

**Exercise** Let  $\{E_{i,j}\}_{i < j}$  be the set of all elementary matrices, of this form. Prove that  $E_{i,j}^{-1} = (b_{l,k})$  is  $E_{i,j} = (a_{l,k})$ , when we substitute  $a_{i,j} = 1$  with  $b_{i,j} = -1$

**Proof** We can see that directly from the fact that if we multiply  $E_{i,j}^{-1}$  by  $E_{i,j}$  from the left then  $E_{i,j}$  is operating on  $E_{i,j}^{-1}$  by adding row  $j$  to row  $i$ . So, in the product matrix,  $(c_{l,k})$ , in order to have 1 on the main diagonal, we need them to exist on the main diagonal of  $E_{i,j}^{-1}$ , to begin with. Now, in order to have  $c_{i,j} = 0$ , we need to have the addition of  $j$  to  $i$  giving  $c_{i,j} = a_{i,j} + b_{i,j} = 0 \Rightarrow b_{i,j} = -a_{i,j} = -1$

**Exercise** Prove that if  $(a_{ij}) = E_{i,j}$ ,  $i < j$  is an elementary matrix, then  $\forall m \in (N)$ ,  $E_{i,j}^m$  is  $E_{i,j}$ , but with  $a_{ij} = m$

$$E_{i,j} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$E_{i,j}^2 = E_{i,j} \cdot E_{i,j}$$

Since  $E_{i,j}$  is an elementary matrix, then it operates on the right matrix as an addition of row  $j$  to row  $i$

So,

$$E_{i,j}^2 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 2 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

We assume this is true for all  $E_{i,j}^m$ , now we prove for  $E_{i,j}^{m+1}$

$$E_{i,j}^{m+1} = E_{i,j} \cdot E_{i,j}^m = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}^m$$

(by the assumption)

$$= \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & m & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & m+1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

## Commutators of elementary matrices

Let  $\{E_{i,j}\}_{i < j}$  be the set of all elementary matrices of this form.

**Exercise**  $(a_{l,m}) = E_{i,j}^{-1}$  is the matrix with 1 on the main diagonal, and  $-1$  in  $a_{i,j}$

**Proof** We can see that directly from the fact that in order to have

$$(c_{l,m}) = (a_{l,m}) \cdot (b_{l,m}) = E_{i,j} \cdot E_{i,j}^{-1} = I,$$

we need to have  $c_{i,j} = 0$ , which means that adding row  $j$  to row  $i$ , in  $E_{i,j}^{-1}$  (by the left multiplication of  $E_{i,j}$ )

must give  $a_{i,j} + b_{i,j} = c_{i,j} = 0 \Rightarrow b_{i,j} = -a_{i,j} = -1$

**Exercise**  $[E_{i,j}, E_{j,k}] = E_{i,k}$

**Proof**  $E_{i,j}$  is operating from left on  $E_{j,k}$  by addition of row  $j$  to row  $i$ , so, the product matrix,  $(a_{l,m}) = E_{i,j} \cdot E_{j,k}$  has 1 on the main diagonal and in  $a_{j,k}, a_{i,j}, a_{i,k}$

$E_{i,j}^{-1}$  is operating from left on  $E_{j,k}^{-1}$  by subtraction of row  $j$  from row  $i$ , so, the product matrix,  $(b_{l,m}) = E_{i,j}^{-1} \cdot E_{j,k}^{-1}$  has 1 on the main diagonal and in  $b_{i,k}$ , and  $-1$  in  $b_{j,k}, b_{i,j}$

Multiplying  $(a_{l,m}) \cdot (b_{l,m})$  yields a product matrix,  $(c_{l,m})$  with 1 on the main diagonal, and,

since  $a_{i,i} = a_{i,j} = a_{i,k} = 1$ , with all other cells in row  $j$  being 0, and since  $b_{i,k} = b_{k,k} = 1$ , and  $b_{j,k} = -1$ , multiplying row  $(a_{l,m})_i$  by column  $(b_{l,m})_k$  yields the value  $c_{i,k} = b_{i,k} + b_{j,k} + b_{k,k} = 1 - 1 + 1 = 1$

