

cases

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Denote  $p^{\min} := p^{\min\{0, v_2 + v_{11}\}}$

Denote  $\rho := (1 - p^{-1})$

Denote  $f(p, t)$  the  $p$  and  $t$  product  $p^{7v_1 + 10v_2 + 11v_3 + 7v_4 + \min\{v_1, v_4\}} t^{4v_1 + 6v_2 + 6v_3 + 4v_4}$

## 1 Case 1

$$v_3 > v_1 \geq v_4 > v_2 \geq 0$$

$$\Rightarrow v_{11} \geq -v_2 - v_1$$

$$\Rightarrow \alpha := \min\{v_2, v_2 + v_{11} + v_1\} = \begin{cases} v_2, & v_{11} \geq -v_1 \\ v_2 - 1, v_2 - 2, \dots, & v_{11} < -v_1 \end{cases}$$

$$\Rightarrow f(p, t) = p^{7v_1 + 10v_2 + 11v_3 + 7v_4 + v_4} t^{4v_1 + 6v_2 + 6v_3 + 4v_4} = p^{7v_1 + 10v_2 + 11v_3 + 8v_4} t^{4v_1 + 6v_2 + 6v_3 + 4v_4}$$

For  $v_{11} \geq -v_2$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho)$ .

For  $-v_1 \leq v_{11} \leq -v_2 - 1$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_1}^{-v_2-1} p^{v_2}\rho(1 + v_2\rho) = p^{v_2}(v_1 - v_2)\rho(1 + v_2\rho) = p^{v_2}(v_1 - v_2)\rho + p^{v_2}v_2(v_1 - v_2)\rho^2$ .

For  $-v_2 - v_1 \leq v_{11} \leq -v_1 - 1$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_1}^{-v_1-1} p^{v_2}\rho(1 + \alpha\rho) = \sum_{i=1}^{v_2} p^{v_2}\rho(1 + (v_2 - i)\rho) = p^{v_2}v_2\rho + p^{v_2}v_2^2\rho^2 - p^{v_2}\binom{v_2+1}{2}\rho^2$ .

So we have  $\sum_{v_{11} \geq -v_2 - v_1} p^{v_2}(1 + v_2\rho + (v_1 - v_2)\rho + v_2(v_1 - v_2)\rho^2 + v_2\rho + v_2^2\rho^2 - \frac{(v_2+1)v_2}{2}\rho^2) = p^{v_2}(1 + v_1\rho + v_2v_1\rho^2 + v_2\rho - \frac{v_2^2}{2}\rho^2 - \frac{v_2}{2}\rho^2)$ . We compute the differences between  $v_1, v_2, v_3, v_4$ ,

$$v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_4 \overset{b}{>} v_2 \overset{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow a + b + c$ ,  $v_2 \rightarrow a$ ,  $v_3 \rightarrow a + b + c + d$ ,  $v_4 \rightarrow a + b$ , and compute series with the new variables as indices, where  $a, c \geq 0$  and  $b, d \geq 1$ .

## 2 Case 2

$$v_1 \geq v_3 \geq 0$$

$$v_1 \geq v_4 > v_2 \geq 0$$

$$\Rightarrow v_{11} \geq -v_2 - v_3$$

$$\Rightarrow \alpha := \min\{v_2, v_2 + v_{11} + v_1\} = \begin{cases} v_2, & v_{11} \geq -v_1 \\ v_2 - 1, v_2 - 2, \dots, & v_{11} < -v_1 \end{cases}$$

$$\Rightarrow f(p, t) = p^{7v_1 + 10v_2 + 11v_3 + 7v_4 + v_4} t^{4v_1 + 6v_2 + 6v_3 + 4v_4} = p^{7v_1 + 10v_2 + 11v_3 + 8v_4} t^{4v_1 + 6v_2 + 6v_3 + 4v_4}$$

We have two sub cases.

## 2.1 Sub case 2.1

$$-v_2 - v_3 \geq -v_1 \Rightarrow v_1 \geq v_2 + v_3$$

Here there is no phase transition, because the phase transition occurs at  $-v_1 - 1 < -v_2 - v_3 \leq v_{11}$

For  $v_{11} \geq -v_2$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho)$ .

For  $-v_2 - v_3 \leq v_{11} \leq -v_2 - 1$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2 - v_3}^{-v_2 - 1} p^{v_2} \rho(1 + v_2\rho) = p^{v_2} v_3 \rho(1 + v_2\rho) = p^{v_2} v_3 \rho + p^{v_2} v_3 v_2 \rho^2$ .

So we have  $\sum_{v_{11} \geq -v_2 - v_3} = p^{v_2}(1 + v_2\rho + v_3\rho + v_2 v_3 \rho^2)$ .

We compute the differences between  $v_1, v_2, v_3, v_4$ , we have several arrangements,

$$1. v_1 \stackrel{d}{\geq} v_2 + v_3 \stackrel{a}{\geq} v_3 \stackrel{c}{\geq} v_4 \stackrel{b}{>} v_2 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow 2a + b + c + d$ ,  $v_2 \rightarrow a$ ,  $v_3 \rightarrow a + b + c$ ,  $v_4 \rightarrow a + b$ , and compute series with the new variables as indices, where  $a, c, d \geq 0$  and  $b \geq 1$ .

$$2. v_1 \stackrel{d}{\geq} v_2 + v_3 \stackrel{c}{\geq} v_4 \stackrel{b}{>} v_3 \stackrel{a}{\geq} v_2 \stackrel{e}{\geq} 0$$

But  $v_2 + v_3 - c - b = v_3 \Rightarrow v_2 = v_2 + v_3 - v_3 = b + c$

Thus, we substitute  $v_1 \rightarrow a + 2b + 2c + d$ ,  $v_2 \rightarrow b + c$ ,  $v_3 \rightarrow a + b + c$ ,  $v_4 \rightarrow a + 2b + c$ , and compute series with the new variables as indices, where  $c, d \geq 0$  and  $a, b \geq 1$ .

$$3. v_1 \stackrel{d}{\geq} v_2 + v_3 \stackrel{c}{\geq} v_4 \stackrel{b}{>} v_2 \stackrel{a}{\geq} v_3 \stackrel{e}{\geq} 0$$

But  $v_2 + v_3 - c - b = v_2 \Rightarrow v_3 = v_2 + v_3 - v_2 = b + c$

Thus, we substitute  $v_1 \rightarrow a + 2b + 2c + d$ ,  $v_2 \rightarrow a + b + c$ ,  $v_3 \rightarrow b + c$ ,  $v_4 \rightarrow a + 2b + c$ , and compute series with the new variables as indices, where  $a, c, d \geq 0$  and  $b \geq 1$ .

$$4. v_1 \stackrel{d}{\geq} v_4 \stackrel{c}{>} v_2 + v_3 \stackrel{a}{\geq} v_2 \stackrel{b}{\geq} v_3 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow 2a + b + c + d$ ,  $v_2 \rightarrow a + b$ ,  $v_3 \rightarrow a$ ,  $v_4 \rightarrow 2a + b + c$ , and compute series with the new variables as indices, where  $a, b, d \geq 0$  and  $c \geq 1$ .

$$5. v_1 \stackrel{d}{\geq} v_4 \stackrel{c}{>} v_2 + v_3 \stackrel{a}{\geq} v_3 \stackrel{b}{>} v_2 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow 2a + b + c + d$ ,  $v_2 \rightarrow a$ ,  $v_3 \rightarrow a + b$ ,  $v_4 \rightarrow 2a + b + c$ , and compute series with the new variables as indices, where  $a, d \geq 0$  and  $b, c \geq 1$ .

## 2.2 Sub case 2.2

$$-v_2 - v_3 < -v_1 \Rightarrow v_1 < v_2 + v_3$$

For  $v_{11} \geq -v_2$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho)$ .

