Your Paper

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Denote $G_5 := G_5(\mathbb{Z}_p)$, and $G_5^+ := G_5^+(\mathbb{Q}_p)$.

$\zeta^{\wedge}_{L_{5,p}}(s) = \int_{G_5^+} |\det g|_p^s d\mu(G_5) = \int_{G_5^+} |\det uh|_p^s d\mu(G_5)$, where $h \in H$ and $u \in N_h$.

Each u is unipotent, hence $\zeta^{\wedge}_{L_{5,p}}(s) = \int_{G_5^+} |\det h|_p^s d\mu(G_5) = \int_{G_5^+} |\lambda_1^4\lambda_2^6\lambda_3^6\lambda_4^4|_p^s d\mu(G_5) =$

$= \int_{G_5^+} \left[|\lambda_1^4|_p|\lambda_2^6|_p|\lambda_3^6|_p|\lambda_4^4|_p\right]^s d\mu(G_5)$, by the inductive formula we have found for every $|h|$.

We denote $v_i := v_p(\lambda_i)$,

and so $\zeta^{\wedge}_{L_{5,p}}(s) = \int_{G_5^+} \left[p^{-4v_1}p^{-6v_2}p^{-6v_3}p^{-4v_4}\right]^s d\mu(G_5) = \int_{G_5^+} p^{-(4v_1+6v_2+6v_3+4v_4)s} d\mu(G_5)$.

We denote $I(\underline{\lambda}) := p^{-(4v_1+6v_2+6v_3+4v_4)s}$. Now we use the natural matrix decomposition of the N_h matrix of Berman's, which means that

$\zeta^{\wedge}_{L_{5,p}}(s) = \int_{G_5^+} I(\underline{\lambda}) d\mu(G_5) = \int_{\underline{\lambda}}\int_{\underline{a}}\int_{\underline{b}}\int_{\underline{c}} I(\underline{\lambda}) d\mu(\underline{c})d\mu(\underline{b})d\mu(\underline{a})d\mu(\underline{\lambda})$. Since $I(\underline{\lambda})$ depends only on $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, which appear only in the computation of the outermost integral, we consider them as constants for all the inner integrals, which means that we have $\zeta^{\wedge}_{L_{5,p}}(s) = \int_{\underline{\lambda}} I(\underline{\lambda}) \int_{\underline{a}}\int_{\underline{b}}\int_{\underline{c}} 1 d\mu(\underline{c})d\mu(\underline{b})d\mu(\underline{a})d\mu(\underline{\lambda})$, hence all the inner integrals evaluate to the measure of their domains of integration. now we compute the innermost integral by considering $\underline{a}$, $\underline{b}$ and $\underline{\lambda}$ as constants, and integrating only over $\underline{c}$. Considering the multiplication uh, we observe that for each element c_j, we must have that $\rho_j = c_j\lambda_1\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p$, which means that $v(\rho_i) = v(c_i\lambda_1\lambda_2\lambda_3\lambda_4) \geq 0 \Rightarrow v(c_i) + v_1 + v_2 + v_3 + v_4 \geq 0 \Rightarrow v(c_i) \geq -(v_1 + v_2 + v_3 + v_4)$. But this means that $c_i \in p^{-(v_1+v_2+v_3+v_4)}\mathbb{Z}_p$, and since the domain of integration for this integral is $\underline{c} = \{c_1, c_2, c_3, c_4\}$, then $\mu(\underline{c}) = |c_j|_p^4 = p^{4(v_1+v_2+v_3+v_4)}$. Denote $I(\underline{\lambda}, \underline{c}) := I(\underline{\lambda})p^{4(v_1+v_2+v_3+v_4)}$, we now have that $\zeta^{\wedge}_{L_{5,p}}(s) = \int_{\underline{\lambda}} I(\underline{\lambda}, \underline{c}) \int_{\underline{a}}\int_{\underline{b}} 1 d\mu(\underline{b})d\mu(\underline{a})d\mu(\underline{\lambda})$.

Denote $\lambda_{13} := \lambda_1\lambda_2\lambda_3$, $\lambda_{24} := \lambda_2\lambda_3\lambda_4$, and $\lambda_{14} := \lambda_1\lambda_2\lambda_3\lambda_4$. We now consider the constraints on $\underline{b}$.

