The pro-isomorphic zeta-functions of some nilpotent Lie algebras over integer rings

Research proposal for a Ph.D. Thesis
Under the supervision of Dr. Michael M. Schein
Department of Mathematics, Bar-Ilan University

Haim Lavi

September 4, 2024

Contents

Abstract

Let G be any group. For any natural number $n \in \mathbb{N}$, let a_n be the number of subgroups $H \leq G$, such that $[G : H] = n$. Assume G is finitely-generated, then $a_n < \infty$, and we can define a Dirichlet series of the form $\zeta_G(s) := \sum_{n=1}^{\infty} a_n n^{-s}$, where $s \in \mathbb{C}$. Assume, in addition, that G is also nilpotent and torsion-free, then this function has some properties of the Riemann ζ-function, such as the Euler decomposition of ζ into a product of local factors indexed by primes. A version of this ζ-function counts pro-isomorphic subgroups, and an analogous function may be defined for appropriate Lie rings. We study here the pro-isomorphic ζ-functions for a family of nilpotent Lie rings of unbounded nilpotency class. We shall compute the automorphism groups of these Lie rings explicitly, prove uniformity of the local factors of the pro-isomorphic ζ-functions, and aim to determine them explicitly.

1 Scientific Background

1.1 Introduction

We start our discussion with the following proposition, which stands at the very foundation of our subject.

Proposition 1.1.1. *Let G be any finitely generated group, and let $n \in \mathbb{N}$ be any natural number. Then there is a finite number of subgroups $H \leq G$, such that $[G : H] = n$*

This proposition gives rise to an entire subject in group theory, called **subgroup growth**. We denote by $a_n(G)$ the number of subgroups of G of index n, and look at the sequence $\{a_n(G)\}_{n=1}^{\infty}$. The subject of subgroup growth aims to relate the properties of this sequence to the algebraic structure of G. For instance, Lubotzky, Mann and Segal showed in [6] that $a_n(G)$ grows polynomially if and only if G is virtually solvable of finite rank, that is, G has a finite-index solvable subgroup, and all finitely-generated subgroups may be generated by a bounded number of generators. This research concentrates on the growth of **pro-isomorphic** subgroups, which we now define.

Definition 1.1.2. *Let G be any group, and let $\mathcal{N}(G) := \{N_k \trianglelefteq G\}_{k \in I}$ be the set of all normal subgroups of G. We define a partial order on $\mathcal{N}(G)$ by* *reverse inclusion, that is for every two indices i, j we say that $i \leq j$ if and only if $N_j \subseteq N_i$, hence for every $i \leq j$ there exists a natural projection map $\pi_{ji} : {}^{G}/_{N_j} \to {}^{G}/_{N_i}$. The inverse limit*

$$\widehat{G} = \underset{\leftarrow}{lim}\{{}^{G}/_{N_k}\}_{k \in I} := \{(h_k)_{k \in I} \in \prod_{k \in I} {}^{G}/_{N_k} : \pi_{ji}(h_j) = h_i, \forall i \leq j\}$$

is called the ***profinite closure*** *of G.*

Definition 1.1.3. *Let G be any group. A subgroup $H \leq G$ is called* ***pro-isomorphic*** *if $\widehat{H} \cong \widehat{G}$.*

Definition 1.1.4. *Let G be any group, and let*

$$\hat{a_n}(G) := \#\{H \leq G : \widehat{H} \cong \widehat{G}, [G : H] = n\}$$

be the number of pro-isomorphic subgroups of G of index n. Assume that $\hat{a_n}(G) < \infty$ for all n. The ***pro-isomorphic ζ-function*** *of G is defined by $\hat{\zeta_G}(s) := \sum_{n=1}^{\infty} \hat{a_n}(G) n^{-s}$ for $s \in \mathbb{C}$.*

If our ζ-function, which is a special case of the Dirichlet series, has some properties of convergence on some subset of $\mathbb{C}$, one may reconstruct its coefficients $\hat{a_n}(G)$, which, in our case, are the number of subgroups of our interest, using the **Perron's forumla**, which is an implementation of a **reverse Mellin transform**, as discussed, for example, in [7]. This method, including the specific properties of convergence required for the reconstruction, is out of the scope of our research, and therefore will not be further discussed, at this stage.

It is known that if $\hat{a_n}(G)$ grows polynomially, then $\hat{\zeta_G}(s)$ converges on some right half-plane of $\mathbb{C}$. For instance, we take the group $G = (\mathbb{Z}, +)$. The group $\mathbb{Z}$ is an abelian group, and every subgroup is of the form $H = n\mathbb{Z} = \langle n \rangle$, for some $n \in \mathbb{N}$, which means that $H \cong \mathbb{Z}$, as both are infinite cyclic groups, and so, $\widehat{H} \cong \widehat{\mathbb{Z}}$. Since we have only one subgroup of index n, for every $n \in \mathbb{N}$, then $a_n(\mathbb{Z}) = \hat{a_n}(\mathbb{Z}) = 1$. Thus, its pro-isomorphic ζ-function is $\hat{\zeta_{\mathbb{Z}}} = \sum_{i=1}^{\infty} n^{-s} = \zeta(s)$, the Riemann ζ-function, which is known to converge for $Re(s) > 1$.

After establishing the basic definitions, we recall a known fact that is a major motivation for this research, which says that the Riemann ζ-function decomposes into an infinite product of local zeta-functions, that is, $\zeta(s) = \prod_p \zeta_p(s) = \prod_p \sum_{k=0}^{\infty} p^{-ks} = \prod_p \frac{1}{1-p^{-s}}$, where the product runs over all the prime numbers. Following this fact regarding the Riemann ζ-function, we observe that for any finitely-generated, nilpotent and torsion-free group G, we have the same decomposition as above for the pro-isomorphic ζ-function: $\hat{\zeta_G}(s) = \prod_p \hat{\zeta_{G,p}}(s)$, where $\hat{\zeta_{G,p}}(s) := \sum_{k=0}^{\infty} \hat{a_{p^k}}(G)p^{-ks}$. The general construction of ζ-functions, as well as their Euler decomposition to local zeta-functions, for the study of subgroup growth, were well established by Grunewald, Segal and Smith, in [5]. We hereby bring several basic definitions of group nilpotency, which are very important for this research.

Definition 1.1.5. *Let G be any group. The* ***lower central series*** *of G is a sequence of subgroups of G defined by the recursive rule $G_k := [G, G_{k-1}]$, for every $k \in \mathbb{N}$, where $G_0 := G$. We recall that $[G, G_k] \leq G$ is the subgroup generated by the collection of commutators $\{gg_kg^{-1}g_k^{-1} : g \in G, g_k \in G_k\}$.*

Definition 1.1.6. *Let G be any group. The* ***nilpotency class*** *of G is $min\{k \in \mathbb{N} : G_k = [G, G_{k-1}] = \{e\}\}$, in words, the smallest natural number k, such that the subgroup of commutators of the form $[G, G_{k-1}]$ is the trivial group. We can extend this definition, and say that the trivial group nilpotency class is 0.*

Definition 1.1.7. *Let G be a group. If G is of finite nilpotency class, then G is said to be a* ***nilpotent*** *group.*

1.2 Linearization

We want to transfer the ideas from the above discussion about groups to a linear context, where we can use tools from linear algebra. Hence, for finitely-generated torsion-free nilpotent groups G, we associate nilpotent Lie algebras over $\mathbb{Z}$. This, in general, is called the **Maltsev correspondance**. If L is a $\mathbb{Z}$-Lie algebra, namely a free $\mathbb{Z}$-module of finite rank with a Lie brackets operation, then consider the number $\hat{a_m}(L)$ of subalgebras $M \leq L$, where $m = [L : M]$, such that $M \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers, for all primes p, and it is also known that $\hat{a_m}(L) < \infty$ for all $m \in \mathbb{N}$. The Dirichlet series $\hat{\zeta_L}(s) := \sum_{m=1}^{\infty} \hat{a_m}(L)m^{-s}$, is called the **pro-isomorphic zeta-function** of L. By the Maltsev correspondence, to every finitely-generated, nilpotent, torsion-free group G, one may associate a Lie algebra $L(G)$, such that $\hat{\zeta_{G,p}}(s) = \hat{\zeta_{L,p}}(s)$, for all but finitely many primes p. If G has nilpotency class 2, one may obtain the equality for all primes. Let $\{L_n\}_{n \in I}$, where $I \subset \mathbb{N}$, be a family of $\mathbb{Z}$-Lie algebras, where each L_n is characterized by n, as we demonstrate in the section of **research goals**. To each L_n we fix a $\mathbb{Z}$-basis $\mathcal{B}_n(\mathbb{Z}) = \{b_1, \ldots, b_r\}$, where $r = \text{rank} L_n$ depends on n. Let $\mathcal{L}_{n,p} = L_n \otimes_{\mathbb{Z}} \mathbb{Q}_p$, for any p. This is a $\mathbb{Q}_p$-Lie algebra, and our choice of basis allows us to identify the automorphism group $G_n(\mathbb{Q}_p) = Aut_{\mathbb{Q}_p}(\mathcal{L}_{n,p})$ with a subgroup of $GL_r(\mathbb{Q}_p)$. Note that $\mathcal{L}_{n,p}$ contains a $\mathbb{Z}_p$-lattice, $L_{n,p} = L_n \otimes_{\mathbb{Z}} \mathbb{Z}_p$. If $\varphi \in G_n(\mathbb{Q}_p)$, then $\varphi(L_{n,p}) = L_{n,p}$ if and only if $\varphi \in G_n(\mathbb{Z}_p) = G_n(\mathbb{Q}_p) \cap GL_r(\mathbb{Z}_p)$. Here $GL_r(\mathbb{Z}_p)$ is the group of $r \times r$ matrices which are invertible over $\mathbb{Z}_p$. Similarly, $\varphi(L_{n,p}) \subseteq L_{n,p}$ if and only if $\varphi \in G_n^+(\mathbb{Q}_p) := G_n(\mathbb{Q}_p) \cap \mathcal{M}_r(\mathbb{Z}_p)$, where $\mathcal{M}_r(\mathbb{Z}_p)$ is the collection of $r \times r$ matrices with entries in $\mathbb{Z}_p$. Note that $G_n^+(\mathbb{Q}_p)$ is a monoid, not a group.

