

cases

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Denote $p^{\min} := p^{\min\{0, v_2 + v_{11}\}}$

Denote $\rho := (1 - p^{-1})$

Denote $f(p, t)$ the p and t product $p^{7v_1 + 10v_2 + 11v_3 + 7v_4 + \min\{v_1, v_4\}} t^{4v_1 + 6v_2 + 6v_3 + 4v_4}$

1 Case 1

$$v_3 > v_1 \geq v_4 > v_2 \geq 0$$

$$\Rightarrow v_{11} \geq -v_2 - v_1$$

$$\Rightarrow \alpha := \min\{v_2, v_2 + v_{11} + v_1\} = \begin{cases} v_2, & v_{11} \geq -v_1 \\ v_2 - 1, v_2 - 2, \dots, & v_{11} < -v_1 \end{cases}$$

$$\Rightarrow f(p, t) = p^{7v_1 + 10v_2 + 11v_3 + 7v_4 + v_4} t^{4v_1 + 6v_2 + 6v_3 + 4v_4} = p^{7v_1 + 10v_2 + 11v_3 + 8v_4} t^{4v_1 + 6v_2 + 6v_3 + 4v_4}$$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho)$.
2. For $-v_1 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_1}^{-v_2-1} p^{v_2}\rho(1 + v_2\rho) = p^{v_2}(v_1 - v_2)\rho(1 + v_2\rho) = p^{v_2}(v_1 - v_2)\rho + p^{v_2}v_2(v_1 - v_2)\rho^2$.
3. For $-v_2 - v_1 \leq v_{11} \leq -v_1 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_1}^{-v_1-1} p^{v_2}\rho(1 + \alpha\rho) = \sum_{i=1}^{v_2} p^{v_2}\rho(1 + (v_2 - i)\rho) = p^{v_2}v_2\rho + p^{v_2}v_2^2\rho^2 - p^{v_2}\binom{v_2+1}{2}\rho^2$.

So we have $\sum_{v_{11} \geq -v_2 - v_1} = p^{v_2}(1 + v_2\rho + (v_1 - v_2)\rho + v_2(v_1 - v_2)\rho^2 + v_2\rho + v_2^2\rho^2 - \frac{(v_2+1)v_2}{2}\rho^2) = p^{v_2}(1 + v_1\rho + v_2\rho + v_2v_1\rho^2 - \frac{v_2^2}{2}\rho^2 - \frac{v_2}{2}\rho^2)$. We compute the differences between v_1, v_2, v_3, v_4 ,

$$v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_4 \overset{b}{>} v_2 \overset{a}{\geq} 0$$

Thus, we substitute $v_1 \rightarrow a + b + c$, $v_2 \rightarrow a$, $v_3 \rightarrow a + b + c + d$, $v_4 \rightarrow a + b$, and compute series with the new variables as indices, where $a, c \geq 0$ and $b, d \geq 1$.

2 Case 2

$$v_1 \geq v_3 \geq 0$$

$$v_1 \geq v_4 > v_2 \geq 0$$

$$\Rightarrow v_{11} \geq -v_2 - v_3$$

$$\Rightarrow \alpha := \min\{v_2, v_2 + v_{11} + v_1\} = \begin{cases} v_2, & v_{11} \geq -v_1 \\ v_2 - 1, v_2 - 2, \dots, & v_{11} < -v_1 \end{cases}$$

$$\Rightarrow f(p, t) = p^{7v_1+10v_2+11v_3+7v_4+v_4} t^{4v_1+6v_2+6v_3+4v_4} = p^{7v_1+10v_2+11v_3+8v_4} t^{4v_1+6v_2+6v_3+4v_4}$$

We have two sub cases.

2.1 Sub case 2.1

$$-v_2 - v_3 \geq -v_1 \Rightarrow v_1 \geq v_2 + v_3$$

Here there is no phase transition, because the phase transition occurs at $-v_1 - 1 < -v_2 - v_3 \leq v_{11}$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho)$.
2. For $-v_2 - v_3 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2 - v_3}^{-v_2 - 1} p^{v_2}\rho(1 + v_2\rho) = p^{v_2}v_3\rho(1 + v_2\rho) = p^{v_2}v_3\rho + p^{v_2}v_3v_2\rho^2$.

So we have $\sum_{v_{11} \geq -v_2 - v_3} = p^{v_2}(1 + v_2\rho + v_3\rho + v_2v_3\rho^2)$.

We compute the differences between v_1, v_2, v_3, v_4 , we have several arrangements,

$$1. v_1 \stackrel{d}{\geq} v_2 + v_3 \stackrel{a}{\geq} v_3 \stackrel{c}{\geq} v_4 \stackrel{b}{>} v_2 \stackrel{a}{\geq} 0$$

Thus, we substitute $v_1 \rightarrow 2a + b + c + d$, $v_2 \rightarrow a$, $v_3 \rightarrow a + b + c$, $v_4 \rightarrow a + b$, and compute series with the new variables as indices, where $a, c, d \geq 0$ and $b \geq 1$.

$$2. v_1 \stackrel{d}{\geq} v_2 + v_3 \stackrel{c}{\geq} v_4 \stackrel{b}{>} v_3 \stackrel{a}{\geq} v_2 \stackrel{e}{\geq} 0$$

$$\text{But } v_2 + v_3 - c - b = v_3 \Rightarrow v_2 = v_2 + v_3 - v_3 = b + c$$

Thus, we substitute $v_1 \rightarrow a + 2b + 2c + d$, $v_2 \rightarrow b + c$, $v_3 \rightarrow a + b + c$, $v_4 \rightarrow a + 2b + c$, and compute series with the new variables as indices, where $c, d \geq 0$ and $a, b \geq 1$.

$$3. v_1 \stackrel{d}{\geq} v_2 + v_3 \stackrel{c}{\geq} v_4 \stackrel{b}{>} v_2 \stackrel{a}{\geq} v_3 \stackrel{e}{\geq} 0$$

$$\text{But } v_2 + v_3 - c - b = v_2 \Rightarrow v_3 = v_2 + v_3 - v_2 = b + c$$

Thus, we substitute $v_1 \rightarrow a + 2b + 2c + d$, $v_2 \rightarrow a + b + c$, $v_3 \rightarrow b + c$, $v_4 \rightarrow a + 2b + c$, and compute series with the new variables as indices, where $a, c, d \geq 0$ and $b \geq 1$.

$$4. v_1 \stackrel{d}{\geq} v_4 \stackrel{c}{\geq} v_2 + v_3 \stackrel{a}{\geq} v_2 \stackrel{b}{\geq} v_3 \stackrel{a}{\geq} 0$$

Thus, we substitute $v_1 \rightarrow 2a + b + c + d$, $v_2 \rightarrow a + b$, $v_3 \rightarrow a$, $v_4 \rightarrow 2a + b + c$, and compute series with the new variables as indices, where $a, b, d \geq 0$ and $c \geq 1$.

$$5. v_1 \stackrel{d}{\geq} v_4 \stackrel{c}{\geq} v_2 + v_3 \stackrel{a}{\geq} v_3 \stackrel{b}{>} v_2 \stackrel{a}{\geq} 0$$

Thus, we substitute $v_1 \rightarrow 2a + b + c + d$, $v_2 \rightarrow a$, $v_3 \rightarrow a + b$, $v_4 \rightarrow 2a + b + c$, and compute series with the new variables as indices, where $a, d \geq 0$ and $b, c \geq 1$.

2.2 Sub case 2.2

$$-v_2 - v_3 < -v_1 \Rightarrow v_1 < v_2 + v_3$$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho)$.
2. For $-v_1 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_1}^{-v_2-1} p^{v_2}\rho(1 + v_2\rho) = p^{v_2}(v_1 - v_2)\rho(1 + v_2\rho) = p^{v_2}(v_1 - v_2)\rho + p^{v_2}(v_1 - v_2)v_2\rho^2$.
3. For $-v_2 - v_3 \leq v_{11} \leq -v_1 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_3}^{-v_1-1} p^{v_2}\rho(1 + \alpha\rho) = \sum_{i=1}^{v_2+v_3-v_1} p^{v_2}\rho(1 + (v_2 - i)\rho) = p^{v_2}(v_2 + v_3 - v_1)\rho + p^{v_2}(v_2 + v_3 - v_1)v_2\rho^2 - p^{v_2}\binom{v_2+v_3-v_1+1}{2}\rho^2$.

