1 The computation of $G_n(\mathbb{Q}_p)$

1.1 The computation of the first block M_{11}

Proposition 1.1.1. *Let $x = \sum_{i=1}^{n-1} \lambda_i e_{i,i+1}$, where $\lambda_i \in \mathbb{Q}_p$ are not all zero. Then $\dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}/_{\gamma_2} = \mathfrak{l}(x) + \mathfrak{m}(x)$, where $\mathfrak{l}(x)$ is the number of sequences of consecutive non-zero coefficients of the form $\lambda_j, \lambda_{j+1}, \ldots, \lambda_{j+k-1}, \lambda_{j+k}$ and $\lambda_{j-1} = \lambda_{j+k+1} = 0$ (that is, the sequences are separated by one of more zero coefficients)*[1]*, and $\mathfrak{m}(x)$ is the number of zero coefficients $\lambda_j = 0$, such that also $\lambda_{j-1} = \lambda_{j+1} = 0$.*

Proof. Let $y = \sum_{i=1}^{n-1} \mu_i e_{i,i+1}$, where $\lambda_i \in \mathbb{Q}_p$, be an element in the quotient ${}^{\gamma_1}/_{\gamma_2}$. For every $1 \leq i \leq n-1$, denote by $(\mathfrak{C}_i)$ the constraint equation $[\lambda_i e_{i,i+1}, \mu_{i+1} e_{i+1,i+2}] - [\lambda_{i+1} e_{i+1,i+2}, \mu_i e_{i,i+1}] = (\lambda_i \mu_{i+1} - \lambda_{i+1}\mu_i) e_{i,i+2} = 0$, and it is clear that $y \in {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}/_{\gamma_2}$ if and only if all the $(\mathfrak{C}_i)$ constraints are satisfied. We observe that each μ_i participates in two constraints, $(\mathfrak{C}_{i-1})$ and $(\mathfrak{C}_i)$, that is, $\lambda_{i-1}\mu_i - \lambda_i \mu_{i-1} = \lambda_i \mu_{i+1} - \lambda_{i+1}\mu_i = 0$. If $\lambda_i = 0$, then $\lambda_i \mu_{i-1} = \lambda_i \mu_{i+1} = 0$, hence by constraint $(\mathfrak{C}_{i-1})$ we have that $\lambda_{i-1}\mu_i = 0$, and by constraint $(\mathfrak{C}_i)$ we have that $\lambda_{i+1}\mu_i = 0$. Hence, if either λ_{i-1} or λ_{i+1} are non-zero, then $\mu_i = 0$. But if $\lambda_{i-1} = \lambda_{i+1} = 0$, then both constraints are satisfied for any choice of μ_i, which increases $\dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}/_{\gamma_2}$ by 1. We need to prove that for any sequence of k consecutive zero coefficients of x, where $k \geq 3$, that is[2] $\lambda_{j+1} = \lambda_{j+2} = \cdots = \lambda_{j+k} = 0$, for $1 \leq j \leq n-2$, we have that the sequence $\mu_{j+2}, \mu_{j+3}, \ldots, \mu_{j+k-1}$ of $k-2$ consecutive coefficients of y is made of scalars of any choice, thus $\dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}/_{\gamma_2}$ is increased by $k-2$. We prove that by simple induction on k. For $k = 3$, we just proved that if $\lambda_{i-1} = \lambda_i = \lambda_{i+1} = 0$, then μ_i can be any scalar. For $k+1$, we look at the sequence of $k+1$ zero coefficients, $\lambda_{j+1} = \lambda_{j+2} = \cdots = \lambda_{j+k} = \lambda_{j+k+1} = 0$. By constraints $(\mathfrak{C}_{j+k-1})$ and $(\mathfrak{C}_{j+k})$, we have that $\lambda_{j+k-1}\mu_{j+k} - \lambda_{j+k}\mu_{j+k-1} = \lambda_{j+k}\mu_{j+k+1} - \lambda_{j+k+1}\mu_{j+k} = 0$, and since $\lambda_{j+k-1} = \lambda_{j+k} = \lambda_{j+k+1} = 0$, we have that μ_{j+k} can be any scalar, as we proved earlier. By the assumption, we have that all $k-2$ previous coefficients, that is $\mu_{j+2}, \ldots, \mu_{j+k-1}$ can also be any scalars, so in total the whole sequence of $k-1 = (k+1)-2$ coefficients of y can be any scalars, which proves the induction step. Suppose that we have m sequences of three or more consecutive zero coefficients in x, whose lengths are $k_1, k_2, \ldots, k_m$,

[1] We extend our notation of indices, to include also the case where $j = 1$ or $j+k = n-1$, and define that $\lambda_{j-1} = \lambda_0 = 0$ or $\lambda_{j+k+1} = \lambda_n = 0$, respectively

[2] Here again, we consider the non-existent $\lambda_0 = \lambda_n = 0$ as part of the sequence