Denote by $G_n(\mathbb{Z}_p)g$, where $g \in G_n^+(\mathbb{Q}_p)$, a right-coset of $G_n(\mathbb{Z}_p)$. One checks that the monoid $G_n^+(\mathbb{Q}_p)$ is a disjoint union of right-cosets of $G_n(\mathbb{Z}_p)$.

The discussion above reveals the construction we base our research upon. We observe that there is a bijection between the set $G_n(\mathbb{Z}_p)\backslash G_n^+(\mathbb{Q}_p)$ of right-cosets of $G_n(\mathbb{Z}_p)$ and the set $\{M \leq L_{n,p} : M \cong L_{n,p}\}$ of $L_{n,p}$-subalgebras which are isomorphic to $L_{n,p}$ itself. This bijection takes $G_n(\mathbb{Z}_p)g$ to $M = \varphi(L_{n,p})$. For any $\varphi \in G_n(\mathbb{Z}_p)g$, this is well-defined. One checks that for every $\psi \in G_n(\mathbb{Z}_p)g$, we have that $\psi(L_{n,p}) = \varphi(L_{n,p}) = M$. We end this part, as a preparation for the final part of this technical background review, with the following result, which states that for each right-coset $G_n(\mathbb{Z}_p)g$, if $M = \varphi(L_{n,p})$, where $\varphi \in G_n(\mathbb{Z}_p)g$, then $[L_{n,p} : M] = |\det \varphi|_p^{-1}$, where $|\det \varphi|_p$ is the p-adic norm of $\det \varphi$, and therefore,

$\hat{\zeta_{L,p}}(s) = \sum_{\substack{M \leq L_{n,p} \\ M \cong L_{n,p}}} [L_{n,p} : M]^{-s} = \sum_{G_n(\mathbb{Z}_p)\varphi \in G_n(\mathbb{Z}_p)\backslash G_n^+(\mathbb{Q}_p)} |\det \varphi|_p^s$. Last for this section, we shall define the lower central series for Lie algebras, which will be our main working tool in the research of the automorphism groups.

Definition 1.2.1. *Let L be a Lie algebra, then the* ***lower central series*** *of L is defined by $\gamma_k L := [L, \gamma_{k-1} L]$ where $k \in \mathbb{N}$, and $\gamma_1 L := L$, same as defined for groups in 1.1.5. A Lie algebra is said to be* ***nilpotent****, if $\gamma_k L = \{0\}$, for some finite k, also as defined above for groups in 1.1.7.*

When the algebra is clear from the context, we may omit the notation of the algebra in $\gamma_k L$, only to stay with the gamma notation γ_k. Sometimes, we will also use the gamma notation for nilpotnecy classes of groups.

1.3 p-adic Integration

In this final part of the technical background review, we finally get to the motivation for all the construction we have presented in the first parts. We now define a very central object for our research. We shall assume, without proof, the existence of such an object, under the prerequisites of the definition.

Definition 1.3.1. *Let Γ be a locally compact topological group, i.e. for all $\gamma \in \Gamma$, there is an open neighborhood of $\gamma \in U_\gamma$ and a compact subset K_γ, such that $U_\gamma \subset K_\gamma$. Then there is a measure μ, with the following property: for any measurable subset $U \subseteq \Gamma$ and any $\gamma \in \Gamma$, $\mu(U\gamma) = \mu(U)$, where $U\gamma := \{u\gamma : u \in U\}$. Such a measure μ is called a **right Haar measure**, and is unique up to multiplication by a non-zero constant.*

Equipped with this definition of a right Haar measure, we can finally make use of the construction from above. We start by claiming, without proof, that for every prime number p, the group $G_n(\mathbb{Q}_p)$ is a locally compact topological group. We also claim that the right Haar measure can be normalized such that $\mu(G_n(\mathbb{Z}_p)) = 1$. The measure of all the right-cosets of $G_n(\mathbb{Z}_p)$ equals to the measure of $G_n(\mathbb{Z}_p)$ itself, i.e. for every $g \in G_n^+(\mathbb{Q}_p)$, we have that $\mu(G_n(\mathbb{Z}_p)g) = \mu(G_n(\mathbb{Z}_p)) = 1$. With this observation, we go directly to the calculation of the p-adic norm of the determinant of every $L_{n,p}$-automorphism, as a p-adic integral over our measure space. First, we observe that given any $L_{n,p}$-automorphism in some right-coset $\varphi \in G_n(\mathbb{Z}_p)\varphi$, we have that $|\det \varphi|_p^s = \int_{G_n(\mathbb{Z}_p)\varphi} |\det \varphi|_p^s d\mu$, because $\mu(G_n(\mathbb{Z}_p)\varphi) = 1$, and $|\det \varphi|_p^{-1}$ is fixed on $G_n(\mathbb{Z}_p)\varphi$.

Going back to our desired function, we observe that

$$\hat{\zeta_{L,p}}(s) = \sum_{G_n(\mathbb{Z}_p)\varphi \in G_n(\mathbb{Z}_p)\backslash G_n^+(\mathbb{Q}_p)} |\det \varphi|_p^s = \sum_{G_n(\mathbb{Z}_p)\varphi \in G_n(\mathbb{Z}_p)\backslash G_n^+(\mathbb{Q}_p)} \int_{G_n(\mathbb{Z}_p)\varphi} |\det \varphi|_p^s d\mu = \int_{G_n^+(\mathbb{Q}_p)} |\det \varphi|_p^s d\mu.$$

This calculation of the local ζ_p-function as a p-adic integral was established by the work of du Sautoy and Lubotzky, in [3]. This integral is the main object we shall study in this research. We end this part, of the technical background review, by a theorem and a couple of definitions, which stand in the center of our research goals.

Theorem 1.3.2. *Let p be a prime number, and let $s \in \mathbb{C}$, then $\hat{\zeta_{L,p}}(s)$ is rational, i.e. there is a rational function in one variable $W_p \in \mathbb{Q}(X)$ such that $\zeta_{L,p}(s) = W_p(p^{-s})$.*

Definition 1.3.3. *Let L be a $\mathbb{Z}$-Lie algebra, and let $s \in \mathbb{C}$. Then $\hat{\zeta_L}(s)$ is called **uniform**, if there exists a rational function in two variables $W \in \mathbb{Q}(X,Y)$ such that for every prime number p, the local function $\zeta_{L,p}(s) = W(p, p^{-s})$, which means that $\zeta_{L,p}(s)$ has exactly the same form for all primes, since its value depends only on p and p^{-s}.*

Definition 1.3.4. *Let L be a $\mathbb{Z}$-Lie algebra, and let $s \in \mathbb{C}$. Then $\hat{\zeta_L}(s)$ is called **finitely uniform**, if there exists a finite set of rational function in two variables $W_1, W_2, \ldots, W_m \in \mathbb{Q}(X,Y)$ such that for every prime number p, the local function $\zeta_{L,p}(s) = W_k(p, p^{-s})$, for some $1 \leq k \leq m$. One observes immediately that this is a weaker condition on L, because a uniform zeta-function is simply a finitely uniform function where $m = 1$.*

The theorem ensures that for any $\mathbb{Z}$-Lie algebra, all the local pro-isomorphic zeta-functions are rational functions in p^{-s}, but their form can vary between different primes. If we can prove uniformity, for some specific $\mathbb{Z}$-Lie algebra L, it means that all its local zeta-functions have exactly the same form as a rational function in p and p^{-s}, and if we can prove finite uniformity, then we can at least show that all the local zeta-functions have only a finite number of different forms. The uniformity is also established in the work of Grunewald, Segal and Smith, see [5]. Proving the uniformity of the pro-isomorphic zeta-functions of the algebras we are studying is one of the main goals of our research.

