1 The computation of $G_n(\mathbb{Q}_p)$

1.1 The computation of the first block M_{11}

Proposition 1.1.1. *Let $x = \sum_{i=1}^{n-1} \lambda_i e_{i,i+1}$, where $\lambda_i \in \mathbb{Q}_p$ not all zero. Then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = l + m$, where **l** is the number of sequences of non-zero coefficients of the form $\lambda_j, \lambda_{j+1}, \ldots, \lambda_{j+k-1}, \lambda_{j+k}$ and $\lambda_{j-1} = \lambda_{j+k+1} = 0$[1], and **m** is the number of zero coefficients $\lambda_j = 0$, such that also $\lambda_{j-1} = \lambda_{j+1} = 0$.*

Proof. Let $y = \sum_{i=1}^{n-1} \mu_i e_{i,i+1} \in \mathcal{C}_{\gamma_3}(x)$, where $\lambda_i \in \mathbb{Q}_p$. For every $1 \leq i \leq n-1$, denote by c_i the constraint equation $[\lambda_i e_{i,i+1}, \mu_{i+1} e_{i+1,i+2}] - [\lambda_{i+1} e_{i+1,i+2}, \mu_i e_{i,i+1}] = (\lambda_i \mu_{i+1} - \lambda_{i+1} \mu_i) e_{i,i+2} = 0$. Let $1 \leq j \leq n-1$ and $1 \leq k \leq n-1-j$ be two indices, such that $\lambda_{j-1} = \lambda_{j+k+1} = 0$, and $\lambda_j, \lambda_{j+1}, \ldots, \lambda_{j+k-1}, \lambda_{j+k}$ are all non-zero, then by constraints $c_j, c_{j+1}, \ldots, c_{m-1}$, we have that $\mu_m = \frac{\lambda_m}{\lambda_{m-1}} \mu_{m-1} = \frac{\lambda_m}{\lambda_{m-1}} \frac{\lambda_{m-1}}{\lambda_{m-2}} \mu_{m-2} = \frac{\lambda_m}{\lambda_{m-2}} \mu_{m-2} = \cdots = \frac{\lambda_m}{\lambda_j} \mu_j$, for every $j+1 \leq m \leq j+k-1$, which means that all μ coefficients of y, with indices from $j+1$ to $j+k$, depend on the first coefficient, namely μ_j. We denote the free choice of μ_j by $\mu_j = *$. One easily checks that we can choose freely any coefficient μ_m from $j+1$ to $j+k$, instead of μ_j, and all other coefficients in that range will depend on our choice of μ_m. By constraint c_{j-1}, we have that $\lambda_{j-1}\mu_j - \lambda_j \mu_{j-1} = 0$, but $\lambda_{j-1} = 0$, hence $\lambda_j \mu_{j-1}$ must vanish, but $\lambda_j \neq 0$, which obviously means that $\mu_{j-1} = 0$. Similarly, we have that $\mu_{j+k+1} = 0$, due to constraint c_{j+k}. By constraint c_{j+k+1}, we have that $\lambda_{k+k+1}\mu_{j+k+2} - \lambda_{j+k+2}\mu_{j+k+1} = 0$, but $\lambda_{j+k+1} = \mu_{j+k+1} = 0$, hence, $\lambda_{j+k+1}\mu_{j+k+2}$ must vanish, but $\lambda_{j+k+1} = 0$, which means that we need to look at constraint c_{j+k+2}, that is, $\lambda_{j+k+2}\mu_{j+k+3} - \lambda_{j+k+3}\mu_{j+k+2} = 0$. We check the different options. If $\lambda_{j+k+2} = 0$, then $\lambda_{j+k+3}\mu_{j+k+2}$ must vanish. Therefore, if $\lambda_{j+k+3} \neq 0$, then $\mu_{j+k+2} = 0$, but if $\lambda_{j+k+3} = 0$, then $\mu_{j+k+2} = *$. If $\lambda_{j+k+2} \neq 0$, then again $\mu_{j+k+2} = *$. If $\lambda_{j+k+2} \neq 0$, then $\mu_{j+k+2} = *$, and we continue the same way as for λ_j and its following coefficients. $\square$

