The pro-isomorphic zeta-functions of some nilpotent Lie algebras over integer rings

Research proposal for a Ph.D. Thesis
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Contents

Abstract

Let G be any group. For any natural number $n \in \mathbb{N}$, let $a_n^{\leq}(G)$ be the number of subgroups $H \leq G$ such that $[G : H] = n$. If G is finitely-generated, then $a_n^{\leq}(G) < \infty$, and we can define a Dirichlet series of the form $\zeta_G(s) := \sum_{n=1}^{\infty} a_n^{\leq}(G) n^{-s}$, where $s \in \mathbb{C}$. Assume, in addition, that G is also nilpotent and torsion-free, then this function has some properties of the Riemann zeta-function ζ, such as the Euler decomposition of ζ into a product of local factors indexed by primes. A version of this zeta-function counts pro-isomorphic subgroups, and an analogous function may be defined for appropriate Lie rings. We study here the pro-isomorphic zeta-functions for a family of nilpotent Lie rings of unbounded nilpotency class. We shall compute the automorphism groups of these Lie rings explicitly and use them to prove uniformity of the local factors of the pro-isomorphic zeta-functions, and aim to determine them explicitly.

1 Scientific Background

1.1 Introduction

Although we will work with Lie algebras, for motivation we first present analogous and more natural questions in the context of groups.

Proposition 1.1.1. *Let G be any finitely generated group, and let $n \in \mathbb{N}$ be any natural number. Then there is a finite number of subgroups $H \leq G$ such that $[G : H] = n$.*

This proposition gives rise to an entire branch of group theory, called **subgroup growth**. We denote by

$$a_n^{\leq}(G) := |\{H \leq G : [G : H] = n\}|$$

the number of subgroups of G of index n and consider the sequence $\{a_n^{\leq}(G)\}_{n=1}^{\infty}$. The subject of subgroup growth aims to relate the properties of this sequence to the algebraic structure of G. We denote by

$$a_n^{\trianglelefteq}(G) := |\{H \trianglelefteq G : [G : H] = n\}|$$

the number of **normal** subgroups of G. We now define another type of subgroups of G.

Definition 1.1.2. *Let G be a group. A subgroup $H \leq G$ is called* ***pro-isomorphic*** *if $\widehat{H} \cong \widehat{G}$, where $\widehat{H}$ and $\widehat{G}$ are the profinite closures of H and G, respectively.*

We denote by

$$\hat{a_n}(G) := |\{H \leq G : \widehat{H} \cong \widehat{G}, [G : H] = n\}|$$

the number of **pro-isomorphic** subgroups of G.

Definition 1.1.3. *Let G be a finitely-generated group, and let $* \in \{\leq, \trianglelefteq, \wedge\}$, then we define the zeta-function $\zeta_G^*(s)$ to be the Dirichlet series $\sum_{n=1}^{\infty} a_n^*(G)n^{-s}$.*

Proposition 1.1.4. *Let G be a $\mathcal{T}$-group, i.e. finitely-generated, nilpotent and torsion-free group, and let $* \in \{\leq, \trianglelefteq, \wedge\}$, then the zeta-functions for G have the following properties:*

Polynomial growth and convergence*. There exist constants b and C such that $a_n^{\leq}(G) \leq Cn^b$ for all $n \in \mathbb{N}$. Hence $\zeta_G^*(s)$ converges on some right half plane $\mathrm{Re}\, s > \alpha$. The abscissa of convergence*

$$\alpha^* := inf\{\alpha : \zeta_G^*(s)\,\mathrm{converges\,on\,Re}\, s > \alpha\}$$

are known to be rational numbers.

