1 The computation of $G_n(\mathbb{Q}_p)$

1.1 The computation of the first block M_{11}

For all the following, we extend our notations of elements in the quotient γ_1/γ_2 by denoting $x = \sum_{i=0}^{n} \lambda_i e_{i,i+1}$, for all $x \in \gamma_1/\gamma_3$, and since $e_{0,1}$ and $e_{n-1,n}$ do not exist in our algebra, it is well understood that $\lambda_0 = \lambda_n = 0$. We also denote by z_j, w_j, v_j or u_z some element in $\gamma_j \mathcal{L}_{n,p}$ whose accurate value is of no interest for the matter of the discussion.

Lemma 1.1.1. *Let* $x = \sum_{i=0}^{n} \lambda_i e_{i,i+1}$, *where* $\lambda_i \in \mathbb{Q}_p$ *are not all zero. Then* $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 = \mathfrak{l}(x) + \mathfrak{m}(x)$, *where* $\mathfrak{l}(x)$ *is the number of sequences of consecutive non-zero coefficients of the form* $\lambda_j, \lambda_{j+1}, \ldots, \lambda_{j+k-1}, \lambda_{j+k}$ *and* $\lambda_{j-1} = \lambda_{j+k+1} = 0$, *and* $\mathfrak{m}(x)$ *is the number of zero coefficients* $\lambda_j = 0$, *such that also* $\lambda_{j-1} = \lambda_{j+1} = 0$.

Proof. Let $y = \sum_{i=1}^{n-1} \mu_i e_{i,i+1}$, where $\lambda_i \in \mathbb{Q}_p$, be an element in the quotient γ_1/γ_2. For every $1 \leq i \leq n-1$, denote by $(\mathfrak{C}_i)$ the constraint equation $[\lambda_i e_{i,i+1}, \mu_{i+1} e_{i+1,i+2}] - [\lambda_{i+1} e_{i+1,i+2}, \mu_i e_{i,i+1}] = (\lambda_i \mu_{i+1} - \lambda_{i+1} \mu_i) e_{i,i+2} = 0$, and it is clear that $y \in \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2$ if and only if all the $(\mathfrak{C}_i)$ constraints are satisfied. We observe that each μ_i participates in two constraints, $(\mathfrak{C}_{i-1})$ and $(\mathfrak{C}_i)$, that is, $\lambda_{i-1} \mu_i - \lambda_i \mu_{i-1} = \lambda_i \mu_{i+1} - \lambda_{i+1} \mu_i = 0$. If $\lambda_i = 0$, then $\lambda_i \mu_{i-1} = \lambda_i \mu_{i+1} = 0$, hence by constraint $(\mathfrak{C}_{i-1})$ we have that $\lambda_{i-1} \mu_i = 0$, and by constraint $(\mathfrak{C}_i)$ we have that $\lambda_{i+1} \mu_i = 0$. Hence, if either λ_{i-1} or λ_{i+1} are non-zero, then $\mu_i = 0$. But if $\lambda_{i-1} = \lambda_{i+1} = 0$, then both constraints are satisfied for any choice of μ_i, which increases $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2$ by 1. We need to prove that for any sequence of k consecutive zero coefficients of x, where $k \geq 3$, that is[1] $\lambda_{j+1} = \lambda_{j+2} = \cdots = \lambda_{j+k} = 0$, for $1 \leq j \leq n-2$, we have that the sequence $\mu_{j+2}, \mu_{j+3}, \ldots, \mu_{j+k-1}$ of $k-2$ consecutive coefficients of y is made of scalars of any choice, thus $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2$ is increased by $k-2$. We prove that by simple induction on k. For $k = 3$, we just proved that if $\lambda_{i-1} = \lambda_i = \lambda_{i+1} = 0$, then μ_i can be any scalar. For $k+1$, we look at the sequence of $k+1$ zero coefficients, $\lambda_{j+1} = \lambda_{j+2} = \cdots = \lambda_{j+k} = \lambda_{j+k+1} = 0$. By constraints $(\mathfrak{C}_{j+k-1})$ and $(\mathfrak{C}_{j+k})$, we have that $\lambda_{j+k-1} \mu_{j+k} - \lambda_{j+k} \mu_{j+k-1} = \lambda_{j+k} \mu_{j+k+1} - \lambda_{j+k+1} \mu_{j+k} = 0$, and since $\lambda_{j+k-1} = \lambda_{j+k} = \lambda_{j+k+1} = 0$, we have that μ_{j+k} can be any scalar, as we proved earlier. By the assumption, we have that all $k-2$ previous coefficients, that is $\mu_{j+2}, \ldots, \mu_{j+k-1}$ can also be any scalars, so in total the

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[1] Here again, we consider the non-existent $\lambda_0 = \lambda_n = 0$ as part of the sequence

