Your Paper

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Denote $G_5 := G_5(\mathbb{Z}_p)$, and $G_5^+ := G_5^+(\mathbb{Q}_p)$.

$\zeta^{\wedge}_{L_{5,p}}(s) = \int_{G_5^+} |\det g|_p^s d\mu(G_5) = \int_{G_5^+} |\det uh|_p^s d\mu(G_5)$, where $h \in H$ and $u \in N_h$.

Each u is unipotent, hence $\zeta^{\wedge}_{L_{5,p}}(s) = \int_{G_5^+} |\det h|_p^s d\mu(G_5) = \int_{G_5^+} |\lambda_1^4\lambda_2^6\lambda_3^6\lambda_4^4|_p^s d\mu(G_5) =$

$= \int_{G_5^+} \left[|\lambda_1^4|_p|\lambda_2^6|_p|\lambda_3^6|_p|\lambda_4^4|_p\right]^s d\mu(G_5)$, by the inductive formula we have found for every $|h|$.

We denote $v_i := v_p(\lambda_i)$,

and so $\zeta^{\wedge}_{L_{5,p}}(s) = \int_{G_5^+} \left[p^{-4v_1}p^{-6v_2}p^{-6v_3}p^{-4v_4}\right]^s d\mu(G_5) = \int_{G_5^+} p^{-(4v_1+6v_2+6v_3+4v_4)s} d\mu(G_5)$.

We denote $I_1 := p^{-(4v_1+6v_2+6v_3+4v_4)s}$. Now we use the natural matrix decomposition of the N_h matrix of Berman's, which means that

$\zeta^{\wedge}_{L_{5,p}}(s) = \int_{G_5^+} I_1 d\mu(G_5) = \int_{\underline{\lambda}}\int_{\underline{a}}\int_{\underline{b}}\int_{\underline{c}} I_1 d\mu(\underline{c})d\mu(\underline{b})d\mu(\underline{a})d\mu(\underline{\lambda})$. Since I_1 depends only on $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, which appear only in the computation of the outermost integral, we consider them as constants for all the inner integrals, which means that we have $\zeta^{\wedge}_{L_{5,p}}(s) = \int_{\underline{\lambda}} I_1 \int_{\underline{a}}\int_{\underline{b}}\int_{\underline{c}} 1 d\mu(\underline{c})d\mu(\underline{b})d\mu(\underline{a})d\mu(\underline{\lambda})$, hence all the inner integrals evaluate to the measure of their domains of integration. now we compute the innermost integral by considering $\underline{a}$, $\underline{b}$ and $\underline{\lambda}$ as constants, and integrating only over $\underline{c}$. Considering the multiplication uh, we observe that for each element c_j, we must have that $\rho_j = c_j\lambda_1\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p$, which means that $v(\rho_i) = v(c_i\lambda_1\lambda_2\lambda_3\lambda_4) \geq 0 \Rightarrow v(c_i) + v_1 + v_2 + v_3 + v_4 \geq 0 \Rightarrow v(c_i) \geq -(v_1 + v_2 + v_3 + v_4)$. But this means that $c_i \in p^{-(v_1+v_2+v_3+v_4)}\mathbb{Z}_p$, and since the domain of integration for this integral is $\underline{c} = \{c_1, c_2, c_3, c_4\}$, then $\mu(\underline{c}) = |c_j|_p^4 = p^{4(v_1+v_2+v_3+v_4)}$. Denote $I_2 := I_1 p^{4(v_1+v_2+v_3+v_4)}$, we now have that $\zeta^{\wedge}_{L_{5,p}}(s) = \int_{\underline{\lambda}} I_2 \int_{\underline{a}}\int_{\underline{b}} 1 d\mu(\underline{b})d\mu(\underline{a})d\mu(\underline{\lambda})$.

Denote $\lambda_{13} := \lambda_1\lambda_2\lambda_3$, $\lambda_{24} := \lambda_2\lambda_3\lambda_4$, and $\lambda_{14} := \lambda_1\lambda_2\lambda_3\lambda_4$. We now consider the constraints on $\underline{b}$.