Euler decomposition*. We have that $\zeta_G^*(s) = \prod_p \zeta_{G,p}^*(s)$, where $\zeta_{G,p}^*(s) = \sum_{k=0}^{\infty} a_{p^k}^*(G)p^{-ks}$.*

Rationality*. For all primes p, there is a rational function $W_p^* \in \mathbb{Q}(X)$ such that $\zeta_{G,p}^*(s) = W_p^*(p^{-s})$.*

Definition 1.1.5. *Let $\zeta_{L,p}^*$ be a* ***finitely-uniform*** *zeta-function, i.e. there are rational functions $W_1^*(X,Y), \dots, W_r^*(X,Y) \in \mathbb{Q}(X,Y)$, such that for all p, there is some $1 \leq i \leq r$ such that $\zeta_{G,p}^*(s) = W_i^*(p, p^{-s})$. We say W_i^* satisfies a* ***functional equation*** *if $W_i^*(X^{-1}, Y^{-1}) = \pm X^a Y^b W_i^*(X,Y)$, where $a, b \in \mathbb{N} \cup \{0\}$, $X^a Y^b$ being called the* ***symmetry factor****.‘*

Proposition 1.1.6. *Let G be a $\mathcal{T}$-group, then $\zeta_{G,p}^{\leq}(s)$ satisfies a functional equation, for all but finitely many p, with the same symmetry factor. If G is a $\mathcal{T}$-group of nilpotency class 2, same is true for $\zeta_{G,p}^{\trianglelefteq}(s)$.*

Remark 1.1.7. *There are specific examples of $\mathcal{T}$-groups of nilpotency class 3, such that $\zeta_{G,p}^{\trianglelefteq}(s)$ is* ***uniform****, i.e. $\zeta_{G,p}^{\trianglelefteq}(s) = W(p, p^{-s})$, but $W(p, p^{-s})$ does not satisfy a functional equation.*

With the discussion on pro-isomorphic zeta-functions, there are results showing that under quite restrictive conditions, $\hat{\zeta_{G,p}}(s)$ is uniform and satisfies a functional equation, see, for instance, [**?**].

If any Dirichlet series, in particular $\zeta_G^*(s)$, has some properties of convergence on some subset of $\mathbb{C}$, one may reconstruct its coefficients $a_n^*(G)$, the number of subgroups of our interest, using the Perron's formula, which is an implementation of a reverse Mellin transform; see, for instance, [7].

As an example, consider the group $G = \mathbb{Z}$, for which every finite-index subgroup $H \leq \mathbb{Z}$ is of the form $H = n\mathbb{Z} = \langle n \rangle$, for some $n \in \mathbb{N}$, which means that $H \cong \mathbb{Z}$, as both are infinite cyclic groups, and so, $\widehat{H} \cong \widehat{\mathbb{Z}}$. Since we have only one subgroup of index n, for every $n \in \mathbb{N}$, then $a_n(\mathbb{Z}) = \hat{a_n}(\mathbb{Z}) = 1$. Thus, its pro-isomorphic zeta-function is $\hat{\zeta_{\mathbb{Z}}}(s) = \sum_{i=1}^{\infty} n^{-s} = \zeta(s)$, the Riemann zeta-function, which is known to converge for $\operatorname{Re} s > 1$. We recall that the Riemann zeta-function decomposes into an infinite product of local zeta-functions, that is, $\zeta(s) = \prod_p \zeta_p(s) = \prod_p \sum_{k=0}^{\infty} p^{-ks} = \prod_p \frac{1}{1-p^{-s}}$, where the product runs over all the prime numbers.

This research concentrates on the growth of **pro-isomorphic** subgroups defined above, hence we shall restrict our further discussion to the pro-isomorphic case only.

1.2 Linearization

We may transfer the ideas from the above discussion about groups to a linear context, where we can use tools from linear algebra. If L is a $\mathbb{Z}$-Lie algebra, namely a free $\mathbb{Z}$-module of finite rank with a Lie bracket operation, then consider the number $\hat{a_n}(L)$ of subalgebras $M \leq L$ of index n such that $M \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong L \otimes_{\mathbb{Z}} \mathbb{Z}_p$, for all primes p. It is known that $\hat{a_n}(L) < \infty$ for all $n \in \mathbb{N}$. The Dirichlet series $\hat{\zeta_L}(s) := \sum_{n=1}^{\infty} \hat{a_n}(L) n^{-s}$, is called the **pro-isomorphic zeta-function** of L. By the Mal'cev correspondence, to every $\mathcal{T}$-group G, one may associate a Lie algebra $L = L(G)$, such that $\hat{\zeta_{G,p}}(s) = \hat{\zeta_{L,p}}(s)$, for all but finitely many primes p. If G has nilpotency class 2, one may obtain the equality for all primes.

