Your Paper

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Denote $G_5 := G_5(\mathbb{Z}_p)$, and $G_5^+ := G_5^+(\mathbb{Q}_p)$.

$\zeta^{\wedge}_{L_{5,p}}(s) = \int_{G_5^+} |\det g|_p^s d\mu(G_5) = \int_{G_5^+} |\det uh|_p^s d\mu(G_5)$, where $h \in H$ and $u \in N_h$.

Each u is unipotent, hence $\zeta^{\wedge}_{L_{5,p}}(s) = \int_{G_5^+} |\det h|_p^s d\mu(G_5) = \int_{G_5^+} |\lambda_1^4\lambda_2^6\lambda_3^6\lambda_4^4|_p^s d\mu(G_5) =$

$= \int_{G_5^+} \left[|\lambda_1^4|_p|\lambda_2^6|_p|\lambda_3^6|_p|\lambda_4^4|_p\right]^s d\mu(G_5)$, by the inductive formula we have found for every $|h|$.

We denote $v_i := v_p(\lambda_i)$,

and so $\zeta^{\wedge}_{L_{5,p}}(s) = \int_{G_5^+} \left[p^{-4v_1}p^{-6v_2}p^{-6v_3}p^{-4v_4}\right]^s d\mu(G_5) = \int_{G_5^+} p^{-(4v_1+6v_2+6v_3+4v_4)s} d\mu(G_5)$.

We denote $I(\underline{\lambda}) := p^{-(4v_1+6v_2+6v_3+4v_4)s}$. Now we use the natural matrix decomposition of the N_h matrix of Berman's, which means that

$\zeta^{\wedge}_{L_{5,p}}(s) = \int_{G_5^+} I(\underline{\lambda}) d\mu(G_5) = \int_{\underline{\lambda}}\int_{\underline{a}}\int_{\underline{b}}\int_{\underline{c}} I(\underline{\lambda}) d\mu(\underline{c})d\mu(\underline{b})d\mu(\underline{a})d\mu(\underline{\lambda})$. Since $I(\underline{\lambda})$ depends only on $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, which appear only in the computation of the outermost integral, we consider them as constants for all the inner integrals, which means that we have $\zeta^{\wedge}_{L_{5,p}}(s) = \int_{\underline{\lambda}} I(\underline{\lambda}) \int_{\underline{a}}\int_{\underline{b}}\int_{\underline{c}} 1 d\mu(\underline{c})d\mu(\underline{b})d\mu(\underline{a})d\mu(\underline{\lambda})$, hence all the inner integrals evaluate to the measure of their domains of integration. now we compute the innermost integral by considering $\underline{a}$, $\underline{b}$ and $\underline{\lambda}$ as constants, and integrating only over $\underline{c}$. Considering the multiplication uh, we observe that for each element c_j, we must have that $\rho_j = c_j\lambda_1\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p$, which means that $v(\rho_i) = v(c_i\lambda_1\lambda_2\lambda_3\lambda_4) \geq 0 \Rightarrow v(c_i) + v_1 + v_2 + v_3 + v_4 \geq 0 \Rightarrow v(c_i) \geq -(v_1 + v_2 + v_3 + v_4)$. But this means that $c_i \in p^{-(v_1+v_2+v_3+v_4)}\mathbb{Z}_p$, and since the domain of integration for this integral is $\underline{c} = \{c_1, c_2, c_3, c_4\}$, then $\mu(\underline{c}) = |c_j|_p^4 = p^{4(v_1+v_2+v_3+v_4)}$. Denote $I(\underline{\lambda}, \underline{c}) := I(\underline{\lambda})p^{4(v_1+v_2+v_3+v_4)}$, we now have that $\zeta^{\wedge}_{L_{5,p}}(s) = \int_{\underline{\lambda}} I(\underline{\lambda}, \underline{c}) \int_{\underline{a}}\int_{\underline{b}} 1 d\mu(\underline{b})d\mu(\underline{a})d\mu(\underline{\lambda})$.

Denote $\lambda_{13} := \lambda_1\lambda_2\lambda_3$, $\lambda_{24} := \lambda_2\lambda_3\lambda_4$, and $\lambda_{14} := \lambda_1\lambda_2\lambda_3\lambda_4$. We now consider the constraints on $\underline{b}$.

