

# 1 The computation of $G_n(\mathbb{Q}_p)$

## 1.1 The computation of the first block $M_{11}$

**Proposition 1.1.1.** *Let  $x = \sum_{i=1}^{n-1} \lambda_i e_{i,i+1}$ , where  $\lambda_i \in \mathbb{Q}_p$  are not all zero. Then  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 = \mathfrak{l}(x) + \mathfrak{m}(x)$ , where  $\mathfrak{l}(x)$  is the number of sequences of consecutive non-zero coefficients of the form  $\lambda_j, \lambda_{j+1}, \dots, \lambda_{j+k-1}, \lambda_{j+k}$  and  $\lambda_{j-1} = \lambda_{j+k+1} = 0$  (that is, the sequences are separated by one or more zero coefficients)<sup>1</sup>, and  $\mathfrak{m}(x)$  is the number of zero coefficients  $\lambda_j = 0$ , such that also  $\lambda_{j-1} = \lambda_{j+1} = 0$ .*

*Proof.* Let  $y = \sum_{i=1}^{n-1} \mu_i e_{i,i+1}$ , where  $\lambda_i \in \mathbb{Q}_p$ , be an element in the quotient  $\gamma_1/\gamma_2$ . For every  $1 \leq i \leq n-1$ , denote by  $(\mathfrak{C}_i)$  the constraint equation  $[\lambda_i e_{i,i+1}, \mu_{i+1} e_{i+1,i+2}] - [\lambda_{i+1} e_{i+1,i+2}, \mu_i e_{i,i+1}] = (\lambda_i \mu_{i+1} - \lambda_{i+1} \mu_i) e_{i,i+2} = 0$ , and it is clear that  $y \in \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2$  if and only if all the  $(\mathfrak{C}_i)$  constraints are satisfied. We observe that each  $\mu_i$  participates in two constraints,  $(\mathfrak{C}_{i-1})$  and  $(\mathfrak{C}_i)$ , that is,  $\lambda_{i-1} \mu_i - \lambda_i \mu_{i-1} = \lambda_i \mu_{i+1} - \lambda_{i+1} \mu_i = 0$ . If  $\lambda_i = 0$ , then  $\lambda_i \mu_{i-1} = \lambda_i \mu_{i+1} = 0$ , hence by constraint  $(\mathfrak{C}_{i-1})$  we have that  $\lambda_{i-1} \mu_i = 0$ , and by constraint  $(\mathfrak{C}_i)$  we have that  $\lambda_{i+1} \mu_i = 0$ . Hence, if either  $\lambda_{i-1}$  or  $\lambda_{i+1}$  are non-zero, then  $\mu_i = 0$ . But if  $\lambda_{i-1} = \lambda_{i+1} = 0$ , then both constraints are satisfied for any choice of  $\mu_i$ , which increases  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2$  by 1. We need to prove that for any sequence of  $k$  consecutive zero coefficients of  $x$ , where  $k \geq 3$ , that is<sup>2</sup>  $\lambda_{j+1} = \lambda_{j+2} = \dots = \lambda_{j+k} = 0$ , for  $1 \leq j \leq n-2$ , we have that the sequence  $\mu_{j+2}, \mu_{j+3}, \dots, \mu_{j+k-1}$  of  $k-2$  consecutive coefficients of  $y$  is made of scalars of any choice, thus  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2$  is increased by  $k-2$ . We prove that by simple induction on  $k$ . For  $k = 3$ , we just proved that if  $\lambda_{i-1} = \lambda_i = \lambda_{i+1} = 0$ , then  $\mu_i$  can be any scalar. For  $k + 1$ , we look at the sequence of  $k + 1$  zero coefficients,  $\lambda_{j+1} = \lambda_{j+2} = \dots = \lambda_{j+k} = \lambda_{j+k+1} = 0$ . By constraints  $(\mathfrak{C}_{j+k-1})$  and  $(\mathfrak{C}_{j+k})$ , we have that  $\lambda_{j+k-1} \mu_{j+k} - \lambda_{j+k} \mu_{j+k-1} = \lambda_{j+k} \mu_{j+k+1} - \lambda_{j+k+1} \mu_{j+k} = 0$ , and since  $\lambda_{j+k-1} = \lambda_{j+k} = \lambda_{j+k+1} = 0$ , we have that  $\mu_{j+k}$  can be any scalar, as we proved earlier. By the assumption, we have that all  $k-2$  previous coefficients, that is  $\mu_{j+2}, \dots, \mu_{j+k-1}$  can also be any scalars, so in total the whole sequence of  $k-1 = (k+1)-2$  coefficients of  $y$  can be any scalars, which proves the induction step. Suppose that we have  $m$  sequences of three or more consecutive zero coefficients in  $x$ , whose lengths are  $k_1, k_2, \dots, k_m$ ,

