

**Exercise** Let  $\{E_{i,j}\}_{i < j}$  be the set of all elementary matrices, of this form. Prove that  $E_{i,j}^{-1} = (b_{l,k})$  is  $E_{i,j} = (a_{l,k})$ , when we substitute  $a_{i,j} = 1$  with  $b_{i,j} = -1$

**Proof** We can see that directly from the fact that if we multiply  $E_{i,j}^{-1}$  by  $E_{i,j}$  from the left then  $E_{i,j}$  is operating on  $E_{i,j}^{-1}$  by adding row  $j$  to row  $i$ . So, in the product matrix,  $(c_{l,k})$ , in order to have 1 on the main diagonal, we need them to exist on the main diagonal of  $E_{i,j}^{-1}$ , to begin with. Now, in order to have  $c_{i,j} = 0$ , we need to have the addition of  $j$  to  $i$  giving  $c_{i,j} = a_{i,j} + b_{i,j} = 0 \Rightarrow b_{i,j} = -a_{i,j} = -1$

**Exercise** Prove that if  $(a_{ij}) = E_{i,j}$ ,  $i < j$  is an elementary matrix, then  $\forall m \in (N)$ ,  $E_{i,j}^m$  is  $E_{i,j}$ , but with  $a_{ij} = m$

$$E_{i,j} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$E_{i,j}^2 = E_{i,j} \cdot E_{i,j}$$

Since  $E_{i,j}$  is an elementary matrix, then it operates on the right matrix as an addition of row  $j$  to row  $i$

So,

$$E_{i,j}^2 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 2 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

We assume this is true for all  $E_{i,j}^m$ , now we prove for  $E_{i,j}^{m+1}$

$$E_{i,j}^{m+1} = E_{i,j} \cdot E_{i,j}^m = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}^m$$

(by the assumption)

$$= \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & m & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & m+1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

## Commutators of elementary matrices

Let  $\{E_{i,j}\}_{i < j}$  be the set of all elementary matrices of this form.

**Exercise**  $(a_{l,m}) = E_{i,j}^{-1}$  is the matrix with 1 on the main diagonal, and  $-1$  in  $a_{i,j}$

**Proof** We can see that directly from the fact that in order to have

$$(c_{l,m}) = (a_{l,m}) \cdot (b_{l,m}) = E_{i,j} \cdot E_{i,j}^{-1} = I,$$

we need to have  $c_{i,j} = 0$ , which means that adding row  $j$  to row  $i$ , in  $E_{i,j}^{-1}$  (by the left multiplication of  $E_{i,j}$ )

must give  $a_{i,j} + b_{i,j} = c_{i,j} = 0 \Rightarrow b_{i,j} = -a_{i,j} = -1$

**Exercise**  $[E_{i,j}, E_{j,k}] = E_{i,k}$

**Proof**  $E_{i,j}$  is operating from left on  $E_{j,k}$  by addition of row  $j$  to row  $i$ , so, the product matrix,  $(a_{l,m} = E_{i,j} \cdot E_{j,k})$  has 1 on the main diagonal and in  $a_{j,k}, a_{i,j}, a_{i,k}$

$E_{i,j}^{-1}$  is operating from left on  $E_{j,k}^{-1}$  by subtraction of row  $j$  from row  $i$ , so, the product matrix,  $(b_{l,m} = E_{i,j}^{-1} \cdot E_{j,k}^{-1})$  has 1 on the main diagonal and in  $b_{i,k}$ , and  $-1$  in  $b_{j,k}, b_{i,j}$

Multiplying  $(a_{l,m} \cdot (b_{l,m}))$  yields a product matrix,  $(c_{l,m})$  with 1 on the main diagonal, and,

since  $a_{i,i} = a_{i,j} = a_{i,k} = 1$ , with all other cells in row  $j$  being 0, and since  $b_{i,k} = b_{k,k} = 1$ , and  $b_{j,k} = -1$ , multiplying row  $(a_{l,m})_i$  by column  $(b_{l,m})_k$  yields the value  $c_{i,k} = b_{i,k} + b_{j,k} + b_{k,k} = 1 - 1 + 1 = 1$

We can see that multiplying  $(a_{l,m})_i \cdot (b_{l,m})_j$  yields  $c_{i,j} = a_{i,i} \cdot b_{i,j} + a_{i,j} \cdot b_{j,j} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

And, we can see that multiplying  $(a_{l,m})_j \cdot (b_{l,m})_k$  yields  $c_{j,k} = a_{j,j} \cdot b_{j,k} + a_{j,k} \cdot b_{k,k} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

For example,  $n = 4$ ,

$$E_{1,2} \cdot E_{2,3} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
E_{1,2}^{-1} \cdot E_{2,3}^{-1} &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
[E_{1,2} \cdot E_{2,3}] &= E_{1,2} \cdot E_{2,3} \cdot E_{1,2}^{-1} \cdot E_{2,3}^{-1} = \\
&= \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{1,3}
\end{aligned}$$

**Exercise**  $[E_{i,j}, E_{l,k}] = I$ , where  $j \neq l$

**Proof**  $E_{i,j}$  is operating from left on  $E_{l,k}$  by addition of row  $j$  to row  $i$ , so, the product matrix,  $(a_{n,m} = E_{i,j} \cdot E_{l,k})$  has 1 on the main diagonal and in  $a_{l,k}, a_{i,j}$

$E_{i,j}^{-1}$  is operating from left on  $E_{l,k}^{-1}$  by subtraction of row  $j$  from row  $i$ , so, the product matrix,  $(b_{n,m} = E_{i,j}^{-1} \cdot E_{l,k}^{-1})$  has 1 on the main diagonal, and  $-1$  in  $b_{l,k}, b_{i,j}$

We can see that multiplying  $(a_{n,m})_i \cdot (b_{n,m})_j$  yields  $c_{i,j} = a_{i,i} \cdot b_{i,j} + a_{i,j} \cdot b_{j,j} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

And, we can see that multiplying  $(a_{n,m})_l \cdot (b_{n,m})_k$  yields  $c_{l,k} = a_{l,l} \cdot b_{l,k} + a_{l,k} \cdot b_{k,k} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$