Corollary 1.1.2. *Let $\mathcal{L}_{n,p}$ be the $\mathbb{Q}_p$-Lie algebra associated with $\mathcal{U}_n(\mathbb{Z})$. If $n \geq 5$, then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \gamma_1/\gamma_3 - 1$ if and only if $x \in \{\lambda e_{12} + \gamma_2 \mathcal{L}_{n,p}\}$ or $x \in \{\lambda e_{n-1,n} + \gamma_2 \mathcal{L}_{n,p}\}$, for a non-zero scalar $\lambda \in \mathbb{Q}_p$. If $n = 4$, then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \gamma_1/\gamma_3 - 1$ if and only if $x \in \{\lambda e_{12} + \mu e_{34} + \gamma_2 \mathcal{L}_{n,p}\}$, for $\lambda, \mu \in \mathbb{Q}_p$ not both zero.*

[1] We extend our notation of indices, to include also the case where $j = 1$ or $j+k = n-1$, and define that $\lambda_{j-1} = \lambda_0 = 0$ or $\lambda_{j+k+1} = \lambda_n = 0$, respectively

Proof. Let $z = \lambda_{j,j+2}e_{j,j+2}$, where $1 \leq j \leq n-2$ and $\lambda_{j,j+2} \in \mathbb{Q}_p$, then for every $w \in {}^{\gamma_1}\!/\!_{\gamma_3}$, either z commutes with w or $[z,w] \in \gamma_3\mathcal{L}_{n,p}$, which means that $\lambda_{j,j+2}e_{j,j+2} \in \mathcal{C}_{\gamma_1/\gamma_3}$, for every $1 \leq j \leq n-2$. Hence, ${}^{\gamma_2}\!/\!_{\gamma_3} = \langle e_{13}, e_{24}, \dots, e_{n-2,n}\rangle \subset \mathcal{C}_{\gamma_1/\gamma_3}(x)$. Therefore, we only need to discuss elements of the quotient ${}^{\gamma_1}\!/\!_{\gamma_2}$, for the purpose of this proof. Suppose that $x = \lambda_1 e_{12} + z$, where $z \in \gamma_2\mathcal{L}_{n,p}$, then we have one sequence of non-zero coefficients, namely λ_1, and we have $n-2$ zero coefficients $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1} = 0$, from which $n-3$ are between two other zeros. Hence, by 1.1.1, we have that $\mathcal{C}_{\gamma_1/\gamma_2}(x) = 1+(n-3) = n-2 = (n-1)-1 = \dim {}^{\gamma_1}\!/\!_{\gamma_2} - 1$. Similarly, the same goes also for $x = \lambda_{n-1}e_{n-1,n} + z$. Suppose that $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = \dim {}^{\gamma_1}\!/\!_{\gamma_2} - 1$, but $x = \sum_{i=1}^{n-1} \lambda_i e_{i,i+1}$, such that either of the following options is true:

1. there is more than one sequence of consecutive non-zero coefficients in the linear combination that forms x.

2. there is one sequence of consecutive non-zero coefficients, but at least one of those coefficients has index $2 \leq j \leq n-2$, meaning it is not λ_1 nor λ_{n-1}.