For  $-v_1 \leq v_{11} \leq -v_2 - 1$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_1}^{-v_2-1} p^{v_2}\rho(1 + v_2\rho) = p^{v_2}(v_1 - v_2)\rho(1 + v_2\rho) = p^{v_2}(v_1 - v_2)\rho + p^{v_2}(v_1 - v_2)v_2\rho^2$ .

For  $-v_2 - v_3 \leq v_{11} \leq -v_1 - 1$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_3}^{-v_1-1} p^{v_2}\rho(1 + \alpha\rho) = \sum_{i=1}^{v_2+v_3-v_1} p^{v_2}\rho(1 + (v_2 - i)\rho) = p^{v_2}(v_2 + v_3 - v_1)\rho + p^{v_2}(v_2 + v_3 - v_1)v_2\rho^2 - p^{v_2}\binom{v_2+v_3-v_1+1}{2}\rho^2$ .

So we have  $\sum_{v_{11} \geq -v_2-v_3} = p^{v_2}(1 + v_2\rho + (v_1 - v_2)\rho + (v_1 - v_2)v_2\rho^2 + (v_2 + v_3 - v_1)\rho + (v_2 + v_3 - v_1)v_2\rho^2 - \frac{(v_2+v_3-v_1+1)(v_2+v_3-v_1)}{2}\rho^2) = p^{v_2}(1 + v_2\rho + v_3\rho - \frac{v_2^2}{2}\rho^2 + \frac{v_2v_1}{2}\rho^2 - \frac{v_3^2}{2}\rho^2 + \frac{v_3v_1}{2}\rho^2 + \frac{v_1v_2}{2}\rho^2 + \frac{v_1v_3}{2}\rho^2 - \frac{v_1^2}{2}\rho^2 - \frac{v_2}{2}\rho^2 - \frac{v_3}{2}\rho^2 + \frac{v_1}{2}\rho^2) = p^{v_2}(1 + v_2\rho + v_3\rho - \frac{v_2^2}{2}\rho^2 + v_2v_1\rho^2 - \frac{v_3^2}{2}\rho^2 + v_3v_1\rho^2 - \frac{v_1^2}{2}\rho^2 - \frac{v_2}{2}\rho^2 - \frac{v_3}{2}\rho^2 + \frac{v_1}{2}\rho^2)$ . We compute the differences between  $v_1, v_2, v_3, v_4$ , we have several arrangements,

$$1. v_2 + v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_3 \overset{b}{\geq} v_4 \overset{a}{>} v_2 \overset{e}{\geq} 0$$

But  $v_2 + v_3 - d - c = v_3 \Rightarrow v_2 = c + d$

Thus, we substitute  $v_1 \rightarrow a + b + 2c + d$ ,  $v_2 \rightarrow c + d$ ,  $v_3 \rightarrow a + b + c + d$ ,  $v_4 \rightarrow a + c + d$ , and compute series with the new variables as indices, where  $b, c \geq 0$  and  $a, d \geq 1$ .

$$2. v_2 + v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_4 \overset{b}{>} v_3 \overset{a}{>} v_2 \overset{e}{\geq} 0$$

But  $v_2 + v_3 - d - c - b = v_3 \Rightarrow v_2 = v_2 + v_3 - v_3 = b + c + d$

Thus, we substitute  $v_1 \rightarrow a + 2b + 2c + d$ ,  $v_2 \rightarrow b + c + d$ ,  $v_3 \rightarrow a + b + c + d$ ,  $v_4 \rightarrow a + 2b + c + d$ , and compute series with the new variables as indices, where  $c \geq 0$  and  $a, b, d \geq 1$ .

$$3. v_2 + v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_4 \overset{b}{>} v_2 \overset{a}{\geq} v_3 \overset{e}{\geq} 0$$

But  $v_2 + v_3 - d - c - b = v_2 \Rightarrow v_3 = v_2 + v_3 - v_2 = b + c + d$

Thus, we substitute  $v_1 \rightarrow a + 2b + 2c + d$ ,  $v_2 \rightarrow a + b + c + d$ ,  $v_3 \rightarrow b + c + d$ ,  $v_4 \rightarrow a + 2b + c + d$ , and compute series with the new variables as indices, where  $a, c \geq 0$  and  $b, d \geq 1$ .

### 3 Case 3

$$v_3 > v_1 \geq 0$$

$$v_4 > v_1 \geq 0$$

$$v_4 > v_2 \geq 0$$

$$\Rightarrow v_{11} \geq -v_2 - v_1$$

$$\Rightarrow \alpha := \min\{v_2, v_2 + v_{11} + v_4\} = \begin{cases} v_2, & v_{11} \geq -v_4 \\ v_2 - 1, v_2 - 2, \dots, & v_{11} < -v_4 \end{cases}$$

$$\Rightarrow f(p, t) = p^{7v_1+10v_2+11v_3+7v_4+v_1} t^{4v_1+6v_2+6v_3+4v_4} = p^{8v_1+10v_2+11v_3+7v_4} t^{4v_1+6v_2+6v_3+4v_4}$$

We have two sub cases

### 3.1 Sub case 3.1

$$-v_2 - v_1 \geq -v_4 \Rightarrow v_2 + v_1 \leq v_4$$

There is no phase transition here, since  $v_{11} \geq -v_2 - v_1 \geq -v_4$

For  $v_{11} \geq -v_2$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho)$ .

For  $-v_2 - v_1 \leq v_{11} \leq -v_2 - 1$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2 - v_1}^{-v_2 - 1} p^{v_2}\rho(1 + v_2\rho) = p^{v_2}v_1\rho + p^{v_2}v_1v_2\rho^2$ .

So we have  $\sum_{v_{11} \geq -v_2 - v_1} p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho + v_1\rho + v_1v_2\rho^2)$ .

We have several arrangements,

$$1. v_4 \overset{d}{\geq} v_1 + v_2 \overset{c}{\geq} v_3 \overset{b}{>} v_2 \overset{a}{>} v_1 \overset{e}{\geq} 0$$

$$\text{But } v_1 + v_2 - c - b = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = b + c$$

Thus, we substitute  $v_1 \rightarrow b + c$ ,  $v_2 \rightarrow a + b + c$ ,  $v_3 \rightarrow a + 2b + c$ ,  $v_4 \rightarrow a + 2b + 2c + d$ , and compute series with the new variables as indices, where  $c, d \geq 0$  and  $a, b \geq 1$ .

$$2. v_4 \overset{d}{\geq} v_1 + v_2 \overset{a}{\geq} v_2 \overset{c}{\geq} v_3 \overset{b}{>} v_1 \overset{a}{\geq} 0$$

We substitute  $v_1 \rightarrow a$ ,  $v_2 \rightarrow a + b + c$ ,  $v_3 \rightarrow a + b$ ,  $v_4 \rightarrow 2a + b + c + d$ , and compute series with the new variables as indices, where  $a, c, d \geq 0$  and  $b \geq 1$ .

$$3. v_4 \overset{d}{\geq} v_1 + v_2 \overset{c}{\geq} v_3 \overset{b}{>} v_1 \overset{a}{\geq} v_2 \overset{e}{\geq} 0$$

$$\text{But } v_1 + v_2 - c - b = v_1 \Rightarrow v_2 = v_1 + v_2 - v_1 = b + c$$

We substitute  $v_1 \rightarrow a + b + c$ ,  $v_2 \rightarrow b + c$ ,  $v_3 \rightarrow a + 2b + c$ ,  $v_4 \rightarrow a + 2b + 2c + d$ , and compute series with the new variables as indices, where  $a, c, d \geq 0$  and  $b \geq 1$ .