$b_{11}\lambda_{13}, b_{31}\lambda_{13}, b_{41}\lambda_{13} \in \mathbb{Z}_p$, and $b_{12}\lambda_{24}, b_{22}\lambda_{24} \in \mathbb{Z}_p$. These constaints are obtained by multiplying elements in block M_{13} with elements in h, but one observes that we have b_{22} also in location $(5,10)$ of the matrix, and b_{31} in location $(7,10)$, which means that $b_{22}\lambda_{14}, b_{31}\lambda_{14} \in \mathbb{Z}_p$. But since we already have $b_{22}\lambda_{24}, b_{31}\lambda_{13} \in \mathbb{Z}_p$, the constraints $b_{22}\lambda_{14}$ and $b_{31}\lambda_{14}$ do not contribute any new information. In addition, we have one of the elements of $\underline{b}$ that forms a constraint together with elements from $\underline{a}$, namely $(a_{11}a_{22} - b_{11})\lambda_{24} \in \mathbb{Z}_p$. The constraints $b_{31}\lambda_{13}, b_{41}\lambda_{13}, b_{12}\lambda_{24}, b_{22}\lambda_{24} \in \mathbb{Z}_p$ from above translate to $p^{-2(v_1+v_2+v_3)}p^{-2(v_2+v_3+v_4)} = p^{-2(v_1+2v_2+2v_3+v_4)}$. On the other hand, b_{11} is a part of two constraints, hence we must have both $b_{11} \in p^{-(v_1+v_2+v_3)}\mathbb{Z}_p$ and $a_{11}a_{22} - b_{11} \in p^{-(v_2+v_3+v_4)}\mathbb{Z}_p \Rightarrow b_{11} \in a_{11}a_{22} + p^{-(v_2+v_3+v_4)}\mathbb{Z}_p$, which means that we need to compute the measure $\mu(A)$, where $A = p^{-(v_1+v_2+v_3)}\mathbb{Z}_p \cap a_{11}a_{22} + p^{-(v_2+v_3+v_4)}\mathbb{Z}_p$. Denote $\alpha := v_1 + v_2 + v_3$, $\beta := v_2 + v_3 + v_4$ and $x := a_{11}a_{22}$, and we need to find a formula for a generic intersection of the form $A = p^{-\alpha}\mathbb{Z}_p \cap x + p^{-\beta}\mathbb{Z}_p$. We need to find a formula for this generic form. Since b_{11} is in the intersection, we have that $b_{11} = z = x + y$ where $y \in p^{-\beta}$ and $z \in p^{-\alpha}\mathbb{Z}_p \Rightarrow z - x \in p^{-\beta}\mathbb{Z}_p$. Assume $\beta \geq \alpha \Rightarrow -\beta \leq -\alpha$, and since $v_p(b_{11}) = v_p(z-x) \geq \min\{v_p(z), v_p(x)\}$, and $v_p(z) \geq -\alpha \geq -\beta$, then we have two cases. If $v_p(x) \geq -\beta$, then $v_p(z-x) \geq \beta \Rightarrow z - x \in p^{-\beta}\mathbb{Z}_p$. But $-\alpha \geq -\beta \Rightarrow p^{-\alpha}\mathbb{Z}_p \subseteq p^{-\beta}\mathbb{Z}_p \Rightarrow A = p^{-\alpha}\mathbb{Z}_p$. If $v_p(x) < -\beta$, then $v_p(z-x) = v_p(x) < -\beta \Rightarrow z - x \notin p^{-\beta}\mathbb{Z}_p$, which means that $A = \varnothing$. One checks that if we assume $\alpha \geq \beta$, then we obtain that $A = p^{-\beta}\mathbb{Z}_p$ if $v_p(x) \geq -\alpha$, and $A = \varnothing$ if $v_p(x) < -\alpha$. Therefore, $\mu(A) = p^{\min\{\alpha,\beta\}}$ for every x such that $v_p(x) \geq \min\{-\alpha, -\beta\} = -\max\{\alpha, \beta\}$, which means, in our case, that $v_p(x) = v_p(a_{11}a_{22}) \geq -\max\{v_1+v_2+v_3, v_2+v_3+v_4\} = -v_2 - v_3 - \max\{v_1, v_4\}$. Thus, denoting $I(\underline{\lambda}, \underline{c}, \underline{b}) := I(\underline{\lambda}, \underline{c})p^{-(v_2+v_3)-\max\{v_1,v_4\}}$, we have that $\zeta^{\wedge}_{L_{5,p}}(s) = \int_{\underline{\lambda}} I(\underline{\lambda}, \underline{c}, \underline{b}) \int_{\underline{a}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda})$. Denote $v_{ij} := v_p(a_{ij})$. For the constraints on $\underline{a}$, we have

$a_{11}\lambda_1\lambda_2, -a_{11}\lambda_2\lambda_3, -a_{11}\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_{11} \geq -(v_1+v_2), v_{11} \geq -(v_2+v_3) \Rightarrow v_{11} \geq -v_2 - \min\{v_1, v_3\}$.