2 Research Goals and Methodology

2.1 The Lie algebras $L_{n,p}$

Let e_{ij} be an $n \times n$ matrix, in which all the elements are zero, except for the element in row i and column j which has 1. On the set $\mathcal{E}_n = \{e_{ij} : 1 \leq i \leq n-1 \wedge i+1 \leq j \leq n\}$ we define a bracket operation: for every $1 \leq k, l \leq n-1$, define $[e_{k,k+1}, e_{l,l+1}] := e_{k,k+1}e_{l,l+1} - e_{l,l+1}e_{k,k+1}$. Let $\mathcal{R}$ be some commutative ring, then the standard operation of $\mathcal{R}$ on $\mathcal{E}_n$, along with the defined bracket operation, form a nilpotent $\mathcal{R}$-Lie algebra. Considering $\mathcal{R} = \mathbb{Z}$, we obtain a nilpotent $\mathbb{Z}$-Lie algebra of strictly upper triangular matrices over $\mathbb{Z}$, which we denote by L_n, with the standard bracket operation as its Lie brackets. As discussed above, this $\mathbb{Z}$-Lie algebra can be extended to a $\mathbb{Z}_p$-algebra, which we denote by $L_{n,p}$, and then to a $\mathbb{Q}_p$-algebra, which we denote by $\mathcal{L}_{n,p}$. It is readily seen that the set of matrices of the form e_{ij} where $i < j$, spans the whole $\mathbb{Z}$-Lie algebra L_n and is $\mathbb{Z}$-linearly independent. Therefore, it forms a basis for L_n as a free module over $\mathbb{Z}$, $\mathcal{B}_n(\mathbb{Z}) := \{e_{12}, e_{13}, \ldots, e_{1n}, e_{23}, \ldots, e_{2,n}, \ldots, e_{n-1,n}\}$, which we call **the standard basis of** L_n. One easily checks that $r = \text{rank} L_n = |\mathcal{B}_n(\mathbb{Z})| = \binom{n}{2}$, which is the number of elements above the main diagonal for every $n \in \mathbb{N}$. To this standard basis we apply a linear order by defining $e_{ij} < e_{kl}$ if $j-i < l-k$ or if $j - i = l - k$ and $i < k$. In other words, we apply an order that divides $\mathcal{B}_n(\mathbb{Z})$ to basis elements of the quotients ${}^{L_n}/_{\gamma_2}, {}^{\gamma_2}/_{\gamma_3}, \ldots, {}^{\gamma_{n-2}}/_{\gamma_{n-1}}, \gamma_{n-1}$

Obviously, the same goes also for the extensions of L_n, namely $L_{n,p}$ and $\mathcal{L}_{n,p}$. The target of our research is studying the $\zeta^{\wedge}_{L,p}$-function on these $\mathbb{Z}_p$-Lie algebras, and the related constructions.

2.2 Research goals

The project consists of two independent parts.

The first part is completing the work of Mark N. Berman on $L_{5,p}$, which we mention later in this paper, and express its $\zeta^{\wedge}_{L,p}$-function by an explicit formula.

The second part consists of three major steps:

1. **Computing the automorphism group of the $\mathbb{Q}_p$-Lie algebras $\mathcal{L}_{n,p}$, for all $n \in \mathbb{N}$ and all primes p.**

2. **Showing that the pro-isomorphic zeta-functions $\zeta^{\wedge}_{L_{n,p}}(s)$ are uniform for all $n \in \mathbb{N}$.**

3. **Giving an explicit uniform formula for the zeta-functions $\zeta^{\wedge}_{L_{n,p}}(s)$ for specific values of n, if not for all $n \in \mathbb{N}$.**

As we elaborate further, steps 1 and 2 are already known entirely for $n \leq 5$, and step 3 is known for $n \leq 4$. We start with the first step of calculating $Aut_{\mathbb{Q}_p}(\mathcal{L}_{n,p})$. These automorphism groups have been studied for decades from a different point of view. There are classical results showing that any automorphism may be expressed as a product of automorphisms of a specific type; see, for instance, the main result of Gibbs [4]. These results are not explicit enough for our purposes; indeed, the submonoid $G_n^+(\mathbb{Q}_p)$ arises for us as the domain of integration of a p-adic integral. In order to calculate this integral, we need to decompose the automorphism group $G_n(\mathbb{Q}_p)$ into a repeated semi-direct product of groups with a simple structure.

After we have analyzed the structure of $G_n(\mathbb{Z}_p)$, we will need to construct the monoid $G_n^+(\mathbb{Q}_p)$ and its $G_n(\mathbb{Z}_p)$ right-cosets, as we have seen above. This will give us both the function to integrate, which is $\det\varphi$ for every $G_n(\mathbb{Z}_p)$ right-coset $G_n(\mathbb{Z}_p)\varphi$, and the domain of integration, which is the monoid $G_n^+(\mathbb{Q}_p)$. We will use this information to analyze the behavior of the p-adic integral we have described above and prove that its calculation depends only on p, thus showing that the $\hat{\zeta_{L,p}}$-function is uniform.

2.3 The Heisenberg Lie algebra L_3

We show the very basic approach to this problem by the simplest example, the **Heisenberg Lie algebra** L_3, which is defined by the set of three matrices $\{e_{12}, e_{13}, e_{23}\}$ with the bracket operation we defined above, to which we define the standard **ordered** basis as $\mathcal{B}_3(\mathbb{Z}) = \{e_{12}, e_{23}, e_{13}\}$. This means that if we analyze some $\varphi \in G_3(\mathbb{Z})$ by its operation on the basis elements, then φ must be some 3×3 matrix over $\mathbb{Z}$, such that multiplying any vector $v = xe_{12} + ye_{23} + ze_{13} \in L_{n,p}$ with φ from the right[1] yields a vector $u = \varphi(v) = x\varphi(e_{12}) + y\varphi(e_{23}) + z\varphi(e_{13}) \in L_{n,p}$, i.e.

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

$$= \begin{pmatrix} xa_{11} + ya_{21} + za_{31} & xa_{12} + ya_{22} + za_{32} & xa_{13} + ya_{23} + za_{33} \end{pmatrix}$$

which means that

$$\varphi(e_{12}) = a_{11}e_{12} + a_{12}e_{23} + a_{13}e_{13}$$

$$\varphi(e_{23}) = a_{21}e_{12} + a_{22}e_{23} + a_{23}e_{13}$$

$$\varphi(e_{13}) = a_{31}e_{12} + a_{32}e_{23} + a_{33}e_{13}$$

Every $\varphi \in G_3(\mathbb{Z})$ must obey the Lie brackets, hence we observe that $a_{31}e_{12} + a_{32}e_{23} + a_{33}e_{13} = \varphi(e_{13}) = \varphi[(e_{12}), (e_{23})] = [\varphi(e_{12}), \varphi(e_{23})] = [a_{11}e_{12} + a_{12}e_{23} +$

[1]We multiply from the right, because we want the rows of φ to represent the images of the basis elements, $\varphi(e_{12})$, $\varphi(e_{23})$ and $\varphi(e_{13})$

$a_{13}e_{13}, a_{21}e_{12}+a_{22}e_{23}+a_{23}e_{13}] = (a_{11}a_{22}-a_{12}a_{21})e_{13}$, which gives the following relations,

$$a_{31} = 0$$

$$a_{32} = 0$$

$$a_{33} = (a_{11}a_{22} - a_{12}a_{21}) \neq 0$$

which means that φ is the following matrix, $\varphi = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & \det A \end{pmatrix}$ where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, and A must be invertible, otherwise φ is not bijective. Based on the construction of $L_{n,p}$ and $\mathcal{L}_{n,p}$ from earlier, all the above applies also for $G_3(\mathbb{Z}_p)$ and for $G_3(\mathbb{Q}_p)$, respectively.

