Annotations
$\mathbb{F}$, an arbitrary field.
$F_{ij}(\alpha)$, where $\alpha \in \mathbb{F}$, and $i < j$, is a $n \times n$ matrix, satisfying,

$$F_{ij}(\alpha) = (a_{kl}) = \begin{cases} \alpha, & k = i \wedge l = j \\ 0, & \text{otherwise} \end{cases}$$

Define $F_{ij} := Fij(1)$
$E_{ij}(\alpha)$, where $\alpha \in \mathbb{F}$, and $i < j$, is a $n \times n$ matrix, satisfying,

$$E_{ij}(\alpha) = (a_{kl}) = \begin{cases} 1, & k = l \\ \alpha, & k = i \wedge l = j \\ 0, & \text{otherwise} \end{cases}$$

Define $E_{ij} := Eij(1)$

1 The group U_n

Proposition 1.1. *Let A be any $n \times n$ matrix. Then multiplying A from the left by any $E_{ij}(\alpha)$, of the same dimensions, yields a result matrix, $B = E_{ij}(\alpha)A$, whose rows are*

$$B_k = \begin{cases} A_i + \alpha A_j, & k = i \\ A_k, & \textit{otherwise} \end{cases}$$

In words, all the rows of B are the rows of A, except for row i of B, which is the addition of row i of A and the multiplication of row j of A by the scalar α.

Proof. Let the elements of A be (a_{kl}), and the elements of $E_{ij}(\alpha)$ be (e_{kl}). Set the result matrix $B = E_{ij}(\alpha)A$, and let its elements be (b_{kl}). For each cell $b_{kl} = \sum_{r=1}^{n} e_{kr}a_{rl}$. For $k = i$, the sum, for each column l, is $0 + \cdots + 0 + eiia_{il} + 0 + \cdots + 0 + e_{ij}a_{jl} + 0 \cdots + 0 = 1a_{il} + \alpha a_{jl} = a_{il} + \alpha a_{jl}$, which proves that $B_i = A_i + \alpha A_j$. The rest is obvious as well. $\square$

Corollary 1.2. *Let $E_{ij}(\alpha)$ be a matrix of the described form. Then $E_{ij}(\alpha)$ has an inverse matrix, and its inverse is $E_{ij}(\alpha)^{-1} = E_{ij}(-\alpha)$*

Proof. Easy to observe that $|E_{ij}(\alpha)| = 1 \neq 0$, so there exists a matrix $B = E_{ij}(\alpha)^{-1}$. Let the elements of $E_{ij}(\alpha)$ be (e_{kl}), and let the elements of B be (b_{kl}). From proposition 1.1, we know that multiplying B from the left by $E_{ij}(\alpha)$ yields a matrix C with all the rows identical to the rows of B, except for row i, which is the addition $B_i + \alpha B_j$. Let the elements of C be (c_{kl}). But $B = E_{ij}(\alpha)^{-1}$, which means $C = I_n$, so

$$c_{kl} = \begin{cases} 1, & k = l \\ 0, & \text{otherwise} \end{cases}$$

This yields the following equations,

$$\begin{cases} c_{ii} = 1 = b_{ii} + \alpha b_{ji} \\ c_{ij} = 0 = b_{ij} + \alpha b_{jj} \end{cases}$$

But, $C_j = B_j$, which means that $c_{jj} = 1 = b_{jj}$. Taking this to the second equation, gives $0 = b_{ij} + \alpha 1 \Rightarrow b_{ij} = \alpha$. Similarly, $c_{ji} = 0 = b_{ji}$. Taking this to the first equation, gives $1 = b_{ii} + 0 \Rightarrow b_{ii} = 1$. It is now clear that $B = E_{ij}(\alpha)^{-1}$ is of the form described above. $\square$

Corollary 1.3. *Let A be any $n \times n$ matrix. Then multiplying A from the left by any $E_{ij}(\alpha)^{-1}$, of the same dimensions, yields a result matrix, $B = E_{ij}(\alpha)^{-1}A$, whose rows are*

$$B_k = \begin{cases} A_i - \alpha A_j, & k = i \\ A_k, & \textit{otherwise} \end{cases}$$

Proof. From 1.2, we have that $E_{ij}(\alpha)^{-1}A = E_{ij}(-\alpha)A$, and from 1.1, the result above is immediate. $\square$

Proposition 1.4. *Let $E_{ij}(\alpha)$ as defined above. then, for any $m \in \mathbb{N}$, we have $E_{ij}(\alpha)^m = E_{ij}(m\alpha)$*