So we have $\sum_{v_{11} \geq -v_2 - v_3} = p^{v_2}(1 + v_2\rho + (v_1 - v_2)\rho + (v_1 - v_2)v_2\rho^2 + (v_2 + v_3 - v_1)\rho + (v_2 + v_3 - v_1)v_2\rho^2 - \frac{(v_2+v_3-v_1+1)(v_2+v_3-v_1)}{2}\rho^2) = p^{v_2}(1 + v_2\rho + v_3\rho - \frac{v_2^2}{2}\rho^2 + \frac{v_2v_1}{2}\rho^2 - \frac{v_3^2}{2}\rho^2 + \frac{v_3v_1}{2}\rho^2 + \frac{v_1v_2}{2}\rho^2 + \frac{v_1v_3}{2}\rho^2 - \frac{v_1^2}{2}\rho^2 - \frac{v_2^2}{2}\rho^2 - \frac{v_3^2}{2}\rho^2 - \frac{v_3}{2}\rho^2 + \frac{v_1}{2}\rho^2) = p^{v_2}(1 + v_2\rho + v_3\rho - \frac{v_2^2}{2}\rho^2 + v_2v_1\rho^2 - \frac{v_3^2}{2}\rho^2 + v_3v_1\rho^2 - \frac{v_1^2}{2}\rho^2 - \frac{v_2^2}{2}\rho^2 - \frac{v_3^2}{2}\rho^2 + \frac{v_1}{2}\rho^2)$. We compute the differences between v_1, v_2, v_3, v_4 , we have several arrangements,

1. $v_2 + v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_3 \overset{b}{\geq} v_4 \overset{a}{>} v_2 \overset{e}{\geq} 0$

$$\text{But } v_2 + v_3 - d - c = v_3 \Rightarrow v_2 = c + d$$

Thus, we substitute $v_1 \rightarrow a + b + 2c + d$, $v_2 \rightarrow c + d$, $v_3 \rightarrow a + b + c + d$, $v_4 \rightarrow a + c + d$, and compute series with the new variables as indices, where $b, c \geq 0$ and $a, d \geq 1$.

2. $v_2 + v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_4 \overset{b}{>} v_3 \overset{a}{>} v_2 \overset{e}{\geq} 0$

$$\text{But } v_2 + v_3 - d - c - b = v_3 \Rightarrow v_2 = v_2 + v_3 - v_3 = b + c + d$$

Thus, we substitute $v_1 \rightarrow a + 2b + 2c + d$, $v_2 \rightarrow b + c + d$, $v_3 \rightarrow a + b + c + d$, $v_4 \rightarrow a + 2b + c + d$, and compute series with the new variables as indices, where $c \geq 0$ and $a, b, d \geq 1$.

3. $v_2 + v_3 \overset{d}{>} v_1 \overset{c}{\geq} v_4 \overset{b}{>} v_2 \overset{a}{\geq} v_3 \overset{e}{\geq} 0$

$$\text{But } v_2 + v_3 - d - c - b = v_2 \Rightarrow v_3 = v_2 + v_3 - v_2 = b + c + d$$

Thus, we substitute $v_1 \rightarrow a + 2b + 2c + d$, $v_2 \rightarrow a + b + c + d$, $v_3 \rightarrow b + c + d$, $v_4 \rightarrow a + 2b + c + d$, and compute series with the new variables as indices, where $a, c \geq 0$ and $b, d \geq 1$.

3 Case 3

$$\begin{aligned} v_3 &> v_1 \geq 0 \\ v_4 &> v_1 \geq 0 \\ v_4 &> v_2 \geq 0 \\ \Rightarrow v_{11} &\geq -v_2 - v_1 \end{aligned}$$

$$\Rightarrow \alpha := \min\{v_2, v_2 + v_{11} + v_4\} = \begin{cases} v_2, & v_{11} \geq -v_4 \\ v_2 - 1, v_2 - 2, \dots, & v_{11} < -v_4 \end{cases}$$

$$\Rightarrow f(p, t) = p^{7v_1+10v_2+11v_3+7v_4+v_1} t^{4v_1+6v_2+6v_3+4v_4} = p^{8v_1+10v_2+11v_3+7v_4} t^{4v_1+6v_2+6v_3+4v_4}$$

We have two sub cases

3.1 Sub case 3.1

$$-v_2 - v_1 > -v_4 \Rightarrow v_4 > v_2 + v_1$$

There is no phase transition here, since $v_{11} \geq -v_2 - v_1 > -v_4$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho)$.
2. For $-v_2 - v_1 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2 - v_1}^{-v_2 - 1} p^{v_2}\rho(1 + v_2\rho) = p^{v_2}v_1\rho + p^{v_2}v_1v_2\rho^2$.

So we have $\sum_{v_{11} \geq -v_2 - v_1} p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho + v_1\rho + v_1v_2\rho^2)$.

We have several arrangements,

1. $v_4 \overset{d}{>} v_1 + v_2 \overset{c}{\geq} v_3 \overset{b}{>} v_2 \overset{a}{>} v_1 \overset{e}{\geq} 0$

$$\text{But } v_1 + v_2 - c - b = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = b + c$$

Thus, we substitute $v_1 \rightarrow b + c$, $v_2 \rightarrow a + b + c$, $v_3 \rightarrow a + 2b + c$, $v_4 \rightarrow a + 2b + 2c + d$, and compute series with the new variables as indices, where $c \geq 0$ and $a, b, d \geq 1$.

2. $v_4 \overset{d}{>} v_1 + v_2 \overset{a}{\geq} v_2 \overset{c}{\geq} v_3 \overset{b}{>} v_1 \overset{a}{\geq} 0$

We substitute $v_1 \rightarrow a$, $v_2 \rightarrow a + b + c$, $v_3 \rightarrow a + b$, $v_4 \rightarrow 2a + b + c + d$, and compute series with the new variables as indices, where $a, c \geq 0$ and $b, d \geq 1$.

3. $v_4 \overset{d}{>} v_1 + v_2 \overset{c}{\geq} v_3 \overset{b}{>} v_1 \overset{a}{\geq} v_2 \overset{e}{\geq} 0$

$$\text{But } v_1 + v_2 - c - b = v_1 \Rightarrow v_2 = v_1 + v_2 - v_1 = b + c$$

We substitute $v_1 \rightarrow a + b + c$, $v_2 \rightarrow b + c$, $v_3 \rightarrow a + 2b + c$, $v_4 \rightarrow a + 2b + 2c + d$, and compute series with the new variables as indices, where $a, c \geq 0$ and $b, d \geq 1$.

4. $v_4 \overset{d}{>} v_3 \overset{c}{>} v_1 + v_2 \overset{a}{\geq} v_1 \overset{b}{\geq} v_2 \overset{a}{\geq} 0$

We substitute $v_1 \rightarrow a + b$, $v_2 \rightarrow a$, $v_3 \rightarrow 2a + b + c$, $v_4 \rightarrow 2a + b + c + d$, and compute series with the new variables as indices, where $a, b \geq 0$ and $c, d \geq 1$.

5. $v_4 \overset{d}{>} v_3 \overset{c}{>} v_1 + v_2 \overset{a}{\geq} v_2 \overset{b}{>} v_1 \overset{a}{\geq} 0$

We substitute $v_1 \rightarrow a$, $v_2 \rightarrow a + b$, $v_3 \rightarrow 2a + b + c$, $v_4 \rightarrow 2a + b + c + d$, and compute series with the new variables as indices, where $a \geq 0$ and $b, c, d \geq 1$.

$$6. v_3 \stackrel{d}{\geq} v_4 \stackrel{c}{>} v_1 + v_2 \stackrel{a}{\geq} v_2 \stackrel{b}{>} v_1 \stackrel{a}{\geq} 0$$

We substitute $v_1 \rightarrow a$, $v_2 \rightarrow a + b$, $v_3 \rightarrow 2a + b + c + d$, $v_4 \rightarrow 2a + b + c$, and compute series with the new variables as indices, where $a, d \geq 0$ and $b, c \geq 1$.

$$7. v_3 \stackrel{d}{\geq} v_4 \stackrel{c}{>} v_1 + v_2 \stackrel{a}{\geq} v_1 \stackrel{b}{\geq} v_2 \stackrel{a}{\geq} 0$$

We substitute $v_1 \rightarrow a + b$, $v_2 \rightarrow a$, $v_3 \rightarrow 2a + b + c + d$, $v_4 \rightarrow 2a + b + c$, and compute series with the new variables as indices, where $a, b, d \geq 0$ and $c \geq 1$.