$$4. v_4 \overset{d}{>} v_3 \overset{c}{\geq} v_1 + v_2 \overset{a}{\geq} v_1 \overset{b}{\geq} v_2 \overset{a}{\geq} 0$$

We substitute  $v_1 \rightarrow a + b$ ,  $v_2 \rightarrow a$ ,  $v_3 \rightarrow 2a + b + c$ ,  $v_4 \rightarrow 2a + b + c + d$ , and compute series with the new variables as indices, where  $a, b, c \geq 0$  and  $d \geq 1$ .

$$5. v_4 \overset{d}{>} v_3 \overset{c}{>} v_1 + v_2 \overset{a}{\geq} v_2 \overset{b}{>} v_1 \overset{a}{\geq} 0$$

We substitute  $v_1 \rightarrow a$ ,  $v_2 \rightarrow a + b$ ,  $v_3 \rightarrow 2a + b + c$ ,  $v_4 \rightarrow 2a + b + c + d$ , and compute series with the new variables as indices, where  $a \geq 0$  and  $b, c, d \geq 1$ .

$$6. v_3 \overset{d}{\geq} v_4 \overset{c}{>} v_1 + v_2 \overset{a}{\geq} v_2 \overset{b}{>} v_1 \overset{a}{\geq} 0$$

We substitute  $v_1 \rightarrow a$ ,  $v_2 \rightarrow a + b$ ,  $v_3 \rightarrow 2a + b + c + d$ ,  $v_4 \rightarrow 2a + b + c$ , and compute series with the new variables as indices, where  $a, d \geq 0$  and  $b, c \geq 1$ .

$$7. v_3 \stackrel{d}{\geq} v_4 \stackrel{c}{>} v_1 + v_2 \stackrel{a}{\geq} v_1 \stackrel{b}{\geq} v_2 \stackrel{a}{\geq} 0$$

We substitute  $v_1 \rightarrow a + b$ ,  $v_2 \rightarrow a$ ,  $v_3 \rightarrow 2a + b + c + d$ ,  $v_4 \rightarrow 2a + b + c$ , and compute series with the new variables as indices, where  $a, b, d \geq 0$  and  $c \geq 1$ .

### 3.2 Sub case 3.2

$$-v_2 - v_1 < -v_4 \Rightarrow v_2 + v_1 > v_4$$

For  $v_{11} \geq -v_2$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho)$ .

For  $-v_4 \leq v_{11} \leq -v_2 - 1$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_4}^{-v_2-1} p^{v_2}\rho(1 + v_2\rho) = p^{v_2}(v_4 - v_2)\rho + p^{v_2}(v_4 - v_2)v_2\rho^2$ .

For  $-v_2 - v_1 \leq v_{11} \leq -v_4 - 1$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_1}^{-v_4-1} p^{v_2}\rho(1 + v_2\rho) = \sum_{i=1}^{v_1+v_2-v_4} p^{v_2}\rho(1 + (v_2-i)\rho) = p^{v_2}(v_1 + v_2 - v_4)\rho + p^{v_2}(v_1 + v_2 - v_4)v_2\rho^2 - p^{v_2}\binom{v_1+v_2-v_4+1}{2}\rho^2$ .

So we have  $\sum_{v_{11} \geq -v_2 - v_1} p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho + (v_4 - v_2)\rho + (v_4 - v_2)v_2\rho^2 + (v_1 + v_2 - v_4)\rho + (v_1 + v_2 - v_4)v_2\rho^2 - \frac{(v_1+v_2-v_4+1)(v_1+v_2-v_4)}{2}\rho^2) = p^{v_2}(1 + v_1\rho + v_2\rho + v_1v_2\rho^2 - \frac{v_1^2}{2}\rho^2 - v_2v_1\rho^2 + v_4v_1\rho^2 - \frac{v_2^2}{2}\rho^2 + v_4v_2\rho^2 - \frac{v_4^2}{2}\rho^2 - \frac{v_1}{2}\rho^2 - \frac{v_2}{2}\rho^2 + \frac{v_4}{2}\rho^2)$

We have several arrangements,

$$1. v_1 + v_2 \stackrel{d}{>} v_4 \stackrel{c}{>} v_3 \stackrel{b}{>} v_2 \stackrel{a}{>} v_1 \stackrel{e}{\geq} 0$$

But  $v_1 + v_2 - d - c - b = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = b + c + d$

Thus, we substitute  $v_1 \rightarrow b + c + d$ ,  $v_2 \rightarrow a + b + c + d$ ,  $v_3 \rightarrow a + 2b + c + d$ ,  $v_4 \rightarrow a + 2b + 2c + d$ , and compute series with the new variables as indices, where  $a, b, c, d \geq 1$

$$2. v_1 + v_2 \stackrel{d}{>} v_4 \stackrel{c}{>} v_2 \stackrel{b}{\geq} v_3 \stackrel{a}{>} v_1 \stackrel{e}{\geq} 0$$

But  $v_1 + v_2 - d - c = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = c + d$

Thus, we substitute  $v_1 \rightarrow c + d$ ,  $v_2 \rightarrow a + b + c + d$ ,  $v_3 \rightarrow a + c + d$ ,  $v_4 \rightarrow a + b + 2c + d$ , and compute series with the new variables as indices, where  $b \geq 0$  and  $a, c, d \geq 1$

$$3. v_1 + v_2 \stackrel{d}{>} v_4 \stackrel{c}{>} v_3 \stackrel{b}{>} v_1 \stackrel{a}{\geq} v_2 \stackrel{e}{\geq} 0$$

But  $v_1 + v_2 - d - c - b = v_1 \Rightarrow v_2 = v_1 + v_2 - v_1 = b + c + d$

Thus, we substitute  $v_1 \rightarrow a + b + c + d$ ,  $v_2 \rightarrow b + c + d$ ,  $v_3 \rightarrow a + 2b + c + d$ ,  $v_4 \rightarrow a + 2b + 2c + d$ , and compute series with the new variables as indices, where  $a \geq 0$  and  $b, c, d \geq 1$

$$4. v_1 + v_2 \stackrel{d}{>} v_3 \stackrel{c}{\geq} v_4 \stackrel{b}{>} v_2 \stackrel{a}{>} v_1 \stackrel{e}{\geq} 0$$

But  $v_1 + v_2 - d - c - b = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = b + c + d$

Thus, we substitute  $v_1 \rightarrow b + c + d$ ,  $v_2 \rightarrow a + b + c + d$ ,  $v_3 \rightarrow a + 2b + 2c + d$ ,  $v_4 \rightarrow a + 2b + c + d$ , and compute series with the new variables as indices, where  $c \geq 0$  and  $a, b, d \geq 1$

$$5. \quad v_1 + v_2 \stackrel{d}{>} v_3 \stackrel{c}{\geq} v_4 \stackrel{b}{>} v_1 \stackrel{a}{\geq} v_2 \stackrel{e}{\geq} 0$$

But  $v_1 + v_2 - d - c - b = v_1 \Rightarrow v_2 = v_1 + v_2 - v_1 = b + c + d$

Thus, we substitute  $v_1 \rightarrow a + b + c + d$ ,  $v_2 \rightarrow b + c + d$ ,  $v_3 \rightarrow a + 2b + 2c + d$ ,  $v_4 \rightarrow a + 2b + c + d$ , and compute series with the new variables as indices, where  $a, c \geq 0$  and  $b, d \geq 1$

$$6. \quad v_3 \stackrel{d}{\geq} v_1 + v_2 \stackrel{c}{>} v_4 \stackrel{b}{>} v_2 \stackrel{a}{>} v_1 \stackrel{e}{\geq} 0$$

But  $v_1 + v_2 - c - b = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = b + c$