Proof. By induction on m. For $m = 1$, $E_{ij}(\alpha)^1 = E_{ij}(1\alpha)$. For $m > 1$, we have $E_{ij}(\alpha)^m = E_{ij}(\alpha)E_{ij}(\alpha)^{m-1}$. By the induction hypothesis, $E_{ij}(\alpha)^{m-1} = E_{ij}((m-1)\alpha)$. We denote $A = E_{ij}((m-1)\alpha)$. By proposition 1.1, $B = E_{ij}(\alpha)A$ is the matrix whose rows are

$$B_k = \begin{cases} A_i + \alpha A_j, & k = i \\ A_k, & \text{otherwise} \end{cases}$$

So $b_{ii} = a_{ii} + \alpha a_{ji} = 1 + \alpha 0 = 1 + 0 = 1$, and $b_{ij} = a_{ij} + \alpha a_{jj} = (m-1)\alpha + \alpha 1 = (m-1+1)\alpha = m\alpha$, which proves the induction step. $\square$

Corollary 1.5. *Let $E_{ij}(\alpha)$ be a matrix as defined above. Then, for any $m \in \mathbb{N}$, $E_{ij}(\alpha)^{-m} = (E_{ij}(\alpha)^m)^{-1} = (E_{ij}(\alpha)^{-1})^m = E_{ji}(-m\alpha)$*

Proof. By 1.2, $E_{ij}(\alpha)^{-1} = E_{ij}(-\alpha)$, and by 1.4, $(E_{ij}(\alpha)^{-1})^m = E_{ij}(-\alpha)^m = E_{ij}(m \cdot -\alpha) = E_{ij}(-m\alpha)$. We can commute the operations, to obtain exactly the same result. $\square$

Corollary 1.6. *Let $E_{ij}(\alpha), E_{ij}(\beta)$ be two matrices as defined above. Then, for any $m, s \in \mathbb{Z}$, $E_{ij}(\alpha)^m E_{ij}(\beta)^s = E_{ij}(m\alpha + s\beta)$*

Proof. Set $A = E_{ij}(\alpha)^m$, and let (a_{kl}) be its elements. If $m > 0$, then by 1.4, $A = E_{ij}(\alpha)^m = E_{ij}(m\alpha)$. If $m < 0$, then by 1.5, we get the same. If $m = 0$, then $A = E_{ij}(\alpha)^0 = I_n$, But the unit matrix has 0 everywhere, except for the main diagonal, which means that also $a_{ij} = 0$, which means that $A = E_{ij}(\alpha)^0 = I_n = E_{ij}(0) = E_{ij}(0\alpha)$, which proves that $E_{ij}(\alpha)^m = E_{ij}(m\alpha)$, for any $m \in \mathbb{Z}$. So, $E_{ij}(\alpha)^m E_{ij}(\beta)^s = E_{ij}(m\alpha)E_{ij}(s\beta)$, and the rest can be concluded from 1.1. $\square$

Corollary 1.7. *Let A be an upper triangular nxn matrix, over $\mathbb{Z}$, with 1 on the main diagonal.*

Proof. Set $A = E_{ij}(\alpha)^m$, and let (a_{kl}) be its elements. If $m > 0$, then by 1.4, $A = E_{ij}(\alpha)^m = E_{ij}(m\alpha)$. If $m < 0$, then by 1.5, we get the same. If $m = 0$, then $A = E_{ij}(\alpha)^0 = I_n$, But the unit matrix has 0 everywhere, except for the main diagonal, which means that also $a_{ij} = 0$, which means that $A = E_{ij}(\alpha)^0 = I_n = E_{ij}(0) = E_{ij}(0\alpha)$, which proves that $E_{ij}(\alpha)^m = E_{ij}(m\alpha)$, for any $m \in \mathbb{Z}$. So, $E_{ij}(\alpha)^m E_{ij}(\beta)^s = E_{ij}(m\alpha)E_{ij}(s\beta)$, and the rest can be concluded from 1.1. $\square$

Proposition 1.8. *Let $E_{ij}(\alpha), E_{st}(\beta)$ be two matrices defined as above, that is, $\alpha, \beta \in \mathbb{F}$, and $i < j$, and $s < t$. Then, $C = E_{ij}(\alpha)E_{st}(\beta)$ is an upper triangular matrix over $\mathbb{F}$, with 1 on the main diagonal.*

Proof. Let (a_{kl}) be the elements of $E_{ij}(\alpha)$, and let (b_{kl}) be the elements of $E_{st}(\beta)$ From 1.1, we have that $C_k = B_k$, for each row $k \neq i$, and $C_i = B_i + \alpha B_j$. But, for any index $r < i < j$, we have $b_{jr} = 0$, so, the sum