Let L be a $\mathbb{Z}$-Lie algebra. Choose a basis $(b_1, \dots, b_r)$, where $r = \operatorname{rank} L$. Let $\mathcal{L}_p = L \otimes_{\mathbb{Z}} \mathbb{Q}_p$, for any p. This is a $\mathbb{Q}_p$-Lie algebra, and our choice of basis allows us to identify the automorphism group $G(\mathbb{Q}_p) = Aut_{\mathbb{Q}_p}(\mathcal{L}_p)$ with a subgroup of $GL_r(\mathbb{Q}_p)$. Note that $\mathcal{L}_p$ contains a $\mathbb{Z}_p$-lattice, $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. If $\varphi \in G(\mathbb{Q}_p)$, then $\varphi(L_p) = L_p$ if and only if $\varphi \in G(\mathbb{Z}_p) = G(\mathbb{Q}_p) \cap GL_r(\mathbb{Z}_p)$.

Similarly, $\varphi(L_p) \subseteq L_p$ if and only if $\varphi \in G^+(\mathbb{Q}_p) := G(\mathbb{Q}_p) \cap M_r(\mathbb{Z}_p)$, where $M_r(\mathbb{Z}_p)$ is the collection of $r \times r$ matrices with entries in $\mathbb{Z}_p$. Note that $G^+(\mathbb{Q}_p)$ is a monoid, not a group.

There is a bijection between the set $G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)$ of right cosets of $G(\mathbb{Z}_p)$ and the set $\{M \leq L_p : M \cong L_p\}$ of L_p-subalgebras which are isomorphic to L_p itself. This bijection takes $G(\mathbb{Z}_p)g$ to $M = \varphi(L_p)$, for $\varphi \in G(\mathbb{Z}_p)g$. Observe that $[L_p : M] = |\det \varphi|_p^{-1}$, where $|\cdot|_p$ is the p-adic norm on $\mathbb{Q}_p$, and therefore,

$$\hat{\zeta_{L,p}}(s) = \sum_{\substack{M \leq L_p \\ M \cong L_p}} [L_p : M]^{-s} = \sum_{G(\mathbb{Z}_p)\varphi \in G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)} |\det \varphi|_p^s \tag{1}$$

.

1.3 p-adic Integration

We fix a specific right Haar measure μ by normalizing $\mu(G(\mathbb{Z}_p)) = 1$. Now we are able to express $\hat{\zeta_{L,p}}(s)$ as a p-adic integral, as first done by du Sautoy and Lubotzky, see [3]. Since $|\det g|_p^s$ is constant on right cosets of $G(\mathbb{Z}_p)$, we have $|\det \varphi|_p^s = \int_{G(\mathbb{Z}_p)\varphi} |\det g|_p^s d\mu(g)$. By (1), we have

$$\hat{\zeta_{L,p}}(s) = \sum_{G(\mathbb{Z}_p)\varphi \in G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)} |\det \varphi|_p^s = \sum_{G(\mathbb{Z}_p)\varphi \in G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)} \int_{G(\mathbb{Z}_p)\varphi} |\det g|_p^s d\mu(g) =$$

$$= \int_{G^+(\mathbb{Z}_p)\varphi} |\det g|_p^s d\mu(g)$$

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2 Research Goals and Methodology

2.1 The Lie algebras L_n

Let R be a commutative ring, and let $L_n(R)$ be the set of strictly upper triangular $n \times n$ matrices with elements in R. The Lie bracket $[A, B] = AB - BA$ gives $L_n(R)$ the structure of a nilpotent R-Lie algebra. Set $L_n = L_n(\mathbb{Z})$ and $\mathcal{L}_n = L_n(\mathbb{Q})$, while $L_{n,p} = L_n(\mathbb{Z}_p)$ and $\mathcal{L}_{n,p} = L_n(\mathbb{Q}_p)$. We will suppress n from the notation, as this should not cause confusion.