whole sequence of $k - 1 = (k + 1) - 2$ coefficients of y can be any scalars, which proves the induction step. Suppose that we have m sequences of three or more consecutive zero coefficients in x, whose lengths are $k_1, k_2, \ldots, k_m$, then $\mathfrak{m}(x) = \sum_{l=1}^{m} k_l - 2m$ is the total number of zero coefficients $\lambda_j = 0$ such that also $\lambda_{j-1} = \lambda_{j+1} = 0$, as proposed. Using again the two consecutive constraints, $(\mathfrak{C}_{i-1})$ and $(\mathfrak{C}_i)$, suppose now that $\lambda_i \neq 0$. If $\lambda_{i-1} = 0$, then by constraint $(\mathfrak{C}_{i-1})$ we must have that $\mu_{i-1} = 0$, but if $\lambda_{i-1} \neq 0$, then by this constraint we have $\mu_i = \frac{\lambda_i \mu_{i-1}}{\lambda_{i-1}}$, which means that μ_i depends on μ_{i-1}. Precisely the same way for constraint $(\mathfrak{C}_i)$, we have that if $\lambda_{i+1} = 0$ then $\mu_{i+1} = 0$, otherwise $\mu_{i+1} = \frac{\lambda_{i+1} \mu_i}{\lambda_i}$, which means that μ_{i+1} depends on μ_i, and if also $\lambda_{i-1} \neq 0$, then $\mu_{i+1} = \frac{\lambda_{i+1} \frac{\lambda_i \mu_{i-1}}{\lambda_{i-1}}}{\lambda_i} = \frac{\lambda_{i+1} \mu_{i-1}}{\lambda_{i-1}}$, which means that both μ_{i+1} and μ_i depend on μ_{i-1}. We need to prove this is true for any sequence of k consecutive non-zero coefficients of x, that is, for a given sequence of coefficients, $\lambda_{j+1}, \lambda_{j+2}, \ldots, \lambda_{j+k}$, where $1 \leq j \leq n - 1$, we need to prove that μ_{j+1} can be any scalar, while $\mu_{j+2}, \mu_{j+3}, \ldots, \mu_{j+k}$ all depend on μ_{j+1}. Here again, we use a simple induction on k. For $k = 1, 2, 3$, we already proved this. For $k + 1$, we look into a sequence of $k + 1$ consecutive non-zero coefficients of x, that is $\lambda_{j+1}, \ldots, \lambda_{j+k}, \lambda_{j+k+1}$. By constraint $(\mathfrak{C}_{j+k})$ we have $\lambda_{j+k} \mu_{j+k+1} - \lambda_{j+k+1} \mu_{j+k} = 0$, and since $\lambda_{j+k} \neq 0$, we have $\mu_{j+k+1} = \frac{\lambda_{j+k+1} \mu_{j+k}}{\lambda_{j+k}}$, but by the assumption, $\mu_{j+k} = \frac{\lambda_{j+k} \mu_{j+k-1}}{\lambda_{j+k-1}} = \frac{\lambda_{j+k}}{\lambda_{j+k-1}} \frac{\lambda_{j+k-1} \mu_{j+k-2}}{\lambda_{j+k-2}} = \frac{\lambda_{j+k}}{\lambda_{j+k-1}} \frac{\lambda_{j+k-1}}{\lambda_{j+k-2}} \frac{\lambda_{j+k-2} \mu_{j+k-3}}{\lambda_{j+k-3}} = \cdots = \frac{\lambda_{j+k} \mu_{j+1}}{\lambda_{j+1}}$, hence $\mu_{j+k+1} = \frac{\lambda_{j+k+1} \mu_{j+k}}{\lambda_{j+k}} = \frac{\lambda_{j+k+1}}{\lambda_{j+k}} \frac{\lambda_{j+k} \mu_{j+1}}{\lambda_{j+1}} = \frac{\lambda_{j+k+1} \mu_{j+1}}{\lambda_{j+1}}$, which proves the induction step. Looking again at $(\mathfrak{C}_{i-1})$ and $(\mathfrak{C}_i)$, we observe that if λ_{i-1} and λ_{i+1} are non-zero and $\lambda_i = 0$, then $\lambda_i \mu_{i-1} = \lambda_i \mu_{i+1} = 0$, hence by constraint $(\mathfrak{C}_{i-1})$ we have that $\lambda_{i-1} \mu_i = 0$, and by constraint $(\mathfrak{C}_i)$ we have that $\lambda_{i+1} \mu_i = 0$, so $\mu_i = 0$ by both constraints, which means that there is no dependency between μ_{i+1} and μ_{i-1}. This shows that any zero coefficient between two non-zero coefficients of x creates two separate sequences, each sequences increases $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2$ by 1, hence, if we have l sequences of consecutive non-zero coefficients of x, they increase $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2$ by l, and we denote $\mathfrak{l}(x) = l$. All this shows that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 = \mathfrak{l}(x) + \mathfrak{m}(x)$, as proposed. $\qquad\square$

Proposition 1.1.2. *Let $\mathcal{L}_{n,p}$ be the $\mathbb{Q}_p$-Lie algebra associated with $\mathcal{U}_n(\mathbb{Z})$, where $n \geq 5$, then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \gamma_1/\gamma_3 - 1$ if and only if $x \in \{\lambda e_{12} + \gamma_2 \mathcal{L}_{n,p}\}$ or $x \in \{\lambda e_{n-1,n} + \gamma_2 \mathcal{L}_{n,p}\}$, for a non-zero scalar $\lambda \in \mathbb{Q}_p$.*

Proof. We recall first that for any algebra $\mathcal{L}_{n,p}$, the first two elements of

the lower central series, namely γ_1/γ_2 and γ_2/γ_3, are of sizes $n-1$ and $n-2$, respectively. Consider the Lie brackets operation $[x_1, x_2]$, where $x_1 \in \gamma_1/\gamma_2$ and $x_2 \in \gamma_2/\gamma_3$, or the opposite, then $[x_1, x_2]$ can be either zero, or $[x_1, x_2] \in \gamma_3$, which means that if we consider the centralizer of some $x \in \mathcal{L}_{n,p}$ in γ_1/γ_3, then any element $y \in \gamma_2/\gamma_3$ would either commute with x or yield an element in γ_3, which also means that it commutes with x in γ_1/γ_3. This shows that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) \geq \dim \gamma_2/\gamma_3 = n-2$. Suppose that $x = \lambda_1 e_{12} + z$ or $x = \lambda_{n-1} e_{n-1,n} + z$, where $z \in \gamma_2 \mathcal{L}_{n,p}$, then looking only at the elements of γ_1/γ_2 in the linear combination that forms x, we have one sequence of one non-zero coefficient, hence $\mathfrak{l}(x) = 1$, and one sequence of $n-1-1 = n-2$ zero coefficients of x, hence we have[2] that $\mathfrak{m}(x) = n-2-1 = n-3$. By 1.1.1 we have that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 = \mathfrak{l}(x) + \mathfrak{m}(x) = 1 + (n-3) = n-2$. Adding all the elements of γ_2/γ_3, we have that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = (n-2) + (n-2) = 2n-4 = (n-1) + (n-2) - 1 = \dim \gamma_1/\gamma_2 + \dim \gamma_2/\gamma_3 - 1 = \dim \gamma_1/\gamma_3 - 1$, as proposed. To prove the opposite direction, we assume that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \gamma_1/\gamma_3 - 1 = 2n-4$, and since we already know that all the elements of $\dim \gamma_2/\gamma_3$ are in the centralizer of x, we have that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 = \dim \mathcal{C}_{\gamma_1/\gamma_3}(x) - \dim \gamma_2/\gamma_3 = (2n-4) - (n-2) = n-2 = (n-1) - 1 = \dim \gamma_1/\gamma_2 - 1$. Suppose that either $x = \lambda_i e_{i,i+1} + z$ where $1 < i < n-1$, or $x = \lambda_{i_1} e_{i_1,i_1+1} + \cdots + \lambda_{i_m} e_{i_m,i_m+1}$ where we denote the number of γ_1/γ_2 coefficients[3] by $2 \leq m \leq n-1$, in both cases $z \in \gamma_2 \mathcal{L}_{n,p}$. In the first case, we have that $\mathfrak{l}(x) = 1$, because there is only one non-zero coefficient of x in γ_1/γ_2, and we have two sequences of zero coefficients of x, that is $\lambda_0 = \lambda_1 = \lambda_2 = \cdots = \lambda_{i-2} = \lambda_{i-1} = 0$ and $\lambda_{i+1} = \lambda_{i+2} = \lambda_{i+3} = \cdots = \lambda_{n-2} = \lambda_{n-1} = \lambda_n = 0$, which means we have a sequence of zeros of length $(i-1) + 1 = i$ and a sequence of length $(n-1) - i + 1 = n-i$, hence by 1.1.1 we have[4] that $\mathfrak{m}(x) \leq (i-2) + (n-i-2) = n-4$, which means that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 = \mathfrak{l}(x) + \mathfrak{m}(x) \leq 1 + (n-4) = n-3 = (n-1) - 2 = \dim \gamma_1/\gamma_2 - 2$. Adding all the elements of γ_2/γ_3, we have that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 + \dim \gamma_2/\gamma_3 \leq (n-3) + (n-2) = 2n-5 < 2n-4$,