Thus, we substitute  $v_1 \rightarrow b + c$ ,  $v_2 \rightarrow a + b + c$ ,  $v_3 \rightarrow a + 2b + 2c + d$ ,  $v_4 \rightarrow a + 2b + c$ , and compute series with the new variables as indices, where  $d \geq 0$  and  $a, b, c \geq 1$

$$7. \quad v_3 \stackrel{d}{\geq} v_1 + v_2 \stackrel{c}{>} v_4 \stackrel{b}{>} v_1 \stackrel{a}{\geq} v_2 \stackrel{e}{\geq} 0$$

But  $v_1 + v_2 - c - b = v_1 \Rightarrow v_2 = v_1 + v_2 - v_1 = b + c$

Thus, we substitute  $v_1 \rightarrow a + b + c$ ,  $v_2 \rightarrow b + c$ ,  $v_3 \rightarrow a + 2b + 2c + d$ ,  $v_4 \rightarrow a + 2b + c$ , and compute series with the new variables as indices, where  $a, d \geq 0$  and  $b, c \geq 1$

## 4 Case 4

$$v_4 > v_1 \geq v_3 \geq 0$$

$$v_4 > v_2 \geq 0$$

$$\Rightarrow v_{11} \geq -v_2 - v_3$$

$$\Rightarrow \alpha := \min\{v_2, v_2 + v_{11} + v_4\} = \begin{cases} v_2, & v_{11} \geq -v_4 \\ v_2 - 1, v_2 - 2, \dots, & v_{11} < -v_4 \end{cases}$$

$$\Rightarrow f(p, t) = p^{7v_1 + 10v_2 + 11v_3 + 7v_4 + v_1} t^{4v_1 + 6v_2 + 6v_3 + 4v_4} = p^{8v_1 + 10v_2 + 11v_3 + 7v_4} t^{4v_1 + 6v_2 + 6v_3 + 4v_4}$$

We have two sub cases

### 4.1 Sub case 4.1

$$-v_2 - v_3 \geq -v_4 \Rightarrow v_2 + v_3 \leq v_4$$

There is no phase transition here, since  $v_{11} \geq -v_2 - v_3 \geq -v_4$

For  $v_{11} \geq -v_2$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho)$ .

For  $-v_2 - v_3 \leq v_{11} \leq -v_2 - 1$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2 - v_3}^{-v_2 - 1} p^{v_2} \rho(1 + v_2\rho) = p^{v_2} v_3 \rho(1 + v_2\rho) = p^{v_2} v_3 \rho + p^{v_2} v_2 v_3 \rho^2$ .

So we have  $\sum_{v_{11} \geq -v_2 - v_3} p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho + v_3\rho + v_2 v_3 \rho^2)$ .

We have several arrangements,

$$1. v_4 \stackrel{d}{\geq} v_2 + v_3 \stackrel{c}{\geq} v_1 \stackrel{b}{\geq} v_3 \stackrel{a}{>} v_2 \stackrel{e}{\geq} 0$$

But  $v_2 + v_3 - c - b = v_3 \Rightarrow v_2 = v_2 + v_3 - v_3 = b + c$

Thus, we substitute  $v_1 \rightarrow a + 2b + c$ ,  $v_2 \rightarrow b + c$ ,  $v_3 \rightarrow a + b + c$ ,  $v_4 \rightarrow a + 2b + 2c + d$ , and compute series with the new variables as indices, where  $b, c, d \geq 0$  and  $a \geq 1$

$$2. v_4 \stackrel{d}{\geq} v_2 + v_3 \stackrel{c}{\geq} v_1 \stackrel{b}{\geq} v_2 \stackrel{a}{\geq} v_3 \stackrel{e}{\geq} 0$$

But  $v_2 + v_3 - c - b = v_2 \Rightarrow v_3 = v_2 + v_3 - v_2 = b + c$

Thus, we substitute  $v_1 \rightarrow a + 2b + c$ ,  $v_2 \rightarrow a + b + c$ ,  $v_3 \rightarrow b + c$ ,  $v_4 \rightarrow a + 2b + 2c + d$ , and compute series with the new variables as indices, where  $a, b, c, d \geq 0$ .

$$3. v_4 \stackrel{d}{\geq} v_2 + v_3 \stackrel{a}{\geq} v_2 \stackrel{c}{>} v_1 \stackrel{b}{\geq} v_3 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow a + b$ ,  $v_2 \rightarrow a + b + c$ ,  $v_3 \rightarrow a$ ,  $v_4 \rightarrow 2a + b + c + d$ , and compute series with the new variables as indices, where  $a, b, d \geq 0$  and  $c \geq 1$ .

$$4. v_4 \stackrel{d}{>} v_1 \stackrel{c}{\geq} v_2 + v_3 \stackrel{a}{\geq} v_2 \stackrel{b}{\geq} v_3 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow 2a + b + c$ ,  $v_2 \rightarrow a + b$ ,  $v_3 \rightarrow a$ ,  $v_4 \rightarrow 2a + b + c + d$ , and compute series with the new variables as indices, where  $a, b, c \geq 0$  and  $d \geq 1$ .

$$5. v_4 \stackrel{d}{>} v_1 \stackrel{c}{\geq} v_2 + v_3 \stackrel{a}{\geq} v_3 \stackrel{b}{>} v_2 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow 2a + b + c$ ,  $v_2 \rightarrow a$ ,  $v_3 \rightarrow a + b$ ,  $v_4 \rightarrow 2a + b + c + d$ , and compute series with the new variables as indices, where  $a, c \geq 0$  and  $b, d \geq 1$ .

## 4.2 Sub case 4.2

$$-v_2 - v_3 < -v_4 \Rightarrow v_2 + v_3 > v_4$$

For  $v_{11} \geq -v_2$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho)$ .

For  $-v_4 \leq v_{11} \leq -v_2 - 1$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_4}^{-v_2-1} p^{v_2}\rho(1 + v_2\rho) = p^{v_2}(v_4 - v_2)\rho(1 + v_2\rho) = p^{v_2}(v_4 - v_2)\rho + p^{v_2}v_2(v_4 - v_2)\rho^2$ .

For  $-v_2 - v_3 \leq v_{11} \leq -v_4 - 1$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_3}^{-v_4-1} p^{v_2}\rho(1 + \alpha\rho) = \sum_{i=-1}^{v_2+v_3-v_4} p^{v_2}\rho(1 + (v_2 - i)\rho) = p^{v_2}(v_2 + v_3 - v_4)\rho + (v_2 + v_3 - v_4)v_2\rho^2 - \binom{v_2+v_3-v_4+1}{2}\rho^2$ .

So we have  $\sum_{v_{11} \geq -v_2 - v_3} p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho + (v_4 - v_2))\rho + v_2(v_4 - v_2)\rho^2 + (v_2 + v_3 - v_4)\rho + (v_2 + v_3 - v_4)v_2\rho^2 - \frac{(v_2+v_3-v_4+1)(v_2+v_3-v_4)}{2}\rho^2 = p^{v_2}(1 + v_2v_4\rho^2 + v_2\rho + v_3\rho + v_3v_2\rho^2 - v_4v_2\rho^2 - \frac{v_2^2}{2}\rho^2 - v_2v_3\rho^2 + v_2v_4\rho^2 - \frac{v_3^2}{2}\rho^2 + v_3v_4\rho^2 - \frac{v_4^2}{2}\rho^2 - \frac{v_2}{2}\rho^2 - \frac{v_3}{2}\rho^2 + \frac{v_4}{2}\rho^2)$ .