$a_{21}\lambda_1\lambda_2, a_{21}\lambda_1\lambda_2\lambda_3, a_{21}\lambda_1\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_{21} \geq -(v_1+v_2)$.

$a_{22}\lambda_2\lambda_3, -a_{22}\lambda_3\lambda_4, a_{22}\lambda_1\lambda_2\lambda_3 \in \mathbb{Z}_p \Rightarrow$

$\Rightarrow v_{22} \geq -(v_2+v_3), v_{22} \geq -(v_3+v_4), v_{22} \geq -(v_1+v_2+v_3) \Rightarrow v_{22} \geq -v_3 - \min\{v_2, v_4\}$.

$a_{33}\lambda_3\lambda_4, a_{33}\lambda_2\lambda_3\lambda_4, a_{33}\lambda_1\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_{33} \geq -(v_3+v_4)$.

$a_{21}a_{22}\lambda_1\lambda_2\lambda_3 \in \mathbb{Z}_p \Rightarrow v_{21} + v_{22} \geq -(v_1+v_2+v_3)$.

$-a_{11}a_{33}\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_{11} + v_{33} \geq -(v_2+v_3+v_4)$.

$a_{21}a_{33}\lambda_1\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_{21} + v_{33} \geq -(v_1+v_2+v_3+v_4)$.

And we also have the constraint found earlier,

$v_{11} + v_{22} \geq -(v_2 + v_3 + \max\{v_1, v_4\})$.

We have three constraints on a_{21}

1. $v_{21} \geq -(v_1+v_2)$
2. $v_{21} \geq -(v_1+v_2+v_3+v_{22})$
3. $v_{21} \geq -(v_1+v_2+v_3+v_4+v_{33})$

But the third constraint does not add new information, because we already have the two separate constraints $v_{21}, v_{33} \geq -(v_1+v_2+v_3+v_4)$.

The two valid constraints translate to

$v_{21} \geq \min\{-(v_1+v_2), -(v_1+v_2+v_3+v_{22})\} = -(v_1+v_2) - \{0, v_3+v_{22}\}$.

In the same way, we obtain the constraint $v_{33} \geq -(v_3+v_4) - \min\{0, v_2+v_{11}\}$.

Thus, we decompose the inner integral for $\underline{a}$ into separate integrals, to obtain

$\zeta^{\wedge}_{L_{5,p}}(s) = \displaystyle\int_{\underline{\lambda}} I(\underline{\lambda}, \underline{c}, \underline{b}) \int_{\underline{a}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda}) =$

$\displaystyle\int_{\underline{\lambda}} I(\underline{\lambda}, \underline{c}, \underline{b}) \int_{a_{11}} \int_{a_{22}} \int_{a_{33}} \int_{a_{21}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda})$.

Hence, we have the measures $\mu(a_{21}) = p^{v_1+v_2+\min\{0,v_3+v_{22}\}}$ and $\mu(a_{33}) = p^{v_3+v_4+\min\{0,v_2+v_{11}\}}$. Denote $I(\underline{\lambda}, \underline{c}, \underline{b}, a_{21}, a_{33}) := I(\underline{\lambda}, \underline{c}, \underline{b}) p^{v_1+v_2} p^{v_3+v_4}$. We have $\zeta^{\wedge}_{L_{5,p}}(s) = \displaystyle\int_{\underline{\lambda}} I(\underline{\lambda}, \underline{c}, \underline{b}) \int_{\underline{a}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda}) =$

$= \displaystyle\int_{\underline{\lambda}} I(\underline{\lambda}, \underline{c}, \underline{b}) \int_{a_{11}} \int_{a_{22}} \int_{a_{33}} \int_{a_{21}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda}) =$

$= \displaystyle\int_{\underline{\lambda}} I(\underline{\lambda}, \underline{c}, \underline{b}, a_{21}, a_{33}) \int_{a_{11}} p^{\min\{0,v_2+v_{11}\}} \int_{a_{22}} p^{\min\{0,v_3+v_{22}\}} d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda})$.