then $\mathfrak{m}(x) = \sum_{l=1}^{m} k_l - 2m$ is the total number of zero coefficients $\lambda_j = 0$ such that also $\lambda_{j-1} = \lambda_{j+1} = 0$, as proposed. Using again the two consecutive constraints, $(\mathfrak{C}_{i-1})$ and $(\mathfrak{C}_i)$, suppose now that $\lambda_i \neq 0$. If $\lambda_{i-1} = 0$, then by constraint $(\mathfrak{C}_{i-1})$ we must have that $\mu_{i-1} = 0$, but if $\lambda_{i-1} \neq 0$, then by this constraint we have $\mu_i = \frac{\lambda_i \mu_{i-1}}{\lambda_{i-1}}$, which means that μ_i depends on μ_{i-1}. Precisely the same way for constraint $(\mathfrak{C}_i)$, we have that if $\lambda_{i+1} = 0$ then $\mu_{i+1} = 0$, otherwise $\mu_{i+1} = \frac{\lambda_{i+1}\mu_i}{\lambda_i}$, which means that μ_{i+1} depends on μ_i, and if also $\lambda_{i-1} \neq 0$, then $\mu_{i+1} = \frac{\lambda_{i+1}\frac{\lambda_i \mu_{i-1}}{\lambda_{i-1}}}{\lambda_i} = \frac{\lambda_{i+1}\mu_{i-1}}{\lambda_{i-1}}$, which means that both μ_{i+1} and μ_i depend on μ_{i-1}. We need to prove this is true for any sequence of k consecutive non-zero coefficients of x, that is, for a given sequence of coefficients, $\lambda_{j+1}, \lambda_{j+2}, \ldots, \lambda_{j+k}$, where $1 \leq j \leq n-1$, we need to prove that μ_{j+1} can be any scalar, while $\mu_{j+2}, \mu_{j+3}, \ldots, \mu_{j+k}$ all depend on μ_{j+1}. Here again, we use a simple induction on k. For $k = 1, 2, 3$, we already proved this. For $k+1$, we look into a sequence of $k+1$ consecutive non-zero coefficients of x, that is $\lambda_{j+1}, \ldots, \lambda_{j+k}, \lambda_{j+k+1}$. By constraint $(\mathfrak{C}_{j+k})$ we have $\lambda_{j+k}\mu_{j+k+1} - \lambda_{j+k+1}\mu_{j+k} = 0$, and since $\lambda_{j+k} \neq 0$, we have $\mu_{j+k+1} = \frac{\lambda_{j+k+1}\mu_{j+k}}{\lambda_{j+k}}$, but by the assumption, $\mu_{j+k} = \frac{\lambda_{j+k}\mu_{j+k-1}}{\lambda_{j+k-1}} = \frac{\lambda_{j+k}}{\lambda_{j+k-1}} \frac{\lambda_{j+k-1}\mu_{j+k-2}}{\lambda_{j+k-2}} = \frac{\lambda_{j+k}}{\lambda_{j+k-1}} \frac{\lambda_{j+k-1}}{\lambda_{j+k-2}} \frac{\lambda_{j+k-2}\mu_{j+k-3}}{\lambda_{j+k-3}} = \cdots = \frac{\lambda_{j+k}\mu_{j+1}}{\lambda_{j+1}}$, hence $\mu_{j+k+1} = \frac{\lambda_{j+k+1}\mu_{j+k}}{\lambda_{j+k}} = \frac{\lambda_{j+k+1}}{\lambda_{j+k}} \frac{\lambda_{j+k}\mu_{j+1}}{\lambda_{j+1}} = \frac{\lambda_{j+k+1}\mu_{j+1}}{\lambda_{j+1}}$, which proves the induction step. Looking again at $(\mathfrak{C}_{i-1})$ and $(\mathfrak{C}_i)$, we observe that if λ_{i-1} and λ_{i+1} are non-zero and $\lambda_i = 0$, then $\lambda_i \mu_{i-1} = \lambda_i \mu_{i+1} = 0$, hence by constraint $(\mathfrak{C}_{i-1})$ we have that $\lambda_{i-1}\mu_i = 0$, and by constraint $(\mathfrak{C}_i)$ we have that $\lambda_{i+1}\mu_i = 0$, so $\mu_i = 0$ by both constraints, which means that there is no dependency between μ_{i+1} and μ_{i-1}. This shows that any zero coefficient between two non-zero coefficients of x creates two separate sequences, each sequences increases $\dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}/_{\gamma_2}$ by 1, hence, if we have l sequences of consecutive non-zero coefficients of x, they increase $\dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}/_{\gamma_2}$ by l, and we denote $\mathfrak{l}(x) = l$. All this shows that $\dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}/_{\gamma_2} = \mathfrak{l}(x) + \mathfrak{m}(x)$, as proposed. $\square$

Corollary 1.1.2. *Let $\mathcal{L}_{n,p}$ be the $\mathbb{Q}_p$-Lie algebra associated with $\mathcal{U}_n(\mathbb{Z})$, where $n \geq 5$, then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim {}^{\gamma_1}/_{\gamma_3} - 1$ if and only if $x \in \{\lambda e_{12} + \gamma_2 \mathcal{L}_{n,p}\}$ or $x \in \{\lambda e_{n-1,n} + \gamma_2 \mathcal{L}_{n,p}\}$, for a non-zero scalar $\lambda \in \mathbb{Q}_p$.*

Proof. We recall first that for any algebra $\mathcal{L}_{n,p}$, the first two elements of the lower central series, namely ${}^{\gamma_1}/_{\gamma_2}$ and ${}^{\gamma_2}/_{\gamma_3}$, are of sizes $n-1$ and $n-2$, respectively. Consider the Lie brackets operation $[x_1, x_2]$, where $x_1 \in {}^{\gamma_1}/_{\gamma_2}$ and $x_2 \in {}^{\gamma_2}/_{\gamma_3}$, or the opposite, then $[x_1, x_2]$ can be either zero, or $[x_1, x_2] \in$

γ_3, which means that if we consider the centralizer of some $x \in \mathcal{L}_{n,p}$ in ${}^{\gamma_1}\!/\!{}_{\gamma_3}$, then any element $y \in {}^{\gamma_2}\!/\!{}_{\gamma_3}$ would either commute with x or yield an element in γ_3, which also means that it commutes with x in ${}^{\gamma_1}\!/\!{}_{\gamma_3}$. This shows that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) \geq \dim {}^{\gamma_2}\!/\!{}_{\gamma_3} = n-2$. Suppose that $x = \lambda_1 e_{12} + z$ or $x = \lambda_{n-1} e_{n-1,n} + z$, where $z \in \gamma_2 \mathcal{L}_{n,p}$, then looking only at the elements of ${}^{\gamma_1}\!/\!{}_{\gamma_2}$ in the linear combination that forms x, we have one sequence of one non-zero coefficient, hence $\mathfrak{l}(x) = 1$, and one sequence of $n-1-1 = n-2$ zero coefficients of x, hence we have[3] that $\mathfrak{m}(x) = n-2-1 = n-3$. By 1.1.1 we have that $\dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}\!/\!{}_{\gamma_2} = \mathfrak{l}(x) + \mathfrak{m}(x) = 1 + (n-3) = n-2$. Adding all the elements of ${}^{\gamma_2}\!/\!{}_{\gamma_3}$, we have that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = (n-2)+(n-2) = 2n-4 = (n-1)+(n-2)-1 = \dim {}^{\gamma_1}\!/\!{}_{\gamma_2} + \dim {}^{\gamma_2}\!/\!{}_{\gamma_3} - 1 = \dim {}^{\gamma_1}\!/\!{}_{\gamma_3} - 1$, as proposed. To prove the opposite direction, we assume that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim {}^{\gamma_1}\!/\!{}_{\gamma_3} - 1 = 2n-4$, and since we already know that all the elements of $\dim {}^{\gamma_2}\!/\!{}_{\gamma_3}$ are in the centralizer of x, we have that $\dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}\!/\!{}_{\gamma_2} = \dim \mathcal{C}_{\gamma_1/\gamma_3}(x) - \dim {}^{\gamma_2}\!/\!{}_{\gamma_3} = (2n-4)-(n-2) = n-2 = (n-1)-1 = \dim {}^{\gamma_1}\!/\!{}_{\gamma_2} - 1$. Suppose that either $x = \lambda_i e_{i,i+1} + z$ where $1 < i < n-1$, or $x = \lambda_{i_1} e_{i_1,i_1+1} + \cdots + \lambda_{i_m} e_{i_m,i_m+1}$ where we denote the number of ${}^{\gamma_1}\!/\!{}_{\gamma_2}$ coefficients[4] by $2 \leq m \leq n-1$, in both cases $z \in \gamma_2 \mathcal{L}_{n,p}$. In the first case, we have that $\mathfrak{l}(x) = 1$, because there is only one non-zero coefficient of x in ${}^{\gamma_1}\!/\!{}_{\gamma_2}$, and we have two sequences of zero coefficients of x, that is $\lambda_0 = \lambda_1 = \lambda_2 = \cdots = \lambda_{i-2} = \lambda_{i-1} = 0$ and $\lambda_{i+1} = \lambda_{i+2} = \lambda_{i+3} = \cdots = \lambda_{n-2} = \lambda_{n-1} = \lambda_n = 0$, which means we have a sequence of zeros of length $(i-1)+1 = i$ and a sequence of length $(n-1)-i+1 = n-i$, hence by 1.1.1 we have[5] that $\mathfrak{m}(x) \leq (i-2)+(n-i-2) = n-4$, which means that $\dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}\!/\!{}_{\gamma_2} = \mathfrak{l}(x) + \mathfrak{m}(x) \leq 1 + (n-4) = n-3 = (n-1)-2 = \dim {}^{\gamma_1}\!/\!{}_{\gamma_2} - 2$. Adding all the elements of ${}^{\gamma_2}\!/\!{}_{\gamma_3}$, we have that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}\!/\!{}_{\gamma_2} + \dim {}^{\gamma_2}\!/\!{}_{\gamma_3} \leq (n-3)+(n-2) = 2n-5 < 2n-4$, which contradicts the assumption. Suppose that x is of the second form, then x may have one sequence of k non-zero coefficients, where $k > 1$, or x may have two or more sequences of one or more non-zero coefficients. In the case

