-1} \cdot E_{3,4}^{-1} =$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

Conclusion

$[E_{i,j},[E_{j,k},E_{k,l}]]=[E_{i,j},E_{j,l}]=E_{i,l}$
$[E_{i,j},[E_{j,k},E_{m,l}]],m\neq k=[E_{i,l},I]=I$
$[E_{i,m},[E_{j,k},E_{k,l}]],m\neq j=[E_{i,m},E_{j,l}]=I$

$$\Rightarrow [E_{i_1,i_2},[E_{i_3,i_4},...[E_{i_{n-2},i_{n-1}},E_{i_{n-1},i_n}]]=\begin{cases}E_{i_1,i_n}, & i_{2k}=i_{2k+1},\forall 1\leq k\leq \frac{n}{2}-1\\ I, & \text{otherwise}\end{cases}$$

Exercise

$\#\{E_{i,j}\in M_n(\mathbb{Z})\}_{i<j}=\binom{n}{2}$

Proof

$(a_{i,j}=E_{i,j})$. We need to count the options for 1 above the main diagonal.
$a_{l,l}=1,\forall 1\leq l\leq n$, so, if $i=l$, we have $n-l=n-i$ options to choose the column index j.
So, the total number of options for i,j is $\sum_{k=1}^{n-1}=\frac{(1+n-1)\cdot(n-1)}{2}=\frac{n\cdot(n-1)}{2}=\binom{n}{2}$

This means that we have $\binom{n}{2}^2$ commutators of the form $[E_{i,j},E_{l,k}]$.

Exercise

$\#\{[E_{i,j},E_{l,k}]\neq I\in M_n(\mathbb{Z})\}_{i<j}=2\cdot\binom{n}{3}$

Proof

As shown above, $[E_{i,j},E_{l,k}]\neq I\Leftrightarrow j=l$
Which means we're counting all the commutators of the form $[E_{i,j},E_{j,k}]$.
So, the count of such commutators is based on the number of options to choose
ordered triples $\{i,j,k\}$ out of the ordered set $[n]=\{1,2,...,n\}$, which is $\binom{n}{3}$
But, as already shown above, $[E_{l,k},E_{i,j}]=[E_{i,j},E_{l,k}]^{-1}$, so, for each triple $\{i,j,k\}$, we have two commutators, $[E_{i,j},E_{j,k}]$ and its inverse, which sum up to $\binom{n}{3}$ pairs of commutators.

For example, $n=5$,

$$(a_{l,k})=E_{i,j}=\begin{pmatrix}1 & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5}\\ 0 & 1 & a_{2,3} & a_{2,4} & a_{2,5}\\ 0 & 0 & 1 & a_{3,4} & a_{3,5}\\ 0 & 0 & 0 & 1 & a_{4,5}\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

Where $a_{i,j} = 1$, and all other $a_{l,k} = 0$
The number of options for choosing i, j, in this case, are $1+2+3+4 = 10 = \binom{5}{2}$,
so, we have $10^2 = 100$ commutators. The number of triples we can choose from $[5] = \{1, 2, 3, 4, 5\}$ is
$\#\{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\}\} = 10 = \binom{5}{3}$,
so we have 10 commutators that are not the unit matrix, and their inverse, total $20 = 2 \cdot 10 = 2 \cdot \binom{5}{3}$.

Exercise
Given the set of commutators of elementary matrices of the form
$\{[E_{i,j}, E_{j,k}] \in M_n(\mathbb{Z})\}_{i<j<k}$,
we can divide this set to subsets of the form
$\{\{[E_{i_1,j_{1,1}}, E_{j_{1,1},k_1}], [E_{i_1,j_{1,2}}, E_{j_{1,2},k_1}], ..., [E_{i_1,j_{1,l_1}}, E_{j_{1,l_1},k_1}]\}, ...,$
$\{[E_{i_m,j_{m,1}}, E_{j_{m,1},k_1}], ..., [E_{i_m,j_{m,l_m}}, E_{j_{m,l_m},k_1}]\}\}$
These subsets are equivalence classes, trivially, since the relation is equality (i.e. $[E_{i_l,j_{l,m_1}}, E_{j_{l,m_1},k_l}] = [E_{i_l,j_{l,m_2}}, E_{j_{l,m_2},k_l}], i_l < j_{l,m_1}, j_{l,m_2} < k_l$).
Fix $i, k, 1 \leq i \leq n-1, 3\ leqk \leq n$, then all the triples of the form $\{i, j, k\}, i \leq i+1 \leq k-1$ are in the same equivalence class,
due to the above equality. So, the number of these equivalence classes is $2 \cdot \binom{n-1}{2}$

Proof

By induction on n. For $n = 3$, we have only one triple, namely $\{1, 2, 3\}$, so $\binom{3-1}{2} = \binom{2}{2} = 1$
For $n + 1$, we shall observe that if we add one to the upper bound (i.e. $n \to n' = n + 1$,
then we add one more equivalence class, for each one of the lower bounds of $n' - 1 = n$ (i.e., the index i).
But we also add a new equivalence class, whose lower bound is $i = n+1-2 = n-1 = n'-2$, which was not in any equivalence class
for $n = n' - 1$, since we consider only the triples where $i \leq n - 2$. So, if we mark m_n as the number of equivalence classes
for n, then we have $m_{n'} = m_{n+1} = m_n + (n-2) + 1 = m_n + n - 1$. But, by the assumption, $m_n = \binom{n-1}{2}$,
so $m_{n'} = m_{n+1} = m_n + n - 1 = \binom{n-1}{2} + n - 1 = \frac{(n-1)\cdot(n-2)}{2} + n - 1 =$

$\frac{n^2-3n+2}{2}+n-1=\frac{n^2-3n+2+2n-2}{2}=\frac{n^2-n}{2}=\frac{n\cdot(n-1)}{2}=\binom{n}{2}=m_{n+1}=m_{n'}$, and we proved the assumption

The group $U_n(\mathbb{Z})$
We have proved several basic facts, regarding elementary matrices, of the form $\{E_{i,j}\}_{i<j}$.
Now, we shall propose a few more basic facts.
Notation
We mark by $U_n(\mathbb{Z})$ the set of all upper triangular matrices $n\times n$ with 1 in the main diagonal, and any integer values above the main diagonal, $a_{i,j}\in\mathbb{Z}$.

Exercise Prove that the set $U_n(\mathbb{Z})$ is a group, with the usual operation of matrix multiplication.

