Abstract

Let G be any group. For any natural number $n \in \mathbb{N}$, let a_n be the number of subgroups $H \leq G$, such that $[G : H] = n$. Assume G is finitely-generated, then $a_n < \infty$, and we can define a ζ-function of the form $\zeta_G(s) := \sum_{i=1}^{\infty} a_n n^{-s}$, where $s \in \mathbb{C}$. Assume, in addition, that G is also nilpotent and torsion-free, then this function has properties of the Riemann ζ-function, mainly the decomposition of ζ to an Euler product of local factors indexed by primes. A version of this ζ-function counts pro-isomorphic subgroups, and an analogous function may be defined for appropriate Lie rings. we study the pro-isomorphic ζ-functions for a family of nilpotent Lie rings of unbounded nilpotency class. We shall compute te automorphism groups of these Lie rings explicitly, prove uniformity of the pro-isomorphic ζ-functions local factors, and aim to determine them explicitly. This study of nilpotent Lie rings can then be reflected in the study of pro-isomorphic subgroups and their associated ζ-functions.

1 Scientific Background

1.1 Introduction

We start our discussion with the following proposition, which stands at the very base of our subject.

Proposition 1.1.1. *Let G be any finitely generated group, and let $n \in \mathbb{N}$ any natural number. Then there is a finite number of subgroups $H \leq G$, such that $[G : H] = n$*

Proof. Let $H \leq G$ be a subgroup, such that $[G : H] = n$, then ${}^{G}\!/_{H} := \{g_1H, g_2H, \ldots, g_nH\}$ is the set containing all left-cosets of H in G. We may consider the action of G by left multiplication on ${}^{G}\!/_{H}$. i.e., for all $g \in G$, and for all left-cosets $g_iH \in {}^{G}\!/_{H}$, we have that $g(g_iH) := (gg_i)H = g_jH$, where $g_jH \in {}^{G}\!/_{H}$ is some left-coset. This means that g maps every index $i \in [n]$ to some index $j \in [n]$, which means that g operates as a permutation on $[n]$. Therefore, there exists a homomorphism $f : G \to \mathcal{S}_n$, from G to the symmetric group of order n. Clearly $g_k \in H$ iff $g_kH = H$, which means that $f(g_k)(k) = k$. This means that $H = \{g \in G : f(g)(k) = k\}$, which clearly shows that the number of subgroups $H \leq G$ of index n is less or equal to the number of maps $f : G \to \mathcal{S}_n$, which are maps from a finitely generated group to a finite group, and since group homomorphisms are uniquely determined by the maps of their generators, it is clear that the number of these maps is finite. $\square$

This proposition gives rise to an entire subject in group theory, called **subgroup growth**. We denote by $a_n(G)$ the number of subgroups of G of index n, and look at the sequence $\{a_n(G)\}$. The subject of subgroup growth aims to relate the properties of this sequence to the algebraic structure of G. For instance, Lubotzky, Mann and Segal showed that $a_n(G)$ grows polynomially iff G is virtually nilpotent of finite rank. That is, G has a finite-index nilpotent subgroup, and all finitely-generated subgroups may be generated by a finite number of generators. This research concentrates on the growth of **pro-isomorphic** subgroups, which we now define.

Definition 1.1.2. *Let G be any group, and let $\mathcal{N} := \{N_i \trianglelefteq G\}_{i\in I}$ the set of all normal subgroups of G. We define a partial order on $\mathcal{N}$, by inclusion, and assign G an infinite set of indices. The inverse limit $\widehat{G} = \varprojlim\{{}^{G}/_{N_k}\}_{k\in I} := \{(h_k)_{k\in I} \in \prod_{k\in I} {}^{G}/_{N_k} : \pi_{ji}(h_j) = h_i, \forall i \leq j\}$, where $\pi_{ji} := {}^{G}/_{N_j} \to {}^{G}/_{N_i}$ is the natural projection, is called the* ***profinite closure*** *of G.*

Definition 1.1.3. *Let G be any group. a subgroup $H \leq G$ is called* ***pro-isomorphic****, if $\widehat{H} \cong \widehat{G}$.*

Definition 1.1.4. *Let G be any group, and let $\hat{a_n}(G) := \#\{H \leq G : \widehat{H} \cong \widehat{G}, [G:H] = n\}$, in words, the number of pro-isomorphic subgroups of G, of index n. Assume $\hat{a_n}(G) < \infty$, for all n. The* ***pro-isomorphic ζ-function*** *of G is defined by $\hat{\zeta_G}(s) := \sum_{n=1}^{\infty} \hat{a_n}(G)n^{-s}$, for $s \in \mathbb{C}$.*

It has been found that if $\hat{a_n}(G)$ grows polynomially, then $\hat{\zeta}_G(s)$ converges on some right half-plane of $\mathbb{C}$. For instance, we take the group $G = (\mathbb{Z}, +)$. $\mathbb{Z}$ is an abelian group, and every $H \leq \mathbb{Z}$ is of the form $H = n\mathbb{Z} = \langle n \rangle$, for some $n \in \mathbb{N}$, which means that $H \cong \mathbb{Z}$, as both are infinite cyclic groups, and so, $\widehat{H} \cong \widehat{\mathbb{Z}}$. Since we have only one $\mathbb{Z}$-subgroup of index n, for every $n \in \mathbb{N}$, then $a_n(\mathbb{Z}) = \hat{a_n}(\mathbb{Z}) = 1$, thus, its pro-isomorphic ζ-function is $\hat{\zeta_{\mathbb{Z}}} = \sum_{i=1}^{\infty} n^{-s} = \zeta(s)$, the Riemann ζ-function, which is known to converge for $Re(s) > 1$.

