

# homework

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**Proposition 0.1.** *Let  $\varphi_r \in N(\mathbb{Q}_p)$  be a unipotent automorphism of  $\mathcal{L}_{n,p}$  such that the matrix upper blocks  $M_{1k}$ , for all  $2 \leq k \leq r-1$ , are zero matrices. Consider the  $(n-1) \times (n-r)$  matrix  $M_{1r} = (a_{ij})$ . Then,*

1. *Let  $2 \leq r < n-2$ . If  $a_{ij} \neq 0$ , then either  $i = j$  or  $i = j + r - 1$ , and we have the relation  $a_{i+r,i+1} = -a_{ii}$ .*
2. *Let  $r = n-2$ . If  $a_{ij} \neq 0$ , then either  $i = j$  or  $i = j + r - 1$  or  $(i, j) \in \{(1, 2), (n-1, 1)\}$ , with the same relation as above.*

*Proof.* From the relation  $[\varphi_r(e_{k,k+1}), \varphi_r(e_{l,l+1})] = 0$  where  $l > k+1$ , we deduce that  $a_{ij} \neq 0$  only if either  $i = j$  or  $i = j + r - 1$  or  $(i, j) \in \{(r+1, 1), (r+2, 1), (n-r-2, n-r), (n-r-1, n-r)\}$ . If  $r < n-2$  then it follows from the conditions

$$\begin{aligned} [\varphi_r(e_{n-r-2, n-r-1}), \varphi_r(e_{n-r-2, n-r})] &= 0 \\ [\varphi_r(e_{n-r-1, n-r}), \varphi_r(e_{n-r-2, n-r})] &= 0 \\ [\varphi_r(e_{r+1, r+2}), \varphi_r(e_{r+1, r+3})] &= 0 \\ [\varphi_r(e_{r+2, r+3}), \varphi_r(e_{r+1, r+3})] &= 0 \end{aligned}$$

that the four exceptional cases cannot occur. When  $r = n-2$ , we have that  $(n-r-2, n-r) = (0, 2)$  and  $(r+2, 1) = (n, 1)$  so these cases do not exist, but so are the four conditions above, which means that the two remaining cases,  $(r+1, 1) = (n-1, 1)$  and  $(n-r-1, n-r) = (1, 2)$ , do not necessarily vanish.  $\square$

**Proposition 0.2.** *Denote by  $N_r := \{\varphi_r : 2 \leq r \leq n-2\} \subset N(\mathbb{Q}_p)$  the set of all automorphisms of the form described in 0.1, then  $N_r \leq N(\mathbb{Q}_p)$ . Note that  $N_2 = N(\mathbb{Q}_p)$ .*

**Proposition 0.3.** *Let  $2 \leq r \leq n-2$ , and let  $0 \leq k \leq n-r$ , and let  $a \in \mathbb{Q}_p$ . We extend our notation of basis elements to include  $e_{01} = e_{n, n+1} = 0$ .*

1. *There is an automorphism  $\varphi_{r,k}(a) \in N(\mathbb{Q}_p)$  determined by*

$$\varphi_{r,k}(a)(e_{i,i+1}) := \begin{cases} e_{k,k+1} + ae_{k,k+r} & : i = k \\ e_{k+r,k+r+1} - ae_{k+1,k+r+1} & : i = k+r \\ e_{i,i+1} & : i \notin \{k, k+r\} \end{cases}$$

2. Suppose that  $r = n - 2$ , let  $(k, l) \in \{(1, 2), (n - 1, 1)\}$ , and let  $a \in \mathbb{Q}_p$ . There is an automorphism  $\varphi_{n-2,k,l}(a) \in G_n^0(\mathbb{Q}_p)$  determined by

$$\varphi_{n-2,k,l}(a)(e_{i,i+1}) := \begin{cases} e_{k,k+1} + ae_{l,l+r} & : i = k \\ e_{i,i+1} & : i \neq k \end{cases}$$

We denote  $\varphi'_{n-2}(a) := \varphi_{n-2,1,2}(a)$  and  $\varphi''_{n-2}(a) := \varphi_{n-2,n-1,1}(a)$ .