Proof

$$(a_{i,j})=A=\begin{pmatrix}1 & a_{1,2} & \dots & a_{1,n-1} & a_{1,n}\\ 0 & 1 & \dots & a_{2,n-1} & a_{2,n}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & a_{k,n-1} & a_{k,n}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & a_{n-1,n}\\ 0 & 0 & \dots & 0 & 1\end{pmatrix},(b_{i,j})=B=\begin{pmatrix}1 & b_{1,2} & \dots & b_{1,n-1} & b_{1,n}\\ 0 & 1 & \dots & b_{2,n-1} & b_{2,n}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & b_{k,n-1} & b_{k,n}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & b_{n-1,n}\\ 0 & 0 & \dots & 0 & 1\end{pmatrix}$$

We need to prove that $(c_{i,j})=A\cdot B\in U_n(\mathbb{Z})$
I. $1\le l\le n, c_{l,l}=\sum_{k=1}^{n}a_{l,k}\cdot b_{k,l}$.
We observe that $a_{l,1}=a_{l,2}=\dots=a_{l,l-1}=0$, and $b_{l+l,l}=b_{l+2,l}=\dots=b_{n,l}=0$.
So, $c_{l,l}=\sum_{k=1}^{n}a_{l,k}\cdot b_{k,l}=0+0+\dots+0+a_{l,l}\cdot b_{l,l}+0+0\dots0=1$.
This proves that each element on the main diagonal of $(c_{i,j})$ is 1.

II. $2\le l\le n, 1\le m\le l-1, c_{l,m}=\sum_{k=1}^{n}a_{l,k}\cdot b_{k,m}$.
We observe that $a_{l,1}=a_{l,2}=\dots=a_{l,l-1}=0$, and $b_{m+1,m}=b_{m+2,m}=\dots=b_{n,m}=0$.
This means, that $a_{l,k}\cdot b_{k,m}=0, 1\le k\le l-1$,
because the first $l-1$ elements of $a_{l,k}$ are 0.
and the last $n-m$ elements of $b_{k,m}$ are also 0.

This proves that each element <u>under</u> the main diagonal of $(c_{i,j})$ is 0.

III. $2 \leq l \leq n, 1 \leq m \leq l-1, c_{m,l} = \sum_{k=1}^{n} a_{m,k} \cdot b_{k,l}$.
$a_{m,k}, b_{k,l} \in \mathbb{Z} \Rightarrow \sum_{k=1}^{n} a_{l,k} \cdot b_{k,m} \in \mathbb{Z}$.
This proves the each element <u>above</u> the main diagonal of $(c_{i,j})$ is an integer.

Thus, we prove that $U_n(\mathbb{Z})$ is closed under matrix multiplication.

Associativity is obvious, from the fact that matrix multiplication is associative.

Obviously, I_n is a matrix of this form, so the unit of $U_n(\mathbb{Z})$ is I_n.

The fact that all matrices of this form have an inverse is obvious by looking at the rank of a matrix of this form, which, clearly, is n, since the matrix is already in a reduced form.

Conclusion: $U_n(\mathbb{Z})$ is a group.

Exercise All the matrices of the form $\{E_{i,i+1}, 1 \leq i \leq n-1\}$ generate $U_n(\mathbb{Z})$.

Proof
We already proved that $[E_{i,j}, E_{j,k}] = E_{i,k}$, specifically, $[E_{i,i+1}, E_{i+1,i+2}] = E_{i,i+2}$.
Also, we proved that $\forall m \in \mathbb{N}, (a_{i,j})^m = E_{i,j}^m = (a_{i,j})$ with $a_{i,j} = m$.
This mean that all the matrices of the form $\{E_{i,i+1}\}$ yield all the matrices which have only one integer above the main diagonal.

Let

$$(a_{i,j}) = A = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{i,j} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, (b_{l,k}) = B = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & b_{l,k} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$|a_{i,j}|, |b_{l,k}| > 1$, and the two pairs of indices, $\{i,j\}, \{l,k\}$, are pairwise disjoint. Then, $(c_{m,p}) = A \cdot B$ is the matrix with 1 on the main diagonal, 0 below the main diagonal,
and $c_{i,j} = a_{i,j}, c_{l,k} = b_{l,k}$.
To prove this, we can observe that since $|a_{i,j}|, |b_{l,k}| > 1$, we can write (based on what we already proved)
$(a_{i,j}) = A = E_{i,j}^{a_{i,j}}$

The free Lie algebra L_n

The group $U_n(\mathbb{Z})$ gives rise to a Lie algebra over some field $\mathbb{F}$, which consists of all matrices of the form

$$(a_{l,k}) = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ 0 & 0 & a_{2,3} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Which means, we have 0 on the main diagonal and under it, and $a_{i,j} \in \mathbb{F}, i < j$,
that is, arbitrary scalars above the main diagonal. These matrices are precisely the matrices of $U_n(\mathbb{Z})$
minus the unit matrix, and with scalars from $\mathbb{F}$, rather than $\mathbb{Z}$.

Exercise This is a Lie algebra over $\mathbb{F}$

Proof

Defining the multiplication (Lie brackets) on L_n by the standard matrix commutator (i.e., $\forall A, B \in M_n(\mathbb{F}), [A, B] := A \cdot B - B \cdot A$, automatically gives all the Lie algebra axioms, for example,
$\forall A, B, C \in M_n(\mathbb{F}), \alpha, \beta \in \mathbb{F}$,
$[\alpha \cdot A + \beta \cdot B, C] = (\alpha \cdot A + \beta \cdot B) \cdot C - C \cdot \alpha \cdot A + \beta \cdot B) =$
$(\alpha \cdot A) \cdot C + (\beta \cdot B) \cdot C - (C \cdot (\alpha \cdot A) + C \cdot (\beta \cdot B) =$
$\alpha \cdot (A \cdot C) + \beta \cdot (B \cdot C) - \alpha \cdot (C \cdot A) - \beta \cdot (C \cdot B) =$
$\alpha \cdot (A \cdot C) - \alpha \cdot (C \cdot A) + \beta \cdot (B \cdot C) - \beta \cdot (C \cdot B) =$
$\alpha \cdot (A \cdot C - C \cdot A) + \beta \cdot (B \cdot C - C \cdot B) = \alpha \cdot [A, C] + \beta \cdot [B, C]$
, Thus we prove linearity in the first component. There should be no need to prove all other axioms,
since they can all be easily proved using the same matrix and scalar multiplications.
We do need to prove that L is closed under Lie brackets defined above.
Let

$$A = (a_{l,k}) = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ 0 & 0 & a_{2,3} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, B = (b_{l,k}) = \begin{pmatrix} 0 & b_{1,2} & b_{1,3} & \dots & b_{1,n} \\ 0 & 0 & b_{2,3} & \dots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & b_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Then,

$$A \cdot B = \begin{pmatrix} 0 & 0 & a_{1,2} \cdot b_{2,3} & \dots & \sum_{k=2}^{n-1} a_{1,k} \cdot b_{k,n} \\ 0 & 0 & 0 & \dots & \sum_{k=3}^{n-1} a_{2,k} \cdot b_{k,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{n-1,n} \cdot b_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 0 & 0 & b_{1,2} \cdot a_{2,3} & \dots & \sum_{k=2}^{n-1} b_{1,k} \cdot a_{k,n} \\ 0 & 0 & 0 & \dots & \sum_{k=3}^{n-1} b_{2,k} \cdot a_{k,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & b_{n-1,n} \cdot a_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

