1 The computation of $G_n(\mathbb{Z}_p)$

Proposition 1.1. *Let $\varphi_r \in N(\mathbb{Q}_p)$ be a unipotent automorphism of $\mathcal{L}_{n,p}$ such that the matrix upper blocks M_{1k}, for all $2 \leq k \leq r-1$, are zero matrices. Consider the $(n-1)\times(n-r)$ matrix $M_{1r} = (a_{ij})$. Then,*

1. *Let $2 \leq r < n-2$. If $a_{ij} \neq 0$, then either $i = j$ or $i = j + r - 1$, and we have the relation $a_{i+r,i+1} = -a_{ii}$.*
2. *Let $r = n-2$. If $a_{ij} \neq 0$, then either $i = j$ or $i = j + r - 1$ or $(i,j) \in \{(1,2),(n-1,1)\}$, with the same relation as above.*

Proof. From the relation $[\varphi_r(e_{k,k+1}), \varphi_r(e_{l,l+1})] = 0$ where $l > k+1$, we deduce that $a_{ij} \neq 0$ only if either $i = j$ or $i = j + r - 1$ or
$(i,j) \in \{(r+1,1),(r+2,1),(n-r-2,n-r),(n-r-1,n-r)\}$. If $r < n-2$ then it follows from the conditions

$$\begin{aligned}
[\varphi_r(e_{n-r-2,n-r-1}), \varphi_r(e_{n-r-2,n-r})] &= 0\\
[\varphi_r(e_{n-r-1,n-r}), \varphi_r(e_{n-r-2,n-r})] &= 0\\
[\varphi_r(e_{r+1,r+2}), \varphi_r(e_{r+1,r+3})] &= 0\\
[\varphi_r(e_{r+2,r+3}), \varphi_r(e_{r+1,r+3})] &= 0
\end{aligned}$$

that the four exceptional cases cannot occur. When $r = n-2$, we have that $(n-r-2, n-r) = (0,2)$ and $(r+2,1) = (n,1)$ so these cases do not exist, but so are the four conditions above, which means that the two remaining cases, $(r+1,1) = (n-1,1)$ and $(n-r-1,n-r) = (1,2)$, do not necessarily vanish. □

Proposition 1.2. *Denote by $N_r := \{\varphi_r : 2 \leq r \leq n-2\} \subset N(\mathbb{Q}_p)$ the set of all automorphisms of the form described in 1.1, then $N_r \leq N(\mathbb{Q}_p)$. Note that $N_2 = N(\mathbb{Q}_p)$.*

Proposition 1.3. *Let $2 \leq r \leq n-2$, and let $0 \leq k \leq n-r$, and let $a \in \mathbb{Q}_p$. We extend our notation of basis elements to include $e_{01} = e_{n,n+1} = 0$.*

1. *There is an automorphism $\varphi_{r,k}(a) \in N(\mathbb{Q}_p)$ determined by*

$$\varphi_{r,k}(a)(e_{i,i+1}) := \begin{cases} e_{k,k+1} + a e_{k,k+r} & : i = k \\ e_{k+r,k+r+1} - a e_{k+1,k+r+1} & : i = k+r \\ e_{i,i+1} & : i \notin \{k, k+r\} \end{cases}$$

2. *Suppose that $r = n-2$, let $(k,l) \in \{(1,2),(n-1,1)\}$, and let $a \in \mathbb{Q}_p$. There is an automorphism $\varphi_{n-2,k,l}(a) \in G_n^0(\mathbb{Q}_p)$ determined by*

$$\varphi_{n-2,k,l}(a)(e_{i,i+1}) := \begin{cases} e_{k,k+1} + a e_{l,l+r} & : i = k \\ e_{i,i+1} & : i \neq k \end{cases}$$

   We denote $\varphi'_{n-2}(a) := \varphi_{n-2,1,2}(a)$ and $\varphi''_{n-2}(a) := \varphi_{n-2,n-1,1}(a)$.