*Proof.* We need to verify that for all  $1 \leq i < j \leq n$  and  $1 \leq l < m \leq n$  we have the following relations

$$[\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{lm})] = \begin{cases} \varphi_{r,k}(a)(e_{im}) & : j = l \\ -\varphi_{r,k}(a)(e_{lj}) & : i = m \\ 0 & : \text{otherwise} \end{cases}$$

We can verify explicitly for  $n = 4$  that these relations are true. Alternatively, Berman did this in [?, §3.3.7]. For  $n > 4$ , let  $m = n$ , then

$$[\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{lm})] = [\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{ln})].$$

If  $i > 1$ , then we consider the inclusion  $\iota : \mathcal{L}_{n-1,p} \hookrightarrow \mathcal{L}_{n,p}$ , mapping each  $e_{i,i+1} \in \mathcal{L}_{n-1,p}$  to  $e_{i+1,i+2} \in \mathcal{L}_{n,p}$  for all  $1 \leq i \leq n - 2$ . By the assumption on  $\mathcal{L}_{n-1,p}$ , we have that

$$\begin{aligned} (\iota \circ \iota^{-1})([\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{ln})]) &= \iota([\iota^{-1}(\varphi_{r,k}(a)(e_{ij})), \iota^{-1}(\varphi_{r,k}(a)(e_{ln}))]) = \\ &= \iota([\varphi_{r,k}(a)(e_{i-1,j-1}), \varphi_{r,k}(a)(e_{l-1,n-1})]) = \\ &= \begin{cases} \iota(\varphi_{r,k}(a)(e_{i-1,n-1})) = \varphi_{r,k}(a)(e_{in}) & : j = l \\ 0 & : j \neq l \end{cases} \end{aligned}$$

If  $i = 1$ , then

$$\begin{aligned} [\varphi_{r,k}(a)(e_{ij}), \varphi_{r,k}(a)(e_{ln})] &= [\varphi_{r,k}(a)(e_{1j}), \varphi_{r,k}(a)(e_{ln})] = \\ &= [\varphi_{r,k}(a)(e_{1j}), [\varphi_{r,k}(a)(e_{l,n-1}), \varphi_{r,k}(a)(e_{n-1,n})]]. \end{aligned}$$

By the Jacobi identity, we have that

$$\begin{aligned} &[\varphi_{r,k}(a)(e_{1j}), [\varphi_{r,k}(a)(e_{l,n-1}), \varphi_{r,k}(a)(e_{n-1,n})]] = \\ &= -[\varphi_{r,k}(a)(e_{n-1,n}), [\varphi_{r,k}(a)(e_{1j}), \varphi_{r,k}(a)(e_{l,n-1})]]. \end{aligned}$$

Now we use the inclusion  $\iota' : \mathcal{L}_{n-1,p} \hookrightarrow \mathcal{L}_{n,p}$ , where  $\iota'(e_{i,i+1}) = e_{i,i+1}$  for all  $1 \leq i \leq n - 1$ , to obtain, same as above, that

$$-[\varphi_{r,k}(a)(e_{n-1,n}), [\varphi_{r,k}(a)(e_{1j}), \varphi_{r,k}(a)(e_{l,n-1})]] = -[\varphi_{r,k}(a)(e_{n-1,n}), \varphi_{r,k}(a)(e_{1,n-1})].$$

To continue, we need to prove the following auxiliary proposition:

**Proposition 0.4.**

$$\begin{aligned}\varphi_{r,k}(a)(e_{k+1-h,k+1}) &= e_{k+1-h,k+1} + ae_{k+1-h,k+r} : h > 0 \\ \varphi_{r,k}(a)(e_{k+r,k+r+h}) &= e_{k+r,k+r+h} - ae_{k+1,k+r+h} : h > 0 \\ \varphi_{r,k}(a)(e_{ij}) &= e_{ij} : i \neq k+r \wedge j \neq k+1\end{aligned}$$