We have several arrangements,

$$1. v_2 + v_3 \stackrel{d}{>} v_4 \stackrel{c}{>} v_1 \stackrel{b}{\geq} v_2 \stackrel{a}{\geq} v_3 \stackrel{e}{\geq} 0$$

$$\text{But } v_2 + v_3 - d - c - b = v_2 \Rightarrow v_3 = v_2 + v_3 - v_2 = b + c + d$$

Thus, we substitute  $v_1 \rightarrow a + 2b + c + d$ ,  $v_2 \rightarrow a + b + c + d$ ,  $v_3 \rightarrow b + c + d$ ,  $v_4 \rightarrow a + 2b + 2c + d$ , and compute series with the new variables as indices, where  $a, b \geq 0$  and  $c, d \geq 1$

$$2. v_2 + v_3 \stackrel{d}{>} v_4 \stackrel{c}{>} v_1 \stackrel{b}{\geq} v_3 \stackrel{a}{>} v_2 \stackrel{e}{\geq} 0$$

$$\text{But } v_2 + v_3 - d - c - b = v_3 \Rightarrow v_2 = v_2 + v_3 - v_3 = b + c + d$$

Thus, we substitute  $v_1 \rightarrow a + 2b + c + d$ ,  $v_2 \rightarrow b + c + d$ ,  $v_3 \rightarrow a + b + c + d$ ,  $v_4 \rightarrow a + 2b + 2c + d$ , and compute series with the new variables as indices, where  $b \geq 0$  and  $a, c, d \geq 1$

$$3. v_2 + v_3 \stackrel{d}{>} v_4 \stackrel{c}{>} v_2 \stackrel{b}{>} v_1 \stackrel{a}{\geq} v_3 \stackrel{e}{\geq} 0$$

$$\text{But } v_2 + v_3 - d - c = v_2 \Rightarrow v_3 = v_2 + v_3 - v_2 = c + d$$

Thus, we substitute  $v_1 \rightarrow a + c + d$ ,  $v_2 \rightarrow a + b + c + d$ ,  $v_3 \rightarrow c + d$ ,  $v_4 \rightarrow a + b + 2c + d$ , and compute series with the new variables as indices, where  $a \geq 0$  and  $b, c, d \geq 1$

## 5 Case 5

$$v_2 \geq v_4 > v_1 > v_3 \geq 0$$

$$\Rightarrow v_{11} \geq -v_2 - v_3$$

$$\Rightarrow \alpha := \min\{v_4, v_2 + v_{11} + v_4\} = \begin{cases} v_4, & v_{11} \geq -v_2 \\ v_4 - 1, v_4 - 2, \dots, & v_{11} < -v_2 \end{cases}$$

$$\Rightarrow f(p, t) = p^{7v_1 + 10v_2 + 11v_3 + 7v_4 + v_1} t^{4v_1 + 6v_2 + 6v_3 + 4v_4} = p^{8v_1 + 10v_2 + 11v_3 + 7v_4} t^{4v_1 + 6v_2 + 6v_3 + 4v_4}$$

$$\text{For } v_{11} \geq -v_2, \text{ we have } p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho).$$

$$\text{For } -v_2 - v_3 \leq v_{11} \leq -v_2 - 1, \text{ we have } p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2 - v_3}^{-v_2 - 1} p^{v_2} \rho(1 + \alpha\rho) = \sum_{i=1}^{v_3} p^{v_2} \rho(1 + (v_4 - i)\rho) = p^{v_2}(v_3\rho + v_3v_4\rho^2 - \binom{v_3+1}{2}\rho^2).$$

$$\text{So we have } \sum_{v_{11} \geq -v_2 - v_3} p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho + v_3\rho + v_3v_4\rho^2 - \frac{(v_3+1)v_3}{2}\rho^2) = p^{v_2}(1 + v_4\rho + v_3\rho + v_3v_4\rho^2 - \frac{v_3^2}{2}\rho^2 - \frac{v_3}{2}\rho^2).$$

$$v_2 \geq v_4 \stackrel{d}{>} v_1 \stackrel{b}{\geq} v_3 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow a + b$ ,  $v_2 \rightarrow a + b + c + d$ ,  $v_3 \rightarrow a$ ,  $v_4 \rightarrow a + b + c$ , and compute series with the new variables as indices, where  $a, b, d \geq 0$  and  $c \geq 1$

## 6 Case 6

$$v_2 \geq v_4 > v_1 \geq 0$$

$$v_3 > v_1 \geq 0$$

$$\Rightarrow v_{11} \geq -v_2 - v_1$$



$$\begin{aligned}
&\Rightarrow \alpha := \min\{v_4, v_2 + v_{11} + v_4\} = \begin{cases} v_4, & v_{11} \geq -v_2 \\ v_4 - 1, v_4 - 2, \dots, & v_{11} < -v_2 \end{cases} \\
&\Rightarrow f(p, t) = p^{7v_1+10v_2+11v_3+7v_4+v_1} t^{4v_1+6v_2+6v_3+4v_4} = p^{8v_1+10v_2+11v_3+7v_4} t^{4v_1+6v_2+6v_3+4v_4} \\
&\text{For } v_{11} \geq -v_2, \text{ we have } p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho). \\
&\text{For } -v_2 - v_1 \leq v_{11} \leq -v_2 - 1, \text{ we have } p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_1}^{-v_2-1} p^{v_2}\rho(1 + \\
&\alpha\rho) = \sum_{i=1}^{v_1} p^{v_2}\rho(1 + (v_4 - i)\rho) = p^{v_2}(v_1\rho + v_1v_4\rho^2 - \binom{v_1+1}{2}\rho^2). \\
&\text{So we have } \sum_{v_{11} \geq -v_2 - v_1} p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho + v_1\rho + v_1v_4\rho^2 - \\
&\frac{(v_1+1)v_1}{2}\rho^2) = p^{v_2}(1 + v_4\rho + v_1\rho + v_1v_4\rho^2 - \frac{v_1^2}{2}\rho^2 - \frac{v_1}{2}\rho^2). \\
&\text{We have several arrangements,}
\end{aligned}$$

$$1. v_2 \stackrel{d}{\geq} v_4 \stackrel{c}{>} v_3 \stackrel{b}{>} v_1 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow a$ ,  $v_2 \rightarrow a + b + c + d$ ,  $v_3 \rightarrow a + b$ ,  $v_4 \rightarrow a + b + c$ , and compute series with the new variables as indices, where  $a, d \geq 0$  and  $b, c \geq 1$

$$2. v_2 \stackrel{d}{\geq} v_3 \stackrel{c}{\geq} v_4 \stackrel{b}{>} v_1 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow a$ ,  $v_2 \rightarrow a + b + c + d$ ,  $v_3 \rightarrow a + b + c$ ,  $v_4 \rightarrow a + b$ , and compute series with the new variables as indices, where  $a, c, d \geq 0$  and  $b \geq 1$

$$3. v_3 \stackrel{d}{>} v_2 \stackrel{c}{\geq} v_4 \stackrel{b}{>} v_1 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow a$ ,  $v_2 \rightarrow a + b + c$ ,  $v_3 \rightarrow a + b + c + d$ ,  $v_4 \rightarrow a + b$ , and compute series with the new variables as indices, where  $a, c \geq 0$  and  $b, d \geq 1$

## 7 Case 7

$$v_2 \geq v_4 \geq 0$$

$$v_3 > v_1 \geq v_4 \geq 0$$

$$\Rightarrow v_{11} \geq -v_2 - v_1$$

$$\Rightarrow f(p, t) = p^{7v_1+10v_2+11v_3+7v_4+v_4} t^{4v_1+6v_2+6v_3+4v_4} = p^{7v_1+10v_2+11v_3+8v_4} t^{4v_1+6v_2+6v_3+4v_4}$$

$$\Rightarrow \alpha := \min\{v_4, v_2 + v_{11} + v_1\} = \begin{cases} v_4, & v_{11} \geq v_4 - v_1 - v_2 \\ v_4 - 1, v_4 - 2, \dots, & v_{11} < v_4 - v_1 - v_2 \end{cases}$$

There are no two sub cases here, because the phase transition occurs at  $v_{11} = v_4 - v_1 - v_2 - 1$ , thus

if  $v_4 = 0$  then the transition would be at  $v_{11} = -v_2 - v_1 - 1 < -v_2 - v_1$ , which is a contradiction.

