

Exercise Let $\{E_{i,j}\}_{i < j}$ be the set of all elementary matrices, of this form. Prove that $E_{i,j}^{-1} = (b_{l,k})$ is $E_{i,j} = (a_{l,k})$, when we substitute $a_{i,j} = 1$ with $b_{i,j} = -1$

Proof We can see that directly from the fact that if we multiply $E_{i,j}^{-1}$ by $E_{i,j}$ from the left then $E_{i,j}$ is operating on $E_{i,j}^{-1}$ by adding row j to row i . So, in the product matrix, $(c_{l,k})$, in order to have 1 on the main diagonal, we need them to exist on the main diagonal of $E_{i,j}^{-1}$, to begin with. Now, in order to have $c_{i,j} = 0$, we need to have the addition of j to i giving $c_{i,j} = a_{i,j} + b_{i,j} = 0 \Rightarrow b_{i,j} = -a_{i,j} = -1$

Exercise Prove that if $(a_{ij}) = E_{i,j}$, $i < j$ is an elementary matrix, then $\forall m \in (N)$, $E_{i,j}^m$ is $E_{i,j}$, but with $a_{ij} = m$

$$E_{i,j} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$E_{i,j}^2 = E_{i,j} \cdot E_{i,j}$$

Since $E_{i,j}$ is an elementary matrix, then it operates on the right matrix as an addition of row j to row i

So,

$$E_{i,j}^2 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 2 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

We assume this is true for all $E_{i,j}^m$, now we prove for $E_{i,j}^{m+1}$

$$E_{i,j}^{m+1} = E_{i,j} \cdot E_{i,j}^m = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}^m$$

(by the assumption)

$$= \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & m & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & m+1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Commutators of elementary matrices

Let $\{E_{i,j}\}_{i < j}$ be the set of all elementary matrices of this form.

Exercise $(a_{l,m}) = E_{i,j}^{-1}$ is the matrix with 1 on the main diagonal, and -1 in $a_{i,j}$

Proof We can see that directly from the fact that in order to have

$$(c_{l,m}) = (a_{l,m}) \cdot (b_{l,m}) = E_{i,j} \cdot E_{i,j}^{-1} = I,$$

we need to have $c_{i,j} = 0$, which means that adding row j to row i , in $E_{i,j}^{-1}$ (by the left multiplication of $E_{i,j}$)

must give $a_{i,j} + b_{i,j} = c_{i,j} = 0 \Rightarrow b_{i,j} = -a_{i,j} = -1$

Exercise $[E_{i,j}, E_{j,k}] = E_{i,k}$

Proof $E_{i,j}$ is operating from left on $E_{j,k}$ by addition of row j to row i , so, the product matrix, $(a_{l,m} = E_{i,j} \cdot E_{j,k})$ has 1 on the main diagonal and in $a_{j,k}, a_{i,j}, a_{i,k}$

$E_{i,j}^{-1}$ is operating from left on $E_{j,k}^{-1}$ by subtraction of row j from row i , so, the product matrix, $(b_{l,m} = E_{i,j}^{-1} \cdot E_{j,k}^{-1})$ has 1 on the main diagonal and in $b_{i,k}$, and -1 in $b_{j,k}, b_{i,j}$

Multiplying $(a_{l,m} \cdot (b_{l,m}))$ yields a product matrix, $(c_{l,m})$ with 1 on the main diagonal, and,

since $a_{i,i} = a_{i,j} = a_{i,k} = 1$, with all other cells in row j being 0, and since $b_{i,k} = b_{j,k} = 1$, and $b_{j,k} = -1$, multiplying row $(a_{l,m})_i$ by column $(b_{l,m})_k$ yields the value $c_{i,k} = b_{i,k} + b_{j,k} + b_{k,k} = 1 - 1 + 1 = 1$

We can see that multiplying $(a_{l,m})_i \cdot (b_{l,m})_j$ yields $c_{i,j} = a_{i,i} \cdot b_{i,j} + a_{i,j} \cdot b_{j,j} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

And, we can see that multiplying $(a_{l,m})_j \cdot (b_{l,m})_k$ yields $c_{j,k} = a_{j,j} \cdot b_{j,k} + a_{j,k} \cdot b_{k,k} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

Conclusion

$$[E_{j,k}, E_{i,j}] = E_{j,k} \cdot E_{i,j} \cdot E_{j,k}^{-1} \cdot E_{i,j}^{-1} = ((E_{i,j}^{-1})^{-1} \cdot (E_{j,k}^{-1})^{-1} \cdot E_{i,j}^{-1} \cdot E_{j,k}^{-1})^{-1} = (E_{i,j} \cdot E_{j,k} \cdot E_{i,j}^{-1} \cdot E_{j,k}^{-1})^{-1} = [E_{i,j}, E_{j,k}]^{-1}$$

For example, $n = 4$,

$$E_{1,2} \cdot E_{2,3} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
E_{1,2}^{-1} \cdot E_{2,3}^{-1} &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
[E_{1,2} \cdot E_{2,3}] &= E_{1,2} \cdot E_{2,3} \cdot E_{1,2}^{-1} \cdot E_{2,3}^{-1} = \\
&= \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{1,3}
\end{aligned}$$

Exercise $[E_{i,j}, E_{l,k}] = I$, where $j \neq l$

Proof $E_{i,j}$ is operating from left on $E_{l,k}$ by addition of row j to row i , so, the product matrix, $(a_{n,m} = E_{i,j} \cdot E_{l,k})$ has 1 on the main diagonal and in $a_{l,k}, a_{i,j}$

$E_{i,j}^{-1}$ is operating from left on $E_{l,k}^{-1}$ by subtraction of row j from row i , so, the product matrix, $(b_{n,m} = E_{i,j}^{-1} \cdot E_{l,k}^{-1})$ has 1 on the main diagonal, and -1 in $b_{l,k}, b_{i,j}$

We can see that multiplying $(a_{n,m})_i \cdot (b_{n,m})_j$ yields $c_{i,j} = a_{i,i} \cdot b_{i,j} + a_{i,j} \cdot b_{j,j} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

And, we can see that multiplying $(a_{n,m})_l \cdot (b_{n,m})_k$ yields $c_{l,k} = a_{l,l} \cdot b_{l,k} + a_{l,k} \cdot b_{k,k} = 1 \cdot -1 + 1 \cdot 1 = 1 - 1 = 0$

For example, $n = 4$,

$$\begin{aligned}
E_{1,2} \cdot E_{3,4} &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
E_{1,2}^{-1} \cdot E_{3,4}^{-1} &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
[E_{1,2} \cdot E_{3,4}] &= E_{1,2} \cdot E_{3,4} \cdot E_{1,2}^{-1} \cdot E_{3,4}^{-1} = \\
&= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I
\end{aligned}$$

Conclusion

$$\begin{aligned}
[E_{i,j}, [E_{j,k}, E_{k,l}]] &= [E_{i,j}, E_{j,l}] = E_{i,l} \\
[E_{i,j}, [E_{j,k}, E_{m,l}]], m \neq k &= [E_{i,l}, I] = I \\
[E_{i,m}, [E_{j,k}, E_{k,l}]], m \neq j &= [E_{i,m}, E_{j,l}] = I
\end{aligned}$$

$$\Rightarrow [E_{i_1, i_2}, [E_{i_3, i_4}, \dots [E_{i_{n-2}, i_{n-1}}, E_{i_{n-1}, i_n}]]] = \begin{cases} E_{i_1, i_n}, & i_{2k} = i_{2k+1}, \forall 1 \leq k \leq \frac{n}{2} - 1 \\ I, & \text{otherwise} \end{cases}$$