$b_{11}\lambda_{13}, b_{31}\lambda_{13}, b_{41}\lambda_{13} \in \mathbb{Z}_p$, and $b_{12}\lambda_{24}, b_{22}\lambda_{24} \in \mathbb{Z}_p$. These constaints are obtained by multiplying elements in block M_{13} with elements in h, but one observes that we have b_{22} also in location $(5,10)$ of the matrix, and b_{31} in location $(7,10)$, which means that $b_{22}\lambda_{14}, b_{31}\lambda_{14} \in \mathbb{Z}_p$. But since we already have $b_{22}\lambda_{24}, b_{31}\lambda_{13} \in \mathbb{Z}_p$, the constraints $b_{22}\lambda_{14}$ and $b_{31}\lambda_{14}$ do not contribute any new information. In addition, we have one of the elements of $\underline{b}$ that forms a constraint together with elements from $\underline{a}$, namely $(a_{11}a_{22} - b_{11})\lambda_{24} \in \mathbb{Z}_p$. The constraints $b_{31}\lambda_{13}, b_{41}\lambda_{13}, b_{12}\lambda_{24}, b_{22}\lambda_{24} \in \mathbb{Z}_p$ from above translate to $p^{-2(v_1+v_2+v_3)}p^{-2(v_2+v_3+v_4)} = p^{-2(v_1+2v_2+2v_3+v_4)}$. On the other hand, b_{11} is a part of two constraints, hence we must have both $b_{11} \in p^{-(v_1+v_2+v_3)}\mathbb{Z}_p$ and $a_{11}a_{22} - b_{11} \in p^{-(v_2+v_3+v_4)}\mathbb{Z}_p \Rightarrow b_{11} \in a_{11}a_{22} + p^{-(v_2+v_3+v_4)}\mathbb{Z}_p$, which means that we need to compute the measure $\mu(A)$, where $A = p^{-(v_1+v_2+v_3)}\mathbb{Z}_p \cap a_{11}a_{22} + p^{-(v_2+v_3+v_4)}\mathbb{Z}_p$. Denote $\alpha := v_1 + v_2 + v_3$, $\beta := v_2 + v_3 + v_4$ and $x := a_{11}a_{22}$, and we need to find a formula for a generic intersection of the form $A = p^{-\alpha}\mathbb{Z}_p \cap x + p^{-\beta}\mathbb{Z}_p$. We need to find a formula for this generic form. Since b_{11} is in the intersection, we have that $b_{11} = z = x + y$ where $y \in p^{-\beta}$ and $z \in p^{-\alpha}\mathbb{Z}_p \Rightarrow z - x \in p^{-\beta}\mathbb{Z}_p$. Assume $\beta \geq \alpha \Rightarrow -\beta \leq -\alpha$, and since $v_p(b_{11}) = v_p(z - x) \geq \min\{v_p(z), v_p(x)\}$, and $v_p(z) \geq -\alpha \geq -\beta$, then we have two cases. If $v_p(x) \geq -\beta$, then $v_p(z - x) \geq \beta \Rightarrow z - x \in p^{-\beta}\mathbb{Z}_p$. But $-\alpha \geq -\beta \Rightarrow p^{-\alpha}\mathbb{Z}_p \subseteq p^{-\beta}\mathbb{Z}_p \Rightarrow A = p^{-\alpha}\mathbb{Z}_p$. If $v_p(x) < -\beta$, then $v_p(z - x) = v_p(x) < -\beta \Rightarrow z - x \notin p^{-\beta}\mathbb{Z}_p$, which means that $A = \varnothing$. One checks that if we assume $\alpha \geq \beta$, then we obtain that $A = p^{-\beta}\mathbb{Z}_p$ if $v_p(x) \geq -\alpha$, and $A = \varnothing$ if $v_p(x) < -\alpha$. Therefore, $\mu(A) = p^{\min\{\alpha,\beta\}}$ for every x such that $v_p(x) \geq \min\{-\alpha, -\beta\} = -\max\{\alpha, \beta\}$, which means, in our case, that $v_p(x) = v_p(a_{11}a_{22}) \geq -\max\{v_1 + v_2 + v_3, v_2 + v_3 + v_4\} = -v_2 - v_3 - \max\{v_1, v_4\}$. Thus, denoting $I_3 := I_2 p^{-(v_2+v_3)-\max\{v_1,v_4\}}$, we have that $\zeta^{\wedge}_{L_{5,p}}(s) = \int_{\underline{\lambda}} I_3 \int_{\underline{a}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda})$. Denote $v_{ij} := v_p(a_{ij})$. For the constraints on $\underline{a}$, we have

