Exercise Let $\{E_{i,j}\}_{i<j}$ be the set of all elementary matrices, of this form. Prove that $E_{i,j}^{-1} = (b_{l,k})$ is $E_{i,j} = (a_{l,k})$, when we substitute $a_{i,j} = 1$ with $b_{i,j} = -1$

Proof We can see that directly from the fact that if we multiply $E_{i,j}^{-1}$ by $E_{i,j}$ from the left then $E_{i,j}$ is operating on $E_{i,j}^{-1}$ by adding row j to row i
So, in the product matrix, $(c_{l,k})$, in order to have 1
on the main diagonal, we need them to exist on the main diagonal of $E_{i,j}^{-1}$, to begin with. Now, in order to have $c_{i,j} = 0$, we need to have the addition of j to i giving $c_{i,j} = a_{i,j} + b_{i,j} = 0 \Rightarrow b_{i,j} = -a_{i,j} = -1$

Exercise Prove that if $(a_{ij}) = E_{i,j}, i < j$ is an elementary matrix, then $\forall m \in (N), E_{i,j}^m$ is $E_{i,j}$, but with $a_{ij} = m$

$$E_{i,j} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$E_{i,j}^2 = E_{i,j} \cdot E_{i,j}$$

Since $E_{i,j}$ is en elemntary matrix, then it operates on the right matrix as an addition of row j to row i

So,

$$E_{i,j}^2 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 2 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

We assume this is true for all $E_{i,j}^m$, now we prove for $E_{i,j}^{m+1}$

$$E_{i,j}^{m+1} = E_{i,j} \cdot E_{i,j}^m = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}^m$$

(by the assumption)

$$= \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & m & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & m+1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

From these two exercises, we obtain an almost trivial corollary
Exercise Prove that if $(a_{ij}) = E_{i,j}, i < j$ is an elementary matrix, then $\forall m \in (N), (E_{i,j}^{-1})^m = (E_{i,j}^m)^{-1} = E_{i,j}^{-m}$ is $E_{i,j}$, but with $a_{ij} = -m$

Proof We immitate both proofs from above (we can either show how the power of m is operating on $E_{i,j}^{-1}$, or show how the inversion is operating on $E_{i,j}^m$).

Commutators of elementary matrices

Let $\{E_{i,j}\}_{i<j}$ be the set of all elementary matrices of this form.

Exercise $(a_{l,m}) = E_{i,j}^{-1}$ is the matrix with 1 on the main diagonal, and -1 in $a_{i,j}$

Proof We can see that directly from the fact that in order to have
$(c_{l,m}) = (a_{l,m}) \cdot (b_{l,m}) = E_{i,j} \cdot E_{i,j}^{-1} = I$,
we need to have $c_{i,j} = 0$, which means that adding row j to row i, in $E_{i,j}^{-1}$ (by the left multiplication of $E_{i,j}$)
must give $a_{i,j} + b_{i,j} = c_{i,j} = 0 \Rightarrow b_{i,j} = -a_{i,j} = -1$

Exercise $[E_{i,j}, E_{j,k}] = E_{i,k}$

Proof $E_{i,j}$ is operating from left on $E_{j,k}$ by addition of row j to row i, so, the product matrix, $(a_{l,m}) = E_{i,j} \cdot E_{j,k}$ has 1 on the main diagonal and in $a_{j,k}, a_{i,j}, a_{i,k}$
$E_{i,j}^{-1}$ is operating from left on $E_{j,k}^{-1}$ by subtraction of row j from row i, so, the product matrix, $(b_{l,m}) = E_{i,j}^{-1} \cdot E_{j,k}^{-1}$ has 1 on the main diagonal and in $b_{i,k}$, and -1 in $b_{j,k}, b_{i,j}$
Multiplying $(a_{l,m}) \cdot (b_{l,m})$ yields a product matrix, $(c_{l,m})$ with 1 on the main diagonal, and,
since $a_{i,i} = a_{i,j} = a_{i,k} = 1$, with all other cells in row j being 0, and since $b_{i,k} = b_{k,k} = 1$, and $b_{j,k} = -1$, multiplying row $(a_{l,m})_i$ by column $(b_{l,m})_k$ yields the value $c_{i,k} = b_{i,k} + b_{j,k} + b_{k,k} = 1 - 1 + 1 = 1$
We can see that multiplying $(a_{l,m})_i \cdot (b_{l,m})_j$ yields $c_{i,j} = a_{i,i} \cdot b_{i,j} + a_{i,j} \cdot b_{j,j} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$
And, we can see that multiplying $(a_{l,m})_j \cdot (b_{l,m})_k$ yields $c_{j,k} = a_{j,j} \cdot b_{j,k} + a_{j,k} \cdot b_{k,k} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