After establishing the basic definitions, we observe a fact that is a major motivation for this research, which says that the Riemann ζ-function decomposes to an infinite product of local ζ_p-functions, that is, $\zeta(s) =$

$\prod_p \zeta_p(s) = \prod_p \sum_{k=0}^{\infty} p^{-ks} = \prod_p \frac{1}{1-p^{-s}}$, where the product runs over all the prime numbers. Following this fact, regarding the Riemann *zeta*-function, we observe that for any finitely-generated, nilpotent and torsion-free group, G, we have the same decomposition as above, for the pro-isomorphic ζ-function, $\hat{\zeta_G}(s) = \prod_p \hat{\zeta_{G,p}}(s)$, where $\hat{\zeta_{G,p}}(s) := \sum_{k=0}^{\infty} a_{p^{ks}}(G)p^{-ks}$ We hereby bring several basic definitions of group nilpotency, which are very important for this research.

Definition 1.1.5. *Let G be any group, then the* ***lower central series*** *of G is a sequence of subgroups of G, defined by the recursive rule, $G_n := [G, G_{n-1}]$, for every $n \in \mathbb{N}$, where $G_0 := G$. We recall that $[G, G_n] \leq G$ is the subgroup of commutators, $\{gg_n g^{-1} g_n^{-1} : g \in G, g_n \in G_n\}$*

Definition 1.1.6. *Let G be any group. the* ***nilpotency class*** *of G is $min\{n \in \mathbb{N} : G_n = [G, G_{n-1}] = \{e\}\}$, in words, the smallest natural number, such that the subgroup of commutators of the form $[G, G_n]$ is the trivial group. We can extend this definition, and say that the trivial group nilpotency class is* 0.

Definition 1.1.7. *Let G be a group. If G if of a finite nilpotency class, $n \in \mathbb{N}$, then G is said to be a* ***nilpotent*** *group.*

1.2 Linearization

For finitely-generated torsion-free nilpotent groups, G, we associate nilpotent Lie algebras over $\mathbb{Z}$. This, in general, is called the **Maltsev correspondance**. If L is a $\mathbb{Z}$-Lie algebra, namely a free $\mathbb{Z}$-module of a finite rank, with a Lie bracket, then consider the (finite) number $\hat{a_n}(L)$ of subalgebras $M \leq L$, such that $M \otimes \mathbb{Z}_p \cong L \otimes \mathbb{Z}_p$, for all primes p. The Dirichlet series, $\hat{\zeta_L}(S) := \sum_{n=1}^{\infty} \hat{a_n}(L)n^{-1}$, is called the **pro-isomorphic ζ-function** of L. By the Maltsev correspondance, to every finitely-generated, nilpotent, torsion-free group, G, one may associate a Lie algebra, $L(G)$, such that $\hat{\zeta_{G,p}}(s) = \hat{\zeta_{L,p}}(s)$, for but finitely many primes, p. If G has nilpotency class 2, one may obtain the equality for all primes. Let L be a Lie algebra, and fix a $\mathbb{Z}$-basis, $\mathcal{B} = \{b_1, \ldots, b_d\}$. Let $\mathcal{L}_p = L \otimes_{\mathbb{Z}} \mathbb{Q}_p$, for any p. This is a $] = \mathbb{Q}_p$-Lie algebra, and our choice of basis allows us to identify the automorphism group $G(\mathbb{Q}_p) = Aut_{\mathbb{Q}_p}(\mathcal{L}_p)$ with a subgroup of $GL_d(\mathbb{Q}_p)$. Note that $\mathcal{L}_p$ contains a $\mathbb{Z}_p$-lattice, $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. If $\varphi \in G(\mathbb{Q}_p)$, then $\varphi(L_p) = L_p$ iff $\varphi \in G(\mathbb{Z}_p) = G(\mathbb{Q}_p) \cap GL_d(\mathbb{Z}_p)$. Here $GL_d(\mathbb{Z}_p)$ is the group of invertible

$d \times d$ matrices over $\mathbb{Z}_p$. Similarly, $\varphi(L_p) \subseteq L_p$ iff $G^+(\mathbb{Q}_p) := G(\mathbb{Q}_p) \cap M_d(\mathbb{Z}_p)$, where $M_d(\mathbb{Z}_p)$ is the collection of $d \times d$ matrices with entries in $\mathbb{Z}_p$. Note that $G^+(\mathbb{Q}_p)$ is a monoid, not a group.

Denote by $G(\mathbb{Z}_p)g$, where $g \in G^+(\mathbb{Q}_p)$, a right-coset of $G(\mathbb{Z}_p)$, one checks that $G^+(\mathbb{Q}_p) = \bigsqcup_{i=1}^{m} G(\mathbb{Z}_p)g_i$, where $[G^+(\mathbb{Q}_p) : G(\mathbb{Z}_p)] = m$, in words, the monoid $G^+(\mathbb{Q}_p)$ is a disjoint union of right-cosets of $G(\mathbb{Z}_p)$.