And then,

$$[A, B] = A \cdot B - B \cdot A = \begin{pmatrix} 0 & 0 & a_{1,2} - b_{1,2} \cdot a_{2,3} & \dots & \sum_{k=2}^{n-1} a_{1,k} \cdot b_{k,n} - \sum_{k=2}^{n-1} b_{1,k} \cdot a_{k,n} \\ 0 & 0 & 0 & \dots & \sum_{k=3}^{n-1} a_{2,k} \cdot b_{k,n} - \sum_{k=3}^{n-1} b_{2,k} \cdot a_{k,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{n-1,n} \cdot b_{n-1,n} - b_{n-1,n} \cdot a_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

So, the Lie brackets operation, defined above, is preserving the form of the matrices in L_n.
In addition, we observe that the second diagonal, above the main diagonal, is also 0, thus we conclude
that $\forall A, B, C, D \in L_n, [[A, B], [C, D]]$ will have the main, the second and the third diagonal all 0.
In other words, L_n is a nilpotent Lie algebra.

After verifying that the Lie brackets operation is defined, as multiplication in L_n, we observe also that L_n
is closed under addition, defined as the standard matrix addition. But this is obvious since the matrix addition is simply $a_{i,j} + b_{i,j}, 1 \leq i, j \leq n$,
and so, the sums of all 0 on the main diagonal and under it, which means $a_{i,j} + b_{i,j}, j \leq i = 0 + 0 = 0$
And the sums of all elements above the main diagonal, $a_{i,j} + b_{i,j}, j > i$, are sums of two scalars in $\mathbb{F}$,.
hence, scalars in the same field.

in the same way, L_n being closed under multiplication by a scalar is also a trivial fact.

Exercise The Lie algebra L_n has a base, which is all the elementary matrices of the form $E_{i,i+1}$ minus the unit matrix, that is, $e_{i,i+1} = E_{i,i+1} - I$.

Exercise The Lie brackets operation on L_n is compatible
with the operation of the U_n group commutators
i.e., $[e_{i,j}, e_{l,k}] \neq 0 \Leftrightarrow j = l \vee i = k$.
Proof
We shall distinguish between the two cases,
$\underline{j = l}$
$A = (a_{i,j})$ a matrix with 1 only in $a_{i,j}$
$B = (b_{j,k})$ a matrix with 1 only in $b_{j,k}$
Then $(c_{i,k}) = A \cdot B$ is a matrix with 1 only in $c_{i,k}$
(we obtain this by multiplying row i of A with column k of B, having only $a_{i,j} \cdot b_{j,k} = 1 \cdot 1 = 1$),
while $(d_{s,t}) = B \cdot A$ is a zero matrix. since multiplying row j of B with column j of A yields $b_{j,k} \cdot a_{k,j} + b_{j,i} \cdot a_{i,j} =$
$1 \cdot 0 + 0 \cdot 1 = 0 + 0 = 0$
Hence, $A \cdot B - B \cdot A = (c_{i,k}) - 0 = (c_{i,k}) = e_{i,k} = E_{i,k} - I$

$\underline{i = k}$
$A = (a_{i,j})$ a matrix with 1 only in $a_{i,j}$
$B = (b_{k,i})$ a matrix with 1 only in $b_{k,i}$
We can observe that this case is exactly the opposite of the first case,
since multiplying $A \cdot B$ yields a zero matrix, while multiplying
$B \cdot A$ yields a matrix $c_{k,j}$ with 1 only in $c_{k,j}$
Hence, $A \cdot B - B \cdot A = 0 - (c_{k,j}) = -(c_{k,j}) = -e_{k,j} = E_{k,j} - I$

Exercise
The Lie algebra $L = L_n$, has its lower central series,
$\gamma_k(L)$ (defined recursively by $\gamma_k(L) := [L, \gamma_{k-1}(L)]$, where $\gamma_1(L) := L$),
$\gamma_k(L) \neq 0, \forall k < n$, and $\gamma_n(L) = 0$

Proof

As seen above, $[e_{i,j}, e_{l,k}] = \begin{cases} e_{i,k} & j = l \\ -e_{l,j} & i = k \\ 0 & otherwise \end{cases}$

So, if the set of generators of L_n is $A(L) = \{e_{i,i+1} | 1 \leq i \leq n-1\}$, we can see that

$\gamma_2(A(L)) = [A(L), A(L)] = \{e_{i,i+2} | 1 \leq i \leq n-1\}$,

containing only elements which come from commutators of the form

$[e_{i,i+1}, e_{i+1,i+2}] = e_{i,i+2}$ (or $[e_{i+1,i+2}, e_{i,i+1}] = -e_{i,i+2}$)

So, $\gamma_3(A(L)) = [A(L), \gamma_2(A(L))] = \{e_{i,i+3} | 1 \leq i \leq n-1\}$, since the only commutators that are not 0, are of the form $[e_{i,i+1}, e_{i+1,(i+1)+2}] = [e_{i,i+1}, e_{i+1,i+1+2}]$ $= [e_{i,i+1}, e_{i+1,i+3}] = e_{i,i+3}$

Obviously, $\gamma_{n-1}(A(L))$ must contain only commutators of the form $e_{i,i+n-1}$, which can only be $e_{1,1+n-1} = e_{1,n}$

That means, for $\gamma_n(A(L))$, we have no valid commutators, since we do not have any element with a second index $n+1$

So, $\gamma_n(A(L)) = 0$

Example $L = L_6$

$\gamma_1(L) = L = \{e_{1,2}, e_{2,3}, e_{3,4}, e_{4,5}, e_{5,6}, e_{1,3}, e_{2,4}, e_{3,5}, e_{4,6}, e_{1,4}, e_{2,5}, e_{3,6}, e_{1,5}, e_{2,6}, e_{1,6}\}$

$\gamma_2(L) = [L, \gamma_1(L)] = [L, L] = \{e_{1,3}, e_{2,4}, e_{3,5}, e_{4,6}, e_{1,4}, e_{2,5}, e_{3,6}, e_{1,5}, e_{2,6}, e_{1,6}\}$

$\gamma_3(L) = [L, \gamma_2(L)] = \{e_{1,4}, e_{2,5}, e_{3,6}, e_{1,5}, e_{2,6}, e_{1,6}\}$