3.2 Sub case 3.2

$$-v_4 \geq -v_2 - v_1 \Rightarrow v_2 + v_1 \geq v_4$$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho)$.
2. For $-v_4 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_4}^{-v_2-1} p^{v_2}\rho(1 + v_2\rho) = p^{v_2}(v_4 - v_2)\rho + p^{v_2}(v_4 - v_2)v_2\rho^2$.
3. For $-v_2 - v_1 \leq v_{11} \leq -v_4 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_1}^{-v_4-1} p^{v_2}\rho(1 + v_2\rho) = \sum_{i=1}^{v_1+v_2-v_4} p^{v_2}\rho(1 + (v_2 - i)\rho) = p^{v_2}(v_1 + v_2 - v_4)\rho + p^{v_2}(v_1 + v_2 - v_4)v_2\rho^2 - p^{v_2}\left(\frac{v_1+v_2-v_4+1}{2}\right)\rho^2$.

So we have $\sum_{v_{11} \geq -v_2 - v_1} p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho + (v_4 - v_2)\rho + (v_4 - v_2)v_2\rho^2 + (v_1 + v_2 - v_4)\rho + (v_1 + v_2 - v_4)v_2\rho^2 - \frac{(v_1+v_2-v_4+1)(v_1+v_2-v_4)}{2}\rho^2) = p^{v_2}(1 + v_1\rho + v_2\rho - \frac{v_1^2}{2}\rho^2 + v_4v_1\rho^2 - \frac{v_2^2}{2}\rho^2 + v_4v_2\rho^2 - \frac{v_4^2}{2}\rho^2 - \frac{v_1}{2}\rho^2 - \frac{v_2}{2}\rho^2 + \frac{v_4}{2}\rho^2)$

We have several arrangements,

$$1. v_1 + v_2 \stackrel{d}{\geq} v_4 \stackrel{c}{>} v_3 \stackrel{b}{>} v_2 \stackrel{a}{>} v_1 \stackrel{e}{\geq} 0$$

$$\text{But } v_1 + v_2 - d - c - b = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = b + c + d$$

Thus, we substitute $v_1 \rightarrow b + c + d$, $v_2 \rightarrow a + b + c + d$, $v_3 \rightarrow a + 2b + c + d$, $v_4 \rightarrow a + 2b + 2c + d$, and compute series with the new variables as indices, where $d \geq 0$ and $a, b, c \geq 1$

$$2. v_1 + v_2 \stackrel{d}{\geq} v_4 \stackrel{c}{>} v_2 \stackrel{b}{\geq} v_3 \stackrel{a}{>} v_1 \stackrel{e}{\geq} 0$$

$$\text{But } v_1 + v_2 - d - c = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = c + d$$

Thus, we substitute $v_1 \rightarrow c + d$, $v_2 \rightarrow a + b + c + d$, $v_3 \rightarrow a + c + d$, $v_4 \rightarrow a + b + 2c + d$, and compute series with the new variables as indices, where $b, d \geq 0$ and $a, c \geq 1$

$$3. v_1 + v_2 \stackrel{d}{\geq} v_4 \stackrel{c}{>} v_3 \stackrel{b}{>} v_1 \stackrel{a}{\geq} v_2 \stackrel{e}{\geq} 0$$

$$\text{But } v_1 + v_2 - d - c - b = v_1 \Rightarrow v_2 = v_1 + v_2 - v_1 = b + c + d$$

Thus, we substitute $v_1 \rightarrow a + b + c + d$, $v_2 \rightarrow b + c + d$, $v_3 \rightarrow a + 2b + c + d$, $v_4 \rightarrow a + 2b + 2c + d$, and compute series with the new variables as indices, where $a, d \geq 0$ and $b, c \geq 1$

$$4. \quad v_1 + v_2 \stackrel{d}{\geq} v_3 \stackrel{c}{\geq} v_4 \stackrel{b}{>} v_2 \stackrel{a}{>} v_1 \stackrel{e}{\geq} 0$$

$$\text{But } v_1 + v_2 - d - c - b = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = b + c + d$$

Thus, we substitute $v_1 \rightarrow b + c + d$, $v_2 \rightarrow a + b + c + d$, $v_3 \rightarrow a + 2b + 2c + d$, $v_4 \rightarrow a + 2b + c + d$, and compute series with the new variables as indices, where $c, d \geq 0$ and $a, b \geq 1$

$$5. \quad v_1 + v_2 \stackrel{d}{>} v_3 \stackrel{c}{\geq} v_4 \stackrel{b}{>} v_1 \stackrel{a}{\geq} v_2 \stackrel{e}{\geq} 0$$

$$\text{But } v_1 + v_2 - d - c - b = v_1 \Rightarrow v_2 = v_1 + v_2 - v_1 = b + c + d$$

Thus, we substitute $v_1 \rightarrow a + b + c + d$, $v_2 \rightarrow b + c + d$, $v_3 \rightarrow a + 2b + 2c + d$, $v_4 \rightarrow a + 2b + c + d$, and compute series with the new variables as indices, where $a, c \geq 0$ and $b, d \geq 1$

$$6. \quad v_3 \stackrel{d}{\geq} v_1 + v_2 \stackrel{c}{\geq} v_4 \stackrel{b}{>} v_2 \stackrel{a}{>} v_1 \stackrel{e}{\geq} 0$$

$$\text{But } v_1 + v_2 - c - b = v_2 \Rightarrow v_1 = v_1 + v_2 - v_2 = b + c$$

Thus, we substitute $v_1 \rightarrow b + c$, $v_2 \rightarrow a + b + c$, $v_3 \rightarrow a + 2b + 2c + d$, $v_4 \rightarrow a + 2b + c$, and compute series with the new variables as indices, where $c, d \geq 0$ and $a, b \geq 1$

$$7. \quad v_3 \stackrel{d}{\geq} v_1 + v_2 \stackrel{c}{\geq} v_4 \stackrel{b}{>} v_1 \stackrel{a}{\geq} v_2 \stackrel{e}{\geq} 0$$

$$\text{But } v_1 + v_2 - c - b = v_1 \Rightarrow v_2 = v_1 + v_2 - v_1 = b + c$$

Thus, we substitute $v_1 \rightarrow a + b + c$, $v_2 \rightarrow b + c$, $v_3 \rightarrow a + 2b + 2c + d$, $v_4 \rightarrow a + 2b + c$, and compute series with the new variables as indices, where $a, c, d \geq 0$ and $b \geq 1$

4 Case 4

$$v_4 > v_1 \geq v_3 \geq 0$$

$$v_4 > v_2 \geq 0$$

$$\Rightarrow v_{11} \geq -v_2 - v_3$$

$$\Rightarrow \alpha := \min\{v_2, v_2 + v_{11} + v_4\} = \begin{cases} v_2, & v_{11} \geq -v_4 \\ v_2 - 1, v_2 - 2, \dots, & v_{11} < -v_4 \end{cases}$$

$$\Rightarrow f(p, t) = p^{7v_1 + 10v_2 + 11v_3 + 7v_4 + v_1} t^{4v_1 + 6v_2 + 6v_3 + 4v_4} = p^{8v_1 + 10v_2 + 11v_3 + 7v_4} t^{4v_1 + 6v_2 + 6v_3 + 4v_4}$$

We have two sub cases

4.1 Sub case 4.1

$$-v_4 < -v_2 - v_3 \Rightarrow v_4 > v_2 + v_3$$

There is no phase transition here, since $v_{11} \geq -v_2 - v_3 > -v_4$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho)$.
2. For $-v_2 - v_3 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2 - v_3}^{-v_2 - 1} p^{v_2}\rho(1 + v_2\rho) = p^{v_2}v_3\rho(1 + v_2\rho) = p^{v_2}v_3\rho + p^{v_2}v_2v_3\rho^2$.