Let $e_{ij} \in L_n$ be the matrix with 1 in the (i,j) entry and 0 elsewhere. Then $B := \{e_{ij} : 1 \le i < j \le n\}$ is a basis of L_n, which we order as follows: $(e_{12}, \ldots, e_{n-1,n}, e_{13}, \ldots, e_{n-2,n}, e_{14}, \ldots, e_{n-3,n}, \ldots, e_{1n})$. In other words we order the elements of B so that $e_{ij} < e_{kl}$ if either $j - i < l - k$ or if $j - i = l - k$ and $i < k$. Note that $\gamma_1 L, \ldots, \gamma_{n-1} L$ are spanned by final segments of B. Let e_{ij} be an $n \times n$ matrix, in which all the elements are zero, except for the element in row i and column j which has 1. On the set $E = \{e_{i,i+1} : 1 \le i \le n-1\}$ we define a bracket operation: for every $1 \le k, l \le n-1$, define $[e_{k,k+1}, e_{l,l+1}] := e_{k,k+1}e_{l,l+1} - e_{l,l+1}e_{k,k+1}$. One checks that applying this bracket operation on E repeatedly, generates $B = \{e_{ij} : 1 \le i < j \le n\}$. Let $\mathcal{R}$ be some commutative ring, then the standard operation of $\mathcal{R}$ on B through scalar multiplication, along with the defined bracket operation, form a nilpotent $\mathcal{R}$-Lie algebra. Considering $\mathcal{R} = \mathbb{Z}$, we obtain a nilpotent $\mathbb{Z}$-Lie algebra of strictly upper triangular matrices over $\mathbb{Z}$, which we denote by L_n, with the bracket operation defined above as its Lie bracket. As discussed above, this $\mathbb{Z}$-Lie algebra can be extended to a $\mathbb{Z}_p$-algebra, which we denote by $L_{n,p}$, and then to a $\mathbb{Q}_p$-algebra, which we denote by $\mathcal{L}_{n,p}$. In all the following, when n and p are clear from the context, we may denote L instead of $L_{n,p}$ and $\mathcal{L}$ instead of $\mathcal{L}_{n,p}$, for abbreviation. One checks that the set of matrices B spans the whole $\mathbb{Z}$-Lie algebra L and is $\mathbb{Z}$-linearly independent, hence it forms a basis for L as a free module over $\mathbb{Z}$, which we call the **standard basis** of L. One also checks easily that $|B| = \sum_{k=1}^{n-1} k = \binom{n}{2}$, for every $n \in \mathbb{N}$. We apply a linear order to B, by defining $e_{ij} < e_{kl}$ if $j - i < l - k$ or if $j - i = l - k$ and $i < k$. In other words, the order we apply divides E to elements of the quotients ${}^{L}\!/_{\gamma_2 L}, {}^{\gamma_2 L}\!/_{\gamma_3 L}, \ldots, {}^{\gamma_{n-2} L}\!/_{\gamma_{n-1} L}, \gamma_{n-1} L$. The same goes also for the extensions of L, namely $L_{n,p}$ and $\mathcal{L}_{n,p}$. The target of our research is studying the zeta-function $\zeta^{\wedge}_{L,p}$ on these $\mathbb{Z}_p$-Lie algebras, and the related constructions.

Remark 2.1.1. *Let $\mathcal{R}$ be a commutative ring, and let $\mathcal{U}_n(\mathcal{R})$ be the group of $n \times n$ upper unitriangular matrices over $\mathcal{R}$, with the standard matrix multiplication as the group operation. Denote by $\mathcal{U}_{n,p} = \mathcal{U}_n(\mathbb{Z}_p)$ the unitriangular matrix group over $\mathbb{Z}_p$, then by Mal'cev correspondence, $\zeta^{\wedge}_{\mathcal{U}_{n,p}}(s) = \zeta^{\wedge}_{L_{n,p}}(s)$, for all but finitely many primes p. This relates the subject of research to the motivation presented at the beginning of this paper.*