Conclusion

$[E_{j,k}, E_{i,j}] = E_{j,k} \cdot E_{i,j} \cdot E_{j,k}^{-1} \cdot E_{i,j}^{-1} = ((E_{i,j}^{-1})^{-1} \cdot (E_{j,k}^{-1})^{-1} \cdot E_{i,j}^{-1} \cdot E_{j,k}^{-1})^{-1} = (E_{i,j} \cdot E_{j,k} \cdot E_{i,j}^{-1} \cdot E_{j,k}^{-1})^{-1} = [E_{i,j}, E_{j,k}]^{-1}$

For example, $n = 4$,

$$E_{1,2} \cdot E_{2,3} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_{1,2}^{-1} \cdot E_{2,3}^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[E_{1,2} \cdot E_{2,3}] = E_{1,2} \cdot E_{2,3} \cdot E_{1,2}^{-1} \cdot E_{2,3}^{-1} =$$

$$= \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{1,3}$$

Exercise $[E_{i,j}, E_{l,k}] = I$, where $j \neq l$

Proof $E_{i,j}$ is operating from left on $E_{l,k}$ by addition of row j to row i, so, the product matrix, $(a_{n,m} = E_{i,j} \cdot E_{l,k}$ has 1 on the main diagonal and in $a_{l,k}, a_{i,j}$

$E_{i,j}^{-1}$ is operating from left on $E_{l,k}^{-1}$ by subtraction of row j from row i, so, the product matrix, $(b_{n,m} = E_{i,j}^{-1} \cdot E_{l,k}^{-1}$ has 1 on the main diagonal,

and -1 in $b_{l,k}, b_{i,j}$

We can see that multiplying $(a_{n,m})_i \cdot (b_{n,m})_j$ yields $c_{i,j} = a_{i,i} \cdot b_{i,j} + a_{i,j} \cdot b_{j,j} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

And, we can see that multiplying $(a_{n,m})_l \cdot (b_{n,m})_k$ yields $c_{l,k} = a_{l,l} \cdot b_{l,k} + a_{l,k} \cdot b_{k,k} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

For example, $n = 4$,

$$E_{1,2} \cdot E_{3,4} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_{1,2}^{-1} \cdot E_{3,4}^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[E_{1,2} \cdot E_{3,4}] = E_{1,2} \cdot E_{3,4} \cdot E_{1,2}^{-1} \cdot E_{3,4}^{-1} =$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

Conclusion
$[E_{i,j},[E_{j,k},E_{k,l}]] = [E_{i,j},E_{j,l}] = E_{i,l}$
$[E_{i,j},[E_{j,k},E_{m,l}]], m \neq k = [E_{i,l},I] = I$
$[E_{i,m},[E_{j,k},E_{k,l}]], m \neq j = [E_{i,m},E_{j,l}] = I$

$$\Rightarrow [E_{i_1,i_2},[E_{i_3,i_4},...[E_{i_{n-2},i_{n-1}},E_{i_{n-1},i_n}]] = \begin{cases} E_{i_1,i_n}, & i_{2k} = i_{2k+1}, \forall 1 \leq k \leq \frac{n}{2}-1 \\ I, & \text{otherwise} \end{cases}$$

Exercise
$\#\{E_{i,j} \in M_n(\mathbb{Z})\}_{i<j} = \binom{n}{2}$

Proof
$(a_{i,j} = E_{i,j})$. We need to count the options for 1 above the main diagonal. $a_{l,l} = 1, \forall 1 \leq l \leq n$, so, if $i = l$, we have $n - l = n - i$ options to choose the column index j.
So, the total number of options for i, j is $\sum_{k=1}^{n-1} = \frac{(1+n-1)\cdot(n-1)}{2} = \frac{n\cdot(n-1)}{2} = \binom{n}{2}$