$a_{11}\lambda_1\lambda_2, -a_{11}\lambda_2\lambda_3, -a_{11}\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_{11} \geq -(v_1 + v_2), v_{11} \geq -(v_2 + v_3) \Rightarrow v_{11} \geq -v_2 - \min\{v_1, v_3\}$.

$a_{21}\lambda_1\lambda_2, a_{21}\lambda_1\lambda_2\lambda_3, a_{21}\lambda_1\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_{21} \geq -(v_1 + v_2)$.

$a_{22}\lambda_2\lambda_3, -a_{22}\lambda_3\lambda_4, a_{22}\lambda_1\lambda_2\lambda_3 \in \mathbb{Z}_p \Rightarrow$

$\Rightarrow v_{22} \geq -(v_2 + v_3), v_{22} \geq -(v_3 + v_4), v_{22} \geq -(v_1 + v_2 + v_3) \Rightarrow v_{22} \geq -v_3 - \min\{v_2, v_4\}$.

$a_{33}\lambda_3\lambda_4, a_{33}\lambda_2\lambda_3\lambda_4, a_{33}\lambda_1\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_{33} \geq -(v_3 + v_4)$.

$a_{21}a_{22}\lambda_1\lambda_2\lambda_3 \in \mathbb{Z}_p \Rightarrow v_{21} + v_{22} \geq -(v_1 + v_2 + v_3)$.

$-a_{11}a_{33}\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_{11} + v_{33} \geq -(v_2 + v_3 + v_4)$.

$a_{21}a_{33}\lambda_1\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_{21} + v_{33} \geq -(v_1 + v_2 + v_3 + v_4)$.

And we also have the constraint found earlier,

$v_{11} + v_{22} \geq -(v_2 + v_3 + \max\{v_1, v_4\})$.

We have three constraints on a_{21}

1. $v_{21} \geq -(v_1 + v_2)$
2. $v_{21} \geq -(v_1 + v_2 + v_3 + v_{22})$
3. $v_{21} \geq -(v_1 + v_2 + v_3 + v_4 + v_{33})$

But the third constraint does not add new information, because we already have the two separate constraints $v_{21}, v_{33} \geq -(v_1 + v_2 + v_3 + v_4)$.

The two valid constraints translate to

$v_{21} \geq \min\{-(v_1+v_2), -(v_1+v_2+v_3+v_{22})\} = -(v_1+v_2) - \{0, v_3+v_{22}\}$.

In the same way, we obtain the constraint $v_{33} \geq -(v_3+v_4) - \min\{0, v_2+v_{11}\}$.

Thus, we decompose the inner integral for $\underline{a}$ into separate integrals, to obtain

$$\zeta^{\wedge}_{L_{5,p}}(s) = \int_{\underline{\lambda}} I_3 \int_{\underline{a}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda}) =$$

$$= \int_{\underline{\lambda}} I_3 \int_{a_{11}} \int_{a_{22}} \int_{a_{33}} \int_{a_{21}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda}).$$