$\gamma_4(L) = [L, \gamma_3(L)] = \{e_{1,5}, e_{2,6}, e_{1,6}\}$

$\gamma_5(L) = [L, \gamma_4(L)] = \{e_{1,6}\}$

$\gamma_6(L) = [L, \gamma_5(L)] = 0$

Exercise Each element of the lower central series, $\gamma_k(L_n), 1 \leq i \leq n$ is a subalgebra of L_n

proof

As seen above, $\gamma_k(L_n) = \{e_{i,j} | j - i \geq k\}$

Let $e_{i,j}, e_{m,l}$, where $j - i, l - m \geq k$

If $j \neq m \wedge i \neq l$, then $[e_{i,j}, e_{m,l}] = [e_{m,l}, e_{i,j}] = 0$

Otherwise, if $j - m$, then $[e_{i,j}, e_{m,l}] = [e_{i,m}, e_{m,l}] = e_{i,l}$

But, $j - i = m - i \geq k$, and $l - m \geq k$, so $l - i = l(-m + m) - i =$ $(l - m) + (m - i) \geq 2 \cdot k > k \Rightarrow [e_{i,j}, e_{m,l}] = e_{i,l} \in \gamma_k(L_n)$

Exercise Let $x \in L_n$ be an element in the Lie algebra of dimension n, then, the centralizer of x in L_n, denoted as
$C_L(x) := \{y \in L | [y, x] = [x, y] = 0\}$
is a subalgebra of L_n.

Proof
Let $y, z \in C_L(x)$, then $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ (Jacobi identity)
But, $y \in C_L(x) \Rightarrow [z, [x, y]] = [z, 0] = 0$
and, $z \in C_L(x) \Rightarrow [y, [z, x]] = [y, 0] = 0$
$\Rightarrow [x, [y, z]] + [z, [x, y]] + [y, [z, x]] = [x, [y, z]] + 0 + 0 = 0$
$\Rightarrow [x, [y, z]] = 0 \Rightarrow [y, z] \in C_L(x)$

Exercise
Let L_n be a Lie algebra, then $Aut(L_n)$ is a group
Proof
Every automorphism can be represented by an invertible $\binom{n}{2} \times \binom{n}{2}$ matrix, so $Aut(L_n) \cong GL_{\binom{n}{2}}(L_n)$ (and $I_{\binom{n}{2}}$ is the trivial automorphism)

Exercise
Let L_n be a Lie algebra. Then,
for each vector of $n-1$ scalars, $\overline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$,
where $\lambda_r \neq 0, 1 \leq r \leq n-1$, the map $\varphi_{\overline{\lambda}} : L_n \to L_n$, defined as,
$\varphi_{\overline{\lambda}}(e_{i,j}) := \Pi_{r=i}^{j-1} \lambda_r \cdot e_{i,j}$ is an automorphism

Proof It is obvious that $\varphi_{\overline{\lambda}}(e_{i,j})$, as a $\binom{n}{n} \times \binom{n}{2}$ matrix is a diagonal matrix, with $\Pi_{r=i}^{j-1} \neq 0$ in the $a_{l,l}$ cell, matching i, j

$$\varphi_{\overline{\lambda}}(e_{i,j}) = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \Pi_{r=1}^{j-1} \lambda_r & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \Pi_{r=1}^{n-2} & 0 \\ 0 & 0 & \ldots & 0 & 0 & \Pi_{r=1}^{n-1} \end{pmatrix},$$

Exercise The Lie algebra L_n, has an automorphism, η, defined by,
$1 \leq i < j \leq n, a_{i,j} \in \mathbb{F}, \eta(a_{i,j} \cdot e_{i,j}) := (-1)^{j-i+1} \cdot (a_{i,j} \cdot e_{n-j+1,n-i+1}) = a_{i,j} \cdot (-1)^{j-i+1} \cdot e_{n-j+1,n-i+1}$

Proof

First, we need to show that η is a homomorphism.
We shall omit the scalars, as the Lie brackets operation is bilinear.
$1 \leq i < j < k \leq n, [\eta(e_{i,j}), \eta(e_{j,k})] =$
$[(-1)^{j-i+1} \cdot e_{n-j+1,n-i+1}, (-1)^{k-j+1} \cdot e_{n-k+1,n-j+1}] =$
$(-1)^{j-i+1} \cdot (-1)^{k-j+1}[e_{n-j+1,n-i+1}, e_{n-k+1,n-j+1}] =$
$(-1)^{j-i+1+k-j+1} \cdot -e_{n-k+1,n-i+1} = (-1)^{k-i+2} \cdot -e_{n-k+1,n-i+1} =$
$(-1)^{k-i} \cdot (-1)^2 \cdot -e_{n-k+1,n-i+1} = (-1)^{k-i} \cdot -e_{n-k+1,n-i+1} =$
$(-1)^{k-i} \cdot (-1) \cdot e_{n-k+1,n-i+1} = (-1)^{k-i+1} \cdot e_{n-k+1,n-i+1} = \eta(e_{i,k}) = \eta([e_{i,j}, e_{j,k}])$
So, we proved that η is a homomorphism, obviously from L_n to itself.
To prove the rest, we can observe that η defines a bijection on the base elements of L_n,
$1 \leq i < n$,
$\eta(e_{i,i+1}) = (-1)^{i+1-i+1} \cdot e_{n-(i+1)+1,n-i+1} =$
$(-1)^2 \cdot e_{n-i-1+1,n-i+1} = 1 \cdot e_{n-i,n-i+1}$
and,
$\eta(e_{n-i,n-i+1}) = (-1)^{n-i+1-(n-i)+1} \cdot e_{n-(n-i+1)+1,n-(n-i)+1} =$
$(-1)^{n-n-i+i+1+1} \cdot e_{n-n+i-1+1,n-n+i+1} = (-1)^2 \cdot e_{i,i+1} = 1 \cdot e_{i,i+1}$
Hence, $e_{i,i+1} \leftrightarrow e_{n-i,n-i+1}$, so η is an automorphism.
η acts on the base elements as the permutation
$(e_{1,2}\ e_{n-1,n}) \cdot (e_{2,3}\ e_{n-2,n-1}) \ldots (e_{i,i+1}\ e_{n-i,n-i+1}) \ldots$