2.2 Research goals

The project consists of three major steps:

1. **Computing the automorphism group of the $\mathbb{Q}_p$-Lie algebras $\mathcal{L}_{n,p}$, for all $n \in \mathbb{N}$ and all primes p.**

2. **Showing that the pro-isomorphic zeta-functions $\zeta^{\wedge}_{L_{n,p}}(s)$ are uniform, or finitely-uniform, for all $n \in \mathbb{N}$.**

3. **Giving an explicit uniform formula for the zeta-functions $\zeta^{\wedge}_{L_{n,p}}(s)$ for specific values of n, if not for all $n \in \mathbb{N}$.** We aim to compute an explicit formula for $\zeta^{\wedge}_{L_{n,p}}(s)$ for all $n \geq 4$ and all primes p in terms of a suitable combinatorial framework that is compatible with the complicated structure of $G^+(\mathbb{Q}_p)$. As a consequence, we will study functional equations. In the event that we do not succeed in this task in general, we still expect to complete Berman's unfinished calculation in the case $n = 5$.

We start with the first step of calculating $Aut_{\mathbb{Q}_p}(\mathcal{L}_{n,p})$. These automorphism groups have been studied before from a different point of view, and there are classical results showing that any automorphism may be expressed as a product of automorphisms of a specific type; see, for instance, the main result of Gibbs [4]. These results are not explicit enough for our purposes; indeed, the submonoid $G^+(\mathbb{Q}_p)$ arises for us as the domain of integration of a p-adic integral. In order to calculate this integral, we need to decompose the automorphism group $G(\mathbb{Q}_p)$ into a repeated semi-direct product of groups with a simple structure.

After we have analyzed the structure of $G(\mathbb{Z}_p)$, we will need to construct the monoid $G^+(\mathbb{Q}_p)$ and its $G(\mathbb{Z}_p)$ right-cosets, as we have seen above. This will give us both the function to integrate, which is $\det \varphi$ for every $G(\mathbb{Z}_p)$ right-coset $G(\mathbb{Z}_p)\varphi$, and the domain of integration, which is the monoid $G^+(\mathbb{Q}_p)$. We will use this information to analyze the behavior of the p-adic integral we have described above and prove that its calculation depends only on p, thus showing that the $\zeta^{\wedge}_{L,p}$-function is uniform.

2.3 $L_{n,p}$-Lie algebras for $n \geq 5$

Mark N. Berman, in his doctoral thesis [1], has displayed an explicit formula for $\zeta^{\wedge}_{L_{4,p}}$, and proved that $\zeta^{\wedge}_{L_{5,p}}$ is indeed uniform. We aim to generalize his work to prove that $\zeta^{\wedge}_{L_{n,p}}$ is uniform for all n. We also aim to compute $\zeta^{\wedge}_{\mathcal{L}_{n,p}}(s)$ explicitly for all n, or at least to obtain explicit formulas for some $n \geq 5$, and specifically for $n = 5$. By analyzing carefully Berman's work on $L_{4,p}$ and

$L_{5,p}$, we gain the basic understanding of the expected structure of the local zeta-functions in the general case. We begin our discussion of the first goal, which is computing $G(\mathbb{Z}_p)$, by first recalling that for every $v \in L_{n,p}$, where $n \geq 4$, we present $\varphi(v)$ as the multiplication of v by a matrix from the right $\varphi(v) = vM$. As stated earlier, M is an $r \times r$ matrix, where $r = \mathrm{rank} L_{n,p}$, whose lines are set by the order we have defined above, i.e. considering the standard ordered basis

$$B = \{e_{12}, e_{23}, \ldots, e_{n-1,n}, e_{13}, \ldots, e_{n-2,n}, \ldots, e_{1n}\}$$

then M is the following matrix,

$$M = \left(\begin{array}{c} \varphi(e_{12}) \\ \varphi(e_{23}) \\ \varphi(e_{n-1,n}) \\ \hdashline \varphi(e_{13}) \\ \vdots \\ \varphi(e_{n-2,n}) \\ \hdashline \vdots \\ \hdashline \varphi(e_{1n}) \end{array}\right)$$