The discussion above reveals the construction we base our research upon. We observe that there is a bijection between $G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)$ and $\{M \leq L_p : M \cong L_p\}$, in words, we have a bijective map between each right-coset of $G(\mathbb{Z}_p)$ in $G^+(\mathbb{Q}_p)$ and each L_p-subalgebra which is isomorphic to L_p itself. This bijection is canonical, by taking $M = \varphi(L_p)$, for some $\varphi \in G(\mathbb{Z}_p)g$. M is a L_p-subalgebra, because $M = \varphi(L_p) \subseteq L_p$. One checks that for every $\psi \in G(\mathbb{Z}_p)g$, we have that $\psi(L_p) = \varphi(L_p) = M$. We end this part, as a preparation for the final part of this technical background review, with the following result, which states that for each right-coset, $G(\mathbb{Z}_p)g$, if $M = \varphi(L_p)$, where $\varphi \in G(\mathbb{Z}_p)g$, then $[L_p : M] = |det\varphi|_p^{-1}$, and therefore,

$$\hat{\zeta_{L,p}}(s) = \sum_{\substack{M \leq L_p \\ M \cong L_p}} [L_p : M]^{-s} = \sum_{G(\mathbb{Z}_p)\varphi \in G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)} |det\varphi|_p^s.$$

1.3 p-adic Integration

In this final part of the technical background review, we finally get to the motivation for all the construction we have presented in the first parts. We now define a very central object for our research. We shall assume, without proof, the existence of such an object, under the prerequisites of the definition.

Definition 1.3.1. *Let Γ be a locally compact topological group, i.e., for all $\gamma \in \Gamma$, there is an open neighborhood of γ, U_γ, and a compact subset K_γ, such that $U_\gamma \subset K_\gamma$. Then there is a measure μ, with the following property: for any measurable subset, $U \subseteq \Gamma$, and any $\gamma \in \Gamma$, $\mu(U\gamma) = \mu(U)$, where $U\gamma := \{u\gamma : u \in U\}$. Such a measure μ is called a **right Haar measure**, and is unique up to multiplication by a non-zero constant.*

Equipped with the right Haar measure, we can finally make use of the construction from above. We start by claiming, without proof, that for every prime number p, the group $G(\mathbb{Q}_p)$ is a locally compact topological group. Therefore, it has a unique right Haar measure. We also claim that the right Haar measure has the property that $\mu(G(\mathbb{Z}_p)) = 1$, and that the measure on all the right-cosets of $G(\mathbb{Z}_p)$ equals to the measure on $G(\mathbb{Z}_p)$ itself, i.e., for every $g \in G^+(\mathbb{Q}_p)$, we have that $\mu(G(\mathbb{Z}_p)g) = \mu(G(\mathbb{Z}_p)) = 1$. With this observation, we go directly to the calculation of the p-adic norm of the determinant of every L_p-automorphism, as a p-adic integral over our measure space. First, we observe that given any L_p-automorphism in some right-coset, $\varphi \in G(\mathbb{Z}_p)\varphi$, we have that $|det\varphi|_p^s = \int_{G(\mathbb{Z}_p)\varphi} |det\varphi|_p^s d\mu$, because $\mu(G(\mathbb{Z}_p)\varphi) = 1$, and $|det\varphi|_p^{-1}$ is fixed on $G(\mathbb{Z}_p)\varphi$.

Going back to our desired function, we observe that $\hat{\zeta_{L,p}}(s) = \sum_{G(\mathbb{Z}_p)\varphi\in G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)} |det\varphi|_p^s = \sum_{G(\mathbb{Z}_p)\varphi\in G(\mathbb{Z}_p)\backslash G^+(\mathbb{Q}_p)} \int_{G(\mathbb{Z}_p)\varphi} |det\varphi|_p^s d\mu = \int_{G^+(\mathbb{Z}_p)} |det\varphi|_p^s d\mu$. This integral is the main object we shall study, in this research. We end this part, of the technical background review, by a definition and a theorem, which stand in the center of our research goals.

Definition 1.3.2. *Let L be a $\mathbb{Z}$-algebra. $\hat{\zeta_L}(s)$ is called **unified**, if there exists a rational function, $W \in \mathbb{Q}(X, Y)$, such that for every prime number p, the local function $\zeta_{L,p}(s) = W(p, p^{-1})$.*

Here, $\mathbb{Q}(X, Y) := \{\frac{f}{g} : f, g \in \mathbb{Q}[X, Y]\}$, the field of rational functions in two variables.

The term unified expresses that the rational function depends, for every p, only on p and its inverse.

Theorem 1.3.3. *Let p be a prime number, $s \in \mathbb{C}$, then there exists a rational function, $w_p(s) := \frac{f(x)}{g(x)}$, where $f, g \in \mathbb{Z}_p[x]$, which satisfies $\hat{\zeta_{L,p}}(s) = w_p(p^{-s})$.*

2 Research Goals and Methodology

2.1 The unipotent group U_n

We start by this following definition.

Definition 2.1.1. *Let $\mathcal{R}$ be a commutative ring. Let $U_n(\mathcal{R}) \leq GL_n(\mathcal{R})$ be the subgroup of upper unitriangular matrices, i.e.* $U_n(\mathcal{R}) = \left\{ \begin{pmatrix} 1 & a_{12} & \\ & \ddots & \ddots \\ & & 1 \end{pmatrix} \right\}$, *then $U_n(\mathcal{R})$ is called a **unipotent group**.*

Looking into the structure of $U_n(\mathcal{R})$, we observe that we can find a set of generators, $\mathcal{U}_n(\mathcal{R})$. Denote by E_{ij}, where $i < j$, an elmentary matrix of the form $\begin{pmatrix} 1 & 0 & \dots & 0 \\ & \ddots & 1 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \in U_n(\mathcal{R})$, where, in addition to the main diagonal, only the element in row i and column j is 1, and all the other elements are 0. One checks that if $i < j$ and $k < l$, then the commutator $[E_{ij}, E_{kl}] = \begin{cases} E_{il}, & j = k \\ -E_{kj}, & i = l \\ I_n, & \text{otherwise} \end{cases}$