Exercise Let $\varphi \in Aut(L_n)$, and let $e_{l,l+1} \in L_n$, then,
$\varphi(e_{k,k+1}) \in C_{L_n}(\varphi(e_{l,l+1})) \Leftrightarrow 1 \leq i \leq n-1, a^{k,k+1}_{i+1,i+2} = a^{k,k+1}_{i,i+1} \cdot \frac{a^{l,l+1}_{i+1,i+2}}{a^{l,l+1}_{i,i+1}}$,
Where,
$m \in \{l,k\}, \varphi(e_{m,m+1}) = \sum_{i=1}^{n-1} a^{m,m+1}_{i,i+1} \cdot e_{i,i+1}$
Proof
$\varphi(e_{k,k+1}) \in C_{L_n}(\varphi(e_{l,l+1}))$ means that $[\varphi(e_{l,l+1}), \varphi(e_{k,k+1})] = [\varphi(e_{k,k+1}), \varphi(e_{l,l+1})] = [\sum_{i=1}^{n-1} a^{k,k+1}_{i,i+1} \cdot e_{i,i+1}, \sum_{j=1}^{n-1} a^{l,l+1}_{j,j+1} \cdot e_{j,j+1}] = 0$
Lie brackets operation is bilinear, so,
$[\varphi(e_{k,k+1}), \varphi(e_{l,l+1})] = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a^{k,k+1}_{i,i+1} \cdot a^{l,l+1}_{j,j+1} \cdot [e_{i,i+1}, e_{j,j+1}] = 0$

For the following exercise, we shall use the annotation,
$\phi_k(e_{i,j}) := \sum_l^n a^{i,j}_{l,l+k} \cdot e_{l,l+k}$, that is, a sum of all the expressions of the form $a_{l,m} \cdot e_{l,m}$, where $m - l = k$ (hence, a partial sum to $\varphi(e_{i,j})$ itself).

Exercise
Let L_n be a Lie algebra
$[\phi_1(e_{1,2}), \phi_1(e_{2,3})] = \phi_1(e_{1,3}) = [\sum_{i=1}^{n-1} a^{1,2}_{i,i+1} \cdot e_{i,i+1}, \sum_{i=1}^{n-1} a^{2,3}_{i,i+1} \cdot e_{i,i+1}] = \sum_{i=1}^{n-2} (a^{1,2}_{i,i+1} \cdot a^{2,3}_{i+1,i+2} - a^{1,2}_{i+1,i+2} \cdot a^{2,3}_{i,i+1}) \cdot [e_{i,i+1}, e_{i+1,1+2}]$
So, if $[\phi_1(e_{1,2}), \phi_1(e_{2,3})] = 0$, it means that all the expressions of the form $(a^{1,2}_{i,i+1} \cdot a^{2,3}_{i+1,i+2} - a^{1,2}_{i+1,i+2} \cdot a^{2,3}_{i,i+1})$ must be 0 (because the $[e_{i,i+1}, e_{i+1,1+2}] = e_{i,i+2}$ are linearly independent).
$a^{1,2}_{i,i+1} \cdot a^{2,3}_{i+1,i+2} - a^{1,2}_{i+1,i+2} \cdot a^{2,3}_{i,i+1} = 0 \Rightarrow a^{1,2}_{i,i+1} \cdot a^{2,3}_{i+1,i+2} = a^{1,2}_{i+1,i+2} \cdot a^{2,3}_{i,i+1} \Rightarrow a^{2,3}_{i+1,i+2} = a^{2,3}_{i,i+1} \cdot \frac{a^{1,2}_{i+1,i+2}}{a^{1,2}_{i,i+1}}$, if $a^{1,2}_{i,i+1} \neq 0$
If $a^{1,2}_{i,i+1} = 0$, we have a few cases,
1. If $a^{1,2}_{i+1,i+2} = 0$, then both $a^{2,3}_{i,i+1}$ and $a^{2,3}_{i+1,i+2}$ can be any integers.
2. If $a^{1,2}_{i+1,i+2} \neq 0$, then $a^{2,3}_{i,i+1}$ must be 0

We can sum up all the above, and say that when given
$\phi_1(e_{1,2}) = \sum_{i=1}^{n-1} a^{1,2}_{i,i+1} \cdot e_{i,i+1}$, we have,
If $a^{1,2}_{i,i+1} \neq 0, 1 \leq i \leq n-1$, then the number of free coefficients in an element of the centralizer of $\phi_1(e_{1,2})$ is 1
Proof
Assume we choose $a^{2,3}_{k,k+1}$ to be any integer, then, according to the above,
$a^{2,3}_{k+1,k+2} = a^{2,3}_{k,k+1} \cdot \frac{a^{1,2}_{k+1,k+2}}{a^{1,2}_{k,k+1}}$, since $a^{1,2}_{l,l+1} \neq 0, 1 \leq l \leq n-1$, so it is easy to

observe that this determines all the $a^{2,3}_{k+l,k+l+1}, 1 \leq l \leq n-k-1$
Precisely in the same way (but running backward),
since $a^{2,3}_{k,k+1} = a^{2,3}_{k-1,k} \cdot \frac{a^{1,2}_{k,k+1}}{a^{1,2}_{k-1,k}} \Rightarrow a^{2,3}_{k-1,k} = a^{2,3}_{k,k+1} \cdot \frac{a^{1,2}_{k-1,k}}{a^{1,2}_{k,k+1}}$, it is easy to observe that the choice of $a^{2,3}_{k,k+1}$ determines also $a^{2,3}_{l,l+1}$, for all $1 \leq l \leq k-1$
(or $a^{2,3}_{k-l,k-l+1}, 1 \leq l \leq k-1$)

Now, assume that we do not have all $a^{1,2}_{l,l+1} \neq 0$,
Take $a^{1,2}_{k,k+1} = 0$, then, obviously $a^{1,2}_{k,k+1} \cdot a^{2,3}_{k+1,k+2} = 0$,
so, in order for the expression $a^{1,2}_{k,k+1} \cdot a^{2,3}_{k+1,k+2} - a^{1,2}_{k+1,k+2} \cdot a^{2,3}_{k,k+1}$ to be 0, we must have either $a^{1,2}_{k+1,k+2} = 0$ or $a^{2,3}_{k,k+1} = 0$
Assume we have $a^{1,2}_{k+1,k+2} = 0$, so, $a^{2,3}_{k,k+1}$, can be a free choice of an integer.
So, for the chain of $a^{1,2}_{k+l,k+l+1} = 0, 1 \leq l \leq m < n-1$, we can have all $a^{2,3}_{k+l,k+l+1}, 1 \leq l < m$ as free integers. Since $a^{1,2}_{m+1,m+2} \neq 0$,
and $a^{1,2}_{m,m+1} = 0 \Rightarrow a^{1,2}_{m,m+1} \cdot a^{2,3}_{m+1,m+2} = 0$, then we must have $a^{2,3}_{m,m+1} = 0$, so, a consecutive chain of m zero coefficients of $\phi_1(e_{1,2})$ will allow $m-1$ free choices of integers for all the coefficients of $\phi_1(e_{2,3})$, with the same indices, except for the last coefficient in the chain.
However, we observe that if $m \leq n-1$, that is, the chain of zeros continues until the last coefficient of $\phi_1(e_{1,2})$ (meaning, $a^{1,2}_{n-1,n}$ is also 0), then $a^{2,3}_{n-1,n}$ is, obviously, also a free choice of an integer. To make a unification of the cases, we can think of $\phi_1(e_{1,2})$ as $\sum_{i=1}^{n} a^{1,2}_{i,i+1} \cdot e_{i,i+1}$, where $a_{n,n+1} = 0$, but this is only semantics.
So, in this case, where the chains of m zeros continues to the end, we have m free choices of integers, for the coefficients $a^{2,3}_{l,l+1}, k \leq l \leq m = n-1$, of $\phi_1(e_{2,3})$

So, if we write down the different options for commutating elements.