[3] Again, we consider also $\lambda_0 = 0$ and $\lambda_n = 0$, hence the sequence of zero coefficients is either $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-2} = \lambda_{n-1} = \lambda_n = 0$ if $\lambda_1 \neq 0$, or $\lambda_0 = \lambda_1 = \lambda_2 = \cdots = \lambda_{n-3} = \lambda_{n-2} = 0$, if $\lambda_{n-1} \neq 0$, which means that the length of the entire sequence of zero coefficients is not $n-2$ but $n-1$, therefore by 1.1.1 we have that $\mathfrak{m}(x) = n-1-2 = n-3$

[4] In words, x has at least two non-zero coefficients in ${}^{\gamma_1}\!/\!{}_{\gamma_2}$

[5] Here, and in all the following options, $\mathfrak{m}(x)$ has an upper bound which depend on the number of consecutive zero coefficients of x, because if any of these sequences is of length k, where $k < 3$, then by 1.1.1 we have that $\mathfrak{m}(x) = k-2 \leq 0$, which means that this particular sequence does not increase $\dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}\!/\!{}_{\gamma_2}$

that x has one sequence of k non-zero coefficients, $\mathfrak{l}(x) = 1$, and the number of consecutive zero coefficients is either less or equal to $(n-1)-k+1 = n-k$, which results in $\mathfrak{m}(x) \leq n-k-2$, if the sequence of non-zero coefficients contains[6] either λ_1 or λ_{n-1}, or the number of consecutive zero coefficients is divided into two separate sequences with a total number of consecutive zeros which is less or equal to $(n-1)-k+2 = n-k+1$, which results in $\mathfrak{m}(x) \leq n-k+1-4 = n-k-3$, in the case that the sequence of non-zero coefficients of x does not contains either λ_1 or λ_{n-1}, which means that $\mathfrak{l}(x) + \mathfrak{m}(x) \leq 1 + (n-k-2) = n-k-1$, and since $k > 1$, we have that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 = \mathfrak{l}(x) + \mathfrak{m}(x) \leq n-2-1 = n-3 = (n-1)-2$, which again contradicts the assumption. Suppose that x has l sequences of non-zero coefficients in γ_1/γ_2, of lengths $k_1, k_2, \ldots, k_l$, then $\mathfrak{l}(x) = l$, and the number of zeros is $n-1-\sum_{j=1}^{l} k_j + 2 = n+1-\sum_{j=1}^{l} k_j$ in $l+1$ sequences[7], considering the two options from earlier, whether any of the sequences of non-zero coefficients contains λ_1 or λ_{n-1}, or does not contain either of them. We observe that $n+1-\sum_{j=1}^{l} k_j$ is bounded by $n+1-l$, which is the case where $k_1 = k_2 = \cdots = k_l = 1$. Suppose that $l = 2$, and that each of the 3 sequences of zero coefficients is of length greater or equal to 3, then $\mathfrak{m}(x) \leq n+1-2-3\cdot 2 = n-7$, hence $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 = \mathfrak{l}(x)+\mathfrak{m}(x) \leq 2+n-7 = n-5$, hence $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 + \dim \gamma_2/\gamma_3 \leq (n-5)+(n-2) = 2n-7 < 2n-2 = \dim \gamma_1/\gamma_3 - 1$. We prove this is true for any number of sequences of non-zero coefficients of x simple induction on l. For $l = 2$ we already proved that, for $l+1$ we take the upper bound case, which means that we add a new non-zero coefficient which splits a sequence of zero coefficients such that each of the two parts of the sequence that we split has at least 3 zero coefficients. Denote by $\mathfrak{l}_l(x) = l$ and $\mathfrak{m}_l(x)$ for l, and denote by $\mathfrak{l}_{l+1}(x) = l+1$ and $\mathfrak{m}_{l+1}(x)$ for $l+1$, then obviously $\mathfrak{l}_{l+1} + \mathfrak{m}_{l+1} \leq \mathfrak{l}_l + \mathfrak{m}_l$, because $\mathfrak{l}_{l+1} - \mathfrak{l}_l = 1$ and $\mathfrak{m}_l - \mathfrak{m}_{l+1} \geq 1$, because the newly added non-zero coefficient does not only remove one zero coefficient from the total count, but may also remove one or two of its neighboring zero coefficients, since they are no longer surrounded by another zero coefficient, which proves the induction step. All this proves that if $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \gamma_1/\gamma_3 - 1$, then the only option for x is either $x = \lambda_1 e_{12} + z$ or $x = \lambda_{n-1} e_{n-1,n} + z$, where $z \in \gamma_2 \mathcal{L}_{n,p}$, as proposed. □