$c_{ir} = b_{ir} + \alpha b_{jr}$, which means that

$$c_{ir} = \begin{cases} 0, & r < i \\ 1, & r = i \\ b_{ir}, & i < r < j \\ b_{ij} + \alpha, & r = j \\ b_{ir} + b_{jr}, & j < r < n \end{cases}$$

, which proves that C is of the form described above. $\square$

Corollary 1.9. *The set of all upper triangular matrices, with* $1 on the main diagonal$

Proof. $\square$

Corollary 1.5
Let $E_{i,j} = (e_{l,k}), i < j \in E_n$, Then,
$\forall m \in \mathbb{N}, (E_{i,j}^{-1})^m = (a_{l,k})$, where $a_{l,l} = 1, 1 \leq l \leq n$, and $a_{i,j} = -m, i < j$, and all other elements are zero

Proof. By induction on m.
$(a_{l,k}) = E_{i,j}^{-1}$ For $m = 2$, we observe that $(E_{i,j}^{-1})^2 = E_{i,j}^{-1} \times E_{i,j}^{-1} = (a_{l,k}) \times (a_{l,k})$, means that $E_{i,j}^{-1}$ operates on itself as the row addition $R_i \leftarrow R_i - R_j$
So, the product matrix $(b_{l,k})$ has $b_{i,i} = a_{i,i} - a_{j,i} = 1 - 0 = 1$, and $b_{i,j} = a_{i,j} - a_{j,j} = -1 - 1 = -2$, and all other elements are zero.
Now, we prove for $m + 1$
$(a_{l,k}) = (E_{i,j}^{-1})^{m+1} = E_{i,j}^{-1} \times (E_{i,j}^{-1})^m$. But, from the induction assumption, $(b_{l,k}) = (E_{i,j}^{-1})^m$, has $b_{l,l} = 1, 1 \leq l \leq n$, and $b_{i,j} = -m, i < j$, and all other elements are zero.
So, $(a_{l,k}) = E_{i,j}^{-1} \times (b_{i,j})$ is the row addition $R_i \leftarrow R_i - R_j$ on $(b_{i,j})$, which means, $a_{i,i} = b_{i,i} - b_{i,j} = 1 = 0 = 1$, and $a_{i,j} = b_{i,j} - b_{j,j} = -m - 1 = -(m+1)$, and all the other elements are zero, thus, we prove the induction step.
$\square$

Corollary 1.6 Let $E_{i,j} = (e_{l,k}), i < j \in E_n$, Then,
$\forall m, r \in \mathbb{Z}, (a_{l,k}) = E_{i,j}^{m+r} = E_{i,j}^{r+m}$ is the matrix where $a_{l,l} = 1, 1 \leq l \leq n$, and $a_{i,j} = m + r = r + m, i < j$, and all other elements are zero.
This shows that multiplying integer powers of matrices, from the set E_n (which means, adding their exponents), is equivalent to adding integer numbers, which means that we have a canonical bijection, $(\mathbb{Z}, +) \leftrightarrow (E_{i,j}^{\mathbb{Z}}, \cdot)$, for any two fixed indices $i < j$, where $1 \leftrightarrow E_{i,j}^1 = E_{i,j}$, and $-1 \leftrightarrow E_{i,j}^{-1}$

Proposition 1.7

Let $(a_{l,k}) = E_{i,j}, t \neq i < j, (b_{l,k}) = E_{s,t}, j \neq s < t \in E_n$, Then, $(c_{l,k}) = E_{i,j} \cdot E_{s,t} = E_{s,t} \cdot E_{i,j}$ is a matrix with $c_{l,l} = 1, 1 \leq l \leq n$, and $c_{i,j} = 1$, and $c_{s,t} = 1$, and, all other elements are zero.

Proof. As seen above, $E_{s,t}$ is operating from the left on $E_{i,j}$ as the addition $R_i, j \leftarrow R_i + R_j$, so, $(c_{l,k})$ is $E_{s,t}$, with row j being added to row i.
So, $c_{i,k} = b_{i,k} + bj, k, 1 \leq k \leq n$. But, since $s \neq j$ the only element in row j of $(b_{l,k})$ which is not zero is $b_{j,j} = 1$, and $b_{i,j} = 0$, so $c_{i,j} = b_{i,j} + b_{j,j} = 0 + 1 = 1$
Also, $b_{j,i} = 0$ (it is below the main diagonal), so, $c_{i,i} = b_{i,i} + b_{j,i} = 1 + 0 = 1$.
It is easy to verify that all the other elements in row i of $(c_{j,k})$ are zero, and that all the other rows of $(c_{l,k})$ remain the same as they are in $(b_{l,k})$
Also, it is easy to verify that, under the condition that $t \neq i$, the multiplication is commuting, and yields the same product matrix. □

Proposition 1.8

Let $(a_{l,k}) = E_{i,j}, i < j, (b_{l,k}) = E_{j,r}, j < r \in E_n$, Then,
1.8.1 $(c_{l,k}) = E_{i,j} \cdot E_{j,r}$ is a matrix with $c_{l,l} = 1, 1 \leq l \leq n$, and $c_{i,j} = 1$, and $c_{j,r} = 1$, and $c_{i,r} = 1$, and, all other elements are zero.

