1. The group U_n

Proposition 1.1
Let $E_n = \{E_{i,j}\}_{i<j}$ be the set of all $n \times n$ matrices, $(e_{l,k})$,
where $a_{l,l} = 1, 1 \le l \le n$, and $a_{i,j} = 1, i < j$, and all other elements are zero. That is, $E_{i,j}$ has 1 only on the main diagonal, and in one element, anywhere above the main diagonal. Let A be any $n \times n$ matrix. Then,
Multiplying A by $E_{i,j}$ (from the left), $E_{i,j} \times A$, is operating as performing the elementary operation $R_i \leftarrow R_i + R_j$ on A

Proof. $A = (a_{l,k}), B = (b_{l,k}) = E_{i_j} \times A = (e_{l,k}) \times (a_{l,k})$
$b_{l,k} = \sum_{r=1}^{n} e_{l,r} \cdot a_{r,k}$
For all the rows, except for row i, $b_{l,k} = \sum_{r=1}^{n} e_{l,r} \cdot a_{r,k} = 0 + 0 + \cdots + e_{l,l} \cdot a_{l,k} + 0 + 0 + \cdots + 0 + 0 = 1 \cdot a_{l,k} = a_{l,k}$
For row i, $b_{i,k} = \sum_{r=1}^{n} e_{i,r} \cdot a_{r,k} = 0 + 0 + \cdots + e_{i,i} \cdot a_{i,k} + 0 + 0 + e_{i,j} \cdot a_{j,k} + 0 + 0 + \cdots + 0 + 0 = 1 \cdot a_{i,k} + 1 \cdot a_{j,k} = a_{i,k} + a_{j,k}$
This shows that the multiplication preserves the rows of A, except for row i, which becomes the sum of rows i, j $\square$

Corollary 1.2
Let $E_{i,j} = (e_{l,k}), i < j \in E_n$, Then,
$E_{i,j}^{-1} = (a_{l,k})$, where $a_{l,l} = 1, 1 \le l \le n$, and $a_{i,j} = -1, i < j$, and all other elements are zero.

Proof. $(b_{l,k}) = E_{i,j} \times (a_{l,k})$
Multiplying $(a_{l,k})$ by $E_{i,j}$ from the left is operating on $(a_{l,k})$ as a row addition, $R_i \leftarrow R_i + R_j$, as seen above.
For all $1 \le k \le n, b_{i,k} = a_{i,k} + a_{j,k}$
But, the only element in row j that is not zero is $a_{j,j=1}$, so, $b_{i,j} = a_{i,j} + a_{j,j} = -1 + 1 = 0$, and, for all the other columns, $a_{j,k} = 0$, so $b_{i,i} = a_{i,i} + a_{j,i} = 1 + 0 = 1$, and $b_{i,k} = a_{i,k} + a_{j,k} = 0 + 0 = 0$, which means that $(b_{l,k}) = I_n$
Easy to verify that also $(a_{l,k}) \times E_{i,j} = I_n$, and that $(a_{l,k})$ is a unique inverse of $E_{i,j}$, since, suppose we have another inverse matrix, $M = E_{i,j}^{-1}$, then $(a_{l,k}) \times E_{i,j} = I_n = M \times E_{i,j} \Rightarrow ((a_{l,k}) \times E_{i,j}) \times M = (M \times E_{i,j}) \times M \Rightarrow (a_{l,k}) \times E_{i,j} \times M = M \times E_{i,j} \times M \Rightarrow (a_{l,k}) \times (E_{i,j} \times M) = M \times (E_{i,j} \times M) \Rightarrow (a_{l,k}) \times I_n = M \times I_n \Rightarrow (a_{l,k}) = M$
So, $(a_{l,k}) = E_{i,j}^{-1}$ is the unique inverse of $E_{i,j}$

$\square$

Corollary 1.3
Let $E_{i,j} = (e_{l,k}), i < j \in E_n$, and $A = (a_{l,k})$ any $n \times n$ matrix. Then,
$E_{i,j}^{-1} = (b_{l,k})$ operates on A as the row addition $R_i \leftarrow R_i - R_j$.