This means that we have $\binom{n}{2}^2$ commutators of the form $[E_{i,j},E_{l,k}]$.
Exercise
$\#\{[E_{i,j},E_{l,k}] \neq I \in M_n(\mathbb{Z})\}_{i<j} = 2 \cdot \binom{n}{3}$

Proof
As shown above, $[E_{i,j},E_{l,k}] \neq I \Leftrightarrow j = l$
Which means we're counting all the commutators of the form $[E_{i,j},E_{j,k}]$.
So, the count of such commutators is based on the number of options to choose
ordered triples $\{i,j,k\}$ out of the ordered set $[n] = \{1,2,...,n\}$, which is $\binom{n}{3}$
But, as already shown above, $[E_{l,k},E_{i,j}] = [E_{i,j},E_{l,k}]^{-1}$, so, for each triple $\{i,j,k\}$, we have two commutators, $[E_{i,j},E_{j,k}]$ and its inverse, which sum up to $\binom{n}{3}$ pairs of commutators.

For example, $n = 5$,

$$(a_{l,k}) = E_{i,j} = \begin{pmatrix} 1 & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ 0 & 1 & a_{2,3} & a_{2,4} & a_{2,5} \\ 0 & 0 & 1 & a_{3,4} & a_{3,5} \\ 0 & 0 & 0 & 1 & a_{4,5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Where $a_{i,j} = 1$, and all other $a_{l,k} = 0$
The number of options for choosing i, j, in this case, are $1+2+3+4 = 10 = \binom{5}{2}$,
so, we have $10^2 = 100$ commutators. The number of triples we can choose from $[5] = \{1, 2, 3, 4, 5\}$ is
$\#\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\} = 10 = \binom{5}{3}$,
so we have 10 commutators that are not the unit matrix, and their inverse, total $20 = 2 \cdot 10 = 2 \cdot \binom{5}{3}$.

Exercise

Given the set of commutators of elementary matrices of the form
$\{[E_{i,j}, E_{j,k}] \in M_n(\mathbb{Z})\}_{i<j<k}$,
we can divide this set to subsets of the form
$\{\{[E_{i_1,j_{1,1}}, E_{j_{1,1},k_1}], [E_{i_1,j_{1,2}}, E_{j_{1,2},k_1}], ..., [E_{i_1,j_{1,l_1}}, E_{j_{1,l_1},k_1}]\}, ...,$
$\{[E_{i_m,j_{m,1}}, E_{j_{m,1},k_1}], ..., [E_{i_m,j_{m,l_m}}, E_{j_{m,l_m},k_1}]\}\}$
These subsets are equivalence classes, trivially, since the relation is equality (i.e. $[E_{i_l,j_{l,m_1}}, E_{j_{l,m_1},k_l}] = [E_{i_l,j_{l,m_2}}, E_{j_{l,m_2},k_l}], i_l < j_{l,m_1}, j_{l,m_2} < k_l$).
Fix $i, k, 1 \leq i \leq n-1, 3\ leqk \leq n$, then all the triples of the form $\{i, j, k\}, i \leq i+1 \leq k-1$ are in the same equivalence class,
due to the above equality. So, the number of these equivalence classes is $2 \cdot \binom{n-1}{2}$

Proof

By induction on n. For $n = 3$, we have only one triple, namely $\{1, 2, 3\}$, so $\binom{3-1}{2} = \binom{2}{2} = 1$
For $n + 1$, we shall observe that if we add one to the upper bound (i.e. $n \to n' = n + 1$,
then we add one more equivalence class, for each one of the lower bounds of $n' - 1 = n$ (i.e., the index i).
But we also add a new eqivalence class, whose lower bound is $i = n+1-2 = n-1 = n'-2$, which was not in any equivalence class
for $n = n' - 1$, since we consider only the triples where $i \leq n - 2$. So, if we mark m_n as the number of equivalence classes
for n, then we have $m_{n'} = m_{n+1} = m_n + (n-2) + 1 = m_n + n - 1$. But, by the assumption, $m_n = \binom{n-1}{2}$,
so $m_{n'} = m_{n+1} = m_n + n - 1 = \binom{n-1}{2} + n - 1 = \frac{(n-1)\cdot(n-2)}{2} + n - 1 =$