So we have $\sum_{v_{11} \geq -v_2 - v_3} p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho + v_3\rho + v_2v_3\rho^2)$.

We have several arrangements,

1. $v_4 \overset{d}{>} v_2 + v_3 \overset{c}{\geq} v_1 \overset{b}{\geq} v_3 \overset{a}{>} v_2 \overset{e}{\geq} 0$

$$\text{But } v_2 + v_3 - c - b = v_3 \Rightarrow v_2 = v_2 + v_3 - v_3 = b + c$$

Thus, we substitute $v_1 \rightarrow a + 2b + c$, $v_2 \rightarrow b + c$, $v_3 \rightarrow a + b + c$, $v_4 \rightarrow a + 2b + 2c + d$, and compute series with the new variables as indices, where $b, c \geq 0$ and $a, d \geq 1$

2. $v_4 \overset{d}{>} v_2 + v_3 \overset{c}{\geq} v_1 \overset{b}{\geq} v_2 \overset{a}{\geq} v_3 \overset{e}{\geq} 0$

$$\text{But } v_2 + v_3 - c - b = v_2 \Rightarrow v_3 = v_2 + v_3 - v_2 = b + c$$

Thus, we substitute $v_1 \rightarrow a + 2b + c$, $v_2 \rightarrow a + b + c$, $v_3 \rightarrow b + c$, $v_4 \rightarrow a + 2b + 2c + d$, and compute series with the new variables as indices, where $a, b, c \geq 0$ and $d \geq 1$.

3. $v_4 \overset{d}{>} v_2 + v_3 \overset{a}{\geq} v_2 \overset{c}{>} v_1 \overset{b}{\geq} v_3 \overset{a}{\geq} 0$

Thus, we substitute $v_1 \rightarrow a + b$, $v_2 \rightarrow a + b + c$, $v_3 \rightarrow a$, $v_4 \rightarrow 2a + b + c + d$, and compute series with the new variables as indices, where $a, b \geq 0$ and $c, d \geq 1$.

4. $v_4 \overset{d}{>} v_1 \overset{c}{\geq} v_2 + v_3 \overset{a}{\geq} v_2 \overset{b}{\geq} v_3 \overset{a}{\geq} 0$

Thus, we substitute $v_1 \rightarrow 2a + b + c$, $v_2 \rightarrow a + b$, $v_3 \rightarrow a$, $v_4 \rightarrow 2a + b + c + d$, and compute series with the new variables as indices, where $a, b, c \geq 0$ and $d \geq 1$.

5. $v_4 \overset{d}{>} v_1 \overset{c}{\geq} v_2 + v_3 \overset{a}{\geq} v_3 \overset{b}{>} v_2 \overset{a}{\geq} 0$

Thus, we substitute $v_1 \rightarrow 2a + b + c$, $v_2 \rightarrow a$, $v_3 \rightarrow a + b$, $v_4 \rightarrow 2a + b + c + d$, and compute series with the new variables as indices, where $a, c \geq 0$ and $b, d \geq 1$.

4.2 Sub case 4.2

$$-v_2 - v_3 \leq -v_4 \Rightarrow v_2 + v_3 \geq v_4$$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho)$.

2. For $-v_4 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_4}^{-v_2-1} p^{v_2}\rho(1 + v_2\rho) = p^{v_2}(v_4 - v_2)\rho(1 + v_2\rho) = p^{v_2}(v_4 - v_2)\rho + p^{v_2}v_2(v_4 - v_2)\rho^2$.
3. For $-v_2 - v_3 \leq v_{11} \leq -v_4 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_3}^{-v_4-1} p^{v_2}\rho(1 + \alpha\rho) = \sum_{i=1}^{v_2+v_3-v_4} p^{v_2}\rho(1 + (v_2 - i)\rho) = p^{v_2}(v_2 + v_3 - v_4)\rho + (v_2 + v_3 - v_4)v_2\rho^2 - \binom{v_2+v_3-v_4+1}{2}\rho^2$.

So we have $\sum_{v_{11} \geq -v_2 - v_3} p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_2\rho + (v_4 - v_2))\rho + v_2(v_4 - v_2)\rho^2 + (v_2 + v_3 - v_4)\rho + (v_2 + v_3 - v_4)v_2\rho^2 - \frac{(v_2+v_3-v_4+1)(v_2+v_3-v_4)}{2}\rho^2 = p^{v_2}(1 + v_2v_4\rho^2 + v_2\rho + v_3\rho + v_3v_2\rho^2 - v_4v_2\rho^2 - \frac{v_2^2}{2}\rho^2 - v_2v_3\rho^2 + v_2v_4\rho^2 - \frac{v_3^2}{2}\rho^2 + v_3v_4\rho^2 - \frac{v_4^2}{2}\rho^2 - \frac{v_2}{2}\rho^2 - \frac{v_3}{2}\rho^2 + \frac{v_4}{2}\rho^2)$.

We have several arrangements,

1. $v_2 + v_3 \stackrel{d}{\geq} v_4 \stackrel{c}{>} v_1 \stackrel{b}{\geq} v_2 \stackrel{a}{\geq} v_3 \stackrel{e}{\geq} 0$

But $v_2 + v_3 - d - c - b = v_2 \Rightarrow v_3 = v_2 + v_3 - v_2 = b + c + d$

Thus, we substitute $v_1 \rightarrow a + 2b + c + d$, $v_2 \rightarrow a + b + c + d$, $v_3 \rightarrow b + c + d$, $v_4 \rightarrow a + 2b + 2c + d$, and compute series with the new variables as indices, where $a, b, d \geq 0$ and $c \geq 1$

2. $v_2 + v_3 \stackrel{d}{\geq} v_4 \stackrel{c}{>} v_1 \stackrel{b}{\geq} v_3 \stackrel{a}{>} v_2 \stackrel{e}{\geq} 0$

But $v_2 + v_3 - d - c - b = v_3 \Rightarrow v_2 = v_2 + v_3 - v_3 = b + c + d$

Thus, we substitute $v_1 \rightarrow a + 2b + c + d$, $v_2 \rightarrow b + c + d$, $v_3 \rightarrow a + b + c + d$, $v_4 \rightarrow a + 2b + 2c + d$, and compute series with the new variables as indices, where $b, d \geq 0$ and $a, c \geq 1$

3. $v_2 + v_3 \stackrel{d}{\geq} v_4 \stackrel{c}{>} v_2 \stackrel{b}{>} v_1 \stackrel{a}{\geq} v_3 \stackrel{e}{\geq} 0$

But $v_2 + v_3 - d - c = v_2 \Rightarrow v_3 = v_2 + v_3 - v_2 = c + d$

Thus, we substitute $v_1 \rightarrow a + c + d$, $v_2 \rightarrow a + b + c + d$, $v_3 \rightarrow c + d$, $v_4 \rightarrow a + b + 2c + d$, and compute series with the new variables as indices, where $a, d \geq 0$ and $b, c \geq 1$

5 Case 5

$$v_2 \geq v_4 > v_1 \geq v_3 \geq 0$$

$$\Rightarrow v_{11} \geq -v_2 - v_3$$

$$\begin{aligned} \Rightarrow \alpha &:= \min\{v_4, v_2 + v_{11} + v_4\} = \begin{cases} v_4, & v_{11} \geq -v_2 \\ v_4 - 1, v_4 - 2, \dots, & v_{11} < -v_2 \end{cases} \\ \Rightarrow f(p, t) &= p^{7v_1+10v_2+11v_3+7v_4+v_1} t^{4v_1+6v_2+6v_3+4v_4} = p^{8v_1+10v_2+11v_3+7v_4} t^{4v_1+6v_2+6v_3+4v_4} \end{aligned}$$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho)$.
2. For $-v_2 - v_3 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_3}^{-v_2-1} p^{v_2}\rho(1 + \alpha\rho) = \sum_{i=1}^{v_3} p^{v_2}\rho(1 + (v_4 - i)\rho) = p^{v_2}(v_3\rho + v_3v_4\rho^2 - \binom{v_3+1}{2}\rho^2)$.

