1 The computation of $G_n(\mathbb{Q}_p)$

1.1 The computation of the first block M_{11}

Proposition 1.1.1. *Let $x = \sum_{i=1}^{n-1} \lambda_i e_{i,i+1}$, where $\lambda_i \in \mathbb{Q}_p$ are not all zero. Then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \mathfrak{l}(x) + \mathfrak{m}(x)$, where $\mathfrak{l}(x)$ is the number of sequences of consecutive non-zero coefficients of the form $\lambda_j, \lambda_{j+1}, \ldots, \lambda_{j+k-1}, \lambda_{j+k}$ and $\lambda_{j-1} = \lambda_{j+k+1} = 0$ (that is, the sequences are separated by one of more zero coefficients)*[1]*, and $\mathfrak{m}(x)$ is the number of zero coefficients $\lambda_j = 0$, such that also $\lambda_{j-1} = \lambda_{j+1} = 0$.*

Proof. Let $y = \sum_{i=1}^{n-1} \mu_i e_{i,i+1}$, where $\lambda_i \in \mathbb{Q}_p$. For every $1 \leq i \leq n-1$, denote by $(\mathfrak{C}_i)$ the constraint equation $[\lambda_i e_{i,i+1}, \mu_{i+1} e_{i+1,i+2}] - [\lambda_{i+1} e_{i+1,i+2}, \mu_i e_{i,i+1}] = (\lambda_i \mu_{i+1} - \lambda_{i+1}\mu_i) e_{i,i+2} = 0$, and it is clear that $y \in \mathcal{C}_{\gamma_1/\gamma_3}(x)$ if and only if all the $(\mathfrak{C}_i)$ constraints are satisfied. We can see that each μ_i participates in two constraints, $(\mathfrak{C}_{i-1})$ and $(\mathfrak{C}_i)$, that is, $\lambda_{i-1}\mu_i - \lambda_i \mu_{i-1} = \lambda_i \mu_{i+1} - \lambda_{i+1}\mu_i = 0$. We have several options. If $\lambda_i = 0$, then $\lambda_i \mu_{i-1} = \lambda_i \mu_{i+1} = 0$, hence by constraint $(\mathfrak{C}_{i-1})$ we have that $\lambda_{i-1}\mu_i = 0$, and by constraint $(\mathfrak{C}_i)$ we have that $\lambda_{i+1}\mu_i = 0$. Hence, if either λ_{i-1} or λ_{i+1} are non-zero, then $\mu_i = 0$. But if $\lambda_{i-1} = \lambda_{i+1} = 0$, then both constraints are satisfied for any choice of μ_i, which increases $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)$ by 1. By simple induction, we prove that for any sequence of k consecutive zero λ coefficients of x, namely $\lambda_{j+1} = \lambda_{j+2} = \cdots = \lambda_{j+k} = 0$, for $1 \leq j \leq n-1$, we have that the sequence $\mu_{j+2}, \mu_{j+3}, \ldots, \mu_{j+k-1}$ of μ coefficients of y is a sequence of any scalars, thus $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x)$ is increased by $k-2$. For $k = 3$, we just proved that if $\lambda_{i-1} = \lambda_i = \lambda_{i+1} = 0$, then μ_i can be any scalar. For $k+1$, we look at the sequence of $k+1$ zero coefficients, $\lambda_{j+1} = \lambda_{j+2} = \cdots = \lambda_{j+k} = \lambda_{j+k+1} = 0$. By constraints $(\mathfrak{C}_{j+k-1})$ and $(\mathfrak{C}_{j+k})$, we have that $\lambda_{j+k-1}\mu_{j+k} - \lambda_{j+k}\mu_{j+k-1} = \lambda_{j+k}\mu_{j+k+1} - \lambda_{j+k+1}\mu_{j+k} = 0$, and since $\lambda_{j+k-1} = \lambda_{j+k} = \lambda_{j+k+1} = 0$, we have that μ_{j+k} can be any scalar, as we proved earlier. By the assumption, we have that $\mu_{j+2}, \ldots, \mu_{j+k-1}$ can be any scalars, and by adding μ_{j+k} to this sequence, we prove the induction step. Let $1 \leq j \leq n-1$ and $1 \leq k \leq n-1-j$ be two indices, such that $\lambda_{j-1} = \lambda_{j+k+1} = 0$, and $\lambda_j, \lambda_{j+1}, \ldots, \lambda_{j+k-1}, \lambda_{j+k}$ are all non-zero, then by constraints $(\mathfrak{C}_j), (\mathfrak{C}_{j+1}), \ldots, (\mathfrak{C}_{m-1})$, we have that $\mu_m = \frac{\lambda_m}{\lambda_{m-1}}\mu_{m-1} = \frac{\lambda_m}{\lambda_{m-1}}\frac{\lambda_{m-1}}{\lambda_{m-2}}\mu_{m-2} = \frac{\lambda_m}{\lambda_{m-2}}\mu_{m-2} = \cdots = \frac{\lambda_m}{\lambda_j}\mu_j$, for every $j+1 \leq m \leq j+k-1$, which means that for any choice of the first coefficient