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[2] Again, we consider also $\lambda_0 = 0$ and $\lambda_n = 0$, hence the sequence of zero coefficients is either $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-2} = \lambda_{n-1} = \lambda_n = 0$ if $\lambda_1 \neq 0$, or $\lambda_0 = \lambda_1 = \lambda_2 = \cdots = \lambda_{n-3} = \lambda_{n-2} = 0$, if $\lambda_{n-1} \neq 0$, which means that the length of the entire sequence of zero coefficients is not $n-2$ but $n-1$, therefore by 1.1.1 we have that $\mathfrak{m}(x) = n-1-2 = n-3$

[3] In words, x has at least two non-zero coefficients in γ_1/γ_2

[4] Here, and in all the following options, $\mathfrak{m}(x)$ has an upper bound which depend on the number of consecutive zero coefficients of x, because if any of these sequences is of length k, where $k < 3$, then by 1.1.1 we have that $\mathfrak{m}(x) = k - 2 \leq 0$, which means that this particular sequence does not increase $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2$

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which contradicts the assumption. Suppose that x is of the second form, then x may have one sequence of k non-zero coefficients, where $k > 1$, or x may have two or more sequences of one or more non-zero coefficients. In the case that x has one sequence of k non-zero coefficients, $\mathfrak{l}(x) = 1$, and the number of consecutive zero coefficients is either less or equal to $(n-1)-k+1 = n-k$, which results in $\mathfrak{m}(x) \leq n - k - 2$, if the sequence of non-zero coefficients contains[5] either λ_1 or λ_{n-1}, or the number of consecutive zero coefficients is divided into two separate sequences with a total number of consecutive zeros which is less or equal to $(n-1) - k + 2 = n - k + 1$, which results in $\mathfrak{m}(x) \leq n - k + 1 - 4 = n - k - 3$, in the case that the sequence of non-zero coefficients of x does not contains either λ_1 or λ_{n-1}, which means that $\mathfrak{l}(x) + \mathfrak{m}(x) \leq 1 + (n - k - 2) = n - k - 1$, and since $k > 1$, we have that $\dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}/_{\gamma_2} = \mathfrak{l}(x) + \mathfrak{m}(x) \leq n - 2 - 1 = n - 3 = (n-1) - 2$, which again contradicts the assumption. Suppose that x has l sequences of non-zero coefficients in γ_1/γ_2, of lengths $k_1, k_2, \ldots, k_l$, then $\mathfrak{l}(x) = l$, and the number of zeros is $n - 1 - \sum_{j=1}^{l} k_j + 2 = n + 1 - \sum_{j=1}^{l} k_j$ in $l+1$ sequences[6], considering the two options from earlier, whether any of the sequences of non-zero coefficients contains λ_1 or λ_{n-1}, or does not contain either of them. We observe that $n + 1 - \sum_{j=1}^{l} k_j$ is bounded by $n + 1 - l$, which is the case where $k_1 = k_2 = \cdots = k_l = 1$. Suppose that $l = 2$, and that each of the 3 sequences of zero coefficients is of length greater or equal to 3, then $\mathfrak{m}(x) \leq n + 1 - 2 - 3 \cdot 2 = n - 7$, hence $\dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}/_{\gamma_2} = \mathfrak{l}(x) + \mathfrak{m}(x) \leq 2 + n - 7 = n - 5$, hence $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim {}^{\mathcal{C}_{\gamma_1/\gamma_3}(x)}/_{\gamma_2} + \dim {}^{\gamma_2}/_{\gamma_3} \leq (n-5) + (n-2) = 2n - 7 < 2n - 2 = \dim {}^{\gamma_1}/_{\gamma_3} - 1$. We prove this is true for any number of sequences of non-zero coefficients of x simple induction on l. For $l = 2$ we already proved that, for $l + 1$ we take the upper bound case, which means that we add a new non-zero coefficient which splits a sequence of zero coefficients such that each of the two parts of the sequence that we split has at least 3 zero coefficients. Denote by $\mathfrak{l}_l(x) = l$ and $\mathfrak{m}_l(x)$ for l, and denote by $\mathfrak{l}_{l+1}(x) = l + 1$ and $\mathfrak{m}_{l+1}(x)$ for $l + 1$, then obviously $\mathfrak{l}_{l+1} + \mathfrak{m}_{l+1} \leq \mathfrak{l}_l + \mathfrak{m}_l$, because $\mathfrak{l}_{l+1} - \mathfrak{l}_l = 1$ and $\mathfrak{m}_l - \mathfrak{m}_{l+1} \geq 1$, because the newly added non-zero coefficient does not only remove one zero coefficient from the total count, but may also remove one or two of its neighboring zero coefficients, since they are no longer surrounded by another zero coefficient, which proves the induction

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[5]That is, $x = \lambda_1 e_{12} + \lambda_2 e_{23} + \cdots + \lambda_{k,k+1} e_{k,k+1}$ or $x = \lambda_{n-k} e_{n-k,n-k+1} + \lambda_{n-k+1} e_{n-k+1,n-k+2} + \cdots + \lambda_{n-1} e_{n-1,n}$

[6]Including $\lambda_0 = 0$ and $\lambda_n = 0$

step. All this proves that if $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \gamma_1/\gamma_3 - 1$, then the only option for x is either $x = \lambda_1 e_{12} + z$ or $x = \lambda_{n-1} e_{n-1,n} + z$, where $z \in \gamma_2 \mathcal{L}_{n,p}$, as proposed. $\qquad\square$

Proposition 1.1.3. *Let $\mathcal{L}_{4,p}$ be the $\mathbb{Q}_p$-Lie algebra associated with $\mathcal{U}_4(\mathbb{Z})$, then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \gamma_1/\gamma_3 - 1$ if and only if $x \in \{\lambda_1 e_{12} + \lambda_3 e_{34} + \gamma_2 \mathcal{L}_{4,p}\}$, for $\lambda_1, \lambda_3 \in \mathbb{Q}_p$ not both zero.*