2.4 $L_{n,p}$-Lie algebras for $n > 3$

Mark N. Berman, in his doctoral thesis [1], has displayed an explicit formula for $\zeta^{\wedge}_{L_{4,p}}$, and proved that $\zeta^{\wedge}_{L_{5,p}}$ is indeed uniform. We aim to generalize his work to prove that $\zeta^{\wedge}_{L_{n,p}}$ is uniform for all n. We also aim to compute $\zeta^{\wedge}_{\mathcal{L}_{n,p}}(s)$ explicitly for all n, or at least to obtain explicit formulas for some $n \geq 5$, and specifically for $n = 5$. By analyzing carefully Berman's work on $L_{4,p}$ and $L_{5,p}$, we gain the basic understanding of the expected structure of the local zeta-functions in the general case. We begin our discussion of the first goal, which is computing $G_n(\mathbb{Z}_p)$, by first recalling that for every $v \in L_{n,p}$, where $n \geq 3$, we present $\varphi(v)$ as the multiplication of v by a matrix from the right $\varphi(v) = vM$. As stated earlier, M is an $r \times r$ matrix, where $r = \text{rank}L_{n,p} = \binom{n}{2}$, whose lines are set by the order we have defined above, i.e. considering the standard ordered basis

$$\mathcal{B}_n = \{e_{12}, e_{23}, \ldots, e_{n-1,n}, e_{13}, \ldots, e_{n-2,n}, \ldots, e_{1n}\}$$

then M is the following matrix,

$$M = \begin{pmatrix} \varphi(e_{12}) \\ \varphi(e_{23}) \\ \varphi(e_{n-1,n}) \\ \hdashline \varphi(e_{13}) \\ \vdots \\ \varphi(e_{n-2,n}) \\ \hdashline \vdots \\ \hdashline \varphi(e_{1n}) \end{pmatrix}$$

Given an $\mathcal{L}_{n,p}$-automorphism φ, we denote by $\varphi_k : \gamma_k\mathcal{L}_{n,p} \to \gamma_k\mathcal{L}_{n,p}$ the operation of φ on all the $n-k$ elements of the lower central series starting from k, that is, we consider only the images

$$\varphi(e_{1,1+k}), \varphi(e_{2,2+k}), \dots, \varphi(e_{n-k,n}), \varphi(e_{1,2+k}), \dots, \varphi(e_{n-k-1,n}), \dots, \varphi(e_{1n})$$

For every φ_k, we have the induced map denoted by φ_{kk}, from the quotient algebra ${}^{\gamma_k}/_{\gamma_{k+1}}$ to itself, defined by $\varphi_{kk}(e_{l,l+k} + \gamma_{k+1}\mathcal{L}_{n,p}) := a_{1,1+k}e_{1,1+k} + a_{2,2+k}e_{2,2+k} + \dots + a_{n-k,n}e_{n-k,n} + z_{k+1}$, where $z_{k+1} \in \gamma_{k+1}\mathcal{L}_{n,p}$, for every $1 \le l \le n-k$. Clearly, φ_{kk} is well-defined, since $\varphi_k(\gamma_k\mathcal{L}_{n,p}) = \gamma_k\mathcal{L}_{n,p}$, for every $1 \le k \le n-1$. Following this division of $\mathcal{L}_{n,p}$ by the lower central series and its quotients, we view M as a block matrix,

$$M = \left(\begin{array}{c|c|c|c|c} M_{11} & M_{12} & \dots & M_{1,n-2} & M_{1,n-1} \\ \hline M_{21} & M_{22} & \dots & M_{2,n-2} & M_{2,n-1} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline M_{n-1,1} & M_{n-1,2} & \dots & M_{n-1,n-2} & M_{n-1,n-1} \end{array}\right)$$

each block is denoted by $M_{kl} \in \mathcal{M}_{m\times r}(\mathbb{Q}_p)$, where $m = \dim {}^{\gamma_k}/_{\gamma_{k+1}}$ and $r = \dim {}^{\gamma_l}/_{\gamma_{l+1}}$. From this, we can understand that the blocks on the main diagonal of M, which are the induced quotient maps defined above, are square matrices $\varphi_{kk} = M_{kk} \in \mathcal{M}_{n-k}(\mathbb{Q}_p)$. We present, in the preliminary results section, our very first conclusions regarding these blocks. Dividing M into blocks is our main strategy for computing the general form of the $\mathcal{L}_{n,p}$-automorphisms, because, as we shall start showing in the preliminary results section, one block determines other blocks related to it. An almost immediate result, regarding these matrix blocks, can be achieved by considering the relation $\varphi(e_{i,i+k}) = [\varphi(e_{i,i+j}), \varphi(e_{i+j,i+k})]$, for all $1 \le j \le k-1$. For $k=2$, $\varphi(e_{i,i+2}) = [\varphi(e_{i,i+1}), \varphi(e_{i+1,i+2})] = \sum_{r=1}^{n-1}\sum_{l=1}^{n-r} a_{l,l+r}e_{l,l+r}$, but any element y in the linear combination that forms $\varphi(e_{i+1,i+2})$ is a multiplication of two elements of $\mathcal{L}_{n,p}$, which means that $y \in \gamma_2\mathcal{L}_{n,p}$, and hence the coefficients $a_{j,j+1}$, for all $1 \le j \le n-1$, must vanish. By simple induction, we can show that for all $2 \le k \le n-1$, if $\varphi(e_{i,i+k}) = \sum_{r=1}^{n-1}\sum_{l=1}^{n-r} a_{l,l+r}e_{l,l+r}$, then the coefficients $a_{l,l+j}$, for all $1 \le j \le k-1$, must vanish, which means that all the matrix blocks M_{kl}, where $k < l$, must be zero, therefore M has the form,

$$M = \left(\begin{array}{c|c|c c|c|c}
M_{11} & M_{12} & M_{13} & \dots & M_{1,n-2} & M_{1,n-1} \\ \hline
0 & M_{22} & M_{23} & \dots & M_{2,n-2} & M_{2,n-1} \\ \hline
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline
0 & 0 & 0 & \dots & M_{2,n-2} & M_{2,n-1} \\ \hline
0 & 0 & 0 & \dots & 0 & M_{n-1,n-1}
\end{array}\right)$$

Relations of the form $[\varphi(e_{l,l+j}), \varphi(e_{l,l+k})] = 0$ and $[\varphi(e_{l+j,l+k}), \varphi(e_{l,l+k})] = 0$, where $2 \le k \le n-1$ and $1 \le j \le k-1$, will also show as critical for our understanding of the matrix blocks above the main diagonal, as we display in details in the preliminary results section.

2.5 Preliminary results

Once we are set to work with the main diagonal blocks M_{kk}, for all $1 \le k \le n-1$, as diagonal operations on ${}^{\gamma_k}/{}_{\gamma_{k+1}}$, that is, for all $e_{ij} \in {}^{\gamma_k}/{}_{\gamma_{k+1}}$, $\varphi(e_{ij}) = a_{ij}e_{ij} + z_{k+1}$, for some $a_{ij} \in \mathcal{L}_{n,p}$ and $z_{k+1} \in \gamma_{k+1}\mathcal{L}_{n,p}$, we can move to the next matrix blocks, above the main diagonal. For this purpose, we have several tools we ought to use. The first one is the relations we mentioned earlier, $[\varphi(e_{i,i+j}), \varphi(e_{i,i+k})] = 0$ and $[\varphi(e_{i+j,i+k}), \varphi(e_{i,i+k})] = 0$, for all $1 \le j \le k-1$. We shall demonstrate a use of these relations for M_{12}, and generalize it in the research itself. Let $x = \varphi(e_{12}) = \lambda_1 e_{12} + \sum_{k=2}^{n-1}\sum_{i=1}^{n-k} a_{i,i+k}e_{i,i+k}$. We want to study the matrix block M_{12}, so we look at the elements $a_{i,i+2}e_{i,i+2}$ in the linear combination that forms x. Let $y = \varphi(e_{23}) = \lambda_2 e_{23} + \sum_{l=2}^{n-1}\sum_{j=1}^{n-l} b_{j,j+l}e_{j,j+k}$, and hence $\varphi(e_{13}) = [x,y] = \lambda_1 e_{12}\lambda_2 e_{23} + z_3 = \lambda_1\lambda_2 e_{13} + z_3$, where $z_3 \in \gamma_3\mathcal{L}_{n,p}$. Looking closer at z_3, we find the expression $-\lambda_2 e_{23}a_{35}e_{35} = \lambda_2 a_{35}e_{13}$, where $\lambda_2 e_{23}$ is coming from y and $a_{35}e_{35}$ is coming from x. Therefore, computing the product $[x,[x,y]]$ yields the expression $-\lambda_1 e_{12}\lambda_2 a_{35}e_{25} - \lambda_1\lambda_2 e_{13}a_{35}e_{35} = -2\lambda_1\lambda_2 a_{35}e_{15}$, we check that there are no other multiplications of e_{15} in

$[x,[x,y]]$, and since $[x,[x,y]] = 0$, we have that $\lambda_1\lambda_2 a_{35} = 0$. We know that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, hence we must have that $a_{35} = 0$.