Given an $\mathcal{L}$-automorphism φ, we denote by $\varphi_k : \gamma_k\mathcal{L} \to \mathcal{L}$ the operation of φ on all the $n-k$ elements of the lower central series starting from k, that is, we consider only the images

$$\varphi(e_{1,1+k}), \varphi(e_{2,2+k}), \ldots, \varphi(e_{n-k,n}), \varphi(e_{1,2+k}), \ldots, \varphi(e_{n-k-1,n}), \ldots, \varphi(e_{1n})$$

Since all the elements of the lower central series of $\mathcal{L}$ are characteristic subalgebras, we have that $\varphi_k(\gamma_k\mathcal{L}) = \gamma_k\mathcal{L}$, for all $1 \leq k \leq n-1$, hence we have an induced map $\varphi_{kk} : {}^{\gamma_k\mathcal{L}}/_{\gamma_{k+1}\mathcal{L}} \to {}^{\gamma_k\mathcal{L}}/_{\gamma_{k+1}\mathcal{L}}$, defined by $\varphi_{kk}(e_{l,l+k} + \gamma_{k+1}\mathcal{L}) := a_{1,1+k}e_{1,1+k} + a_{2,2+k}e_{2,2+k} + \cdots + a_{n-k,n}e_{n-k,n} + z_{k+1}$, where $z_{k+1} \in \gamma_{k+1}\mathcal{L}$, for all $1 \leq l \leq n-k$, one checks easily that since we also have that $\varphi(\gamma_{k+1}\mathcal{L}) = \gamma_{k+1}\mathcal{L}$, then φ_{kk} is well-defined and maps ${}^{\gamma_k\mathcal{L}}/_{\gamma_{k+1}\mathcal{L}}$ onto itself. Following the division of $\mathcal{L}$ into quotients of elements of the lower central series, we view M as a block matrix,

$$M = \left(\begin{array}{c|c|c|c|c} M_{11} & M_{12} & \ldots & M_{1,n-2} & M_{1,n-1} \\ \hline M_{21} & M_{22} & \ldots & M_{2,n-2} & M_{2,n-1} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline M_{n-1,1} & M_{n-1,2} & \ldots & M_{n-1,n-2} & M_{n-1,n-1} \end{array}\right)$$

each block is denoted by $M_{kl} \in M_{m\times r}(\mathbb{Q}_p)$, where $m = \dim {}^{\gamma_k\mathcal{L}}/_{\gamma_{k+1}\mathcal{L}}$ and $r = \dim {}^{\gamma_l\mathcal{L}}/_{\gamma_{l+1}\mathcal{L}}$. From this we understand that the blocks on the main diagonal of M, which are the induced maps defined above, are square matrices $\varphi_{kk} = M_{kk} \in M_{n-k}(\mathbb{Q}_p)$. We also observe that for all $l < k$, since $\varphi_k({}^{\gamma_l\mathcal{L}}/_{\gamma_k\mathcal{L}}) = 0$, as implied above, then $M_{kl} = 0$, and therefore M has the form,

$$M = \left(\begin{array}{c|c|c c|c|c} M_{11} & M_{12} & M_{13} & \dots & M_{1,n-2} & M_{1,n-1} \\ \hline 0 & M_{22} & M_{23} & \dots & M_{2,n-2} & M_{2,n-1} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \dots & M_{2,n-2} & M_{2,n-1} \\ \hline 0 & 0 & 0 & \dots & 0 & M_{n-1,n-1} \end{array}\right)$$

2.4 Preliminary results

We have made progress towards step 1 of our research program, namely determining the automorphism groups $G(\mathbb{Q}_p)$. The first result in this direction appears already in the thesis of M. N. Berman[1, Prop. 3.6].