[6] That is, $x = \lambda_1 e_{12} + \lambda_2 e_{23} + \cdots + \lambda_{k,k+1} e_{k,k+1}$ or $x = \lambda_{n-k} e_{n-k,n-k+1} + \lambda_{n-k+1} e_{n-k+1,n-k+2} + \cdots + \lambda_{n-1} e_{n-1,n}$

[7] Including $\lambda_0 = 0$ and $\lambda_n = 0$

Corollary 1.1.3. *Let $\mathcal{L}_{4,p}$ be the $\mathbb{Q}_p$-Lie algebra associated with $\mathcal{U}_4(\mathbb{Z})$, then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim {}^{\gamma_1}/_{\gamma_3} - 1$ if and only if $x \in \{\lambda_1 e_{12} + \lambda_3 e_{34} + \gamma_2 \mathcal{L}_{4,p}\}$, for $\lambda_1, \lambda_3 \in \mathbb{Q}_p$ not both zero.*

Proof. Suppose that $x = \lambda e_{12} + \mu e_{34} + z$, where $x \in \gamma_2 \mathcal{L}_{4,p}$, then by 1.1.1 if $\lambda \neq 0$ and $\mu \neq 0$, we have that $\mathfrak{l}(x) = 2$, and $\mathfrak{m}(x) = 0$ because there are three isolated zero coefficients, $\lambda_0 = \lambda_2 = \lambda_4 = 0$, hence $\dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}/_{\gamma_2} = \mathfrak{l}(x) + \mathfrak{m}(x) = 2$, which means that $\dim \mathfrak{C}_{\gamma_1/\gamma_3}(x) = \dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}/_{\gamma_2} + \dim {}^{\gamma_2}/_{\gamma_3} = 2 + (4-2) = 4 = (4-1) + (4-2) - 1 = \dim {}^{\gamma_1}/_{\gamma_2} + \dim {}^{\gamma_2}/_{\gamma_3} - 1 = \dim {}^{\gamma_1}/_{\gamma_3} - 1$. If $\lambda_1 = 0$ or $\lambda_3 = 0$, that is $x = \lambda_1 e_{12} + z$ or $x = \lambda$, where $z \in \gamma_2 \mathcal{L}_{4,p}$, then $\mathfrak{l}(x) = 1$ and we have a sequence of 3 consecutive zero coefficients, either $\lambda_2 = \lambda_3 = \lambda_4 = 0$ or $\lambda_0 = \lambda_1 \lambda_2 = 0$, which means that $\mathfrak{l}(x) = 3 - 2 = 1$, hence $\mathfrak{l}(x) + \mathfrak{m}(x) = 1 + 1 = 2$, hence $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim {}^{\gamma_1}/_{\gamma_3} - 1$, as proposed. To prove the opposite, we review the different options. Suppose that $x \in \gamma_2 \mathcal{L}_{4,p}$, which means that $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, hence $\mathfrak{l}(x) = 0$ and $\mathfrak{m}(x) = 5 - 2 = 3$, so $\dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}/_{\gamma_2} = \mathfrak{l}(x) + \mathfrak{m}(x) = 0 + 3$, and therefore $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}/_{\gamma_2} + \dim {}^{\gamma_2}/_{\gamma_3} = 3 + (4-2) = 5 > \dim {}^{\gamma_1}/_{\gamma_3} - 1$. The other option is that $x = \lambda_1 e_{12} + \lambda_2 e_{23} + \lambda_3 e_{34} + z$, where $\lambda_2 \neq 0$, where λ_1 and λ_3 may be either zero or non-zero, and where $z \in \gamma_2 \mathcal{L}_{4,p}$. Since $\lambda_2 \neq 0$ there is no sequence of zero coefficients of x which is greater than 2, and also, we observe that whether $\lambda_1 \neq 0$ or $\lambda_3 \neq 0$ or both are non-zero, they are part of the sequence of non-zero coefficients which has λ_2, hence $\mathfrak{l}(x) = 1$ and $\mathfrak{m}(x) = 0$, so $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}/_{\gamma_2} + \dim {}^{\gamma_2}/_{\gamma_3} = \mathfrak{l}(x) + \mathfrak{m} + 2 = 1 + 2 = 3 < 4 = \dim {}^{\gamma_1}/_{\gamma_3} - 1$, which means that if x is not as proposed, then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) \neq \dim {}^{\gamma_1}/_{\gamma_3} - 1$. $\square$

Corollary 1.1.4. *Let $\varphi \in G_n(\mathbb{Q}_p)$, where $n \geq 5$, then M_{11} the first block of M, the coefficient matrix of φ, is either diagonal or anti-diagonal.*

Proof. We first look at $\varphi(e_{12})$. We observe[8] that

$$\mathcal{C}_{\gamma_1/\gamma_3}(e_{12}) = \{e_{12}, e_{34}, e_{45}, \ldots, e_{n-1,n}, e_{13}, e_{23}, e_{34}, \ldots, e_{n-2,n-1}, e_{n-2,n}\}$$

hence $\dim \mathcal{C}_{\gamma_1/\gamma_3}(e_1 2) = (\dim {}^{\gamma_1}/_{\gamma_2} - 1) + \dim {}^{\gamma_2}/_{\gamma_3} = (n-1-1) + (n-2) = n - 4 = \dim {}^{\gamma_1}/_{\gamma_3} - 1$, which means that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(\varphi(e_{12})) = \dim {}^{\gamma_1}/_{\gamma_3} - 1$ as well,

[8] Since we review the elements in the centralizer of e_{12} in the quotient ${}^{\gamma_1}/_{\gamma_3}$, we have that $[e_{12}, e_{23}] = e_{13} \in {}^{\gamma_1}/_{\gamma_3}$ hence e_{23} does not commute with e_{12} in this quotient, but $[e_{12}, e_{24}] = e_{14} \in \gamma_3 \mathcal{L}_{n,p}$ which is zero in the quotient, so e_{24} is commuting with e_{12}, and so it is an element of the centralizer