For  $v_4 \geq 1$  the transition would be at  $a_{11} \geq -v_2 - v_1$ , which is always true for this case.

$$\text{For } v_{11} \geq -v_2, \text{ we have } p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho).$$

$$\begin{aligned}
&\text{For } v_4 - v_1 - v_2 \leq v_{11} \leq -v_2 - 1, \text{ we have } p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{v_4-v_1-v_2}^{-v_2-1} p^{v_2}\rho(1 + \\
&v_4\rho) = \sum_{i=1}^{v_1-v_4} p^{v_2}\rho(1 + v_4\rho) = p^{v_2}(v_1 - v_4)\rho(1 + v_4\rho) = p^{v_2}(v_1 - v_4)\rho + p^{v_2}(v_1 - \\
&v_4)v_4\rho^2.
\end{aligned}$$

For  $-v_1 - v_2 \leq v_{11} \leq v_4 - v_1 - v_2 - 1$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_1-v_2}^{v_4-v_1-v_2-1} p^{v_2}\rho(1 + \alpha\rho) = \sum_{i=1}^{v_4} p^{v_2}\rho(1 + (v_4 - i)\rho) = p^{v_2}v_4\rho + p^{v_2}v_4^2\rho^2 - p^{v_2}\binom{v_4+1}{2}\rho^2$ .

So we have  $\sum_{v_{11} \geq -v_2-v_1} p^{\min}\mu(a_{11}) = p^{v_2}(1 + v_4\rho + (v_4 - v_1)\rho + (v_4 - v_1)v_4\rho^2 + v_4\rho + v_4^2\rho^2 - \frac{v_4(v_4+1)}{2}\rho^2) = p^{v_2}(1 + 3v_4\rho - v_1\rho + 2v_4^2\rho^2 - v_1v_4\rho^2 - \frac{v_4^2}{2}\rho^2 + \frac{v_4}{2}\rho^2)$

We have several arrangements,

$$1. v_2 \stackrel{d}{\geq} v_3 \stackrel{c}{>} v_1 \stackrel{b}{\geq} v_4 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow a + b$ ,  $v_2 \rightarrow a + b + c + d$ ,  $v_3 \rightarrow a + b + c$ ,  $v_4 \rightarrow a$ , and compute series with the new variables as indices, where  $a, b, d \geq 0$  and  $c \geq 1$

$$2. v_3 \stackrel{d}{>} v_2 \stackrel{c}{>} v_1 \stackrel{b}{\geq} v_4 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow a + b$ ,  $v_2 \rightarrow a + b + c$ ,  $v_3 \rightarrow a + b + c + d$ ,  $v_4 \rightarrow a$ , and compute series with the new variables as indices, where  $a, b \geq 0$  and  $c, d \geq 1$

$$3. v_3 \stackrel{d}{>} v_1 \stackrel{c}{\geq} v_2 \stackrel{b}{\geq} v_4 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow a + b + c$ ,  $v_2 \rightarrow a + b$ ,  $v_3 \rightarrow a + b + c + d$ ,  $v_4 \rightarrow a$ , and compute series with the new variables as indices, where  $a, b, c \geq 0$  and  $d \geq 1$

## 8 Case 8

$$v_2 \geq v_4 \geq 0$$

$$v_1 \geq v_4 \geq 0$$

$$v_1 \geq v_3 \geq 0$$

$$\Rightarrow v_{11} \geq -v_2 - v_3$$

$$\Rightarrow \alpha := \min\{v_4, v_2 + v_{11} + v_1\} = \begin{cases} v_4, & v_{11} \geq v_4 - v_1 - v_2 \\ v_4 - 1, v_4 - 2, \dots, & v_{11} < v_4 - v_1 - v_2 \end{cases}$$

$$\Rightarrow f(p, t) = p^{7v_1+10v_2+11v_3+7v_4+v_4} t^{4v_1+6v_2+6v_3+4v_4} = p^{7v_1+10v_2+11v_3+8v_4} t^{4v_1+6v_2+6v_3+4v_4}$$

There are two sub cases here

### 8.1 Sub case 8.1

$$-v_2 - v_3 \geq v_4 - v_1 - v_2 \Rightarrow -v_3 \geq v_4 - v_1 \Rightarrow v_1 \geq v_3 + v_4$$

For  $v_{11} \geq -v_2$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho)$ .

For  $-v_2 - v_3 \leq v_{11} \leq -v_2 - 1$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_3}^{-v_2-1} p^{v_2}\rho(1 + v_4\rho) = \sum_{i=1}^{v_3} p^{v_2}\rho(1 + v_4\rho) = p^{v_2}(v_3\rho + v_3v_4\rho^2)$ .

So we have  $\sum_{v_{11} \geq -v_2-v_3} p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho + v_3\rho + v_3v_4\rho^2)$ .

We have several arrangements,

$$1. v_1 \stackrel{d}{\geq} v_3 + v_4 \stackrel{a}{\geq} v_3 \stackrel{c}{>} v_2 \stackrel{b}{\geq} v_4 \stackrel{a}{\geq} 0$$

But  $e = v_2 + v_3 - v_3 = v_2 = a + b$ .

Thus, we substitute  $v_1 \rightarrow 2a + b + c + d$ ,  $v_2 \rightarrow a + b$ ,  $v_3 \rightarrow a + b + c$ ,  $v_4 \rightarrow a$ , and compute series with the new variables as indices, where  $a, b, d \geq 0$  and  $c \geq 1$

$$2. v_1 \stackrel{d}{\geq} v_3 + v_4 \stackrel{c}{\geq} v_2 \stackrel{b}{\geq} v_3 \stackrel{a}{\geq} v_4 \stackrel{e}{\geq} 0$$

But  $v_3 = v_3 + v_4 - c - b \Rightarrow v_4 = b + c$

Thus, we substitute  $v_1 \rightarrow a + 2b + 2c + d$ ,  $v_2 \rightarrow a + 2b + c$ ,  $v_3 \rightarrow a + b + c$ ,  $v_4 \rightarrow b + c$ , and compute series with the new variables as indices, where  $a, b, c, d \geq 0$ .

$$3. v_1 \stackrel{d}{\geq} v_3 + v_4 \stackrel{c}{\geq} v_2 \stackrel{b}{\geq} v_4 \stackrel{a}{>} v_3 \stackrel{e}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow a + 2b + 2c + d$ ,  $v_2 \rightarrow a + 2b + c$ ,  $v_3 \rightarrow a + b + c$ ,  $v_4 \rightarrow b + c$ , and compute series with the new variables as indices, where  $b, c, d \geq 0$  and  $a \geq 1$

$$4. v_1 \stackrel{d}{\geq} v_2 \stackrel{c}{>} v_3 + v_4 \stackrel{a}{\geq} v_4 \stackrel{b}{>} v_3 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow 2a + b + c + d$ ,  $v_2 \rightarrow 2a + b + c$ ,  $v_3 \rightarrow a$ ,  $v_4 \rightarrow a + b$ , and compute series with the new variables as indices, where  $a, d \geq 0$  and  $b, c \geq 1$

$$5. v_1 \stackrel{d}{\geq} v_2 \stackrel{c}{>} v_3 + v_4 \stackrel{a}{\geq} v_3 \stackrel{b}{\geq} v_4 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow 2a + b + c + d$ ,  $v_2 \rightarrow 2a + b + c$ ,  $v_3 \rightarrow a + b$ ,  $v_4 \rightarrow a$ , and compute series with the new variables as indices, where  $a, b, d \geq 0$  and  $c \geq 1$

$$6. v_2 \stackrel{d}{>} v_1 \stackrel{c}{\geq} v_3 + v_4 \stackrel{a}{\geq} v_3 \stackrel{b}{\geq} v_4 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow 2a + b + c$ ,  $v_2 \rightarrow 2a + b + c + d$ ,  $v_3 \rightarrow a + b$ ,  $v_4 \rightarrow a$ , and compute series with the new variables as indices, where  $a, b, c \geq 0$  and  $d \geq 1$

$$7. v_2 \stackrel{d}{>} v_1 \stackrel{c}{\geq} v_3 + v_4 \stackrel{a}{\geq} v_4 \stackrel{b}{>} v_3 \stackrel{a}{\geq} 0$$