In addition, it is easy to observe that $E_{ij}^m = \begin{pmatrix} 1 & 0 & \dots & 0 \\ & \ddots & m & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$, for every $m \in \mathbb{Z}$, by simple induction. All this means that we can generate $U_n(\mathcal{R})$ entirely by the set $\mathcal{U}_n := \{E_{1,2}, \dots, E_{n-1,n}\}$. From this, it is also clear that $U_n(\mathcal{R})$ is nilpotent, of nilpotency class n, because the longest chain of commutators in $U_n(\mathcal{R})$, i.e., chaining all the elements in the set of generators, $\mathcal{U}_n$, by the order of their indices, yields $[E_{1,2}, [E_{2,3}, [\dots, [E_{n-1,n}]]]] = E_{1,n}$, which means that $\gamma_{n-1} U_n(\mathcal{R}) = \{E_{1,n}\}$, which means that $\gamma_n U_n(\mathcal{R}) = I_n$.

This also shows that $U_n(\mathcal{R})$ is torsion-free, iff $\mathcal{R}$ itself is torsion-free, as can be readily seen from the fact that E_{ij}^m has an m in row i and column j, and $m \neq 0$ iff $\mathcal{R}$ is torsion-free, for instance, $\mathcal{R} = \mathbb{Z}$.

These facts, regarding $U_n(\mathbb{Z})$ place it as a group of our interest, for this research, and bring us next to its associated Lie algebra.

2.2 The algebra $L_{p,n}$

We start with E_{ij} from above and define matrices of the form $e_{ij} = E_{ij} - I_n$, in words, e_{ij} is obtained by replacing all the 1 on the main diagonal with 0. Then clearly, $e_{ij}e_{kl} = e_{il}$ where $j = k$, $e_{ij}e_{kl} = -e_{kj}$ where $i = j$, and $e_{ij}e_{kl} = 0$ in any other case.

Corollary 2.2.1. *Let $\mathcal{B}_n$ be the set $\{e_{ij} : i < j\}$, of all the matrices of the form described in 2.2.1. Then $\mathcal{B}_n$, with the standard matrix addition, and a multiplication operation $*$, defined by $e_{ij} * e_{jk} = e_{ij}e_{jk} - e_{jk}e_{ij}$, is a* *basis for a Lie algerba over $\mathbb{Z}_p$, which shall be denoted by $L_n(\mathbb{Z}_p)$, or L_n, for abbreviation, which is the $\mathbb{Z}_p$-algebra of all the matrices $A \in \mathcal{M}_n(\mathbb{Z}_p)$, with 0 on the main diagonal. The multiplication operation $*$ shall be denoted by Lie Brackets, that is, $e_{ij} * e_{jk} = [e_{ij}, e_{jk}]$.*

Proof. Since we have defined the multiplication operation as the standard Lie brackets, for matrix Lie algebras, i.e., $[A, B] = AB - BA$, for all the matrices A, B in the algebra, one easily checks that all the axioms of a Lie algebra hold for this definition. Obviously, L_n is a $\mathbb{Z}_p$-span of $\mathcal{B}_n$, since every matrix of the form $A = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1n} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is a linear combination of matrices of $\mathcal{B}_n$, i.e., $A = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_{ij} e_{ij}$. We can observe that if $B = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} b_{ij} e_{ij} = 0_n$, then clearly all the b_{ij} are 0. We conclude that $\mathcal{B}_n$ is a basis for L_n. □

Proposition 2.2.2. *Let p be a prime number, and let $n \in \mathbb{N}$ be any natural number, then dim $L_n(\mathbb{Z}_p) = \binom{n}{2}$*

Proof. From 2.2.2, we have that a basis for $L_n(\mathbb{Z}_p)$ is the set of all e_{ij}, where $i < j$. For each row $1 \leq i \leq n-1$, we have $n - i$ elements of the form e_{ij}, which gives, in total, $\frac{n(n-1)}{2} = \frac{n!}{2!(n-2)!} = \binom{n}{2}$ elements of the the basis. □

Proposition 2.2.3. *Let p be a prime number, and let $n \in \mathbb{N}$ be any natural number, then $L_n(\mathbb{Z}_p)$ is a nilpotent Lie algebra.*

Proof. It is followed directly from 2.2.1, and from 2.1.9, since $[e_{ij}, e_{jk}] = [E_{ij}, E_{jk}] - I_n = E_{ik} - I_n = e_{ik}$, where the first brackets are Lie Brackets of $L_n(\mathbb{Z}_p)$, and the second brackets are a group commutator of $U_n(\mathbb{Z}_p)$. □

By considering the behavior of $\mathcal{L}_n(\mathbb{Q}_p)$ under the Lie brackets, we can learn about the structure of $Aut_{\mathbb{Q}_p}(\mathcal{L}_n)$. As a basic fact, every $\mathcal{L}_n(\mathbb{Q}_p)$-automorphism φ must obey the $\mathcal{L}_n$ Lie brackets, meaning that for all $x, y \in \mathcal{L}_n$, we must have that $\varphi([x, y]) = [\varphi(x), \varphi(y)]$. Let $B = \{b_1, b_2, \dots, b_m\}$ be a basis for $\mathcal{L}_n$, we have that $x = \sum_{i=1}^{m} \lambda_i b_i$, and $y = \sum_{i=1}^{m} \rho_i b_i$, so $\varphi([x, y]) = [\varphi(\sum_{i=1}^{m} \lambda_i b_i), \varphi(\sum_{i=1}^{m} \rho_i b_i)] = [\sum_{i=1}^{m} \varphi(\lambda_i b_i), \sum_{i=1}^{m} \varphi(\rho_i b_i)] = [\sum_{i=1}^{m} \lambda_i \varphi(b_i), \sum_{i=1}^{m} \rho_i \varphi(b_i)] = \sum_{i=1}^{m} \sum_{j=1}^{m} [\lambda_i \varphi(b_i), \rho_j \varphi(b_j)] = \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_i \rho_j [\varphi(b_i), \varphi(b_j)]$. This technique can be demonstrated in the most simple case, which is the Heisenberg group.