hence by 1.1.2 we have that either $\varphi(e_{12}) = \lambda_1 e_{12} + z$ or $\varphi(e_{12}) = \lambda_{n-1} e_{n-1,n} + z$, where $z \in \gamma_2 \mathcal{L}_{n,p}$. Symmetrically, we have that either $\varphi(e_{n-1}) = \rho_1 e_{12} + w$ or $\rho_{n-1} e_{n-1,n} + w$, where $w \in \gamma_2 \mathcal{L}_{n,p}$. Assume[9] that $\varphi(e_{12}) = \lambda 1 e_{12} + z$ and $\varphi(e_{n-1}) = \rho_1 e_{12} + w$, where $\rho_1 \neq \lambda_1$, then denote the inverse map by $\psi = \varphi^{-1}$, and we have that $e_{12} = \psi\varphi(e_{12}) = \psi(\lambda_1 e_{12} + z) = \lambda_1 \psi(e_{12} + z) = \lambda_1 \psi(e_{12}) + \lambda_1 \psi(z)$, and we know that for any $x \in \gamma_i \mathcal{L}_{n,p}$, we have that $\psi(x) \in \gamma_i \mathcal{L}_{n,p}$, hence the only coefficients of e_{12} in the image of ψ are in $\lambda_1 \psi(e_{12})$, which means that $e_{12} = \lambda_1 \psi(e_{12})$, thus we must have that $\psi(e_{12}) = \frac{1}{\lambda_1} e_{12} + \psi(z)$. We now look at $e_{n-1,n} = \psi\varphi(e_{n-1,n}) = \psi(\rho_1 e_{12} + w) = \rho_1 \psi(e_{12}) + \rho_1 \psi(w)$, here again, we look only at the image of $e_{n-1,n}$, and so we have that $e_{n-1,n} = \rho_1 \psi(e_{12}) = \frac{\rho_1}{\lambda_1} e_{12}$, which is obviously a contradiction, therefore, if we assume that $\varphi(e_{12}) = \lambda_1 e_{12} + z$ then we must have that $\varphi(e_{n-1,n}) = \lambda_{n-1} e_{n-1,n} + w$, and clearly if $\varphi(e_{12}) = \lambda_{n-1} e_{n-1,n} + w$ then $\varphi(e_{n-1,n}) = \lambda_1 e_{12} + z$. For convenience, we take, throughout this proof, the assumption[10] that $\varphi(e_{12}) = \lambda_1 e_{12} + z$. We now look at $\varphi(e_{23})$. We observe that

$$\mathcal{C}_{\gamma_1/\gamma_3}(e_{23}) = \{e_{23}, e_{45}, e_{56}, \ldots, e_{n-2,n-1}, e_{n-1,n}, e_{13}, e_{24}, e_{35}, \ldots, e_{n-3,n-1}, e_{n-2,n}\}$$

that is, the only elements that do not commute with e_{23} in the quotient γ_1/γ_2 are e_{12} and e_{34}, but $[\varphi(e_{12}), \varphi(e_{23})] = \varphi([e_{12}, e_{23}]) = \varphi(e_{13})$, and $e_{13} \in \mathcal{C}_{\gamma_1/\gamma_3}(e_{23})$, so $\varphi(e_{13}) \in \mathcal{C}_{\gamma_1/\gamma_3}(\varphi(e_{23}))$, which means that $\varphi(e_{12})$ does not commute with $\varphi(e_{23})$, but $\varphi(e_{12}) = \lambda_1 e_{12} + z$ or $\varphi(e_{12}) = \lambda_{n-1} e_{n-1,n} + z$, where $z \in \gamma_2 \mathcal{L}_{n,p}$, hence $\varphi(e_{23})$ must have $\lambda_2 e_{23}$ or $\lambda_{n-2} e_{n-2,n-1}$, respectively. Following our choice from above, we have that $\varphi(e_{12}) = \lambda_1 e_{12} + z$, hence the linear combination that forms $\varphi(e_{23})$ must have $\lambda_2 e_{23}$, and it is trivial that if $\varphi(e_{23}) = \lambda_2 e_{23} + z$, then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(\varphi(e_{23}))/\gamma_2 = \dim \mathcal{C}_{\gamma_1/\gamma_3}(e_{23})/\gamma_2$. Assume that $\varphi(e_{23}) = \lambda_2 e_{23} + \lambda_i e_{i,i+1} + z$, where $1 \leq i \leq n-1$ and $i \neq 2$. Assume $i = 3$, that is, $\varphi(e_{23}) = \lambda_2 e_{23} + \lambda_3 e_{34} + z$, then following 1.1.1 notations, we have that $\mathfrak{l}(\lambda_2 e_{23}) = 1$, because we have only one sequence of consecutive coefficients, and we have[11] that $\mathfrak{m}(\lambda_2 e_{23}) = (n-1) - 2 + 1 - 2 = n - 4$, but for

[9] In words, assume that both e_{12} and $e_{n-1,n}$ map into scalar multiplications of e_{12} in the quotient γ_1/γ_2

[10] We shall prove later that this assumption can always be taken as the general case

[11] A detailed explanation: We have $n-1$ coefficients in the first block. Omit λ_2 which is non-zero, and omit λ_1, which is a zero coefficient, but together with λ_0 is not a part of a consecutive sequence of three zero coefficients. Consider also λ_n, which is part of a sequence of at least three zero coefficients, and by 1.1.1 subtract 2 to obtain the total number of zero coefficients which are surrounded by other zeros