Proof. Suppose that $x = \lambda e_{12} + \mu e_{34} + z$, where $x \in \gamma_2 \mathcal{L}_{4,p}$, then by 1.1.1 if $\lambda \neq 0$ and $\mu \neq 0$, we have that $\mathfrak{l}(x) = 2$, and $\mathfrak{m}(x) = 0$ because there are three isolated zero coefficients, $\lambda_0 = \lambda_2 = \lambda_4 = 0$, hence $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 = \mathfrak{l}(x) + \mathfrak{m}(x) = 2$, which means that $\dim \mathfrak{C}_{\gamma_1/\gamma_3}(x) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 + \dim \gamma_2/\gamma_3 = 2 + (4-2) = 4 = (4-1) + (4-2) - 1 = \dim \gamma_1/\gamma_2 + \dim \gamma_2/\gamma_3 - 1 = \dim \gamma_1/\gamma_3 - 1$. If $\lambda_1 = 0$ or $\lambda_3 = 0$, that is $x = \lambda_1 e_{12} + z$ or $x = \lambda$, where $z \in \gamma_2 \mathcal{L}_{4,p}$, then $\mathfrak{l}(x) = 1$ and we have a sequence of 3 consecutive zero coefficients, either $\lambda_2 = \lambda_3 = \lambda_4 = 0$ or $\lambda_0 = \lambda_1 \lambda_2 = 0$, which means that $\mathfrak{l}(x) = 3 - 2 = 1$, hence $\mathfrak{l}(x) + \mathfrak{m}(x) = 1 + 1 = 2$, hence $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \gamma_1/\gamma_3 - 1$, as proposed. To prove the opposite, we review the different options. Suppose that $x \in \gamma_2 \mathcal{L}_{4,p}$, which means that $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, hence $\mathfrak{l}(x) = 0$ and $\mathfrak{m}(x) = 5 - 2 = 3$, so $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 = \mathfrak{l}(x) + \mathfrak{m}(x) = 0 + 3$, and therefore $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 + \dim \gamma_2/\gamma_3 = 3 + (4-2) = 5 > \dim \gamma_1/\gamma_3 - 1$. The other option is that $x = \lambda_1 e_{12} + \lambda_2 e_{23} + \lambda_3 e_{34} + z$, where $\lambda_2 \neq 0$, where λ_1 and λ_3 may be either zero or non-zero, and where $z \in \gamma_2 \mathcal{L}_{4,p}$. Since $\lambda_2 \neq 0$ there is no sequence of zero coefficients of x which is greater than 2, and also, we observe that whether $\lambda_1 \neq 0$ or $\lambda_3 \neq 0$ or both are non-zero, they are part of the sequence of non-zero coefficients which has λ_2, hence $\mathfrak{l}(x) = 1$ and $\mathfrak{m}(x) = 0$, so $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(x)/\gamma_2 + \dim \gamma_2/\gamma_3 = \mathfrak{l}(x) + \mathfrak{m} + 2 = 1 + 2 = 3 < 4 = \dim \gamma_1/\gamma_3 - 1$, which means that if x is not as proposed, then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) \neq \dim \gamma_1/\gamma_3 - 1$. $\qquad\square$

Proposition 1.1.4. *Let $\varphi \in G_n(\mathbb{Q}_p)$, where $n \geq 5$, then M_{11} the first block of M, the coefficient matrix of φ, is either diagonal or anti-diagonal.*

Proof. We first look at $\varphi(e_{12})$. We observe that

$$\mathcal{C}_{\gamma_1/\gamma_3}(e_{12}) = \langle e_{12}, \cancel{e_{23}}, e_{34}, e_{45}, \ldots, e_{n-1,n}, e_{13}, e_{24}, e_{35}, \ldots, e_{n-2,n} \rangle$$

hence $\dim \mathcal{C}_{\gamma_1/\gamma_3}(\varphi(e_{12})) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(e_{12}) = \dim \gamma_1/\gamma_3 - 1$, hence by 1.1.2 we have that either $\varphi(e_{12}) = \lambda_1 e_{12} + z_2$ or $\varphi(e_{12}) = \lambda_{n-1} e_{n-1,n} + w_2$. From the same considerations, we have that either $\varphi(e_{n-1}) = \rho_1 e_{12} + v_2$ or $\rho_{n-1} e_{n-1,n} +$

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u_2. Clearly, if $\varphi(e_{12}) = \lambda_1 e_{12} + z_2$ then $\varphi(e_{n-1,n}) = \lambda_{n-1,n} e_{n-1,n} + w_2$ and vice versa, because φ is a monomorphism. We take, throughout this proof, $\varphi(e_{12}) = \lambda_1 e_{12} + z_2$, and we shall see later that this convention holds for our computational purposes. We now look at $\varphi(e_{23})$, and observe that

$$\mathcal{C}_{\gamma_1/\gamma_3}(e_{23}) = \langle \cancel{e_{12}}, e_{23}, \cancel{e_{34}}, e_{45}, \ldots, e_{n-1,n}, e_{13}, e_{24}, e_{35}, \ldots, e_{n-2,n} \rangle$$

but $e_{13} \in \mathcal{C}_{\gamma_1/\gamma_3}(e_{23})$ if and only if $\varphi(e_{13}) \in \mathcal{C}_{\gamma_1/\gamma_3}(\varphi(e_{23}))$, because φ is an automorphism, which means that $\varphi(e_{13}) = \varphi([e_{12}, e_{23}]) = [\varphi(e_{12}), \varphi(e_{23})] \neq 0$ in the quotient, hence $[\varphi(e_{12}), \varphi(e_{23})] = [\lambda_1 e_{12} + z_2, \sum_{i=1}^{n-1} \rho_i e_{i,i+1} + w_2] = \sum_{i=1}^{n-1} \lambda_1 \rho_i [e_{12}, e_{i,i+1}] + \lambda_1 w_2 + z_2 \sum_{i=1}^{n-1} \rho_i e_{i,i+1} + z_2 w_2 \neq 0$, and we observe that since we are in the quotient γ_1/γ_3, and any element of $\gamma_2 \mathcal{L}_{n,p}$ commutes with any element of the algebra in this quotient, then we must have $\sum_{i=1}^{n-1} \lambda_1 \rho_i [e_{12}, e_{i,i+1}] \neq 0$, but this is true only if $\rho_2 \neq 0$, thus $\sum_{i=1}^{n-1} \lambda_1 \rho_i [e_{12}, e_{i,i+1}] = \lambda_1 \rho_2 [e_{12}, e_{23}] = \lambda_1 \rho_2 e_{13} \neq 0$, hence the linear combination that forms $\varphi(e_{23})$ must have $\lambda_2 e_{23}$. Assume that $\varphi(e_{23}) = \lambda_2 e_{23} + \sum_{i=1}^{n-1} \lambda_i e_{i,i+1} + z_2$, where $i \neq 2$ and not all λ_i are zero. We observe that if $\varphi(e_{23}) = x = \lambda_1 e_{12} + \lambda_2 e_{23} + z_2$, then by 1.1.1 $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(\lambda_2 e_{23} + w_2)$. Therefore, we need to consider a larger centralizer,

$$\mathcal{C}_{\gamma_1/\gamma_4}(e_{23}) = \langle \cancel{e_{12}}, e_{23}, \cancel{e_{34}}, e_{45}, \ldots, e_{n-1,n}, e_{13}, e_{24}, \cancel{e_{35}}, e_{45}, \ldots, e_{n-2,n}, e_{14}, \ldots, e_{n-3,n} \rangle$$

hence $\mathrm{codim}\mathcal{C}_{\gamma_1/\gamma_4}(e_{23}) = 3$, and following the assumption on $\varphi(e_{23})$ we have that $\varphi(e_{23}) = x = \lambda_1 e_{12} + \lambda_2 e_{23} + \sum_{i=1}^{n-2} a_{i,i+2} e_{i,i+2} + z_3$, and we look into an arbitrary element $y \in \mathcal{C}_{\gamma_1/\gamma_4}(x)$, that is $y = \sum_{i=1}^{n-1} \mu_i e_{i,i+1} + \sum_{i=1}^{n-2} b_{i,i+2} e_{i,i+2} + w_3$, so we have the following equations ① $(\lambda_1 \mu_2 - \lambda_2 \mu_1)e_{13} = 0$, ② $\lambda_2 \mu_3 e_{24} = 0 \Rightarrow \mu_3 = 0$, ③ $(\lambda_1 b_{24} - a_{24}\mu_1)e_{14} = 0$, ④ $(\lambda_2 b_{35} - a_{35}\mu_2 + a_{24}\mu_4)e_{25} = 0$, and we observe that none of these equations is a linear combination of one or more of the other equations. This means that we have four independent constraints, which means that $\mathrm{codim}\mathcal{C}_{\gamma_1/\gamma_4}(x) < \mathrm{codim}\mathcal{C}_{\gamma_1/\gamma_4}(e_{23})$. These four equations can be formed in a 4×7 matrix