Another tool we are using is considering dimensions of centralizers of the images $\varphi(e_{i,i+1})$ in the different quotients. Since obviously $\dim \mathcal{C}_{\gamma_k/\gamma_{k+1}}(e_{ij}) = \dim \mathcal{C}_{\gamma_k/\gamma_{k+1}}(\varphi(e_{ij}))$, because the $\mathcal{L}_{n,p}$-automorphisms preserve all the quotient elements features on their images, then let $x = \varphi(e_{12}) = \lambda_1 e_{12} + z_2$ and let $y = \sum_{l=1}^{2} b_{j,j+l} e_{j,j+l} + w_2 \in \mathcal{C}_{\mathcal{L}_{n,p}/\gamma_3}(x)$, where $z_2 \in \gamma_2\mathcal{L}_{n,p}$ and $w_3 \in \gamma_3\mathcal{L}_{n,p}$, so $[x,y] = 0$. We know that all the elements in $\mathcal{L}_{n,p}/\gamma_2$ commute with each other, but taking $\mathcal{L}_{n,p}/\gamma_3$, we have only one element that **does not** commute with e_{12} which is e_{23}, because $[e_{12}, e_{23}] = e_{13} \in \mathcal{L}_{n,p}/\gamma_3$ and $[e_{12}, e_{r,r+1}] = 0$, for all $r \neq 2$. Computing the Lie product $[x,y] = \lambda_1 e_{12} b_{23} e_{23} + z_3 = \lambda_1 b_{23} e_{13} + z_3 = 0$ does give us one constraint on $\mathcal{C}_{\mathcal{L}_{n,p}/\gamma_3}(x)$, because $\lambda_1 b_{23} = 0$ means that $b_{23} = 0$, since $\lambda_1 \neq 0$, which means that $\mathrm{codim}\mathcal{C}_{\mathcal{L}_{n,p}/\gamma_3}(e_{12}) = \mathrm{codim}\mathcal{C}_{\mathcal{L}_{n,p}/\gamma_3}(\varphi(e_{12})) = 1$. Considering $\mathcal{C}_{\mathcal{L}_{n,p}/\gamma_4}(e_{12})$, we observe that the only elements that do not commute with e_{12} in $\mathcal{L}_{n,p}/\gamma_4$ are e_{23} and e_{24}, which means that we must have another constraint for $\mathcal{C}_{\mathcal{L}_{n,p}/\gamma_4}(\varphi(e_{12}))$ as well. For that, we have the equation $a_{24} b_{45} e_{25} = 0$, but if $b_{45} = 0$, then it means that $\mathrm{codim}\mathcal{C}_{\mathcal{L}_{n,p}/\gamma_3}(\varphi(e_{12})) = 2 \neq \mathrm{codim}\mathcal{C}_{\mathcal{L}_{n,p}/\gamma_3}(e_{12})$, and hence we have that also $a_{24} = 0$. We prove that $a_{j,j+2} = 0$, for all $1 \leq j \leq n-2$, in the linear combination that forms $\varphi(e_{12})$. Now let $\varphi(e_{i,i+1) = \lambda_i e_{i,i+1}} + a_{j,j+2} e_{j,j+2} + z_3$, for all $1 \leq j \leq n-2$, we prove, by the very same method, that $a_{j,j+2} = 0$, for all $i+1 \leq j \leq n-2$, in the linear combination that forms $\varphi(e_{i,i+1})$. Again, by the same method, we prove that $a_{j,j+2} = 0$ in the linear combination that forms $\varphi(e_{i,i+1})$, for all $1 \leq i \leq n-1$ and $1 \leq j \leq i-2$. Thus, we obtain the structure of the matrix block $M_{12} = (m_{kl})$, as a $(n-1)\times(n-2)$ matrix, for which all but the two diagonal sequences $\{m_{11}, m_{22}, m_{33}, \ldots, m_{n-2,n-2}\}$ and $\{m_{21}, m_{32}, m_{43}, \ldots, m_{n-1,n-2}\}$, are zero. Using exactly the same tools, we continue to study the next matrix blocks, as shall be detailed in the research itself.

A major step towards the computation of $G_n(\mathbb{Q}_p)$ would be to observe some invariants of elements of $\mathcal{L}_{n,p}$, which must be preserved under any $\mathcal{L}_{n,p}$-automorphism $\varphi \in G_n(\mathbb{Q}_p)$, and analyze the constraints on the structure of the $\mathcal{L}_{n,p}$-automorphisms that come out of this observation. We shall start by examining dimensions of centralizers of basis elements relative to subalgebras of $\mathcal{L}_{n,p}$ and their quotients. As we saw earlier, dividing the matrix M into blocks corresponds to considering the operation of φ on subalgebras of $\mathcal{L}_{n,p}$ and their quotients. For instance, the block M_{11} is the matrix of the induced automorphism $\varphi_{11} : \gamma_1/\gamma_2 \to \gamma_1/\gamma_2$. We start the block analysis with the following proposition,

Proposition 2.5.1. *Let $\mathcal{L}_{n,p}$ be the $\mathbb{Q}_p$-Lie algebra associated with $\mathcal{U}_n(\mathbb{Z})$. If $n \geq 5$, then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim {}^{\gamma_1}/_{\gamma_3} - 1$ if and only if $x \in \{\lambda e_{12} + \gamma_2 \mathcal{L}_{n,p}\}$ or $x \in \{\lambda e_{n-1,n} + \gamma_2 \mathcal{L}_{n,p}\}$, for a non-zero scalar $\lambda \in \mathbb{Q}_p$. If $n = 4$, then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim {}^{\gamma_1}/_{\gamma_3} - 1$ if and only if $x \in \{\lambda e_{12} + \mu e_{34} + \gamma_2 \mathcal{L}_{n,p}\}$, for $\lambda, \mu \in \mathbb{Q}_p$ not both zero.*

The proof of this proposition requires understanding the pattern of the linear combination which forms every $x \in \mathcal{L}_{n,p}$, and shall be fully detailed in the research paper. Applying this proposition, we can now start computing the first block on the main diagonal M_{11}. Since φ induces an automorphism on ${}^{\gamma_1}/_{\gamma_2}$, then the codimension of the centralizer of each basis element e_{ij} in ${}^{\gamma_1}/_{\gamma_3}$ must be preserved by the image $\varphi(e_{12})$ in ${}^{\gamma_1}/_{\gamma_3}$. This means that for the basis elements in both edges of ${}^{\gamma_1}/_{\gamma_2}$, namely e_{12} and $e_{n-1,n}$ whose centralizers codimension is 1, we have that $\varphi(e_{12})$ and $\varphi(e_{n-1,n})$ centralizers codimension is also 1, which means that $\varphi_(e_{12}) = \lambda_1 e_{12} + z_1$ or $\varphi(e_{12}) = \lambda_{n-1} e_{n-1,n} + z_{n-1}$, hence $\varphi_{11}(e_{12}) = \lambda_1 e_{12}$ or $\varphi_{11}(e_{12}) = \lambda_{n-1} e_{n-1,n}$, for some $\lambda_1, \lambda_{n-1} \in \mathbb{Q}_p$ and $z_1, z_{n-1} \in \gamma_2 \mathcal{L}_{n,p}$, and since φ is an $\mathcal{L}_{n,p}$-automorphism, it means that $\varphi_{11}(e_{n-1,n}) = \lambda_{n-1} e_{n-1,n}$ or $\varphi_{11}(e_{n-1,n}) = \lambda_1 e_{12}$, respectively. We now show that we can assume $\varphi_{11}(e_{12}) = \lambda_1 e_{12}$, and hence $\varphi_{11}(e_{n-1,n}) = \lambda_{n-1} e_{n-1,n}$, without loss of generality. For this, we have the following proposition,

Proposition 2.5.2. *Let $\mathcal{L}_{n,p}$ be the $\mathbb{Q}$-Lie algebra associated with $\mathcal{U}_n(\mathbb{Z}_p)$. We define a map $\eta_n : \mathcal{L}_{n,p} \to \mathcal{L}_{n,p}$ by its operation on elements of the standard basis of $\mathcal{L}_{n,p}$:*

$$\eta_n(e_{ij}) := (-1)^{j-i-1} e_{n+1-j,n+1-i}$$

for every element $e_{ij} \in \mathcal{B}_{n,p}$. Then, η is an involution, hence, also an $\mathcal{L}_{n,p}$-automorphism.