Hence, we have the measures $\mu(a_{21}) = p^{v_1+v_2+\min\{0,v_3+v_{22}\}}$ and $\mu(a_{33}) = p^{v_3+v_4+\min\{0,v_2+v_{11}\}}$. Denote $I_4 := I_3 p^{v_1+v_2} p^{v_3+v_4}$. We have

$$\zeta^{\wedge}_{L_{5,p}}(s) = \int_{\underline{\lambda}} I_3 \int_{\underline{a}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda}) =$$

$$= \int_{\underline{\lambda}} I_3 \int_{a_{11}} \int_{a_{22}} \int_{a_{33}} \int_{a_{21}} 1 d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda}) =$$

$$= \int_{\underline{\lambda}} I_4 \int_{a_{11}} p^{\min\{0,v_2+v_{11}\}} \int_{a_{22}} p^{\min\{0,v_3+v_{22}\}} d\mu(\underline{b}) d\mu(\underline{a}) d\mu(\underline{\lambda}).$$

By the constraints we found earlier on a_{22}, we have the following.

1. $v_{22} \geq -v_3 - \min\{v_2, v_4\}$
2. $v_{22} \geq -(v_2+v_3) - \max\{v_1, v_4\} - v_{11}$

which translates into $v_{22} \geq -v_3 - \min\{\min\{v_2, v_4\}, v_2 + \max\{v_1, v_4\} + v_{11}\} =$

$= -v_3 - \min\{v_2, v_4, v_2 + \max\{v_1, v_4\} + v_{11}\}$.

Denote $\alpha := v_2 + \max\{v_1, v_4\} + v_{11}$ and $\beta := \min\{v_2, v_4, \alpha\}$. We already have the constraint $v_{11} \geq -(v_2 + \min\{v_1, v_3\})$, which means that, in either case, $v_{11} \geq -(v_1+v_2) \geq -\min\{v_1, v_4\} - v_2$

$\Rightarrow \alpha = v_2 + \max\{v_1, v_4\} + v_{11} \geq \max\{v_1, v_4\} - \min\{v_1, v_4\} \geq 0$

$\Rightarrow \beta = \min\{v_2, v_4, \alpha\} \geq 0 \Rightarrow v_3 + \alpha > 0 \Rightarrow v_{22} \geq -(v_3+\beta)$.

For the inner integral $\int_{a_{22}} p^{\min\{0,v_3+v_{22}\}} d\mu(a_{22})$, we have two cases. If $v_3 + v_{22} \geq 0$, then $\min\{0, v_3+v_{22}\} = 0 \Rightarrow \int_{a_{22}} p^{\min\{0,v_3+v_{22}\}} d\mu(a_{22}) = \int_{v_{22} \geq -v_3} 1 d\mu(a_{22}) = p^{v_3}$.

If $v_3 + v_{22} < 0$, then $\int_{a_{22}} p^{\min\{0,v_3+v_{22}\}} d\mu(a_{22}) = \int_{v_{22} < -v_3} p^{v_3+v_{22}} d\mu(a_{22})$. But we saw earlier that $v_{22} \geq -(v_3+\beta)$, hence $-v_3 - \beta \leq v_{22} \leq -v_3 - 1 \Rightarrow -\beta \leq v_3 + v_{22} \leq -1$, which means that we can compute the integral over a_{22} as a sum of β integrals, $\int_{-v_3-\beta \leq v_{22} \leq -v_3-1} p^{v_3+v_{22}} d\mu(a_{22}) = \sum_{\tau=1}^{\beta} \int_{v_{22}=-v_3-\tau} p^{-\tau} d\mu(a_{22})$. To evaluate each integral in the sum, we need to calculate the measure of its domain, namely $\mu(\{v_{22} = -(v_3+\tau)\}) = \mu(\{v_{22} \geq -(v_3+\tau+1)\} \setminus \{v_{22} \geq -(v_3+\tau)\}) = \mu(p^{-(v_3+\tau+1)}\mathbb{Z}_p \setminus p^{-(v_3+\tau)}\mathbb{Z}_p) = p^{v_3+\tau+1} - p^{v_3+\tau} = p^{v_3+\tau}(p-1)$, which means that each integral evaluates as $p^{v_3+\tau}(p-1)p^{\tau} = p^{v_3}(p-1)$, and the sum is over β such integrals, so we have that $\int_{a_{22}} p^{\min\{0,v_3+v_{22}\}} d\mu(a_{22}) = p^{v_3} + \beta p^{v_3}(p-1)$, where β depends also on v_{11}.