$\lambda_1 e_{12} + \lambda_2 e_{23}$ we have that $\mathfrak{l}(\lambda_1 e_{12} + \lambda_2 e_{23}) = 1 = \mathfrak{l}(\lambda_1 e_{12})$ because $\lambda_1 e_{12}$ an $\lambda_2 e_{23}$ are both part of the same sequence of consecutive non-zero coefficients, but $\mathfrak{m}(\lambda_1 e_{12} + \lambda_2 e_{23}) = (n-1) - 3 + 1 - 2 = n - 5 < n - 4 = \mathfrak{m}(\lambda_1 e_{12})$, which means that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(\lambda_1 e_{12} + \lambda_2 e_{23}) < \dim \mathcal{C}_{\gamma_1/\gamma_3}(\lambda_2 e_{23}) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(e_{23})$. Assume that $n > 5$ and $i > 3$, then we observe that $\mathfrak{l}(\lambda_1 e_{12} + \lambda_i e_{i,i+1}) = 2$ because we have two sequences of non-zero coefficients, but[12] then $\mathfrak{m}(\lambda_1 e_{12} + \lambda_i e_{i,i+1}) \leq \mathfrak{m}(\lambda_1 e_{12}) - 2$, which means that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(\lambda_1 e_{12} + \lambda_i e_{i,i+1}) = \mathfrak{l}(\lambda_1 e_{12} + \lambda_i e_{i,i+1}) + \mathfrak{m}(\lambda_1 e_{12} + \lambda_i e_{i,i+1}) \leq \mathfrak{l}(\lambda_2 e_{23}) + 1 + \mathfrak{m}(\lambda_2 e_{23}) - 2 < \mathfrak{l}(\lambda_2 e_{23}) + \mathfrak{m}(\lambda_i e_{i,i+1}) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(\lambda_2 e_{23}) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(e_{23})$. Assume that $i = 1$, that is, $\varphi(e_{23}) = \lambda_1 e_{12} + \lambda_2 e_{23} + z$. We observe that in this case, $\mathfrak{l}(\lambda_1 e_{12} + \lambda_2 e_{23}) = \mathfrak{l}(\lambda_2 e_{23})$ because in either case we have only one sequence of consecutive non=zero coefficients, and that also $\mathfrak{m}(\lambda_1 e_{12} + \lambda_2 e_{23}) = \mathfrak{m}(\lambda_2 e_{23})$, because when $\lambda_1 = 0$ it is not part of a sequence of three consecutive zero coefficients hence turning it into a non-zero coefficient does not change change the total count of zero coefficients that are surrounded by other zeros, thus we have[13] that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(\lambda_1 e_{12} + \lambda_2 e_{23}) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(\lambda_2 e_{23})$. Therefore, we need to consider a larger centralizer, $\mathcal{C}_{\gamma_1/\gamma_4}(\lambda_1 e_{23}) = \mathcal{C}_{\gamma_1/\gamma_4}(e_{23}) = \langle e_{23}, e_{45}, e_{56}, \ldots, e_{n-1,n}, e_{13}, e_{24}, e_{46}, e_{57}, \ldots, e_{n-2,n}, e_{14}, e_{25}, e_{47}, e_{58}, \ldots, e_{n-3,n} \rangle$, hence $\text{codim}\mathcal{C}_{\gamma_1/\gamma_4}(e_{23}) = 3$, and following the assumption on $\varphi(e_{23})$ we have that $x = \varphi(e_{23}) = \lambda_1 e_{12} + \lambda_2 e_{23} + \sum_{i=1}^{n-2} a_{i,i+2} e_{i,i+2} + z_3$, where $z_3 \in \gamma_3 \mathcal{L}_{n,p}$, and we look into an arbitrary element $y \in \mathcal{C}_{\gamma_1/\gamma_4}(x)$, that is $y = \sum_{i=1}^{n-1} \mu_i e_{i,i+1} + \sum_{i=1}^{n-2} b_{i,i+2} e_{i,i+2} + w_3$, where $w_3 \in \gamma_3 \mathcal{L}_{n,p}$, so we have that

$$(\lambda_1 \mu_2 - \lambda_2 \mu_1) e_{13} = 0$$

$$\lambda_2 \mu_3 e_{24} = 0 \Rightarrow \mu_3 = 0$$

$$(\lambda_1 b_{24} - a_{24} \mu_1) e_{14} = 0$$

$$(\lambda_2 b_{35} - a_{35} \mu_2 + a_{24} \mu_4) e_{25} = 0$$

and we observe that none of these equations is a linear combination of one or more of the other equations. This means that we have four independent constraints, which means that $\text{codim}\mathcal{C}_{\gamma_1/\gamma_4}(x) = 4 < 3 = \text{codim}\mathcal{C}_{\gamma_1/\gamma_4}(e_{23})$, we observe that since $\lambda_2 \neq 0$, if we assume that $\lambda_1 = 0$ then the first equation becomes $\lambda_2 \mu_1 = 0 \Rightarrow \mu_1 = 0$, thus the equation $\lambda_1 b_{24} - a_{24} \mu_1 = 0$ does

[12] The accurate difference is 2 if $i = n-1$ because we omit both λ_{n-1} and λ_{n-2}, and 3 if $i < n-1$ because we omit λ_{i-1}, λ_i and λ_{i+1}

[13] To show this explicitly, $\mathcal{C}_{\gamma_1/\gamma_3}(\lambda_2 e_{23}) = \{\mu_2 e_{23}, \mu_4 e_{45}, \mu_5 e_{56}, \ldots, \mu_{n-1}, e_{n-1,n}, b_{13} e_{13}, \ldots, b_{n-2,n} e_{n-2,n}\}$, and $\mathcal{C}_{\gamma_1/\gamma_3}(\lambda_1 e_{12} + \lambda_2 e_{23}) = \{\mu_1 e_{12} + \frac{\lambda_1}{\lambda_2} \mu_1 e_{23}, \mu_4 e_{45}, \mu_5 e_{56}, \ldots, \mu_{n-1} e_{n-1,n}, b_{13} e_{13}, \ldots, b_{n-2,n} e_{n-2,n}\}$

not form any constraint on y because $\lambda_1 = \mu_1 = 0$, so the co dimension of the centralizer, in this case of $x = \varphi(e_{23}) = \lambda_2 e_{23} + \sum_{i=1}^{n-2} a_{i,i+2} e_{i,i+2} + z_3$ is correct, which means that $\varphi(e_{23}) \neq \lambda_1 e_{12} + \lambda_2 e_{23} + z$. We also observe that when $n = 5$, if $i = 4$, that is $\varphi(e_{23}) = \lambda_2 e_{23} + \lambda_4 e_{45}$, we have that $\dim \mathcal{C}_{\gamma_1/\gamma_2}(\lambda_2 e_{23} + \lambda_4 e_{45}) = \dim \mathcal{C}_{\gamma_1/\gamma_2}(\lambda_2 e_{23})$, and therefore we need to consider a larger centralizer to rule out this option as well, hence we conclude that $\varphi(e_{23}) = \lambda_1 e_{23} + z$, where $z \in \gamma_2 \mathcal{L}_{n,p}$. We observe that this is true for the entire first block, that is, $\varphi(e_{j,j+1}) = \lambda_j e_{j,j+1}$, where $1 \leq j \leq n-1$, which means that the block M_{11} is diagonal, if we assume that $\varphi(e_{12}) = \lambda_1 e_{12}$, and it is immediate to follow the course of this proof with the assumption that $\varphi(e_{12}) = \lambda_{n-1} e_{n-1,n} + w$, where $w \in \gamma_2 \mathcal{L}_{n,p}$, and prove that in this case the block M_{11} is anti-diagonal. $\square$

Corollary 1.1.5. *Let $\varphi \in G_4(\mathbb{Q}_p)$, then M_{11} is either diagonal or anti-diagonal.*

