# 1 The computation of $G_n(\mathbb{Q}_p)$

## 1.1 The computation of the first block $M_{11}$

**Proposition 1.1.1.** *Let $x = \sum_{i=1}^{n-1} \lambda_i e_{i,i+1}$, where $\lambda_i \in \mathbb{Q}_p$ are not all zero. Then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \mathfrak{l}(x) + \mathfrak{m}(x)$, where $\mathfrak{l}(x)$ is the number of sequences of consecutive non-zero coefficients of the form $\lambda_j, \lambda_{j+1}, \dots, \lambda_{j+k-1}, \lambda_{j+k}$ and $\lambda_{j-1} = \lambda_{j+k+1} = 0$ (that is, the sequences are separated by one of more zero coefficients)*[1]*, and $\mathfrak{m}(x)$ is the number of zero coefficients $\lambda_j = 0$, such that also $\lambda_{j-1} = \lambda_{j+1} = 0$.*

*Proof.* Let $y = \sum_{i=1}^{n-1} \mu_i e_{i,i+1}$, where $\lambda_i \in \mathbb{Q}_p$. For every $1 \le i \le n-1$, denote by $(\mathfrak{C}_i)$ the constraint equation $[\lambda_i e_{i,i+1}, \mu_{i+1} e_{i+1,i+2}] - [\lambda_{i+1} e_{i+1,i+2}, \mu_i e_{i,i+1}] = (\lambda_i \mu_{i+1} - \lambda_{i+1} \mu_i) e_{i,i+2} = 0$, and it is clear that $y \in \mathcal{C}_{\gamma_1/\gamma_3}(x)$ if and only if all the $(\mathfrak{C}_i)$ constraints are satisfied. Let $1 \le j \le n-1$ and $1 \le k \le n-1-j$ be two indices, such that $\lambda_{j-1} = \lambda_{j+k+1} = 0$, and $\lambda_j, \lambda_{j+1}, \dots, \lambda_{j+k-1}, \lambda_{j+k}$ are all non-zero, then by constraints $(\mathfrak{C}_j), (\mathfrak{C}_{j+1}), \dots, (\mathfrak{C}_{m-1})$, we have that $\mu_m = \frac{\lambda_m}{\lambda_{m-1}} \mu_{m-1} = \frac{\lambda_m}{\lambda_{m-1}} \frac{\lambda_{m-1}}{\lambda_{m-2}} \mu_{m-2} = \frac{\lambda_m}{\lambda_{m-2}} \mu_{m-2} = \cdots = \frac{\lambda_m}{\lambda_j} \mu_j$, for every $j+1 \le m \le j+k-1$, which means that for any choice of the first coefficient in the sequence, namely $\mu_j$, all the next $\mu$ coefficients of the sequence, with indices from $j+1$ to $j+k$, depend on $\mu_j$. By constraint $(\mathfrak{C}_{j-1})$, we have that $\lambda_{j-1}\mu_j - \lambda_j \mu_{j-1} = 0$, but $\lambda_{j-1} = 0$, hence $\lambda_j \mu_{j-1}$ must vanish, and since $\lambda_j \neq 0$, we must have that $\mu_{j-1} = 0$. Similarly, we have that $\mu_{j+k+1} = 0$, due to constraint $(\mathfrak{C}_{j+k})$. This shows that for any sequence of $k$ consecutive non-zero $\lambda$ coefficients of $x$, we have a sequence of $k$ consecutive non-zero $\mu$ coefficients in $y$, but since they all depend on the first $\lambda$ coefficient in the sequence, then this sequence increases $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)$ by 1, regardless of its length. By constraint $(\mathfrak{C}_{j+k+1})$, we have that $\lambda_{j+k+1}\mu_{j+k+2} - \lambda_{j+k+2}\mu_{j+k+1} = 0$, but $\lambda_{j+k+1} = \mu_{j+k+1} = 0$, hence, $\lambda_{j+k+2}$ and $\mu_{j+k+2}$ can be any scalar, in order for constraint $(\mathfrak{C}_{j+k+1})$ to be satisfied. If $lambda_{j+k+2} \neq 0$, then we have a new sequence of consecutive non-zero $\lambda$ coefficients, a case we have already handled. Therefore, we assume $\lambda_{j+k+2} = 0$, and we look into the constraint $(\mathfrak{C}_{j+k+2})$. Since $\lambda_{j+k+2}\mu_{j+k+3} - \lambda_{j+k+3}\mu_{j+k+2} = 0$ and $\lambda_{j+k+2}\mu_{j+k+3}$ vanishes, then $\lambda_{j+k+3}\mu_{j+k+2}$ must vanish as well. Hence, if $\lambda_{j+k+3} \neq 0$, we must have $\mu_{j+k+2} = 0$, but if $\lambda_{j+k+3} = 0$, then $\mu_{j+k+2}$ can be any scalar. So, we have that for $\lambda_{j+k+2} = 0$, where $\lambda_{j+k+1} = \lambda_{j+k+3} = 0$, we have that $\mu_{j+k+2}$ can be