Proposition 2.4.1. *Let $\varphi \in G(\mathbb{Q}_p)$ be a $\mathcal{L}$-automorphism, and M its representing matrix, divided into matrix blocks, as shown earlier. Then, $M_{11} \in M_{n-1}(\mathbb{Q}_p)$ is either diagonal or anti-diagonal.*

Define an involution $\eta \in G(\mathbb{Q}_p)$ by $\eta(e_{ij}) := (-1)^{j-i-1}e_{n+1-j,n+1-i}$, for $1 \le i \le n-1$ and $i+1 \le j \le n$. Replacing φ by $\varphi \circ \eta$, we assume without loss of generality that M_{11} is diagonal. We check, for any $\lambda_1, \lambda_2, \dots, \lambda_{n-1} \in \mathbb{Q}_p^*$, that the diagonal matrix

$$h = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_1\lambda_2, \lambda_2\lambda_3 \dots, \lambda_{n-2}\lambda_{n-1}, \dots, \lambda_1\lambda_2\cdots\lambda_{n-2}\lambda_{n-1})$$

represents an automorphism of $\mathcal{L}$. Thus, multiplying φ from the right by a unique such automorphism, we may now assume that M has 1's on the diagonal. By this print we know $H(\mathbb{Q}_p) := \{\operatorname{diag}(\lambda_1, \lambda_2, \dots) : \lambda_1, \lambda_2, \dots \in \mathbb{Q}_p^*\}$ is the **reductive part** of $\mathbb{Q}_p$, while $N(\mathbb{Q}_p) := \{\varphi \in G(\mathbb{Q}_p)$ with 1's in diagonal $\}$ is the **unipotent radical** of $G(\mathbb{Q}_p)$. Every $g \in G(\mathbb{Q}_p)$ has a unique decomposition $g = uh$, with $u \in N(\mathbb{Q}_p)$, $h \in H(\mathbb{Q}_p)$. We aim to determine the structure of the unipotent radical $N(\mathbb{Q}_p)$ by decomposing it into a semidirect product of abelian subgroups. We can simplify the domain of integration, for the p-adic integral that we aim to calculate, at the price of replacing a single integral by multiple integrals. As we saw earlier, the calculation of

$\hat{\zeta}_L(s)$ requires computing $G(\mathbb{Z}_p)$ and $G^+(\mathbb{Q}_p)$ first. Assuming we have already computed $G(\mathbb{Q}_p)$, based on the strategy that we have presented above, we need to identify $G(\mathbb{Z}_p)$ as a subgroup of $G(\mathbb{Q}_p)$, which is expected not to be difficult, and continue from there to identify the monoid $G^+(\mathbb{Q}_p)$, which is expected to be a substantial challenge. By applying **Fubini's theorem** for semidirect products of topological groups, we have that

$$\hat{\zeta}_{\mathcal{L}}(s) = \int_{G^+(\mathbb{Q}_p)} |\det \varphi|_p^s d\mu_{G(\mathbb{Z}_p)\varphi} = \int_{H^+(\mathbb{Q}_p)} \left(\int_{N_h^+} |\det uh|_p^s d\mu_{N(\mathbb{Q}_p)} \right) d\mu_{H(\mathbb{Q}_p)}$$

where $H^+(\mathbb{Q}_p)$ consists of all $h \in H(\mathbb{Q}_p)$ that appear in the decomposition $\varphi = uh$ for some $\varphi \in G^+(\mathbb{Q}_p)$, and, for a given $h \in H^+(\mathbb{Q}_p)$, we set $N_h^+(\mathbb{Q}_p) := \{u \in N(\mathbb{Q}_p) : uh \in G^+(\mathbb{Q}_p)\}$. The integrand of the inner integral is constant, so the integral amounts to computing the measure of the set $N_h^+(\mathbb{Q}_p)$. The advantage that we gain by this decomposition is that it simplifies the calculation of the integral. The integral function $|\det uh|_p^s = |\det h|_p^s$ depends only on h, so computing the inner integral amounts to finding the measure of $N_h^+(\mathbb{Q}_p)$. For the unipotent matrix u, the determinant is 1, for the digonal matrix h, we have the following proposition,

Proposition 2.4.2. *Let $n \geq 2$, and let*

$$h = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_1\lambda_2, \lambda_2\lambda_3 \ldots, \lambda_{n-2}\lambda_{n-1}, \ldots, \lambda_1\lambda_2 \cdots \lambda_{n-2}\lambda_{n-1})$$

as defined above, then $\det h = \prod_{i=1}^{n-1} \lambda_i^{i(n-i)}$.