By the constraints we found earlier on a_{22}, we have the following.

1. $v_{22} \geq -v_3 - \min\{v_2, v_4\}$
2. $v_{22} \geq -(v_2+v_3) - \max\{v_1, v_4\} - v_{11}$

which translates into $v_{22} \geq -v_3 - \min\{\min\{v_2, v_4\}, v_2 + \max\{v_1, v_4\} + v_{11}\} =$

$= -v_3 - \min\{v_2, v_4, v_2 + \max\{v_1, v_4\} + v_{11}\}$.

Denote $\alpha := v_2 + \max\{v_1, v_4\} + v_{11}$ and $\beta := \min\{v_2, v_4, \alpha\}$. We already have the constraint $v_{11} \geq -(v_2 + \min\{v_1, v_3\})$, which means that, in either case, $v_{11} \geq -(v_1+v_2) \geq -\min\{v_1, v_4\} - v_2$

$\Rightarrow \alpha = v_2 + \max\{v_1, v_4\} + v_{11} \geq \max\{v_1, v_4\} - \min\{v_1, v_4\} \geq 0$

$\Rightarrow \beta = \min\{v_2, v_4, \alpha\} \geq 0 \Rightarrow v_3 + \alpha > 0 \Rightarrow v_{22} \geq -(v_3+\beta)$.

For the inner integral $\displaystyle\int_{a_{22}} p^{\min\{0,v_3+v_{22}\}} d\mu(a_{22})$, we have two cases. If $v_3 + v_{22} \geq 0$, then $\min\{0, v_3+v_{22}\} = 0 \Rightarrow \displaystyle\int_{a_{22}} p^{\min\{0,v_3+v_{22}\}} d\mu(a_{22}) = \int_{v_{22}\geq -v_3} 1 d\mu(a_{22}) = p^{v_3}$.

If $v_3 + v_{22} < 0$, then $\displaystyle\int_{a_{22}} p^{\min\{0,v_3+v_{22}\}} d\mu(a_{22}) = \int_{v_{22}<-v_3} p^{v_3+v_{22}} d\mu(a_{22})$. But we saw earlier that $v_{22} \geq -(v_3+\beta)$, hence $-v_3 - \beta \leq v_{22} \leq -v_3 - 1 \Rightarrow -\beta \leq v_3 + v_{22} \leq -1$, which means that we can compute the integral over a_{22} as a sum of β integrals, $\displaystyle\int_{-v_3-\beta\leq v_{22}\leq -v_3-1} p^{v_3+v_{22}} d\mu(a_{22}) = \sum_{\tau=1}^{\beta} \int_{v_{22}=-v_3-\tau} p^{-\tau} d\mu(a_{22})$.

To evaluate each integral in the sum, we need to calculate the measure of its domain, namely $\mu(\{v_{22} = -(v_3+\tau)\}) = \mu(\{v_{22} \geq -(v_3+\tau+1)\} \setminus \{v_{22} \geq -(v_3+\tau)\}) = \mu(p^{-(v_3+\tau+1)}\mathbb{Z}_p \setminus p^{-(v_3+\tau)}\mathbb{Z}_p) = p^{v_3+\tau+1} - p^{v_3+\tau} = p^{v_3+\tau}(p-1)$, which means that each integral evaluates as $p^{v_3+\tau}(p-1)p^{\tau} = p^{v_3}(p-1)$, and the sum is over

β such integrals, so we have that $\displaystyle\int_{a_{22}} p^{\min\{0,v_3+v_{22}\}} d\mu(a_{22}) = p^{v_3} + \beta p^{v_3}(p-1)$, where β depends also on v_{11}.