2.3 The Heisenberg group

Definition 2.3.1. *The* ***Heisenberg group*** *is the unipotent group of* 3×3 *matrices, over* $\mathbb{Q}_p$*, namely* $U_3(\mathbb{Q}_p)$*. Every matrix* $A \in U_3$ *is of the form*

$$\begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

where $a_1, a_2, a_3 \in \mathbb{Q}_p$*.*

The $\mathbb{Q}_p$-algebra associated with U_3 consists of matrices of the form

$$A - I_3 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} = a_{12}e_{12} + a_{13}e_{13} + a_{23}e_{23}$$

. Let $\varphi \in Aut_{\mathbb{Q}_p}(\mathcal{L}_p)$ be an $\mathcal{L}_{p,n}$-automorphism. The image of every $A \in \mathcal{L}_{p,n}$, as a linear combination of elements of the basis, is a linear combination of the images of these elements. So, let $v = (x, y, z) = xe_{12} + y_{23} + z_{13}$, where $x, y, z \in \mathbb{Q}_p$, we have that $\varphi(v) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$ $\begin{pmatrix} a_{11}x + a_{12}x + a_{13}x & a_{21}y + a_{22}y + a_{23}y & a_{31}z + a_{32}z + a_{33}z \end{pmatrix} =$ $\begin{pmatrix} (a_{11} + a_{12} + a_{13})x & (a_{21} + a_{22} + a_{23})y & (a_{31} + a_{32} + a_{33})z \end{pmatrix} = \begin{pmatrix} \varphi(x) & \varphi(y) & \varphi(z) \end{pmatrix}$, which means that

$$\varphi(e_{12}) = a_{11}e_{12} + a_{12}e_{23} + a_{13}e_{13}$$

$$\varphi(e_{23}) = a_{21}e_{12} + a_{22}e_{23} + a_{23}e_{13}$$

$$\varphi(e_{13}) = a_{31}e_{12} + a_{32}e_{23} + a_{33}e_{13}$$

. We want to find relations between the elements of φ. Considering the fact that $[\varphi(x), \varphi(y)] = \varphi([x, y]) =$, we observe that the Lie brackets on images of any two commuting elements of the basis give 0, as they are images of 0, i.e., for every $x, y \in \mathcal{L}_n$, such that $[x, y] = 0$, we have that $[\varphi(x), \varphi(y)] = \varphi([x, y]) = \varphi(0) = 0$. Hence, the only images that do not vanish under Lie brackets are $[\varphi(e_{12}), \varphi(e_{23})] = [a_{11}e_{12} + a_{12}e_{23} + a_{13}e_{13}, a_{21}e_{12} + a_{22}e_{23} + a_{23}e_{13}] = a_{11}a_{21}[e_{12}, e_{12}] + a_{11}a_{22}[e_{12}, e_{23}] + \cdots + a_{13}a_{23}[e_{13}, e_{13}] = \varphi([e_{12}, e_{23}]) = \varphi(e_{13}) = a_{31}e_{12} + a_{32}e_{23} + a_{33}e_{13}$. Considering again only the non-vanishing

Lie brackets, we have that $[\varphi(e_{12}), \varphi(e_{23})] = a_{11}a_{22}[e_{12}, e_{23}] + a_{12}a_{21}[e_{23}, e_{12}] = a_{11}a_{22}e_{13} - a_{12}a_{21}e_{13} = (a_{11}a_{22} - a_{12}a_{21})e_{13} = a_{31}e_{12} + a_{32}e_{23} + a_{33}e_{13} = \varphi(e_{13})$. Comparing the scalars, for the three elements of the basis, gives the following relations,

$$a_{31} = 0$$

$$a_{32} = 0$$

$$a_{33} = (a_{11}a_{22} - a_{12}a_{21}) \neq 0$$

which gives the following matrix,

$$\varphi(v) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & det(A) \end{pmatrix}$$

where A is the minor

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

We can observe that for every $v \in \mathcal{L}_{p,n}$, writing $M = \varphi(v)$ lines in the following way,

$$M = \begin{pmatrix} \varphi(e_{12}) \\ \varphi(e_{23}) \\ \varphi(e_{n-1n}) \\ \varphi(e_{13}) \\ \vdots \\ \varphi(e_{n-1n}) \\ \vdots \\ \varphi(e_{1n}) \end{pmatrix}$$

where $m = \binom{n}{2}$, divides M to a block matrix,

$$M = \left(\begin{array}{c|c|c|c|c} M_{11} & M_{12} & \dots & M_{1n-1} & M_{1n} \\ \hline M_{21} & M_{22} & \dots & M_{2n-1} & M_{2n} \\ \hline \vdots & \vdots & \dots & \vdots & \vdots \\ \hline M_{n1} & M_{n2} & \dots & M_{nn-1} & M_{nn} \end{array}\right)$$

where $M_{ij} \in \mathcal{M}_{k \times l}(\mathbb{Q}_p)$, $k =$dim$(\gamma_i\mathcal{L}), l =dim(\gamma_j\mathcal{L})$. From this, we can understand that the blocks on the main diagonal of M are squared matrices,