Hence, we need to compute the integral $\int_{a_{11}} p^{\min\{0,v_2+v_{11}\}} p^{v_3}[1+\beta(p-1)]d\mu(a_{11})$. We denote $I_5 := I_4 p^{v_3}$, so $\zeta^{\wedge}_{L_{5,p}}(s) = = \int_{\underline{\lambda}} I_5 \int_{a_{11}} p^{\min\{0,v_2+v_{11}\}}[1+\beta(p-1)]d\mu(a_{11})$. But similar to what we saw earlier, $p^{\min\{0,v_2+v_{11}\}}$ has two cases. If $v_{11} \geq -v_2$, then $p^{\min\{0,v_2+v_{11}\}} = p^0$. If $v_{11} < -v_2$, then $p^{\min\{0,v_2+v_{11}\}} = p^{v_2+v_{11}}$. We saw earlier that $v_{11} \geq -(v_2+v_3) \Rightarrow v_{11}+v_2 \geq -v_3$, so for this case, we have that $-v_3 \leq v_{11}+v_2 \leq 0$, which means that $\int_{a_{11}} p^{\min\{0,v_2+v_{11}\}}[1+\beta(p-1)]d\mu(a_{11}) = \int_{v_{11}\leq -v_2} 1 + \beta(p-1)d\mu(a_{11}) + \int_{v_{11}+v_2 \geq -v_3} p^{v_2+v_{11}}[1+\beta(p-1)]d\mu(a_{11}) = \int_{v_{11}\leq -v_2} 1+\beta(p-1)d\mu(a_{11}) + \sum_{\tau=1}^{v_3} \int_{v_{11}\geq -(v_3+v_2)} p^{-\tau}[1+\beta(p-1)]d\mu(a_{11})$. Now we need to resolve $\beta = \min\{v_2, v_4, v_2+\max\{v_1,v_4\}+v_{11}\}$, hence we need to divide the inner integral to different orderings of v_1, v_2, v_3, v_4.

Case 1: $v_1 \geq v_2 \geq v_3 \geq v_4$. For this case, we have that $\beta = \min\{v_2, v_4, v_2 + \max\{v_1, v_4\} + v_{11}\} = \min\{v_4, v_1+v_2+v_{11}\}$.

The two possible minimum values are equal when $v_4 = v_1+v_2+v_{11}$, that is, when $v_2+v_{11} = -(v_1-v_4)$. But for this case we have two subcases. If $v_1 - v_4 \leq v_3$, then, since $v_{11} \geq -(v_2+v_3)$, we have that $v_{11}+v_2 \geq -v_3 \geq -(v_1-v_4) \Rightarrow v_1+v_2+v_{11} \geq v_4 \Rightarrow \beta = v_4$, hence $\int_{\underline{\lambda}} I_5 \int_{a_{11}} p^{\min\{0,v_2+v_{11}\}}(1+v_4(1-p^{-1}))d\mu(a_{11}) =$

$= \int_{\underline{\lambda}} I_5(1+v_4(1-p^{-1})) \int_{v_{11}\geq -(v_2+v_3)} p^{\min\{0,v_2+v_{11}\}} d\mu(a_{11})$, but same as earlier $\int_{v_{11}\geq -(v_2+v_3)} p^{\min\{0,v_2+v_{11}\}} d\mu(a_{11}) = p^0 \mu(\{v_{11}+v_2 \geq 0\}) + \int_{v_2+v_{11}<0} p^{v_2+v_{11}} d\mu(a_{11}) = 1\mu(\{v_{11} \geq -v_2\}) + \int_{v_2+v_{11}<0} p^{v_2+v_{11}} d\mu(a_{11}) = p^{v_2} + \int_{-v_3 \leq v_{11} < -v_2} p^{v_2+v_{11}} d\mu(a_{11}) = p^{v_2} + \sum_{\tau=-v_3}^{-(v_2+1)} \int_{v_{11}=\tau} p^{v_2+\tau} d\mu(a_{11}) =$