*Proof.* For  $h = 1$ ,  $\varphi_{r,k}(a)(e_{k,k+1}) = e_{k,k+1} + ae_{k,k+r}$ . For  $h' = h > 1$ ,  $\varphi_{r,k}(a)(e_{k+1-h',k+1}) = [\varphi_{r,k}(a)(e_{k-h,k-h+1}), \varphi_{r,k}(a)(e_{k-h+1,k+1})]$ . By the assumption, we have that  $[\varphi_{r,k}(a)(e_{k-h,k-h+1}), \varphi_{r,k}(a)(e_{k-h+1,k+1})] =$

$$= [\varphi_{r,k}(a)(e_{k-h,k-h+1}), e_{k+1-h,k+1} + ae_{k+1-h,k+r}].$$

But for all  $h > 0$ , we have that  $k-h \neq k$  and  $k-h+1 \neq k+1$ , and hence

$$\begin{aligned}[\varphi_{r,k}(a)(e_{k-h,k-h+1}), e_{k+1-h,k+1} + ae_{k+1-h,k+r}] &= \\ = [e_{k-h,k-h+1}, e_{k+1-h,k+1} + ae_{k+1-h,k+r}] &= e_{k-h,k+1} + ae_{k-h,k+r} = \\ = e_{k+1-h',k+1} + ae_{k+1-h',k+r}. &\end{aligned}$$

We prove the two other cases in the same way.  $\square$

By 0.4 we have that

$$\varphi_{r,k}(a)(e_{1,n-1}) = \begin{cases} e_{1,n-1} + ae_{1,n} & : r = 2 \wedge k = n-2 \\ e_{1,n-1} & : \text{otherwise} \end{cases}$$

while  $k+r \neq 1$  for all  $r, k$ . Thus,  $-\varphi_{2,n-2}(a)(e_{n-1,n}), \varphi_{2,n-2}(a)(e_{1,n-1})] = -[\varphi_{2,n-2}(a)(e_{n-1,n}), e_{1,n-1} + ae_{1,n}] = [e_{n-1,n}, e_{1,n-1} + ae_{1,n}] = e_{1n}$ .

For  $(r, k) \notin (2, n-2)$ , if  $k+r = n-1$  then  $-\varphi_{r,k}(a)(e_{n-1,n}), \varphi_{r,k}(a)(e_{1,n-1})] = -[e_{n-1,n} - ae_{n-r,n}, e_{1,n-1}] = e_{1n}$ , otherwise  $-\varphi_{r,k}(a)(e_{n-1,n}), \varphi_{r,k}(a)(e_{1,n-1})] = -[e_{n-1,n}, e_{1,n-1}] = e_{1n}$ .  $\square$

Fix the two parameters  $2 \leq r \leq n-2$  and  $0 \leq k \leq n-r$ , and denote by  $N_{r,k} := \{\varphi_{r,k}(a) : a \in \mathbb{Q}_p\} \subset N(\mathbb{Q}_p)$  the set of all automorphisms of this form. Also denote  $N'_{n-2} := \{\varphi'_{n-2}(a) : a \in \mathbb{Q}_p\}$  and  $N''_{n-2} := \{\varphi''_{n-2}(a) : a \in \mathbb{Q}_p\}$ .

**Proposition 0.0.1.** *Let  $N_{r,k}$ ,  $N'_{n-2}$  and  $N''_{n-2}$  be the subsets defined above, then*

1.  $N_{r,k}, N'_{n-2}, N''_{n-2} \leq N(\mathbb{Q}_p)$ .
2.  $N_{r,k}, N'_{n-2}, N''_{n-2} \cong \mathbb{Q}_p$ .

*Proof.* A simple check shows that these subsets are subgroups of  $N(\mathbb{Q}_p)$ . Define  $\tau_{r,k} : \mathbb{Q}_p \rightarrow N_{r,k}$ . For every  $a, b \in \mathbb{Q}_p$ , it is easy to see that the image of the sum,  $\tau_{r,k}(a+b) = \tau_{r,k}(a) \cdot \tau_{r,k}(b)$ , is the product of the images of  $a$  and  $b$ , and that  $\tau_{r,k}^{-1}(I) = \{0\}$ .  $\square$

The following proposition follows from a simple computation.