Thus, we substitute  $v_1 \rightarrow 2a + b + c$ ,  $v_2 \rightarrow 2a + b + c + d$ ,  $v_3 \rightarrow a$ ,  $v_4 \rightarrow a + b$ , and compute series with the new variables as indices, where  $a, c \geq 0$  and  $b, d \geq 1$

## 8.2 Sub case 8.2

$$-v_2 - v_3 < v_4 - v_1 - v_2 \Rightarrow -v_3 < v_4 - v_1 \Rightarrow v_1 < v_3 + v_4$$

For  $v_{11} \geq -v_2$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho)$ .

For  $v_4 - v_1 - v_2 \leq v_{11} \leq -v_2 - 1$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{v_4 - v_1 - v_2}^{-v_2 - 1} p^{v_2} \rho(1 + v_4\rho) = p^{v_2}((v_1 - v_4)\rho + (v_1 - v_4)v_4\rho^2)$ .

For  $-v_2 - v_3 \leq v_{11} \leq v_4 - v_1 - v_2 - 1$ , we have  $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2 - v_3}^{v_4 - v_1 - v_2 - 1} p^{v_2} \rho(1 + \alpha\rho) = \sum_{i=1}^{v_3 + v_4 - v_1} p^{v_2} \rho(1 + (v_4 - i)\rho) = p^{v_2}((v_3 + v_4 - v_1)\rho + (v_3 + v_4 - v_1)v_4\rho^2 - \binom{v_3 + v_4 - v_1 + 1}{2}\rho^2)$ .

So we have  $\sum_{v_{11} \geq -v_2 - v_3} \rho(1 + \alpha\rho) = p^{v_2}(1 + v_4\rho + (v_1 - v_4)\rho + (v_1 - v_4)v_4\rho^2 + (v_3 + v_4 - v_1)\rho + (v_3 + v_4 - v_1)v_4\rho^2 - \frac{(v_3 + v_4 - v_1 + 1)(v_3 + v_4 - v_1)}{2}\rho^2) = p^{v_2}(1 + v_3\rho + v_4\rho - \frac{v_3^2}{2}\rho^2 + v_1v_3\rho^2 - \frac{v_4^2}{2} + v_1v_4\rho^2 - \frac{v_1^2}{2} - v_3\rho^2 - v_4\rho^2 + v_1\rho^2)$ .

We have several arrangements,

$$1. v_3 + v_4 \overset{d}{>} v_1 \overset{c}{\geq} v_3 \overset{b}{>} v_2 \overset{a}{\geq} v_4 \overset{e}{\geq} 0$$

But  $v_3 + v_4 - d - c = v_3 \Rightarrow v_4 = c + d$ .

Thus, we substitute  $v_1 \rightarrow a + b + 2c + d$ ,  $v_2 \rightarrow a + c + d$ ,  $v_3 \rightarrow a + b + c + d$ ,  $v_4 \rightarrow c + d$ , and compute series with the new variables as indices, where  $a, c \geq 0$  and  $b, d \geq 1$

$$2. v_3 + v_4 \overset{d}{>} v_1 \overset{c}{\geq} v_2 \overset{b}{\geq} v_3 \overset{a}{\geq} v_4 \overset{e}{\geq} 0$$

But  $v_3 + v_4 - d - c - b = v_3 \Rightarrow v_4 = b + c + d$ .

Thus, we substitute  $v_1 \rightarrow a + 2b + 2c + d$ ,  $v_2 \rightarrow a + 2b + c + d$ ,  $v_3 \rightarrow a + b + c + d$ ,  $v_4 \rightarrow b + c + d$ , and compute series with the new variables as indices, where  $a, b, c \geq 0$  and  $d \geq 1$ .

$$3. v_3 + v_4 \overset{d}{>} v_1 \overset{c}{\geq} v_2 \overset{b}{\geq} v_4 \overset{a}{>} v_3 \overset{e}{\geq} 0$$

But  $v_3 + v_4 - d - c - b = v_4 \Rightarrow v_3 = b + c + d$ .

Thus, we substitute  $v_1 \rightarrow a + 2b + 2c + d$ ,  $v_2 \rightarrow a + 2b + c + d$ ,  $v_3 \rightarrow b + c + d$ ,  $v_4 \rightarrow a + b + c + d$ , and compute series with the new variables as indices, where  $b, c \geq 0$  and  $a, d \geq 1$

$$4. v_3 + v_4 \overset{d}{\geq} v_2 \overset{c}{>} v_1 \overset{b}{\geq} v_4 \overset{a}{>} v_3 \overset{e}{\geq} 0$$

But  $v_3 + v_4 - d - c - b = v_4 \Rightarrow v_3 = b + c + d$ .

Thus, we substitute  $v_1 \rightarrow a + 2b + c + d$ ,  $v_2 \rightarrow a + 2b + 2c + d$ ,  $v_3 \rightarrow b + c + d$ ,  $v_4 \rightarrow a + b + c + d$ , and compute series with the new variables as indices, where  $b, d \geq 0$  and  $a, c \geq 1$

$$5. v_3 + v_4 \overset{d}{>} v_2 \overset{c}{>} v_1 \overset{b}{\geq} v_3 \overset{a}{\geq} v_4 \overset{e}{\geq} 0$$

But  $v_3 + v_4 - d - c - b = v_3 \Rightarrow v_4 = b + c + d$ .

Thus, we substitute  $v_1 \rightarrow a + 2b + c + d$ ,  $v_2 \rightarrow a + 2b + 2c + d$ ,  $v_3 \rightarrow a + b + c + d$ ,  $v_4 \rightarrow b + c + d$ , and compute series with the new variables as indices, where  $a, b \geq 0$  and  $c, d \geq 1$

$$6. v_2 \overset{d}{>} v_3 + v_4 \overset{c}{>} v_1 \overset{b}{\geq} v_4 \overset{a}{>} v_3 \overset{e}{\geq} 0$$

But  $v_3 + v_4 - c - b = v_4 \Rightarrow v_3 = b + c$ .

Thus, we substitute  $v_1 \rightarrow a + 2b + c$ ,  $v_2 \rightarrow a + 2b + 2c + d$ ,  $v_3 \rightarrow b + c$ ,  $v_4 \rightarrow a + b + c$ , and compute series with the new variables as indices, where  $b, d \geq 0$  and  $a, c \geq 1$

$$7. v_2 \overset{d}{\geq} v_3 + v_4 \overset{c}{>} v_1 \overset{b}{\geq} v_3 \overset{a}{\geq} v_4 \overset{e}{\geq} 0$$

But  $v_3 + v_4 - c - b = v_3 \Rightarrow v_4 = b + c$ .