$A_{ii} \in \mathcal{M}_{n-i}$. From the calculation on $\mathcal{L}_{p,3}$, we understand also that any element $e_{ii+k} \in \gamma_k \mathcal{L}_{p,n}$ must vanish in the images of elements from higher nilpotency classes, i.e. $\varphi(e_{i,i+l})$, where $l > k$, which means that all the elements under every squared block on the main diagonal must be zero, so M has the form,

$$M = \left(\begin{array}{c|c|c|c|c}
M_{11} & M_{12} & M_{13} \quad \dots & M_{1m-1} & M_{1m} \\ \hline
0 & M_{22} & M_{23} \quad \dots & M_{2m-1} & M_{2m} \\ \hline
\vdots & \vdots & \vdots \quad \dots & \vdots & \vdots \\ \hline
0 & 0 & 0 \quad \dots & M_{2m-1} & M_{2m} \\ \hline
0 & 0 & 0 \quad \dots & 0 & M_{mm}
\end{array}\right)$$

We observe that the matrix M_{ij} blocks represent quotients of the form ${}^{\gamma_i \mathcal{L}_{p,n}}/_{\gamma_{i+1}\mathcal{L}_{p,n}}, {}^{\gamma_j \mathcal{L}_{p,n}}/_{\gamma_{j+1}\mathcal{L}_{p,n}}$. We shall state, as a fact, that the block M_{11} is either diagonal or anti-diagonal, i.e.,

$$M_{11} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

or

$$M_{11} = \begin{pmatrix} & & & \lambda_1 \\ & & \lambda_2 & \\ & \iddots & & \\ \lambda_n & & & \end{pmatrix}$$

In the case of an anti-diagonal block, we have the following proposition,

Proposition 2.3.2. *Let p be a prime number, and let $n \in \mathbb{N}$. $B_n = \{e_1, \dots, e_{m-1}\}$, where $m = \binom{n}{2}$. Then, the map $\eta_n : B_n \to B_n$, defined by $\eta_n(e_i) := e_{m-i}$ is a $\mathcal{L}_{p,n}$-automorphism, which is also an involution.*

Proof. Clearly, η_n is the anti-diagonal $m \times m$ matrix,

$$\eta_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$$

η_n is an invertible matrix, which operates on any vector

$$v = (a_1, a_2, \ldots, a_{m-1}) = \sum_{i=1}^{m-1} a_i e_i$$

in the following way,

$$\eta_n(v) = \eta_n\left(\sum_{i=1}^{m-1} a_i e_i\right) = \begin{pmatrix} a_1 & a_2 & \ldots & a_{m-1} \end{pmatrix} \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} = \begin{pmatrix} a_{m-1} & a_{m-2} & \ldots & a_1 \end{pmatrix} =$$

$$\begin{pmatrix} \eta_n(a_1) & \eta_n(a_2) & \ldots & \eta_n(a_{m-1}) \end{pmatrix} = \sum_{i=1}^{m-1} \eta_n(a_i e_i) = \sum_{i=1}^{m-1} a_i \eta_n(e_i)$$

And, $\eta_n^2(v) = \eta_n(\eta_n(v)) = \eta_n\left(\eta_n\left(\sum_{i=1}^{m-1} a_i e_i\right)\right) = \eta_n\left(\sum_{i=1}^{m-1} a_i \eta_n(e_i)\right) = \sum_{i=1}^{m-1} a_i \eta_n^2(e_i) = \sum_{i=1}^{m-1} a_i \eta_n(e_{n-i}) = \sum_{i=1}^{m-1} a_i e_i$ □

From this proposition, we realize that if M_{11} is anti-diagonal, then $\eta_n \varphi$ is the automorphism which has that M_{11} is diagonal.

Proposition 2.3.3. *Let p be a prime number, and let $n \in \mathbb{N}$, and let $M = \varphi \in \mathcal{L}_{p,n}$. Then, all the blocks on the main diagonal, $M_{ii}, \ldots, M_{n-1n-1}$, are diagonal, of the form,*

$$M = \varphi = \begin{pmatrix} \lambda_1 & & & & & & & & & \\ & \lambda_2 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & \lambda_n & & & & & & \\ & & & & \lambda_1\lambda_2 & & & & & \\ & & & & & \lambda_2\lambda_3 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \lambda_{n-1}\lambda_n & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \lambda_1\lambda_2\cdots\lambda_n \end{pmatrix}$$

Proof. By simple induction. We have already assumed that M_{11} is diagonal. Every sequential block M_{ii} contains the coefficients of elements of ${}^{\gamma_i \mathcal{L}_{p,n}}\!/\!_{\gamma_{i+1}\mathcal{L}_{p,n}}$ as summands in images of elements of the same quotient algebra. So, $\varphi(e_{ii+2}) = \sum_{i=1}^{n-2} a_{ii+2}e_{ii+2}$, but $e_{ii+2} = [e_{ii+1}, e_{i+1i+2}]$, so $\varphi(e_{ii+2}) = [\varphi(e_{ii+1}), \varphi(e_{i+1i+2})]$, hence, $\lambda_{i+2} = a_{ii+2} = a_{ii+1}a_{i+1i+2} = \lambda_i\lambda_{i+1}$, which proves the proposition. □

Proposition 2.3.4. *Let $n \in \mathbb{N}$, and let*

$$A_n = \begin{pmatrix} \lambda_1 & & & & & & & & & \\ & \lambda_2 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & \lambda_n & & & & & & \\ & & & & \lambda_1\lambda_2 & & & & & \\ & & & & & \lambda_2\lambda_3 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \lambda_{n-1}\lambda_n & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \lambda_1\lambda_2\cdots\lambda_n \end{pmatrix}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{Q}_p$, then, $det(A_n) = \prod_{i=1}^{n} \lambda_i^{i(n+1-i)}$.