Hence, we need to compute the integral $\displaystyle\int_{a_{11}} p^{\min\{0,v_2+v_{11}\}} p^{v_3}[1+\beta(p-1)]d\mu(a_{11})$. We denote $I(\underline{\lambda}, \underline{c}, \underline{b}, a_{21}, a_{33}, a_{22}) := I(\underline{\lambda}, \underline{c}, \underline{b}, a_{21}, a_{33}, a_{22})p^{v_3}$, so $\zeta^{\wedge}_{L_{5,p}}(s) =$

$= \displaystyle\int_{\underline{\lambda}} I(\underline{\lambda}, \underline{c}, \underline{b}, a_{21}, a_{33}, a_{22}) \int_{a_{11}} p^{\min\{0,v_2+v_{11}\}}[1+\beta(p-1)]d\mu(a_{11})$. But similar to what we saw earlier, $p^{\min\{0,v_2+v_{11}\}}$ has two cases. If $v_{11} \geq -v_2$, then $p^{\min\{0,v_2+v_{11}\}} = p^0$. If $v_{11} < -v_2$, then $p^{\min\{0,v_2+v_{11}\}} = p^{v_2+v_{11}}$. We saw earlier that $v_{11} \geq -(v_2+v_3) \Rightarrow v_{11}+v_2 \geq -v_3$, so for this case, we have that $-v_3 \leq v_{11}+v_2 \leq 0$, which means that $\displaystyle\int_{a_{11}} p^{\min\{0,v_2+v_{11}\}}[1+\beta(p-1)]d\mu(a_{11}) = \int_{v_{11}\leq -v_2} 1+\beta(p-1)d\mu(a_{11}) + \int_{v_{11}+v_2\geq -v_3} p^{v_2+v_{11}}[1+\beta(p-1)]d\mu(a_{11}) = \int_{v_{11}\leq -v_2} 1+ \beta(p-1)d\mu(a_{11}) + \sum_{\tau=1}^{v_3}\int_{v_{11}\geq -(v_3+v_2)} p^{-\tau}[1+\beta(p-1)]d\mu(a_{11})$. Now we need to resolve $\beta = \min\{v_2, v_4, v_2+\max\{v_1,v_4\}+v_{11}\}$, hence we need to divide the inner integral to different orderings of v_1, v_2, v_3, v_4.

Case 1: $v_1 \geq v_2 \geq v_3 \geq v_4$. For this case, we have that $\beta = \min\{v_2, v_4, v_2 + \max\{v_1,v_4\}+v_{11}\} = \min\{v_4, v_1+v_2+v_{11}\}$.

The two possible minimum values are equal when $v_4 = v_1+v_2+v_{11}$, that is, when $v_2+v_{11} = -(v_1-v_4)$. But for this case we have two subcases. If $v_1 - v_4 \leq v_3$, then, since $v_{11} \geq -(v_2+v_3)$, we have that $v_{11}+v_2 \geq -v_3 \geq -(v_1-v_4) \Rightarrow v_1+v_2+v_{11} \geq v_4 \Rightarrow \beta = v_4$, hence

$\displaystyle\int_{\underline{\lambda}} I(\underline{\lambda}, \underline{b}, a_{33}, a_{33}, a_{33}) \int_{a_{11}} p^{\min\{0,v_2+v_{11}\}}(1+v_4(1-p^{-1}))d\mu(a_{11}) =$

$= \displaystyle\int_{\underline{\lambda}} I(\underline{\lambda}, \underline{b}, a_{33}, a_{33}, a_{33})(1+v_4(1-p^{-1})) \int_{v_{11}\geq -(v_2+v_3)} p^{\min\{0,v_2+v_{11}\}}d\mu(a_{11})$,

but same as earlier $\displaystyle\int_{v_{11}\geq -(v_2+v_3)} p^{\min\{0,v_2+v_{11}\}}d\mu(a_{11}) = p^0\mu(\{v_{11}+v_2 \geq 0\}) + \int_{v_2+v_{11}<0} p^{v_2+v_{11}}d\mu(a_{11}) = 1\mu(\{v_{11} \geq -v_2\}) + \int_{v_2+v_{11}<0} p^{v_2+v_{11}}d\mu(a_{11}) = p^{v_2} + \int_{-v_3\leq v_{11}<-v_2} p^{v_2+v_{11}}d\mu(a_{11}) = p^{v_2} + \sum_{\tau=-v_3}^{-(v_2+1)}\int_{v_{11}=\tau} p^{v_2+\tau}d\mu(a_{11}) =$

$= p^{v_2} + \sum_{\tau=1}^{v_3}\displaystyle\int_{v_{11}=-(v_2+\tau)} p^{\tau}d\mu(a_{11}) = p^{v_2} + \sum_{\tau=1}^{v_3} p^{-\tau}(p^{v_2+\tau} - p^{v_2+\tau-1}) = p^{v_2} + \sum_{\tau=1}^{v_3} p^{v_2}(1-p^{-1}) = p^2+v_3 = p^{v_2}(1+v_3(1-p^{-1}))$