Proof. As seen above, $E_{i,j}^{-1} = (b_{l,k})$ is a matrix where $b_{l,l} = 1, 1 \leq l \leq n$, and $b_{i,j} = -1, i < j$, and all other elements are zero.
So, the multiplication $(c_{l,k}) = E_{i,j}^{-1} \times A = (b_{l,k}) \times (a_{l,k})$ is exactly the same as $E_{i,j} \times A$, for all the rows except for row i
For row i, we have $c_{i,k}, 1 \leq k \leq n = \sum_{r=1}^{n} b_{i,r} \cdot a_{r,k}$
But, the only elements that are not zero, in row i of $(b_{l,k})$ are $b_{i,i} = 1$, and $b_{i,j} = -1$, so, $c_{i_k} = b_{i,1} \cdot a_{1,k} + b_{i,2} \cdot a_{2,k} + \cdots + b_{i,i} \cdot a_{i,k} + \cdots + b_{i,j} \cdot a_{j,k} + \cdots + b_{i,n-1} \cdot a_{n-1,k} + b_{i,n} \cdot a_{n,k} = 0 \cdot a_{1,k} + 0 \cdot a_{2,k} + \cdots + 1 \cdot a_{i,k} + \cdots + (-1) \cdot a_{j,k} + \cdots + 0 \cdot a_{n-1,k} + 0 \cdot a_{n,k} = 0 + 0 + \cdots + a_{i,k} + \cdots + (-a_{j,k}) = a_{i,k} - a_{j,k}$, which shows that in the product matrix, $(c_{l,k})$, R_i turns into $R_i - R_j$

$\square$

Proposition 1.4
Let $E_{i,j} = (e_{l,k}), i < j \in E_n$, Then,
$\forall m \in \mathbb{N}, E_{i,j}^{m} = (a_{l,k})$, where $a_{l,l} = 1, 1 \leq l \leq n$, and $a_{i,j} = m, i < j$, and all other elements are zero.

Proof. By induction on m.
For $m = 2$, we observe that $E_{i,j}^2 = E_{i,j} \times E_{i,j}$, which means that the multiplication from the left of $E_{i,j}$ by itself is operating on $E_{i,j}$ as the row addition $R_i \leftarrow R_i + R_j$, so, $a_{i,i} = a_{i,i} + aj, i = 1 + 0 = 1$, and, $a_{i,j} = a_{i,j} + a_{j,j} = 1 + 1 = 2$, and, all the other elements are zero (easy to verify).
So, $(a_{l,k}) = E_{i,j}^2$, where $a_{l,l} = 1, 1 \leq l \leq n$, and $a_{i,j} = 2, i < j$, and all other elements are zero.
Now, we prove for $m + 1$
$(a_{l,k}) = E_{i,j}^{m+1} = E_{i,j} \times E_{i,j}^{m}$. But, from the induction assumption, $(b_{l,k}) = E_{i,j}^{m}$, $b_{l,l} = 1, 1 \leq l \leq n$, and $b_{i,j} = m, i < j$, and all other elements are zero.
Multiplying from the left $(b_{l,k})$ by $E_{i,j}$ is operating as the row addition $R_i \leftarrow R_i + R_j$, so, $a_{i,i} = b_{i,i} + b_{j,i} = 1 + 0 = 1$, and, $a_{i,j} = b_{i,j} + b_{j,j} = m + 1$, and easy to verify that all the other elements are zero, thus, we prove the induction step.

$\square$

Corollary 1.5
Let $E_{i,j} = (e_{l,k}), i < j \in E_n$, Then,
$\forall m \in \mathbb{N}, (E_{i,j}^{-1})^m = (a_{l,k})$, where $a_{l,l} = 1, 1 \leq l \leq n$, and $a_{i,j} = -m, i < j$, and all other elements are zero

Proof. By induction on m.
$(a_{l,k}) = E_{i,j}^{-1}$ For $m = 2$, we observe that $(E_{i,j}^{-1})^2 = E_{i,j}^{-1} \times E_{i,j}^{-1} = (a_{l,k}) \times (a_{l,k})$, means that $E_{i,j}^{-1}$ operates on itself as the row addition $R_i \leftarrow R_i - R_j$
So, the product matrix $(b_{l,k})$ has $b_{i,i} = a_{i,i} - a_{j,i} = 1 - 0 = 1$, and $b_{i,j} = a_{i,j} - a_{j,j} = -1 - 1 = -2$, and all other elements are zero.
Now, we prove for $m + 1$
$(a_{l,k}) = (E_{i,j}^{-1})^{m+1} = E_{i,j}^{-1} \times (E_{i,j}^{-1})^m$. But, from the induction assumption, $(b_{l,k}) = (E_{i,j}^{-1})^m$, has $b_{l,l} = 1, 1 \leq l \leq n$, and $b_{i,j} = -m, i < j$, and all other elements are zero.
So, $(a_{l,k}) = E_{i,j}^{-1} \times (b_{i,j})$ is the row addition $R_i \leftarrow R_i - R_j$ on $(b_{i,j})$, which means, $a_{i,i} = b_{i,i} - b_{i,j} = 1 = 0 = 1$, and $a_{i,j} = b_{i,j} - b_{j,j} = -m - 1 = -(m+1)$, and all the other elements are zero, thus, we prove the induction step.