<sup>1</sup>We extend our notation of indices, to include also the case where  $j = 1$  or  $j+k = n-1$ , and define that  $\lambda_{j-1} = \lambda_0 = 0$  or  $\lambda_{j+k+1} = \lambda_n = 0$ , respectively

<sup>2</sup>Here again, we consider the non-existent  $\lambda_0 = \lambda_n = 0$  as part of the sequence

then  $\mathbf{m}(x) = \sum_{l=1}^m k_l - 2m$  is the total number of zero coefficients  $\lambda_j = 0$  such that also  $\lambda_{j-1} = \lambda_{j+1} = 0$ , as proposed. Using again the two consecutive constraints,  $(\mathfrak{C}_{i-1})$  and  $(\mathfrak{C}_i)$ , suppose now that  $\lambda_i \neq 0$ . If  $\lambda_{i-1} = 0$ , then by constraint  $(\mathfrak{C}_{i-1})$  we must have that  $\mu_{i-1} = 0$ , but if  $\lambda_{i-1} \neq 0$ , then by this constraint we have  $\mu_i = \frac{\lambda_i \mu_{i-1}}{\lambda_{i-1}}$ , which means that  $\mu_i$  depends on  $\mu_{i-1}$ . Precisely the same way for constraint  $(\mathfrak{C}_i)$ , we have that if  $\lambda_{i+1} = 0$  then  $\mu_{i+1} = 0$ , otherwise  $\mu_{i+1} = \frac{\lambda_{i+1} \mu_i}{\lambda_i}$ , which means that  $\mu_{i+1}$  depends on  $\mu_i$ ,