$$A = \begin{pmatrix} -\lambda_2 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ -a_{24} & 0 & 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & -a_{35} & 0 & a_{24} & 0 & 0 & \lambda_2 \end{pmatrix}$$

which is ordered in relation to the vector

$$v = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ b_{13} \\ b_{24} \\ b_{35} \end{pmatrix}$$

such that $Av = 0$. We observe that since $\mathrm{codim}\,\mathcal{C}_{\gamma_1/\gamma_4}(e_{23}) = 3$, then we must have that any 4×4 minor B of A has that $\det B = 0$. Let

$$B_1 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ -a_{35} & 0 & 0 & \lambda_2 \end{pmatrix}$$

be such a minor, so $\det B_1 = \lambda_1^2 \lambda_2^2 = 0$, but we know that $\lambda_2 \neq 0$, hence we must have that $\lambda_1 = 0$. we observe that since $\lambda_2 \neq 0$, if we assume that $\lambda_1 = 0$ then the first equation becomes $\lambda_2 \mu_1 = 0 \Rightarrow \mu_1 = 0$, thus the equation $\lambda_1 b_{24} - a_{24}\mu_1 = 0$ does not form any constraint on y because $\lambda_1 = \mu_1 = 0$, so the co dimension of the centralizer, in this case of $x = \varphi(e_{23}) = \lambda_2 e_{23} + \sum_{i=1}^{n-2} a_{i,i+2} e_{i,i+2} + z_3$ is correct, which means that $\varphi(e_{23}) \neq \lambda_1 e_{12} + \lambda_2 e_{23} + z$. We also observe that when $n = 5$, if $i = 4$, that is $\varphi(e_{23}) = \lambda_2 e_{23} + \lambda_4 e_{45}$, we have that $\dim \mathcal{C}_{\gamma_1/\gamma_2}(\lambda_2 e_{23} + \lambda_4 e_{45}) = \dim \mathcal{C}_{\gamma_1/\gamma_2}(\lambda_2 e_{23})$, and therefore we need to consider a larger centralizer to rule out this option as well, hence we conclude that $\varphi(e_{23}) = \lambda_1 e_{23} + z$, where $z \in \gamma_2 \mathcal{L}_{n,p}$. Continue to $\varphi(e_{34})$, we observe that since $[\varphi(e_{23}), \varphi(e_{34})] = \varphi(e_{24}) \neq 0$, we must have that the linear combination that forms $\varphi(e_{34})$ contains $\lambda_3 e_{34}$. Suppose that $\varphi(e_{34}) = \lambda_3 e_{34} + \lambda_i e_{i,i+1}$. Assume $i = 1$, that is, $\varphi(e_{34}) = x = \lambda_1 e_{12} + \lambda_3 e_{34} + z_2$, where $z_2 \in \gamma_2 \mathcal{L}_{n,p}$, we have that $\mathfrak{l}(\lambda_1 e_{12} + \lambda_3 e_{34}) = \mathfrak{l}(\lambda_3 e_{34}) + 1$, but $\mathfrak{m}(\lambda_1 e_{12} + \lambda_3 e_{34}) = \mathfrak{m}(\lambda_3 e_{34}) - 1$, hence $\dim \mathcal{C}_{\gamma_1/\gamma_3}(\lambda_1 e_{12} + \lambda_3 e_{34}) = \mathfrak{l}(\lambda_1 e_{12} + \lambda_3 e_{34}) + \mathfrak{m}(\lambda_1 e_{12} + \lambda_3 e_{34}) = (\mathfrak{l}(\lambda_3 e_{34}) + 1) + (\mathfrak{m}(\lambda_3 e_{34}) - 1) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(\lambda_3 e_{34})$, so we need to take a larger quotient, as we did for $\varphi(e_{23})$. Computing $\mathcal{C}_{\gamma_1/\gamma_4}(x)$, we have the following equations: ① $\lambda_1 \mu_2 e_{13} = 0$, ② $-\lambda_3 \mu_2 e_{24} = 0$, ③ $\lambda_3 \mu_4 e_{35} = 0$, ④ $(\lambda_1 b_{24} - a_{24}\mu_1 + a_{13}\mu_3 - \lambda_3 b_{13})e_{14} = 0$, ⑤ $(\lambda_3 b_{46} - a_{46}\mu_3 + a_{35}\mu_5)e_{36} = 0$, but obviously, if $n = 5$ then e_{36} does not exist in $\mathcal{L}_{5,p}$ hence equation ⑤ is omitted. We observe that for any $n \geq 6$ we have that $\mathcal{C}_{\gamma_1/\gamma_4}(e_{34}) =$

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$\{e_{12}, e_{34}, e_{56}, e_{67}, \ldots, e_{n,n-1}, e_{24}, e_{35}, e_{57}, e_{68}, \ldots, e_{n-2,n}, e_{14}, \ldots, e_{n-3,n}\}$, which means that $\mathrm{codim}\mathcal{C}_{\gamma_1/\gamma_4 \mathcal{L}_{n,p}}(e_{34}) = 4$, but if $n = 5$ then e_{46} does not exist in $\mathcal{L}_{5,p}$, hence $\mathrm{codim}\mathcal{C}_{\gamma_1/\gamma_4 \mathcal{L}_{5,p}}(e_{34}) = 3$. We observe that equations ① and ② yield the same constraint $\mu_2 = 0$, hence if $n \geq 6$ we have equations ①, ③, ④ and ⑤ which means that we have 4 constraints on the dimension of the centralizer, but if $n = 5$ we omit equation ⑤ which means that we have 3 constraints on the dimension of the centralizer, but this shows that for every $n \geq 5$, we have that $\dim \mathcal{C}_{\gamma_1/\gamma_4}(\lambda_2 e_{12} + \lambda_3 e_{34} + \sum_{i=1}^{n-2} a_{i,i+2} e_{i,i+2} + z_3) = \dim \mathcal{C}_{\gamma_1/\gamma_4}(e_{34})$. Therefore, we take a larger quotient γ_1/γ_5 and add more equations: ⑥ $(\lambda_1 b_{25} + a_{25}\mu_1 + a_{13}b_{35} - a_{35}b_{13})e_{15} = 0$, ⑦ $(-a_{36}\mu_2 + a_{24}b_{46} - a_{46}b_{24} + a_{25}\mu_5)e_{26} = 0$. Here again, we observe that if $n = 5$ then e_{26} does not exist in $\mathcal{L}_{5,p}$ hence equation ⑦ is omitted, hence and if $n \geq 6$ then the co dimension of the centralizer is $4 + 2 = 6$, and if $n = 5$ then the co dimension of the centralizer is $3 + 1 = 4$, but the complement set to the centralizer of e_{34} in γ_1/γ_5 for any $n \geq 6$ is $\{e_{23}, e_{45}, e_{13}, e_{46}, e_{47}\}$ hence for $n \geq 6$ we have that the co dimension of the centralizer is $5 < 6$, and for $n = 5$ we have that e_{46} and e_{47} are omitted from the complement set, hence the co dimension of the centralizer is $3 < 4$, which shows that for any $n \geq 5$ we have that $\dim \mathcal{C}_{\gamma_1/\gamma_5}(\lambda_2 e_{12} + \lambda_3 e_{34} + \sum_{i=1}^{n-2} a_{i,i+2} e_{i,i+2} + \sum_{i=1}^{n-3} a_{i,i+3} e_{i,i+3} + z_4) = \dim \mathcal{C}_{\gamma_1/\gamma_5}(e_{34}) - 1$. We realize that in order to match the number of constraints to the dimension of the centralizer, we need to remove one equation or present it as a scalar multiplication of another constraint. We observe that since there is a match, for the quotient γ_1/γ_4, then the equation that must be removed or compromised is equation ⑥, because it is an equation on the coefficients of $e_{15} \notin \gamma_1/\gamma_4$. We prove that by setting the four equations in the following 4×9 coefficient matrix

