

Your Paper

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Denote $G_5 := G_5(\mathbb{Z}_p)$, and $G_5^+ := G_5^+(\mathbb{Q}_p)$.

$\zeta_{L_{5,p}}^\wedge(s) = \int_{G_5^+} |\det g|_p^s d\mu(G_5) = \int_{G_5^+} |\det uh|_p^s d\mu(G_5)$, where $h \in H$ and $u \in N_h$.

Each u is unipotent, hence $\zeta_{L_{5,p}}^\wedge(s) = \int_{G_5^+} |\det h|_p^s d\mu(G_5) = \int_{G_5^+} |\lambda_1^4 \lambda_2^6 \lambda_3^6 \lambda_4^4|_p^s d\mu(G_5) = \int_{G_5^+} \left[|\lambda_1^4|_p |\lambda_2^6|_p |\lambda_3^6|_p |\lambda_4^4|_p \right]^s d\mu(G_5)$, by the inductive formula we have found for every $|h|$.

We denote $v_i := v_p(\lambda_i)$,

and so $\zeta_{L_{5,p}}^\wedge(s) = \int_{G_5^+} \left[p^{-4v_1} p^{-6v_2} p^{-6v_3} p^{-4v_4} \right]^s d\mu(G_5) = \int_{G_5^+} p^{-(4v_1+6v_2+6v_3+4v_4)s} d\mu(G_5)$.

We denote $I(\underline{\lambda}) := p^{-(4v_1+6v_2+6v_3+4v_4)s}$. Now we use the natural matrix decomposition of the N_h matrix of Berman's, which means that

$\zeta_{L_{5,p}}^\wedge(s) = \int_{G_5^+} I(\underline{\lambda}) d\mu(G_5) = \int_{\underline{\lambda}} \int_{\underline{a}} \int_{\underline{b}} \int_{\underline{c}} I(\underline{\lambda}) d\mu(\underline{c}) d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda})$. Since $I(\underline{\lambda})$ depends only on $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, which appear only in the computation of the outermost integral, we consider them as constants for all the inner integrals, which means that we have $\zeta_{L_{5,p}}^\wedge(s) = \int_{\underline{\lambda}} I(\underline{\lambda}) \int_{\underline{a}} \int_{\underline{b}} \int_{\underline{c}} 1 d\mu(\underline{c}) d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda})$,

hence all the inner integrals evaluate to the measure of their domains of integration. now we compute the innermost integral by considering \underline{a} , \underline{b} and $\underline{\lambda}$ as constants, and integrating only over \underline{c} . Considering the multiplication uh , we observe that for each element c_j , we must have that $\rho_j = c_j \lambda_1 \lambda_2 \lambda_3 \lambda_4 \in \mathbb{Z}_p$, which means that $v(\rho_i) = v(c_i \lambda_1 \lambda_2 \lambda_3 \lambda_4) \geq 0 \Rightarrow v(c_i) + v_1 + v_2 + v_3 + v_4 \geq 0 \Rightarrow v(c_i) \geq -(v_1 + v_2 + v_3 + v_4)$. But this means that $c_i \in p^{-(v_1+v_2+v_3+v_4)} \mathbb{Z}_p$, and since the domain of integration for this integral is $\underline{c} = \{c_1, c_2, c_3, c_4\}$, then $\mu(\underline{c}) = |c_j|_p^4 = p^{4(v_1+v_2+v_3+v_4)}$. Denote $I(\underline{\lambda}, \underline{c}) := I(\underline{\lambda}) p^{4(v_1+v_2+v_3+v_4)s}$, we now

have that $\zeta_{L_{5,p}}^\wedge(s) = \int_{\underline{\lambda}} I(\underline{\lambda}, \underline{c}) \int_{\underline{a}} \int_{\underline{b}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda})$.

Denote $\lambda_{13} := \lambda_1 \lambda_2 \lambda_3$, $\lambda_{24} := \lambda_2 \lambda_3 \lambda_4$, and $\lambda_{14} := \lambda_1 \lambda_2 \lambda_3 \lambda_4$. We now consider the constraints on \underline{b} .

$b_{11} \lambda_{13}, b_{31} \lambda_{13}, b_{41} \lambda_{13} \in \mathbb{Z}_p$, and $b_{12} \lambda_{24}, b_{22} \lambda_{24} \in \mathbb{Z}_p$. These constraints are obtained by multiplying elements in block M_{13} with elements in h , but one

observes that we have b_{22} also in location $(5, 10)$ of the matrix, and b_{31} in location $(7, 10)$, which means that $b_{22}\lambda_{14}, b_{31}\lambda_{14} \in \mathbb{Z}_p$. But since we already have $b_{22}\lambda_{24}, b_{31}\lambda_{13} \in \mathbb{Z}_p$, the constraints $b_{22}\lambda_{14}$ and $b_{31}\lambda_{14}$ do not contribute any new information. In addition, we have one of the elements of \underline{b} that forms a constraint together with elements from \underline{a} , namely $(a_{11}a_{22} - b_{11})\lambda_{24} \in \mathbb{Z}_p$. The constraints $b_{31}\lambda_{13}, b_{41}\lambda_{13}, b_{12}\lambda_{24}, b_{22}\lambda_{24} \in \mathbb{Z}_p$ from above translate to $p^{-2(v_1+v_2+v_3)}p^{-2(v_2+v_3+v_4)} = p^{-2(v_1+2v_2+2v_3+v_4)}$. On the other hand, b_{11} is a part of two constraints, hence we must have both $b_{11} \in p^{-(v_1+v_2+v_3)}\mathbb{Z}_p$ and $a_{11}a_{22} - b_{11} \in p^{-(v_2+v_3+v_4)}\mathbb{Z}_p \Rightarrow b_{11} \in a_{11}a_{22} + p^{-(v_2+v_3+v_4)}\mathbb{Z}_p$, which means that we need to compute the measure of the intersection of the two modules $\mu\left(p^{-(v_1+v_2+v_3)}\mathbb{Z}_p \cap a_{11}a_{22} + p^{-(v_2+v_3+v_4)}\mathbb{Z}_p\right)$. Denote $\alpha := v_1 + v_2 + v_3$, $\beta := v_2 + v_3 + v_4$ and $x := a_{11}a_{22}$, so $b_{11} \in p^{-\alpha}\mathbb{Z}_p \cap x + p^{-\beta}\mathbb{Z}_p$. Assume $\beta \geq \alpha \Rightarrow -\beta \leq -\alpha \Rightarrow p^{-\alpha}\mathbb{Z}_p \subset p^{-\beta}\mathbb{Z}_p \Rightarrow b_{11} \in p^{-\beta}\mathbb{Z}_p \cap x + p^{-\beta}\mathbb{Z}_p$.