□

Corollary 1.6 Let $E_{i,j} = (e_{l,k}), i < j \in E_n$, Then,
$\forall m, r \in \mathbb{Z}, (a_{l,k}) = E_{i,j}^{m+r} = E_{i,j}^{r+m}$ is the matrix where $a_{l,l} = 1, 1 \leq l \leq n$, and $a_{i,j} = m + r = r + m, i < j$, and all other elements are zero.
This shows that multiplying integer powers of matrices, from the set E_n (which means, adding their exponents), is equivalent to adding integer numbers, which means that we have a canonical bijection, $(\mathbb{Z}, +) \leftrightarrow (E_{i,j}^{\mathbb{Z}}, \cdot)$, for any two fixed indices $i < j$, where $1 \leftrightarrow E_{i,j}^1 = E_{i,j}$, and $-1 \leftrightarrow E_{i,j}^{-1}$

Proposition 1.7

Let $(a_{l,k}) = E_{i,j}, t \neq i < j, (b_{l,k}) = E_{s,t}, j \neq s < t \in E_n$, Then, $(c_{l,k}) = E_{i,j} \cdot E_{s,t} = E_{s,t} \cdot E_{i,j}$ is a matrix with $c_{l,l} = 1, 1 \leq l \leq n$, and $c_{i,j} = 1$, and $c_{s,t} = 1$, and, all other elements are zero.

Proof. As seen above, $E_{s,t}$ is operating from the left on $E_{i,j}$ as the addition $R_i, j \leftarrow R_i + R_j$, so, $(c_{l,k})$ is $E_{s,t}$, with row j being added to row i.
So, $c_{i,k} = b_{i,k} + bj, k, 1 \leq k \leq n$. But, since $s \neq j$ the only element in row j of $(b_{l,k})$ which is not zero is $b_{j,j} = 1$, and $b_{i,j} = 0$, so $c_{i,j} = b_{i,j} + b_{j,j} = 0 + 1 = 1$
Also, $b_{j,i} = 0$ (it is below the main diagonal), so, $c_{i,i} = b_{i,i} + b_{j,i} = 1 + 0 = 1$.
It is easy to verify that all the other elements in row i of $(c_{j,k})$ are zero, and that all the other rows of $(c_{l,k})$ remain the same as they are in $(b_{l,k})$
Also, it is easy to verify that, under the condition that $t \neq i$, the multiplication is commuting, and yields the same product matrix. □

Proposition 1.8

Let $(a_{l,k}) = E_{i,j}, i < j, (b_{l,k}) = E_{j,r}, j < r \in E_n$, Then,
1.8.1 $(c_{l,k}) = E_{i,j} \cdot E_{j,r}$ is a matrix with $c_{l,l} = 1, 1 \leq l \leq n$, and $c_{i,j} = 1$, and $c_{j,r} = 1$, and $c_{i,r} = 1$, and, all other elements are zero.

Proof. The multiplication from the left of $E_{j,r}$ by $E_{j,r}$ is the addition on row j to row i of the matrix $E_{j,}$, which gives $c_{i,k} = b_{i,k} + b_{j,k}, 1 \leq k \leq n$, so $c_{i,i} = b_{i,i} + b_{j,i} = 1 + 0 = 1$, and $c_{i,j} = b_{i,j} + b_{j,j} = 0 + 1 = 1$, and $c_{i,r} = b_{i,r} + b_{j,r} = 0 + 1 = 1$, and, it is easy to verify that all other $c_{i,k}$ are zero.

□

On the other hand,
1.8.2 $(d_{l,k}) = E_{j,r} \cdot E_{i,j}$ is a matrix with $d_{l,l} = 1, 1 \leq l \leq n$, and $d_{i,j} = 1$, and $c_{j,r} = 1$, and, all other elements are zero.

Proof. The multiplication from the left of $E_{i,j}$ by $E_{s,t}$ is the addition on row j to row i of the matrix $E_{s,t}$, which gives $c_{i,k} = b_{i,k} + b_{j,k}, 1 \leq k \leq n$, so $c_{i,i} = b_{i,i} + b_{j,i} = 1 + 0 = 1$, and $c_{i,j} = b_{i,j} + b_{j,j} = 0 + 1 = 1$, and $c_{i,r} = b_{i,r} + b_{j,r} = 0 + 1 = 1$, and, it is easy to verify that all other $c_{i,k}$ are zero. □

1.8.3 Let $(c_{l,k}) = E_{i,j}^{-1}, (d_{l,k}) = E_{j,r}^{-1}$
$(f_{l,k}) = E_{i,j}^{-1} \cdot E_{j,r}^{-1}$ is a matrix with $f_{l,l} = 1, 1 \leq l \leq n$, and $f_{i,j} = -1$, and $f_{j,r} = -1$, and $f_{i,r} = 1$ and, all other elements are zero.