Suppose we have $\varphi(e_{i,i+1}) = \sum_{j=1}^{n-1} a^{i,i+1}_{j,j+1}$, then
If $\varphi(e_{1,2}) = a^{1,2}_{l,l+1} \cdot e_{l,l+1}$ (and all other coefficients are 0),
then $\varphi(e_{2,3})$ must have $a^{2,3}_{l+1,l+2} = 0$, otherwise, $a^{1,2}_{l,l+1} \cdot e_{l,l+1} \cdot a^{2,3}_{l+1,l+2} \cdot e_{l+1,l+2} = a^{1,2}_{l,l+1} \cdot a^{2,3}_{l+1,l+2} \cdot [e_{l,l+1}, e_{l+1,l+2}] = a^{1,2}_{l,l+2} \cdot a^{2,3}_{l+1,l+2} \cdot e_{l,l+2} \neq 0$

If $\varphi(e_{1,2}) = a^{1,2}_{l,l+1} \cdot e_{l,l+1} + a^{1,2}_{i+1,i+2} \cdot e_{i+1,i+2}$ (and all other coefficients are 0),
then either $\varphi(e_{2,3})$ must have $a^{2,3}_{l,l+1} = a^{2,3}_{l+1,l+2} = 0$, or $a^{2,3}_{l,l+1}, a^{2,3}_{l+1,l+2} \neq 0$

Construction

Let L_n be a Lie algebra of dimension n, and $\varphi \in Aut(L_n)$
Let $A_\varphi = A$ be the matrix representing φ,
Clearly, A is an $n \times n$ invertible squared matrix (all because $\varphi \in Aut(L_n)$), i.e. $A \in GL_n(L_n)$
Given a vector v, representing an element in L_n, that is
$v = (a_{1,2} \cdot e_{1,2}, a_{2,3} \cdot e_{2,3}, \ldots, a_{1,n} \cdot e_{1,n}), a_i, j \in \mathbb{F}, e_{i,j} \in B(L_n)$, where $B(L_n)$ is the basis of L_n,
we shall want this vector to be divided and ordered by the quotient algebras that are formed by the lower central series $\gamma_m(L_n)$, that is,

$$
\begin{aligned}
v = \\
&(a_{1,2} \cdot e_{1,2}, a_{2,3} \cdot e_{2,3}, \ldots, a_{n-1,n} \cdot e_{n-1,n}, 0, \ldots, 0) && + \\
&(0, \ldots, 0, a_{1,3} \cdot e_{1,3}, a_{2,4} \cdot e_{2,4}, \ldots, a_{n-2,n} \cdot e_{n-2,n}, 0, \ldots, 0) && + \\
&\vdots \\
&(0, \ldots, 0, a_{1,1+k} \cdot e_{1,1+k}, a_{2,2+k} \cdot e_{2,2+k}, \ldots, a_{n-k,n} \cdot e_{n-k,n}, 0, \ldots, 0) && + \\
&\vdots \\
&(0, \ldots, 0, a_{1,n} \cdot e_{1,n})
\end{aligned}
$$

where the first $n-1$ elements are all in ${}^{L}/_{\gamma_2(L)}$

Exercise

Let L_n be a Lie algebra, and let $\varphi \in Aut(L_n)$, then
The first $n \times n$ squared block of φ is a diagonal matrix.

Proof

The first $n-1$ rows of φ are $\varphi_1 = \varphi(e_{1,2}), \varphi_2 = \varphi(e_{2,3}), \ldots, \varphi_{n-1} = \varphi(e_{n-1,n})$
And, each column j represents the coefficient of $e_{j,j+1}$ in $\varphi(e_{i,i+1}) = \sum_j^{n-1} a_{j,j+1}^{i,i+1} \cdot e_{j,j+1} + \gamma_j(L_n)$

Exercise

Let L_n be a Lie algebra, $\varphi \in Aut(L_n)$, and $A = A_\varphi = (a_{i,j})$ the matrix representing φ

Assume that the block A_1 of $n-1 \times n-1$ elements, starting from $a_{1,1}$ is diagonal, then,

All the blocks on the main diagonal of A, that is,

A_2, the block of $n-2 \times n-2$ elements, starting from $a_{n,n}$

A_3, the block of $n-3 \times n-3$ elements, starting from $a_{2n-2,2n-2}$

$\vdots$

$A_{n-1} = a_{\binom{n}{2},\binom{n}{2}}$ (a block of 1×1, i.e., a single element)

are diagonal, as well,

Proof

By induction on k, where k is the level of the lower central series, modulu the lower central series of the next level, ${}^{\gamma_k(L_n)}/_{\gamma_{k+1}(L_n)}$

We have assumed this is true for A_1, which is the block representing ${}^{L_n}/_{\gamma_2(L_n)} = {}^{\gamma_1(L_n)}/_{\gamma_2(L_n)}$

So, assume for k, we shall prove for $k+1$

As seen above, ${}^{\gamma_k+1(L_n)}/_{\gamma_{k+2}(L_n)} = \{e_{i,j} | j-i = k+1\} = {}^{[L_n,\gamma_k(L_n)]}/_{\gamma_k+2(L_n)}$

So, let $e_{i,j} \in {}^{\gamma_k+1(L_n)}/_{\gamma_{k+2}(L_n)}$, that means, $e_{i,j} = [e_{i,i+1}, e_{i+1,j}]$

So, $\varphi(e_{i,j}) = \varphi([e_{i,i+1}, e_{i+1,j}]) = [\varphi(e_{i,i+1}), \varphi(e_{i+1,j})]$

But, $\varphi(e_{i+1,j}) \in {}^{\gamma_k(L_n)}/_{\gamma_{k+1}(L_n)}$

But, according to the assumption, if $e_{i+1,j} \in {}^{\gamma_k(L_n)}/_{\gamma_{k+1}(L_n)}$, then $\varphi(e_{i+1,j}) = \lambda_{i+1,j} \cdot e_{i+1,j}$, and, if $e_{i,i+1} \in {}^{L_n}/_{\gamma_2(L_n)}$, then $\varphi(e_{i,i+1}) = \lambda i, i+1 \cdot e_{i,i+1}$