**Proposition 0.5.** Consider  $\varphi_r \in N_r$ .

1. If  $r < n - 2$ , denote by  $\psi_r$  the automorphism

$$\psi_r := \varphi_r \circ \varphi_{r,n-r}(-a_{n-r,n-r}) \circ \cdots \circ \varphi_{r,1}(-a_{11}) \circ \varphi_{r,0}(-a_{r+1,1}).$$

Then  $\psi_r \in N_{r+1}$ .

2. If  $r = n - 2$ , denote by  $\psi_{n-2}$  the automorphism

$$\begin{aligned} \psi_{n-2} &:= \varphi_r \circ \varphi'_{n-2}(-a_{12}) \circ \varphi''_{n-2}(-a_{n-1,1}) \circ \\ &\circ \varphi_{n-2,2}(-a_{n-2,2}) \circ \varphi_{n-2,1}(-a_{n-2,1}) \circ \varphi_{n-2,0}(-a_{n-2,0}). \end{aligned}$$

Then  $\psi_{n-2} \in N_{n-1}$ .

*Proof.* By the definition,  $\varphi_r$  has 1 on the main diagonal and all the upper blocks  $M_{1k}$ , for  $2 \leq k \leq r - 1$  are zero matrices. One checks that the composition of  $\varphi_r$  and the chain of compositions  $\prod_{k=1}^{n-r} \varphi_{r,k}(-a_{kk}) \circ \varphi_{r,0}(-a_{r+1,1})$  yields also a matrix with 1 on the main diagonal whose upper blocks  $M_{1k}$ , for all  $2 \leq k \leq r$ , are zero matrices, thus  $\psi_r \in N_{r+1}$ . Same applies for the case  $r = n - 2$ , considering the specific structure of  $M_{1,n-2}$ , as described in the second part of 0.1.  $\square$

**Corollary 0.6.** We have the following decompositions:

1. For all  $2 \leq r < n - 2$ , we have

$$N_r = N_{r+1} \rtimes (N_{r,0} \rtimes (\cdots (N_{r,n-r-1} \rtimes N_{r,n-r}) \cdots)).$$

2. For  $r = n - 2$ , we have

$$N_{n-2} = N_{n-1} \rtimes (N_{n-2,0} \rtimes (N_{n-2,1} \rtimes (N_{n-2,2} \rtimes (N''_{n-2} \rtimes N'_{n-2}))))).$$

*Proof.* This is immediate from Proposition 0.5, since we have that

$$\varphi_r = \psi_r \circ \varphi_{r,0}(-a_{r+1,1}) \circ \varphi_{r,1}(-a_{11}) \circ \cdots \circ \varphi_{r,n-r}(-a_{n-r,n-r}).$$

$\square$

Corollary 0.6 provides a recursive decomposition of the unipotent radical  $N(\mathbb{Q}_p)$  as an iterated semidirect product of  $N_{n-1}$  and subgroups isomorphic to  $\mathbb{Q}_p$ .

As we saw earlier, the calculation of  $\zeta_{L_{n,p}}^\wedge(s)$  requires understanding  $G_n(\mathbb{Z}_p)$  and  $G_n^+(\mathbb{Q}_p)$  first. As  $G_n(\mathbb{Z}_p)$  is a group, its structure is easily deduced from the above.

**Proposition 0.7.** *For all  $n \geq 4$  the group  $G_n^0(\mathbb{Z}_p)$  has the decomposition  $G_n^0(\mathbb{Z}_p) = N(\mathbb{Z}_p) \rtimes H(\mathbb{Z}_p)$ , where*

$$N(\mathbb{Z}_p) := M_{\binom{n}{2}}(\mathbb{Z}_p) \cap N(\mathbb{Q}_p),$$

$$H(\mathbb{Z}_p) := \{\text{diag}(\lambda_1, \lambda_2, \dots, ) : \lambda_1, \dots, \lambda_{n-1} \in \mathbb{Z}_p^*\}.$$