Proof. The multiplication from the left of $E_{j,r}^{-1}$ by $E_{i,j}^{-1}$ is the subtraction of row j from row i of the matrix $E_{j,r}^{-1}$, which gives $f_{i,k} = d_{i,k} - d_{j,k}, 1 \leq k \leq n$, so $f_{i,i} = d_{i,i} - d_{j,i} = 1 - 0 = 1$, and $f_{i,j} = d_{i,j} - d_{j,j} = 0 - 1 = -1$, and $f_{i,r} = d_{i,r} - d_{j,r} = 0 - (-1) = 0 + 1 = 1$, and, it is easy to verify that all other $f_{i,k}$ are zero. □

1.8.4 Let $(c_{l,k}) = E_{i,j}^{-1}, (d_{l,k}) = E_{j,r}^{-1}$
$(g_{l,k}) = E_{j,r}^{-1} \cdot E_{i,j}^{-1}$ is a matrix with $f_{l,l} = 1, 1 \leq l \leq n$, and $f_{i,j} = -1$, and $f_{j,r} = -1$, and, all other elements are zero.

Proof. The multiplication from the left of $E_{i,j}^{-1}$ by $E_{j,r}^{-1}$ is the subtraction of row r from row j of the matrix $E_{i,j}^{-1}$, which gives $g_{i,k} = c_{i,k} - c_{j,k}, 1 \leq k \leq n$, so $g_{j,j} = c_{j,j} - c_{r,j} = 1 - 0 = 1$, and $g_{i,j} = c_{i,j} - c_{r,j} = -1 - 0 = -1$, and $g_{j,r} = c_{j,r} - c_{r,r} = 0 - 1 = -1$, and, it is easy to verify that all other $g_{j,k}, g_{i,k}$ are zero. □

Corollary 1.9 Let $(a_{l,k}) = E_{i,j}, (b_{l,k}) = E_{j,r}$, Then
1.9.1 $(c_{l,k}) = [E_{i,j}, E_{j,r}] = E_{i,j} \cdot E_{j,r} \cdot E_{i,j}^{-1} \cdot E_{j,r}^{-1} = E_{i,r}$

Proof. By associativity, $E_{i,j} \cdot E_{j,r} \cdot E_{i,j}^{-1} \cdot E_{j,r}^{-1} = (E_{i,j} \cdot E_{j,r}) \cdot (E_{i,j}^{-1} \cdot E_{j,r}^{-1})$, and we have already calculated these matrix products.
$(f_{l,k}) = E_{i,j} \cdot E_{j,r} = I + F_{i,j} + F_{i,r} + F_{j,r}$
$(g_{l,k}) = E_{i,j}^{-}1 \cdot E_{j,r}^{-}1 = I - F_{i,j} + Fi, r - F_{j,r}$
So, in the product matrix, $(c_{l,k})$, $c_{i,j} = \sum_{k=1}^{n} f_{i,k} \cdot g_{k,j} = 0 \cdot 0 + \cdots + 0 \cdot 0 + f_{i,i} \cdot g_{i,j} + f_{i,j} \cdot g_{j,j} + 0 \cdot 0 \ldots 0 \cdot 0 = 1 \cdot -1 + 1 \cdot 1 = -1 + 1 = 0$
So, $c_{i,j}$ is cancelled by the multiplication. Easy to verify that the same goes also for $c_{j,r}$, but $c_{i,r} = \sum_{k=1}^{n} f_{i,k} \cdot g_{k,r} = 0 \cdot 0 + \cdots + 0 \cdot 0 + f_{i,i} \cdot g_{i,r} + f_{i,j} \cdot g_{j,r} + f_{i,r} \cdot g_{r,r} + 0 \cdot 0 \ldots 0 \cdot 0 = 1 \cdot 1 + 1 \cdot -1 + 1 \cdot 1 = 1 + (-1) + 1 = 1 - 1 + 1 = 1$
Which means that $[E_{i,j}, E_{j,r}] = E_{i,j} \cdot E_{j,r} \cdot E_{i,j}^{-1} \cdot E_{j,r}^{-1} = I + F_{i,r} = E_{i,r}$ □