So we have $\sum_{v_{11} \geq -v_2 - v_3} p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho + v_3\rho + v_3v_4\rho^2 - \frac{(v_3+1)v_3}{2}\rho^2) = p^{v_2}(1 + v_4\rho + v_3\rho + v_3v_4\rho^2 - \frac{v_3^2}{2}\rho^2 - \frac{v_3}{2}\rho^2)$.

$$v_2 \stackrel{d}{\geq} v_4 \stackrel{c}{>} v_1 \stackrel{b}{\geq} v_3 \stackrel{a}{\geq} 0$$

Thus, we substitute $v_1 \rightarrow a + b$, $v_2 \rightarrow a + b + c + d$, $v_3 \rightarrow a$, $v_4 \rightarrow a + b + c$, and compute series with the new variables as indices, where $a, b, d \geq 0$ and $c \geq 1$

6 Case 6

$$v_2 \geq v_4 > v_1 \geq 0$$

$$v_3 > v_1 \geq 0$$

$$\Rightarrow v_{11} \geq -v_2 - v_1$$

$$\Rightarrow \alpha := \min\{v_4, v_2 + v_{11} + v_4\} = \begin{cases} v_4, & v_{11} \geq -v_2 \\ v_4 - 1, v_4 - 2, \dots, & v_{11} < -v_2 \end{cases}$$

$$\Rightarrow f(p, t) = p^{7v_1+10v_2+11v_3+7v_4+v_1} t^{4v_1+6v_2+6v_3+4v_4} = p^{8v_1+10v_2+11v_3+7v_4} t^{4v_1+6v_2+6v_3+4v_4}$$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho)$.

2. For $-v_2 - v_1 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_1}^{-v_2-1} p^{v_2}\rho(1 + \alpha\rho) = \sum_{i=1}^{v_1} p^{v_2}\rho(1 + (v_4 - i)\rho) = p^{v_2}(v_1\rho + v_1v_4\rho^2 - \binom{v_1+1}{2}\rho^2)$.

So we have $\sum_{v_{11} \geq -v_2 - v_1} p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho + v_1\rho + v_1v_4\rho^2 - \frac{(v_1+1)v_1}{2}\rho^2) = p^{v_2}(1 + v_4\rho + v_1\rho + v_1v_4\rho^2 - \frac{v_1^2}{2}\rho^2 - \frac{v_1}{2}\rho^2)$.

We have several arrangements,

$$1. v_2 \stackrel{d}{\geq} v_4 \stackrel{c}{>} v_3 \stackrel{b}{>} v_1 \stackrel{a}{\geq} 0$$

Thus, we substitute $v_1 \rightarrow a$, $v_2 \rightarrow a + b + c + d$, $v_3 \rightarrow a + b$, $v_4 \rightarrow a + b + c$, and compute series with the new variables as indices, where $a, d \geq 0$ and $b, c \geq 1$

$$2. v_2 \stackrel{d}{\geq} v_3 \stackrel{c}{\geq} v_4 \stackrel{b}{>} v_1 \stackrel{a}{\geq} 0$$

Thus, we substitute $v_1 \rightarrow a$, $v_2 \rightarrow a + b + c + d$, $v_3 \rightarrow a + b + c$, $v_4 \rightarrow a + b$, and compute series with the new variables as indices, where $a, c, d \geq 0$ and $b \geq 1$

$$3. v_3 \stackrel{d}{>} v_2 \stackrel{c}{\geq} v_4 \stackrel{b}{>} v_1 \stackrel{a}{\geq} 0$$

Thus, we substitute $v_1 \rightarrow a$, $v_2 \rightarrow a + b + c$, $v_3 \rightarrow a + b + c + d$, $v_4 \rightarrow a + b$, and compute series with the new variables as indices, where $a, c \geq 0$ and $b, d \geq 1$

7 Case 7

$$v_2 \geq v_4 \geq 0$$

$$\begin{aligned}
v_3 &> v_1 \geq v_4 \geq 0 \\
\Rightarrow v_{11} &\geq -v_2 - v_1 \\
\Rightarrow f(p, t) &= p^{7v_1+10v_2+11v_3+7v_4+v_4} t^{4v_1+6v_2+6v_3+4v_4} = p^{7v_1+10v_2+11v_3+8v_4} t^{4v_1+6v_2+6v_3+4v_4} \\
\Rightarrow \alpha &:= \min\{v_4, v_2 + v_{11} + v_1\} = \begin{cases} v_4, & v_{11} \geq v_4 - v_1 - v_2 \\ v_4 - 1, v_4 - 2, \dots, & v_{11} < v_4 - v_1 - v_2 \end{cases}
\end{aligned}$$

There are no two sub cases here, because the phase transition occurs at $v_{11} = v_4 - v_1 - v_2 - 1$, thus

if $v_4 = 0$ then the transition would be at $v_{11} = -v_2 - v_1 - 1 < -v_2 - v_1$, which is a contradiction.

For $v_4 \geq 1$ the transition would be at $a_{11} \geq -v_2 - v_1$, which is always true for this case.

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho)$.
2. For $v_4 - v_1 - v_2 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{v_4 - v_1 - v_2}^{-v_2 - 1} p^{v_2} \rho(1 + v_4\rho) = \sum_{i=1}^{v_1 - v_4} p^{v_2} \rho(1 + v_4\rho) = p^{v_2}(v_1 - v_4)\rho + p^{v_2}(v_1 - v_4)v_4\rho^2$.
3. For $-v_1 - v_2 \leq v_{11} \leq v_4 - v_1 - v_2 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_1 - v_2}^{v_4 - v_1 - v_2 - 1} p^{v_2} \rho(1 + \alpha\rho) = \sum_{i=1}^{v_4} p^{v_2} \rho(1 + (v_4 - i)\rho) = p^{v_2}v_4\rho + p^{v_2}v_4^2\rho^2 - p^{v_2}\binom{v_4+1}{2}\rho^2$.

So we have $\sum_{v_{11} \geq -v_2 - v_1} p^{\min}\mu(a_{11}) = p^{v_2}(1 + v_4\rho + (v_4 - v_1)\rho + (v_4 - v_1)v_4\rho^2 + v_4\rho + v_4^2\rho^2 - \frac{v_4(v_4-1)}{2}\rho^2) = p^{v_2}(1 + 3v_4\rho - v_1\rho + 2v_4^2\rho^2 - v_1v_4\rho^2 - \frac{v_4^2}{2}\rho^2 + \frac{v_4}{2}\rho^2)$

We have several arrangements,

1. $v_2 \stackrel{d}{\geq} v_3 \stackrel{c}{>} v_1 \stackrel{b}{\geq} v_4 \stackrel{a}{\geq} 0$

Thus, we substitute $v_1 \rightarrow a + b$, $v_2 \rightarrow a + b + c + d$, $v_3 \rightarrow a + b + c$, $v_4 \rightarrow a$, and compute series with the new variables as indices, where $a, b, d \geq 0$ and $c \geq 1$

2. $v_3 \stackrel{d}{>} v_2 \stackrel{c}{>} v_1 \stackrel{b}{\geq} v_4 \stackrel{a}{\geq} 0$

Thus, we substitute $v_1 \rightarrow a + b$, $v_2 \rightarrow a + b + c$, $v_3 \rightarrow a + b + c + d$, $v_4 \rightarrow a$, and compute series with the new variables as indices, where $a, b \geq 0$ and $c, d \geq 1$

3. $v_3 \stackrel{d}{>} v_1 \stackrel{c}{\geq} v_2 \stackrel{b}{\geq} v_4 \stackrel{a}{\geq} 0$

Thus, we substitute $v_1 \rightarrow a + b + c$, $v_2 \rightarrow a + b$, $v_3 \rightarrow a + b + c + d$, $v_4 \rightarrow a$, and compute series with the new variables as indices, where $a, b, c \geq 0$ and $d \geq 1$