$\Rightarrow [\varphi(e_{i,i+1}), \varphi(e_{i+1,j})] = [\lambda_{i,i+1} \cdot e_{i,i+1}, \lambda_{i+1,j} \cdot e_{i+1,j}] = \lambda_{i,i+1} \cdot \lambda_{i+1,j} \cdot [e_{i,i+1}, e_{i+1,j}] = \lambda_{i,i+1} \cdot \lambda_{i+1,j} \cdot e_{i,j}$, which proves the induction step, for $k+1$

(to be convinced of the modulu calculations, we only need to observe that if $a \in {}^{\gamma_l(L_n)}/_{\gamma_k(L_n)}, b \in {}^{\gamma_m(L_n)}/_{\gamma_r(L_n)}, l < k, m < r$, it means that

$a = \lambda_{i,j} \cdot e_{i,j} + \gamma_k(L), j-i \geq l$, and

$b = \lambda_{s,t} \cdot e_{s,t} + \gamma_r(L), t-s \geq m$,

which means that,

$a = \lambda_{i,j} \cdot e_{i,j} + \lambda_{f,g} \cdot e_{f,g}, j-i \geq l, g-f \geq k$, and

$b = \lambda_{s,t} \cdot e_{s,t} + \lambda_{c,d} \cdot e_{c,d}, t-s \geq m, d-c \geq r$

So,

$[a,b] = [\lambda_{i,j} \cdot e_{i,j} + \lambda_{f,g} \cdot e_{f,g}, \lambda_{s,t} \cdot e_{s,t} + \lambda_{c,d} \cdot e_{c,d}] =$

$[\lambda_{i,j} \cdot e_{i,j}, \lambda_{s,t} \cdot e_{s,t}] + [\lambda_{i,j} \cdot e_{i,j}, \lambda_{c,d} \cdot e_{c,d}] + [\lambda_{f,g} \cdot e_{f,g}, \lambda_{s,t} \cdot e_{s,t}] + [\lambda_{f,g} \cdot e_{f,g}, \lambda_{c,d} \cdot e_{c,d}] =$

$\lambda_{i,j} \cdot \lambda_{s,t} \cdot [e_{i,j}, e_{s,t}] + \lambda_{i,j} \cdot \lambda_{c,d} \cdot [e_{i,j}, e_{c,d}] + \lambda_{f,g} \cdot \lambda_{s,t} \cdot [e_{f,g}, e_{s,t}] + \lambda_{f,g} \cdot \lambda_{c,d} \cdot [e_{f,g}, e_{c,d}]$

We have several options, for each of the commutators in the product,
if $j = c \Rightarrow d - i = d - c + c - i = d - c + j - i \geq r + l$
$\Rightarrow [e_{i,j}, e_{c,d}] = e_{i,d} \in \gamma_{r+l}(L_n) \subseteq \gamma_r(L_n), \gamma_l(L_n)$
if $i = d \Rightarrow j - c = j - i + i - d = j - i + d - c \geq l + r$
$\Rightarrow [e_{i,j}, e_{c,d}] = -e_{c,j} \in \gamma_{l+r}(L_n) \subseteq \gamma_r(L_n), \gamma_l(L_n)$
if $j \neq c \wedge i \neq d \Rightarrow [e_{i,j}, e_{c,d}] = 0$

if $g = s \Rightarrow t - f = t - s + s - f = t - s + g - f \geq m + k$
$\Rightarrow [e_{f,g}, e_{s,t}] = e_{f,t} \in \gamma_{m+k}(L_n) \subseteq \gamma_m(L_n), \gamma_k(L_n)$
if $f = t \Rightarrow g - s = g - f + f - s = g - f + t - s \geq k + m$
$\Rightarrow [e_{f,g}, e_{s,t}] = -e_{s,g} \in \gamma_{k+m}(L_n) \subseteq \gamma_k(L_n), \gamma_m(L_n)$
if $g \neq s \wedge f \neq t \Rightarrow [e_{f,g}, e_{s,t}] = 0$

if $g = c \Rightarrow d - f = d - c + c - f = d - c + g - f \geq r + k$
$\Rightarrow [e_{f,g}, e_{c,d}] = e_{f,d} \in \gamma_{r+k}(L_n) \subseteq \gamma_r(L_n), \gamma_k(L_n)$
if $f = d \Rightarrow g - c = g - f + f - c = g - f + d - c \geq k + r$
$\Rightarrow [e_{f,g}, e_{c,d}] = -e_{c,g} \in \gamma_{k+r}(L_n) \subseteq \gamma_k(L_n), \gamma_r(L_n)$
if $g \neq c \wedge f \neq d \Rightarrow [e_{f,g}, e_{c,d}] = 0$

so, we have,
$[e_{i,j}, e_{c,d}] = 0 \vee [e_{i,j}, e_{c,d}] \in \gamma_r(L_n)$
$[e_{f,g}, e_{s,t}] = 0 \vee [e_{f,g}, e_{s,t}] \in \gamma_k(L_n)$
$[e_{f,g}, e_{c,d}] = 0 \vee [e_{f,g}, e_{c,d}] \in \gamma_k(L_n), \gamma_r(L_n)$

So, there are several options, here,
$r < k \Rightarrow \gamma_k \subset \gamma_r$
$k < r \Rightarrow \gamma_r \subset \gamma_k$
$r = k \Rightarrow \gamma_k = \gamma_r$
WLOG, assume $k < r$, so, if $\alpha = [e_{i,j}, e_{c,d}] + [e_{f,g}, e_{s,t}] + [e_{f,g}, e_{c,d}]$, then $\alpha = 0 \vee \alpha \in \gamma_k(L_n)$, but, if $\alpha \in \gamma_k(L_n) \Rightarrow \overline{\alpha} = 0 \in {}^{L_n}/_{\gamma_k(L_n)}$
Going back to our $[a, b]$ product, we only need to observe that,
if $j = s \Rightarrow [e_{i,j}, e_{s,t}] = e_{i,t}$
if $i = t \Rightarrow [e_{i,j}, e_{s,t}] = -e_{s,j}$
if $j \neq s \wedge i \neq t \Rightarrow [e_{i,j}, e_{s,t}] = 0$

Exercise

Let $A \in GL_n((F))$, an upper triangular matrix, with no zeros on the main diagonal, over some field, $\mathbb{F}$, Then,
A has a unique decomposition to a product, $A = A_N \times A_H$, where A_N is a unipotent matrix (that is, an upper triangular matrix, with 1 on the main diagonal), and A_H is a diagonal matrix.