[1] We extend our notation of indices, to include also the case where $j = 1$ or $j+k = n-1$, and define that $\lambda_{j-1} = \lambda_0 = 0$ or $\lambda_{j+k+1} = \lambda_n = 0$, respectively

any scalar, which means that it increases $\mathcal{C}_{\gamma_1/\gamma_3}(x)$ by 1. By simple induction, we prove that given a sequence of $l$ consecutive zero coefficients of $x$, $\lambda_{j+1} = \lambda_{j+2} = \cdots = \lambda_{j+l-1} = \lambda_{j+l} = 0$, it increases $\mathcal{C}_{\gamma_1/\gamma_3}(x)$ by $l-2$, and all the $\mu$ coefficients, $\mu_{j+2}, \mu_{j+3}, \ldots, \mu_{j+l-2}, \mu_{j+l-1}$ can be any scalar. For $l = 3$, in our previous notations, $\lambda_{j+k+1} = \lambda_{j+k+2} = \lambda_{j+k+3} = 0$, we have that $\mu_{j+k+2}$ can be any scalar, and so it increases the dimension of the centralizer by $l-2 = 1$. To prove this is true for $l+1$, we look into the sequence of $l+1$ zero coefficients of $x$, $\lambda_{j+1}, \ldots, \lambda_{j+l+1}$. By the assumption, $\mu_{j+2}, \ldots, \mu_{j+l-1}$ can be any scalars. By constraint $\mathfrak{C}_{j+l-1}$, we have that $\lambda_{j+l-1}\mu_{j+l} - \lambda_{j+l}\mu_{j+l-1} = 0$, but $\lambda_{j+l-1} = \lambda_{j+l} = 0$, which means that $\mu_{j+l}$ can be any scalar. By constraint $\mathfrak{C}_{j+l}$, we have that $\lambda_{j+l}\mu_{j+l+1} - \lambda_{j+l+1}\mu_{j+l} = 0$, and since $\lambda_{j+l} = \lambda_{j+l+1} = 0$, it is clear that $\mu_{j+l}$ is not constrained either by $\mathfrak{C}_{j+l-1}$ nor by $\mathfrak{C}_{j+l}$, which means that $\mu_{j+l}$ can be any scalar, and thus increases $\mathcal{C}_{\gamma_1/\gamma_3}(x)$ by 1, so the $l$ zero coefficients increase $\mathcal{C}_{\gamma_1/\gamma_3}(x)$ by $l-1 = l+1-2$, which proves the induction step. □