[1] We extend our notation of indices, to include also the case where $j = 1$ or $j+k = n-1$, and define that $\lambda_{j-1} = \lambda_0 = 0$ or $\lambda_{j+k+1} = \lambda_n = 0$, respectively

in the sequence, namely μ_j, all the next μ coefficients of the sequence, with indices from $j+1$ to $j+k$, depend on μ_j. $\square$

Corollary 1.1.2. *Let $\mathcal{L}_{n,p}$ be the $\mathbb{Q}_p$-Lie algebra associated with $\mathcal{U}_n(\mathbb{Z})$. If $n \geq 5$, then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim {}^{\gamma_1}/_{\gamma_3} - 1$ if and only if $x \in \{\lambda e_{12} + \gamma_2 \mathcal{L}_{n,p}\}$ or $x \in \{\lambda e_{n-1,n} + \gamma_2 \mathcal{L}_{n,p}\}$, for a non-zero scalar $\lambda \in \mathbb{Q}_p$. If $n = 4$, then $\dim \mathcal{C}_{\gamma_1/\gamma_3}(x) = \dim {}^{\gamma_1}/_{\gamma_3} - 1$ if and only if $x \in \{\lambda e_{12} + \mu e_{34} + \gamma_2 \mathcal{L}_{n,p}\}$, for $\lambda, \mu \in \mathbb{Q}_p$ not both zero.*

Proof. Let $z = \lambda_{j,j+2} e_{j,j+2}$, where $1 \leq j \leq n-2$ and $\lambda_{j,j+2} \in \mathbb{Q}_p$, then for every $w \in {}^{\gamma_1}/_{\gamma_3}$, either z commutes with w or $[z,w] \in \gamma_3 \mathcal{L}_{n,p}$, which means that $\lambda_{j,j+2} e_{j,j+2} \in \mathcal{C}_{\gamma_1/\gamma_3}$, for every $1 \leq j \leq n-2$. Hence, ${}^{\gamma_2}/_{\gamma_3} = \langle e_{13}, e_{24}, \ldots, e_{n-2,n} \rangle \subset \mathcal{C}_{\gamma_1/\gamma_3}(x)$. Therefore, we only need to discuss elements of the quotient ${}^{\gamma_1}/_{\gamma_2}$, for the purpose of this proof. Suppose that $x = \lambda_1 e_{12} + z$, where $z \in \gamma_2 \mathcal{L}_{n,p}$, then we have one sequence of non-zero coefficients, namely λ_1, and we have $n-2$ zero coefficients $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1} = 0$, from which $n-3$ are between two other zeros. Hence, by 1.1.1, we have that $\mathcal{C}_{\gamma_1/\gamma_2}(x) = 1 + (n-3) = n-2 = (n-1)-1 = \dim {}^{\gamma_1}/_{\gamma_2} - 1$. Similarly, the same goes also for $x = \lambda_{n-1} e_{n-1,n} + z$. Suppose that $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = \dim {}^{\gamma_1}/_{\gamma_2} - 1$, but $x = \sum_{i=1}^{n-1} \lambda_i e_{i,i+1}$, such that either of the following options is true:

1. there is more than one sequence of consecutive non-zero coefficients in the linear combination that forms x.

2. there is one sequence of consecutive non-zero coefficients, but at least one of those coefficients has index $2 \leq j \leq n-2$, meaning it is not λ_1 nor λ_{n-1}.