Proof. We observe that the determinants, for $n = 1, 2, 3, \dots$, form a recursive sequence,

$$det(A_1) = \lambda_1$$

$$det(A_2) = det(A_1)\lambda_1\lambda_2^2$$

$$det(A_3) = det(A_2)\lambda_1\lambda_2^2\lambda_3^3$$

$$\vdots$$

$$det(A_n) = det(A_{n-1})\lambda_1\lambda_2^2\lambda_3^3\cdots\lambda_n^n$$

Calculating the general element, $a_n = det(A_n)$, we see that we have n times λ_1, $n-1$ times λ_2^2, $n-2$ times λ_3^3, and so forth. In general, we have $n-i+1$ times λ_i^i, which means that we have $i(n-i+1)$ times λ_i, and in total, $a_n = det(A_n) = \prod_{i=1}^{n} \lambda_i^{i(n+1-i)}$. □

This means that every $M = \varphi \in \mathcal{L}_{p,n}$ is of the form,

$$M = \varphi = \left(\begin{array}{cccc|cccc|cc|c|c}
\lambda_1 & & & & & & & & & & & \\
& \lambda_2 & & & & & & & & & & \\
& & \ddots & & & M_{12} & & & M_{13} & \dots & M_{1m-1} & M_{1m} \\
& & & \lambda_n & & & & & & & & \\
\hline
& & & & \lambda_1\lambda_2 & & & & & & & \\
& & & & & \lambda_2\lambda_3 & & & & & & \\
& 0 & & & & & \ddots & & M_{23} & \dots & M_{2m-1} & M_{2m} \\
& & & & & & & \lambda_{n-1}\lambda_n & & & & \\
\hline
& \vdots & & & & \vdots & & & \vdots & \dots & \vdots & \vdots \\
\hline
& 0 & & & & 0 & & & 0 & \dots & M_{2m-1} & M_{2m} \\
\hline
& 0 & & & & 0 & & & 0 & \dots & 0 & \lambda 1\lambda 2\cdots\lambda_n
\end{array}\right)$$

The above discussion gives rise to the decomposition of each $\varphi \in \mathcal{L}_{p,n}$ to two matrices, one is the diagonal matrix

$$h = \begin{pmatrix}
\lambda_1 & & & & & & & & & \\
& \lambda_2 & & & & & & & & \\
& & \ddots & & & & & & & \\
& & & \lambda_n & & & & & & \\
& & & & \lambda_1\lambda_2 & & & & & \\
& & & & & \lambda_2\lambda_3 & & & & \\
& & & & & & \ddots & & & \\
& & & & & & & \lambda_{n-1}\lambda_n & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & \lambda_1\lambda_2\cdots\lambda_n
\end{pmatrix}$$

and the other matrix is

$$n = \begin{pmatrix} 1 & * & * & * & * & * & * & * \\ & 1 & * & * & * & * & * & * \\ & & \ddots & * & * & * & * & * \\ & & & 1 & * & * & * & * \\ & & & & 1 & * & * & * \\ & & & & & 1 & * & * \\ & & & & & & \ddots & * \\ & & & & & & & 1 \end{pmatrix}$$

So, we have the following proposition,

Proposition 2.3.5. *Let p be a prime number, and let $n \in \mathbb{N}$, and let $M = \varphi \in \mathcal{L}_{p,n}$. Then, $M = \varphi = nh$, where n and h are of the above form.*

Proof. Trivially, h is an invertible matrix, and its inverse is the matrix

$$h^{-1} = \begin{pmatrix} \lambda_1^{-1} & & & & & & & & & \\ & \lambda_2^{-1} & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & \lambda_n^{-1} & & & & & & \\ & & & & (\lambda_1\lambda_2)^{-1} & & & & & \\ & & & & & (\lambda_2\lambda_3)^{-1} & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & (\lambda_{n-1}\lambda_n)^{-1} & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & (\lambda_1\lambda_2\cdots\lambda_n)^{-1} \end{pmatrix}$$

Easy to check that $n = Mh^{-1}$ is also an invertible matrix, with 1 on the main diagonal, and 0 below it. $\square$

We observe that all the matrices with non-zero elements on the main diagonal, and 0 everywhere else form an abelian subgroup of $G_n(\mathbb{Q}_p)$, since multiplying such matrices yields a matrix of the same specification. Let

$$h_\alpha = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_m \end{pmatrix}, h_\beta = \begin{pmatrix} \beta_1 & & & \\ & \beta_2 & & \\ & & \ddots & \\ & & & \beta_m \end{pmatrix}$$

Then,

$$h_\alpha h_\beta = \begin{pmatrix} \alpha_1\beta_1 & & & \\ & \alpha_2\beta_2 & & \\ & & \ddots & \\ & & & \alpha_m\beta_m \end{pmatrix} = \begin{pmatrix} \beta_1\alpha_1 & & & \\ & \beta_2\alpha_2 & & \\ & & \ddots & \\ & & & \beta_m\alpha_m \end{pmatrix} = h_\beta h_\alpha$$