We show this by induction on n. We conclude this section with the observation that the sets $N_h^+(\mathbf{Q_p})$ arising in our computation are quite complicated. We will decompose $N(\mathbb{Q}_p)$ into an iterated semidirect product of a large number of subgroups, each abelian with a simple structure. This will decompose the integral over $N_h^+(\mathbb{Q}_p)$ into a multiple integral that can be computed explicitly, given a suitable combinatorical framework. Hence, we strive to decompose $N(\mathbb{Q}_p)$ itself into a product of finitely many simpler subgroups, $N(\mathbb{Q}_p) = N(\mathbb{Q}_p)_1 \rtimes_\phi N(\mathbb{Q}_p)_2 \rtimes_\phi \cdots \rtimes_\phi N(\mathbb{Q}_p)_m$, where m is the number of subgroups in the decomposition of $N(\mathbb{Q}_p)$, which means that

$$\int_{G^+(\mathbb{Q}_p)} |\det \varphi|_p^s d\mu_{G(\mathbb{Q}_p)} =$$

$$= \int_{H^+(\mathbb{Q}_p)} \left(\int_{\mathcal{N}_m^+} \cdots \left(\int_{\mathcal{N}_3^+} \left(\int_{\mathcal{N}_2^+} \left(\int_{\mathcal{N}_1^+} |\det \varphi|_p^s d\mu_{\mathcal{N}_1} \right) d\mu_{\mathcal{N}_2} \right) d\mu_{\mathcal{N}_3} \right) \cdots d\mu_{\mathcal{N}_m} \right) d\mu_{H(\mathbb{Q}_p)}$$

where we denote $\mathcal{N}_i := N(\mathbb{Q}_p)_i$ and $\mathcal{N}_i^+ := N^+(\mathbb{Q}_p)_i$, for every $1 \leq i \leq m$. One checks that every $\mathcal{N}_i^+$ depends on $h, u_1, u_2, \ldots, u_{i-1}$, if $\varphi = u_m \cdots u_2 u_1 h$, where $h \in H^+(\mathbb{Q}_p)$ and $u_k \in \mathcal{N}_k$, for every $1 \leq k \leq i-1$. All the subgroups in the decomposition of $N(\mathbb{Q}_p)$ are obviously unipotent as well, which means that their determinants are also 1. This means that computing the inner integrals amounts to determining the measure of the sets $\mathcal{N}_i^+$ in terms of $h, u_1, u_2, \ldots, u_{n-1}$.

2.5 Base Extension

Let K be a number field of degree $d = [K : \mathbb{Q}]$, and let $\mathcal{O}_K$ be its ring of integers. Let L be a $\mathbb{Z}$-Lie algebra of rank r. By base extension we can consider $L \otimes_{\mathbb{Z}} \mathcal{O}_K$ as a $\mathbb{Z}$-Lie algebra of rank rd, and by extension of scalars we can consider also $L_{K,p} = (L \otimes_{\mathbb{Z}} \mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ as a $\mathbb{Q}_p$-Lie algebra of the same rank. Berman-Glazer-Schein give a criterion in [2], under which the pro-isomorphic zeta-function of $L_{K,p}$ can be calculated without a significant extra effort relative to that of L_p itself. Note that the criterion does not necessarily apply for all p. We shall research whether the criterion applies to the $\mathbb{Q}_p$-Lie algebras $\mathcal{L}$ of our work. If so, then $\hat{\zeta}_{\mathcal{L}\otimes\mathcal{O}_K,p}$ will be finitely-uniform. In other words, for each of the finitely many decomposition types of a prime in $\mathcal{O}_K$, there is a rational function in two variables $W \in \mathbb{Q}(X,Y)$ such that $\hat{\zeta}_{\mathcal{L}\otimes\mathcal{O}_K,p}(s) = W(p, p^{-s})$ for all p of that decomposition type.

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