$= p^{v_2} + \sum_{\tau=1}^{v_3} \int_{v_{11}=-(v_2+\tau)} p^{\tau} d\mu(a_{11}) = p^{v_2} + \sum_{\tau=1}^{v_3} p^{-\tau}(p^{v_2+\tau} - p^{v_2+\tau-1}) = p^{v_2} + \sum_{\tau=1}^{v_3} p^{v_2}(1-p^{-1}) = p^2 + v_3 = p^{v_2}(1+v_3(1-p^{-1}))$. Denote $I_6 := I_5 p^{v_2}(1+v_3(1-p^{-1}))$, thus we have $\zeta^{\wedge}_{L_{5,p}}(s) = \int_{\underline{\lambda}} I_6 d\mu(\lambda)$. We compute the complete expression $I_6 = p^{4(v_1+v_2+v_3+v_4)} p^{-(v_2+v_3)-\max\{v_1,v_4\}} p^{v_1+v_2} p^{v_3+v_4} p^{v_3} p^{v_2} (1+v_3(1-p^{-1})) p^{-(4v_1+6v_2+6v_3+4v_4)s} = p^{7v_1+11v_2+11v_3+8v_4}(1+v_3(1-p^{-1}))(1+v_4(1+p^{-1})) p^{-(4v_1+6v_2+6v_3+4v_4)s} = p^{(7-4s)v_1} p^{(11-6s)v_2} p^{(11-6s)v_3} p^{(8-4s)v_4}(1+v_3(1-p^{-1}))(1+v_4(1-p^{-1}))$ and integrate it over $\underline{\lambda}$, which translates to the infinite sum

$S := \sum_{v_1\geq v_2\geq v_3\geq v_4} p^{(7-4s)v_1}p^{(11-6s)v_2}p^{(11-6s)v_3}p^{(8-4s)v_4}(1+v_3(1-p^{-1}))(1+v_4(1+p^{-1}))$.

We notice that we have no constraint which dictates an order relation between v_1 and v_2, thus we have the following constraints

$v_2 \geq v_3 \geq v_4$ and $v_1 \geq v_3+v_4 \geq v_3 \geq v_4$. But this allows us to break the computed sum into separate sums, where the index of summation must preserve the constraints between v_1, v_2, v_3, v_4.

Denote $u_4 := v_4$, $u_3 := v_3 - v_4$, $u_2 := v_2 - v_3$ and $u_1 := v_1 - (v_3 + v_4)$. With these notations, we have

$S = \sum_{u_1=0}^{\infty}\sum_{u_2=0}^{\infty}\sum_{u_3=0}^{\infty}\sum_{u_4=0}^{\infty} p^{(7-4s)(u_1+u_3+2u_4)}p^{(11-6s)(u_2+u_3+u_4)}p^{(11-6s)(u_3+u_4)}p^{(8-4s)u_4}(1+(u_3+u_4)(1-p^{-1}))(1+u_4(1-p^{-1})) =$

$\sum_{u_1=0}^{\infty} p^{(7-4s)u_1}\sum_{u_2=0}^{\infty} p^{(11-6s)u_2}\sum_{u_3=0}^{\infty} p^{(29-16s)u_3}\sum_{u_4=0}^{\infty} p^{(44-24s)u_4}(1+(u_3+2u_4)(1-p^{-1})+(u_3u_4+u_4^2(1-p^{-1})))$. Denote $w_1 := p^{(7-4s)}$, $w_2 := p^{(11-6s)}$, $w_3 := p^{(29-16s)}$ and $w_4 := p^{(44-24s)}$. We shall compute each summand separately.

$S_1 := \sum_{u_1}^{\infty} w_1^{u_1}\sum_{u_2}^{\infty} w_2^{u_2}\sum_{u_3}^{\infty} w_3^{u_3}\sum_{u_4}^{\infty} w_4^{u_4} = \frac{1}{1-w_1^{u_1}}\frac{1}{1-w_2^{u_2}}\frac{1}{1-w_3^{u_3}}\frac{1}{1-w_4^{u_4}}$.