and if also  $\lambda_{i-1} \neq 0$ , then  $\mu_{i+1} = \frac{\lambda_{i+1} \frac{\lambda_i \mu_{i-1}}{\lambda_{i-1}}}{\lambda_i} = \frac{\lambda_{i+1} \mu_{i-1}}{\lambda_{i-1}}$ , which means that both  $\mu_{i+1}$  and  $\mu_i$  depend on  $\mu_{i-1}$ . We need to prove this is true for any sequence of  $k$  consecutive non-zero coefficients of  $x$ , that is, for a given sequence of coefficients,  $\lambda_{j+1}, \lambda_{j+2}, \dots, \lambda_{j+k}$ , where  $1 \leq j \leq n-1$ , we need to prove that  $\mu_{j+1}$  can be any scalar, while  $\mu_{j+2}, \mu_{j+3}, \dots, \mu_{j+k}$  all depend on  $\mu_{j+1}$ . Here again, we use a simple induction on  $k$ . For  $k = 1, 2, 3$ , we already proved this. For  $k+1$ , we look into a sequence of  $k+1$  consecutive non-zero coefficients of  $x$ , that is  $\lambda_{j+1}, \dots, \lambda_{j+k}, \lambda_{j+k+1}$ . By constraint  $(\mathfrak{C}_{j+k})$  we have  $\lambda_{j+k} \mu_{j+k+1} - \lambda_{j+k+1} \mu_{j+k} = 0$ , and since  $\lambda_{j+k} \neq 0$ , we have  $\mu_{j+k+1} = \frac{\lambda_{j+k+1} \mu_{j+k}}{\lambda_{j+k}}$ , but by the assumption,  $\mu_{j+k} = \frac{\lambda_{j+k} \mu_{j+k-1}}{\lambda_{j+k-1}} = \frac{\lambda_{j+k}}{\lambda_{j+k-1}} \frac{\lambda_{j+k-1} \mu_{j+k-2}}{\lambda_{j+k-2}} = \frac{\lambda_{j+k}}{\lambda_{j+k-1}} \frac{\lambda_{j+k-1}}{\lambda_{j+k-2}} \frac{\lambda_{j+k-2} \mu_{j+k-3}}{\lambda_{j+k-3}} = \dots = \frac{\lambda_{j+k} \mu_{j+1}}{\lambda_{j+1}}$ , hence  $\mu_{j+k+1} = \frac{\lambda_{j+k+1} \mu_{j+k}}{\lambda_{j+k}} = \frac{\lambda_{j+k+1} \lambda_{j+k} \mu_{j+1}}{\lambda_{j+k} \lambda_{j+1}} = \frac{\lambda_{j+k+1} \mu_{j+1}}{\lambda_{j+1}}$ , which proves the induction step. Looking again at  $(\mathfrak{C}_{i-1})$  and  $(\mathfrak{C}_i)$ , we observe that if  $\lambda_{i-1}$  and  $\lambda_{i+1}$  are non-zero and  $\lambda_i = 0$ , then  $\lambda_i \mu_{i-1} = \lambda_i \mu_{i+1} = 0$ , hence by constraint  $(\mathfrak{C}_{i-1})$  we have that  $\lambda_{i-1} \mu_i = 0$ , and by constraint  $(\mathfrak{C}_i)$  we have that  $\lambda_{i+1} \mu_i = 0$ , so  $\mu_i = 0$  by both constraints, which means that there is no dependency between  $\mu_{i+1}$  and  $\mu_{i-1}$ . This shows that any zero coefficient between two non-zero coefficients of  $x$  creates two separate sequences, each sequences increases  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2$  by 1, hence, if we have  $l$  sequences of consecutive non-zero coefficients of  $x$ , they increase  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2$  by  $l$ , and we denote  $\mathfrak{l}(x) = l$ . All this shows that  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 = \mathfrak{l}(x) + \mathbf{m}(x)$ , as proposed.  $\square$

**Corollary 1.1.2.** *Let  $\mathcal{L}_{n,p}$  be the  $\mathbb{Q}_p$ -Lie algebra associated with  $\mathcal{U}_n(\mathbb{Z})$ . If  $n \geq 5$ , then  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \gamma_1/\gamma_3 - 1$  if and only if  $x \in \{\lambda e_{12} + \gamma_2 \mathcal{L}_{n,p}\}$  or  $x \in \{\lambda e_{n-1,n} + \gamma_2 \mathcal{L}_{n,p}\}$ , for a non-zero scalar  $\lambda \in \mathbb{Q}_p$ . If  $n = 4$ , then  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \gamma_1/\gamma_3 - 1$  if and only if  $x \in \{\lambda e_{12} + \mu e_{34} + \gamma_2 \mathcal{L}_{n,p}\}$ , for  $\lambda, \mu \in \mathbb{Q}_p$  not both zero.*

*Proof.* We recall first that for any algebra  $\mathcal{L}_{n,p}$ , the first two elements of the lower central series, namely  $\gamma_1/\gamma_2$  and  $\gamma_2/\gamma_3$ , are of sizes  $n-1$  and  $n-2$ ,