Proof. We need to verify that for all $1 \leq i < j \leq n$ and $1 \leq l < m \leq n$ we have the following relations

$$[\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{lm})] = \begin{cases} \varphi_{r,k}(a)(e_{im}) & : j = l \\ -\varphi_{r,k}(a)(e_{lj}) & : i = m \\ 0 & : \text{otherwise} \end{cases}$$

We can verify explicitly for $n = 4$ that these relations are true. Alternatively, Berman did this in [**?**, §3.3.7]. For $n > 4$, let $m = n$, then

$$[\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{lm})] = [\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{ln})].$$

If $i > 1$, then we consider the inclusion $\iota : \mathcal{L}_{n-1,p} \hookrightarrow \mathcal{L}_{n,p}$, mapping each $e_{i,i+1} \in \mathcal{L}_{n-1,p}$ to $e_{i+1,i+2} \in \mathcal{L}_{n,p}$ for all $1 \leq i \leq n-2$. By the assumption on $\mathcal{L}_{n-1,p}$, we have that

$$(\iota \circ \iota^{-1})([\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{ln})]) = \iota([\iota^{-1}(\varphi_{r,k}(a)(e_{ij})), \iota^{-1}(\varphi_{r,k}(a)(e_{ln}))]) =$$

$$= \iota([\varphi_{r,k}(a)(e_{i-1,j-1}), \varphi_{r,k}(a)(e_{l-1,n-1})]) =$$

$$= \begin{cases} \iota(\varphi_{r,k}(a)(e_{i-1,n-1})) = \varphi_{r,k}(a)(e_{in}) & : j = l \\ 0 & : j \neq l \end{cases}$$

If $i = 1$, then

$$[\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{ln})] = [\varphi_{r,k}(a)(e_{1j}), \varphi_{r,k}(a)(e_{ln})] =$$

$$= [\varphi_{r,k}(a)(e_{1j}), [\varphi_{r,k}(a)(e_{l,n-1}), \varphi_{r,k}(a)(e_{n-1,n})]].$$

By the Jacobi identity, we have that

$$[\varphi_{r,k}(a)(e_{1j}), [\varphi_{r,k}(a)(e_{l,n-1}), \varphi_{r,k}(a)(e_{n-1,n})]] =$$

$$= -[\varphi_{r,k}(a)(e_{n-1,n}), [\varphi_{r,k}(a)(e_{1j}), \varphi_{r,k}(a)(e_{l,n-1})]].$$

Now we use the inclusion $\iota' : \mathcal{L}_{n-1,p} \hookrightarrow \mathcal{L}_{n,p}$, where $\iota'(e_{i,i+1}) = e_{i,i+1}$ for all $1 \leq i \leq n-1$, to obtain, same as above, that

$$-[\varphi_{r,k}(a)(e_{n-1,n}), [\varphi_{r,k}(a)(e_{1j}), \varphi_{r,k}(a)(e_{l,n-1})]] = -[\varphi_{r,k}(a)(e_{n-1,n}), \varphi_{r,k}(a)(e_{1,n-1})].$$

To continue, we need to prove the following auxiliary proposition:

Proposition 1.4.

$$\varphi_{r,k}(a)(e_{k+1-h,k+1}) = e_{k+1-h,k+1} + ae_{k+1-h,k+r} : h > 0$$

$$\varphi_{r,k}(a)(e_{k+r,k+r+h}) = e_{k+r,k+r+h} - ae_{k+1,k+r+h} : h > 0$$

$$\varphi_{r,k}(a)(e_{ij}) = e_{ij} : i \neq k + r \wedge j \neq k + 1$$

Proof. For $h = 1$, $\varphi_{r,k}(a)(e_{k,k+1}) = e_{k,k+1} + ae_{k,k+r}$. For $h' = h > 1$, $\varphi_{r,k}(a)(e_{k+1-h',k+1}) = [\varphi_{r,k}(a)(e_{k-h,k-h+1}), \varphi_{r,k}(a)(e_{k-h+1,k+1})]$. By the assumption, we have that $[\varphi_{r,k}(a)(e_{k-h,k-h+1}), \varphi_{r,k}(a)(e_{k-h+1,k+1})] =$