One easily checks that the proposition comes directly from the definition of η. This means that if $\varphi_{11}(e_{12}) = \lambda_{n-1} e_{n-1,n}$ and $\varphi_{11}(e_{n-1,n}) = \lambda_1 e_{12}$, then we can compose φ with the above automorphism η_n to obtain a new $\mathcal{L}_{n,p}$-automorphism $\psi = \varphi \eta_n$ such that $\psi(e_{12}) = \varphi \eta_n(e_{12}) = \varphi(e_{n-1,n}) = \lambda_1 e_{12} + z_1$ and $\psi(e_{n-1,n}) = \lambda_{n-1} e_{n-1,n} + z_{n-1}$, where $z_1, z_{n-1} \in \gamma_2 \mathcal{L}_{n,p}$, which means that $\psi_{11}(e_{12}) = \lambda_1 e_{12}$ and $\psi_{11}(e_{n-1,n}) = \lambda_{n-1} e_{n-1,n}$. Following the assumption for φ, we now look at e_{23}. We observe that every linear combination which has $\lambda_1 e_{12}$ or $\lambda_3 e_{34}$ or both, for $\lambda_1, \lambda_3 \in \mathbb{Q}_p$, does not commute with e_{23}, hence $[\varphi(e_{12}), \varphi(e_{23})] = [\lambda_1 e_{12} + z, a_{12} e_{12} + \cdots + a_{n-1,n} e_{n-1,n} + w] = [\lambda_1 e_{12}, a_{23} e_{23}] + q = \lambda_1 a_{23} e_{13} + q$, for $a_{12}, a_{23}, \ldots, a_{n-1,n} \in \mathbb{Q}_p$ and $z, w, q \in \gamma_3 \mathcal{L}_{n,p}$, which means that $a_{23} \neq 0$, and we denote $\lambda_2 = a_{23}$. Therefore, we must have $\lambda_2 e_{23}$ in the linear combination that forms the image $\varphi(e_{23})$. Assume we are in the case that $x = \varphi(e_{23}) = \sum_{k=1}^{n-1} \rho_k e_{k,k+1}$, where $\rho_k \in \mathbb{Q}_p$ are all non-zero. We choose an element in the centralizer of x, denoted by $y = \sum_{k=1}^{n-1} \mu_k e_{k,k+1}$, where $\mu_k \in \mathbb{Q}_p$. The same way we did for the Heisenberg group associated algebra, we observe that $[x, y] = [\sum_{k=1}^{n-1} \rho_k e_{k,k+1}, \sum_{k=1}^{n-1} \mu_k e_{k,k+1}] = \sum_{k=1}^{n-2} (\rho_k \mu_{k+1} - \rho_{k+1}\mu_k) e_{k,k+2}$. For x and y to commute, we need the $n-2$ expressions $\rho_1\mu_2 - \rho_2\mu_1, \rho_2\mu_3 - \rho_3\mu_2, \ldots, \rho_{n-2}\mu_{n-1} - \rho_{n-1}\mu_{n-2}$ to vanish, which means that we have $n-2$ linear constraints of the form $\rho_k\mu_{k+1} - \rho_{k+1}\mu_k$ on y. Given fixed coefficients for x, if we choose an arbitrary μ_1, then $\mu_2 = \frac{\rho_2\mu_1}{\rho_1}$, in order for $\rho_1\mu_2 - \rho_2\mu_1$ to vanish. But then $\mu_3 = \frac{\rho_3\mu_2}{\rho_2} = \frac{\rho_3\rho_2\mu_1}{\rho_2\rho_1} = \frac{\rho_3\mu_1}{\rho_1}$, in order for $\rho_2\mu_3 - \rho_3\mu_2$ to vanish. Continue this, to obtain that $\mu_k = \frac{\rho_k\mu_1}{\rho_1}$, for every $2 \leq k \leq n-1$, which clearly shows that $\mu_2, \mu_3, \ldots, \mu_{n-1}$ all depend on the choice of μ_1, as we consider all the coefficients of x, namely $\rho_1, \rho_2, \ldots, \rho_{n-1}$, as constants, for the computation of y. Computing explicitly the dimensions of the centralizers of e_{23} and x in the quotient ${}^{\gamma_1}\!/_{\gamma_3}$ shows that they are different. We have that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(e_{12}) = n-1-2+n-2 = 2n-5$, because $\dim {}^{\gamma_1}\!/_{\gamma_2} = n-1$ and $\dim {}^{\gamma_2}\!/_{\gamma_3} = n-2$, but there are two basis elements in ${}^{\gamma_1}\!/_{\gamma_3}$ that do not commute with e_{23}. And we have that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = 1 + n - 2 = n-1$, because we just showed that all the coefficients μ_k of x in the quotient ${}^{\gamma_1}\!/_{\gamma_2}$ depend on one coefficient, namely μ_1. A simple induction shows that $n-1 < 2n-5$, for every $n \geq 5$. Setting $\rho_{k_1} = \rho_{k_2} = \cdots = \rho_{k_l} = 0$, for a subset of $1 \leq l \leq n-2$ indices, none of them is 2, yields diverse results. We shall not review all the different options here, but we shall partially demonstrate the basic technique with a very specific example, while the full proof shall be reviewed in the research itself. Suppose that $x = \varphi(e_{23}) = \rho_2 e_{23} + \rho_l e_{i,i+1} + z$, where $1 \leq i \leq n-1$ and $z \in \gamma_2\mathcal{L}_{n,p}$, and let $y = \sum_{k=1}^{n-1} \mu_k e_{k,k+1} + w$, where $w \in \gamma_2\mathcal{L}_{n,p}$, be an element in the centralizer of x. Loyal to the rule $[\rho_k e_{k,k+1} + \rho_{k+1}e_{k+1,k+2}, \mu_k e_{k,k+1} + \mu_{k+1}e_{k+1,k+2}] = [\rho_k e_{k,k+1}, \mu_{k+1}e_{k+1,k+2}] + [\rho_{k+1}e_{k+1,k+2}, \mu_k e_{k,k+1}] = (\rho_k\mu_{k+1} - \rho_{k+1}\mu_k)e_{k,k+2}$, then in order for the coefficient of $e_{k,k+2}$ to vanish, we must have $\rho_k\mu_{k+1} - \rho_{k+1}\mu_k = 0$, which means, as seen above, that $\mu_{k+1} = \frac{\rho_{k+1}\mu_k}{\rho_k}$, which means that if $x = \varphi(e_{23}) = \rho_2 e_{23} + \rho_3 e_{34} + z$ or if $x = \varphi(e_{23}) = \rho_1 e_{12} + \rho_2 e_{23} + z$ the product $[\rho_1 e_{12} + \rho_2 e_{23}, \mu_1 e_{12} + \mu_2 e_{23}]$ makes $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)$ decrease in 1, because of the dependency $\mu_2 = \frac{\rho_2\mu_1}{\rho_1}$ or $\mu_3 = \frac{\rho_3\mu_2}{\rho_2}$, respectively. Then, we have the product $[\rho_2 e_{23} + \rho_i e_{i,i+1}, \mu_3 e_{34}] = \rho_2\mu_3 e_{24} - \rho_i\mu_3 e_{3,i+1}$. In case $i > 4$, $e_{3,i+1}$ vanishes in the quotient ${}^{\gamma_1}\!/_{\gamma_3}$, so we need to annihilate only the scalar multiplication $\rho_2\mu_3 e_{24}$, which obviously means that $\mu_3 = 0$. If $i = 4$, we need to annihilate $[\rho_2 e_{23} + \rho_4 e_{45}, \mu_3 e_{34}] = \rho_2\mu_3 e_{24} - \rho_4\mu_3 e_{35}$, which means that we need to annihilate each of the expressions $\rho_2\mu_3 e_{24}$ and $\rho_4\mu_3 e_{35}$ separately. But this also means that $\mu_3 = 0$, so this is true for all $i \geq 4$. We now observe that if either $k > i+1$ or $3 < k < i$, then the product $[\rho_2 e_{23} + \rho_i e_{i,i+i}, \mu_k e_{k,k+1}] = \rho_2\mu_k e_{2,k} + \rho_i\mu_k e_{k,i}$, but since $k > 3$ either way, then $e_{2,k} = 0$, and since $k < i$ or $k > i+1$, then $e_{k,i} = 0$, both because we are in the quotient ${}^{\gamma_1}\!/\!_{\gamma_3}$. This means that the product $[\rho_2 e_{23} + \rho_i e_{i,i+i}, \mu_k e_{k,k+1}]$ vanishes regardless of the coefficient μ_k, which clearly means that we can choose any $\mu_k \in \mathbb{Q}_p$ and it is independent of any other coefficient of y. With all the above, one verifies that if we have $x = \varphi_{e_{23}} = \rho_2 e_{23} + \rho_i e_{i,i+1} + z$, then the dimension of the centralizer of x, relative to the quotient, is different than the centralizer of e_{23} itself, which means that the only linear combination allowed for $\varphi(e_{23})$ is $\rho_2 e_{23} + z$, which was already denoted above by $\lambda_2 e_{23} + z$. We continue to prove that $\varphi(e_{i,i+1}) = \lambda_i e_{i,i+1} + z_i$, for all $3 \leq i \leq n-2$, by induction on i, using this technique to prove the induction step. And so, we have the next rows of the block, namely $\varphi_{11}(e_{34}), \varphi_{11}(e_{45}), \ldots, \varphi_{11}(e_{n-2,n-1})$, while the last row of the block, namely $\varphi_{11}(e_{n-1,n})$, is already known from earlier. We obtain that for every $n \geq 5$, the block M_{11} is diagonal,

$$M_{11} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{n-1} \end{pmatrix}$$

In the research, we will show this is true also for $n = 4$, hence it is true for all $n \geq 4$. The importance of this fact is well explained in the following proposition,