$\frac{n^2-3n+2}{2}+n-1=\frac{n^2-3n+2+2n-2}{2}=\frac{n^2-n}{2}=\frac{n\cdot(n-1)}{2}=\binom{n}{2}=m_{n+1}=m_{n'}$, and we proved the assumption

The group $U_n(\mathbb{Z})$

We have proved several basic facts, regarding elementary matrices, of the form $\{E_{i,j}\}_{i<j}$.

Now, we shall propose a few more basic facts.

Notation

We mark by $U_n(\mathbb{Z})$ the set of all upper triangular matrices $n \times n$ with 1 in the main diagonal, and any integer values above the main diagonal, $a_{i,j} \in \mathbb{Z}$.

Exercise Prove that the set $U_n(\mathbb{Z})$ is a group, with the usual operation of matrix multiplication.

Proof

$$(a_{i,j}) = A = \begin{pmatrix} 1 & a_{1,2} & \dots & a_{1,n-1} & a_{1,n} \\ 0 & 1 & \dots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{k,n-1} & a_{k,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_{n-1,n} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, (b_{i,j}) = B = \begin{pmatrix} 1 & b_{1,2} & \dots & b_{1,n-1} & b_{1,n} \\ 0 & 1 & \dots & b_{2,n-1} & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & b_{k,n-1} & b_{k,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b_{n-1,n} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

We need to prove that $(c_{i,j}) = A \cdot B \in U_n(\mathbb{Z})$

I. $1 \leq l \leq n, c_{l,l} = \sum_{k=1}^{n} a_{l,k} \cdot b_{k,l}$.

We observe that $a_{l,1} = a_{l,2} = \dots = a_{l,l-1} = 0$, and $b_{l+l,l} = b_{l+2,l} = \dots = b_{n,l} = 0$.

So, $c_{l,l} = \sum_{k=1}^{n} a_{l,k} \cdot b_{k,l} = 0 + 0 + \dots + 0 + a_{l,l} \cdot b_{l,l} + 0 + 0 \dots 0 = 1$.

This proves that each element <u>on</u> the main diagonal of $(c_{i,j})$ is 1.

II. $2 \leq l \leq n, 1 \leq m \leq l-1, c_{l,m} = \sum_{k=1}^{n} a_{l,k} \cdot b_{k,m}$.

We observe that $a_{l,1} = a_{l,2} = \dots = a_{l,l-1} = 0$, and $b_{m+1,m} = b_{m+2,m} = \dots = b_{n,m} = 0$.

This means, that $a_{l,k} \cdot b_{k,m} = 0, 1 \leq k \leq l-1$,

because the first $l-1$ elements of $a_{l,k}$ are 0.

and the last $n-m$ elements of $b_{k,m}$ are also 0.

This proves that each element <u>under</u> the main diagonal of $(c_{i,j})$ is 0.

III. $2 \leq l \leq n, 1 \leq m \leq l-1, c_{m,l} = \sum_{k=1}^{n} a_{m,k} \cdot b_{k,l}$.
$a_{m,k}, b_{k,l} \in \mathbb{Z} \Rightarrow \sum_{k=1}^{n} a_{l,k} \cdot b_{k,m} \in \mathbb{Z}$.
This proves the each element <u>above</u> the main diagonal of $(c_{i,j})$ is an integer.

Thus, we prove that $U_n(\mathbb{Z})$ is closed under matrix multiplication.

Associativity is obvious, from the fact that matrix multiplication is associative.

Obviously, I_n is a matrix of this form, so the unit of $U_n(\mathbb{Z})$ is I_n.

The fact that all matrices of this form have an inverse is obvious by looking at the rank of a matrix of this form, which, clearly, is n, since the matrix is already in a reduced form.

Conclusion: $U_n(\mathbb{Z})$ is a group.