$$A = \begin{pmatrix} 0 & -\lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 & 0 & 0 & 0 & 0 \\ -a_{24} & 0 & a_{13} & 0 & -\lambda_3 & \lambda_1 & 0 & 0 & 0 \\ -a_{25} & 0 & 0 & 0 & -a_{35} & 0 & a_{13} & 0 & \lambda_1 \end{pmatrix}$$

which is ordered in relation to the vector

$$v = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ b_{13} \\ b_{24} \\ b_{35} \\ b_{14} \\ b_{25} \end{pmatrix}$$

such that $Av = 0$. But $\text{codim}\mathcal{C}_{\gamma_1/\gamma_5}(e_{34}) = 3$ hence $\text{rank}A = 3$, which means that any 4×4 minor B of matrix A has that $\det B = 0$. Take the minor

$$B_1 = \begin{pmatrix} -\lambda_3 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & -\lambda_3 & 0 \\ 0 & 0 & -a_{35} & \lambda_1 \end{pmatrix}$$

which means that we take from A the columns 2, 4, 5 and 9, and since $\det B_1 = \lambda_3^3 \lambda_1 = 0$, but we know that $\lambda_3 \neq 0$, hence we must have that $\lambda_1 = 0$. Replacing column 9 by column 7, we get another minor

$$B_2 = \begin{pmatrix} -\lambda_3 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & -\lambda_3 & 0 \\ 0 & 0 & -a_{35} & a_{13} \end{pmatrix}$$

which leads to the conclusion that $a_{13} = 0$. We can already observe here that taking the minor

$$B_3 = \begin{pmatrix} -\lambda_3 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & -\lambda_3 & -a_{24} \\ 0 & 0 & -a_{35} & -a_{25} \end{pmatrix}$$

by taking columns 2, 4, 5 and 1, and computing the equation $\det B_3 = -\lambda_3^2(\lambda_3 a_{25} - a_{24}a_{35}) = 0$, based on the fact that $\lambda_3 \neq 0$, yields the relation $\lambda_3 a_{25} - a_{24}a_{35} = 0$ between coefficients of x, which gives us information about coefficients in the next blocks, but we shall analyze these relations

later. For $\varphi(e_{45})$, we already know from the same considerations we made earlier for e_{23} and e_{34} that we need to have $\lambda_4 e_{45}$ in the linear combination that forms $\varphi(e_{45})$. To rule out any other option for $\varphi(e_{45})$, we first observe that since we have 4 coefficient indices before index 4, including $\lambda_0 = 0$, then if $\lambda_1 \neq$ we remove both λ_1 and its neighboring λ_2 from the count of zero coefficients that are surrounded by two zeros, if $\lambda_2 \neq 0$ then we remove both λ_2 and λ_1 from this count, and if $\lambda_3 \neq 0$ we remove λ_2 alone from this count, which means that if either $x = \varphi(e_{45}) = \lambda_1 e_{12} + \lambda_4 e_{45} + z_2$ or $x = \varphi(e_{45}) = \lambda_2 e_{23} + \lambda_4 e_{45} + z_2$, where $z_2 \in \gamma_2 \mathcal{L}_{n,p}$, then $\mathfrak{m}(x) = \mathfrak{m}(e_{45}) - 1$, and if $x = \varphi(e_{45}) = \lambda_3 e_{34} + \lambda_4 e_{45} + z_2$ then $\mathfrak{m}(x) = \mathfrak{m}(e_{45}) - 2$. We also observe that if either $x = \varphi(e_{45}) = \lambda_1 e_{12} + \lambda_4 e_{45} + z_2$ or $x = \varphi(e_{45}) = \lambda_2 e_{23} + \lambda_4 e_{45} + z_2$ then $\mathfrak{l}(x) = \mathfrak{l}(e_{45}) + 1$, and if $x = \varphi(e_{45}) = \lambda_3 e_{34} + \lambda_4 e_{45} + z_2$ then $\mathfrak{l}(x) = \mathfrak{l}(e_{45})$, since λ_3 and λ_4 are part of the same sequence of non-zero coefficients of x, which means in either of these three cases, either $\mathfrak{l}(x) + \mathfrak{m}(x) = (\mathfrak{l}(e_{34}) + 1) + (\mathfrak{m}(e_{45}) - 2) = \mathfrak{l}(e_{34}) + \mathfrak{m}(e_{45}) - 1$ or $\mathfrak{l}(x) + \mathfrak{m}(x) = \mathfrak{l}(e_{34}) + (\mathfrak{m}(e_{45}) - 1)$, which means that none of these cases can hold for $x = \varphi(e_{45})$. One easily checks that there is no linear combination of the form $x_4 = \lambda_1 e_{12} + \lambda_2 e_{23} + \lambda_3 e_{34}$, where at least two coefficients are non-zero, such that $x = \varphi(e_{45}) = x_4 + \lambda_4 e_{45} + z_2$ satisfies $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(e_{45})$

We observe that this is true for the entire first block, that is, $\varphi(e_{j,j+1}) = \lambda_j e_{j,j+1}$, where $1 \leq j \leq n - 1$, which means that the block M_{11} is diagonal, if we assume that $\varphi(e_{12}) = \lambda_1 e_{12}$, and it is immediate to follow the course of this proof with the assumption that $\varphi(e_{12}) = \lambda_{n-1} e_{n-1,n} + w$, where $w \in \gamma_2 \mathcal{L}_{n,p}$, and prove that in this case the block M_{11} is anti-diagonal. $\qquad \square$

Proposition 1.1.5. *Let $\varphi \in G_4(\mathbb{Q}_p)$, then M_{11} is either diagonal or anti-diagonal.*