$$= [\varphi_{r,k}(a)(e_{k-h,k-h+1}), e_{k+1-h,k+1} + ae_{k+1-h,k+r}].$$

But for all $h > 0$, we have that $k - h \neq k$ and $k - h + 1 \neq k + 1$, and hence

$$[\varphi_{r,k}(a)(e_{k-h,k-h+1}), e_{k+1-h,k+1} + ae_{k+1-h,k+r}] =$$

$$= [e_{k-h,k-h+1}, e_{k+1-h,k+1} + ae_{k+1-h,k+r} = e_{k-h,k+1} + ae_{k-h,k+r} =$$

$$= e_{k+1-h',k+1} + ae_{k+1-h',k+r}.$$

We prove the two other cases in the same way. □

By 1.4 we have that

$$\varphi_{r,k}(a)(e_{1,n-1}) = \begin{cases} e_{1,n-1} + ae_{1,n} & : r = 2 \land k = n-2 \\ e_{1,n-1} & : \text{otherwise} \end{cases}$$

while $k + r \neq 1$ for all r, k. Thus, $-[\varphi_{2,n-2}(a)(e_{n-1,n}), \varphi_{2,n-2}(a)(e_{1,n-1})] = -[\varphi_{2,n-2}(a)(e_{n-1,n}), e_{1,n-1} + ae_{1,n}] = [e_{n-1,n}, e_{1,n-1} + ae_{1,n}] = e_{1n}$.

For $(r, k) \notin (2, n-2)$, if $k+r = n-1$ then $-[\varphi_{r,k}(a)(e_{n-1,n}), \varphi_{r,k}(a)(e_{1,n-1})] = -[e_{n-1,n} - ae_{n-r,n}, e_{1,n-1}] = e_{1n}$, otherwise $-[\varphi_{r,k}(a)(e_{n-1,n}), \varphi_{r,k}(a)(e_{1,n-1})] = -[e_{n-1,n}, e_{1,n-1}] = e_{1n}$ □

Fix the two parameters $2 \leq r \leq n-2$ and $0 \leq k \leq n-r$, and denote by $N_{r,k} := \{\varphi_{r,k}(a) : a \in \mathbb{Q}_p\} \subset N(\mathbb{Q}_p)$ the set of all automorphisms of this form. Also denote $N'_{n-2} := \{\varphi'_{n-2}(a) : a \in \mathbb{Q}_p\}$ and $N''_{n-2} := \{\varphi''_{n-2}(a) : a \in \mathbb{Q}_p\}$.

Proposition 1.5. *Let $N_{r,k}$, N'_{n-2} and N''_{n-2} be the subsets defined above, then*

1. $N_{r,k}, N'_{n-2}, N''_{n-2} \leq N(\mathbb{Q}_p)$.
2. $N_{r,k}, N'_{n-2}, N''_{n-2} \cong \mathbb{Q}_p$.

Proof. A simple check shows that these subsets are subgroups of $N(\mathbb{Q}_p)$. Define $\tau_{r,k} : \mathbb{Q}_p \to N_{r,k}$. For every $a, b \in \mathbb{Q}_p$, it is easy to see that the image of the sum, $\tau_{r,k}(a + b) = \tau_{r,k}(a) \cdot \tau_{r,k}(b)$, is the product of the images of a and b, and that $\tau_{r,k}^{-1}(I) = \{0\}$. □

The following proposition follows from a simple computation.