**Corollary 1.1.2.** *Let $\mathcal{L}_{n,p}$ be the $\mathbb{Q}_p$-Lie algebra associated with $\mathcal{U}_n(\mathbb{Z})$. If $n \geq 5$, then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim {}^{\gamma_1}/{}_{\gamma_3} - 1$ if and only if $x \in \{\lambda e_{12} + \gamma_2\mathcal{L}_{n,p}\}$ or $x \in \{\lambda e_{n-1,n} + \gamma_2\mathcal{L}_{n,p}\}$, for a non-zero scalar $\lambda \in \mathbb{Q}_p$. If $n = 4$, then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim {}^{\gamma_1}/{}_{\gamma_3} - 1$ if and only if $x \in \{\lambda e_{12} + \mu e_{34} + \gamma_2\mathcal{L}_{n,p}\}$, for $\lambda, \mu \in \mathbb{Q}_p$ not both zero.*

*Proof.* Let $z = \lambda_{j,j+2}e_{j,j+2}$, where $1 \leq j \leq n-2$ and $\lambda_{j,j+2} \in \mathbb{Q}_p$, then for every $w \in {}^{\gamma_1}/{}_{\gamma_3}$, either $z$ commutes with $w$ or $[z, w] \in \gamma_3\mathcal{L}_{n,p}$, which means that $\lambda_{j,j+2}e_{j,j+2} \in \mathcal{C}_{\gamma_1/\gamma_3}$, for every $1 \leq j \leq n-2$. Hence, ${}^{\gamma_2}/{}_{\gamma_3} = \langle e_{13}, e_{24}, \ldots, e_{n-2,n}\rangle \subset \mathcal{C}_{\gamma_1/\gamma_3}(x)$. Therefore, we only need to discuss elements of the quotient ${}^{\gamma_1}/{}_{\gamma_2}$, for the purpose of this proof. Suppose that $x = \lambda_1 e_{12} + z$, where $z \in \gamma_2\mathcal{L}_{n,p}$, then we have one sequence of non-zero coefficients, namely $\lambda_1$, and we have $n-2$ zero coefficients $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1} = 0$, from which $n-3$ are between two other zeros. Hence, by 1.1.1, we have that $\mathcal{C}_{\gamma_1/\gamma_2}(x) = 1 + (n-3) = n-2 = (n-1)-1 = \dim {}^{\gamma_1}/{}_{\gamma_2} - 1$. Similarly, the same goes also for $x = \lambda_{n-1}e_{n-1,n} + z$. Suppose that $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = \dim {}^{\gamma_1}/{}_{\gamma_2} - 1$, but $x = \sum_{i=1}^{n-1} \lambda_i e_{i,i+1}$, such that either of the following options is true:

1. there is more than one sequence of consecutive non-zero coefficients in the linear combination that forms $x$.

2. there is one sequence of consecutive non-zero coefficients, but at least one of those coefficients has index $2 \leq j \leq n-2$, meaning it is not $\lambda_1$ nor $\lambda_{n-1}$.