For the second option, we start by fixing one index $2 \leq j \leq n-2$, and assume that $x = \lambda_j e_{j,j+1}$. The number of zero coefficients in x is $n-1-1 = n-2$, but λ_j and the zeros in indices $j-1, j+1$ are neighboring, hence $m_1 = n-2-2 = n-4$, and then $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = l_1 + m_1 = 1 + n - 4 = n-3 < n-2 = \dim {}^{\gamma_1}\!/\!_{\gamma_2} - 1$. We denote by k the length of the sequence of consecutive non-zero parameters, and prove that for any $k > 0$, where at least one non-zero coefficient λ_j lies in $2 \leq j \leq n-2$, $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) < n-2$, by simple induction on k. For $k = 1$, we have just shown that. For $k > 1$, there are $k-1$ additional zeros that are replaced by non-zero coefficients, where except for λ_{j-1} and λ_{j+1}, all the other zeros were originally lying between two other zeros. If the original sequence was $\lambda_2 e_{23}$ or $\lambda_{n-2}e_{n-2,n-1}$, and the new sequence is $\lambda_1 e_{12}, \lambda_2 e_{23}$ or $\lambda_{n-2}e_{n-2,n-1}, \lambda_{n-1}e_{n-1,n}$, respectively, then $m_k = m_1$, but clearly, in any other case, $m_k < m_1$, while $l_k = l_1 = 1$ at any case. by the assumption, for the original sequence, $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = l_1 + m_1 < n-2$, hence for the new sequence, $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = l_k + m_k \leq l_1 + m_1 = n-3 < n-2$. Now we check the first option, starting from the case where $x = \lambda_1 e_{12} + \lambda_{n-1}e_{n-1,n}$. In this case, $l_2 = 2$ and the number of zeros is $n-1-2 = n-3$, but λ_1 and the zero in index 2 are neighboring, and so are λ_{n-1} and the zero in index $n-2$, hence $m_2 = n-3-2 = n-5$ zeros are lying between two other zeros, therefore $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = l_2 + m_2 = n-5+2 = n-3 < n-2$. if we add another non-zero coefficient, then it must lie in some index $2 \leq j \leq n-2$, for which we have already proved that $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) < n-2$, which completes the proof for $n \geq 5$. For $n = 4$, we can check explicitly. Assume $x = \lambda e_{12} + \mu e_{34}$, denote an element in the centralizer of x by $y = \rho e_{12} + \tau e_{23} + \nu e_{34}$, and we observe that $[x, y] = [\lambda e_{12}, \tau e_{23}] + [\mu e_{34}, \tau e_{23}] = \lambda\tau e_{13} - \tau\mu e_{24} = 0$, hence $\tau = 0$, while $\rho = *$ and $\nu = *$, so $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = 2 = \dim {}^{\gamma_1}/_{\gamma_2} - 1$, as requested, and it is readily seen that even if either $\lambda = 0$ or $\mu = 0$, but not both, then τ still has to be zero, in order to satisfy either $\tau\mu = 0$ or $\lambda\tau = 0$, respectively, and ρ, ν can still be anything, which means that in either case, where the coefficient of e_{23} is zero but $x \neq 0$, we have $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = 2$. Assume $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = \dim {}^{\gamma_1}/_{\gamma_2} - 1 = 3 - 1 = 2$, then if x is not of the suggested form, it means that $x = \lambda e_{12} + \sigma e_{23} + \mu e_{34}$, where $\sigma \neq 0$ and either λ or μ or both can be zero. If $x = \lambda e_{12} + \sigma e_{23} + \mu e_{34}$ and all coefficients are non-zero, then for every $y \in \mathcal{C}_{\gamma_1/\gamma_2}(x)$ denoted by $y = \rho e_{12} + \tau e_{23} + \nu e_{34}$, we have $[x, y] = [\lambda e_{12}, \tau e_{23}] + [\sigma e_{23}, \rho e_{12}] + [\sigma e_{23}, \nu e_{34}] + [\mu e_{34}, \tau e_{23}] = (\lambda\tau - \sigma\rho)e_{13} + (\sigma\nu - \mu\tau)e_{24}$, hence $\tau = \frac{\sigma}{\lambda}\rho$ and $\nu = \frac{\mu}{\sigma}\tau = \frac{\mu}{\sigma}\frac{\sigma}{\lambda}\rho = \frac{\mu}{\lambda}\rho$, but this means that $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = 1$, because both τ and ν depend on ρ. If either λ or μ or both are zero, then either $\sigma\rho$ or $\sigma\mu$ or both are zero, which means that ρ or ν or both are zero, since $\sigma \neq 0$, but this means that either $y = \tau e_{23} + \frac{\mu}{\sigma}\tau e_{34}$ or $y = \frac{\lambda}{\sigma}\tau e_{12} + \tau e_{23}$ or $y = \tau e_{23}$, respectively. Therefore, in either case, where $\sigma \neq 0$, we have $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = 1$, which completes the proof for $n = 4$. □

Corollary 1.1.3. *Let $\mathcal{L}_{n,p}$ be a $\mathbb{Q}_p$-Lie algebra, where $n \geq 4$, and let $\varphi \in G_n(\mathbb{Q}_p)$ be an $\mathcal{L}_{n,p}$-automorphism, then $\varphi_{11}(e_{12}) = \lambda_1 e_{12}$ and $\varphi_{11}(e_{n,n-1}) = \lambda_{n-1}e_{n-1,n}$, or $\varphi_{11}(e_{12}) = \lambda_{n-1}e_{n-1,n}$ and $\varphi_{11}(e_{n,n-1}) = \lambda_1 e_{1,2}$.*