Moreover,  $N(\mathbb{Z}_p)$  itself has the decomposition:

$$\begin{aligned} N(\mathbb{Z}_p) = \tilde{N}_2(\mathbb{Z}_p) = \tilde{N}_{n-1} \rtimes (\tilde{N}_{n-2,0} \rtimes (\tilde{N}_{n-2,1} \rtimes (\tilde{N}_{n-2,2} \rtimes (\tilde{N}_{n-2}'' \rtimes \tilde{N}_{n-2}')))) \rtimes \dots \\ \dots \rtimes (\tilde{N}_{2,0} \rtimes (\dots (\tilde{N}_{2,n-3} \rtimes \tilde{N}_{2,n-2}) \dots)), \end{aligned}$$

where  $\tilde{N}_r = N_r \cap N(\mathbb{Z}_p)$  and  $\tilde{N}_{r,k} = \{\varphi_{r,k}(a) : a \in \mathbb{Z}_p\}$ .

By contrast, describing the structure of the monoid  $G_n^+(\mathbb{Q}_p)$  is expected to be a substantial challenge.

By applying Fubini's theorem for semidirect products of topological groups [?, Proposition 28], we have that

$$\zeta_{L_{n,p}}^\wedge(s) = \int_{G_n^+(\mathbb{Q}_p)} |\det \varphi|_p^s d\mu_{G_n(\mathbb{Z}_p)} \varphi = \int_{H^+(\mathbb{Q}_p)} \left( \int_{N_h^+} |\det uh|_p^s d\mu_{N(\mathbb{Q}_p)} \right) d\mu_{H(\mathbb{Q}_p)},$$

where

$$H^+(\mathbb{Q}_p) := \{\text{diag}(\lambda_1, \dots, \lambda_{n-1}, \lambda_1 \lambda_2, \dots, \lambda_1 \lambda_2 \dots \lambda_{n-2} \lambda_{n-1}) : \lambda_i \in \mathbb{Z}_p \setminus \{0\}\},$$

that is,  $H^+(\mathbb{Q}_p)$  consists of all  $h \in H(\mathbb{Q}_p)$  that appear in the decomposition  $\varphi = uh$  for some  $\varphi \in G_n^+(\mathbb{Q}_p)$ , and, for a given  $h \in H^+(\mathbb{Q}_p)$ , we set  $N_h^+ := \{u \in N(\mathbb{Q}_p) : uh \in G_n^+(\mathbb{Q}_p)\}$ . The integrand is constant on  $N_h^+$ , so computing the inner integral amounts to finding the measure of  $N_h^+$ , which is complicated, but using the decomposition from Corollary 0.6, we can simplify  $N_h^+$  at the price of replacing a single integral by multiple integrals.

Let  $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_m$ , where  $m = \binom{n}{2}$ , be an enumeration of the subgroups

$$\begin{aligned} N_{2,n-2}, N_{2,n-3}, \dots, N_{2,0}, N_{3,n-3}, \dots, N_{3,0}, \dots \\ \dots, N_{n-2,2}, N_{n-2,1}, N_{n-2,0}, N_{n-2}', N_{n-2}'', N_{n-1}. \end{aligned}$$

Every  $\varphi \in G_n^0(\mathbb{Q}_p)$  can be written uniquely as  $\varphi = u_m \dots u_1 h$ , where  $u_i \in \mathcal{N}_i$ . Thus, by Fubini

$$\zeta_{L_{n,p}}^\wedge(s) = \int_{H^+} \int_{\mathcal{N}_1^+(h)} \int_{\mathcal{N}_2^+(h, u_1)} \dots \int_{\mathcal{N}_m^+(h, u_1, \dots, u_{m-1})} |\det h|_p^s d\mu_H d\mu_{\mathcal{N}_1} \dots d\mu_{\mathcal{N}_m},$$

where each  $\mu_{\mathcal{N}_i}$  is Haar measure on  $\mathcal{N}_i(\mathbb{Q}_p)$  normalized so that  $\mu_{\mathcal{N}_i}(\mathcal{N}_i(\mathbb{Z}_p)) = 1$ , and

$$\mathcal{N}_i^+(h, u_1, \dots, u_{i-1}) := \{u_i \in \mathcal{N}_i : \exists u_{i+1}, \dots, u_m \text{ such that } u_m \dots u_1 h \in G_n^+(\mathbb{Q}_p)\}.$$