Proof. We observe[14] that $\mathcal{C}_{\gamma_1/\gamma_3}(e_{12}) = \langle e_{12}, e_{34}, e_{13}, e_{24} \rangle$, hence $\dim \mathcal{C}_{\gamma_1/\gamma_3}(\varphi(e_{12})) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(e_{12}) = 4 = (4-1)+(4-2)-1 = \dim \gamma_1/\gamma_3 - 1$, but by 1.1.3, we have that $\varphi(e_{12}) = \lambda_1 e_{12} + \lambda_3 e_{34}$. Looking at a larger centralizer, $\mathcal{C}_{\gamma_1/\gamma_4}(\lambda_1 e_{12}) = \mathcal{C}_{\gamma_1/\gamma_4}(e_{12}) = \langle e_{12}, e_{34}, e_{13}, e_{14} \rangle$, but $\mathcal{C}_{\gamma_1/\gamma_4}(\lambda_1 e_{12} + \lambda_3 e_{34}) = \langle e_{12}, e_{34}, e_{14} \rangle$, hence $\dim \mathcal{C}_{\gamma_1/\gamma_4}(\lambda_1 e_{12} + \lambda_3 e_{34}) < \dim \mathcal{C}_{\gamma_1/\gamma_4}(e_{12})$, hence we must have that $\varphi(e_{12}) = \lambda_1 e_{12} + z$ or, symmetrically, $\varphi(e_{12}) = \lambda_3 e_{34} + z$, where $z \in \gamma_2 \mathcal{L}_{4,p}$. Again, symmetrically we have that if $\varphi(e_{12}) = \lambda_1 e_{12} + z$ then $\varphi(e_{34}) = \lambda_3 e_{34} + w$, and if $\varphi(e_{12}) = \lambda_3 e_{34} + w$ then $\varphi(e_{34}) = \lambda_1 e_{12} + z$, where $z, w \in \gamma_2 \mathcal{L}_{4,p}$. Assume that $\varphi(e_{12}) = \lambda_1 e_{12} + z$ and $\varphi(e_{34}) = \lambda_3 e_{34} + w$, then following the identity $[\varphi(e_{12}), \varphi(e_{23})] = \varphi([e_{12}, e_{23}]) = \varphi(e_{13})$, we denote by $x = \varphi(e_{23}) = \lambda_1 e_{12} + \lambda_2 e_{23} + \lambda_3 e_{34} + w$, where $w \in \gamma_2 \mathcal{L}_{4,p}$ and λ_1, λ_2, λ_3 can be zero, hence $\varphi(e_{13}) = [\varphi(e_{12}), \varphi(e_{23})] = [e_{12} + z, \lambda_1 e_{12} + \lambda_2 e_{23} + \lambda_3 e_{34} + w] = \lambda_2 e_{13} + y$, where $y \in \gamma_3 \mathcal{L}_{4,p}$, hence, if we assume that λ_2 we have that $\varphi(e_{13})$ is zero in the quotient γ_1/γ_3, but φ is an automorphism, so this is a contradiction. We now look at $\mathcal{C}_{\mathcal{L}_{4,p}}(e_{23}) = \langle e_{23}, e_{13}, e_{24}, e_{14} \rangle$, but $\mathcal{C}_{\mathcal{L}_{4,p}}(\lambda_1 e_{12} + \lambda_2 e_{23}) = \langle \mu_1 e_{12} + \frac{\lambda_1}{\lambda_2} \mu_1 e_{23}, e_{13}, e_{14} \rangle$, which means that $\dim \mathcal{C}_{\mathcal{L}_{4,p}}(\lambda_1 e_{12} + \lambda_2 e_{23}) < \dim \mathcal{C}_{\mathcal{L}_{4,p}}(e_{23})$, hence we must have that $\lambda_1 = 0$ and symmetrically we also have that $\lambda_3 = 0$, therefore we have that M_{11} is diagonal. We observe that if $\varphi(e_{12}) = \lambda_3 e_{34} + z$ and $\varphi(e_{34}) = \lambda_1 e_{12} + w$, then also in this case $\varphi(e_{23}) = \lambda_2 e_{23} + y$, where $z, w, y \in \gamma_2 \mathcal{L}_{4,p}$, which means that M_{11} is anti-diagonal. $\square$

[14] Here again $e_{24} \in \mathcal{C}_{\gamma_1/\gamma_3}(e_{12})$ because $[e_{12}, e_{24}] = e_{14} \in \gamma_3 \mathcal{L}_{4,p}$, hence it is zero in the quotient γ_1/γ_3

Proposition 1.1.6. *Let $\mathcal{L}_{n,p}$ be the $\mathbb{Q}$-Lie algebra associated with $\mathcal{U}_n(\mathbb{Z}_p)$, and let $\mathcal{B}_n = \{e_{12}, e_{23}, \ldots, e_{1n}\}$ be its basis. Then, the map $\eta : \mathcal{B}_n \to \mathcal{B}_n$, defined by*

$$\eta(e_{ij}) := (-1)^{j-i-1} e_{n+1-j,n+1-i}$$

is an involution, hence, also an $\mathcal{L}_{n,p}$-automorphism.

Proof. Clearly, we calculate $\eta^2(e_{ij})$ by taking $k = n+1-j$ and $l = n+1-i$. Hence,

$$\eta(\eta(e_{ij})) = \eta((-1)^{j-i-1}e_{kl}) = (-1)^{j-i-1}\eta(e_{ij}) = (-1)^{j-i-1}(-1)^{l-k-1}e_{n+1-l,n+1-k} =$$
$$= (-1)^{j-i-1}(-1)^{n+1-i-(n+1-j)-1}e_{n+1-(n+1-i),n+1-(n+1-j)} = (-1)^{j-i-1}(-1)^{j-i-1}e_{ij} =$$
$$= (-1)^{2(j-i-1)}e_{ij} = e_{ij}$$

To complete the proof, we need to show that η is indeed a homomorphism,

$$[\eta(e_{ij}), \eta(e_{kl})] = [(-1)^{j-i-1}e_{n+1-j,n+1-i}, (-1)^{l-k-1}e_{n+1-l,n+1-k}] =$$
$$= (-1)^{j-i-1}(-1)^{l-k-1}[e_{n+1-j,n+1-i}, e_{n+1-l,n+1-k}] =$$
$$(-1)^{j-i+l-k-2}[e_{n+1-j,n+1-i}, e_{n+1-l,n+1-k}]$$

We observe that if $n+1-i = n+1-l$ or $n+1-j = n+1-k$ then the Lie brackets evaluate to a non-zero element, but this happens if and only if $i = l$ or $j = k$, respectively. Assume $j = k$, then

$$(-1)^{j-i+l-k-2}[e_{n+1-j,n+1-i}, e_{n+1-l,n+1-k}] = (-1)^{l-i-2}[e_{n+1-j,n+1-i}, e_{n+1-l,n+1-j}] =$$
$$(-1)^{l-i-2} \cdot -e_{n+1-l,n+1-i} = (-1)^{l-i-1}e_{n+1-l,n+1-i} = \eta(e_{il}) = \eta[e_{ij}, e_{jl}]$$

□

Corollary 1.1.7. *Let $\mathcal{L}_{n,p}$, where $n \geq 4$, then the block M_{11} is diagonal.*