For the second option, we start by fixing one index $2 \leq j \leq n-2$, and assume that $x = \lambda_j e_{j,j+1}$. The number of zero coefficients in x is $n-1-1 = n-2$, but λ_j and the zeros in indices $j-1, j+1$ are neighboring, hence $m_1 = n-2-2 = n-4$, and then $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = l_1 + m_1 = 1 + n - 4 = n-3 < n-2 = \dim {}^{\gamma_1}/_{\gamma_2} - 1$. We denote by k the length of the sequence of consecutive non-zero parameters, and prove that for any $k > 0$, where at least one non-zero coefficient λ_j lies in $2 \leq j \leq n-2$, $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) < n-2$, by simple induction on k. For $k = 1$, we have just shown that. For $k > 1$, there are $k-1$ additional zeros that are replaced by non-zero coefficients, where except for λ_{j-1} and λ_{j+1}, all the other zeros were originally lying between two other zeros. If the original sequence was $\lambda_2 e_{23}$ or $\lambda_{n-2} e_{n-2,n-1}$, and the new sequence is $\lambda_1 e_{12}, \lambda_2 e_{23}$ or $\lambda_{n-2} e_{n-2,n-1}, \lambda_{n-1} e_{n-1,n}$, respectively, then $m_k = m_1$, but clearly, in any other case, $m_k < m_1$, while $l_k = l_1 = 1$ at any case. by the assumption, for the original sequence, $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = l_1 + m_1 < n-2$, hence for the new sequence, $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = l_k + m_k \leq l_1 + m_1 = n-3 < n-2$. Now we check the first option, starting from the case where $x = \lambda_1 e_{12} + \lambda_{n-1} e_{n-1,n}$. In this case, $l_2 = 2$ and the number of zeros is $n-1-2 = n-3$, but λ_1 and the zero in index 2 are neighboring, and so are λ_{n-1} and the zero in index $n-2$, hence $m_2 = n-3-2 = n-5$ zeros are lying between two other zeros, therefore $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = l_2 + m_2 = n-5+2 = n-3 < n-2$. if we add another non-zero coefficient, then it must lie in some index $2 \leq j \leq n-2$, for which we have already proved that $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) < n-2$, which completes the proof for $n \geq 5$. For $n = 4$, we can check explicitly. Assume $x = \lambda e_{12} + \mu e_{34}$, denote an element in the centralizer of x by $y = \rho e_{12} + \tau e_{23} + \nu e_{34}$, and we observe that $[x, y] = [\lambda e_{12}, \tau e_{23}] + [\mu e_{34}, \tau e_{23}] = \lambda \tau e_{13} - \tau \mu e_{24} = 0$, hence $\tau = 0$, while $\rho = *$ and $\nu = *$, so $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = 2 = \dim {}^{\gamma_1}/_{\gamma_2} - 1$, as requested, and it is readily seen that even if either $\lambda = 0$ or $\mu = 0$, but not both, then τ still has to be zero, in order to satisfy either $\tau\mu = 0$ or $\lambda\tau = 0$, respectively, and ρ, ν can still be anything, which means that in either case, where the coefficient of e_{23} is zero but $x \neq 0$, we have $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = 2$. Assume $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = \dim {}^{\gamma_1}/_{\gamma_2} - 1 = 3 - 1 = 2$, then if x is not of the suggested form, it means that $x = \lambda e_{12} + \sigma e_{23} + \mu e_{34}$, where $\sigma \neq 0$ and either λ or μ or both can be zero. If $x = \lambda e_{12} + \sigma e_{23} + \mu e_{34}$ and all coefficients are non-zero, then for every $y \in \mathcal{C}_{\gamma_1/\gamma_2}(x)$ denoted by $y = \rho e_{12} + \tau e_{23} + \nu e_{34}$, we have $[x, y] = [\lambda e_{12}, \tau e_{23}] + [\sigma e_{23}, \rho e_{12}] + [\sigma e_{23}, \nu e_{34}] + [\mu e_{34}, \tau e_{23}] = (\lambda\tau - \sigma\rho) e_{13} + (\sigma\nu - \mu\tau) e_{24}$, hence $\tau = \frac{\sigma}{\lambda}\rho$ and $\nu = \frac{\mu}{\sigma}\tau = \frac{\mu}{\sigma}\frac{\sigma}{\lambda}\rho = \frac{\mu}{\lambda}\rho$, but this means that $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = 1$, because both τ and ν depend on ρ. If either λ or μ or both are zero, then either $\sigma\rho$ or $\sigma\mu$ or both are zero, which means that ρ or ν or both are zero, since $\sigma \neq 0$, but this means that either $y = \tau e_{23} + \frac{\mu}{\sigma}\tau e_{34}$ or $y = \frac{\lambda}{\sigma}\tau e_{12} + \tau e_{23}$ or $y = \tau e_{23}$, respectively. Therefore, in either case, where $\sigma \neq 0$, we have $\dim \mathcal{C}_{\gamma_1/\gamma_2}(x) = 1$, which completes the proof for $n = 4$. $\square$