respectively. Consider the Lie brackets operation  $[x_1, x_2]$ , where  $x_1 \in \gamma_1/\gamma_2$  and  $x_2 \in \gamma_2/\gamma_3$ , or the opposite, then  $[x_1, x_2] \in \gamma_3$ , which means that if we consider the centralizer of some  $x \in \mathcal{L}_{n,p}$  in  $\gamma_1/\gamma_3$ , then any element  $y \in \gamma_2/\gamma_3$  would either commute with  $x$  or yield an element in  $\gamma_3$ , which also means that it commutes with  $x$  in  $\gamma_1/\gamma_3$ . This shows that  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) \geq \dim \gamma_2/\gamma_3 = n - 2$ . Suppose that  $x = \lambda_1 e_{12} + z$  or  $x = \lambda_{n-1} e_{n-1,n} + z$ , where  $z \in \gamma_2 \mathcal{L}_{n,p}$ , then looking only at the elements of  $\gamma_1/\gamma_2$  in the linear combination that forms  $x$ , we have one sequence of one non-zero coefficient, hence  $\mathbf{l}(x) = 1$ , and one sequence of  $n - 1 - 1 = n - 2$  zero coefficients of  $x$ , hence we have<sup>3</sup> that  $\mathbf{m}(x) = n - 2 - 1 = n - 3$ . By 1.1.1 we have that  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 = \mathbf{l}(x) + \mathbf{m}(x) = 1 + (n - 3) = n - 2$ . Adding all the elements of  $\gamma_2/\gamma_3$ , we have that  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = (n - 2) + (n - 2) = 2n - 4 = (n - 1) + (n - 2) - 1 = \dim \gamma_1/\gamma_2 + \dim \gamma_2/\gamma_3 - 1 = \dim \gamma_1/\gamma_3 - 1$ , as proposed. To prove the opposite direction, we assume that  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \gamma_1/\gamma_3 - 1 = 2n - 4$ , and since we already know that all the elements of  $\dim \gamma_2/\gamma_3$  are in the centralizer of  $x$ , we have that  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 = \dim \mathcal{C}_{\gamma_1/\gamma_3}(x) - \dim \gamma_2/\gamma_3 = (2n - 4) - (n - 2) = n - 2 = (n - 1) - 1 = \dim \gamma_1/\gamma_2 - 1$ . Suppose that either  $x = \lambda_i e_{i,i+1} + z$  where  $1 < i < n - 1$ , or  $x = \lambda_{i_1} e_{i_1,i_1+1} + \dots + \lambda_{i_m} e_{i_m,i_m+1}$  where we denote the number of  $\gamma_1/\gamma_2$  coefficients<sup>4</sup> by  $2 \leq m \leq n - 1$ , in both cases  $z \in \gamma_2 \mathcal{L}_{n,p}$ . In the first case, we have that  $\mathbf{l}(x) = 1$ , because there is only one non-zero coefficient of  $x$  in  $\gamma_1/\gamma_2$ , and we have two sequences of zero coefficients of  $x$ , that is  $\lambda_0 = \lambda_1 = \lambda_2 = \dots = \lambda_{i-2} = \lambda_{i-1} = 0$  and  $\lambda_{i+1} = \lambda_{i+2} = \lambda_{i+3} = \dots = \lambda_{n-2} = \lambda_{n-1} = \lambda_n = 0$ , which means we have a sequence of zeros of length  $(i - 1) + 1 = i$  and a sequence of length  $(n - 1) - i + 1 = n - i$ , hence by 1.1.1 we have that  $\mathbf{m}(x) = (i - 2) + (n - i - 2) = n - 4$ , which means that  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 = \mathbf{l}(x) + \mathbf{m}(x) = 1 + (n - 4) = n - 3 = (n - 1) - 2 = \dim \gamma_1/\gamma_2 - 2$ . Adding all the elements of  $\gamma_2/\gamma_3$ , we have that  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 + \dim \gamma_2/\gamma_3 = (n - 3) + (n - 2) = 2n - 5 < 2n - 4$ , which contradicts the assumption. Suppose that  $x$  is of the second form, then  $x$  can have one sequence of  $k$  non-zero coefficients, where  $k > 1$ . In that case,  $\mathbf{l}(x) = 1$ , and the number of zero coefficients is either  $(n - 1) - k + 1 = n - k$ , which results in  $\mathbf{m}(x) = n - k - 2$ , if the sequence of non-zero coefficients contains either  $\lambda_1$  or  $\lambda_{n-1}$ , or the number of consecutive zero coefficients is

<sup>3</sup>Again, we consider also  $\lambda_0 = 0$  and  $\lambda_n = 0$ , hence the sequence of zero coefficients is either  $\lambda_2 = \lambda_3 = \dots = \lambda_{n-2} = \lambda_{n-1} = \lambda_n = 0$  if  $\lambda_1 \neq 0$ , or  $\lambda_0 = \lambda_1 = \lambda_2 = \dots = \lambda_{n-3} = \lambda_{n-2} = 0$ , if  $\lambda_{n-1} \neq 0$ , which means that the length of the entire sequence of zero coefficients is not  $n - 2$  but  $n - 1$ , therefore by 1.1.1 we have that  $\mathbf{m}(x) = n - 1 - 2 = n - 3$