Proof. By 1.1.4 and 1.1.5, we have that M_{11} is either diagonal or anti-diagonal. Assume that M_{11} is anti-diagonal, that is, $\varphi(e_{12}) = \lambda_{n-1}e_{n-1,n} + z_{n-1}, \varphi(e_{23}) = \lambda_{n-2}e_{n-2,n-1} + z_{n-2}, \ldots, \varphi(e_{n-1,n}) = \lambda_1 e_{12} + z_1$, where $z_1, z_2, \ldots, z_{n-1} \in \gamma_2\mathcal{L}_{n,p}$, then we define a new automorphism by the composition $\psi = \varphi\eta$. Let $e_{i,i+1} \in \gamma_1/\gamma_2$, where $1 \leq i \leq n-1$, then $\psi(e_{i,i+1}) = \varphi\eta(e_{i,i+1}) = \varphi((-1)^{i+1-i-1}e_{n+1-(i+1),n+1-i}) = \varphi(e_{n-i,n-i+1})$, hence $\psi(e_{12}) = \varphi(e_{n-1,n}) = \lambda_1 e_{12} + z_1, \psi(e_{23}) = \varphi(e_{n-2,n-1}) = \lambda_2 e_{23} + z_2, \ldots, \psi(e_{n-1,n}) = \varphi(e_{12}) = \lambda_{n-1}e_{n-1,n} + z_{n-1}$, which means that M_{11}, as the first block of the matrix representing ψ, is diagonal. □

Corollary 1.1.8. *Let $\mathcal{L}_{n,p}$, where $n \geq 4$, then all the blocks $M_{11}, M_{22}, \ldots, M_{n-1,n-1}$ are diagonal.*

Proof. By simple induction on the index $1 \leq k \leq n-1$ of the diagonal block. For $k = 1$ we proved this in 1.1.7, for $k+1$ we observe that for every $1 \leq i \leq n-k$, we have the identity $\varphi(e_{i,i+k+1}) = \varphi([e_{i,i+j}, e_{i+j,i+k+1}]) = [\varphi(e_{i,i+j}), \varphi(e_{i+j,i+k+1})]$, for $1 \leq j \leq k$, but by the assumption we have that $\varphi(e_{i,i+j}) = a_{i,i+j}e_{i,i+j} + z_{j+1}$ and $\varphi(e_{i+j,i+k+1}) = a_{i+j,i+k+1}e_{i+j,i+k+1} + z_{k+2-j}$, where $z_{j+1} \in \gamma_{j+1}\mathcal{L}_{n,p}$ and $z_{k+2-j} \in \gamma_{k+2-j}\mathcal{L}_{n,p}$, hence $\varphi(e_{i,i+k+1}) = [a_{i,i+j}e_{i,i+j}+z_{j+1}, a_{i+j,i+k+1}e_{i+j,i+k+1}+z_{k+2-j}] = a_{i,i+j}a_{i+j,i+k+1}e_{i,i+k+1}+a_{i,i+j}e_{i,i+j}z_{k+2-j}+ a_{i+j,i+k+1}e_{i+j,i+k+1}z_{j+1} + z_{k+2-j}z_{j+1} = a_{i,i+j}a_{i+j,i+k+1}e_{i,i+k+1} + z_{k+2-j}$, where $z_{k+2-j} \in \gamma_{k+2-j}\mathcal{L}_{n,p}$, which proves the induction step. $\square$

Corollary 1.1.9. *Let $\mathcal{L}_{n,p}$, where $n \geq 4$, then the main diagonal is*

$$\begin{pmatrix} \lambda_1 &&&&&&&& \\ & \lambda_2 &&&&&&& \\ && \ddots &&&&&& \\ &&& \lambda_{n-1} &&&&& \\ &&&& \lambda_1\lambda_2 &&&& \\ &&&&& \ddots &&& \\ &&&&&& \lambda_{n-1}\lambda_{n-2} && \\ &&&&&&& \ddots & \\ &&&&&&&& \lambda_1\lambda_2\cdots\lambda_{n-2}\lambda_{n-1} \end{pmatrix}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{Q}_p$, such that for every $1 \leq k \leq n-1$ and for every $1 \leq i \leq n-k$, we have that $\varphi(e_{i,i+k}) = \lambda_i\lambda_{i+1}\lambda_{i+2}\cdots\lambda_{i+k-2}\lambda_{i+k-1}e_{i,i+k}+z_{k+1}$, where $z_{k+1} \in \gamma_{k+2}\mathcal{L}_{n,p}$

Proof. Following the proof of 1.1.8, we have that for every $1 \leq k \leq n-1$ and for every $1 \leq i \leq n-k$, $\varphi(e_{i,i+k}) = \lambda_{i,i+k}e_{i,i+k} + z_{k+1}$, where $z_{k+1} \in \gamma_{k+1}\mathcal{L}_{n,p}$, and we use the same method of induction on k. For $k = 1$, we already proved in 1.1.4 that $\varphi(e_{12}) = \lambda_1 e_{12} + z_1, \varphi(e_{23}) = \lambda_2 e_{23} + z_2, \ldots, \varphi(e_{n-1,n}) = \lambda_{n-1}e_{n-1,n} + z_{n-1}$, where $z_i \in \gamma_i\mathcal{L}_{n,p}$, for every $1 \leq i \leq n-1$. For $k+1$, we have that $\varphi(e_{i,i+k+1}) = \varphi([e_{i,i+j}, e_{i+j,i+k+1}]) = [\varphi(e_{i,i+j}), \varphi(e_{i+j,i+k+1})]$, for every $1 \leq j \leq k$, but by the assumption we have that $\varphi(e_{i,i+j}) = \lambda_i\lambda_{i+1}\cdots\lambda_{i+j} + z_{i+j+1}$ and $\varphi(e_{i+j,i+k+1}) = \lambda_{i+j}\lambda_{i+j+1}\cdots\lambda_{i+k+1} + z_{i+k+2}$, where $z_{i+j+1} \in \gamma_{i+j+1}\mathcal{L}_{n,p}$ and $z_{i+k+2} \in \gamma_{i+k+2}\mathcal{L}_{n,p}$, hence $\varphi(e_{i,i+k+1}) =$

$\varphi([e_{i,i+j}, e_{i+j,i+k+1}]) = [\varphi(e_{i,i+j}), \varphi(e_{i+j,i+k+1})] = [\lambda_i\lambda_{i+1}\cdots\lambda_{i+j}+z_{i+j+1}, \lambda_{i+j}\lambda_{i+j+1}\cdots\lambda_{i+k+1}+z_{i+k+2}] = \lambda_i\lambda_{i+1}\lambda_{i+2}\lambda_{i+k}\cdots\lambda_{i+k+1}+z_{i+k+2}$, which proves the induction step.

$\square$