Proposition 2.5.3. *Let $\mathcal{L}_{n,p}$ be the $\mathbb{Q}$-Lie algebra associated with $\mathcal{U}_n(\mathbb{Z}_p)$, where $n \geq 4$, and let $\varphi \in G_n(\mathbb{Q}_p)$ be an $\mathcal{L}_{n,p}$-automorphism. Then the* *coefficient matrix of φ denoted by M is of the form:*

$$M = \begin{pmatrix}
\lambda_1 & 0 & \dots & 0 & * & * & \dots & * & \dots & * \\
0 & \lambda_2 & \vdots & \vdots & * & * & \vdots & * & \dots & * \\
\vdots & \dots & \ddots & 0 & \vdots & \dots & \ddots & \vdots & \dots & * \\
0 & \dots & 0 & \lambda_{n-1} & * & * & \dots & * & \dots & * \\
 & & & & \lambda_1\lambda_2 & 0 & \dots & 0 & \dots & * \\
 & & & & 0 & \lambda_2\lambda_3 & \dots & \vdots & \dots & * \\
 & & & & \vdots & \dots & \ddots & 0 & \dots & \vdots \\
 & & & & 0 & \dots & 0 & \lambda_{n-2}\lambda_{n-1} & \dots & * \\
 & & & & & & & & \ddots & \\
 & & & & & & & & & \lambda_1\lambda_2\cdots\lambda_{n-1}
\end{pmatrix}$$

where $\lambda_1, \lambda_2, \dots, \lambda_{n-1} \in \mathbb{Q}_p^*$. In other words, each diagonal block M_{ii} is itself the $(n-i) \times (n-i)$ diagonal matrix

$$M_{ii} = \begin{pmatrix}
\lambda_1\lambda_2\cdots\lambda_i & & & \\
 & \lambda_2\lambda_3\cdots\lambda_{i+1} & & \\
 & & \ddots & \\
 & & & \lambda_{n-i}\lambda_{n-i+1}\cdots\lambda_{n-1}
\end{pmatrix}$$

This means that every $M = \varphi \in \mathcal{L}_{n,p}$ has the form,

$$M = \left(\begin{array}{c|c|c|c|c}
\begin{matrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{n-1} \end{matrix} & M_{12} & \dots & M_{1,n-2} & M_{1,n-1} \\
\hline
0 & \begin{matrix} \lambda_1\lambda_2 & & & \\ & \lambda_2\lambda_3 & & \\ & & \ddots & \\ & & & \lambda_{n-2}\lambda_{n-1} \end{matrix} & \dots & M_{2,n-2} & M_{2,n-1} \\
\hline
\vdots & \vdots & \dots & \vdots & \vdots \\
\hline
0 & 0 & \dots & \ddots & M_{n-2,n-1} \\
\hline
0 & 0 & \dots & 0 & \lambda_1\lambda_2\cdots\lambda_{n-1}
\end{array}\right)$$

Before we continue with some conclusions from this highly important observation, we need to go back to the Heisenberg group, where $n = 3$, and explain why it does not follow this rule, as found for the groups where $n > 3$. The reasons is that for $\mathcal{L}_{3,p}$, both e_{12} and e_{23} have that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(e_{12}) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(e_{23}) = \dim \mathcal{L}_{3,p} - 1 = 2$, so our lead of finding an element whose centralizer is larger does not apply for this case. We can show this explicitly, by observing that $\mathcal{C}_{\mathcal{L}_{3,p}}(e_{12}) = \langle e_{12}, e_{13}\rangle$, which means that $\dim \mathcal{C}_{\mathcal{L}_{3,p}}(e_{12}) = 2$, but, assuming that $\varphi(e_{12}) = \lambda e_{12} + \mu e_{23} + \rho e_{13}$, where $\lambda, \mu, \rho \in \mathbb{Q}_p$ and at least λ and μ are non-zero, then $\mathcal{C}_{\mathcal{L}_{3,p}}(\varphi(e_{12})) = \langle \tau e_{12} + \nu e_{23}, e_{13}\rangle$, where $\lambda\nu = \mu\tau$, which means that $\dim \mathcal{C}_{\mathcal{L}_{3,p}}(\varphi(e_{12})) = 2$ as well. The second row $\varphi(e_{23})$ is obtained exactly in the same way. The above discussion gives rise to the basic strategy of decomposing each automorphism $\varphi \in G(\mathbb{Q}_p)$ into a product of two automoprhisms $\varphi = \mathfrak{n}\mathfrak{h}$, one is represented by the diagonal matrix

$$\mathfrak{h} = \left(\begin{array}{cccc|cccc|c|c}
\lambda_1 & & & & & & & & & \\
& \lambda_2 & & & & & & & & \\
& & \ddots & & & & & & & \\
& & & \lambda_{n-1} & & & & & & \\
\hline
& & & & \lambda_1\lambda_2 & & & & & \\
& & & & & \lambda_2\lambda_3 & & & & \\
& & & & & & \ddots & & & \\
& & & & & & & \lambda_{n-2}\lambda_{n-1} & & \\
\hline
& & & & & & & & \ddots & \\
\hline
& & & & & & & & & \lambda_1\lambda_2\cdots\lambda_{n-1}
\end{array}\right)$$

and the other is represented by a unipotent matrix of the form,

$$\mathfrak{n} = \left(\begin{array}{cccc|ccc|c}
1 & 0 & \dots & 0 & * & \dots & * & * \\
0 & 1 & \vdots & 0 & * & \dots & * & * \\
\vdots & \dots & \ddots & 0 & * & \dots & * & * \\
0 & \dots & 0 & 1 & * & \dots & * & * \\
\hline
0 & \dots & 0 & 0 & 1 & \dots & 0 & * \\
0 & \dots & 0 & 0 & 0 & 1 & 0 & * \\
0 & \dots & 0 & 0 & 0 & \dots & 1 & * \\
\hline
0 & \dots & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

where the main diagonal blocks of $\mathfrak{n}$ are $I_{n-1}, I_{n-2}, \ldots, I_1$. It is readily seen that the decomposition $\varphi = \mathfrak{n}\mathfrak{h}$ is unique, because, let $N_n(\mathbb{Q}_p)$ be the set of all matrices of the same form of $\mathfrak{n}$ and let $H_n(\mathbb{Q}_p)$ be the set of all matrices of the same form of $\mathfrak{h}$, and assume that we have $\varphi = \mathfrak{n}\mathfrak{h} = \mathfrak{n}'\mathfrak{h}'$, where $\mathfrak{n}' \neq \mathfrak{n}$ and $\mathfrak{h}' \neq \mathfrak{h}$. This means that $\mathfrak{n}'^{-1}\mathfrak{n} = \mathfrak{h}'\mathfrak{h}^{-1}$, but then $\mathfrak{n}'^{-1}\mathfrak{n}, \mathfrak{h}'\mathfrak{h}^{-1} \in N_n(\mathbb{Q}_p) \cap H_n(\mathbb{Q}_p) = \{I\}$, hence $\mathfrak{n} = \mathfrak{n}'$ and $\mathfrak{h}' = \mathfrak{h}$, which contradicts the assumption. This observation gives the following set of propositions, one easily checks that all of them are correct,

Proposition 2.5.4. *Let $H_n(\mathbb{Q}_p)$ be the set of all matrices of the same form of $\mathfrak{h}$, then $H_n(\mathbb{Q}_p)$ is an abelian subgroup which forms the **reductive** part of $G_n(\mathbb{Q}_p)$*

Proposition 2.5.5. *Let $N_n(\mathbb{Q}_p)$ be the set of all matrices of the same form of $\mathfrak{n}$, then $N_n(\mathbb{Q}_p)$ is a normal subgroup which forms the **unipotent radical** of $G_n(\mathbb{Q}_p)$*

The unique decomposition $\varphi = \mathfrak{n}\mathfrak{h}$ suggests that the $\mathcal{L}_{n,p}$-automorphism group itself has the unique decomposition $G_n(\mathbb{Q}_p) = N_n(\mathbb{Q}_p)H_n(\mathbb{Q}_p)$ and that $N_n(\mathbb{Q}_p) \cap H_n(\mathbb{Q}_p) = \{I\}$. Also, it is readily seen that $H_n(\mathbb{Q}_p)$ it not a normal subgroup of $G_n(\mathbb{Q}_p)$. This leads us to the following result,

Corollary 2.5.6. *Let $G_n(\mathbb{Q}_p)$ be the group of $\mathcal{L}_{n,p}$-automorphisms, and let $\phi : H_n(\mathbb{Q}_p) \to Aut(N_n(\mathbb{Q}_p))$ be the homomorphism defined by $\phi(\mathfrak{h})(\mathfrak{n}) := \mathfrak{h}\mathfrak{n}\mathfrak{h}^{-1}$, for every $\mathfrak{h} \in H_n(\mathbb{Q}_p)$ and every $\mathfrak{n} \in N_n(\mathbb{Q}_p)$. Then, $G_n(\mathbb{Q}_p) = N_n(\mathbb{Q}_p) \rtimes_\phi H_n(\mathbb{Q}_p)$, in words, $G_n(\mathbb{Q}_p)$ decomposes into an inner semidirect product of $N_n(\mathbb{Q}_p)$ and $H_n(\mathbb{Q}_p)$, induced by conjugation.*