Proof. We look at the centralizer of e_{12} in the quotient ${}^{\gamma_1}/_{\gamma_3}$, namely $\mathcal{C}_{\gamma_1/\gamma_3}(e_{12})$. Clearly, for any $e_{i,i+2} \in {}^{\gamma_2}/_{\gamma_3}$, we have that $[e_{12}, e_{i,i+2}]$ is either zero, or $i = 2$ and then $[e_{12}, e_{24}] = e_{14} \in \gamma_3\mathcal{L}_{n,p}$, which vanishes in the quotient ${}^{\gamma_1}/_{\gamma_3}$, which means that in either case it is zero in this quotient. Therefore, we look only at elements $e_{i,i+1} \in {}^{\gamma_1}/_{\gamma_2}$. It is readily seen that every element of the form $e_{i,i+1}$ where $i \neq 2$ commutes with e_{12}, hence $\mathcal{C}_{\gamma_1/\gamma_2}(e_{12}) = \langle e_{12}, e_{34}, e_{45}, \ldots, e_{n-2,n-1}, e_{n-1,n}\rangle$, so $\dim \mathcal{C}_{\gamma_1/\gamma_2}(e_{12}) = \dim {}^{\gamma_1}/_{\gamma_2} - 1$, but since φ_{11} is an automorphism, it must preserve the dimension of the centralizer, meaning $\dim \mathcal{C}_{\gamma_1/\gamma_2}(\varphi_{11}(e_{12})) = \dim \mathcal{C}_{\gamma_1/\gamma_2}(e_{12}) = \dim {}^{\gamma_1}/_{\gamma_2} - 1$. But by corollary 1.1.2, if $n \geq 5$, then $\varphi_{11}(e_{12}) = \lambda e_{12}$ or $\varphi_{11}(e_{12}) = \lambda e_{n-1,n}$, and it is readily seen that the same applies also for $\varphi_{11}(e_{n-1,n})$, and since φ is injective, then clearly, if $\varphi_{11}(e_{12}) = \lambda e_{12}$ then $\varphi_{11}(e_{n-1,n}) = \lambda e_{n-1,n}$, and if $\varphi_{11}(e_{12}) = \lambda e_{n-1,n}$ then $\varphi_{11}(e_{n-1,n}) = \lambda e_{12}$. If $n = 4$, then by the same corollary, $\varphi_{11}(e_{12}) = \lambda e_{12} + \mu e_{34}$, where λ and μ are not both zero, and the same applies for $\varphi_{11}(e_{34})$. Assume that λ and μ are both non-zero, and denote by $\psi = \varphi^{-1}$, the inverse of φ. Obviously, ψ itself must be of the same form $\psi_{11}(e_{12}) = \rho e_{12} + \tau e_{34}$, where ρ and τ are not both zero, and $\psi_{11}(e_{34}) = \nu e_{12} + \sigma e_{34}$, where ν and σ are not both zero. Then $e_{12} = \psi_{11}(\varphi_{11}(e_{12})) = \psi_{11}(\lambda e_{12} + \mu e_{34}) = \lambda\psi_{11}(e_{12}) + \mu\psi_{11}(e_{34}) = \lambda\rho e_{12} + \lambda\tau e_{34} + \mu\nu e_{12} + \mu\sigma e_{34} = (\lambda\rho + \mu\nu)e_{12} + (\lambda\tau + \mu\sigma)e_{34}$. Clearly, the coefficient $\lambda\tau + \mu\sigma$ of e_{34} must vanish, so the equation can be satisfied. We observe that τ and σ must be either both zero or both non-zero, otherwise $\lambda\tau + \mu\sigma \neq 0$. Assume $\tau = \sigma = 0$, then $\psi_{11}(e_{12}) = \rho e_{12}$ and $\psi_{11}(e_{34}) = \nu e_{12}$. But then $e_{12} = \varphi_{11}(\psi_{11}(e_{12})) = \varphi_{11}(\rho e_{12}) = \rho\varphi_{11}(e_{12}) = \rho\lambda e_{12} + \rho\mu e_{34}$, which means that $\rho\mu$ must vanish, hence $\rho = 0$, but then $\psi_{11}(e_{12}) = 0$. $\square$