<sup>4</sup>In words,  $x$  has at least two non-zero coefficients in  $\gamma_1/\gamma_2$

divided into two separate sequences with a total of  $(n-1)-k+2 = n-k+1$ , which results in  $\mathbf{m} = n-k+1-4 = n-k-3$ , in the case that the sequence of non-zero coefficients of  $x$  does not contains either  $\lambda_1$  or  $\lambda_{n-1}$ , which means that  $\mathbf{l}(x) + \mathbf{m}(x) \leq 1 + (n-k-2) = n-k-1$ , and since  $k > 1$ , we have that  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 = \mathbf{l}(x) + \mathbf{m}(x) \leq n-2-1 = n-3 = (n-1)-2$ , which again contradicts the assumption.

Looking into the elements of  $\gamma_1/\gamma_2$  itself, then by 1.1.1, we have that  $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 =$

Let  $z = \lambda_{j,j+2}e_{j,j+2}$ , where  $1 \leq j \leq n-2$  and  $\lambda_{j,j+2} \in \mathbb{Q}_p$ , then for every  $w \in \gamma_1/\gamma_3$ , either  $z$  commutes with  $w$  or  $[z, w] \in \gamma_3\mathcal{L}_{n,p}$ , which means that  $\lambda_{j,j+2}e_{j,j+2} \in \mathcal{C}_{\gamma_1/\gamma_3}$ , for every  $1 \leq j \leq n-2$ . Hence,  $\gamma_2/\gamma_3 = \langle e_{13}, e_{24}, \dots, e_{n-2,n} \rangle \subset \mathcal{C}_{\gamma_1/\gamma_3}(x)$ . Therefore, we only need to discuss elements of the quotient  $\gamma_1/\gamma_2$ , for the purpose of this proof. Suppose that  $x = \lambda_1 e_{12} + z$ , where  $z \in \gamma_2\mathcal{L}_{n,p}$ , then we have one sequence of non-zero coefficients, namely  $\lambda_1$ , and we have  $n-2$  zero coefficients  $\lambda_2 = \lambda_3 = \dots = \lambda_{n-1} = 0$ , from which  $n-3$  are between two other zeros. Hence, by 1.1.1, we have that  $\mathcal{C}_{\gamma_1/\gamma_2}(x) = 1 + (n-3) = n-2 = (n-1)-1 = \dim \gamma_1/\gamma_2 - 1$ . Similarly, the same goes also for  $x = \lambda_{n-1}e_{n-1,n} + z$ . Suppose that  $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = \dim \gamma_1/\gamma_2 - 1$ , but  $x = \sum_{i=1}^{n-1} \lambda_i e_{i,i+1}$ , such that either of the following options is true:

1. there is more than one sequence of consecutive non-zero coefficients in the linear combination that forms  $x$ .
2. there is one sequence of consecutive non-zero coefficients, but at least one of those coefficients has index  $2 \leq j \leq n-2$ , meaning it is not  $\lambda_1$  nor  $\lambda_{n-1}$ .

For the second option, we start by fixing one index  $2 \leq j \leq n-2$ , and assume that  $x = \lambda_j e_{j,j+1}$ . The number of zero coefficients in  $x$  is  $n-1-1 = n-2$ , but  $\lambda_j$  and the zeros in indices  $j-1, j+1$  are neighboring, hence  $m_1 = n-2-2 = n-4$ , and then  $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = l_1 + m_1 = 1 + n-4 = n-3 < n-2 = \dim \gamma_1/\gamma_2 - 1$ . We denote by  $k$  the length of the sequence of consecutive non-zero parameters, and prove that for any  $k > 0$ , where at least one non-zero coefficient  $\lambda_j$  lies in  $2 \leq j \leq n-2$ ,  $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) < n-2$ , by simple induction on  $k$ . For  $k = 1$ , we have just shown that. For  $k > 1$ , there are  $k-1$  additional zeros that are replaced by non-zero coefficients, where except for  $\lambda_{j-1}$  and  $\lambda_{j+1}$ , all the other zeros were originally lying between two other zeros. If the original sequence was  $\lambda_2 e_{23}$  or  $\lambda_{n-2} e_{n-2,n-1}$ , and the new sequence is  $\lambda_1 e_{12}, \lambda_2 e_{23}$  or  $\lambda_{n-2} e_{n-2,n-1}, \lambda_{n-1} e_{n-1,n}$ , respectively, then  $m_k =$

