1. The group U_n

Proposition 1.1
Let $E_n = \{E_{i,j}\}_{i<j}$ be the set of all $n \times n$ matrices, $(e_{l,k})$,
where $a_{l,l} = 1, 1 \leq l \leq n$, and $a_{i,j} = 1, i < j$, and all other elements are zero. That is, $E_{i,j}$ has 1 only on the main diagonal, and in one element, anywhere above the main diagonal. Let A be any $n \times n$ matrix. Then,
Multiplying A by $E_{i,j}$ (from the left), $E_{i,j} \times A$, is operating as performing the elementary operation $R_i \leftarrow R_i + R_j$ on A

Proof. $A = (a_{l,k}), B = (b_{l,k}) = E_{i_j} \times A = (e_{l,k}) \times (a_{l,k})$
$b_{l,k} = \sum_{r=1}^{n} e_{l,r} \cdot a_{r,k}$
For all the rows, except for row i, $b_{l,k} = \sum_{r=1}^{n} e_{l,r} \cdot a_{r,k} = 0 + 0 + \cdots + e_{l,l} \cdot a_{l,k} + 0 + 0 + \cdots + 0 + 0 = 1 \cdot a_{l,k} = a_{l,k}$
For row i, $b_{i,k} = \sum_{r=1}^{n} e_{i,r} \cdot a_{r,k} = 0 + 0 + \cdots + e_{i,i} \cdot a_{i,k} + 0 + 0 + e_{i,j} \cdot a_{j,k} + 0 + 0 + \cdots + 0 + 0 = 1 \cdot a_{i,k} + 1 \cdot a_{j,k} = a_{i,k} + a_{j,k}$
This shows that the multiplication preserves the rows of A, except for row i, which becomes the sum of rows i, j □

Corollary 1.2
Let $E_{i,j} = (e_{l,k}), i < j \in E_n$, Then,
$E_{i,j}^{-1} = (a_{l,k})$, where $a_{l,l} = 1, 1 \leq l \leq n$, and $a_{i,j} = -1, i < j$, and all other elements are zero.

Proof. $(b_{l,k}) = E_{i,j} \times (a_{l,k})$
Multiplying $(a_{l,k})$ by $E_{i,j}$ from the left is operating on $(a_{l,k})$ as a row addition, $R_i \leftarrow R_i + R_j$, as seen above.
For all $1 \leq k \leq n, b_{i,k} = a_{i,k} + a_{j,k}$
But, the only element in row j that is not zero is $a_{j,j=1}$, so, $b_{i,j} = a_{i,j} + a_{j,j} = -1 + 1 = 0$, and, for all the other columns, $a_{j,k} = 0$, so $b_{i,i} = a_{i,i} + a_{j,i} = 1 + 0 = 1$, and $b_{i,k} = a_{i,k} + a_{j,k} = 0 + 0 = 0$, which means that $(b_{l,k}) = I_n$
Easy to verify that also $(a_{l,k}) \times E_{i,j} = I_n$, and that $(a_{l,k})$ is a unique inverse of $E_{i,j}$, since, suppose we have another inverse matrix, $M = E_{i,j}^{-1}$, then $(a_{l,k}) \times E_{i,j} = I_n = M \times E_{i,j} \Rightarrow ((a_{l,k}) \times E_{i,j}) \times M = (M \times E_{i,j}) \times M \Rightarrow (a_{l,k}) \times E_{i,j} \times M = M \times E_{i,j} \times M \Rightarrow (a_{l,k}) \times (E_{i,j} \times M) = M \times (E_{i,j} \times M) \Rightarrow (a_{l,k}) \times I_n = M \times I_n \Rightarrow (a_{l,k}) = M$
So, $(a_{l,k}) = E_{i,j}^{-1}$ is the unique inverse of $E_{i,j}$

□

Corollary 1.3
Let $E_{i,j} = (e_{l,k}), i < j \in E_n$, and $A = (a_{l,k})$ any $n \times n$ matrix. Then, $E_{i,j}^{-1} = (b_{l,k})$ operates on A as the row addition $R_i \leftarrow R_i - R_j$.