We can see that multiplying  $(a_{l,m})_i \cdot (b_{l,m})_j$  yields  $c_{i,j} = a_{i,i} \cdot b_{i,j} + a_{i,j} \cdot b_{j,j} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

And, we can see that multiplying  $(a_{l,m})_j \cdot (b_{l,m})_k$  yields  $c_{j,k} = a_{j,j} \cdot b_{j,k} + a_{j,k} \cdot b_{k,k} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

### **Conclusion**

$$[E_{j,k}, E_{i,j}] = E_{j,k} \cdot E_{i,j} \cdot E_{j,k}^{-1} \cdot E_{i,j}^{-1} = ((E_{i,j}^{-1})^{-1} \cdot (E_{j,k}^{-1})^{-1} \cdot E_{i,j}^{-1} \cdot E_{j,k}^{-1})^{-1} = (E_{i,j} \cdot E_{j,k} \cdot E_{i,j}^{-1} \cdot E_{j,k}^{-1})^{-1} = [E_{i,j}, E_{j,k}]^{-1}$$

For example,  $n = 4$ ,

$$E_{1,2} \cdot E_{2,3} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
E_{1,2}^{-1} \cdot E_{2,3}^{-1} &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
[E_{1,2} \cdot E_{2,3}] &= E_{1,2} \cdot E_{2,3} \cdot E_{1,2}^{-1} \cdot E_{2,3}^{-1} = \\
&= \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{1,3}
\end{aligned}$$

**Exercise**  $[E_{i,j}, E_{l,k}] = I$ , where  $j \neq l$

**Proof**  $E_{i,j}$  is operating from left on  $E_{l,k}$  by addition of row  $j$  to row  $i$ , so, the product matrix,  $(a_{n,m} = E_{i,j} \cdot E_{l,k})$  has 1 on the main diagonal and in  $a_{l,k}, a_{i,j}$

$E_{i,j}^{-1}$  is operating from left on  $E_{l,k}^{-1}$  by subtraction of row  $j$  from row  $i$ , so, the product matrix,  $(b_{n,m} = E_{i,j}^{-1} \cdot E_{l,k}^{-1})$  has 1 on the main diagonal, and  $-1$  in  $b_{l,k}, b_{i,j}$

We can see that multiplying  $(a_{n,m})_i \cdot (b_{n,m})_j$  yields  $c_{i,j} = a_{i,i} \cdot b_{i,j} + a_{i,j} \cdot b_{j,j} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

And, we can see that multiplying  $(a_{n,m})_l \cdot (b_{n,m})_k$  yields  $c_{l,k} = a_{l,l} \cdot b_{l,k} + a_{l,k} \cdot b_{k,k} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

For example,  $n = 4$ ,

$$\begin{aligned}
E_{1,2} \cdot E_{3,4} &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
E_{1,2}^{-1} \cdot E_{3,4}^{-1} &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
[E_{1,2} \cdot E_{3,4}] &= E_{1,2} \cdot E_{3,4} \cdot E_{1,2}^{-1} \cdot E_{3,4}^{-1} = \\
&= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I
\end{aligned}$$

### Conclusion

$$\begin{aligned} [E_{i,j}, [E_{j,k}, E_{k,l}]] &= [E_{i,j}, E_{j,l}] = E_{i,l} \\ [E_{i,j}, [E_{j,k}, E_{m,l}]], m \neq k &= [E_{i,l}, I] = I \\ [E_{i,m}, [E_{j,k}, E_{k,l}]], m \neq j &= [E_{i,m}, E_{j,l}] = I \end{aligned}$$

$$\Rightarrow [E_{i_1, i_2}, [E_{i_3, i_4}, \dots [E_{i_{n-2}, i_{n-1}}, E_{i_{n-1}, i_n}]]] = \begin{cases} E_{i_1, i_n}, & i_{2k} = i_{2k+1}, \forall 1 \leq k \leq \frac{n}{2} - 1 \\ I, & \text{otherwise} \end{cases}$$