Proof. We observe that $\mathcal{C}_{\gamma_1/\gamma_3}(e_{12}) = \langle e_{12}, e_{34}, e_{13}, e_{24} \rangle$, hence $\dim \mathcal{C}_{\gamma_1/\gamma_3}(\varphi(e_{12})) = \dim \mathcal{C}_{\gamma_1/\gamma_3}(e_{12}) = 4 = (4 - 1) + (4 - 2) - 1 = \dim \gamma_1/\gamma_3 - 1$, but by 1.1.3, we have that $\varphi(e_{12}) = x = \lambda_1 e_{12} + \lambda_3 e_{34} + z_2$, where λ_1 and λ_3 are not both zero. Looking at the centralizer of a larger quotient, we have that $\mathcal{C}_{\gamma_1/\gamma_4}(e_{12}) = \langle e_{12}, e_{34}, e_{13}, e_{14} \rangle$. For any $y = \mu_1 e_{12} + \mu_2 e_{23} + \mu_3 e_{34} + a_{13} e_{13} + a_{24} e_{24} + z_3$, such that $[x, y] = 0$, we have the following equations: ① $\lambda_1 \mu_2 e_{13} = 0$, ② $-\lambda_3 \mu_2 e_{24} = 0$, ③ $(\lambda_1 b_{24} - a_{24} \mu_1 + a_{13} \mu_3 - \lambda_3 b_{13}) e_{14} = 0$, but both equations ① and ② yield the same constraint $\mu_2 = 0$, which means that we have two constraints on the centralizer of x, hence $\mathrm{codim}\mathcal{C}_{\gamma_1/\gamma_4}(x) = \mathrm{codim}\mathcal{C}_{\gamma_1/\gamma_4}(e_{12}) = \mathrm{codim}\mathcal{C}_{\gamma_1/\gamma_4}(\lambda_1 e_{12}) = 2$, which means that we cannot determine $\varphi(e_{12})$ by dimension considerations alone. We observe that $\mathcal{C}_{\gamma_1/\gamma_3}(e_{23}) = \langle e_{23}, e_{13}, e_{24} \rangle$,

which means that $\dim \mathcal{C}_{\gamma_1/\gamma_3}(e_{23}) = 3 < 4 = \dim {}^{\gamma_1}/\gamma_3 - 1$, hence by 1.1.3 we have that $\varphi(e_{23}) \neq \lambda_1 e_{12} + \lambda_3 e_{34} + z_2$ where λ_1 and λ_3 are not both zero, which means that we must have $\lambda_2 e_{23}$ in the linear combination that forms $\varphi(e_{23})$. Suppose that $\varphi(e_{23}) = x = \lambda_1 e_{12} + \lambda_2 e_{23} + z_2$, we have the constraints ① $\lambda_2 \mu_3 e_{23} = 0$ and ② $(\lambda_1 \mu_2 - \lambda_2 \mu_1) e_{13} = 0$, which means that $\mu_3 = 0$ and $\mu_1 = \frac{\lambda_1}{\lambda_2} \mu_2$, hence $\mathrm{codim}\mathcal{C}_{\gamma_1/\gamma_3}(x) = 2 = \mathrm{codim}\mathcal{C}_{\gamma_1/\gamma_3}(e_{23}) = \mathrm{codim}\mathcal{C}_{\gamma_1/\gamma_3}(\lambda_2 e_{23} + z_2)$. We take the centralizer of a larger quotient, $\mathcal{C}_{\gamma_1/\gamma_4}(e_{23}) = \langle e_{23}, e_{13}, e_{24}, e_{14} \rangle$, which means that $\mathrm{codim}\mathcal{C}_{\gamma_1/\gamma_4}(e_{23}) = 2$, but then we have an additional equation ④ $(\lambda_1 b_{24} - a_{24} \mu_1 + a_{13} \mu_3) e_{14} = 0$, but since we already know that $\mu_3 = 0$, then equation ④ turns into $(\lambda_1 b_{24} - a_{24} \mu_1) e_{14} = 0$, which means that $b_{24} = \frac{a_{24}}{\lambda_1} \mu_1 = \frac{a_{24}}{\lambda_1} \frac{\lambda_1}{\lambda_2} \mu_2 = \frac{a_{24}}{\lambda_2} \mu_2$, hence the number of constraints remains 2, hence $\mathrm{codim}\mathcal{C}_{\gamma_1/\gamma_4}(x) = 2 = \mathrm{codim}\mathcal{C}_{\gamma_1/\gamma_4}(e_{23}) = \mathrm{codim}\mathcal{C}_{\gamma_1/\gamma_4}(\lambda_2 e_{23})$. This means that we cannot determine also $\varphi(e_{23})$ only by computing dimensions of centralizers. $\qquad\square$

Proposition 1.1.6. *Let $\mathcal{L}_{n,p}$ be the $\mathbb{Q}$-Lie algebra associated with $\mathcal{U}_n(\mathbb{Z}_p)$, and let $\mathcal{B}_n = \{e_{12}, e_{23}, \ldots, e_{1n}\}$ be its basis. Then, the map $\eta : \mathcal{B}_n \to \mathcal{B}_n$, defined by*

$$\eta(e_{ij}) := (-1)^{j-i-1} e_{n+1-j,n+1-i}$$

is an involution, hence, also an $\mathcal{L}_{n,p}$-automorphism.

Proof. Clearly, we calculate $\eta^2(e_{ij})$ by taking $k = n+1-j$ and $l = n+1-i$. Hence,

$$\eta(\eta(e_{ij})) = \eta((-1)^{j-i-1} e_{kl}) = (-1)^{j-i-1} \eta(e_{ij}) = (-1)^{j-i-1}(-1)^{l-k-1} e_{n+1-l,n+1-k} =$$

$$= (-1)^{j-i-1}(-1)^{n+1-i-(n+1-j)-1} e_{n+1-(n+1-i),n+1-(n+1-j)} = (-1)^{j-i-1}(-1)^{j-i-1} e_{ij} =$$

$$= (-1)^{2(j-i-1)} e_{ij} = e_{ij}$$

To complete the proof, we need to show that η is indeed a homomorphism,

$$[\eta(e_{ij}), \eta(e_{kl})] = [(-1)^{j-i-1} e_{n+1-j,n+1-i}, (-1)^{l-k-1} e_{n+1-l,n+1-k}] =$$

$$= (-1)^{j-i-1}(-1)^{l-k-1}[e_{n+1-j,n+1-i}, e_{n+1-l,n+1-k}] =$$

$$(-1)^{j-i+l-k-2}[e_{n+1-j,n+1-i}, e_{n+1-l,n+1-k}]$$

We observe that if $n + 1 - i = n + 1 - l$ or $n + 1 - j = n + 1 - k$ then the Lie brackets evaluate to a non-zero element, but this happens if and only if $i = l$ or $j = k$, respectively. Assume $j = k$, then

$$(-1)^{j-i+l-k-2}[e_{n+1-j,n+1-i}, e_{n+1-l,n+1-k}] = (-1)^{l-i-2}[e_{n+1-j,n+1-i}, e_{n+1-l,n+1-j}] =$$