Proof

Given a matrix,

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n-1} & a_{1,n} \\ 0 & a_{2,2} & \dots & a_{2,n-1} & a_{2,n} \\ 0 & 0 & \ddots & a_{k,n-1} & a_{k,n} \\ 0 & 0 & \dots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \dots & 0 & a_{n,n} \end{pmatrix}$$

For simplicity, we shall write $\lambda_k = a_{k,k}, 1 \le k \le n$
We shall write two canonical matrices,

$$A_N = \begin{pmatrix} 1 & \frac{a_{1,2}}{\lambda_2} & \dots & \frac{a_{1,n-1}}{\lambda_{n-1}} & \frac{a_{1,n}}{\lambda_n} \\ 0 & 1 & \dots & \frac{a_{2,n-1}}{\lambda_{n-1}} & \frac{a_{2,n}}{\lambda_n} \\ 0 & 0 & \ddots & \frac{a_{k,n-1}}{\lambda_{n-1}} & a_{k,n} \\ 0 & 0 & \dots & 1 & \frac{a_{n-1,n}}{\lambda_n} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad A_H = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & \lambda_{n-1} & 0 \\ 0 & 0 & \dots & 0 & \lambda_n \end{pmatrix}$$

Obviously, $A_N \times A_H = A$

Although this decomposition is canonical, we still need to prove that it is unique.
Suppose that $A = B_N \times B_H$, where B_N is a unipotent matrix, and B_H is a diagonal matrix.
So, $A_N \times A_H = A = B_N \times B_H$
These matrices are all in $GL_n(\mathbb{F})$, so, each has its inverse matrix.
So, we can multiply,
$A_N^{-1} \times (A_N \times A_H) = A_N^{-1} \times (B_N \times B_H) \Rightarrow (A_N^{-1} \times A_N) \times A_H =$
$A_N^{-1} \times (B_N \times B_H) \Rightarrow A_H = A_N^{-1} \times B_N \times B_H \Rightarrow A_H \times B_H^{-1} =$
$A_N^{-1} \times B_N \times B_H \times B_H^{-1} \Rightarrow A_H = A_N^{-1} \times B_N \times B_H \Rightarrow A_H \times B_H^{-1} =$
$A_N^{-1} \times B_N \times (B_H \times B_H^{-1}) \Rightarrow A_H \times B_H^{-1} = A_N^{-1} \times B_N$

To continue, we need to prove two auxiliary propositions.

Exercise
Let $\mathbb{F}$ be any field, and

$$B = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & \lambda_{n-1} & 0 \\ 0 & 0 & \dots & 0 & \lambda_n \end{pmatrix}$$

, where $\lambda_k \in \mathbf{F}, 1 \leq k \leq n$, Then,

$$B' = B^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \dots & \frac{1}{\lambda_{n-1}} & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{\lambda_n} \end{pmatrix}$$

A simple check would show us that $B \times B' = B' \times B = I_n$, and, since we have found an inverse matrix to B, it is, obviously, unique
(because, if B'' is also an inverse of B, then, $B \times B' = B \times B'' \Rightarrow B' \times B \times B' = B' \times B \times B'' \Rightarrow (B' \times B) \times B' = (B' \times B) \times B'' \Rightarrow I_n \times B' = I_n \times B'' \Rightarrow B' = B''$)
This means that the inverse of a diagonal matrix is also a diagonal matrix.

Exercise
Let $\mathbb{F}$ be any field, and

$$C = \begin{pmatrix} 1 & a_{1,2} & \dots & a_{1,n-1} & a_{1,n} \\ 0 & 1 & \dots & a_{2,n-1} & a_{2,n} \\ 0 & 0 & \ddots & a_{k,n-1} & a_{k,n} \\ 0 & 0 & \dots & 1 & a_{n-1,n} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

, Then,

$$C^{-1} = \begin{pmatrix} 1 & a'_{1,2} & \dots & a'_{1,n-1} & a'_{1,n} \\ 0 & 1 & \dots & a'_{2,n-1} & a'_{2,n} \\ 0 & 0 & \ddots & a'_{k,n-1} & a'_{k,n} \\ 0 & 0 & \dots & 1 & a'_{n-1,n} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Proof

For this, we will need to use (and briefly prove) a well know fact.

Exercise

Let $\mathbb{F}$ be any field, and $D \in GL_n(\mathbb{F})$ an invertible squared matrix over $\mathbb{F}$
Let $e_1, e_2, \dots, e_m$ be a sequence of elementary operations on D, such that $e_m(e_{m-1}(\dots(e_2(e_1(D))\dots)) = I_n$, Then,
$e_m(e_{m-1}(\dots(e_2(e_1(I_n))\dots)) = D^{-1}$

Proof

The three elementary operations translate into a left multiplication, of the operand matrix, A, by an elementary matrix.

1. $R_i \leftarrow R_i + R_j$ translates into $E_{i,j} \cdot A$, which we have defined above
2. $R_i \leftarrow \lambda \cdot R_i$ translates into

$$E_{\lambda_i} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda_i & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

3. $R_i \leftrightarrow R_j$ translates into

$$E_{\lambda_i} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

, that is, I_n, with the operation $R_i \leftrightarrow R_j$

So, $e_m(e_{m-1}(\dots(e_2(e_1(D))\dots)) = I_n$ translates into,
$e_m \times (e_{m-1} \times (\dots \times (e_2 \times (e_1 \times D)\dots)) = e_m \times e_{m-1} \times \dots \times e_2 \times e_1 \times D = I_n$
$\Rightarrow D^{-1} = I_n \times D^{-1} = (e_m \times e_{m-1} \times \dots \times e_2 \times e_1 \times D) \times D^{-1} =$
$e_m \times e_{m-1} \times \dots \times e_2 \times e_1 \times D \times D^{-1} = e_m \times e_{m-1} \times \dots \times e_2 \times e_1 \times (D \times D^{-1}) =$

$e_m \times e_{m-1} \times \cdots \times e_2 \times e_1 \times I_n = e_m(e_{m-1}(\ldots(e_2(e_1(I_n))\ldots))$
Going back to finding C^{-1}, we observe that, since $(c_{i,j}) = C$ is already in a canonical echelon form, with 1 on the main diagonal, then, to bring C to I_n, using elementary operations, would be to perform the sequence of operations, on each row, R_i,
For each element in row R_i, $c_{i,j} \neq 0, j > i$,
1. $R_j \leftarrow -c_{i,j} \cdot R_j$
2. $R_i \leftarrow R_i + R_j$
3. $R_j \leftarrow -\frac{1}{c_{i,j}} \cdot R_j$
So, in order to bring C to I_n, we need a sequence of up to $\binom{n-1}{2} \times 3$ elementary operations on C, but none of them is $R_i \leftrightarrow R_j$, and, the triple sequence of operations, on each element, preserves the canonical form of C
This means that performing this sequence of operations on I_n will preserve the canonical form of I_n, which shows that C^{-1} is an upper triangular matrix, with 1 on the main diagonal, as well. And, since we have found an inverse matrix of C, it must be unique.
Going back to the proof that the decomposition of $A = A_H \times A_N$ is unique.