8 Case 8

$$v_2 \geq v_4 \geq 0$$

$$\begin{aligned}
v_1 &\geq v_4 \geq 0 \\
v_1 &\geq v_3 \geq 0 \\
\Rightarrow v_{11} &\geq -v_2 - v_3 \\
\Rightarrow \alpha &:= \min\{v_4, v_2 + v_{11} + v_1\} = \begin{cases} v_4, & v_{11} \geq v_4 - v_1 - v_2 \\ v_4 - 1, v_4 - 2, \dots, & v_{11} < v_4 - v_1 - v_2 \end{cases} \\
\Rightarrow f(p, t) &= p^{7v_1+10v_2+11v_3+7v_4+v_4} t^{4v_1+6v_2+6v_3+4v_4} = p^{7v_1+10v_2+11v_3+8v_4} t^{4v_1+6v_2+6v_3+4v_4}
\end{aligned}$$

There are two sub cases here

8.1 Sub case 8.1

$-v_2 - v_3 \geq v_4 - v_1 - v_2 \Rightarrow -v_3 \geq v_4 - v_1 \Rightarrow v_1 \geq v_3 + v_4$
For $v_{11} \geq -v_2$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho)$.
For $-v_2 - v_3 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2-v_3}^{-v_2-1} p^{v_2}\rho(1 + v_4\rho) = \sum_{i=1}^{v_3} p^{v_2}\rho(1 + v_4\rho) = p^{v_2}(v_3\rho + v_3v_4\rho^2)$.
So we have $\sum_{v_{11} \geq -v_2-v_3} p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho + v_3\rho + v_3v_4\rho^2)$.
We have several arrangements,

$$1. v_1 \stackrel{d}{\geq} v_3 + v_4 \stackrel{a}{\geq} v_3 \stackrel{c}{>} v_2 \stackrel{b}{\geq} v_4 \stackrel{a}{\geq} 0$$

But $e = v_2 + v_3 - v_3 = v_2 = a + b$.

Thus, we substitute $v_1 \rightarrow 2a + b + c + d$, $v_2 \rightarrow a + b$, $v_3 \rightarrow a + b + c$, $v_4 \rightarrow a$, and compute series with the new variables as indices, where $a, b, d \geq 0$ and $c \geq 1$

$$2. v_1 \stackrel{d}{\geq} v_3 + v_4 \stackrel{c}{\geq} v_2 \stackrel{b}{\geq} v_3 \stackrel{a}{\geq} v_4 \stackrel{e}{\geq} 0$$

But $v_3 = v_3 + v_4 - c - b \Rightarrow v_4 = b + c$

Thus, we substitute $v_1 \rightarrow a + 2b + 2c + d$, $v_2 \rightarrow a + 2b + c$, $v_3 \rightarrow a + b + c$, $v_4 \rightarrow b + c$, and compute series with the new variables as indices, where $a, b, c, d \geq 0$.

$$3. v_1 \stackrel{d}{\geq} v_3 + v_4 \stackrel{c}{\geq} v_2 \stackrel{b}{\geq} v_4 \stackrel{a}{>} v_3 \stackrel{e}{\geq} 0$$

Thus, we substitute $v_1 \rightarrow a + 2b + 2c + d$, $v_2 \rightarrow a + 2b + c$, $v_3 \rightarrow a + b + c$, $v_4 \rightarrow b + c$, and compute series with the new variables as indices, where $b, c, d \geq 0$ and $a \geq 1$

$$4. v_1 \stackrel{d}{\geq} v_2 \stackrel{c}{>} v_3 + v_4 \stackrel{a}{\geq} v_4 \stackrel{b}{>} v_3 \stackrel{a}{\geq} 0$$

Thus, we substitute $v_1 \rightarrow 2a + b + c + d$, $v_2 \rightarrow 2a + b + c$, $v_3 \rightarrow a$, $v_4 \rightarrow a + b$, and compute series with the new variables as indices, where $a, d \geq 0$ and $b, c \geq 1$

$$5. v_1 \stackrel{d}{\geq} v_2 \stackrel{c}{>} v_3 + v_4 \stackrel{a}{\geq} v_3 \stackrel{b}{\geq} v_4 \stackrel{a}{\geq} 0$$

Thus, we substitute $v_1 \rightarrow 2a + b + c + d$, $v_2 \rightarrow 2a + b + c$, $v_3 \rightarrow a + b$, $v_4 \rightarrow a$, and compute series with the new variables as indices, where $a, b, d \geq 0$ and $c \geq 1$

$$6. v_2 \overset{d}{>} v_1 \overset{c}{\geq} v_3 + v_4 \overset{a}{\geq} v_3 \overset{b}{\geq} v_4 \overset{a}{\geq} 0$$

Thus, we substitute $v_1 \rightarrow 2a + b + c$, $v_2 \rightarrow 2a + b + c + d$, $v_3 \rightarrow a + b$, $v_4 \rightarrow a$, and compute series with the new variables as indices, where $a, b, c \geq 0$ and $d \geq 1$

$$7. v_2 \overset{d}{>} v_1 \overset{c}{\geq} v_3 + v_4 \overset{a}{\geq} v_4 \overset{b}{>} v_3 \overset{a}{\geq} 0$$

Thus, we substitute $v_1 \rightarrow 2a + b + c$, $v_2 \rightarrow 2a + b + c + d$, $v_3 \rightarrow a$, $v_4 \rightarrow a + b$, and compute series with the new variables as indices, where $a, c \geq 0$ and $b, d \geq 1$

@@@@ CHECKED @@@@

8.2 Sub case 8.2

$$-v_2 - v_3 < v_4 - v_1 - v_2 \Rightarrow -v_3 < v_4 - v_1 \Rightarrow v_1 < v_3 + v_4$$

1. For $v_{11} \geq -v_2$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = p^{v_2}(1 + v_4\rho)$.
2. For $v_4 - v_1 - v_2 \leq v_{11} \leq -v_2 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{v_4 - v_1 - v_2}^{-v_2 - 1} p^{v_2}\rho(1 + v_4\rho) = p^{v_2}((v_1 - v_4)\rho + (v_1 - v_4)v_4\rho^2)$.
3. For $-v_2 - v_3 \leq v_{11} \leq v_4 - v_1 - v_2 - 1$, we have $p^{\min}(1 + \alpha\rho)\mu(a_{11}) = \sum_{-v_2 - v_3}^{v_4 - v_1 - v_2 - 1} p^{v_2}\rho(1 + \alpha\rho) = \sum_{i=1}^{v_3 + v_4 - v_1} p^{v_2}\rho(1 + (v_4 - i)\rho) = p^{v_2}((v_3 + v_4 - v_1)\rho + (v_3 + v_4 - v_1)v_4\rho^2 - \binom{v_3 + v_4 - v_1 + 1}{2}\rho^2)$.

So we have $\sum_{v_{11} \geq -v_2 - v_3} \rho(1 + \alpha\rho) = p^{v_2}(1 + v_4\rho + (v_1 - v_4)\rho + (v_1 - v_4)v_4\rho^2 + (v_3 + v_4 - v_1)\rho + (v_3 + v_4 - v_1)v_4\rho^2 - \frac{(v_3 + v_4 - v_1 + 1)(v_3 + v_4 - v_1)}{2}\rho^2) = p^{v_2}(1 + v_3\rho + v_4\rho - \frac{v_3^2}{2}\rho^2 + v_1v_3\rho^2 - \frac{v_4^2}{2}\rho^2 + v_1v_4\rho^2 - \frac{v_1^2}{2}\rho^2 - v_3\rho^2 - v_4\rho^2 + v_1\rho^2)$.

We have several arrangements,

$$1. v_3 + v_4 \overset{d}{>} v_1 \overset{c}{\geq} v_3 \overset{b}{>} v_2 \overset{a}{\geq} v_4 \overset{e}{\geq} 0$$

But $v_3 + v_4 - d - c = v_3 \Rightarrow v_4 = c + d$.

