Exercise Let $\{E_{i,j}\}_{i<j}$ be the set of all elementary matrices, of this form. Prove that $E_{i,j}^{-1} = (b_{l,k})$ is $E_{i,j} = (a_{l,k})$, when we substitute $a_{i,j} = 1$ with $b_{i,j} = -1$

Proof We can see that directly from the fact that if we multiply $E_{i,j}^{-1}$ by $E_{i,j}$ from the left then $E_{i,j}$ is operating on $E_{i,j}^{-1}$ by adding row j to row i
So, in the product matrix, $(c_{l,k})$, in order to have 1
on the main diagonal, we need them to exist on the main diagonal of $E_{i,j}^{-1}$, to begin with. Now, in order to have $c_{i,j} = 0$, we need to have the addition of j to i giving $c_{i,j} = a_{i,j} + b_{i,j} = 0 \Rightarrow b_{i,j} = -a_{i,j} = -1$

Exercise Prove that if $(a_{ij}) = E_{i,j}, i < j$ is an elementary matrix,
then $\forall m \in (N), E_{i,j}^m$ is $E_{i,j}$, but with $a_{ij} = m$

$$E_{i,j} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$E_{i,j}^2 = E_{i,j} \cdot E_{i,j}$$

Since $E_{i,j}$ is en elemntary matrix, then it operates on the right matrix as an addition of row j to row i

So,

$$E_{i,j}^2 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 2 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

We assume this is true for all $E_{i,j}^m$, now we prove for $E_{i,j}^{m+1}$

$$E_{i,j}^{m+1} = E_{i,j} \cdot E_{i,j}^m = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}^m$$

(by the assumption)

$$= \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & m & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & m+1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Commutators of elementary matrices

Let $\{E_{i,j}\}_{i<j}$ be the set of all elementary matrices of this form.

Exercise $(a_{l,m}) = E_{i,j}^{-1}$ is the matrix with 1 on the main diagonal, and -1 in $a_{i,j}$

Proof We can see that directly from the fact that in order to have
$(c_{l,m}) = (a_{l,m}) \cdot (b_{l,m}) = E_{i,j} \cdot E_{i,j}^{-1} = I$,
we need to have $c_{i,j} = 0$, which means that adding row j to row i, in $E_{i,j}^{-1}$ (by the left multiplication of $E_{i,j}$)
must give $a_{i,j} + b_{i,j} = c_{i,j} = 0 \Rightarrow b_{i,j} = -a_{i,j} = -1$

Exercise $[E_{i,j}, E_{j,k}] = E_{i,k}$

Proof $E_{i,j}$ is operating from left on $E_{j,k}$ by addition of row j to row i, so, the product matrix, $(a_{l,m} = E_{i,j} \cdot E_{j,k}$ has 1 on the main diagonal and in $a_{j,k}, a_{i,j}, a_{i,k}$
$E_{i,j}^{-1}$ is operating from left on $E_{j,k}^{-1}$ by subtraction of row j from row i, so, the product matrix, $(b_{l,m} = E_{i,j}^{-1} \cdot E_{j,k}^{-1}$ has 1 on the main diagonal and in $b_{i,k}$, and -1 in $b_{j,k}, b_{i,j}$
Multiplying $(a_{l,m} \cdot (b_{l,m})$ yields a product matrix, $(c_{l,m}$ with 1 on the main diagonal, and,
since $a_{i,i} = a_{i,j} = a_{i,k} = 1$, with all other cells in row j being 0, and since $b_{i,k} = b_{k,k} = 1$, and $b_{j,k} = -1$, multiplying row $(a_{l,m})_i$ by column $(b_{l,m})_k$ yields the value $c_{i,k} = b_{i,k} + b_{j,k} + b_{k,k} = 1 - 1 + 1 = 1$
We can see that multiplying $(a_{l,m})_i \cdot (b_{l,m})_j$ yields $c_{i,j} = a_{i,i} \cdot b_{i,j} + a_{i,j} \cdot b_{j,j} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$
And, we can see that multiplying $(a_{l,m})_j \cdot (b_{l,m})_k$ yields $c_{j,k} = a_{j,j} \cdot b_{j,k} + a_{j,k} \cdot b_{k,k} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

For example, $n = 4$,

$$E_{1,2} \cdot E_{2,3} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_{1,2}^{-1} \cdot E_{2,3}^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[E_{1,2} \cdot E_{2,3}] = E_{1,2} \cdot E_{2,3} \cdot E_{1,2}^{-1} \cdot E_{2,3}^{-1} =$$

$$= \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{1,3}$$

Exercise $[E_{i,j}, E_{l,k}] = I$, where $j \neq l$

Proof $E_{i,j}$ is operating from left on $E_{l,k}$ by addition of row j to row i, so, the product matrix, $(a_{n,m} = E_{i,j} \cdot E_{l,k}$ has 1 on the main diagonal and in $a_{l,k}, a_{i,j}$

$E_{i,j}^{-1}$ is operating from left on $E_{l,k}^{-1}$ by subtraction of row j from row i, so, the product matrix, $(b_{n,m} = E_{i,j}^{-1} \cdot E_{l,k}^{-1}$ has 1 on the main diagonal,

and -1 in $b_{l,k}, b_{i,j}$

We can see that multiplying $(a_{n,m})_i \cdot (b_{n,m})_j$ yields $c_{i,j} = a_{i,i} \cdot b_{i,j} + a_{i,j} \cdot b_{j,j} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

And, we can see that multiplying $(a_{n,m})_l \cdot (b_{n,m})_k$ yields $c_{l,k} = a_{l,l} \cdot b_{l,k} + a_{l,k} \cdot b_{k,k} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

For example, $n = 4$,

$$E_{1,2} \cdot E_{3,4} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E_{1,2}^{-1} \cdot E_{3,4}^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[E_{1,2} \cdot E_{3,4}] = E_{1,2} \cdot E_{3,4} \cdot E_{1,2}^{-1} \cdot E_{3,4}^{-1} =$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

Conclusion

$[E_{i,j},[E_{j,k},E_{k,l}]]=[E_{i,j},E_{j,l}]=E_{i,l}$
$[E_{i,j},[E_{j,k},E_{m,l}]],m\neq k=[E_{i,l},I]=I$
$[E_{i,m},[E_{j,k},E_{k,l}]],m\neq j=[E_{i,m},E_{j,l}]=I$

$$\Rightarrow [E_{i_1,i_2},[E_{i_3,i_4},...[E_{i_{n-2},i_{n-1}},E_{i_{n-1},i_n}]]=\begin{cases}E_{i_1,i_n}, & i_{2k}=i_{2k+1},\forall 1\leq k\leq \frac{n}{2}-1\\ I, & \text{otherwise}\end{cases}$$