Proposition 1.6. *Consider $\varphi_r \in N_r$.*

1. *If $r < n-2$, denote by ψ_r the automorphism*

$$\psi_r := \varphi_r \circ \varphi_{r,n-r}(-a_{n-r,n-r}) \circ \cdots \circ \varphi_{r,1}(-a_{11}) \circ \varphi_{r,0}(-a_{r+1,1}).$$

Then $\psi_r \in N_{r+1}$.

2. *If $r = n-2$, denote by ψ_{n-2} the automorphism*

$$\psi_{n-2} := \varphi_r \circ \varphi'_{n-2}(-a_{12}) \circ \varphi''_{n-2}(-a_{n-1,1}) \circ$$
$$\circ \varphi_{n-2,2}(-a_{n-2,2}) \circ \varphi_{n-2,1}(-a_{n-2,1}) \circ \varphi_{n-2,0}(-a_{n-2,0}).$$

Then $\psi_{n-2} \in N_{n-1}$.

Proof. By the definition, φ_r has 1 on the main diagonal and all the upper blocks M_{1k}, for $2 \leq k \leq r-1$ are zero matrices. One checks that the composition of φ_r and the chain of compositions $\prod_{k=1}^{n-r} \varphi_{r,k}(-a_{kk}) \circ \varphi_{r,0}(-a_{r+1,1})$ yields also a matrix with 1 on the main diagonal whose upper blocks M_{1k}, for all $2 \leq k \leq r$, are zero matrices, thus $\psi_r \in N_{r+1}$. Same applies for the case $r = n-2$, considering the specific structure of $M_{1,n-2}$, as described in the second part of 1.1. □

Corollary 1.7. *We have the following decompositions:*

1. *For all $2 \leq r < n-2$, we have*

$$N_r = N_{r+1} \rtimes (N_{r,0} \rtimes (\cdots (N_{r,n-r-1} \rtimes N_{r,n-r}) \cdots)).$$

2. *For $r = n-2$, we have*

$$N_{n-2} = N_{n-1} \rtimes (N_{n-2,0} \rtimes (N_{n-2,1} \rtimes (N_{n-2,2} \rtimes (N''_{n-2} \rtimes N'_{n-2})))).$$

Proof. This is immediate from Proposition 1.6, since we have that

$$\varphi_r = \psi_r \circ \varphi_{r,0}(-a_{r+1,1}) \circ \varphi_{r,1}(-a_{11}) \circ \cdots \circ \varphi_{r,n-r}(-a_{n-r,n-r}).$$

□

Corollary 1.7 provides a recursive decomposition of the unipotent radical $N(\mathbb{Q}_p)$ as an iterated semidirect product of N_{n-1} and subgroups isomorphic to $\mathbb{Q}_p$.

As we saw earlier, the calculation of $\zeta^{\wedge}_{L_{n,p}}(s)$ requires understanding $G_n(\mathbb{Z}_p)$ and $G_n^+(\mathbb{Q}_p)$ first. As $G_n(\mathbb{Z}_p)$ is a group, its structure is easily deduced from the above.

Proposition 1.8. *For all $n \geq 4$ the group $G_n^0(\mathbb{Z}_p)$ has the decomposition $G_n^0(\mathbb{Z}_p) = N(\mathbb{Z}_p) \rtimes H(\mathbb{Z}_p)$, where*

$$N(\mathbb{Z}_p) := M_{\binom{n}{2}}(\mathbb{Z}_p) \cap N(\mathbb{Q}_p),$$

$$H(\mathbb{Z}_p) := \{\mathrm{diag}(\lambda_1, \lambda_2, \ldots,) : \lambda_1, \ldots, \lambda_{n-1} \in \mathbb{Z}_p^*\}.$$

Moreover, $N(\mathbb{Z}_p)$ itself has the decomposition:

$$N(\mathbb{Z}_p) = \tilde{N}_2(\mathbb{Z}_p) = \tilde{N}_{n-1} \rtimes (\tilde{N}_{n-2,0} \rtimes (\tilde{N}_{n-2,1} \rtimes (\tilde{N}_{n-2,2} \rtimes (\tilde{N}''_{n-2} \rtimes \tilde{N}'_{n-2})))) \rtimes \cdots$$
$$\cdots \rtimes (\tilde{N}_{2,0} \rtimes (\cdots (\tilde{N}_{2,n-3} \rtimes \tilde{N}_{2,n-2}) \cdots)),$$

where $\tilde{N}_r = N_r \cap N(\mathbb{Z}_p)$ and $\tilde{N}_{r,k} = \{\varphi_{r,k}(a) : a \in \mathbb{Z}_p\}$.