Obviously, this subgroup, which we shall denote as $H < G_n(\mathbb{Q}_p)$ is not normal, as we observe by taking the n matrix described above, and multiplying $A = nhn^{-1}$, clearly $A \notin H$. On the other hand, the set of all n matrices if a normal subgroup of $G_n(\mathbb{Q}_p)$, because if

$$n_\alpha = \begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1m} \\ & 1 & \alpha_{23} & \dots & \alpha_{2m} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & \alpha_{m-1m} \\ & & & & 1 \end{pmatrix}, n_\beta = \begin{pmatrix} 1 & \beta_{12} & \beta_{13} & \dots & \beta_{1m} \\ & 1 & \beta_{23} & \dots & \beta_{2m} \\ & & \ddots & \vdots & \vdots \\ & & & 1 & \beta_{m-1m} \\ & & & & 1 \end{pmatrix}$$

Then

$$n_\alpha n_\beta = \begin{pmatrix} 1 & \alpha_{12}+\beta_{12} & * & \dots & * \\ & 1 & * & \dots & * \\ & & \ddots & \vdots & \vdots \\ & & & 1 & \alpha_{m-1m}+\beta_{m-1m} \\ & & & & 1 \end{pmatrix}$$

which proves that all the n matrices form a subgroup, which we shall denote by $N \in G_n(\mathbb{Q}_p)$. taking any matrix, $g \in G_n(\mathbb{Q}_p)$, and taking the product $A = gng^{-1}$, if we look at the main diagonals, we see that the product is of the general form

$$gng^{-1} = \begin{pmatrix} \lambda_1 & a_{12} & \dots & a_{1m} \\ 0 & \lambda_2 & \dots & a_{2m} \\ & & \ddots & * \\ & & & \lambda_m \end{pmatrix} \begin{pmatrix} 1 & b_{12} & \dots & b_{1m} \\ 0 & 1 & \dots & b_{2m} \\ & & \ddots & * \\ & & & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^{-1} & c_{12} & \dots & c_{1m} \\ 0 & \lambda_2^{-1} & \dots & c_{2m} \\ & & \ddots & * \\ & & & \lambda_m^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} 1 & d_{12} & \dots & d_{1m} \\ 0 & 1 & \dots & d_{2m} \\ & & \ddots & * \\ & & & 1 \end{pmatrix} \in N$$

So, $N \lhd G_n(\mathbb{Q}_p)$ is a normal subgroup. This discussion gives rise to the decomposition of $G_n(\mathbb{Q}_p)$. Since only N is a normal subgroup of $G_n(\mathbb{Q}_p)$, we decompose $G_n(\mathbb{Q}_p)$ to a semi-direct product, $G \cong N \rtimes H$, where the map $\phi : H \to Aut(N)$, given by $\phi(h)(n) := hnh^{-1}$, for every $h \in H$, and $n \in N$, is a homomorphism, as we can see by the fact that for every $h_1, h_2 \in H$, and for every $n \in N$, $\phi(h_1)\phi(h_2)(n) = h_1 n(h_2 n h_2^{-1})h_1^{-1} = h_1 h_2 n h_2^{-1} h_1^{-1} = (h_1 h_2)n(h_1 h_2)^{-1} = \phi(h1h2)(n)$ This means that calculating the integral, for $G_n(\mathbb{Q}_p)$, reduces to calculating a double integral, $\displaystyle\int_{N \rtimes H}$. We mean to show in the research that the normal subgroup N can itself be decomposed to a semi-direct product of several subgroups, thus simplyifing the integration.

By 1.2.3, we have that any $L_p(\mathbb{Z}_p)$-automorphism must be in $G_n(\mathbb{Z}_p)$, in words, any $\varphi \in G(\mathbb{Z}_p)$ is an invertible matrix with elements in $\mathbb{Z}_p$. Our goal is to find a way to compute $G(\mathbb{Z}_p)$, the automorphism group of $L_n(\mathbb{Z}_p)$, for any $n \in \mathbb{N}$. After finding a general formula for this calculation, we shall be able to show a way to compute the n-multiple p-adic integral of the form $\displaystyle\int\int\cdots\int\int_{D_1 \times D_2 \cdots \times D_{n-1} \times D_n} f(h_1, h_2, \dots, h_{n-1}, h_n) d(\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n)$, where D_i is the set of $G(\mathbb{Z}_p)$-cosets, for $G(\mathbb{Z}_p)$, the group of $\mathbb{Z}_p$-automorphisms on the algebra $L_i(\mathbb{Z}_p)$, and h_i is any element of this group, and μ_i is the Haar measure on this group. By Fubini, this multiple integral can be calculated as the iterated integral

$$\int_{D_n} \left(\int_{D_{n-1}} \cdots \left(\int_{D_2} \left(\int_{D_1} f(h_1, h_2, \dots, h_{n-1}, h_n) d\mu_1 \right) d\mu_2 \right) \dots d\mu_{n-1} \right) d\mu_n$$

. Alternatively, if we do not find an explicit formula for this calculation, we will show the general approach for this calculation, and prove the necessary conditions for its validity.

3 Notations

- $\mathbb{Z}_p$, the ring of p-adic integers.
- $\mathbb{Q}_p$, the fraction field of $\mathbb{Z}_p$.
- L_p, a $\mathbb{Z}_p$-algebra over the ring of p-adic integers.
- $\mathcal{L}_p$, a $\mathbb{Q}_p$-algebra, over the fraction field of $\mathbb{Z}_p$.

- $G(L_p) := Aut_{\mathbb{Z}_p}(L_p)$, the group of $\mathbb{Z}_p$-automorphisms of L_p.
- $G(\mathcal{L}_p) := Aut_{\mathbb{Q}_p}(\mathcal{L}_p)$, the group of $\mathbb{Q}_p$-automorphisms of $\mathcal{L}_p$.