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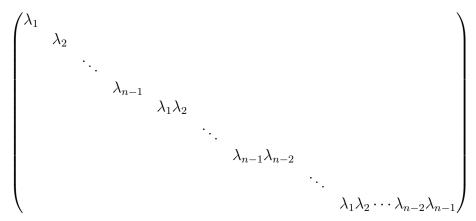$$(-1)^{l-i-2} \cdot -e_{n+1-l,n+1-i} = (-1)^{l-i-1}e_{n+1-l,n+1-i} = \eta(e_{il}) = \eta[e_{ij}, e_{jl}]$$

$\square$

Corollary 1.1.7. *Let $\mathcal{L}_{n,p}$, where $n \geq 4$, then the block M_{11} is diagonal.*

Proof. By 1.1.4 and 1.1.5, we have that M_{11} is either diagonal or anti-diagonal. Assume that M_{11} is anti-diagonal, that is, $\varphi(e_{12}) = \lambda_{n-1}e_{n-1,n} + z_{n-1}, \varphi(e_{23}) = \lambda_{n-2}e_{n-2,n-1}+z_{n-2}, \ldots, \varphi(e_{n-1,n}) = \lambda_1 e_{12}+z_1$, where $z_1, z_2, \ldots, z_{n-1} \in \gamma_2\mathcal{L}_{n,p}$, then we define a new automorphism by the composition $\psi = \varphi\eta$. Let $e_{i,i+1} \in \gamma_1/\gamma_2$, where $1 \leq i \leq n-1$, then $\psi(e_{i,i+1}) = \varphi\eta(e_{i,i+1}) = \varphi((-1)^{i+1-i-1}e_{n+1-(i+1),n+1-i}) = \varphi(e_{n-i,n-i+1})$, hence $\psi(e_{12}) = \varphi(e_{n-1,n}) = \lambda_1 e_{12} + z_1, \psi(e_{23}) = \varphi(e_{n-2,n-1}) = \lambda_2 e_{23} + z_2, \ldots, \psi(e_{n-1,n}) = \varphi(e_{12}) = \lambda_{n-1}e_{n-1,n} + z_{n-1}$, which means that M_{11}, as the first block of the matrix representing ψ, is diagonal. $\square$

Corollary 1.1.8. *Let $\mathcal{L}_{n,p}$, where $n \geq 4$, then all the blocks $M_{11}, M_{22}, \ldots, M_{n-1,n-1}$ are diagonal.*

Proof. By simple induction on the index $1 \leq k \leq n-1$ of the diagonal block. For $k = 1$ we proved this in 1.1.7, for $k+1$ we observe that for every $1 \leq i \leq n-k$, we have the identity $\varphi(e_{i,i+k+1}) = \varphi([e_{i,i+j}, e_{i+j,i+k+1}]) = [\varphi(e_{i,i+j}), \varphi(e_{i+j,i+k+1})]$, for $1 \leq j \leq k$, but by the assumption we have that $\varphi(e_{i,i+j}) = a_{i,i+j}e_{i,i+j} + z_{j+1}$ and $\varphi(e_{i+j,i+k+1}) = a_{i+j,i+k+1}e_{i+j,i+k+1} + z_{k+2-j}$, where $z_{j+1} \in \gamma_{j+1}\mathcal{L}_{n,p}$ and $z_{k+2-j} \in \gamma_{k+2-j}\mathcal{L}_{n,p}$, hence $\varphi(e_{i,i+k+1}) = [a_{i,i+j}e_{i,i+j}+z_{j+1}, a_{i+j,i+k+1}e_{i+j,i+k+1}+z_{k+2-j}] = a_{i,i+j}a_{i+j,i+k+1}e_{i,i+k+1}+a_{i,i+j}e_{i,i+j}z_{k+2-j}+ a_{i+j,i+k+1}e_{i+j,i+k+1}z_{j+1} + z_{k+2-j}z_{j+1} = a_{i,i+j}a_{i+j,i+k+1}e_{i,i+k+1} + z_{k+2-j}$, where $z_{k+2-j} \in \gamma_{k+2-j}\mathcal{L}_{n,p}$, which proves the induction step. $\square$

Corollary 1.1.9. *Let $\mathcal{L}_{n,p}$, where $n \geq 4$, then the main diagonal is*

$$\begin{pmatrix} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_{n-1} & & & & \\ & & & & \lambda_1\lambda_2 & & & \\ & & & & & \ddots & & \\ & & & & & & \lambda_{n-1}\lambda_{n-2} & \\ & & & & & & & \ddots & \\ & & & & & & & & \lambda_1\lambda_2\cdots\lambda_{n-2}\lambda_{n-1} \end{pmatrix}$$

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where $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{Q}_p$, such that for every $1 \leq k \leq n-1$ and for every $1 \leq i \leq n-k$, we have that $\varphi(e_{i,i+k}) = \lambda_i \lambda_{i+1} \lambda_{i+2} \cdots \lambda_{i+k-2} \lambda_{i+k-1} e_{i,i+k} + z_{k+1}$, where $z_{k+1} \in \gamma_{k+2} \mathcal{L}_{n,p}$

Proof. Following the proof of 1.1.8, we have that for every $1 \leq k \leq n-1$ and for every $1 \leq i \leq n-k$, $\varphi(e_{i,i+k}) = \lambda_{i,i+k} e_{i,i+k} + z_{k+1}$, where $z_{k+1} \in \gamma_{k+1} \mathcal{L}_{n,p}$, and we use the same method of induction on k. For $k = 1$, we already proved in 1.1.4 that $\varphi(e_{12}) = \lambda_1 e_{12} + z_1, \varphi(e_{23}) = \lambda_2 e_{23} + z_2, \ldots, \varphi(e_{n-1,n}) = \lambda_{n-1} e_{n-1,n} + z_{n-1}$, where $z_i \in \gamma_i \mathcal{L}_{n,p}$, for every $1 \leq i \leq n-1$. For $k+1$, we have that $\varphi(e_{i,i+k+1}) = \varphi([e_{i,i+j}, e_{i+j,i+k+1}]) = [\varphi(e_{i,i+j}), \varphi(e_{i+j,i+k+1})]$, for every $1 \leq j \leq k$, but by the assumption we have that $\varphi(e_{i,i+j}) = \lambda_i \lambda_{i+1} \cdots \lambda_{i+j} + z_{i+j+1}$ and $\varphi(e_{i+j,i+k+1}) = \lambda_{i+j} \lambda_{i+j+1} \cdots \lambda_{i+k+1} + z_{i+k+2}$, where $z_{i+j+1} \in \gamma_{i+j+1} \mathcal{L}_{n,p}$ and $z_{i+k+2} \in \gamma_{i+k+2} \mathcal{L}_{n,p}$, hence $\varphi(e_{i,i+k+1}) = \varphi([e_{i,i+j}, e_{i+j,i+k+1}]) = [\varphi(e_{i,i+j}), \varphi(e_{i+j,i+k+1})] = [\lambda_i \lambda_{i+1} \cdots \lambda_{i+j} + z_{i+j+1}, \lambda_{i+j} \lambda_{i+j+1} \cdots \lambda_{i+k+1} + z_{i+k+2}] = \lambda_i \lambda_{i+1} \lambda_{i+2} \lambda_{i+k} \cdots \lambda_{i+k+1} + z_{i+k+2}$, which proves the induction step. $\square$