$m_1$ , but clearly, in any other case,  $m_k < m_1$ , while  $l_k = l_1 = 1$  at any case. by the assumption, for the original sequence,  $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = l_1 + m_1 < n - 2$ , hence for the new sequence,  $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = l_k + m_k \leq l_1 + m_1 = n - 3 < n - 2$ . Now we check the first option, starting from the case where  $x = \lambda_1 e_{12} + \lambda_{n-1} e_{n-1,n}$ . In this case,  $l_2 = 2$  and the number of zeros is  $n - 1 - 2 = n - 3$ , but  $\lambda_1$  and the zero in index 2 are neighboring, and so are  $\lambda_{n-1}$  and the zero in index  $n - 2$ , hence  $m_2 = n - 3 - 2 = n - 5$  zeros are lying between two other zeros, therefore  $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = l_2 + m_2 = n - 5 + 2 = n - 3 < n - 2$ . if we add another non-zero coefficient, then it must lie in some index  $2 \leq j \leq n - 2$ , for which we have already proved that  $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) < n - 2$ , which completes the proof for  $n \geq 5$ . For  $n = 4$ , we can check explicitly. Assume  $x = \lambda e_{12} + \mu e_{34}$ , denote an element in the centralizer of  $x$  by  $y = \rho e_{12} + \tau e_{23} + \nu e_{34}$ , and we observe that  $[x, y] = [\lambda e_{12}, \tau e_{23}] + [\mu e_{34}, \tau e_{23}] = \lambda \tau e_{13} - \tau \mu e_{24} = 0$ , hence  $\tau = 0$ , while  $\rho = *$  and  $\nu = *$ , so  $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = 2 = \dim \gamma_1/\gamma_2 - 1$ , as requested, and it is readily seen that even if either  $\lambda = 0$  or  $\mu = 0$ , but not both, then  $\tau$  still has to be zero, in order to satisfy either  $\tau \mu = 0$  or  $\lambda \tau = 0$ , respectively, and  $\rho, \nu$  can still be anything, which means that in either case, where the coefficient of  $e_{23}$  is zero but  $x \neq 0$ , we have  $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = 2$ . Assume  $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = \dim \gamma_1/\gamma_2 - 1 = 3 - 1 = 2$ , then if  $x$  is not of the suggested form, it means that  $x = \lambda e_{12} + \sigma e_{23} + \mu e_{34}$ , where  $\sigma \neq 0$  and either  $\lambda$  or  $\mu$  or both can be zero. If  $x = \lambda e_{12} + \sigma e_{23} + \mu e_{34}$  and all coefficients are non-zero, then for every  $y \in \mathcal{C}_{\gamma_1/\gamma_2}(x)$  denoted by  $y = \rho e_{12} + \tau e_{23} + \nu e_{34}$ , we have  $[x, y] = [\lambda e_{12}, \tau e_{23}] + [\sigma e_{23}, \rho e_{12}] + [\sigma e_{23}, \nu e_{34}] + [\mu e_{34}, \tau e_{23}] = (\lambda \tau - \sigma \rho) e_{13} + (\sigma \nu - \mu \tau) e_{24}$ , hence  $\tau = \frac{\sigma}{\lambda} \rho$  and  $\nu = \frac{\mu}{\sigma} \tau = \frac{\mu}{\sigma} \frac{\sigma}{\lambda} \rho = \frac{\mu}{\lambda} \rho$ , but this means that  $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = 1$ , because both  $\tau$  and  $\nu$  depend on  $\rho$ . If either  $\lambda$  or  $\mu$  or both are zero, then either  $\sigma \rho$  or  $\sigma \mu$  or both are zero, which means that  $\rho$  or  $\nu$  or both are zero, since  $\sigma \neq 0$ , but this means that either  $y = \tau e_{23} + \frac{\mu}{\sigma} \tau e_{34}$  or  $y = \frac{\lambda}{\sigma} \tau e_{12} + \tau e_{23}$  or  $y = \tau e_{23}$ , respectively. Therefore, in either case, where  $\sigma \neq 0$ , we have  $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = 1$ , which completes the proof for  $n = 4$ .  $\square$