For the second option, we start by fixing one index $2 \le j \le n-2$, and assume that $x = \lambda_j e_{j,j+1}$. The number of zero coefficients in $x$ is $n-1-1 = n-2$, but $\lambda_j$ and the zeros in indices $j-1, j+1$ are neighboring, hence $m_1 = n-2-2 = n-4$, and then $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = l_1 + m_1 = 1 + n - 4 = n-3 < n-2 = \dim {}^{\gamma_1}\!/\!_{\gamma_2} - 1$. We denote by $k$ the length of the sequence of consecutive non-zero parameters, and prove that for any $k > 0$, where at least one non-zero coefficient $\lambda_j$ lies in $2 \le j \le n-2$, $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) < n-2$, by simple induction on $k$. For $k = 1$, we have just shown that. For $k > 1$, there are $k-1$ additional zeros that are replaced by non-zero coefficients, where except for $\lambda_{j-1}$ and $\lambda_{j+1}$, all the other zeros were originally lying between two other zeros. If the original sequence was $\lambda_2 e_{23}$ or $\lambda_{n-2} e_{n-2,n-1}$, and the new sequence is $\lambda_1 e_{12}, \lambda_2 e_{23}$ or $\lambda_{n-2} e_{n-2,n-1}, \lambda_{n-1} e_{n-1,n}$, respectively, then $m_k = m_1$, but clearly, in any other case, $m_k < m_1$, while $l_k = l_1 = 1$ at any case. by the assumption, for the original sequence, $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = l_1 + m_1 < n-2$, hence for the new sequence, $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = l_k + m_k \le l_1 + m_1 = n-3 < n-2$. Now we check the first option, starting from the case where $x = \lambda_1 e_{12} + \lambda_{n-1} e_{n-1,n}$. In this case, $l_2 = 2$ and the number of zeros is $n-1-2 = n-3$, but $\lambda_1$ and the zero in index 2 are neighboring, and so are $\lambda_{n-1}$ and the zero in index $n-2$, hence $m_2 = n-3-2 = n-5$ zeros are lying between two other zeros, therefore $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = l_2 + m_2 = n-5+2 = n-3 < n-2$. if we add another non-zero coefficient, then it must lie in some index $2 \le j \le n-2$, for which we have already proved that $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) < n-2$, which completes the proof for $n \ge 5$. For $n = 4$, we can check explicitly. Assume $x = \lambda e_{12} + \mu e_{34}$, denote an element in the centralizer of $x$ by $y = \rho e_{12} + \tau e_{23} + \nu e_{34}$, and we observe that $[x,y] = [\lambda e_{12}, \tau e_{23}] + [\mu e_{34}, \tau e_{23}] = \lambda\tau e_{13} - \tau\mu e_{24} = 0$, hence $\tau = 0$, while $\rho = *$ and $\nu = *$, so $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = 2 = \dim {}^{\gamma_1}\!/\!_{\gamma_2} - 1$, as requested, and it is readily seen that even if either $\lambda = 0$ or $\mu = 0$, but not both, then $\tau$ still has to be zero, in order to satisfy either $\tau\mu = 0$ or $\lambda\tau = 0$, respectively, and $\rho, \nu$ can still be anything, which means that in either case, where the coefficient of $e_{23}$ is zero but $x \ne 0$, we have $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = 2$. Assume $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = \dim {}^{\gamma_1}\!/\!_{\gamma_2} - 1 = 3 - 1 = 2$, then if $x$ is not of the suggested form, it means that $x = \lambda e_{12} + \sigma e_{23} + \mu e_{34}$, where $\sigma \ne 0$ and either $\lambda$ or $\mu$ or both can be zero. If $x = \lambda e_{12} + \sigma e_{23} + \mu e_{34}$ and all coefficients are non-zero, then for every $y \in \mathcal{C}_{\gamma_1/\gamma_2}(x)$ denoted by $y = \rho e_{12} + \tau e_{23} + \nu e_{34}$, we have $[x,y] = [\lambda e_{12}, \tau e_{23}] + [\sigma e_{23}, \rho e_{12}] + [\sigma e_{23}, \nu e_{34}] + [\mu e_{34}, \tau e_{23}] = (\lambda\tau - \sigma\rho)e_{13} + (\sigma\nu - \mu\tau)e_{24}$, hence $\tau = \frac{\sigma}{\lambda}\rho$ and $\nu = \frac{\mu}{\sigma}\tau = \frac{\mu}{\sigma}\frac{\sigma}{\lambda}\rho = \frac{\mu}{\lambda}\rho$, but this means that $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = 1$, because both $\tau$ and $\nu$ depend on $\rho$. If either $\lambda$ or $\mu$ or both are zero, then either $\sigma\rho$ or $\sigma\mu$ or both are zero, which means that $\rho$ or