Thus, we substitute $v_1 \rightarrow a + b + 2c + d$, $v_2 \rightarrow a + c + d$, $v_3 \rightarrow a + b + c + d$, $v_4 \rightarrow c + d$, and compute series with the new variables as indices, where $a, c \geq 0$ and $b, d \geq 1$

$$2. v_3 + v_4 \overset{d}{>} v_1 \overset{c}{\geq} v_2 \overset{b}{\geq} v_3 \overset{a}{\geq} v_4 \overset{e}{\geq} 0$$

But $v_3 + v_4 - d - c - b = v_3 \Rightarrow v_4 = b + c + d$.

Thus, we substitute $v_1 \rightarrow a + 2b + 2c + d$, $v_2 \rightarrow a + 2b + c + d$, $v_3 \rightarrow a + b + c + d$, $v_4 \rightarrow b + c + d$, and compute series with the new variables as indices, where $a, b, c \geq 0$ and $d \geq 1$.

$$3. v_3 + v_4 \overset{d}{>} v_1 \overset{c}{\geq} v_2 \overset{b}{\geq} v_4 \overset{a}{>} v_3 \overset{e}{\geq} 0$$

But $v_3 + v_4 - d - c - b = v_4 \Rightarrow v_3 = b + c + d$.

Thus, we substitute $v_1 \rightarrow a + 2b + 2c + d$, $v_2 \rightarrow a + 2b + c + d$, $v_3 \rightarrow b + c + d$, $v_4 \rightarrow a + b + c + d$, and compute series with the new variables as indices, where $b, c \geq 0$ and $a, d \geq 1$

$$4. v_3 + v_4 \stackrel{d}{\geq} v_2 \stackrel{c}{>} v_1 \stackrel{b}{\geq} v_4 \stackrel{a}{>} v_3 \stackrel{e}{\geq} 0$$

But $v_3 + v_4 - d - c - b = v_4 \Rightarrow v_3 = b + c + d$.

Thus, we substitute $v_1 \rightarrow a + 2b + c + d$, $v_2 \rightarrow a + 2b + 2c + d$, $v_3 \rightarrow b + c + d$, $v_4 \rightarrow a + b + c + d$, and compute series with the new variables as indices, where $b, d \geq 0$ and $a, c \geq 1$

$$5. v_3 + v_4 \stackrel{d}{>} v_2 \stackrel{c}{>} v_1 \stackrel{b}{\geq} v_3 \stackrel{a}{\geq} v_4 \stackrel{e}{\geq} 0$$

But $v_3 + v_4 - d - c - b = v_3 \Rightarrow v_4 = b + c + d$.

Thus, we substitute $v_1 \rightarrow a + 2b + c + d$, $v_2 \rightarrow a + 2b + 2c + d$, $v_3 \rightarrow a + b + c + d$, $v_4 \rightarrow b + c + d$, and compute series with the new variables as indices, where $a, b \geq 0$ and $c, d \geq 1$

$$6. v_2 \stackrel{d}{\geq} v_3 + v_4 \stackrel{c}{>} v_1 \stackrel{b}{\geq} v_4 \stackrel{a}{>} v_3 \stackrel{e}{\geq} 0$$

But $v_3 + v_4 - c - b = v_4 \Rightarrow v_3 = b + c$.

Thus, we substitute $v_1 \rightarrow a + 2b + c$, $v_2 \rightarrow a + 2b + 2c + d$, $v_3 \rightarrow b + c$, $v_4 \rightarrow a + b + c$, and compute series with the new variables as indices, where $b, d \geq 0$ and $a, c \geq 1$

$$7. v_2 \stackrel{d}{\geq} v_3 + v_4 \stackrel{c}{>} v_1 \stackrel{b}{\geq} v_3 \stackrel{a}{\geq} v_4 \stackrel{e}{\geq} 0$$

But $v_3 + v_4 - c - b = v_3 \Rightarrow v_4 = b + c$.

Thus, we substitute $v_1 \rightarrow a + 2b + c$, $v_2 \rightarrow a + 2b + 2c + d$, $v_3 \rightarrow a + b + c$, $v_4 \rightarrow b + c$, and compute series with the new variables as indices, where $a, b, d \geq 0$ and $c \geq 1$

Checking all the cases by the 4! arrangements.

$$1. v_1 \geq v_2 \geq v_3 \geq v_4 \geq 0$$

$$v_1 \geq v_3 + v_4 \geq v_2 \geq v_3 \geq v_4 \geq 0 \quad (8.1.2)$$

$$v_1 \geq v_2 > v_3 + v_4 \geq v_3 \geq v_4 \geq 0 \quad (8.1.5)$$

$$v_3 + v_4 > v_1 \geq v_2 \geq v_3 \geq v_4 \geq 0 \quad (8.2.2)$$

$$2. v_1 \geq v_2 \geq v_4 > v_3 \geq 0$$

$$v_1 \geq v_3 + v_4 \geq v_2 \geq v_4 > v_3 \geq 0 \quad (8.1.3)$$

$$v_1 \geq v_2 > v_3 + v_4 \geq v_4 > v_3 \geq 0 \quad (8.1.4)$$

$$v_3 + v_4 > v_1 \geq v_2 \geq v_4 > v_3 \geq 0 \quad (8.2.3)$$

$$3. v_1 \geq v_3 > v_2 \geq v_4 \geq 0$$

$$v_1 \geq v_3 + v_4 \geq v_3 > v_2 \geq v_4 \geq 0 \quad (8.1.1)$$

$$v_3 + v_4 > v_1 \geq v_3 > v_2 \geq v_4 \geq 0 \quad (8.2.1)$$

4. $v_1 \geq v_3 \geq v_4 > v_2 \geq 0$
 $v_1 \geq v_2 + v_3 \geq v_3 \geq v_4 > v_2 \geq 0$ (2.1.1)
 $v_2 + v_3 > v_1 \geq v_3 \geq v_4 > v_2 \geq 0$ (2.2.1)
5. $v_1 \geq v_4 > v_2 \geq v_3 \geq 0$
 $v_1 \geq v_2 + v_3 \geq v_4 > v_2 \geq v_3 \geq 0$ (2.1.3)
 $v_1 \geq v_4 > v_2 + v_3 \geq v_2 \geq v_3 \geq 0$ (2.1.4)
 $v_2 + v_3 > v_1 \geq v_4 > v_2 \geq v_3 \geq 0$ (2.2.3)
6. $v_1 \geq v_4 > v_3 > v_2 \geq 0$
 $v_1 \geq v_2 + v_3 \geq v_4 > v_3 > v_2 \geq 0$ (2.1.2)
 $v_2 + v_3 > v_1 \geq v_4 > v_3 > v_2 \geq 0$ (2.2.2)
 $v_1 \geq v_4 > v_2 + v_3 \geq v_3 > v_2 \geq 0$ (2.1.5)
7. $v_2 > v_1 \geq v_3 \geq v_4 \geq 0$
 $v_2 > v_1 \geq v_3 + v_4 \geq v_3 \geq v_4 \geq 0$ (8.1.6)
 $v_2 \geq v_3 + v_4 > v_1 \geq v_3 \geq v_4 \geq 0$ (8.2.7)
 $v_3 + v_4 > v_2 > v_1 \geq v_3 \geq v_4 \geq 0$ (8.2.5)
8. $v_2 > v_1 \geq v_4 > v_3 \geq 0$
 $v_2 > v_1 \geq v_3 + v_4 \geq v_4 > v_3 \geq 0$ (8.1.7)
 $v_2 \geq v_3 + v_4 > v_1 \geq v_4 > v_3 \geq 0$ (8.2.6)
 $v_3 + v_4 \geq v_2 > v_1 \geq v_4 > v_3 \geq 0$ (8.2.4)
9. $v_2 \geq v_3 > v_1 \geq v_4 \geq 0$ (7.1)
10. $v_2 \geq v_3 \geq v_4 > v_1 \geq 0$ (6.2)
11. $v_2 \geq v_4 > v_1 \geq v_3 \geq 0$ (5)
12. $v_2 \geq v_4 > v_3 > v_1 \geq 0$ (6.1)
13. $v_3 > v_1 \geq v_2 \geq v_4 \geq 0$ (7.3)
14. $v_3 > v_1 \geq v_4 > v_2 \geq 0$ (1)
15. $v_3 > v_2 > v_1 \geq v_4 \geq 0$ (7.2)
16. $v_3 > v_2 \geq v_4 > v_1 \geq 0$ (6.3)
17. $v_3 \geq v_4 > v_1 \geq v_2 \geq 0$
 $v_3 \geq v_4 > v_1 + v_2 \geq v_1 \geq v_2 \geq 0$ (3.1.7)
 $v_3 \geq v_1 + v_2 \geq v_4 > v_1 \geq v_2 \geq 0$ (3.2.7)
 $v_1 + v_2 > v_3 \geq v_4 > v_1 \geq v_2 \geq 0$ (3.2.5)