Thus, we substitute  $v_1 \rightarrow a + 2b + c$ ,  $v_2 \rightarrow a + 2b + 2c + d$ ,  $v_3 \rightarrow a + b + c$ ,  $v_4 \rightarrow b + c$ , and compute series with the new variables as indices, where  $a, b, d \geq 0$  and  $c \geq 1$

Checking all the cases by the  $4!$  arrangements.

$$1. v_1 \geq v_2 \geq v_3 \geq v_4 \geq 0$$

$$v_1 \geq v_3 + v_4 \geq v_2 \geq v_3 \geq v_4 \geq 0 \quad (8.1.2)$$

$$v_1 \geq v_2 > v_3 + v_4 \geq v_3 \geq v_4 \geq 0 \quad (8.1.5)$$

$$v_3 + v_4 > v_1 \geq v_2 \geq v_3 \geq v_4 \geq 0 \quad (8.2.2)$$

$$2. v_1 \geq v_2 \geq v_4 > v_3 \geq 0$$

$$v_1 \geq v_3 + v_4 \geq v_2 \geq v_4 > v_3 \geq 0 \quad (8.1.3)$$

$$v_1 \geq v_2 > v_3 + v_4 \geq v_4 > v_3 \geq 0 \quad (8.1.4)$$

$$v_3 + v_4 > v_1 \geq v_2 \geq v_4 > v_3 \geq 0 \quad (8.2.3)$$

$$3. v_1 \geq v_3 > v_2 \geq v_4 \geq 0$$

$$v_1 \geq v_3 + v_4 \geq v_3 > v_2 \geq v_4 \geq 0 \quad (8.1.1)$$

$$v_3 + v_4 > v_1 \geq v_3 > v_2 \geq v_4 \geq 0 \quad (8.2.1)$$

$$4. v_1 \geq v_3 \geq v_4 > v_2 \geq 0$$

$$v_1 \geq v_2 + v_3 \geq v_3 \geq v_4 > v_2 \geq 0 \quad (2.1.1)$$

$$v_2 + v_3 > v_1 \geq v_3 \geq v_4 > v_2 \geq 0 \quad (2.2.1)$$

$$5. v_1 \geq v_4 > v_2 \geq v_3 \geq 0$$

$$v_1 \geq v_2 + v_3 \geq v_4 > v_2 \geq v_3 \geq 0 \quad (2.1.3)$$

$$v_1 \geq v_4 > v_2 + v_3 \geq v_2 \geq v_3 \geq 0 \quad (2.1.4)$$

$$v_2 + v_3 > v_1 \geq v_4 > v_2 \geq v_3 \geq 0 \quad (2.2.3)$$

$$6. v_1 \geq v_4 > v_3 > v_2 \geq 0$$

$$v_1 \geq v_2 + v_3 \geq v_4 > v_3 > v_2 \geq 0 \quad (2.1.2)$$

$$v_2 + v_3 > v_1 \geq v_4 > v_3 > v_2 \geq 0 \quad (2.2.2)$$

$$v_1 \geq v_4 > v_2 + v_3 \geq v_3 > v_2 \geq 0 \quad (2.1.5)$$

7.  $v_2 > v_1 \geq v_3 \geq v_4 \geq 0$   
 $v_2 > v_1 \geq v_3 + v_4 \geq v_3 \geq v_4 \geq 0$  (8.1.6)  
 $v_2 \geq v_3 + v_4 > v_1 \geq v_3 \geq v_4 \geq 0$  (8.2.7)  
 $v_3 + v_4 > v_2 > v_1 \geq v_3 \geq v_4 \geq 0$  (8.2.5)  
@@@CHECKED@@@
8.  $v_2 > v_1 \geq v_4 > v_3 \geq 0$   
 $v_2 > v_1 \geq v_3 + v_4 \geq v_4 > v_3 \geq 0$  (8.1.7)  
 $v_3 + v_4 \geq v_2 > v_1 \geq v_4 > v_3 \geq 0$  (8.2.4)  
 $v_2 > v_3 + v_4 > v_1 \geq v_4 > v_3 \geq 0$  (8.2.6)
9.  $v_2 \geq v_3 > v_1 \geq v_4 \geq 0$  (7.1)
10.  $v_2 \geq v_3 \geq v_4 > v_1 \geq 0$  (6.2)
11.  $v_2 \geq v_4 > v_1 \geq v_3 \geq 0$  (5)
12.  $v_2 \geq v_4 > v_3 > v_1 \geq 0$  (6.1)
13.  $v_3 > v_1 \geq v_2 \geq v_4 \geq 0$  (7.3)
14.  $v_3 > v_1 \geq v_4 > v_2 \geq 0$  (1)
15.  $v_3 > v_2 > v_1 \geq v_4 \geq 0$  (7.2)
16.  $v_3 > v_2 \geq v_4 > v_1 \geq 0$  (6.3)
17.  $v_3 \geq v_4 > v_1 \geq v_2 \geq 0$   
 $v_3 \geq v_4 > v_1 + v_2 \geq v_1 \geq v_2 \geq 0$  (3.1.7)  
 $v_1 + v_2 > v_3 \geq v_4 > v_1 \geq v_2 \geq 0$  (3.2.5)  
 $v_3 \geq v_1 + v_2 > v_4 > v_1 \geq v_2 \geq 0$  (3.2.7)
18.  $v_3 \geq v_4 > v_2 > v_1 \geq 0$   
 $v_3 \geq v_4 > v_1 + v_2 \geq v_2 > v_1 \geq 0$  (3.1.6)  
 $v_1 + v_2 > v_3 \geq v_4 > v_2 > v_1 \geq 0$  (3.2.4)  
 $v_3 \geq v_1 + v_2 > v_4 > v_2 > v_1 \geq 0$  (3.2.6)
19.  $v_4 > v_1 \geq v_2 \geq v_3 \geq 0$   
 $v_4 \geq v_2 + v_3 \geq v_1 \geq v_2 \geq v_3 \geq 0$  (4.1.2)  
 $v_4 > v_1 \geq v_2 + v_3 \geq v_2 \geq v_3 \geq 0$  (4.1.4)  
 $v_2 + v_3 > v_4 > v_2 \geq v_3 \geq 0$  (4.2.1)
20.  $v_4 > v_1 \geq v_3 > v_2 \geq 0$   
 $v_4 \geq v_2 + v_3 \geq v_1 \geq v_3 > v_2 \geq 0$  (4.1.1)  
 $v_4 > v_1 \geq v_2 + v_3 \geq v_3 > v_2 \geq 0$  (4.1.5)  
 $v_2 + v_3 > v_4 > v_1 \geq v_3 > v_2 \geq 0$  (4.2.2)

21.  $v_4 > v_2 > v_1 \geq v_3 \geq 0$   
 $v_4 \geq v_2 + v_3 \geq v_2 > v_1 \geq v_3 \geq 0$  (4.1.3)  
 $v_2 + v_3 > v_4 > v_2 > v_1 \geq v_3 \geq 0$  (4.2.3)
22.  $v_4 > v_2 \geq v_3 > v_1 \geq 0$   
 $v_4 \geq v_1 + v_2 \geq v_2 \geq v_3 > v_1 \geq 0$  (3.1.2)  
 $v_1 + v_2 > v_4 > v_2 \geq v_3 > v_1 \geq 0$  (3.2.2)
23.  $v_4 > v_3 > v_1 \geq v_2 \geq 0$   
 $v_4 \geq v_1 + v_2 \geq v_3 > v_1 \geq v_2 \geq 0$  (3.1.3)  
 $v_4 > v_3 \geq v_1 + v_2 \geq v_1 \geq v_2 \geq 0$  (3.1.4)  
 $v_1 + v_2 > v_4 > v_3 > v_1 \geq v_2 \geq 0$  (3.2.3)
24.  $v_4 > v_3 > v_2 > v_1 \geq 0$   
 $v_4 \geq v_1 + v_2 \geq v_3 > v_2 > v_1 \geq 0$  (3.1.1)  
 $v_4 > v_3 \geq v_1 + v_2 \geq v_2 > v_1 \geq 0$  (3.1.5)  
 $v_1 + v_2 > v_4 > v_3 > v_2 > v_1 \geq 0$  (3.2.1)