$\nu$ or both are zero, since $\sigma \neq 0$, but this means that either $y = \tau e_{23} + \frac{\mu}{\sigma}\tau e_{34}$ or $y = \frac{\lambda}{\sigma}\tau e_{12} + \tau e_{23}$ or $y = \tau e_{23}$, respectively. Therefore, in either case, where $\sigma \neq 0$, we have $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = 1$, which completes the proof for $n = 4$. $\square$

**Corollary 1.1.3.** *Let $\mathcal{L}_{n,p}$ be a $\mathbb{Q}_p$-Lie algebra, where $n \geq 4$, and let $\varphi \in G_n(\mathbb{Q}_p)$ be an $\mathcal{L}_{n,p}$-automorphism, then $\varphi_{11}(e_{12}) = \lambda_1 e_{12}$ and $\varphi_{11}(e_{n,n-1}) = \lambda_{n-1}e_{n-1,n}$, or $\varphi_{11}(e_{12}) = \lambda_{n-1}e_{n-1,n}$ and $\varphi_{11}(e_{n,n-1}) = \lambda_1 e_{1,2}$.*

*Proof.* We look at the centralizer of $e_{12}$ in the quotient $\gamma_1/\gamma_3$, namely $\mathcal{C}_{\gamma_1/\gamma_3}(e_{12})$. Clearly, for any $e_{i,i+2} \in \gamma_2/\gamma_3$, we have that $[e_{12}, e_{i,i+2}]$ is either zero, or $i = 2$ and then $[e_{12}, e_{24}] = e_{14} \in \gamma_3\mathcal{L}_{n,p}$, which vanishes in the quotient $\gamma_1/\gamma_3$, which means that in either case it is zero in this quotient. Therefore, we look only at elements $e_{i,i+1} \in \gamma_1/\gamma_2$. It is readily seen that every element of the form $e_{i,i+1}$ where $i \neq 2$ commutes with $e_{12}$, hence $\mathcal{C}_{\gamma_1/\gamma_2}(e_{12}) = \langle e_{12}, e_{34}, e_{45}, \ldots, e_{n-2,n-1}, e_{n-1,n}\rangle$, so $\dim \mathcal{C}_{\gamma_1/\gamma_2}(e_{12}) = \dim \gamma_1/\gamma_2 - 1$, but since $\varphi_{11}$ is an automorphism, it must preserve the dimension of the centralizer, meaning $\dim \mathcal{C}_{\gamma_1/\gamma_2}(\varphi_{11}(e_{12})) = \dim \mathcal{C}_{\gamma_1/\gamma_2}(e_{12}) = \dim \gamma_1/\gamma_2 - 1$. But by corollary 1.1.2, if $n \geq 5$, then $\varphi_{11}(e_{12}) = \lambda e_{12}$ or $\varphi_{11}(e_{12}) = \lambda e_{n-1,n}$, and it is readily seen that the same applies also for $\varphi_{11}(e_{n-1,n})$, and since $\varphi$ is injective, then clearly, if $\varphi_{11}(e_{12}) = \lambda e_{12}$ then $\varphi_{11}(e_{n-1,n}) = \lambda e_{n-1,n}$, and if $\varphi_{11}(e_{12}) = \lambda e_{n-1,n}$ then $\varphi_{11}(e_{n-1,n}) = \lambda e_{12}$. If $n = 4$, then by the same corollary, $\varphi_{11}(e_{12}) = \lambda e_{12} + \mu e_{34}$, where $\lambda$ and $\mu$ are not both zero, which means that the same proof does not hold. Therefore, we now look at the centralizer of $e_{12}$ in the algebra $\mathcal{L}_{4,p}$ itself. We denote by $\mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$ the centralizer of $e_{12}$ in the algebra, which is $\mathcal{C}_{\mathcal{L}_{4,p}}(e_{12}) = \langle e_{12}, e_{34}, e_{13}, e_{14}\rangle$, so $\dim \mathcal{C}_{\mathcal{L}_{4,p}}(e_{12}) = 4$. Denote by $x = \varphi(e_{12}) = \lambda_{12}e_{12} + \lambda_{23}e_{23} + \lambda_{34}e_{34} + \lambda_{13}e_{13} + \lambda_{24}e_{24} + \lambda_{14}e_{14} \in \mathcal{L}_{4,p}$, and denote by $y = \mu_{12}e_{12} + \mu_{23}e_{23} + \mu_{34}e_{34} + \mu_{13}e_{13} + \mu_{24}e_{24} + \mu_{14}e_{14} \in \mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$, an element in the centralizer of $e_{12}$, hence $[x, y] = (\lambda_{12}\mu_{23} - \lambda_{23}\mu_{12})e_{13} + (\lambda_{23}\mu_{34} - \lambda_{34}\mu_{23})e_{24} + (\lambda_{12}\mu_{24} - \lambda_{24}\mu_{12} + \lambda_{13}\mu_{34} - \lambda_{34}\mu_{13})e_{14} = 0$. Assume all the coefficients of the linear combination that forms $x$ are non-zero. Then, as seen earlier, we have that $\mu_{23} = \frac{\lambda_{23}}{\lambda_{12}}\mu_{12}$, and $\mu_{34} = \frac{\lambda_{34}}{\lambda_{23}}\mu_{23} = \frac{\lambda_{34}}{\lambda_{23}}\frac{\lambda_{23}}{\lambda_{12}}\mu_{12} = \frac{\lambda_{34}}{\lambda_{12}}\mu_{12}$, and also $\lambda_{12}\mu_{24} - \lambda_{24}\mu_{12} + \lambda_{13}\mu_{34} - \lambda_{34}\mu_{13} = 0$, which means that $\mu_{24} = \frac{\lambda_{24}\mu_{12} + \lambda_{13}\mu_{34} - \lambda_{34}\mu_{13}}{\lambda_{12}} = \frac{\lambda_{24}\mu_{12} + \lambda_{13}\frac{\lambda_{34}}{\lambda_{12}}\mu_{12} - \lambda_{34}\mu_{13}}{\lambda_{12}}$, hence we can choose freely $\mu_{12}$, $\mu_{13}$ and $\mu_{14}$, while $\mu_{23}$ and $\mu_{34}$ depend on $\mu_{12}$, and $\mu_{24}$ depends on $\mu_{12}$ and $\mu_{13}$, which means that $\dim \mathcal{C}_{\mathcal{L}_{4,p}}(y) = 3 < 4 = \dim \mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$. Assume that all the coefficients of $x$ are non-zero, except for $\lambda_{23} = 0$, then $\lambda_{12}\mu_{23}$ and $\lambda_{34}\mu_{23}$ must vanish, hence $\mu_{23} = 0$, but then $\mu_{34}$ does not depend on $\mu_{23}$, which implies that it does not depend on $\mu_{12}$ either, and can be chosen freely, hence there is no change in the dimension of $\mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$ from the general case. Now we assume $x = \lambda_{12}e_{12} + z$, where $z \in \gamma_2\mathcal{L}_{4,p}$, and observe the three equations from above with the current assumption. The second equation $\lambda_{23}\mu_{34} - \lambda_{34}\mu_{23} = 0$ completely falls, which from the other two we obtain that $\lambda_{12}\mu_{23}$ and $\lambda_{12}\mu_{24}$ must vanish, which means that $\mu_{23} = \mu_{24} = 0$, while $\mu_{12}$, $\mu_{34}$, $\mu_{13}$ and $\mu_{14}$ can be chosen freely, which means that $\dim \mathcal{C}_{\mathcal{L}_{4,p}}(y) = 4 = \dim \mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$. One checks that the same applies also for $\varphi(e_{12}) = \lambda_{34}e_{34} + z$, and that no other linear combination of $x$ satisfies that $\dim \mathcal{C}_{\mathcal{L}_{4,p}}(\varphi(e_{12})) = \dim \mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$. $\square$