Corollary 1.1.3. *Let $\mathcal{L}_{n,p}$ be a $\mathbb{Q}_p$-Lie algebra, where $n \geq 4$, and let $\varphi \in G_n(\mathbb{Q}_p)$ be an $\mathcal{L}_{n,p}$-automorphism, then $\varphi_{11}(e_{12}) = \lambda_1 e_{12}$ and $\varphi_{11}(e_{n,n-1}) = \lambda_{n-1} e_{n-1,n}$, or $\varphi_{11}(e_{12}) = \lambda_{n-1} e_{n-1,n}$ and $\varphi_{11}(e_{n,n-1}) = \lambda_1 e_{1,2}$.*

Proof. We look at the centralizer of e_{12} in the quotient ${}^{\gamma_1}/_{\gamma_3}$, namely $\mathcal{C}_{\gamma_1/\gamma_3}(e_{12})$. Clearly, for any $e_{i,i+2} \in {}^{\gamma_2}/_{\gamma_3}$, we have that $[e_{12}, e_{i,i+2}]$ is either zero, or $i = 2$ and then $[e_{12}, e_{24}] = e_{14} \in \gamma_3 \mathcal{L}_{n,p}$, which vanishes in the quotient ${}^{\gamma_1}/_{\gamma_3}$, which means that in either case it is zero in this quotient. Therefore, we look only at elements $e_{i,i+1} \in {}^{\gamma_1}\!/\!_{\gamma_2}$. It is readily seen that every element of the form $e_{i,i+1}$ where $i \neq 2$ commutes with e_{12}, hence $\mathcal{C}_{\gamma_1/\gamma_2}(e_{12}) = \langle e_{12}, e_{34}, e_{45}, \ldots, e_{n-2,n-1}, e_{n-1,n} \rangle$, so $\dim \mathcal{C}_{\gamma_1/\gamma_2}(e_{12}) = \dim {}^{\gamma_1}\!/\!_{\gamma_2} - 1$, but since φ_{11} is an automorphism, it must preserve the dimension of the centralizer, meaning $\dim \mathcal{C}_{\gamma_1/\gamma_2}(\varphi_{11}(e_{12})) = \dim \mathcal{C}_{\gamma_1/\gamma_2}(e_{12}) = \dim {}^{\gamma_1}\!/\!_{\gamma_2} - 1$. But by corollary 1.1.2, if $n \geq 5$, then $\varphi_{11}(e_{12}) = \lambda e_{12}$ or $\varphi_{11}(e_{12}) = \lambda e_{n-1,n}$, and it is readily seen that the same applies also for $\varphi_{11}(e_{n-1,n})$, and since φ is injective, then clearly, if $\varphi_{11}(e_{12}) = \lambda e_{12}$ then $\varphi_{11}(e_{n-1,n}) = \lambda e_{n-1,n}$, and if $\varphi_{11}(e_{12}) = \lambda e_{n-1,n}$ then $\varphi_{11}(e_{n-1,n}) = \lambda e_{12}$. If $n = 4$, then by the same corollary, $\varphi_{11}(e_{12}) = \lambda e_{12} + \mu e_{34}$, where λ and μ are not both zero, which means that the same proof does not hold. Therefore, we now look at the centralizer of e_{12} in the algebra $\mathcal{L}_{4,p}$ itself. We denote by $\mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$ the centralizer of e_{12} in the algebra, which is $\mathcal{C}_{\mathcal{L}_{4,p}}(e_{12}) = \langle e_{12}, e_{34}, e_{13}, e_{14} \rangle$, so $\dim \mathcal{C}_{\mathcal{L}_{4,p}}(e_{12}) = 4$. Denote by $x = \varphi(e_{12}) = \lambda_{12}e_{12}+\lambda_{23}e_{23}+\lambda_{34}e_{34}+\lambda_{13}e_{13}+\lambda_{24}e_{24}+\lambda_{14}e_{14} \in \mathcal{L}_{4,p}$, and denote by $y = \mu_{12}e_{12}+\mu_{23}e_{23}+\mu_{34}e_{34}+\mu_{13}e_{13}+\mu_{24}e_{24}+\mu_{14}e_{14} \in \mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$, an element in the centralizer of e_{12}, hence $[x, y] = (\lambda_{12}\mu_{23} - \lambda_{23}\mu_{12})e_{13} + (\lambda_{23}\mu_{34} - \lambda_{34}\mu_{23})e_{24} + (\lambda_{12}\mu_{24} - \lambda_{24}\mu_{12} + \lambda_{13}\mu_{34} - \lambda_{34}\mu_{13})e_{14} = 0$. Assume all the coefficients of the linear combination that forms x are non-zero. Then, as seen earlier, we have that $\mu_{23} = \frac{\lambda_{23}}{\lambda_{12}}\mu_{12}$, and $\mu_{34} = \frac{\lambda_{34}}{\lambda_{23}}\mu_{23} = \frac{\lambda_{34}}{\lambda_{23}}\frac{\lambda_{23}}{\lambda_{12}}\mu_{12} = \frac{\lambda_{34}}{\lambda_{12}}\mu_{12}$, and also $\lambda_{12}\mu_{24} - \lambda_{24}\mu_{12} + \lambda_{13}\mu_{34} - \lambda_{34}\mu_{13} = 0$, which means that $\mu_{24} = \frac{\lambda_{24}\mu_{12}+\lambda_{13}\mu_{34}-\lambda_{34}\mu_{13}}{\lambda_{12}} = \frac{\lambda_{24}\mu_{12}+\lambda_{13}\frac{\lambda_{34}}{\lambda_{12}}\mu_{12}-\lambda_{34}\mu_{13}}{\lambda_{12}}$, hence we can choose freely μ_{12}, μ_{13} and μ_{14}, while μ_{23} and μ_{34} depend on μ_{12}, and μ_{24} depends on μ_{12} and μ_{13}, which means that $\dim \mathcal{C}_{\mathcal{L}_{4,p}}(y) = 3 < 4 = \dim \mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$. Assume that all the coefficients of x are non-zero, except for $\lambda_{23} = 0$, then $\lambda_{12}\mu_{23}$ and $\lambda_{34}\mu_{23}$ must vanish, hence $\mu_{23} = 0$, but then μ_{34} does not depend on μ_{23}, which implies that it does not depend on μ_{12} either, and can be chosen freely, hence there is no change in the dimension of $\mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$ from the general case. Now we assume $x = \lambda_{12}e_{12} + z$, where $z \in \gamma_2\mathcal{L}_{4,p}$, and observe the three equations from above with the current assumption. The second equation $\lambda_{23}\mu_{34} - \lambda_{34}\mu_{23} = 0$ completely falls, which from the other two we obtain that $\lambda_{12}\mu_{23}$ and $\lambda_{12}\mu_{24}$ must vanish, which means that $\mu_{23} = \mu_{24} = 0$, while μ_{12}, μ_{34}, μ_{13} and μ_{14} can be chosen freely, which means that $\dim \mathcal{C}_{\mathcal{L}_{4,p}}(y) = 4 = \dim \mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$. One checks that the same applies also for $\varphi(e_{12}) = \lambda_{34}e_{34} + z$, and that no other linear combination of x satisfies that $\dim \mathcal{C}_{\mathcal{L}_{4,p}}(\varphi(e_{12})) = \dim \mathcal{C}_{\mathcal{L}_{4,p}}(e_{12})$. $\square$