$S_2 := \sum_{u_1}^{\infty} w_1^{u_1}\sum_{u_2}^{\infty} w_2^{u_2}\sum_{u_3}^{\infty} w_3^{u_3}\sum_{u_4}^{\infty} w_4^{u_4}(u_3+2u_4)(1-p^{-1}) =$
$= (1-p^{-1})\sum_{u_1}^{\infty} w_1^{u_1}\sum_{u_2}^{\infty} w_2^{u_2}\sum_{u_3}^{\infty} w_3^{u_3}\sum_{u_4}^{\infty} w_4^{u_4}(u_3+2u_4) =$
$= (1-p^{-1})\sum_{u_1}^{\infty} w_1^{u_1}\sum_{u_2}^{\infty} w_2^{u_2}\sum_{u_3}^{\infty} w_3^{u_3}\sum_{u_4}^{\infty} w_4^{u_4}(u_3+2u_4) =$
$= (1-p^{-1})\Big[\sum_{u_1}^{\infty} w_1^{u_1}\sum_{u_2}^{\infty} w_2^{u_2}\sum_{u_3}^{\infty} u_3w_3^{u_3}\sum_{u_4}^{\infty} w_4^{u_4}+$
$+\sum_{u_1}^{\infty} w_1^{u_1}\sum_{u_2}^{\infty} w_2^{u_2}\sum_{u_3}^{\infty} w_3^{u_3}\sum_{u_4}^{\infty} 2u_4w_4^{u_4}\Big] = (1-p^{-1})\Big[\frac{1}{1-w_1}\frac{1}{1-w_2}\frac{w_3}{(1-w_3)^2}\frac{1}{1-w_4}+2\frac{1}{1-w_1}\frac{1}{1-w_2}\frac{1}{1-w_3}\frac{w_4}{(1-w_4)^2}\Big]$.

$S_3 := \sum_{u_1}^{\infty} w_1^{u_1}\sum_{u_2}^{\infty} w_2^{u_2}\sum_{u_3}^{\infty} w_3^{u_3}\sum_{u_4}^{\infty} w_4^{u_4}u_4(1-p^{-1}) =$
$= (1-p^{-1})\sum_{u_1}^{\infty} w_1^{u_1}\sum_{u_2}^{\infty} w_2^{u_2}\sum_{u_3}^{\infty} u_3w_3^{u_3}\sum_{u_4}^{\infty} w_4^{u_4}u_4 =$
$= \sum_{u_1}^{\infty} w_1^{u_1}\sum_{u_2}^{\infty} w_2^{u_2}\sum_{u_3}^{\infty} w_3^{u_3}\sum_{u_4}^{\infty}\frac{1}{1-w_1}\frac{1}{1-w_2}\frac{1}{1-w_3}\frac{w_4}{(1-w_4)^2}$.

The second sub case is where $\beta = \min\{v_4, \alpha\} = \alpha < v_4$, where $\alpha = v_1 + v_2 + v_{11}$. But $v_{11} \geq -(v_2+v_3) \Rightarrow v_1+v_2+v_{11} \geq v_1 - v_3$, which means that $v_4 > v_1 - v_3$ is a necessary condition for this sub case. We use a strong inequality here, because we have already counted the case where $v_4 = v_1 + v_2 + v_{11}$.

Thus, $v_1 - v_3 \leq \alpha < v_4$, which means that for this case, the value of β is not constant, but rather $\beta \in \{v_1-v_3, v_1-v_3+1, v_1-v_3+2, \dots, v_4-2, v_4-1\}$.