Proof. The multiplication from the left of $E_{j,r}$ by $E_{j,r}$ is the addition on row j to row i of the matrix $E_{j,}$, which gives $c_{i,k} = b_{i,k} + b_{j,k}, 1 \leq k \leq n$, so $c_{i,i} = b_{i,i} + b_{j,i} = 1 + 0 = 1$, and $c_{i,j} = b_{i,j} + b_{j,j} = 0 + 1 = 1$, and $c_{i,r} = b_{i,r} + b_{j,r} = 0 + 1 = 1$, and, it is easy to verify that all other $c_{i,k}$ are zero.

□

On the other hand,
1.8.2 $(d_{l,k}) = E_{j,r} \cdot E_{i,j}$ is a matrix with $d_{l,l} = 1, 1 \leq l \leq n$, and $d_{i,j} = 1$, and $c_{j,r} = 1$, and, all other elements are zero.

Proof. The multiplication from the left of $E_{i,j}$ by $E_{s,t}$ is the addition on row j to row i of the matrix $E_{s,t}$, which gives $c_{i,k} = b_{i,k} + b_{j,k}, 1 \leq k \leq n$, so $c_{i,i} = b_{i,i} + b_{j,i} = 1 + 0 = 1$, and $c_{i,j} = b_{i,j} + b_{j,j} = 0 + 1 = 1$, and $c_{i,r} = b_{i,r} + b_{j,r} = 0 + 1 = 1$, and, it is easy to verify that all other $c_{i,k}$ are zero. □

1.8.3 Let $(c_{l,k}) = E_{i,j}^{-1}, (d_{l,k}) = E_{j,r}^{-1}$
$(f_{l,k}) = E_{i,j}^{-1} \cdot E_{j,r}^{-1}$ is a matrix with $f_{l,l} = 1, 1 \leq l \leq n$, and $f_{i,j} = -1$, and $f_{j,r} = -1$, and $f_{i,r} = 1$ and, all other elements are zero.

Proof. The multiplication from the left of $E_{j,r}^{-1}$ by $E_{i,j}^{-1}$ is the subtraction of row j from row i of the matrix $E_{j,r}^{-1}$, which gives $f_{i,k} = d_{i,k} - d_{j,k}, 1 \leq k \leq n$, so $f_{i,i} = d_{i,i} - d_{j,i} = 1 - 0 = 1$, and $f_{i,j} = d_{i,j} - d_{j,j} = 0 - 1 = -1$, and $f_{i,r} = d_{i,r} - d_{j,r} = 0 - (-1) = 0 + 1 = 1$, and, it is easy to verify that all other $f_{i,k}$ are zero. $\square$

1.8.4 Let $(c_{l,k}) = E_{i,j}^{-1}, (d_{l,k}) = E_{j,r}^{-1}$
$(g_{l,k}) = E_{j,r}^{-1} \cdot E_{i,j}^{-1}$ is a matrix with $f_{l,l} = 1, 1 \leq l \leq n$, and $f_{i,j} = -1$, and $f_{j,r} = -1$, and, all other elements are zero.

Proof. The multiplication from the left of $E_{i,j}^{-1}$ by $E_{j,r}^{-1}$ is the subtraction of row r from row j of the matrix $E_{i,j}^{-1}$, which gives $g_{i,k} = c_{i,k} - c_{j,k}, 1 \leq k \leq n$, so $g_{j,j} = c_{j,j} - c_{r,j} = 1 - 0 = 1$, and $g_{i,j} = c_{i,j} - c_{r,j} = -1 - 0 = -1$, and $g_{j,r} = c_{j,r} - c_{r,r} = 0 - 1 = -1$, and, it is easy to verify that all other $g_{j,k}, g_{i,k}$ are zero. $\square$

Corollary 1.9 Let $(a_{l,k}) = E_{i,j}, (b_{l,k}) = E_{j,r}$, Then
1.9.1 $(c_{l,k}) = [E_{i,j}, E_{j,r}] = E_{i,j} \cdot E_{j,r} \cdot E_{i,j}^{-1} \cdot E_{j,r}^{-1} = E_{i,r}$