This corollary enables us to simplify the domain of integration, for the p-adic integral that we aim to calculate, at the price of replacing a single integral by multiple integrals. As we saw earlier, the calculation of $\hat{\zeta_{L_{n,p}}}$ requires computing $G_n(\mathbb{Z}_p)$ and $G_n^+(\mathbb{Q}_p)$ first. Assuming we have already computed $G_n(\mathbb{Q}_p)$, based on the strategy that we have presented above, we need to identify $G_n(\mathbb{Z}_p)$ as a subgroup of $G_n(\mathbb{Q}_p)$, which is expected not to be difficult, and continue from there to identify the monoid $G_n^+(\mathbb{Q}_p)$, which is expected to be a substantial challenge. By applying **Fubini's theorem** for semidirect products of topological groups, we have that

$$\hat{\zeta}_{\mathcal{L}_{n,p}}(s) = \int_{G_n^+(\mathbb{Q}_p)} |\det \varphi|_p^s d\mu_{G_n(\mathbb{Z}_p)\varphi} = \int_{H_n^+(\mathbb{Q}_p)} \left(\int_{N_{\mathfrak{h}}^+} |\det \mathfrak{n}\mathfrak{h}|_p^s d\mu_{N_n(\mathbb{Q}_p)} \right) d\mu_{H_n(\mathbb{Q}_p)}$$

where $H_n^+(\mathbb{Q}_p)$ consists of all $\mathfrak{h} \in H_n(\mathbb{Q}_p)$ that appear in the decomposition $\varphi = \mathfrak{n}\mathfrak{h}$ for some $\varphi \in G_n^+(\mathbb{Q}_p)$, and, for a given $\mathfrak{h} \in H_n^+(\mathbb{Q}_p)$, we set $N_{\mathfrak{h}}^+(\mathbb{Q}_p) := \{\mathfrak{n} \in N_n(\mathbb{Q}_p) : \mathfrak{n}\mathfrak{h} \in G_n^+(\mathbb{Q}_p)\}$. The integrand of the inner integral is constant, so the integral amounts to computing the measure of the set $N_{\mathfrak{h}}^+(\mathbb{Q}_p)$. The advantage that we gain by this decomposition is that it simplifies the calculation of the integral. The integral function $|\det \mathfrak{n}\mathfrak{h}|_p^s = |\det \mathfrak{h}|_p^s$ depends only on $\mathfrak{h}$, so computing the inner integral amounts to finding the measure of $N_{\mathfrak{h}}^+(\mathbb{Q}_p)$. For the unipotent matrix $\mathfrak{n}$, the determinant is 1, for the digonal matrix $\mathfrak{h}$, we have the following proposition,

Proposition 2.5.7. *Let $n = 2, 3, 4, \ldots$, and let A_n be the diagonal $m \times m$ matrix where $m = \binom{n}{2}$, of the same form of $\mathfrak{h}$ from above for $n-1$ scalars, then* $\det A_n = \prod_{i=1}^{n-1} \lambda_i^{i(n-i)}$.

One easily checks this by simple induction on n, based on the increase in multiplicity of each factor λ_i when transitioning from n to $n+1$. We conclude this section with the observation that the sets $N_{\mathfrak{h}}^+(\mathbf{Q_p})$ arising in our computation are quite complicated. We will decompose $N_n(\mathbb{Q}_p)$ into an iterated semidirect product of a large number of subgroups, each abelian with a simple structure. This will decompose the integral over $N_{\mathfrak{h}}^+(\mathbb{Q}_p)$ into a multiple integral that can be computed explicitly, given a suitable combinatorical framework. Hence, we strive to decompose $N_n(\mathbb{Q}_p)$ itself into a product of finitely many simpler subgroups, $N_n(\mathbb{Q}_p) = N_n(\mathbb{Q}_p)_1 \rtimes_\phi N_n(\mathbb{Q}_p)_2 \rtimes_\phi \cdots \rtimes_\phi N_n(\mathbb{Q}_p)_{m_n}$, where m_n is the number of subgroups in the decomposition of $N_n(\mathbb{Q}_p)$, for every $n \in \mathbb{N}$, which means that $\displaystyle\int_{G_n^+(\mathbb{Q}_p)} |\det \varphi|_p^s d\mu_{G_n(\mathbb{Q}_p)} =$

$$\int_{H_n^+(\mathbb{Q}_p)} \left(\int_{\mathcal{N}_{m_n}^+} \cdots \left(\int_{\mathcal{N}_3^+} \left(\int_{\mathcal{N}_2^+} \left(\int_{\mathcal{N}_1^+} |\det \varphi|_p^s d\mu_{\mathcal{N}_1} \right) d\mu_{\mathcal{N}_2} \right) d\mu_{\mathcal{N}_3} \right) \cdots d\mu_{\mathcal{N}_{m_n}} \right) d\mu_{H_n(\mathbb{Q}_p)},$$

where we denote $\mathcal{N}_i := N_n(\mathbb{Q}_p)_i$ and $\mathcal{N}_i^+ := N_n^+(\mathbb{Q}_p)_i$, for every $1 \leq i \leq m_n$. One checks that every $\mathcal{N}_i^+$ depends on $\mathfrak{h}, \mathfrak{n}_1, \mathfrak{n}_2, \ldots, \mathfrak{n}_{i-1}$, if $\varphi = \mathfrak{n}_{m_n} \cdots \mathfrak{n}_2\mathfrak{n}_1\mathfrak{h}$, where $\mathfrak{h} \in H_n^+(\mathbb{Q}_p)$ and $\mathfrak{n}_k \in \mathcal{N}_k$, for every $1 \leq k \leq i-1$. All the subgroups in the decomposition of $N_n(\mathbb{Q}_p)$ are obviously unipotent as well, which means that their determinants are also 1. This means that computing the inner integrals amounts to determining the measure of the sets $\mathcal{N}_i^+$ in terms of $\mathfrak{h}, \mathfrak{n}_1, \mathfrak{n}_2, \ldots, \mathfrak{n}_{n-1}$.

2.6 Base Extension

Let K be a number field of degree $d = [K : \mathbb{Q}]$, and let $\mathcal{O}_K$ be its ring of integers. Let L be a $\mathbb{Z}$-Lie algebra of rank r. By base extension we can consider $L \otimes_{\mathbb{Z}} \mathcal{O}_K$ as a $\mathbb{Z}$-Lie algebra of rank rd, and by extension of scalars we can consider also $L_{K,p} = (L \otimes_{\mathbb{Z}} \mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ as a $\mathbb{Q}_p$-Lie algebra of the same rank. Berman-Glazer-Schein give a criterion in [2], under which the pro-isomorphic zeta-function of $L_{K,p}$ can be calculated without a significant extra effort relative to that of L_p itself. Note that the criterion does not necessarily apply for all p. We shall research whether the criterion applies to the $\mathbb{Q}_p$-Lie algebras $\mathcal{L}_{n,p}$ of our work. If so, then $\hat{\zeta}_{\mathcal{L}_n \otimes \mathcal{O}_K, p}$ will be finitely uniform (see 1.3.4). In other words, for each of the finitely many decomposition types of a prime in $\mathcal{O}_K$, there is a rational function in two variables $W \in \mathbb{Q}(X, Y)$ such that $\hat{\zeta}_{\mathcal{L}_n \otimes \mathcal{O}_K, p}(s) = W(p, p^{-s})$ for all p of that decomposition type.

References

[1] Mark. N. Berman, Proisomorphic zeta functions of groups , Ph.D. thesis, University of Oxford, 2005.

[2] Mark N. Berman, Itay Glazer, and Michael M. Schein, Pro-isomorphic zeta functions of nilpotent groups and lie rings under base extension, Trans. Amer. Math. Soc. 375 (2022), 1051–1100.

[3] M.P.F. du Sautoy and A. Lubotzky, Functional equations and uniformity for local zeta functions of nilpotent groups, Amer. J. Math. 118 (1996), no. 1, 39– 90.

[4] John A. Gibbs, Automorphisms of certain unipotent groups, J. Algebra 14 (1970), 203-228.

[5] F. J. Grunewald, D. Segal, and G. C. Smith, Subgroups of finite index in nilpotent groups, Invent. Math. 93 (1988), no. 1, 185–223.

[6] Alexander Lubotzky, Avinoam Mann, and Dan Segal, Finitely generated groups of polynomial subgroup growth , Israel J. Math. 82 (1993), no. 1-3, 363–371.

[7] Hugh L. Montgomery and Robert C. Vaughan, Multiplicative Number Theory I. Classical Theory, Cambridge Studies in Advanced Mathematics 97.