Proof. As seen above, $E_{i,j}^{-1} = (b_{l,k})$ is a matrix where $b_{l,l} = 1, 1 \leq l \leq n$, and $b_{i,j} = -1, i < j$, and all other elements are zero.
So, the multiplication $(c_{l,k}) = E_{i,j}^{-1} \times A = (b_{l,k}) \times (a_{l,k})$ is exactly the same as $E_{i,j} \times A$, for all the rows except for row i
For row i, we have $c_{i,k}, 1 \leq k \leq n = \sum_{r=1}^{n} b_{i,r} \cdot a_{r,k}$
But, the only elements that are not zero, in row i of $(b_{l,k})$ are $b_{i,i} = 1$, and $b_{i,j} = -1$, so, $c_{i_k} = b_{i,1} \cdot a_{1,k} + b_{i,2} \cdot a_{2,k} + \cdots + b_{i,i} \cdot a_{i,k} + \cdots + b_{i,j} \cdot a_{j,k} + \cdots + b_{i,n-1} \cdot a_{n-1,k} + b_{i,n} \cdot a_{n,k} = 0 \cdot a_{1,k} + 0 \cdot a_{2,k} + \cdots + 1 \cdot a_{i,k} + \cdots + (-1) \cdot a_{j,k} + \cdots + 0 \cdot a_{n-1,k} + 0 \cdot a_{n,k} = 0 + 0 + \cdots + a_{i,k} + \cdots + (-a_{j,k}) = a_{i,k} - a_{j,k}$, which shows that in the product matrix, $(c_{l,k})$, R_i turns into $R_i - R_j$

□

Proposition 1.4
Let $E_{i,j} = (e_{l,k}), i < j \in E_n$, Then,
$\forall m \in \mathbb{N}, E_{i,j}^{m} = (a_{l,k})$, where $a_{l,l} = 1, 1 \leq l \leq n$, and $a_{i,j} = m, i < j$, and all other elements are zero.

Proof. By induction on m.
For $m = 2$, we observe that $E_{i,j}^2 = E_{i,j} \times E_{i,j}$, which means that the multiplication from the left of $E_{i,j}$ by itself is operating on $E_{i,j}$ as the row addition $R_i \leftarrow R_i + R_j$, so, $a_{i,i} = a_{i,i} + aj, i = 1 + 0 = 1$, and, $a_{i,j} = a_{i,j} + a_{j,j} = 1 + 1 = 2$, and, all the other elements are zero (easy to verify).
So, $(a_{l,k}) = E_{i,j}^2$, where $a_{l,l} = 1, 1 \leq l \leq n$, and $a_{i,j} = 2, i < j$, and all other elements are zero.
Now, we prove for $m + 1$
$(a_{l,k}) = E_{i,j}^{m+1} = E_{i,j} \times E_{i,j}^{m}$. But, from the induction assumption, $(b_{l,k}) = E_{i,j}^{m}$, $b_{l,l} = 1, 1 \leq l \leq n$, and $b_{i,j} = m, i < j$, and all other elements are zero.
Multiplying from the left $(b_{l,k})$ by $E_{i,j}$ is operating as the row addition $R_i \leftarrow R_i + R_j$, so, $a_{i,i} = b_{i,i} + b_{j,i} = 1 + 0 = 1$, and, $a_{i,j} = b_{i,j} + b_{j,j} = m + 1$, and easy to verify that all the other elements are zero, thus, we prove the induction step.

□

Corollary 1.5

Let $E_{i,j} = (e_{l,k}), i < j \in E_n$, Then,
$\forall m \in \mathbb{N}, (E_{i,j}^{-1})^m = (a_{l,k})$, where $a_{l,l} = 1, 1 \leq l \leq n$, and $a_{i,j} = -m, i < j$, and all other elements are zero

Proof. By induction on m.
$(a_{l,k}) = E_{i,j}^{-1}$ For $m = 2$, we observe that $(E_{i,j}^{-1})^2 = E_{i,j}^{-1} \times E_{i,j}^{-1} = (a_{l,k}) \times (a_{l,k})$, means that $E_{i,j}^{-1}$ operates on itself as the row addition $R_i \leftarrow R_i - R_j$
So, the product matrix $(b_{l,k})$ has $b_{i,i} = a_{i,i} - a_{j,i} = 1 - 0 = 1$, and $b_{i,j} = a_{i,j} - a_{j,j} = -1 - 1 = -2$, and all other elements are zero.
Now, we prove for $m + 1$
$(a_{l,k}) = (E_{i,j}^{-1})^{m+1} = E_{i,j}^{-1} \times (E_{i,j}^{-1})^m$. But, from the induction assumption, $(b_{l,k}) = (E_{i,j}^{-1})^m$, has $b_{l,l} = 1, 1 \leq l \leq n$, and $b_{i,j} = -m, i < j$, and all other elements are zero.
So, $(a_{l,k}) = E_{i,j}^{-1} \times (b_{i,j})$ is the row addition $R_i \leftarrow R_i - R_j$ on $(b_{i,j})$, which means, $a_{i,i} = b_{i,i} - b_{i,j} = 1 = 0 = 1$, and $a_{i,j} = b_{i,j} - b_{j,j} = -m - 1 = -(m+1)$, and all the other elements are zero, thus, we prove the induction step.

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