$b_{11}\lambda_{13}, b_{31}\lambda_{13}, b_{41}\lambda_{13} \in \mathbb{Z}_p$, and $b_{12}\lambda_{24}, b_{22}\lambda_{24} \in \mathbb{Z}_p$. These constaints are obtained by multiplying elements in block M_{13} with elements in h, but one observes that we have b_{22} also in location $(5,10)$ of the matrix, and b_{31} in location $(7,10)$, which means that $b_{22}\lambda_{14}, b_{31}\lambda_{14} \in \mathbb{Z}_p$. But since we already have $b_{22}\lambda_{24}, b_{31}\lambda_{13} \in \mathbb{Z}_p$, the constraints $b_{22}\lambda_{14}$ and $b_{31}\lambda_{14}$ do not contribute any new information. In addition, we have one of the elements of $\underline{b}$ that forms a constraint together with elements from $\underline{a}$, namely $(a_{11}a_{22} - b_{11})\lambda_{24} \in \mathbb{Z}_p$. The constraints $b_{31}\lambda_{13}, b_{41}\lambda_{13}, b_{12}\lambda_{24}, b_{22}\lambda_{24} \in \mathbb{Z}_p$ from above translate to $p^{-2(v_1+v_2+v_3)}p^{-2(v_2+v_3+v_4)} = p^{-2(v_1+2v_2+2v_3+v_4)}$. On the other hand, b_{11} is a part of two constraints, hence we must have both $b_{11} \in p^{-(v_1+v_2+v_3)}\mathbb{Z}_p$ and $a_{11}a_{22} - b_{11} \in p^{-(v_2+v_3+v_4)}\mathbb{Z}_p \Rightarrow b_{11} \in a_{11}a_{22} + p^{-(v_2+v_3+v_4)}\mathbb{Z}_p$, which means that we need to compute the measure $\mu(A)$, where $A = p^{-(v_1+v_2+v_3)}\mathbb{Z}_p \cap a_{11}a_{22} + p^{-(v_2+v_3+v_4)}\mathbb{Z}_p$. Denote $\alpha := v_1 + v_2 + v_3$, $\beta := v_2 + v_3 + v_4$ and $x := a_{11}a_{22}$, and we need to find a formula for a generic intersection of the form $A = p^{-\alpha}\mathbb{Z}_p \cap x + p^{-\beta}\mathbb{Z}_p$. We need to find a formula for this generic form. Since b_{11} is in the intersection, we have that $b_{11} = z = x + y$ where $y \in p^{-\beta}$ and $z \in p^{-\alpha}\mathbb{Z}_p \Rightarrow z - x \in p^{-\beta}\mathbb{Z}_p$. Assume $\beta \geq \alpha \Rightarrow -\beta \leq -\alpha$, and since $v_p(b_{11}) = v_p(z - x) \geq \min\{v_p(z), v_p(x)\}$, and $v_p(z) \geq -\alpha \geq -\beta$, then we have two cases. If $v_p(x) \geq -\beta$, then $v_p(z - x) \geq \beta \Rightarrow z - x \in p^{-\beta}\mathbb{Z}_p$. But $-\alpha \geq -\beta \Rightarrow p^{-\alpha}\mathbb{Z}_p \subseteq p^{-\beta}\mathbb{Z}_p \Rightarrow A = p^{-\alpha}\mathbb{Z}_p$. If $v_p(x) < -\beta$, then $v_p(z - x) = v_p(x) < -\beta \Rightarrow z - x \notin p^{-\beta}\mathbb{Z}_p$, which means that $A = \varnothing$. One checks that if we assume $\alpha \geq \beta$, then we obtain that $A = p^{-\beta}\mathbb{Z}_p$ if $v_p(x) \geq -\alpha$, and $A = \varnothing$ if $v_p(x) < -\alpha$. Therefore, $\mu(A) = p^{\min\{\alpha,\beta\}}$ for every x such that $v_p(x) \geq \min\{-\alpha, -\beta\} = -\max\{\alpha, \beta\}$, which means, in our case, that $v_p(x) = v_p(a_{11}a_{22}) \geq -\max\{v_1+v_2+v_3, v_2+v_3+v_4\} = -v_2-v_3-\max\{v_1, v_4\}$. Thus, denoting $I(\underline{\lambda}, \underline{c}, \underline{b}) := I(\underline{\lambda}, \underline{c})p^{-(v_2+v_3)-\max\{v_1,v_4\}}$, we have that $\zeta^{\wedge}_{L_{5,p}}(s) = \int_{\underline{\lambda}} I(\underline{\lambda}, \underline{c}, \underline{b}) \int_{\underline{a}} 1 d\mu(\underline{b})d\mu(\underline{a})d\mu(\underline{\lambda})$. For the constraints on $\underline{a}$, we have

$a_{11}\lambda_1\lambda_2, -a_{11}\lambda_2\lambda_3, -a_{11}\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_p(a_{11}) \geq -(v_1 + v_2), v_p(a_{11}) \geq -(v_2 + v_3) \Rightarrow v_p(a_{11}) \geq -v_2 - \min\{v_1, v_3\}$.

$a_{21}\lambda_1\lambda_2, a_{21}\lambda_1\lambda_2\lambda_3, a_{21}\lambda_1\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p \Rightarrow v_p(a_{21}) \geq -(v_1 + v_2)$,

$a_{22}\lambda_2\lambda_3, -a_{22}\lambda_3\lambda_4, a_{22}\lambda_1\lambda_2\lambda_3$,

$a_{33}\lambda_3\lambda_4, a_{33}\lambda_2\lambda_3\lambda_4, a_{33}\lambda_1\lambda_2\lambda_3\lambda_4$,

$a_{21}a_{22}\lambda_1\lambda_2, -a_{11}a_{33}\lambda_2\lambda_3\lambda_4, a_{21}a_{33}\lambda_1\lambda_2\lambda_3\lambda_4 \in \mathbb{Z}_p$.