**Corollary 1.1.3.** *Let  $\mathcal{L}_{n,p}$  be a  $\mathbb{Q}_p$ -Lie algebra, where  $n \geq 4$ , and let  $\varphi \in G_n(\mathbb{Q}_p)$  be an  $\mathcal{L}_{n,p}$ -automorphism, then  $\varphi_{11}(e_{12}) = \lambda_1 e_{12}$  and  $\varphi_{11}(e_{n,n-1}) = \lambda_{n-1} e_{n-1,n}$ , or  $\varphi_{11}(e_{12}) = \lambda_{n-1} e_{n-1,n}$  and  $\varphi_{11}(e_{n,n-1}) = \lambda_1 e_{1,2}$ .*

*Proof.* We look at the centralizer of  $e_{12}$  in the quotient  $\gamma_1/\gamma_3$ , namely  $\mathcal{C}_{\gamma_1/\gamma_3}(e_{12})$ . Clearly, for any  $e_{i,i+2} \in \gamma_2/\gamma_3$ , we have that  $[e_{12}, e_{i,i+2}]$  is either zero, or  $i = 2$  and then  $[e_{12}, e_{24}] = e_{14} \in \gamma_3 \mathcal{L}_{n,p}$ , which vanishes in the quotient  $\gamma_1/\gamma_3$ , which means that in either case it is zero in this quotient. Therefore,