### Exercise

$$\#\{E_{i,j} \in M_n(\mathbb{Z})\}_{i < j} = \binom{n}{2}$$

### Proof

$(a_{i,j} = E_{i,j})$ . We need to count the options for 1 above the main diagonal.  
 $a_{l,l} = 1, \forall 1 \leq l \leq n$ , so, if  $i = l$ , we have  $n - l = n - i$  options to choose the column index  $j$ .

So, the total number of options for  $i, j$  is  $\sum_{k=1}^{n-1} = \frac{(1+n-1) \cdot (n-1)}{2} = \frac{n \cdot (n-1)}{2} = \binom{n}{2}$

This means that we have  $\binom{n}{2}^2$  commutators of the form  $[E_{i,j}, E_{l,k}]$ .

### Exercise

$$\#\{[E_{i,j}, E_{l,k}] \neq I \in M_n(\mathbb{Z})\}_{i < j} = 2 \cdot \binom{n}{3}$$

### Proof

As shown above,  $[E_{i,j}, E_{l,k}] \neq I \Leftrightarrow j = l$

Which means we're counting all the commutators of the form  $[E_{i,j}, E_{j,k}]$ .

So, the count of such commutators is based on the number of options to choose

ordered triples  $\{i, j, k\}$  out of the ordered set  $[n] = \{1, 2, \dots, n\}$ , which is  $\binom{n}{3}$   
 But, as already shown above,  $[E_{l,k}, E_{i,j}] = [E_{i,j}, E_{l,k}]^{-1}$ , so, for each triple  $\{i, j, k\}$ , we have two commutators,  $[E_{i,j}, E_{j,k}]$  and its inverse, which sum up to  $\binom{n}{3}$  pairs of commutators.

For example,  $n = 5$ ,

$$(a_{l,k}) = E_{i,j} = \begin{pmatrix} 1 & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ 0 & 1 & a_{2,3} & a_{2,4} & a_{2,5} \\ 0 & 0 & 1 & a_{3,4} & a_{3,5} \\ 0 & 0 & 0 & 1 & a_{4,5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Where  $a_{i,j} = 1$ , and all other  $a_{i,k} = 0$

The number of options for choosing  $i, j$ , in this case, are  $1 + 2 + 3 + 4 = 10 = \binom{5}{2}$ ,

so, we have  $10^2 = 100$  commutators. The number of triples we can choose from  $[5] = \{1, 2, 3, 4, 5\}$  is

$\#\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\} = 10 = \binom{5}{3}$ ,

so we have 10 commutators that are not the unit matrix, and their inverse, total  $20 = 2 \cdot 10 = 2 \cdot \binom{5}{3}$ .

### Exercise

Given the set of commutators of elementary matrices of the form

$\{[E_{i,j}, E_{j,k}] \in M_n(\mathbb{Z})\}_{i < j < k}$ ,

we can divide this set to subsets of the form

$\{[E_{i_1, j_{1,1}}, E_{j_{1,1}, k_1}], [E_{i_1, j_{1,2}}, E_{j_{1,2}, k_1}], \dots, [E_{i_1, j_{1, l_1}}, E_{j_{1, l_1}, k_1}]\}, \dots,$   
 $\{[E_{i_m, j_{m,1}}, E_{j_{m,1}, k_1}], \dots, [E_{i_m, j_{m, l_m}}, E_{j_{m, l_m}, k_1}]\}$

These subsets are equivalence classes, trivially, since the relation is equation (i.e.  $[E_{i_l, j_{l, m_1}}, E_{j_{l, m_1}, k_l}] = [E_{i_l, j_{l, m_2}}, E_{j_{l, m_2}, k_l}], i_l < j_{l, m_1}, j_{l, m_2} < k_l$ ).