Hence, the lower bound for v_{11} remains $-(v_2+v_3)$, but our upper bound comes from either $\min\{0, v_2+v_{11}\}$ or α. We have

$$\min\{0, v_2+v_{11}\} \leq 0 \Rightarrow v_2+v_{11} \leq 0 \Rightarrow v_{11} \leq -v_2$$

and

$$\alpha < v_4 \Rightarrow v_1+v_2+v_{11} \leq v_4 - 1 \Rightarrow v_{11} \leq v_4 - (v_1+v_2+1)$$

But $v_4 \leq v_1 \Rightarrow v_4 - v_1 \leq 0$, and hence $v_4 - v_1 - v_2 - 1 \leq -v_2 - 1$, which means that $-(v_2+v_3) \leq v_{11} \leq v_4 - (v_1+v_2+1)$.

Therefore, we have

$$S_{11}=\int_{a_{11}}p^{\min\{0,v_2+v_{11}\}}[1+\beta(1-p^{-1})]d\mu(a_{11})=\int_{a_{11}}p^{\min\{0,v_2+v_{11}\}}[1+\alpha(1-p^{-1})]d\mu(a_{11})=$$

$$=\sum_{v_{11}=-(v_2+v_3)}^{v_4-(v_1+v_2+1)}p^{\min\{0,v_2+v_{11}\}}[1+(v_1+v_2+v_{11})(1-p^{-1})]d\mu(a_{11})$$

We notice that for this case we do not have $\min\{0,v_2+v_{11}\}=0$, because $\max v_{11}=v_4-(v_1+v_2+1)$, hence if $v_2+v_{11}=0$, then $v_{11}=-v_2=v_4-v_1-v_2-1\Rightarrow v_4-v_1-1=0\Rightarrow v_4-v_1=1$, which contradicts $v_1\geq v_4$. Therefore,

$$S_{11}=\sum_{v_{11}=-(v_2+v_3)}^{v_4-(v_1+v_2+1)}p^{v_2+v_{11}}[1+(v_1+v_2+v_{11})(1-p^{-1})]d\mu(a_{11})=$$

$$\begin{aligned}
&=p^{-v_3}[1+(v_1-v_3)(1-p^{-1})]\mu(p^{-(v_2+v_3)}\mathbb{Z}_p\setminus p^{-(v_2+v_3-1)}\mathbb{Z}_p)+\\
&+p^{-v_3+1}[1+(v_1-v_3+1)(1-p^{-1})]\mu(p^{-(v_2+v_3-1)}\mathbb{Z}_p\setminus p^{-(v_2+v_3-2)}\mathbb{Z}_p)+\\
&\vdots\\
&+p^{v_4-v_1-1}[1+(v_4-1)(1-p^{-1})]\mu(p^{v_4-(v_1+v_2+1)}\mathbb{Z}_p\setminus p^{v_4-(v_1+v_2)}\mathbb{Z}_p)=\\
&=p^{-v_3}[1+(v_1-v_3)(1-p^{-1})][p^{v_2+v_3}-p^{v_2+v_3-1}]+\\
&+p^{-v_3+1}[1+(v_1-v_3+1)(1-p^{-1})][p^{v_2+v_3-1}-p^{v_2+v_3-2}]+\\
&\vdots\\
&+p^{v_4-v_1-1}[1+(v_4-1)(1-p^{-1})][p^{v_1+v_2+1-v_4}-p^{v_1+v_2-v_4}]=\\
&=p^{-v_3}[1+(v_1-v_3)(1-p^{-1})][p^{v_2+v_3}(1-p^{-1})]+\\
&+p^{-v_3+1}[1+(v_1-v_3+1)(1-p^{-1})][p^{v_2+v_3-1}(1-p^{-1})]+\\
&\vdots\\
&+p^{v_4-v_1-1}[1+(v_4-1)(1-p^{-1})][p^{v_1+v_2+1-v_4}(1-p^{-1})]=\\
&=[1+(v_1-v_3)(1-p^{-1})][p^{v_2}(1-p^{-1})]+\\
&+p^{-v_3+1}[1+(v_1-v_3+1)(1-p^{-1})][p^{v_2}(1-p^{-1})]+\\
&\vdots\\
&+p^{v_4-v_1-1}[1+(v_4-1)(1-p^{-1})][p^{v_2}(1-p^{-1})]=
\end{aligned}$$