we look only at elements  $e_{i,i+1} \in \gamma_1/\gamma_2$ . It is readily seen that every element of the form  $e_{i,i+1}$  where  $i \neq 2$  commutes with  $e_{12}$ , hence  $\mathcal{C}_{\gamma_1/\gamma_2}(e_{12}) = \langle e_{12}, e_{34}, e_{45}, \dots, e_{n-2,n-1}, e_{n-1,n} \rangle$ , so  $\dim \mathcal{C}_{\gamma_1/\gamma_2}(e_{12}) = \dim \gamma_1/\gamma_2 - 1$ , but since  $\varphi_{11}$  is an automorphism, it must preserve the dimension of the centralizer, meaning  $\dim \mathcal{C}_{\gamma_1/\gamma_2}(\varphi_{11}(e_{12})) = \dim \mathcal{C}_{\gamma_1/\gamma_2}(e_{12}) = \dim \gamma_1/\gamma_2 - 1$ . But by corollary 1.1.2, if  $n \geq 5$ , then  $\varphi_{11}(e_{12}) = \lambda e_{12}$  or  $\varphi_{11}(e_{12}) = \lambda e_{n-1,n}$ , and it is readily seen that the same applies also for  $\varphi_{11}(e_{n-1,n})$ , and since  $\varphi$  is injective, then clearly, if  $\varphi_{11}(e_{12}) = \lambda e_{12}$  then  $\varphi_{11}(e_{n-1,n}) = \lambda e_{n-1,n}$ , and if  $\varphi_{11}(e_{12}) = \lambda e_{n-1,n}$  then  $\varphi_{11}(e_{n-1,n}) = \lambda e_{12}$ . If  $n = 4$ , then by the same corollary,  $\varphi_{11}(e_{12}) = \lambda e_{12} + \mu e_{34}$ , where  $\lambda$  and  $\mu$  are not both zero, which means that the same proof does not hold. Therefore, we now look at the centralizer of  $e_{12}$  in the algebra  $\mathcal{L}_{4,p}$  itself. We denote by  $\mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$  the centralizer of  $e_{12}$  in the algebra, which is  $\mathcal{C}_{\mathcal{L}_{4,p}}(e_{12}) = \langle e_{12}, e_{34}, e_{13}, e_{14} \rangle$ , so  $\dim \mathcal{C}_{\mathcal{L}_{4,p}}(e_{12}) = 4$ . Denote by  $x = \varphi(e_{12}) = \lambda_{12}e_{12} + \lambda_{23}e_{23} + \lambda_{34}e_{34} + \lambda_{13}e_{13} + \lambda_{24}e_{24} + \lambda_{14}e_{14} \in \mathcal{L}_{4,p}$ , and denote by  $y = \mu_{12}e_{12} + \mu_{23}e_{23} + \mu_{34}e_{34} + \mu_{13}e_{13} + \mu_{24}e_{24} + \mu_{14}e_{14} \in \mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$ , an element in the centralizer of  $e_{12}$ , hence  $[x, y] = (\lambda_{12}\mu_{23} - \lambda_{23}\mu_{12})e_{13} + (\lambda_{23}\mu_{34} - \lambda_{34}\mu_{23})e_{24} + (\lambda_{12}\mu_{24} - \lambda_{24}\mu_{12} + \lambda_{13}\mu_{34} - \lambda_{34}\mu_{13})e_{14} = 0$ . Assume all the coefficients of the linear combination that forms  $x$  are non-zero. Then, as seen earlier, we have that  $\mu_{23} = \frac{\lambda_{23}}{\lambda_{12}}\mu_{12}$ , and  $\mu_{34} = \frac{\lambda_{34}}{\lambda_{23}}\mu_{23} = \frac{\lambda_{34}}{\lambda_{23}}\frac{\lambda_{23}}{\lambda_{12}}\mu_{12} = \frac{\lambda_{34}}{\lambda_{12}}\mu_{12}$ , and also  $\lambda_{12}\mu_{24} - \lambda_{24}\mu_{12} + \lambda_{13}\mu_{34} - \lambda_{34}\mu_{13} = 0$ , which means that  $\mu_{24} = \frac{\lambda_{24}\mu_{12} + \lambda_{13}\mu_{34} - \lambda_{34}\mu_{13}}{\lambda_{12}} = \frac{\lambda_{24}\mu_{12} + \lambda_{13}\frac{\lambda_{34}}{\lambda_{12}}\mu_{12} - \lambda_{34}\mu_{13}}{\lambda_{12}}$ , hence we can choose freely  $\mu_{12}$ ,  $\mu_{13}$  and  $\mu_{14}$ , while  $\mu_{23}$  and  $\mu_{34}$  depend on  $\mu_{12}$ , and  $\mu_{24}$  depends on  $\mu_{12}$  and  $\mu_{13}$ , which means that  $\dim \mathcal{C}_{\mathcal{L}_{4,p}}(y) = 3 < 4 = \dim \mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$ . Assume that all the coefficients of  $x$  are non-zero, except for  $\lambda_{23} = 0$ , then  $\lambda_{12}\mu_{23}$  and  $\lambda_{34}\mu_{23}$  must vanish, hence  $\mu_{23} = 0$ , but then  $\mu_{34}$  does not depend on  $\mu_{23}$ , which implies that it does not depend on  $\mu_{12}$  either, and can be chosen freely, hence there is no change in the dimension of  $\mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$  from the general case. Now we assume  $x = \lambda_{12}e_{12} + z$ , where  $z \in \gamma_2\mathcal{L}_{4,p}$ , and observe the three equations from above with the current assumption. The second equation  $\lambda_{23}\mu_{34} - \lambda_{34}\mu_{23} = 0$  completely falls, which from the other two we obtain that  $\lambda_{12}\mu_{23}$  and  $\lambda_{12}\mu_{24}$  must vanish, which means that  $\mu_{23} = \mu_{24} = 0$ , while  $\mu_{12}$ ,  $\mu_{34}$ ,  $\mu_{13}$  and  $\mu_{14}$  can be chosen freely, which means that  $\dim \mathcal{C}_{\mathcal{L}_{4,p}}(y) = 4 = \dim \mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$ . One checks that the same applies also for  $\varphi(e_{12}) = \lambda_{34}e_{34} + z$ , and that no other linear combination of  $x$  satisfies that  $\dim \mathcal{C}_{\mathcal{L}_{4,p}}(\varphi(e_{12})) = \dim \mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$ .  $\square$