18. $v_3 \geq v_4 > v_2 > v_1 \geq 0$
 $v_3 \geq v_4 > v_1 + v_2 \geq v_2 > v_1 \geq 0$ (3.1.6)
 $v_3 \geq v_1 + v_2 \geq v_4 > v_2 > v_1 \geq 0$ (3.2.6)
 $v_1 + v_2 \geq v_3 \geq v_4 > v_2 > v_2 \geq 0$ (3.2.4)
19. $v_4 > v_1 \geq v_2 \geq v_3 \geq 0$
 $v_4 > v_2 + v_3 \geq v_1 \geq v_2 \geq v_3 \geq 0$ (4.1.2)
 $v_2 + v_3 \geq v_4 > v_1 \geq v_2 \geq v_3 \geq 0$ (4.2.1)
 $v_4 > v_1 \geq v_2 + v_3 \geq v_2 \geq v_3 \geq 0$ (4.1.4)
20. $v_4 > v_1 \geq v_3 > v_2 \geq 0$
 $v_4 > v_2 + v_3 \geq v_1 \geq v_3 > v_2 \geq 0$ (4.1.1)
 $v_2 + v_3 \geq v_4 > v_1 \geq v_3 > v_2 \geq 0$ (4.2.2)
 $v_4 > v_1 \geq v_2 + v_3 \geq v_3 > v_2 \geq 0$ (4.1.5)
21. $v_4 > v_2 > v_1 \geq v_3 \geq 0$
 $v_4 > v_2 + v_3 \geq v_2 > v_1 \geq v_3 \geq 0$ (4.1.3)
 $v_2 + v_3 \geq v_4 > v_2 > v_1 \geq v_3 \geq 0$ (4.2.3)
22. $v_4 > v_2 \geq v_3 > v_1 \geq 0$
 $v_4 > v_1 + v_2 \geq v_2 \geq v_3 > v_1 \geq 0$ (3.1.2)
 $v_1 + v_2 \geq v_4 > v_2 \geq v_3 > v_1 \geq 0$ (3.2.2)
23. $v_4 > v_3 > v_1 \geq v_2 \geq 0$
 $v_4 > v_1 + v_2 \geq v_3 > v_1 \geq v_2 \geq 0$ (3.1.3)
 $v_1 + v_2 \geq v_4 > v_3 > v_1 \geq v_2 \geq 0$ (3.2.3)
 $v_4 > v_3 > v_1 + v_2 \geq v_1 \geq v_2 \geq 0$ (3.1.4)
24. $v_4 > v_3 > v_2 > v_1 \geq 0$
 $v_4 > v_1 + v_2 \geq v_3 > v_2 > v_1 \geq 0$ (3.1.1)
 $v_1 + v_2 \geq v_4 > v_3 > v_2 > v_1 \geq 0$ (3.2.1)
 $v_4 > v_3 > v_1 + v_2 \geq v_2 > v_1 \geq 0$ (3.1.5)

9 Sum of Series

$$\sum_{a \geq 0} x^a = \frac{1}{1-x}$$

$$\sum_{a \geq 1} x^a = \sum_{a \geq 0} x^a - x^0 = \frac{1}{1-x} - 1 = \frac{x}{1-x}$$

$$\sum_{a \geq 0} ax^a = x \sum_{a \geq 0} ax^{a-1} = x \left(\sum_{a \geq 0} x^a \right)' = x((1-x)^{-1})' = -1 \cdot x(1-x)^{-2} \cdot -1 = \frac{x}{(1-x)^2}$$

$$\sum_{a \geq 1} ax^a = \sum_{a \geq 0} ax^a - 0x^0 = \sum_{a \geq 0} ax^a = \frac{x}{(1-x)^2}$$

$$\begin{aligned} \left(\sum_{a \geq 0} x^a \right)'' &= \left(\sum_{a \geq 0} ax^{a-1} \right)' = \sum_{a \geq 0} a(a-1)x^{a-2} = \frac{1}{x^2} \sum_{a \geq 0} a(a-1)x^a = \\ &= \frac{1}{x^2} \sum_{a \geq 0} a^2 x^a - \frac{1}{x^2} \sum_{a \geq 0} ax^a = \frac{1}{x^2} \sum_{a \geq 0} a^2 x^a - \frac{1}{x^2} \frac{x}{(1-x)^2} \end{aligned}$$

But

$$\left(\sum_{a \geq 0} x^a \right)'' = \left[\frac{1}{(1-x)^2} \right]' = ((1-x)^{-2})' = -2 \cdot (1-x)^{-3} \cdot -1 = \frac{2}{(1-x)^3} \Rightarrow$$

$$\begin{aligned} \Rightarrow \frac{1}{x^2} \sum_{a \geq 0} a^2 x^a &= \frac{2}{(1-x)^3} + \frac{x}{x^2(1-x)^2} \Rightarrow \sum_{a \geq 0} a^2 x^a = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \\ &= \frac{2x^2 + x - x^2}{(1-x)^3} = \frac{x^2 + x}{(1-x)^3} \end{aligned}$$

$$\sum_{a \geq 1} a^2 x^a = \sum_{a \geq 0} a^2 x^a - 0^2 x^0 = \sum_{a \geq 0} a^2 x^a = \frac{x^2 + x}{(1-x)^3}$$

Case 1

$$f(v_1, v_2, v_3, v_4) = p^{7v_1+11v_2+11v_3+8v_4} t^{4v_1+6v_2+6v_3+4v_4}$$

$$g(v_1, v_2, v_3, v_4) = 1 + v_1\rho + v_2\rho + v_2v_1\rho^2 - \frac{v_2^2}{2}\rho^2 - \frac{v_2}{2}\rho^2.$$

$$v_1 \rightarrow a + b + c, v_2 \rightarrow a, v_3 \rightarrow a + b + c + d, v_4 \rightarrow a + b.$$

$$a, c \geq 0, b, d \geq 1.$$

$$f(a, b, c, d) = (p^{37}t^{20})^a (p^{26}t^{14})^b (p^{18}t^{10})^c (p^{11}t^6)^d = x_0^a x_1^b x_2^c x_3^d.$$

$$g(a, b, c, d) = 1 + (2a + b + c)\rho + \left(\frac{a^2}{2} + ab + ac - \frac{a}{2}\right)\rho^2.$$

$$\begin{aligned} (fg)(a, b, c, d) &= \\ &= \frac{x_1 x_3}{(1-x_0)(1-x_1)(1-x_2)(1-x_3)} + \\ &+ \frac{2\rho x_0 x_1 x_3}{(1-x_0)^2(1-x_1)(1-x_2)(1-x_3)} + \\ &+ \frac{\rho x_1 x_3}{(1-x_0)(1-x_1)^2(1-x_2)(1-x_3)} + \end{aligned}$$

$$\begin{aligned}
& + \frac{\rho x_2 x_1 x_3}{(1-x_0)(1-x_1)(1-x_2)^2(1-x_3)} + \\
& + \frac{\rho^2(x_0^2 + x_0)x_1 x_3}{2(1-x_0)^3(1-x_1)(1-x_2)(1-x_3)} + \\
& + \frac{\rho^2 x_0 x_1 x_3}{(1-x_0)^2(1-x_1)^2(1-x_2)(1-x_3)} + \\
& + \frac{\rho^2 x_0 x_2 x_1 x_3}{(1-x_0)^2(1-x_1)(1-x_2)^2(1-x_3)} - \\
& - \frac{\rho^2 x_0 x_1 x_3}{2(1-x_0)^2(1-x_1)(1-x_2)(1-x_3)}
\end{aligned}$$