Proof. By associativity, $E_{i,j} \cdot E_{j,r} \cdot E_{i,j}^{-1} \cdot E_{j,r}^{-1} = (E_{i,j} \cdot E_{j,r}) \cdot (E_{i,j}^{-1} \cdot E_{j,r}^{-1})$, and we have already calculated these matrix products.
$(f_{l,k}) = E_{i,j} \cdot E_{j,r} = I + F_{i,j} + F_{i,r} + F_{j,r}$
$(g_{l,k}) = E_{i,j}^{-}1 \cdot E_{j,r}^{-}1 = I - F_{i,j} + Fi, r - F_{j,r}$
So, in the product matrix, $(c_{l,k})$, $c_{i,j} = \sum_{k=1}^{n} f_{i,k} \cdot g_{k,j} = 0 \cdot 0 + \cdots + 0 \cdot 0 + f_{i,i} \cdot g_{i,j} + f_{i,j} \cdot g_{j,j} + 0 \cdot 0 \ldots 0 \cdot 0 = 1 \cdot -1 + 1 \cdot 1 = -1 + 1 = 0$
So, $c_{i,j}$ is canceled by multiplication. Easy to verify that the same goes also for $c_{j,r}$, but $c_{i,r} = \sum_{k=1}^{n} f_{i,k} \cdot g_{k,r} = 0 \cdot 0 + \cdots + 0 \cdot 0 + f_{i,i} \cdot g_{i,r} + f_{i,j} \cdot g_{j,r} + f_{i,r} \cdot g_{r,r} + 0 \cdot 0 \ldots 0 \cdot 0 = 1 \cdot 1 + 1 \cdot -1 + 1 \cdot 1 = 1 + (-1) + 1 = 1 - 1 + 1 = 1$
Which means that $[E_{i,j}, E_{j,r}] = E_{i,j} \cdot E_{j,r} \cdot E_{i,j}^{-1} \cdot E_{j,r}^{-1} = I + F_{i,r} = E_{i,r}$ $\square$

1.9.2 $(d_{l,k}) = [E_{j,r}, E_{i,j}] = E_{j,r} \cdot E_{i,j} \cdot E_{j,r}^{-1} \cdot E_{i,j}^{-1} = E_{i,r}$

Proof. By associativity, $E_{j,r} \cdot E_{i,j} \cdot E_{j,r}^{-1} \cdot E_{i,j}^{-1} = (E_{j,r} \cdot E_{i,j}) \cdot (E_{j,r}^{-1} \cdot E_{i,j}^{-1})$, and we have already calculated these matrix products.
$(f_{l,k}) = E_{j,r} \cdot E_{i,j} = I + F_{i,j} + F_{j,r}$
$(g_{l,k}) = E_{j,r}^{-}1 \cdot E_{i,j}^{-}1 = I - F_{i,j} - F_{j,r}$
So, in the product matrix, $(d_{l,k})$, $d_{i,j} = \sum_{k=1}^{n} f_{i,k} \cdot g_{k,j} = 0 \cdot 0 + \cdots + 0 \cdot 0 + f_{i,i} \cdot g_{i,j} + f_{i,j} \cdot g_{j,j} + 0 \cdot 0 \ldots 0 \cdot 0 = 1 \cdot -1 + 1 \cdot 1 = -1 + 1 = 0$
So, $d_{i,j}$ is canceled by multiplication. Easy to verify that the same goes also

for $d_{j,r}$, but $d_{i,r} = \sum_{k=1}^{n} f_{i,k} \cdot g_{k,r} = 0 \cdot 0 + \cdots + 0 \cdot 0 + f_{i,i} \cdot g_{i,r} + f_{i,j} \cdot g_{j,r} + f_{i,r} \cdot g_{r,r} + 0 \cdot 0 \ldots 0 \cdot 0 = 0 \cdot 0 + \cdots + 0 \cdot 0 + f_{i,i} \cdot 0 + f_{i,j} \cdot g_{j,r} + 0 \cdot g_{r,r} + 0 \cdot 0 \ldots 0 \cdot 0 = 1 \cdot 0 + 1 \cdot -1 + 0 \cdot 1 = 0 + (-1) + 0 = 0 - 1 + 0 = -1$
Which means that $[E_{j,r}, E_{i,j}] = E_{j,r} \cdot E_{i,j} \cdot E_{j,r}^{-1} \cdot E_{i,j}^{-1} = I - F_{i,r} = E_{i,r}^{-1}$ $\square$