By contrast, describing the structure of the monoid $G_n^+(\mathbb{Q}_p)$ is expected to be a substantial challenge.

By applying Fubini's theorem for semidirect products of topological groups [**?**, Proposition 28], we have that

$$\zeta^{\wedge}_{L_{n,p}}(s) = \int_{G_n^+(\mathbb{Q}_p)} |\det \varphi|_p^s d\mu_{G_n(\mathbb{Z}_p)\varphi} = \int_{H^+(\mathbb{Q}_p)} \left(\int_{N_h^+} |\det uh|_p^s d\mu_{N(\mathbb{Q}_p)} \right) d\mu_{H(\mathbb{Q}_p)},$$

where

$$H^+(\mathbb{Q}_p) := \{\mathrm{diag}(\lambda_1, \dots, \lambda_{n-1}, \lambda_1\lambda_2, \dots, \lambda_1\lambda_2 \cdots \lambda_{n-2}\lambda_{n-1}) : \lambda_i \in \mathbb{Z}_p \setminus \{0\}\},$$

that is, $H^+(\mathbb{Q}_p)$ consists of all $h \in H(\mathbb{Q}_p)$ that appear in the decomposition $\varphi = uh$ for some $\varphi \in G_n^+(\mathbb{Q}_p)$, and, for a given $h \in H^+(\mathbb{Q}_p)$, we set $N_h^+ := \{u \in N(\mathbb{Q}_p) : uh \in G_n^+(\mathbb{Q}_p)\}$. The integrand is constant on N_h^+, so computing the inner integral amounts to finding the measure of N_h^+, which is complicated, but using the decomposition from Corollary 1.7, we can simplify N_h^+ at the price of replacing a single integral by multiple integrals.

Let $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_m$, where $m = \binom{n}{2}$, be an enumeration of the subgroups

$$N_{2,n-2}, N_{2,n-3}, \dots, N_{2,0}, N_{3,n-3}, \dots, N_{3,0}, \dots$$

$$\dots, N_{n-2,2}, N_{n-2,1}, N_{n-2,0}, N'_{n-2}, N''_{n-2}, N_{n-1}.$$

Every $\varphi \in G_n^0(\mathbb{Q}_p)$ can be written uniquely as $\varphi = u_m \cdots u_1 h$, where $u_i \in \mathcal{N}_i$. Thus, by Fubini

$$\zeta^{\wedge}_{L_{n,p}}(s) = \int_{H^+} \int_{\mathcal{N}_1^+(h)} \int_{\mathcal{N}_2^+(h,u_1)} \cdots \int_{\mathcal{N}_m^+(h,u_1,\dots,u_{m-1})} |\det h|_p^s d\mu_H d\mu_{\mathcal{N}_1} \cdots d\mu_{\mathcal{N}_m},$$

where each $\mu_{\mathcal{N}_i}$ is Haar measure on $\mathcal{N}_i(\mathbb{Q}_p)$ normalized so that $\mu_{\mathcal{N}_i}(\mathcal{N}_i(\mathbb{Z}_p)) = 1$, and

$$\mathcal{N}_i^+(h, u_1, \dots, u_{i-1}) := \{u_i \in \mathcal{N}_i : \exists u_{i+1}, \dots, u_m \,\text{such that}\, u_m \cdots u_1 h \in G_n^+(\mathbb{Q}_p)\}.$$